THERMODYNAMIC FORMALISM FOR INTERVAL MAPS: INDUCING SCHEMES

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ABSTRACT. This survey article concerns inducing schemes in the context of interval maps. We explain how the study of these induced systems allows for the fine description of, not only, the thermodynamic formalism for certain multimodal maps, but also of its multifractal structure.

1. INTRODUCTION

A major breakthrough in theory of dynamical systems was the realisation that complicated behaviour exhibited by certain systems can be studied using probabilistic methods. This approach is based on what is usually called *ergodic theory* and at its core is the study of existence and properties of dynamically relevant measures, the so-called *invariant measures*. It was Poincaré [Poi] who realised that the simple existence of a finite invariant measure yields non-trivial information on the orbit structure. Invariant measures exist under very weak assumptions on the phase space and on the system [W, Corollary 6.9.1], hence the power of the approach.

There is however a difficulty, in that a large class of interesting dynamical systems have many invariant measures. Indeed, in simple settings such as the full-shift on two symbols the set of invariant probability measures is a Poulsen simplex [GW, LOS], that is, an infinite dimensional, convex and compact set for which the extreme points are dense on the whole set. It is, therefore, an important problem to find criteria to choose *relevant* invariant measures. This is where the *thermodynamic formalism* comes into play. Indeed, given a dynamical system $T : X \to X$ and a continuous function $\varphi : X \to \mathbb{R}$ (the *potential*) the *topological pressure* can be defined in the following way

$$P(\varphi) = \sup \left\{ h(\mu) + \int \varphi \, d\mu : \mu \in \mathcal{M}_T \text{ and } - \int \varphi \, d\mu < \infty \right\}, \qquad (1)$$

where $h(\mu)$ denotes the entropy of the measure μ (see [W, Chapter 4]) and $\mathcal{M}_T$ denotes the space of $T-$invariant probability measures. A measure $\nu \in \mathcal{M}_T$ attaining the above supremum is called an *equilibrium state*. Proving existence and

uniqueness of equilibrium states is one of the major problems in the theory of thermodynamic formalism. This approach to choosing relevant measures as equilibrium states is a set of ideas and techniques which derive from statistical mechanics and that was brought into dynamics in the early seventies by Ruelle and Sinai among others [Dobr, Si, Bo, K3, Ru, W].

Properties and regularity of thermodynamic formalism depend, essentially, on two factors. On the one hand the regularity of the potential considered; and on the other the hyperbolicity/expansiveness of the underlying dynamical system. If the potential φ is regular enough (say Hölder continuous) and the system is sufficiently hyperbolic (say uniformly expanding) then there exists a unique equilibrium state for φ and it has strong ergodic properties. Moreover, the pressure function $t \mapsto P(t\varphi)$ is real analytic on $\mathbb{R}$ (see [Ru]).

The main focus of this survey article will be to study thermodynamic formalism for interesting systems that are far from being uniformly hyperbolic and for natural potentials that are not even continuous. That is, the two main features which govern the thermodynamic formalism behave in a poor way. However, we will show that even in that setting we are able to describe the pressure function in great the detail. We will survey results from the papers [IT1] and [IT2]. The main technique we use is that of *inducing*. The idea is to associate to our dynamical system a generalisation of the first return map (the *induced system*) that not only is uniformly expanding but also is a full-shift on a countable alphabet. A price one has to pay in order to obtain a Markov structure and expansiveness is that our dynamical system is no longer defined over a compact set. This is serious matter since in that context the thermodynamic formalism, even for regular potentials, can be irregular (see Section 2). Another issue is that it is possible that the inducing procedure 'throws away' parts of the system which are important in obtaining thermodynamic quantities. We will discuss the advantages and disadvantages of this approach, explain the main steps in implementing this, and state the results proved in the above papers.

The main class of dynamical systems we consider here, denoted by $\mathcal{F}$, is the collection of C^2 multimodal interval maps $f : I \to I$, where $I = [0, 1]$, satisfying:

(a) the critical set $\mathcal{C}r = \mathcal{C}r(f)$ consists of finitely many critical points c with critical order $1 < \ell_c < \infty$, i.e., there exists a neighbourhood U_c of c and a C^2 diffeomorphism $g_c : U_c \to g_c(U_c)$ with $g_c(c) = 0$, $f(x) = f(c) \pm |g_c(x)|^{\ell_c}$;
(b) f has negative Schwarzian derivative, i.e., $1/\sqrt{|Df|}$ is convex;
(c) f is topologically transitive on I (i.e., there exists a dense orbit);
(d) $f^n(\mathcal{C}r) \cap f^m(\mathcal{C}r) = \emptyset$ for $m \neq n$.

