

PARABOLIC SUSPENSION FLOWS

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ABSTRACT. This paper is devoted to the study of thermodynamic formalism for suspension flows over countable Markov shifts with roof functions not necessarily bounded away from zero. This models, for example, billiards with cusps. We establish conditions to ensure the existence of equilibrium measures for regular potentials. We define the notions of recurrence and transience of a potential in this setting. We define the so called *renewal flow*, which is a symbolic model for flows in the boundary of hyperbolicity. We study the corresponding thermodynamic formalism establishing conditions for the existence of equilibrium measures and phase transitions. Finally, applications are given to suspension flows defined over interval maps having parabolic fixed points.

1. INTRODUCTION

In this paper we study suspension flows, that is a discrete dynamical system on the ‘base’ along with a ‘roof’ function which determines the time the flow takes to return to this base. In particular we consider suspension flows over Markov shifts. The ergodic theory of suspension flows with a Markov structure has been studied extensively in the context of Axiom A flows [Bo, BR, Ra], geodesic flows on surfaces of negative curvature [Mo, Ar, Se] (both finite Markov shift cases), and billiard flows [BS1, BS2, BChS1]. The main novelty here, aside from the facts that we develop a more general theory of thermodynamic formalism than in the works above and that we study countable Markov shifts, is that we do not assume that the roof function is uniformly bounded away from zero, which leads to significant technical difficulties. We call such a system a *parabolic suspension flow*: where the roof function tends to zero we see a *cusp*. For example, such a system was considered in the billiard case in [BM], where the interest was in statistical properties of the physical measure.

Thermodynamic formalism for suspension flows over countable Markov shifts began with the work of Savchenko [Sav]. He gave a definition of topological entropy in the case that the roof function depends only on the first coordinate, but it is not necessarily bounded away from zero. Barreira and Iommi [BI1] proposed a definition of topological pressure when the roof function is bounded away from zero and established the Variational Principle in this case. Recently, Kempton [Ke] and independently Jaerisch, Kesseböhmer and Lamei [JKL] gave a definition of

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pressure in the case that the roof function is not necessarily bounded away from zero. As opposed to the work in [BI1, Sav] this definition is not given implicitly. The regularity of the pressure function for this type of flow (with roof function bounded away from zero) was studied by Iommi and Jordan [IJ]. Conditions in order for the pressure to be real analytic or to exhibit phase transitions were found.

To develop the theory in the parabolic case, we consider conditions which guarantee that measures which maximise ‘free energy’, equilibrium measures, exist (Theorem 3.3). There are several difficulties when considering this problem. To start with, the phase space is not compact, therefore the classical functional analytic approach can not be used directly. Moreover, there is no bijection between the set of invariant measures for the flow and the corresponding one for the base map. This prevents us from reducing the study of the thermodynamic formalism for the flow to that of the shift (this was the strategy used by Bowen and Ruelle [BR] in the compact setting). We also extend to this continuous time setting the definitions of recurrence and transience of a potential introduced in the discrete time by Sarig [Sa1]. We prove that despite the difference in the setting, our definitions imply the same ergodic properties obtained in the shift case.

In order to give a relatively straightforward family of examples exhibiting the different types of behaviour in the presence of a cusp, we define and study in some detail the so called *renewal flow*. This is a suspension flow with base map the renewal shift (see Section 6 for a precise definition). It provides a symbolic model for suspension flows defined over the Manneville Pomeau [MP] map having a cusp. We study the corresponding thermodynamic formalism establishing conditions for the existence of equilibrium measures and phase transitions. As pointed out by Chernov and Markarian [ChM2, p.728], the ergodic theory and the statistical properties dispersing billiards with cusps seems to be similar to that of expanding interval maps with parabolic fixed points. The renewal flow seems to be a good testing ground for these ideas.

Returning to the broad motivations for our work, the suspension flows here provide models for various non-uniformly hyperbolic flows where the thermodynamic formalism is not well developed. These are systems which behave like Axiom A systems in most of the phase space, but not in all of it. It is possible for these systems to exhibit pathological behaviour in small parts of the domain. Interest in these systems is partially due to the novel dynamical features they exhibit (see for example the statistical laws and rates of decay of correlation that can occur in [FMT, MT, M1, BM, M2]). Also, these systems have great importance in the program aimed at obtaining a global description of the space of dynamical systems (see [BDV]). While these systems still preserve some of the good properties of Axiom A (uniformly hyperbolic) systems, this is not enough to retain their regular dynamical properties. Suspension flows over countable Markov shifts serve as symbolic models for some of these flows. For example, Bufetov and Gurevich [BG] and Hamenstädt [Ha] have coded Teichmüller flows in this way and have used this symbolic representation to prove the uniqueness of the measure of maximal entropy. Another classical example of a flow that is modeled by this type of suspension flows is the geodesic flow over the modular surface (see [Ar]). Recent results by Sarig

[Sa5] suggest that this should also be the case for a wide range of non-uniformly hyperbolic flows.

Finally we describe the layout of this paper. In Section 2 we give the necessary definitions and results from the setting of countable Markov shifts; we also introduce suspension flows over countable shifts and the notion of topological entropy for these flows. Section 3 begins with the definition of pressure for these flows, which has been introduced in [Sav, BI1, Ke, JKL]. We then state and prove our first result, Theorem 3.3 which characterises the potentials for which there exists an equilibrium measure. Section 4 looks at inducing to a full shift and how the pressure for the induced potential can be related to the pressure for the original potential. In Section 5 we define the notions of recurrence and transience for suspension flows and relate these notions to the thermodynamics formalism for the shift map, proving a Ruelle-Perron-Frobenius-type theorem, Theorem 5.1. The specific case of the renewal shift is studied in Section 6 and in particular we look at the existence of equilibrium measures (including the existence of measures of maximal entropy) and phase transitions for the pressure function. Finally in Section 7 we apply our results to the setting where the base map f is the non-uniformly expanding Manneville-Pomeau map and the roof function is $\log |f'|$.

2. PRELIMINARIES

In this section we collect all the definitions and results for countable Markov shifts and for suspension flows over countable Markov shifts that will be used in the latter sections. We will also take the opportunity to fix some notation:

Notation: Given sequences $(A_n)_n, (B_n)_n \subset [-\infty, \infty]$, we write $A_n \asymp B_n$ if there exists a constant $C > 0$ such that $\frac{1}{C} \leq \frac{A_n}{B_n} \leq C$ for all $n \in \mathbb{N}$. We will use the same notation if the ‘sequences’ are simply constants, which will be the case when we are determining if a quantity is finite or not.

2.1. Recurrence, Transience and Thermodynamic formalism for countable Markov shifts. Here we recall some results, mostly due to Sarig, that will be of use in the following sections. We note that Mauldin and Urbański also developed a theory in this context [MU1, MU2, MU3]. However, the combinatorial restrictions they impose in the shifts are too strong for what we will require here.

Let B be a transition matrix defined on the alphabet of natural numbers. That is, the entries of the matrix $B = B(i, j)_{\mathbb{N}_0 \times \mathbb{N}_0}$ are zeros and ones (with no row and no column made entirely of zeros). The countable Markov shift (Σ, σ) is the set

$$\Sigma := \{(x_n)_{n \in \mathbb{N}_0} : B(x_n, x_{n+1}) = 1 \text{ for every } n \in \mathbb{N}_0\},$$

together with the shift map $\sigma : \Sigma \rightarrow \Sigma$ defined by $\sigma(x_0, x_1, \dots) = (x_1, x_2, \dots)$. We will always assume that this system is *topologically mixing*, i.e., for each pair $x, y \in \mathbb{N}_0$, there exists $N \in \mathbb{N}$ such that for every $n > N$ there is a word $(i_0, \dots, i_{n-1}) \in \mathbb{N}_0^n$ such that $i_0 = x, i_{n-1} = y$ and $B(i_k, i_{k+1}) = 1$ for all $0 \leq k \leq n-2$.

Let $C_{i_0 \dots i_{n-1}}$ denote the n -cylinder consisting of all sequences $(x_0, x_1, \dots)$ where $x_k = i_k$ for $0 \leq k \leq n-1$. The n -th variation of $\phi : \Sigma \rightarrow \mathbb{R}$ is defined by

$$V_n(\phi) := \sup\{|\phi(x) - \phi(y)| : x, y \in \Sigma, x_i = y_i, 0 \leq i \leq n-1\}.$$

We say that ϕ is of *summable variations* if $\sum_{n=1}^{\infty} V_n(\phi) < \infty$. We say that it is *locally Hölder* (with parameter θ) if there exists $\theta \in (0, 1)$ such that for all $n \geq 1$ we have $V_n(\phi) \leq O(\theta^n)$. The first return time map to C_i is defined by

$$r_i(x) := \mathbb{1}_{C_i}(x) \inf\{n \geq 1 : \sigma^n x \in C_i\}, \quad (1)$$

where $\mathbb{1}_{C_i}$ is the indicator function of the cylinder C_i . Let

$$X_n^i := \{x \in \Sigma : r_i(x) = n\}.$$

When it is clear what i is, we will often drop the superscript. Given ϕ a potential of summable variations and a 1-cylinder C_i , we define the partition functions

$$Z_n(\phi, C_i) := \sum_{\sigma^n x = x} \exp\left(\sum_{k=0}^{n-1} \phi(\sigma^k x)\right) \mathbb{1}_{C_i}(x),$$

and

$$Z_n^*(\phi, C_i) := \sum_{\sigma^n x = x} \exp\left(\sum_{k=0}^{n-1} \phi(\sigma^k x)\right) \mathbb{1}_{X_n^i}(x).$$

The *Gurevich Pressure* of ϕ was introduced by Sarig in [Sa1], generalising previous results by Gurevich [Gu1, Gu2]. It is defined by

$$P(\phi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log Z(\phi, C_{i_0}).$$

If the system is topologically mixing then its value does not depend on the cylinder C_{i_0} considered. The Gurevich pressure is convex and if $\mathcal{K} := \{K \subset \Sigma : K \text{ compact and } \sigma\text{-invariant}, K \neq \emptyset\}$ then

$$P(\phi) = \sup\{P(\phi|K) : K \in \mathcal{K}\}, \quad (2)$$

where $P(\phi|K)$ is the topological pressure of ϕ restricted to the compact set K (for definition and properties see [Wa, Chapter 9]). Moreover, this notion of pressure satisfies the Variational Principle (see [Sa1]):

Theorem 2.1. *Let (Σ, σ) be a countable Markov shift and $\phi : \Sigma \rightarrow \mathbb{R}$ be a function of summable variations, then*

$$P_\sigma(\phi) = \sup\left\{h(\nu) + \int \phi d\nu : \nu \in \mathcal{M}_\sigma \text{ and } -\int \phi d\nu < \infty\right\},$$

where $\mathcal{M}_\sigma$ denotes the set of σ -invariant probability measures and $h(\nu)$ denotes the entropy of the measure ν (for a precise definition see [Wa, Chapter 4]).

The quantity $h(\nu) + \int \phi d\nu$ is sometimes called the *free energy* w.r.t. ϕ , see [K]. A measure $\nu \in \mathcal{M}_\sigma$ attaining the supremum of the free energies, that is

$$P_\sigma(\phi) = h(\nu) + \int \phi d\nu, \quad (3)$$

is called an *equilibrium measure* for ϕ . Buzzi and Sarig [BS] proved that a potential of summable variations has at most one equilibrium measure.

Under certain combinatorial assumptions on the shift (for example if (Σ, σ) is a full-shift) the equilibrium measure also satisfies the Gibbs property. We define this property here for general measures (i.e., not necessarily equilibrium measures).

Definition 2.1. *We say that μ is a Gibbs measure for ϕ if there exist $K, P \in \mathbb{R}$ such that for every $n \geq 1$, given an n -cylinder $C_{i_0 \dots i_{n-1}}$,*

$$\frac{1}{K} \leq \frac{\mu(C_{i_0 \dots i_{n-1}})}{e^{S_n \phi(x) - nP}} \leq K$$

for any $x \in C_{i_0 \dots i_{n-1}}$.

Note that we will usually have $P = P_\sigma(\phi)$, for example in the full shift example mentioned above.

Potentials can be classified according to their recurrence properties as follows (see [Sa1, Sa2]); note that by topological mixing and summable variations, these definitions are independent of the choice of cylinder C_i .

Definition 2.2. *Let ϕ be a potential of summable variations with finite Gurevich pressure $P_\sigma(\phi) = \log \lambda$. We say that ϕ is*

- recurrent if

$$\sum_{n \geq 1} \lambda^{-n} Z_n(\phi, C_i) = \infty,$$

and transient otherwise. Moreover, we say that ϕ is

- positive recurrent if it is recurrent and

$$\sum_{n \geq 1} n \lambda^{-n} Z_n^*(\phi, C_i) < \infty;$$

- null recurrent if it is recurrent and

$$\sum_{n \geq 1} n \lambda^{-n} Z_n^*(\phi, C_i) = \infty.$$

Consider the Ruelle operator defined formally in some space of functions by:

$$L_\phi g(x) := \sum_{\sigma y = x} \exp(\phi(y)) g(y).$$

Sarig [Sa2] generalises the Ruelle Perron Frobenius Theorem to countable Markov shifts.

Theorem 2.2 (RPF Theorem). *Let (Σ, σ) be a countable Markov shift and ϕ a potential of summable variations of finite Gurevich pressure $\log \lambda$. If ϕ is*

1. positive recurrent then there exists a conservative conformal measure m and a continuous function h such that $L_\phi^* m = \lambda m$, $L_\phi h = \lambda h$ and $\int h dm < \infty$;
2. null recurrent then there exists a conservative conformal measure m and a continuous function h such that $L_\phi^* m = \lambda m$, $L_\phi h = \lambda h$ and $\int h dm = \infty$;
3. transient then there is no conservative conformal measure.