Note that the class $\mathcal{F}$ includes transitive *Collet-Eckmann maps*, that is maps where $|Df^n(f(c))|$ grows exponentially fast in n. In particular, the set of quadratic Collet-Eckmann maps in $\mathcal{F}$ has positive Lebesgue measure in the parameter space of quadratic maps (see [BC, J]). Our main application will be to maps in this family, although we will also remark on extensions (see Section 1.3). It will sometimes be useful to restrict our family further. For an interval map $f : I \to I$, the *metric attractor* is a set $A \subset I$ such that the corresponding set $B(A) := \{x \in I : \omega(x) \subset A\}$ has positive Lebesgue measure and there is no proper subset of A with this property (here $\omega(x)$ is the set of accumulation points of the orbit of x). On the other hand,

A is a *topological attractor* if $B(A)$ is residual. The map f has a *wild attractor* if the there is a metric attractor which is not a topological attractor[1]. We call the set of interval maps without wild attractors *tame*, and denote those in $\mathcal{F}$ by $\mathcal{F}_T$. We will also be interested in $f \in \mathcal{F}$ which admit a measure $\mu \in \mathcal{M}_f$ which is absolutely continuous w.r.t. Lebesgue, which we call an *acip*. If f has an acip μ, then f must be in $\mathcal{F}_T$. This can be seen using [SV, Theorem E] in the C^3 case, or by using topological transitivity to show that the density of μ is positive everywhere: then Poincaré recurrence implies that a metric attractor must include the whole of I.

We will be particularly interested in the *geometric potentials* $x \mapsto -t \log |Df(x)|$. The equilibrium states corresponding to the geometric potentials capture important geometric features that allow us to study the fractal geometry of dynamically relevant subsets of the phase space (see Theorem B). These measures are supported on sets for which the expansion properties of the system are simple, in the sense that expansion rates are constant for almost every point. This is of particular importance because when there is strong expansion the system behaves as in the classical uniformly hyperbolic theory. The geometric potentials are also related to the existence of an acip as we will see in Section 1.1. The presence of critical points, which leads to significant distortion problems with both the map and the geometric potentials, as well as the non-Markov structure make these systems particularly interesting, and particularly suited to study using inducing methods.

Considering the above discussion, we define the various kinds of Lyapunov exponents. Note that we will often suppress the dependence of these quantities on the map f in the notation. Given a smooth interval map f, the *lower (pointwise) Lyapunov exponent* and *upper (pointwise) Lyapunov exponent* at $x \in I$ are defined as

$$\underline{\lambda}(x) = \underline{\lambda}_f(x) := \liminf_{n\to\infty} \frac{1}{n} \log |Df^n(x)| \text{ and } \overline{\lambda}(x) = \overline{\lambda}_f(x) := \limsup_{n\to\infty} \frac{1}{n} \log |Df^n(x)|,$$

respectively. When these values are equal, we denote their common value, the *(pointwise) Lyapunov exponent* at x, by $\lambda_f(x)$. Note that in some cases, we will deal with only piecewise smooth functions, in which case we disregard points where the derivative is not defined. We define the global infimum and supremum of these quantities by

$$\lambda_{\inf} = \lambda_{\inf}(f) := \inf\{\lambda_f(x) : x \in I \text{ and this limit exists}\},$$

and

$$\lambda_{\sup} = \lambda_{\sup}(f) := \sup\{\lambda_f(x) : x \in I \text{ and this limit exists}\}.$$

We also define the *Lyapunov exponent* of $\mu \in \mathcal{M}_f$ to be

$$\lambda(\mu) = \lambda_f(\mu) := \int \log |Df| \ d\mu.$$

Note that if μ is ergodic then $\lambda(x) = \lambda(\mu)$ for μ-a.e. $x \in I$. We also define

$$\lambda_m = \lambda_m(f) := \inf\{\lambda_f(\mu) : \mu \in \mathcal{M}_f\} \text{ and } \lambda_M = \lambda_M(f) := \sup\{\lambda_f(\mu) : \mu \in \mathcal{M}_f\}.$$

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[1] By part (c) of the definition of the class $\mathcal{F}$, the topological attractor of a map $f \in \mathcal{F}$ is always the whole of I.

In this setting, [Pr] shows that $\lambda_m(f) \geqslant 0$ for all $f \in \mathcal{F}$. As in [IT2, Lemma 4.1], $\lambda_M(f) = \lambda_{\sup}(f)$ and $\lambda_m(f) \geqslant \lambda_{\inf}(f)$ for $f \in \mathcal{F}$. We would expect, by analogue with the complex setting in [GPR, Lemma 6], that actually $\lambda_m(f) = \lambda_{\inf}(f)$ in general: in fact aside from the possibility that $\lambda_{\inf}(f) = 0$, but $\lambda_m(f) > 0$, this can now be shown using recent work of Rivera-Letelier.[2] The main part of the argument is to show that in the case $\lambda_{\inf}(f) > 0$, if there is a point $x \in I$ with $\lambda(x) > 0$, then we can fix a small $\delta > 0$ such that for each $\varepsilon > 0$, the size of the pullback to x by f^{-n} of $B(f^n(x), \delta)$ is larger than $2\delta(\lambda(x) + \varepsilon)^{-n}$ for all large n. This dominance of the shrinking by the pointwise Lyapunov exponent follows since the orbit cannot approach the critical set too quickly, so we have a bounded distortion condition. The Main Theorem of [R] then shows that the exponential rate of shrinking of such pullbacks can not be smaller than $\lambda_{\inf}(f)$, so $\lambda(x) \geqslant \lambda_{\inf}(f)$. A similar argument implies that $\lambda_m(f) \geqslant 0$ also.