In the recurrent case, we call the measure $\nu = hdm$ the *Ruelle-Perron-Frobenius (RPF)* measure of ϕ . In the case where ϕ is positive recurrent, the total measure of the space is finite, and we rescale to make ν a probability measure. This is an equilibrium measure for ϕ provided $-\int \phi d\nu < \infty$.

The following result was established in [Sa2, Theorem 2] and provides a version of the Variational Principle for infinite invariant measures.

Theorem 2.3. *Let (Σ, σ) be a countable Markov shift and $\phi: \Sigma \rightarrow \mathbb{R}$ be a recurrent potential of summable variations and of finite Gurevich pressure.*

1. *For every conservative ergodic invariant measure ν which is finite on cylinders, if $\int (P_\sigma(\phi) - \phi) d\nu < \infty$ then $h(\nu) \leq \int (P_\sigma(\phi) - \phi) d\nu$.*
2. *Let h and m be the density and the conformal measure provided by the Ruelle Peron Frobenius Theorem and $\nu = hm$. If $\int (P_\sigma(\phi) - \phi) d\nu < \infty$ then $h(\nu) = \int (P_\sigma(\phi) - \phi) d\nu$.*

2.2. Suspension semi-flows. Let (Σ, σ) be a countable Markov shift and $\tau: \Sigma \rightarrow \mathbb{R}^+$ be a positive continuous function such that for every $x \in \Sigma$ we have

$$\sum_{i=0}^{\infty} \tau(\sigma^i x) = \infty. \quad (4)$$

Consider the space

$$Y = \{(x, t) \in \Sigma \times \mathbb{R}: 0 \leq t \leq \tau(x)\} / \sim$$

where $(x, \tau(x)) \sim (\sigma(x), 0)$ for each $x \in \Sigma$.

The *suspension semi-flow* over σ with *roof function* τ is the semi-flow $\Phi = (\varphi_t)_{t \geq 0}$ on Y defined by

$$\varphi_t(x, s) = (x, s + t) \text{ whenever } s + t \in [0, \tau(x)].$$

In particular,

$$\varphi_{\tau(x)}(x, 0) = (\sigma(x), 0).$$

In the case of two-sided Markov shifts we can define a suspension flow $(\varphi_t)_{t \in \mathbb{R}}$ in a similar manner.

Remark 2.1. *Note that the condition assumed in equation (4) implies that the flow is well defined for every positive time $t > 0$. It is possible to drop this assumption and to consider flows for which some orbits hit the singularity in finite time. In that case, dissipative measures should also be considered (see Section 2.3 for more details). The assumption given in (4) can also be understood in the context of billiards. Indeed, the trajectory of a particle in a billiard table is defined for every $t \in \mathbb{R}$ unless the particle hits a corner or the sequence of collision times $(t_n)_n$ has an accumulation point in $\mathbb{R}$. The issue of accumulation points for the collision times has been studied in certain detail and corresponds exactly to the case of $\sum_{n \geq 0} \tau(\sigma^n x) < \infty$. Suppose that the collision times have an accumulation point and that the particle converges to a corner, then it must be a cusp [ChM1, Section 2.4]. Moreover, it must be a cusp with one side of negative curvature and other of positive curvature. To our knowledge, there are no fully developed examples with this phenomenon. On the other hand, if the particle accumulates in a regular point*

of the boundary then it can not lie in a flat or dispersing component. Moreover, it was shown by Halpern [Hal], that such type of collision is impossible on any wall that is focusing with a bounded third derivative and nowhere vanishing curvature.

2.3. Invariant measures. In this section we discuss the relation between invariant measures for the flow and invariant measures for the base map.

Definition 2.3. A probability measure μ on Y is Φ -invariant if $\mu(\varphi_t^{-1}A) = \mu(A)$ for every $t \geq 0$ and every measurable set $A \subset Y$. Denote by $\mathcal{M}_\Phi$ the space of Φ -invariant probability measures on Y .

The space $\mathcal{M}_\Phi$ is closely related to the space $\mathcal{M}_\sigma$ of σ -invariant probability measures on Σ . Let us consider the space of σ -invariant measures for which τ is integrable,

$$\mathcal{M}_\sigma(\tau) := \left\{ \mu \in \mathcal{M}_\sigma : \int \tau d\mu < \infty \right\}. \quad (5)$$

Let m denote one dimensional Lebesgue measure. As the flow direction is one-dimensional and m is the unique measure which is invariant under all translations, it follows that a Φ -invariant probability measure will be of the form $C\mu \times m$ for $\mu \in \mathcal{M}_\sigma(\tau)$ and some $C > 0$. Indeed, it follows directly from classical results by Ambrose and Kakutani [AK] that

$$\frac{(\mu \times m)|_Y}{(\mu \times m)(Y)} \in \mathcal{M}_\Phi.$$

This suggests that the study of the map $R: \mathcal{M}_\sigma \rightarrow \mathcal{M}_\Phi$, defined by

$$R(\mu) = \frac{(\mu \times m)|_Y}{(\mu \times m)(Y)} \quad (6)$$

should allow for the translation of problems from the flow onto the shift map. This is indeed the case:

1. When (Σ, σ) is a sub-shift of finite type defined over a finite alphabet, a compact case, the map R is a bijection.
2. If (Σ, σ) is a countable Markov shift and $\tau: \Sigma \rightarrow \mathbb{R}$ is not bounded above then it is possible for there to be a measure $\nu \in \mathcal{M}_\sigma \setminus \mathcal{M}_\sigma(\tau)$, i.e., such that $\int \tau d\nu = \infty$. In this situation the measure $\nu \times m$ is an infinite invariant measure for Φ . Hence, the map $R(\cdot)$ is not well defined. Nevertheless, it follows directly from the results by Ambrose and Kakutani [AK] that if τ is uniformly bounded away from zero then the map $R: \mathcal{M}_\sigma(\tau) \rightarrow \mathcal{M}_\Phi$ is bijective.
3. If (Σ, σ) is a countable Markov shift and $\tau: \Sigma \rightarrow \mathbb{R}$ is not bounded away from zero then it is possible (see Section 6.3) that for an infinite (sigma-finite) σ -invariant measure ν we have $\int \tau d\nu < \infty$. In this case the measure $(\nu \times m)|_Y / (\nu \times m)(Y) \in \mathcal{M}_\Phi$. In such a situation, the map R is not surjective.

The situation in the first two cases is somehow simpler since every measure in $\mathcal{M}_\Phi$ can be written as $\nu \times m$, where $\nu \in \mathcal{M}_\sigma$. Therefore, the ergodic properties of the flow can be reduced to the ergodic properties of the base. If the roof function is not bounded away from zero this is no longer the case. However:

Remark 2.2. *The time assumption given by equation (4) implies by a result of Hopf (see [Aa, Proposition 1.1.6] and [H]) that every sigma-finite measure $\nu \in \mathcal{M}_\sigma$ defined on the base such that $\int \tau d\nu < \infty$ is conservative.*

Given a continuous function $g: Y \rightarrow \mathbb{R}$ we define the function $\Delta_g: \Sigma \rightarrow \mathbb{R}$ by

$$\Delta_g(x) = \int_0^{\tau(x)} g(x, t) dt.$$

The function Δ_g is also continuous, moreover if $\mu \in \mathcal{M}_\Phi$ is the normalisation of $\nu \times m$ (note ν can be an infinite measure, as long as the $\int \tau d\nu < \infty$) then

$$\int_Y g dR(\nu) = \frac{\int_\Sigma \Delta_g d\nu}{\int_\Sigma \tau d\nu}. \quad (7)$$

Remark 2.3 (Extension of potentials defined on the base). *Let $\phi: \Sigma \rightarrow \mathbb{R}$ be a locally Hölder potential. It is shown in [BRW] that there exists a continuous function $g: Y \rightarrow \mathbb{R}$ such that $\Delta_g = \phi$.*

2.4. Abramov's formula. The entropy of a flow with respect to an invariant measure, denoted $h_\Phi(\mu)$, can be defined as the entropy of the corresponding time one map. The following classical result obtained by Abramov [Ab] relates the entropy of a measure for the flow with the entropy of a measure for the base map.

Proposition 2.1 (Abramov). *Let $\mu \in \mathcal{M}_\Phi$ be such that $\mu = (\nu \times m)|_Y / (\nu \times m)(Y)$, where $\nu \in \mathcal{M}_\sigma$ then*

$$h_\Phi(\mu) = \frac{h_\sigma(\nu)}{\int \tau d\nu}. \quad (8)$$

The result of Abramov holds for any suspension flow with positive (not necessarily bounded away from zero) roof function and for any invariant (not necessarily ergodic) finite measure for the flow μ that can be written as $\mu = \nu \times m$, where ν is an invariant probability measure for the base with $\int \tau d\nu < \infty$. This settles the case when the roof function is bounded away from zero, since every invariant measure for the flow is of that form. When the roof function is not bounded away from zero there are invariant measures for the flow μ that are not of that form. But instead, $\mu = \nu \times m$, where ν is an infinite invariant measure for the shift. Savchenko [Sav, Theorem 1] proved that if μ is ergodic then Abramov's formula still holds. Let $\mathcal{E}_\Phi$ be the set of ergodic Φ -invariant measures.

Proposition 2.2 (Savchenko). *Let Φ be a suspension semi-flow defined over a countable Markov shift with positive roof function τ . Let $\mu \in \mathcal{E}_\Phi$ be such that $\mu = (\nu \times m)|_Y / (\nu \times m)(Y)$, where ν is a sigma-finite (infinite) invariant measure for the shift with $\int \tau d\nu < \infty$. Then*

$$h_\Phi(\mu) = \frac{h_\sigma(\nu)}{\int \tau d\nu}. \quad (9)$$

Corollary 2.1. *Let $\mu \in \mathcal{M}_\Phi$ be such that $\mu = (\nu \times m)|_Y / (\nu \times m)(Y)$ for $\nu \in \mathcal{M}_\sigma$. Then $h_\Phi(\mu) = \infty$ if and only if $h_\sigma(\nu) = \infty$.*

When the phase space is non-compact there are several different notions of topological entropy of a flow, we will consider the following,

Definition 2.4. *The topological entropy of the suspension flow (Y, Φ) denoted by $h(\Phi)$ is defined by*

$$h(\Phi) := \sup \{h_\Phi(\mu) : \mu \in \mathcal{E}_\Phi\},$$

where $\mathcal{E}_\Phi$ is the set of ergodic Φ -invariant measures.

A measure $\mu \in \mathcal{E}_\Phi$ such that $h(\Phi) = h_\Phi(\mu)$ is called a *measure of maximal entropy*. Since the phase space is not compact, there exist suspension flows of finite entropy with no measure of maximal entropy (see Example 6.2). Moreover, there are suspension flows for which the measure of maximal entropy μ is of the form $\mu = \nu \times m$, where ν is an infinite invariant measure for the shift map (see Example 7.1). In Corollary 3.1 we establish criteria to determine when suspension flows have (or do not have) measures of maximal entropy.

2.5. Flows and semi-flows. Sinai remarked that it is possible to translate problems regarding thermodynamic formalism from flows to semi-flows. Indeed, denote by (Σ^*, σ) a two-sided countable Markov shift. Two continuous functions $\phi, \gamma \in C(\Sigma^*)$ are said to be *cohomologous* if there exists a continuous function $\psi \in C(\Sigma^*)$ such that $\phi = \gamma + \psi \circ \sigma - \psi$. The pressure function is invariant under cohomology and so are the thermodynamic properties, such as the existence of equilibrium measures. The following result is due to Daon [Da, Theorem 7.1] and generalises previous results by Sinai [PP, Proposition 1.2] and Coelho and Quas [CQ],

Proposition 2.3. *If $\phi \in C(\Sigma^*)$ has summable variation, then there exists $\gamma \in C(\Sigma^*)$ of summable variation cohomologous to ϕ such that $\gamma(x) = \gamma(y)$ whenever $x_i = y_i$ for all $i \geq 0$ (that is, γ depends only on the future coordinates).*

Daon also proves the more general statement that above result holds under the Walters regularity assumption (for precise definitions see [Da]). Proposition 2.3 implies that thermodynamic formalism for suspension flows can be studied by considering suspension semi-flows.

3. EXISTENCE OF EQUILIBRIUM MEASURES

The definition of pressure for suspension flows over countable Markov shifts has been given with different degrees of generality by Savchenko [Sav], Barreira and Iommi [BI1], Kempton [Ke] and Jaerisch, Kesseböhmer and Lamei [JKL]. These definitions can be summarised as follows

Theorem 3.1. *Let (Σ, σ) a topologically mixing countable Markov shift and $\tau : \Sigma \rightarrow \mathbb{R}$ a positive function of summable variations satisfying (4). Let (Y, Φ) be the suspension semi-flow over (Σ, σ) with roof function τ . Let $g : Y \rightarrow \mathbb{R}$ be a function*

such that $\Delta_g : \Sigma \rightarrow \mathbb{R}$ is of summable variations. Then the following equalities hold

$$\begin{aligned} P_\Phi(g) &:= \lim_{t \rightarrow \infty} \frac{1}{t} \log \left(\sum_{\phi_s(x,0)=(x,0), 0 < s \leq t} \exp \left(\int_0^s g(\phi_k(x,0)) dk \right) \mathbb{1}_{C_{i_0}}(x) \right) \\ &= \inf \{ t \in \mathbb{R} : P_\sigma(\Delta_g - t\tau) \leq 0 \} = \sup \{ t \in \mathbb{R} : P_\sigma(\Delta_g - t\tau) \geq 0 \} \\ &= \sup \{ P_{\sigma|K}(\phi) : K \in \mathcal{K} \}, \end{aligned}$$

where $\mathcal{K}$ is the set of all compact and Φ -invariant sets and P_K is the classical topological pressure of the potential ϕ restricted to the compact and σ -invariant set K .

The Variational Principle has been proved in the context of suspension flows defined over countable Markov shifts (analogous to Theorem 2.1) by different people with different degrees of generality (see [BI1, JKL, Ke, Sav]). The version we will be interested here is the following:

Theorem 3.2 (Variational Principle). *Under the same assumptions of Theorem 3.1 we have*

$$P_\Phi(g) = \sup \left\{ h_\mu(\Phi) + \int_Y g d\mu : \mu \in \mathcal{E}_\Phi \text{ and } - \int_Y g d\mu < \infty \right\},$$

where $\mathcal{E}_\Phi$ is the set of ergodic Φ -invariant measures.

Note that the set of measures considered in the Variational Principle is that of ergodic flow-invariant measures and not the set of flow-invariant probability measures. If the roof function is bounded away from zero the Variational Principle holds in complete generality [BI1]. However, at present time the available proofs in the case that the roof function is not bounded away from zero [JKL, Ke] only hold for ergodic measures. The reason for this being that Abramov formula only holds for these measures (see Subsection 2.4). Also note that it follows from our definitions that $P_\Phi(0) = h(\Phi)$.

Similarly to (3), we define:

Definition 3.1. *A measure $\mu \in \mathcal{E}_\Phi$ is called an equilibrium measure for g if*

$$P_\Phi(g) = h(\mu) + \int g d\mu.$$

The next Theorem is our first main result. In this general context, with roof function not necessarily bounded away from zero, we establish conditions to ensure the existence of equilibrium measures.

Theorem 3.3. *Let Φ be a finite entropy suspension semi-flow on Y defined over a countable Markov shift (Σ, σ) and roof function τ of summable variations satisfying (4). Let $g : Y \rightarrow \mathbb{R}$ be a continuous function such that Δ_g is of summable variations. In the following cases there exists an equilibrium measure for g ;*

1. *If $P_\sigma(\Delta_g - P_\Phi(g)\tau) = 0$ and $\Delta_g - P_\Phi(g)\tau$ is positive recurrent with equilibrium measure ν_g satisfying $\int \tau d\nu_g < \infty$;*

2. If $P_\sigma(\Delta_g - P_\Phi(g)\tau) = 0$ and the potential $\Delta_g - P_\Phi(g)\tau$ is null recurrent with infinite RPF measure ν_g and $\int \tau d\nu_g < \infty$.

In any other case the potential g does not have an equilibrium measure. Thus there is no equilibrium measure for g when

1. $P_\sigma(\Delta_g - P_\Phi(g)\tau) < 0$;
2. $P_\sigma(\Delta_g - P_\Phi(g)\tau) = 0$ and the potential $\Delta_g - P_\Phi(g)\tau$ is positive recurrent with equilibrium measure ν_g such that $\int \tau d\nu_g = \infty$;
3. $P_\sigma(\Delta_g - P_\Phi(g)\tau) = 0$ and the potential $\Delta_g - P_\Phi(g)\tau$ is null recurrent with infinite RPF measure ν_g and $\int \tau d\nu_g = \infty$;
4. If $\Delta_g - P_\Phi(g)\tau$ is transient.

Simplifying to the case of the potential which is constant zero, and the corresponding measures of maximal entropy, we have the following.

Corollary 3.1 (Measures of maximal entropy). *Let Φ be a finite entropy suspension semi-flow on Y defined over a countable Markov shift (Σ, σ) and roof function τ of summable variations satisfying (4).*

1. If $P_\sigma(-h(\Phi)\tau) = 0$ and $-h(\Phi)\tau$ is positive recurrent with equilibrium measure ν satisfying $\int \tau d\nu < \infty$ then there exists a measure of maximal entropy.
2. If $P_\sigma(-h(\Phi)\tau) = 0$ and the potential $h(\Phi)\tau$ is null recurrent with infinite RPF measure ν and $\int \tau d\nu < \infty$ then there exists a measure of maximal entropy.

In any other case the flow does not have a measure of maximal entropy.

To prove Theorem 3.3, we require two lemmas.

Lemma 3.1. *If $P_\sigma(\Delta_g - P_\Phi(g)\tau) < 0$ then there are no equilibrium measures for g .*

Proof. We will show that for any measure $\mu \in \mathcal{M}_\Phi$ we have $h_\Phi(\mu) + \int g d\mu < P_\Phi(g)$. Assume first that $\mu = \nu \times m$ where $\nu \in \mathcal{M}_\sigma$, i.e., ν is a probability measure. Since $P_\sigma(\Delta_g - P_\Phi(g)\tau) < 0$, Theorem 3.3 implies

$$h_\sigma(\nu) + \int \Delta_g d\nu - P_\Phi(g) \int \tau d\nu < 0,$$

thus

$$\frac{h_\sigma(\nu)}{\int \tau d\nu} + \frac{\int \Delta_g d\nu}{\int \tau d\nu} = h_\Phi(\mu) + \int g d\mu < P_\Phi(g).$$

Therefore, no measure $\mu \in \mathcal{M}_\Phi$ of the form $\mu = \nu \times m$, where $\nu \in \mathcal{M}_\sigma$, can be an equilibrium measure for g .

Let us assume now that $\mu = \nu \times m$ where ν is an infinite invariant measure such that $\int \tau d\nu < \infty$. Note that since the flow has finite entropy, $h(\mu) = h(\nu \times m) < \infty$, so Abramov's formula implies that $h(\nu) < \infty$. Assume by way of contradiction that

the measure μ is an equilibrium measure for g . In particular, since $h(\mu) < \infty$, this implies that $\int g d\mu < \infty$. Since $\tau \in L^1(\nu)$ we have $\Delta_g \in L^1(\nu)$ and

$$P_\Phi(g) = h_\Phi(\mu) + \int g d\mu = \frac{h_\sigma(\nu)}{\int \tau d\nu} + \frac{\int \Delta_g d\nu}{\int \tau d\nu}.$$

This implies,

$$h_\sigma(\nu) + \int \Delta_g d\nu - P_\Phi(g) \int \tau d\nu = 0. \quad (10)$$

A direct application of [Sa2, Theorem 2] gives

$$h_\sigma(\nu) \leq \int \left(P_\sigma(\Delta_g - P_\Phi(g)\tau) - \Delta_g + P_\Phi(g)\tau \right) d\nu$$

However, since by (10),

$$h_\sigma(\nu) = - \int (\Delta_g + P_\Phi(g)\tau) d\nu,$$

we obtain

$$P_\sigma(\Delta_g - P_\Phi(g)\tau) = 0.$$

This contradiction proves the statement. $\square$

Lemma 3.2. *If $P_\sigma(\Delta_g - P_\Phi(g)\tau) = 0$ and the potential $\Delta_g - P_\Phi(g)\tau$ is null-recurrent with corresponding infinite measure ν satisfying $\tau \in L^1(\nu)$ then there exists an equilibrium measure for g .*

Proof. In the proof of Lemma 3.1 we showed that if ν is the infinite RPF measure associated to $\Delta_g - P_\Phi(g)\tau$ satisfying $\tau \in L^1(\nu)$, and the flow is of finite entropy, then $\Delta_g \in L^1(\nu)$. It is a consequence of [Sa2, Theorem 2] that

$$h_\sigma(\nu) = \int \left(P_\sigma(\Delta_g - P_\Phi(g)\tau) - \Delta_g + P_\Phi(g)\tau \right) d\nu.$$

Since $P(\Delta_g - P_\Phi(g)\tau) = 0$ we obtain

$$h_\sigma(\nu) = - \int \Delta_g d\nu + P_\Phi(g) \int \tau d\nu.$$

That is

$$P_\Phi(g) = \frac{h_\sigma(\nu)}{\int \tau d\nu} + \frac{\int \Delta_g d\nu}{\int \tau d\nu}.$$

Therefore, for $\mu = \nu \times m \in \mathcal{M}_\Phi$,

$$P_\Phi(g) = h_\Phi(\mu) + \int g d\mu,$$

so μ is an equilibrium measure for g , as required. $\square$

Proof of Theorem 3.3. Case 1 of the theorem follows from [BI1, Theorem 4]. Case 2 of the theorem follows by Lemma 3.2.

To complete the proof, we will show that cases 1 and 2 are the only cases in which there is an equilibrium measure for g . Suppose that g has a finite equilibrium measure μ . Then μ is of the form $\nu \times m$ where ν is a σ -invariant measure, which can be either finite or infinite and $\int \tau d\nu < \infty$. If ν is finite then it is an equilibrium measure for $\Delta_g - P_\Phi(g)\tau$ and $P(\Delta_g - P_\Phi(g)\tau) = 0$. Thus $\Delta_g - P_\Phi(g)\tau$ is either

recurrent or positive recurrent and so we are in case 1 or 2. If ν is infinite then it satisfies the Variational Principle for invariant measures (see Theorem 2.3) and is a fixed point for $L_{\Delta_g - P_\Phi(g)\tau}^*$ and thus $\Delta_g - P_\Phi(g)\tau$ is null recurrent, as in case 2. $\square$

Remark 3.1. Let μ be an equilibrium measure for the potential g which can be written as $\mu = \nu \times m$, where ν is the equilibrium measure for $\Delta_g - P_\Phi(g)\tau$. Techniques developed by Melbourne and Török [MT] to obtain statistical limit theorems can be applied in this setting. Indeed, let $\psi : Y \rightarrow \mathbb{R}$ be a zero mean potential, that is $\int \psi \, d\mu = 0$. Assume that $\tau \in L^a(\nu)$ and that $\psi \in L^b(\mu)$ with

$$\left(1 - \frac{1}{a}\right) \left(1 - \frac{1}{b}\right) \geq \frac{1}{2}.$$

If Δ_ψ and τ satisfy the Central Limit Theorem then ψ satisfies the Central limit Theorem.

4. INDUCING SCHEMES

The combinatorial structure of a countable Markov shift has several important consequences in the properties of its corresponding thermodynamic formalism. For example, if (Σ, σ) is a full-shift then locally Hölder potentials ϕ of finite entropy have corresponding Gibbs measures and the pressure function $t \mapsto P(t\phi)$ (when finite) is real analytic (see [Sa3]). Moreover, if Σ does not satisfy a certain combinatorial assumption (the so called BIP property, see [Sa4] for a precise definition) then locally Hölder potentials do not have corresponding Gibbs measures [Sa4, Theorem 1]. The inducing procedure in the context of topologically mixing countable Markov shifts consists of associating to any system (Σ, σ) a full-shift on a countable alphabet. The idea being to solve problems in this new (better behaved) system and then to translate them back into the original system. While this theory is well developed in the context of maps, in our context has not been thoroughly studied. It was used in [IJ, Section 6.2] to establish the existence of phase transitions for arbitrary topologically mixing shifts and potentials satisfying certain growth conditions.

Let (Σ, σ) be a topologically mixing countable Markov shift. Fix a symbol in the alphabet, say $a \in \mathbb{N}$. The *induced system* over the cylinder C_a , denoted by $(\bar{\Sigma}, \bar{\sigma})$, is the full-shift defined on the alphabet

$$\{C_{ai_1 \dots i_m} : i_j \neq a \text{ and } C_{ai_1 \dots i_m a} \neq \emptyset\}.$$

As defined in Section 2.1 the *first return time map* to the cylinder C_a is

$$r_a(x) := \mathbb{1}_{C_a}(x) \inf \{n \geq 1 : \sigma^n(x) \in C_a\}.$$

For every potential $\phi : \Sigma \rightarrow \mathbb{R}$ we define the induced potential by

$$\bar{\phi} := \left(\sum_{k=0}^{r_a(x)-1} \phi \circ \sigma^k \right).$$

Note that if ϕ has summable variations then it is also the case for $\bar{\phi}$. In the same way, if ϕ is locally Hölder then so is $\bar{\phi}$. Sarig [Sa3, Lemma 2] showed that for topologically mixing systems the pressure $P(\bar{\phi})$ does not depend on the symbol we

induce on. One reason to induce is that if $\inf\{\tau(x) : x \in C_a\} > 0$ then the induced roof function $\bar{\tau}$ will be bounded away from 0.

There is a relation between invariant measures for the induced system $\bar{\mu}$ and measures on the original system. Indeed, an invariant probability measure $\bar{\mu}$ on the induced systems such that $\int r_a d\bar{\mu} < \infty$ can be projected onto an invariant probability measure μ on the original system in the following way

$$\mu(A) = \frac{1}{\int r_a d\bar{\mu}} \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \bar{\mu}(\sigma^{-k}(A) \cap X_n^a), \quad (11)$$

where the sets X_n were defined in Section 2.1. We say that $\bar{\mu}$ is the *lift* of μ and that μ is the *projection* of $\bar{\mu}$. Also note that the integral of a potential with respect to a measure and the integral of the induced potential with respect to the lifted measure are related by the Kac formula:

Remark 4.1 (Kac's formula). *Let $\nu \in \mathcal{M}_\sigma$ be ergodic and satisfy that $\nu(C_a) > 0$. Denote by $\bar{\nu} \in \mathcal{M}_{\bar{\sigma}}$ its lift to the induced system. Let $\phi : \Sigma \rightarrow \mathbb{R}$ be a potential of summable variations and let $\bar{\phi}$ be the corresponding induced potential. If $\int r_a d\bar{\nu} < \infty$ then*

$$\int \phi d\nu = \frac{\int \bar{\phi} d\bar{\nu}}{\int r_a d\bar{\nu}}.$$

On the other hand note that if $\bar{\nu} \in \mathcal{M}_{\bar{\sigma}}$ with $\int \bar{\tau} d\bar{\nu} < \infty$ but $\int r_a d\bar{\nu} = \infty$ then the projection of $\bar{\nu}$ is an infinite σ -invariant measure but $\mu = \bar{\nu} \times m$ will be a finite Φ -invariant measure. For a potential $g : Y \rightarrow \mathbb{R}$ we will have that

$$\int g d\mu = \frac{\int \bar{\Delta}_g d\bar{\nu}}{\int \bar{\tau} d\bar{\nu}}.$$

Note that inducing corresponds to choosing a different base map to suspend the flow over. The first important remark is that the pressure for the flow can be computed using the induced system. This fact was implicit in [IJ, Lemma 6.1]. However as well as this if $\inf\{\tau(x) : x \in C_a\} > 0$ then the existence of equilibrium measures can be determined by the induced system.

Lemma 4.1. *Let (Y, Φ) be a suspension semi-flow defined over a topologically mixing countable Markov shift (Σ, σ) with roof function τ of summable variations. Let $g : Y \rightarrow \mathbb{R}$ be such that Δ_g is of summable variations. Then*

$$P_\Phi(g) = \inf \{s \in \mathbb{R} : P_{\bar{\sigma}}(\bar{\Delta}_g - s\bar{\tau}) \leq 0\}.$$

Moreover if $\inf\{\tau(x) : x \in C_a\} > 0$ then g has an equilibrium measure if and only if $P_{\bar{\sigma}}(\bar{\Delta}_g - P_\Phi(g)\bar{\tau}) = 0$ and $\bar{\Delta}_g - P_\Phi(g)\bar{\tau}$ has an equilibrium measure with respect to which $\bar{\tau}$ is integrable.

Proof. By the definition of $P_\Phi(g)$ it suffices to show that

$$\sup\{s \in \mathbb{R} : P_\sigma(\Delta_g - s\tau) > 0\} = \sup\{s \in \mathbb{R} : P_{\bar{\sigma}}(\bar{\Delta}_g - s\bar{\tau}) > 0\}.$$

To start let $t \in \mathbb{R}$ satisfy that $P_{\bar{\sigma}}(\bar{\Delta}_g - t\bar{\tau}) > 0$ and note that there must exist a compactly supported $\bar{\sigma}$ -ergodic probability measure $\bar{\mu}$ such that

$$h_{\bar{\sigma}}(\bar{\mu}) + \int \bar{\Delta}_g d\bar{\mu} - t \int \bar{\tau} d\bar{\mu} > 0.$$

By considering the projection of $\bar{\mu}$ and applying the Variational Principle we can deduce that $P_\sigma(\Delta_g - t\tau) > 0$.

On the other hand if we fix $t \in \mathbb{R}$ such that $P_\sigma(\Delta_g - t\tau) > 0$ then we can find a compactly supported ergodic measure, μ such that $h(\mu) + \int \Delta_g d\mu - t \int \tau d\mu > 0$ and induce to observe that $P_{\bar{\sigma}}(\bar{\Delta}_g - t\bar{\tau}) > 0$. The first part of the result now follows.

To prove the second part suppose that $P_{\bar{\sigma}}(\bar{\Delta}_g - P_\Phi(g)\bar{\tau}) = 0$ and $\bar{\Delta}_g - P_\Phi(g)\bar{\tau}$ has an ergodic equilibrium measure $\bar{\mu}$ such that $\int \bar{\tau} d\bar{\mu} < \infty$. Thus $\bar{\mu}$ can be pushed down to a (possibly infinite) σ -invariant measure μ where $\int \tau d\mu < \infty$ and

$$0 = h(\bar{\mu}) + \int (\bar{\Delta}_g - P_\Phi(g)\bar{\tau}) d\bar{\mu}$$

which means

$$P_\Phi(g) = \frac{h(\mu)}{\int \tau d\mu} + \frac{\int \Delta_g d\mu}{\int \tau d\mu}.$$

Therefore, $\mu \times m$ is a finite equilibrium measure for g .

On the other hand if g has an equilibrium measure, then by Theorem 3.3 and Theorem 2.2 it has an equilibrium measure of the form $\nu \times m$ where ν must be a RPF measure for $\Delta_g - P_\Phi(g)\tau$ (NB: it can be infinite). Thus we can induce to yield $\bar{\nu}$ which will be an equilibrium measure for $\bar{\Delta}_g - P_\Phi(g)\bar{\tau}$. $\square$

Remark 4.2. *In the proof of the second part of the lemma, the assumption $\int \bar{\tau} d\bar{\mu} < \infty$ follows immediately in some cases. For example, if $P_\Phi(g) \neq 0$ and Δ_g and τ are not asymptotically comparable (i.e., to rule out $|\int \bar{\Delta}_g d\bar{\mu}|$ and $|\int \bar{\tau} d\bar{\mu}|$ being simultaneously infinite, but $|\int \bar{\Delta}_g - P_\Phi(g)\bar{\tau} d\bar{\mu}| < \infty$), then the fact that $\bar{\mu}$ is an equilibrium measure for $\bar{\Delta}_g - P_\Phi(g)\bar{\tau}$ implies that $|\int \bar{\Delta}_g - P_\Phi(g)\bar{\tau} d\bar{\mu}| < \infty$ and thus $\int \bar{\tau} d\bar{\mu} < \infty$.*

We let

$$s_\infty := \inf\{s : P(-s\bar{\tau}) < \infty\}.$$

This number plays an important role in the thermodynamic formalism of the associated suspension flow.

Remark 4.3. *We collect some basic facts about s_∞ and $s \mapsto P_{\bar{\sigma}}(-s\bar{\tau})$.*

1. *The constant $s_\infty = \infty$ if and only if $h(\Phi) = \infty$.*
2. *Since $\tau \geq 0$, this is also true for the induced version. Hence $s \mapsto P_{\bar{\sigma}}(-s\bar{\tau})$ is a non-increasing function. In particular, since $P_{\bar{\sigma}}(0) = \infty$, this means that $s_\infty \geq 0$.*

The inducing procedure and these remarks will be used in the study of the renewal flow (see Section 6).

5. RECURRENCE AND TRANSIENCE FOR SUSPENSION FLOWS

This section is devoted to extend the notions of recurrence and transience to potentials defined over suspension flows. This notions were given in the context of

countable Markov shifts by Sarig in [Sa1] and allow for the classification of a potential according to its recurrence properties. These definitions have also been extended beyond the realm of Markov systems in [IT].

We begin by defining the relevant partition function.

Definition 5.1. *Let $g : Y \rightarrow \mathbb{R}$ be a potential such that $\Delta_g : \Sigma \rightarrow \mathbb{R}$ is of summable variations and $P_\Phi(g) < \infty$. Given $C_{i_0} \in \Sigma$, let*

$$Z_n(g, C_{i_0}) := \sum_{x \in C_{i_0}, \phi_s(x, 0) = (x, 0), \text{ for } n-1 < s \leq n} e^{\int_0^s g(\phi_t(0, x)) dt}.$$

We say that g is recurrent if

$$\sum_{n=1}^{\infty} Z_n(g - P_\Phi(g), C_{i_0}) = \infty.$$

Otherwise, g is transient.

The following result establishes the relationship between recurrence and transience with equilibrium measures.

Theorem 5.1. *Let $g : Y \rightarrow \mathbb{R}$ be a potential such that $\Delta_g : \Sigma \rightarrow \mathbb{R}$ is of summable variations and $P_\Phi(g) < \infty$ and suppose τ satisfies (4).*

- (a) *The definition of recurrence is independent of the cylinder C_{i_0} .*
- (b) *$P_\Phi(g) = \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n(g, C_{i_0})$.*
- (c) *The potential g is recurrent if and only if $\Delta_g - \tau P_\Phi(g)$ is recurrent and $P(\Delta_g - P_\Phi(g)\tau) = 0$.*
- (d) *If g is recurrent then there exists a conservative measure ν_g which can be obtained as $\mu \times m$ where μ is the RPF measure for $\Delta_g - \tau P_\Phi(g)$.*

Remark 5.1. *The condition $P_\sigma(\Delta_g - P_\Phi(g)\tau) = 0$ in (c) is crucial. In Example 6.2 below we construct a case where $\Delta_g - \tau P_\Phi(g)$ is recurrent, but $P_\sigma(\Delta_g - P_\Phi(g)\tau) < 0$ and thus g is transient.*

Proof. The proof of (a) follows from the proof of (c). The proof of (b) follows from [JKL].

For part (c), we will shortly prove the following:

Claim 1.

$$\sum_{n=1}^{\infty} Z_n(g - P_\Phi(g), C_{i_0}) = \sum_{n=1}^{\infty} Z_n(\Delta_g - \tau P_\Phi(g), C_{i_0}).$$

This can be added to the following claim, which follows immediately from the definition of pressure being the radius of convergence of the power series defined by partition functions.

Claim 2. *$P_\sigma(\Delta_g - P_\Phi(g)\tau) < 0$ implies that $\sum_{n=1}^{\infty} Z_n(\Delta_g - \tau P_\Phi(g), C_{i_0}) < \infty$.*

Together the claims show that g is recurrent implies that $\Delta_g - \tau P_\Phi(g)$ is recurrent and $P(\Delta_g - P_\Phi(g)\tau) = 0$. Moreover, Claim 1 alone shows that if $\Delta_g - \tau P_\Phi(g)$ is recurrent and $P(\Delta_g - P_\Phi(g)\tau) = 0$ then g is recurrent. So (c) holds,

Proof of Claim 1. First for $x \in C_{i_0}$ such that $\phi_s(x, 0) = (x, 0)$ where $n-1 < s \leq n$, set $k_{x,n}$ to be the number of times that $(x, 0)$ has returned to the base at time s , i.e., let $k \in \mathbb{N}$ such that $\tau^k(x) = s$. Then

$$\begin{aligned} Z_n(g - P_\Phi(g), C_{i_0}) &= \sum_{x \in C_{i_0}, \phi_s(x, 0) = (x, 0), \text{ for } n-1 < s \leq n} e^{\int_0^s (g - P_\Phi(g))(\phi_t(0, x)) dt} \\ &= \sum_{x \in C_{i_0}, \phi_s(x, 0) = (x, 0), \text{ for } n-1 < s \leq n} e^{\sum_{i=0}^{k_{x,n}-1} \int_{\tau^i(x)}^{\tau^{i+1}(x)} s(g - P_\Phi(g))(\phi_t(0, x)) dt} \\ &= \sum_{x \in C_{i_0}, \phi_s(x, 0) = (x, 0), \text{ for } n-1 < s \leq n} e^{\left(\sum_{i=0}^{k_{x,n}-1} \Delta_g(\sigma^i(x))\right) - s P_\Phi(g)}. \end{aligned}$$

Since

$$Z_n(\Delta_g - \tau P_\Phi(g), C_{i_0}) = \sum_{\{x \in C_{i_0} : \sigma^n x = x\}} e^{(\sum_{i=0}^{n-1} \Delta_g(\sigma^i(x))) - n P_\Phi(g)},$$

each term $e^{(\sum_{i=0}^{k_{x,n}-1} \Delta_g(\sigma^i(x))) - s P_\Phi(g)}$ in the sum for $Z_n(g - P_\Phi(g), C_{i_0})$ is counted as part of the sum for $Z_{k_{x,n}}(\Delta_g - \tau P_\Phi(g), C_{i_0})$. Thus the sum $\sum_{n=1}^{\infty} Z_n(g - P_\Phi(g), C_{i_0})$ is simply a reordering of the series of positive terms $\sum_{n=1}^{\infty} Z_n(\Delta_g - \tau P_\Phi(g), C_{i_0})$. $\square$

For (d), the existence of the measure described follows from (c) plus the RPF Theorem (see Theorem 2.2) applied to $\Delta_g - \tau P_\Phi(g)$. Conservativity follows from the fact that since the RPF measure is conservative then so is $\mu \times m$. $\square$

In order to give a criterion for the existence of equilibrium measures for the flow, we consider the following,

Definition 5.2. Let $g : Y \rightarrow \mathbb{R}$ be a potential such that $\Delta_g : \Sigma \rightarrow \mathbb{R}$ is of summable variations and $P_\Phi(g) < \infty$. Let

$$P_n := \{x \in C_{i_0} : \exists s \in (n-1, n] \text{ s.t. } \phi_s(x, 0) = (x, 0), \text{ but } \phi_t(x, 0) \notin C_{i_0} \forall t \in (0, s)\}.$$

For $x \in P_n$, let $s_x > 0$ be minimal such that $\phi_{s_x}(x, 0) = (x, 0)$. Now let

$$Z_{n,\tau}^*(g, C_{i_0}) := \sum_{x \in P_n} e^{\int_0^{s_x} g(\phi_t(x)) dt}.$$

Remark 5.2. For each $x \in P_n$ there exists $n_x \in \mathbb{N}$ such that $s_x = \tau^{n_x}(x)$, so for a potential $\tilde{g}$ such that $\Delta_{\tilde{g}} : \Sigma \rightarrow \mathbb{R}$ is of summable variations, $P(\Delta_{\tilde{g}}) < \infty$ and $\overline{\Delta_{\tilde{g}}}$ is locally Hölder we have

$$\int_0^{s_x} \tilde{g}(\phi_t) dt = S_{n_x} \Delta_{\tilde{g}} = \overline{\Delta_{\tilde{g}}}(x).$$

Let Y_x be the $(n_x - 1)$ -cylinder Y around x w.r.t. the dynamics σ , so $\sigma^{n_x}(Y) = C_{i_0}$. This set is a 1-cylinder for the induced map $\bar{\sigma}$. We will use the fact below that if $\hat{\mu}$ is a Gibbs measure for $\overline{\Delta_{\tilde{g}}}$ then $\bar{\mu}(Y) \asymp e^{\overline{\Delta_{\tilde{g}}}(x)}$. Note that this term is a summand in the sum for $Z_{n,\tau}^*$.

Definition 5.3. Let $g : Y \rightarrow \mathbb{R}$ be a potential such that $\Delta_g : \Sigma \rightarrow \mathbb{R}$ is of summable variations and $P_\Phi(g) < \infty$. Suppose that g is recurrent. If

$$\sum_n n Z_{n,\tau}^*(g) e^{-nP_\Phi(g)} < \infty$$

we say that g is positive recurrent. If

$$\sum_n n Z_{n,\tau}^*(g) e^{-nP_\Phi(g)} = \infty$$

we say that g is null recurrent.

Note that definition above is independent of the cylinder C_{i_0} .

Theorem 5.2. Let $g : Y \rightarrow \mathbb{R}$ be a positive recurrent potential, and suppose τ satisfies (4). Then there exists an equilibrium measure ν_g .

Proof. Define $\overline{\Delta_g - \tau P_\Phi(g)}$ to be the induced version of $\Delta_g - \tau P_\Phi(g)$ on C_{i_0} . Also define $\bar{\tau}$ to be the induced roof function. Since g is recurrent and, by Theorem 5.1, $P_\sigma(\Delta_g - \tau P_\Phi(g)) = 0$, the proof of [Sa3, Lemma 3] implies that there is an equilibrium measure $\bar{\mu}$ for $\overline{\Delta_g - \tau P_\Phi(g)}$, which projects to the flow if $\int \bar{\tau} d\bar{\mu} < \infty$ (see Remark 4.1). Since by the same result (see also [Sa4]), $\bar{\mu}$ is a Gibbs measure, as in Remark 5.2 the value of $\int \bar{\tau} d\bar{\mu}$ can be bounded by a constant times $\sum_n n Z_{n,\tau}^*(g) e^{-nP_\Phi(g)}$. Since g is positive recurrent, these values are bounded and the result is proved. $\square$

Remark 5.3. In the spirit of [IT], we now characterise recurrence in a way which extends beyond semi-flows over shifts. Let $g : Y \rightarrow \mathbb{R}$ be a potential such that $\Delta_g : \Sigma \rightarrow \mathbb{R}$ is of summable variations and $P_\Phi(g) < \infty$. Summarising, we say that g is

1. Positive recurrent if it has an equilibrium measure
2. Null-recurrent if $P_\sigma(\Delta_g - P_\Phi(g)\tau) = 0$, the potential $\Delta_g - P_\Phi(g)\tau$ is recurrent with corresponding measure ν and $\int \tau d\nu = \infty$.
3. Transient in any other case.

In particular, our definitions extend the corresponding ones given by Sarig [Sa1] for countable Markov shifts.

6. THE RENEWAL FLOW

In this section we define and study a class of suspension flows of particular interest, since they serve as symbolic models for a class of flows belonging to the boundary of hyperbolic flows. Namely, suspension flows defined over the Manneville-Pomeau maps. We study the corresponding thermodynamic formalism establishing conditions for the existence of equilibrium measures and phase transitions.

6.1. The renewal shift and the renewal flow. For the alphabet $\mathbb{N}_0$, consider the transition matrix $A = (a_{ij})_{i,j \in \mathbb{N}_0}$ with $a_{0,0} = a_{0,n} = a_{n,n-1} = 1$ for each $n \geq 1$ and with all other entries equal to zero. The *renewal shift* is the Markov shift (Σ_R, σ) defined by the transition matrix A , that is, the shift map σ on the space

$$\Sigma_R = \{(x_i)_{i \geq 0} : x_i \in \mathbb{N}_0 \text{ and } a_{x_i x_{i+1}} = 1 \text{ for each } i \geq 0\}.$$

Remark 6.1. Let (Σ_2, σ) be the full-shift on the alphabet $\{0, 1\}$. There exists a topological conjugacy between the renewal shift (Σ_R, σ) and $(\Sigma_2 \setminus \bigcup_{i=0}^{\infty} \sigma^{-i}(\bar{0}), \sigma)$, where $\bar{0} = (000\cdots)$. Indeed, denote by $(0\cdots 01)_n$ the cylinder $C_{0\cdots 01}$ with n zeros, and consider the alphabet $\{(0\cdots 01)_n : n \geq 1\} \cup \{C_0\}$. The possible transitions on this alphabet are

$$(0\cdots 01)_n \rightarrow (0\cdots 01)_{n-1}, \quad C_0 \rightarrow C_0, \quad \text{and } C_0 \rightarrow (0\cdots 01)_n \text{ for } n \geq 1.$$

Note that this is simply a recoding of $(\Sigma_2 \setminus \bigcup_{i=0}^{\infty} \sigma^{-i}(\bar{0}), \sigma)$.

Let $\mathcal{R}$ be the class of functions $\phi : \Sigma_R \rightarrow \mathbb{R}$ such that:

1. the function ϕ has summable variation and is bounded from above;
2. the function ϕ has finite Gurevich pressure;
3. the induced function $\bar{\phi}$ is locally Hölder continuous.

We observe that $\mathcal{R}$ includes the class of Hölder continuous functions. Nevertheless, there are non-Hölder continuous functions that belong to $\mathcal{R}$.

Thermodynamic formalism is well understood in this setting for potentials of summable variations. Indeed, Sarig [Sa3] proved the following (the version of this result for every $q \in \mathbb{R}$, and not just for positive values, appears in [BI2, Proposition 3]).

Proposition 6.1. Let (Σ_R, σ) be the renewal shift. For each bounded $\phi \in \mathcal{R}$ there exist $q_c^+ \in (0, +\infty]$ and $q_c^- \in [-\infty, 0)$ such that:

1. $q \mapsto P_\sigma(q\phi)$ is strictly convex and real analytic in (q_c^-, q_c^+) .
2. $P_\sigma(q\phi) = mq$ for $q < q_c^-$, and $P_G(q\phi) = Mq$ for $q > q_c^+$. Here $m := \inf \left\{ \int_{\Sigma_R} \phi d\mu : \mu \in \mathcal{M}_R \right\}$ and $M := \sup \left\{ \int_{\Sigma_R} \phi d\mu : \mu \in \mathcal{M}_R \right\}$.
3. At q_c^- and q_c^+ the function $q \mapsto P_\sigma(q\phi)$ is continuous but not analytic.
4. For each $q \in (q_c^-, q_c^+)$ there is a unique equilibrium measure μ_q for $q\phi$.
5. For each $q \notin [q_c^-, q_c^+]$ there is no equilibrium measure for $q\phi$.
6. The critical values q_c^+ and q_c^- are never simultaneously finite.

If $q \in (q_c^-, q_c^+)$ the potential $q\phi$ is positive recurrent and for $q < q_c^-$ or $q > q_c^+$ the potential $q\phi$ is transient. At the critical values the potential can have any recurrence mode: $q_c\phi$ can be positive recurrent, null-recurrent or transient (see [Sa3, Example 2]).

Definition 6.1. Let $\tau : \Sigma_R \rightarrow \mathbb{R}$ be a positive potential with $\tau \in \mathcal{R}$ and satisfying (4). The suspension semi-flow $\Phi_R = \varphi_t(x, s)$ defined in the canonical way over the (non-compact) space

$$Y_R = \{(x, t) \in \Sigma_R \times \mathbb{R} : 0 \leq t \leq \tau(x)\}.$$

is called a renewal semi-flow.

If $\lim_{x \rightarrow \bar{0}} \tau(x) = 0$ then we can think of this flow as one having a *cusp* at $(\bar{0}, 0)$. This, of course, has several dynamical consequences. For instance,

Example 6.1 (An infinite entropy renewal flow). *Here we present an example of a semi-flow where the presence of a cusp causes the flow to have infinite topological entropy. Clearly for this to be interesting, the base dynamics should have finite entropy: we consider the renewal shift, which has topological entropy $\log 2$.*

Consider the renewal semi-flow with roof function τ , to be defined later. By Propositions 2.1 and 2.2,

$$h(\Phi) = \sup_{\nu \in \mathcal{E}_\sigma} \left\{ \frac{h_\sigma(\nu)}{\int \tau \, d\nu} \right\}.$$

Consider the induced system $(\bar{\Sigma}, \bar{\sigma}, \bar{\tau})$ given by the first return map to C_0 . As usual, for each $n \in \mathbb{N}$, denote the domain with first return time n by X_n . Then by the Abramov formula and by approximating measures by compactly supported ones,

$$h(\Phi) = \sup_{\bar{\nu} \in \mathcal{E}_{\bar{\sigma}}(\bar{\tau})} \left\{ \frac{h_{\bar{\sigma}}(\bar{\nu})}{\int \bar{\tau} \, d\bar{\nu}} \right\} = \sup_{\bar{\nu} \in \mathcal{E}_{\bar{\sigma}}} \left\{ \frac{h_{\bar{\sigma}}(\bar{\nu})}{\int \bar{\tau} \, d\bar{\nu}} \right\}.$$

(We don't actually use the second equality here.) If $\bar{\nu}$ is a Markov measure (see [Wa, p.22] for the definition) for $(\bar{\Sigma}, \bar{\sigma})$, then

$$\frac{h_{\bar{\sigma}}(\bar{\nu})}{\int \bar{\tau} \, d\bar{\nu}} = - \frac{\sum_n \nu(X_n) \log \nu(X_n)}{\sum_n \nu(X_n) s_n},$$

where $s_n = \bar{\tau}|_{X_n}$.

Setting

$$\tau(x) = \begin{cases} \log \log(1 + e) & \text{if } x \in C_0 \\ \log \log(1 + e + n) - \log \log(1 + e + n - 1) & \text{if } x \in C_n, \end{cases}$$

we obtain $s_n = \log \log(e + n)$. So in this case,

$$\frac{h_{\bar{\sigma}}(\bar{\nu})}{\int \bar{\tau} \, d\bar{\nu}} = - \frac{\sum_n \nu(X_n) \log \nu(X_n)}{\sum_n \nu(X_n) \log \log(e + n)}.$$

So for example, for $N \in \mathbb{N}$, the measure $\bar{\nu}_N$ giving mass $1/N$ to X_n if $1 \leq n \leq N$ and zero mass otherwise has

$$\frac{h_{\bar{\sigma}}(\bar{\nu}_N)}{\int \bar{\tau} \, d\bar{\nu}_N} = \frac{N \log N}{\sum_{n=1}^N \log \log(e + n)} \geq \frac{\log N}{\log \log(e + N)} \rightarrow \infty \text{ as } N \rightarrow \infty.$$

Thus $h(\Phi) = \infty$. This argument implies that we have some freedom to alter τ , but so long as it is chosen so that $\log N/s_N \rightarrow \infty$ as $N \rightarrow \infty$, the entropy of the flow will still be infinite.

6.2. Equilibrium measures for the renewal flow. In the next proposition we characterise bounded potentials having equilibrium measures.

Proposition 6.2. *Let Φ_R be a renewal semi-flow of finite entropy and $g : Y_R \rightarrow \mathbb{R}$ a bounded potential such that $\Delta_g : \Sigma_R \rightarrow \mathbb{R}$ is locally Hölder. There exists an equilibrium measure for g if and only if the potential $\Delta_g - P_\Phi(g)\tau$ is recurrent, and the RPF measure ν_g has $\int \tau \, d\nu_g < \infty$. Here $\mu = \nu_g \times m$.*

Proof. First we will show that the equation in the variable $q \in \mathbb{R}$ given by

$$P_\sigma(\Delta_g - q(g)\tau) = 0,$$

always has a root. Indeed, since the potential g is bounded there exist constants $K_1, K_2 \in \mathbb{R}$ such that $K_2\tau \leq \Delta_g \leq K_1\tau$. Therefore,

$$P_\sigma((K_2 - q)\tau) \leq P_\sigma(\Delta_g - q\tau) \leq P_\sigma((K_1 - q)\tau).$$

It is a direct consequence of Proposition 6.1 and of the fact that $h(\Phi) < \infty$ that $P_\sigma(-h(\Phi)\tau) = 0$. Therefore there exist $q_1, q_2 \in \mathbb{R}$ such that

$$0 \leq P_\sigma((K_2 - q_1)\tau) \leq P_\sigma(\Delta_g - q\tau) < \infty,$$

and

$$P_\sigma(\Delta_g - q_2\tau) \leq P_\sigma((K_1 - q_2)\tau) \leq 0.$$

Since the pressure is a continuous function of the variable q we obtain the desired result.

By virtue of Theorem 3.3, if the potential $\Delta_g - P_\Phi(g)\tau$ is transient then there are no equilibrium measures for g .

If $\Delta_g - P_\Phi(g)\tau$ is positive recurrent then there exist a measure $\nu_g \in \Sigma_R$ which is an equilibrium measure for $\Delta_g - P_\Phi(g)\tau$. Since the roof function $\tau \in \mathcal{R}$, it is bounded, so $\tau \in L^1(\nu_g)$, therefore there exists an equilibrium measure for g .

The remaining case is that when the potential $\Delta_g - P_\Phi(g)\tau$ is null recurrent. Denote by ν_g the corresponding infinite RPF measure. Theorem 3.3 together with the fact that $\tau \in L^1(\nu_g)$ yield the desired result. $\square$

Remark 6.2. *If the renewal flow Φ_R has infinite entropy, as in Example 6.1, then it is a direct consequence of the Variational Principle that bounded potentials on Y_R do not have equilibrium measures.*

Remark 6.3. *In the proof of Proposition 6.2 we obtained that if $\Delta_g - P_\Phi(g)\tau$ is positive recurrent and ν_g is the corresponding measure then $\tau \in L^1(\nu_g)$. Therefore, in the positive recurrent case that assumption is not needed.*

Remark 6.4. *Let Φ_R be a renewal flow of finite entropy and $g : Y_R \rightarrow \mathbb{R}$ a bounded potential such that $\Delta_g : \Sigma_R \rightarrow \mathbb{R}$ is locally Hölder. Recall that regular potentials of finite pressure defined on the full-shift on a countable alphabet are positive recurrent (see [Sa4, Corollary 2]). By Kac's formula obtained in [Sa3, Lemma 3] we have that if $P_\sigma(\Delta_g - P_\Phi(g)\tau) = 0$ then $\Delta_g - P_\Phi(g)\tau$ is recurrent. Denote by ν_g the corresponding equilibrium measure. So as above, to find an equilibrium measure for g , it suffices to check that for the RPF measure ν_g , we have $\tau \in L^1(\nu_g)$.*

6.3. Measures of maximal entropy: Hofbauer type roof functions. In this subsection we look at when suspension flows over the renewal shift have measures of maximal entropy. We will assume that

$$\lim_{n \rightarrow \infty} \sup_{x \in C_n} \{\tau(x)\} = 0.$$

By lemma 4.1 it follows that there exists a measure of maximal entropy if and only if the roof function τ satisfies that $P_\sigma(-h(\Phi)\bar{\tau}) = 0$ and $-h(\Phi)\bar{\tau}$ has an equilibrium measure. This means that in particular if $P_\sigma(-s_\infty\bar{\tau}) > 0$ then there

exists a measure of maximal entropy however if $P_{\bar{\sigma}}(-s_{\infty}\bar{\tau}) < 0$ there exists no measure of maximal entropy.

We can also say that if $P_{\bar{\sigma}}(-s_{\infty}\bar{\tau}) > 0$ then $-h(\Phi)\tau$ will be positive recurrent and will have an equilibrium measure if and only if $\int r_0 d\bar{\mu} < \infty$ where $\bar{\mu}$ is the equilibrium measure for $-h(\Phi)\bar{\tau}$. Otherwise $-h(\Phi)\tau$ will be null-recurrent. Specific cases of this where $-h(\Phi)\tau$ is null-recurrent are given in Lemma 7.1.

In the case where $P_{\bar{\sigma}}(-s_{\infty}\bar{\tau}) = 0$ and thus $h(\Phi) = s_{\infty}$ then it is possible that the Gibbs measure $\bar{\mu}$ for $-s_{\infty}\bar{\tau}$ will satisfy that $\int \bar{\tau} d\bar{\mu} = \infty$. In this case $-h(\Phi)\tau$ is null recurrent and there is no measure of maximal entropy for Φ . Finally if $P_{\bar{\sigma}}(-s_{\infty}\bar{\tau}) < 0$ and thus $h(\Phi) = s_{\infty}$ then $P(-s_{\infty}\tau) = 0$ and $-s_{\infty}\tau$ is transient. We can adapt the techniques used by Hofbauer in [Ho] to produce examples where there is no measure of maximal entropy and $-h(\Phi)\tau$ is either transient or null-recurrent (In this setting if $-h(\Phi)\tau$ is positive recurrent then there must be a measure of maximal entropy for Φ .)

Example 6.2. We fix $k > 0$ such that $\sum_{n=0}^{\infty} \frac{1}{(n+k)(\log(n+k))^2} < 1$ and for $n \geq 0$ let $a_n = \frac{1}{(n+k)(\log(n+k))^2}$. and consider the locally constant potential $\tau : \Sigma_R \rightarrow \mathbb{R}$ defined on each cylinder C_n with $n \geq 1$ by $\tau|_{C_n} = -\log\left(\frac{a_n}{a_{n-1}}\right)$ and with $\tau|_{C_0} = -\log a_0$. This gives that the induced roof function $\bar{\tau}$ will also be locally constant with $\bar{\tau}|_{\bar{C}_n} = -\log a_n$ for $n \geq 0$. Thus

$$P_{\bar{\sigma}}(-t\bar{\tau}) = \log \left(\sum_{n=0}^{\infty} \left(\frac{1}{(n+k)(\log(n+k))^2} \right)^t \right)$$

when this is finite and otherwise $P_{\bar{\sigma}}(-t\bar{\tau}) = \infty$. So we have that $s_{\infty} = 1$ and $P_{\bar{\sigma}}(-\bar{\tau}) < 0$. Thus $h(\Phi) = 1$ and there is no measure of maximal entropy for Φ and $-\tau$ is transient. This example also relates to the construction of infinite iterate function systems with no measure of maximal dimension considered by Mauldin and Urbański in [MU1].

Now consider the case when $a_0 = 1 - \sum_{n=1}^{\infty} a_n$. Define τ as above and note that we now have that $P(-\bar{\tau}) = 0$. We have that

$$-\sum_{n=0}^{\infty} a_n \log a_n = \infty$$

and so the Gibbs measure $\bar{\mu}$ for $\bar{\tau}$ satisfies $\int \bar{\tau} d\bar{\mu} = \infty$ and we are in the case when there is no measure of maximal entropy for Φ and $-\tau$ is null-recurrent.

6.4. Phase transitions for the renewal flow. Bowen and Ruelle [BR] showed that the pressure function, $t \mapsto P_{\Phi}(tg)$, for suspension flows defined over (finite state) sub-shifts of finite type with Hölder roof function is real analytic when considering potentials g such that Δ_g is Hölder. In particular, the pressure is real analytic for Axiom A flows. Note that since the pressure is convex it is differentiable in every point of the domain, except for at most a countable set. We say that the pressure exhibits a *phase transition* at the point $t_0 \in \mathbb{R}$ if the function $P_{\Phi}(tg)$ is *not* real analytic at $t = t_0$. In [IJ] the regularity of the pressure was studied and conditions in order for the pressure to be real analytic or to exhibit

phase transitions were found in the context of BIP shifts in the base (see [IJ] for precise definitions, but roughly speaking these are shifts that combinatorially are close to the full-shift). In the discrete time setting, the renewal shift is fairly well understood (see Proposition 6.1 and [Sa3]). The pressure exhibits at most one phase transition after which the pressure takes the form $P(t\psi) = At$. Phase transitions of these type are called phase transition of *zero entropy*, since the line At passes through zero. Recently, in the context of maps phase transitions of *positive entropy* have been constructed (in this cases the pressure function takes the form $P(t\psi) = At + B$, with $B \neq 0$). Indeed, examples have been obtained in [DGR, IT] and most notably in [BT], where $B = h_{top}(T)$.

For the renewal flow the situation is richer than in the renewal shift setting [Sa3]. In particular, the pressure function can exhibit two phase transitions (see Example 6.4) as opposed to the discrete time case, where at most there exists one phase transition. Moreover it can exhibit phase transitions of zero and positive entropy depending on the value of s_∞ (see Subsection 4 for a precise definition). Indeed, the case in which $s_\infty = 0$ can be thought of as zero entropy phase transitions and $s_\infty > 0$ correspond to positive entropy phase transitions (see Example 6.4 for a positive entropy phase transition). In this subsection we establish conditions in order for the pressure function $t \mapsto P_\Phi(tg)$ to be real analytic or to exhibit phase transitions.

We fix a positive function $\tau \in -\mathcal{R}$ as our roof function and let $g : Y_R \rightarrow \mathbb{R}$ be a function such that $\Delta_g \in \pm\mathcal{R}$. In addition we will assume that there exists $\alpha \in \mathbb{R}$ such that

$$\lim_{n \rightarrow \infty} \sup_{x \in C_n} \left\{ \frac{\Delta_g(x)}{\tau(x)} \right\} = \lim_{n \rightarrow \infty} \inf_{x \in C_n} \left\{ \frac{\Delta_g(x)}{\tau(x)} \right\} = \alpha. \quad (12)$$

Recall that we proved in Lemma 4.1 that we can calculate the pressure of g using the induced map. We have that

$$P_\Phi(g) = \inf\{s : P_{\bar{\sigma}}(\overline{\Delta_g} - s\bar{\tau}) \leq 0\}.$$

Remark 6.5. Recall that $s_\infty = \inf\{s : P(-s\bar{\tau}) < \infty\}$. Under condition (12), and assuming $h(\Phi) < \infty$, there must exist a sequence of measures $(\bar{\nu}_n)_n$ such that $\int \bar{\tau} d\bar{\nu}_n \rightarrow \infty$. This follows from the Variational Principle and the two observations that $h(\bar{\sigma}) = \infty$; and for each $s > s_\infty \geq 0$ there exists $C > 0$ such that $h_{\bar{\sigma}}(\bar{\nu}) - s \int \bar{\tau} d\bar{\nu} < C$ for any $\bar{\sigma}$ -invariant measure $\bar{\nu}$.

Lemma 6.1. For $t \in \mathbb{R}$ we have that

$$\inf\{s : P_{\bar{\sigma}}(t\overline{\Delta_g} - s\bar{\tau}) < \infty\} = s_\infty + t\alpha.$$

Proof. For $t = 0$ the result is obvious so we will assume throughout the proof that $t \neq 0$. If we let $s < s_\infty + t\alpha$ then since, as in Remark 6.5, there exists a sequence of $\bar{\sigma}$ -invariant probability measures $\bar{\nu}_n$ such that $\lim_{n \rightarrow \infty} \int \bar{\tau} d\bar{\nu}_n = \infty$ and thus $\lim_{n \rightarrow \infty} \frac{\int \overline{\Delta_g} d\bar{\nu}_n}{\int \bar{\tau} d\bar{\nu}_n} = \alpha$ it follows that $P_{\bar{\sigma}}(-s\bar{\tau} + t\overline{\Delta_g}) = \infty$. Hence $\inf\{s : P_{\bar{\sigma}}(t\overline{\Delta_g} - s\bar{\tau}) < \infty\} \geq s_\infty + t\alpha$.

Let $\epsilon > 0$ and let $\bar{\nu}$ be a $\bar{\sigma}$ -invariant probability measure where $\int \bar{\tau} d\bar{\nu} < \infty$ and $\left| \frac{\int \bar{\Delta}_g d\bar{\nu}}{\int \bar{\tau} d\bar{\nu}} - \alpha \right| < \frac{\epsilon}{2}$. We then have that

$$\begin{aligned} h_{\bar{\sigma}}(\bar{\nu}) - (s_{\infty} + t\alpha + |t\epsilon|) \int \bar{\tau} d\bar{\nu} + t \int \bar{\Delta}_g d\bar{\nu} &\leq h_{\bar{\sigma}}(\bar{\nu}) - \left(s_{\infty} + \frac{|t\epsilon|}{2} \right) \int \bar{\tau} d\bar{\nu} \\ &\leq P_{\bar{\sigma}}(-(s_{\infty} + |t\epsilon|/2)\bar{\tau}) < \infty. \end{aligned}$$

If on the other hand

$$\left| \frac{\int \bar{\Delta}_g d\bar{\nu}}{\int \bar{\tau} d\bar{\nu}} - \alpha \right| > \frac{\epsilon}{2}$$

then by (12) there exists a uniform constant C such that $\int \bar{\tau} d\bar{\nu} < C$ and so again $h_{\bar{\sigma}}(\bar{\nu}) - (s_{\infty} + t\alpha + |t\epsilon|) \int \bar{\tau} d\bar{\nu} + t \int \bar{\Delta}_g d\bar{\nu}$ is uniformly bounded. Therefore the proof now easily follows by the Variational Principle. $\square$

The lemma immediately implies that $P_{\Phi}(tg) \geq s_{\infty} + t\alpha$. We now let

$$I = \{t : P_{\bar{\sigma}}(-(s_{\infty} + t\alpha)\bar{\tau} + t\bar{\Delta}_g) \leq 0\}$$

and note that by the convexity of pressure this is either the empty set or an interval.

Proposition 6.3. *The regularity of the pressure function is given by*

1. *For $t \in I$ we have that $P_{\Phi}(tg) = s_{\infty} + t\alpha$. Moreover if $t \in \text{int}(I)$ then either g is cohomologous to the constant function α or tg is transient.*
2. *For $t \in \mathbb{R} \setminus I$ the function $t \rightarrow P_{\Phi}(tg)$ varies analytically and tg is positive recurrent.*

Proof. We prove the two items in separate ways.

1. Fix $t \in I$. Then Lemma 6.1 implies that if $s < s_{\infty} + t\alpha$ then $P_{\bar{\sigma}}(-s\bar{\tau} + t\bar{\Delta}_g) = \infty$. Since $t \in I$, $P_{\bar{\sigma}}(-(s_{\infty} + t\alpha)\bar{\tau} + t\bar{\Delta}_g) \leq 0$, so by Lemma 4.1 we have that $P_{\Phi}(tg) = s_{\infty} + t\alpha$. Furthermore since $\bar{\tau}$ and $\bar{\Delta}_g$ are locally Hölder the function $t \rightarrow P_{\bar{\sigma}}(-s_{\infty}\bar{\tau} + t(\alpha\bar{\tau} + \bar{\Delta}_g))$ is analytic and convex for $t \in \text{int}(I)$ (see [Sa4]). Thus either $P_{\bar{\sigma}}(-(s_{\infty} + t\alpha)\bar{\tau} + t\bar{\Delta}_g) < 0$ for all $t \in \text{int}I$ in which case tg is transient, or $P_{\bar{\sigma}}(-(s_{\infty} + t\alpha)\bar{\tau} + t\bar{\Delta}_g) = 0$ for all $t \in I$.

To complete the proof of 1, suppose that $\text{int}(I) \neq \emptyset$ and $P_{\bar{\sigma}}(-(s_{\infty} + t\alpha)\bar{\tau} + t\bar{\Delta}_g) = 0$ for all $t \in I$. Since $P_{\bar{\sigma}}(-s_{\infty}\bar{\tau} + t(\alpha\bar{\tau} + \bar{\Delta}_g)) = 0$ for all $t \in \text{int}(I)$ the associated Gibbs state μ_t has $\int \bar{\Delta}_g - \alpha\bar{\tau} d\mu_t = 0$. Thus μ_t is an equilibrium measure for $-(s_{\infty} + t\alpha)\bar{\tau} + t\bar{\Delta}_g$ for all $t \in \text{int}(I)$. By [MU3, Theorem 2.2.7] this implies that $\bar{\Delta}_g - \alpha\bar{\tau}$ is cohomologous to a constant which must be 0. Thus for all Φ -invariant probability measures $\int g d\mu = \alpha$ and thus g is cohomologous to the constant function α .

2. For this part we can follow the method from Proposition 6 in [BI1]. Note that the function $s \rightarrow P_{\bar{\sigma}}(-s\bar{\tau} + t\bar{\Delta}_g)$ is real analytic and decreasing for $s > s_{\infty} + t\alpha$. So if $J \subseteq \mathbb{R}$ is an open interval for which $J \cap I = \emptyset$ then for $t \in J$ we can define $P_{\Phi}(tg)$ implicitly by

$$P_{\bar{\sigma}}(-P_{\Phi}(tg)\bar{\tau} + t\bar{\Delta}_g) = 0.$$

If we let $\bar{\nu}_t$ denote the equilibrium measure for $-P_\Phi(tg)\bar{\tau} + t\bar{\Delta}_g$ then it follows that

$$\frac{\partial}{\partial s} P_{\bar{\tau}}(-s\bar{\tau} - t\bar{\Delta}_g) \big|_{s=P_\Phi(tg)} = - \int \bar{\tau} d\mu_t < 0.$$

Thus we can apply the implicit function theorem to show that $t \rightarrow P_\Phi(tg)$ is analytic on J . Hence $\nu = \mu_t \times m$ will be the equilibrium measure for tg and so tg is positive recurrent.

□

Thus we have phase transitions if and only if $I \neq \mathbb{R}$ and $I \neq \emptyset$.

Example 6.3. First of all we give an example of a potential g where $t \rightarrow P_\Phi(tg)$ is analytic for the whole of $\mathbb{R}$. We define

$$\tau(x) = \log(n+2) - \log(n+1) \text{ if } x \in C_n$$

and g to satisfy

$$\Delta_g(x) = -\log \log \log(n+2) + \log \log \log(n+1) \text{ if } x \in C_n.$$

This gives that

$$\bar{\tau}(x) = \log(n+1) \text{ if } x \in \bar{C}_n$$

and

$$\bar{\Delta}_g(x) = -\log \log \log(n+1) \text{ if } x \in \bar{C}_n.$$

Using the notation above this means that $s_\infty = 1$ and $\alpha = 0$. For any $t \in \mathbb{R}$ we have that

$$P_{\bar{\tau}}(-s_\infty \bar{\tau} + t\bar{\Delta}_g) = \log \left(\sum_{n=1}^{\infty} \frac{(\log \log(n+2))^{-t}}{n+2} \right) = \infty.$$

Thus we have that $I = \emptyset$ and by Proposition 6.3 the function $t \rightarrow P_\Phi(tg)$ is analytic.

Example 6.4. We now give an example with two phase transitions. We choose $K > 2$ to satisfy $\sum_{n=0}^{\infty} ((n+K)(\log(n+K))^2)^{-1} < \frac{1}{9}$. We set

$$\tau(x) = \begin{cases} \log 2 & \text{if } x \in C_0 \\ \log(K) - \log 2 & \text{if } x \in C_1 \\ \log(K+n-1) - \log(K+n-2) & \text{if } x \in C_n \text{ for } n \geq 2 \end{cases}$$

and g to satisfy that

$$\Delta_g(x) = \begin{cases} \log(4/3) & \text{if } x \in \bar{C}_1 \\ -\log \log(K) - \log(4/3) & \text{if } x \in C_1 \\ -\log \log(K+n-1) + \log \log(K+n-2) & \text{if } x \in C_n \text{ for } n \geq 2 \end{cases}$$

This gives that

$$\exp(-\bar{\tau}(x) + t\bar{\Delta}_g(x)) = \frac{1}{2} \left(\frac{4}{3} \right)^t \text{ if } x \in C_0$$

and for $n \geq 2$

$$\exp(-\bar{\tau}(x) + t\bar{\Delta}_g(x)) = \frac{1}{\log(K+n-2)} (\log(K+n-2))^{-t} \text{ if } x \in C_n.$$

As in the previous example we have that $s_\infty = 1$ and $\alpha = 0$. For $t \in \mathbb{R}$ we have that

$$P_{\bar{\sigma}}(-s_\infty \bar{\tau} + t \bar{\Delta}_g) = \log \left(\frac{1}{2} \left(\frac{4}{3} \right)^t + \sum_{n=2}^{\infty} \frac{1}{K+n-1} (\log(K+n-1))^{-t} \right).$$

Now for $t \leq 1$ this is divergent and for $t > 1$ this is convergent. If we take $t = 2$ then

$$P_{\bar{\sigma}}(-s_\infty \bar{\tau} + 2 \bar{\Delta}_g) = \log \left(\frac{8}{9} + \sum_{n=0}^{\infty} ((n+K)(\log(n+K))^2)^{-1} \right) < 0,$$

and so $\underline{t} = \inf\{t : P_{\bar{\sigma}}(-\bar{\tau}(x) + t \bar{\Delta}_g) \leq 0\} \in (1, 2)$. Furthermore for $t = 3$, we have that $\frac{1}{2} \left(\frac{4}{3} \right)^t > 1$ and so $P_{\bar{\sigma}}(-s_\infty \bar{\tau} + 3 \bar{\Delta}_g) > 0$. Thus $\bar{t} = \sup\{t : P_{\bar{\sigma}}(-\bar{\tau}(x) + t \bar{\Delta}_g) \leq 0\} \in (2, 3)$. Therefore $I = [\underline{t}, \bar{t}]$ and for $t \in I$ we will have that $P_\Phi(tg) = s_\infty = 1$. There will be phase transitions at $\underline{t}$ and $\bar{t}$ and outside of I the function $t \rightarrow P_\Phi(tg)$ will vary analytically and be strictly greater than s_∞ .

6.5. Improving (or not) recurrence properties. In this sub-section, we discuss the idea that by suspending a system with a roof function not bounded away from zero, we can speed up the return times improving the mixing properties and therefore obtaining better thermodynamics. The example we consider is the renewal flow. We show that no general statement can be made. In order to be more precise, let Φ_R be a renewal flow with roof function τ not bounded away from zero and let $g : Y \rightarrow \mathbb{R}$ be a potential. We say that the recurrence properties of the potential $\Delta_g : \Sigma \rightarrow \mathbb{R}$ *improve* if Δ_g is transient and g is recurrent or if Δ_g is null-recurrent and g is positive recurrent.

Example 6.5. Let $\tau \in \mathcal{R}$ be a roof function such that

$$P_\sigma(-t\tau) = \begin{cases} \text{positive} & t < 1; \\ 0 & t \geq 1. \end{cases}$$

By virtue of Proposition 6.1 we have that for $t < 1$ the potential $-t\tau$ is positive recurrent and for $t > 1$ the potential $-t\tau$ is transient. Moreover, for $t = 1$ the potential can be positive recurrent, null-recurrent or transient (see Section 6.3 and [Sa3, Example 2]).

Consider the suspension flow Φ_R with roof function τ . The constant potential, $g = C$, defined over Y_R is positive recurrent. Indeed note that the corresponding potential $\Delta_g = C\tau$ is such that

$$P_\Phi(tg) = \inf\{q \in \mathbb{R} : P_\sigma(tC\tau - q\tau) \leq 0\}.$$

Since $P_\Phi(tC\tau - q\tau) = P_\sigma((tC - q)\tau)$ we obtain that

$$P_\sigma(tg) = tC + 1.$$

The above, of course, could have been obtained from the Variational Principle for the flow. In both cases below we are able to make our conclusion using Proposition 6.2.

1. Let us consider first the case in which the potential $-\tau$ is transient. In this case the potential $t\Delta_g - P_\Phi(g)\tau = -\tau$ is transient, therefore the potential tg is transient for every $t \in \mathbb{R}$. However, if $t < 1/C$ the potential $t\Delta_g$ is positive recurrent while tg is transient.

2. Assume that $-\tau$ is null-recurrent with infinite measure ν such that $\int \tau d\nu < \infty$ (see Lemma 7.1 for an example of this). If $t > 1/C$ then the potential $t\Delta_g$ is transient and tg is positive recurrent.

The simple example above shows that the recurrence properties and the thermodynamic formalism can either improve or get worse by suspending with a roof function not bounded away from zero.

7. SUSPENSION FLOWS OVER MANNEVILLE-POMEAU MAPS

7.1. Manneville-Pomeau flows. In this section we study suspension flows over a simple non-uniformly hyperbolic interval map, namely the Manneville-Pomeau map [MP]. We give the form studied in [LSV]. For $\alpha > 0$ the map is defined by

$$f(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{if } x \in [0, 1/2), \\ 2x - 1 & \text{if } x \in [1/2, 1). \end{cases}$$

The pressure function of the potential $-\log |f'|$ satisfies the following (see [Lo, Sa3]),

$$P_f(-t \log |f'|) = \begin{cases} \text{strictly convex and real analytic} & \text{if } t < 1, \\ 0 & \text{if } t \geq 1. \end{cases}$$

It is well known (see [Lo, Sa3]) that if $\alpha \in (0, 1)$ then there exists an absolutely continuous invariant measure. This measure together with the Dirac delta at zero are the equilibrium measures for $-\log |f'|$. If $\alpha > 1$ then there is no absolutely continuous invariant measure and the only equilibrium measure for $-\log |f'|$ is the Dirac delta at zero. However, there exists an infinite f -invariant measure ν absolutely continuous with respect to Lebesgue. This measure is such that (see [N, p.849])

$$\int \log |f'| d\nu < \infty. \quad (13)$$

If we remove the parabolic fixed point at zero and its preimages, then the (non-compact) dynamical system that is left can be coded by the renewal shift (see [Sa3]). More precisely, if we denote by $\Omega = [0, 1] \setminus \cup_{n=0}^{\infty} f^{-n}(0)$, then the map f restricted to Ω can be coded by the renewal shift.

We define the *Manneville-Pomeau flow*, that we denote by Φ_{mp} , as the suspension semi-flow with base $f(x)$ and roof-function $\log |f'|$. We denote by Y its phase space. This flow has a singularity at $(0, 0)$ and there exists an atomic invariant measure supported on it. If we remove the point $(0, 0)$ and all its pre-images, then the non-compact semi-flow that is left can be coded as a renewal flow with roof function the symbolic representation of $\log |f'|$. We denote this renewal flow by Φ_R . The set of invariant measure for the Manneville-Pomeau flow is in one to one correspondence with the set of invariant measures for the renewal flow, denoted by $\mathcal{M}_{\Phi_R}$, together with the atomic measure supported at $(0, 0)$.

Remark 7.1. Melbourne [M2] studied the decay of correlations for this class of semi-flows with respect to the measure of maximal entropy for the flow. The decay rates are related to that of the f -invariant absolutely continuous measure. This

is also closely related to the study of the thermodynamics of Rovella flows, see [PT]. Moreover, Holland, Nicol and Török have studied Extreme Value Theory for Manneville-Pomeau flows in [HNT].

Let $g : Y \rightarrow \mathbb{R}$ be a bounded potential and consider the map $\Delta_g : [0, 1] \rightarrow \mathbb{R}$ defined by

$$\Delta_g(x) = \int_0^{\log |f'(x)|} g(t, x) dt.$$

Denote by Δ_g^r the symbolic representation of $\Delta_g(x)$ in the renewal shift. We will develop a thermodynamic formalism for the following class of potentials:

$$\mathcal{MP} := \{g : Y \rightarrow \mathbb{R} : g \text{ is bounded and } \Delta_g^r(x) \in \mathcal{R}\}.$$

We define the pressure of a potential $g \in \mathcal{MP}$ by

$$P_{\Phi_{mp}}(g) := \sup \left\{ h(\mu) + \int g d\mu : \mu \in \mathcal{E}_{mp} \right\},$$

where $\mathcal{E}_{mp}$ denotes the set of ergodic Φ_{mp} -invariant probability measures. We stress that $\mathcal{E}_{mp}$ is in one to one correspondence with the set $\mathcal{E}_{\Phi_R} \cup \delta_{(0,0)}$.

Lemma 7.1. *Every Manneville-Pomeau flow has entropy equal to one and there exists a unique measure of maximal entropy.*

Proof. Recall that $h(\Phi_{mp}) = \inf\{s \in \mathbb{R} : P(-s \log |f'|) \leq 0\}$. Since

$$P_f(-\log |f'|) = 0,$$

and for every $t < 0$ we have that $P_f(-t \log |f'|) > 0$ the entropy of the flow is equal to 1. The potential $-\log |f'|$ is recurrent and because of the property stated in equation (13) the flow has a measure of maximal entropy. It is unique since regular potentials over countable Markov shifts do have at most one equilibrium measure [BS] and the atomic measure supported at $(0, 0)$ has zero entropy. $\square$

Note that if $\alpha \geq 1$ then this measure of maximal entropy will be finite for the flow but will project to an infinite invariant measure for the Manneville-Pomeau map.

In our next result we discuss the existence and uniqueness of equilibrium measures for potentials in $\mathcal{MP}$. Denote by $\log^r |f'|$ the symbolic representation of $\log |f'|$ in the renewal shift. First note that it follows from Proposition 6.2 that the equation $P(\Delta_g^r - s \log^r |f'|) = 0$, always has a root, that we denote by $P_{\Phi_{mp}}^r(g)$.

Proposition 7.1. *Let $g \in \mathcal{MP}$.*

1. *If $\Delta_g^r - P_{\Phi_{mp}}^r(g) \log^r |f'|$ is positive recurrent then there exists an equilibrium measure for g . Moreover,*
 - (a) *If $P_{\Phi_{mp}}^r(g) \neq \int g \delta_{(0,0)}$ then the equilibrium measure for g is unique.*
 - (b) *If $P_{\Phi_{mp}}^r(g) = \int g \delta_{(0,0)}$ then there exist two equilibrium measures for g .*
2. *If $\Delta_g^r - P_{\Phi_{mp}}^r(g) \log^r |f'|$ is null recurrent with infinite measure ν and we have that $\log^r |f'| \in L^1(\mu)$ then there exists an equilibrium measure for g . Moreover,*
 - (a) *If $P_{\Phi_{mp}}^r(g) \neq \int g \delta_{(0,0)}$ then the equilibrium measure for g is unique.*

- (b) If $P_{\Phi_{mp}}^r(g) = \int g \delta_{(0,0)}$ then there exist two equilibrium measures for g .
3. Assume that $\Delta_g^r - P_{\Phi_{mp}}^r(g) \log^r |f'|$ is null recurrent with infinite measure ν and $\log^r |f'| \notin L^1(\mu)$. If $P_{\Phi_{mp}}^r(g) > \int g \delta_{(0,0)}$ then there is no equilibrium measure for g . On the other hand, if $P_{\Phi_{mp}}^r(g) < \int g \delta_{(0,0)}$ then $\delta_{(0,0)}$ is the unique equilibrium measure for g .
4. Assume that $\Delta_g^r - P_{\Phi_{mp}}^r(g) \log^r |f'|$ is transient. If $P_{\Phi_{mp}}^r(g) > \int g \delta_{(0,0)}$ then there is no equilibrium measure for g . On the other hand, if $P_{\Phi_{mp}}^r(g) < \int g \delta_{(0,0)}$ then $\delta_{(0,0)}$ is the unique equilibrium measure for g .

Proof. The proof follows directly from Proposition 6.2 and the observation that

$$P_{\Phi_{mp}}(g) = \max \left\{ P_{\Phi_{mp}}^r(g), \int g \delta_{(0,0)} \right\}.$$

□

Remark 7.2. It is possible to use Proposition 6.3 to obtain potentials where the pressure function for the Manneville-Pomeau flow has phase transitions. If we consider the induced potential $\bar{\tau}$ we have that $s_\infty = \frac{\alpha}{\alpha+1}$. We now take a negative function g where $\overline{\Delta g}$ satisfies that

$$\lim_{n \rightarrow \infty} \frac{\sup_{x \in C_n} \{\overline{\Delta g}(x)\}}{\inf_{x \in C_n} \{\bar{\tau}(x)\}} = 0$$

and

$$\lim_{n \rightarrow \infty} \frac{\inf_{x \in C_n} \{-\overline{\Delta g}(x)\}}{\log \log n} = \infty.$$

This means that for all $t > 0$ we have $P_{\bar{\tau}}(-s_\infty \bar{\tau} + t \overline{\Delta g}) < \infty$ and there will exist $t^* \in \mathbb{R}$ such that $P_{\bar{\tau}}(-s_\infty \bar{\tau} + t \overline{\Delta g}) = 0$ and for all $t \geq t^*$, $P_{\bar{\tau}}(-s_\infty \bar{\tau} + t \overline{\Delta g}) < 0$. Thus $t \rightarrow P_\Phi(tg)$ will have a (positive entropy) phase transition at t^* .

7.2. On the boundary of hyperbolicity. The class of suspension semi-flows studied in the previous section belong to the boundary of hyperbolic systems. Indeed, consider a family of height functions $\tau_l : [0, 1] \rightarrow \mathbb{R}$ of positive, bounded away from zero Hölder functions such that the family τ_l converges uniformly to $\tau(x) = \log |f'(x)|$ as l converges to zero.

The suspension semi-flows Φ_l defined over

$$Y_l = \{(x, t) \in [0, 1] \times \mathbb{R} : 0 \leq t \leq \tau_l(x)\}, \quad (14)$$

in the canonical way. These flows are hyperbolic (see [BR]). The pressure function

$$t \rightarrow P(tg)$$

is real analytic for any potential g such that the symbolic representation of Δ_g belongs to $\mathcal{R}$. Moreover, there exists a unique equilibrium measure for g .

The Manneville-Pomeau semi-flow Φ_{mp} is obtained from Φ_l when l tends to zero. In dynamical terms, we have a Hopf bifurcation at the Manneville Pomeau flow. That is, a periodic orbit is collapsed into a singularity. The results in Subsection 7.1 show that the thermodynamic formalism is completely different at the bifurcation than at the hyperbolic component.

REFERENCES

- [Aa] J. Aaronson, *An introduction to infinite ergodic theory*. Mathematical Surveys and Monographs, 50. American Mathematical Society, Providence, RI, 1997.
- [Ab] L.M. Abramov, *On the entropy of a flow*, Dokl. Akad. Nauk SSSR **128** (1959), 873–875.
- [AK] W. Ambrose and S. Kakutani, *Structure and continuity of measurable flows*, Duke Math. J. **9** (1942), 25–42.
- [Ar] E. Artin, *Ein Mechanisches System mit quasi-ergodischen Bahnen*. Collected papers (Addison Wesley, Reading, Mass., 1965), pp. 499–501.
- [BM] P. Bálint, I. Melbourne, *Decay of correlations and invariance principles for dispersing billiards with cusps, and related planar billiard flows*, J. Stat. Phys. **133** (2008), 435–447.
- [BI1] L. Barreira and G. Iommi, *Suspension flows over countable Markov shifts*, J. Stat. Phys. **124** (2006), 207–230.
- [BI2] L. Barreira and G. Iommi, *Multifractal analysis and phase transitions for hyperbolic and parabolic horseshoes*, Israel J. Math. **181** (2011), 347–379.
- [BRW] L. Barreira, L. Radu and C. Wolf, *Dimension of measures for suspension flows*, Dyn. Syst. **19** (2004), 89–107.
- [BDV] Ch. Bonatti, L. Díaz and M. Viana, *Dynamics beyond uniform hyperbolicity. A global geometric and probabilistic perspective*. Encyclopaedia of Mathematical Sciences, 102. Mathematical Physics, III. Springer-Verlag, Berlin, 2005. xviii+384 pp.
- [Bo] R. Bowen, *Symbolic dynamics for hyperbolic flows*, Amer. J. Math. **95** (1973), 429–460.
- [BR] R. Bowen and D. Ruelle, *The ergodic theory of Axiom A flows*, Invent. Math. **29** (1975), 181–202.
- [BT] H. Bruin and M. Todd, *Transience and thermodynamic formalism for infinitely branched interval maps*. J. London Math. Soc. **86** (2012), 171–194.
- [BG] A.I. Bufetov and B.M. Gurevich, *Existence and uniqueness of the measure of maximal entropy for the Teichmüller flow on the moduli space of Abelian differentials*, Sbornik Mathematics **202** (2011), 935–970.
- [BS1] L.A. Bunimovich and Y.G. Sinai, *Markov partitions for dispersed billiards*, Comm. Math. Phys. **78** (1980/81), 247–280.
- [BS2] L.A. Bunimovich and Y.G. Sinai, *Statistical properties of Lorentz gas with periodic configuration of scatterers*, Comm. Math. Phys. **78** (1980/81), 479–497.
- [BChS1] L.A. Bunimovich, Y.G. Sinai and N.I. Chernov, *Markov partitions for two-dimensional hyperbolic billiards*, (Russian) Uspekhi Mat. Nauk **45** (1990), no. 3(273), 97–134, 221; translation in Russian Math. Surveys **45** (1990), no. 3, 105–152.
- [BS] J. Buzzi and O. Sarig, *Uniqueness of equilibrium measures for countable Markov shifts and multidimensional piecewise expanding maps*, Ergodic Theory Dynam. Systems **23** (2003), 1383–1400.
- [ChM1] N.I. Chernov and R. Markarian, *Chaotic billiards*. Mathematical Surveys and Monographs, 127. American Mathematical Society, Providence, RI, 2006. xii+316 pp.
- [ChM2] N.I. Chernov and R. Markarian, *Dispersing billiards with cusps: slow decay of correlations*, Comm. Math. Phys. **270** (2007), 727–758.
- [CQ] Z. Coelho and A. Quas, *Criteria for d -continuity*, Trans. Amer. Math. Soc. **350** (1998), 3257–3268.
- [Da] Y. Daon, *Bernoullicity of equilibrium measures on countable Markov shifts*, Preprint (arXiv:1206.4160).
- [DGR] L. Díaz, K. Gelfert and M. Rams, *Rich phase transitions in step skew products*, Nonlinearity **24** (2011), 3391–3412.
- [FMT] M. Field, I. Melbourne and A. Török, *Stability of mixing and rapid mixing for hyperbolic flows*, Ann. of Math. **166** (2007), 269–291.
- [Gu1] B.M. Gurevič, *Topological entropy for denumerable Markov chains*, Dokl. Akad. Nauk SSSR **10** (1969), 911–915.
- [Gu2] B.M. Gurevič, *Shift entropy and Markov measures in the path space of a denumerable graph*, Dokl. Akad. Nauk SSSR **11** (1970), 744–747.
- [Hal] B. Halpern, *Strange billiard tables*, Trans. Amer. Math. Soc. **232** (1977), 297–305.
- [Ha] U. Hamenstädt, *Symbolic dynamics for the Teichmüller flow*, Preprint (arXiv:1112.6107).
- [Ho] F. Hofbauer, *Examples for the nonuniqueness of the equilibrium state*, Trans. Amer. Math. Soc. **228** (1977), 223–241.

- [HNT] M. P. Holland, M. Nicol and A. Torok, *Extreme value theory for non-uniformly expanding dynamical systems*, Trans. Amer. Math. Soc. **364** (2012), 661–688.
- [H] E. Hopf, *Ergodentheorie* Ergeb. Mat. vol. 5 Springer, Berlin 1937.
- [IJ] G. Iommi and T. Jordan, *Phase transitions for suspension flows*, To appear in Comm. Math. Phys. (arXiv:1202.0849).
- [IT] G. Iommi and M. Todd, *Transience in Dynamical Systems*, ‘FirstView’ article at Ergodic Theory Dynam. Systems.
- [JKL] J. Jaerisch, M. Kesseböhmer and S. Lamei, *Induced topological pressure for countable state Markov shifts*, Preprint (arXiv:1010.2162).
- [K] G. Keller, *Equilibrium states in ergodic theory*, London Mathematical Society Student Texts, 42. Cambridge University Press, Cambridge, 1998.
- [Ke] T. Kempton, *Thermodynamic formalism for suspension flows over countable Markov shifts*, Nonlinearity **24** (2011), 2763–2775.
- [LSV] C. Liverani, B. Saussol and S. Vaienti, *Probabilistic approach to intermittency*, Ergodic Theory Dynam. Systems **19** (1999), 671–685.
- [Lo] A.O. Lopes, *The zeta function, nondifferentiability of pressure, and the critical exponent of transition*, Adv. Math. **101** (1993), 133–165.
- [MP] P. Manneville and Y. Pomeau, *Intermittent transition to turbulence in dissipative dynamical systems*, Comm. Math. Phys. **74** (1980), 189–197.
- [MU1] R. Mauldin and M. Urbański, *Dimensions and measures in infinite iterated function systems*, Proc. London Math. Soc. (3) **73** (1996), 105–154.
- [MU2] R. Mauldin and M. Urbański, *Gibbs states on the symbolic space over an infinite alphabet*, Israel J. Math. **125** (2001), 93–130.
- [MU3] R. Mauldin and M. Urbański, *Graph directed Markov systems: geometry and dynamics of limit sets*, Cambridge tracts in mathematics 148, Cambridge University Press, Cambridge 2003.
- [M1] I. Melbourne, *Rapid decay of correlations for nonuniformly hyperbolic flows*, Trans. Amer. Math. Soc. **359** (2007), 2421–2441.
- [M2] I. Melbourne, *Decay of correlations for slowly mixing flows*, Proc. London Math. Soc. **98** (2009) 163–190.
- [MT] I. Melbourne and A. Török, *Statistical limit theorems for suspension flows*, Israel J. Math. **144** (2004), 191–209.
- [Mo] H.M. Morse, *Recurrent geodesics on a surface of negative curvature*. Trans. Amer. Math. Soc. **22** (1921), 84–100.
- [N] K. Nakaishi, *Multifractal formalism for some parabolic maps*, Ergodic Theory Dynam. Systems **20**, (2000) 843–857.
- [PT] M.J. Pacifico and M. Todd, *Thermodynamic formalism for contracting Lorenz flows*, J. Stat. Phys. **139** (2010), 159–176.
- [PP] W. Parry and M. Pollicott, *Zeta Functions and the Periodic Orbit Structure of Hyperbolic Dynamics*, Astérisque **187–188**, 1990.
- [Ra] M. Ratner, *Markov partitions for Anosov flows on n -dimensional manifolds*, Israel J. Math. **15** (1973), 92–114.
- [Sa1] O. Sarig, *Thermodynamic formalism for countable Markov shifts*, Ergodic Theory Dynam. Systems **19** (1999), 1565–1593.
- [Sa2] O. Sarig, *Thermodynamic formalism for null recurrent potentials*, Israel J. Math. **121** (2001), 285–311.
- [Sa3] O. Sarig, *Phase transitions for countable Markov shifts*, Comm. Math. Phys. **217** (2001), 555–577.
- [Sa4] O. Sarig, *Existence of Gibbs measures for countable Markov shifts*, Proc. Amer. Math. Soc. **131** (2003), 1751–1758.
- [Sa5] O. Sarig, *Symbolic dynamics for surface diffeomorphisms with positive topological entropy*, Preprint (arXiv:1105.1650).
- [Sav] S. Savchenko, *Special flows constructed from countable topological Markov chains*, Funct. Anal. Appl. **32** (1998), 32–41.
- [Se] C. Series, *Geometrical Markov coding of geodesics on surfaces of constant negative curvature*, Ergodic Theory Dynam. Systems **6** (1986), 601–625.
- [Wa] P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics 79, Springer, 1981.

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