Define the 'expanding' and 'good' elements of $\mathcal{F}$ by

$$F_{\exp} := \{f \in \mathcal{F} : \lambda_m(f) > 0\} \text{ and } F_{\mathrm{acip}} := \{f \in \mathcal{F} : \text{ there exists an acip}\}.$$

Some of the key behaviour here can be determined by the growth along the critical orbits. In the unimodal case, where there is only one critical point c, the main result of [NS] implies that $f \in F_{\exp}$ if and only if $\underline{\lambda}_f(c) > 0$. In [BRSS], it was shown that even in the multimodal case $|Df^n(f(c))| \to \infty$ for all $c \in \mathcal{C}r$ implies $f \in F_{\mathrm{acip}}$. Moreover, all unimodal maps with critical order 2 are tame, see [Ly, GSS]. However, when there is more than one critical point, the situation is obscured by the possibility of mixed behaviour between different critical points.

1.1. Conformal measures.
We now fix some notation and give an important definition. Denote $\varphi_t := -t \log|Df|$. The pressure function will be denoted by $p(t) = p_f(t) := P(-t \log|Df|)$. Given a potential $\varphi : I \to \mathbb{R}$, we say that the measure (not necessarily invariant) m is a φ-*conformal measure* if for any measurable set $A \subset I$ such that $f : A \to f(A)$ is a bijection, we have

$$m(f(A)) = \int_A e^{-\varphi} \, dm.$$

We also write $dm(f(x)) = e^{-\varphi(x)} dm(x)$. Before stating our first main Theorem, we observe that if $f \in \mathcal{F}$ then f has no periodic attractors and so $p(1) = 0$ and Lebesgue measure m_1 is $-\log|f'|$-conformal, see [Le, NS]. Furthermore, by [Le], if $\mu \in \mathcal{M}_f$ has $\lambda(\mu) > 0$, then $\mu \ll m_1$ (i.e. μ is an acip) if and only if μ is an equilibrium state for $-\log|f'|$. Theorem A below concerns a more general set of measures.

1.2. Main results.
Let $t^+ = t^+(f) := \sup\{t : p_f(t) > -\lambda_m(f)t\}$. The following result was proved in [IT1].

Theorem A. *Let $f \in \mathcal{F}$. Then $t^+ \in (0, \infty]$ and:*

(a) *for each $t < t^+$, there exists a unique equilibrium state μ_t for φ_t;*
(b) *for each $t < t^+$, there exists a unique $(\varphi_t - p(t))$-conformal measure m_t. Moreover, $\mu_t \ll m_t$;*

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[2]We are grateful to J. Rivera-Letelier for pointing this out to us.

(c) $p(t)$ is strictly decreasing, strictly convex and C^1 on $(-\infty, t^+)$;

Furthermore,

(1) if $f \in F_{exp}$ then $t^+ > 1$;
(2) if $f \in F_{acip} \setminus F_{exp}$, then $t^+ = 1$, $D^- p(1) < 1$ and there exists an acip;
(3) if $f \in \mathcal{F}_T$ then $t^+ = 1$. If moreover $f \in \mathcal{F}_T \setminus F_{acip}$ then $p'(1) = 0$;

Remark 1.1. *To the best of our knowledge, the behaviour of the pressure function is not known for $f \in \mathcal{F} \setminus \mathcal{F}_T$. However, the works [AL] and [BT3] suggest that $t^+ < 1$, $p'(t^+) = 0$ and there is no equilibrium state for $-t^+ \log |Df|$.*

It is an interesting fact that despite the strong lack of expansiveness of the system (principally due to the presence of critical points), we are able to describe the pressure function $p(t)$ in great detail. Note that for $t > t^+$ the pressure function $p(t)$ is linear. The point t^+ has been called *freezing point* [PR]. The language here comes from statistical mechanics; for example, comparable transitions occur at a freezing point in Fischer-Felderhof models, see [F]. We stress that if $t^+ < \infty$ then the pressure function exhibits a *phase transition* at $t = t^+$, that is, the pressure function is not real analytic at $t = t^+$.

Theorem A is a generalisation of the following results. [BK] deals with the case of unimodal Collet-Eckmann maps (recall that this means $\underline{\lambda}_f(f(c)) > 0$) for a small range of t near 1, one advantage of their results being analyticity of pressure in that range. [PS] considers a subset of Collet-Eckmann maps, but for all t in a neighbourhood of $[0, 1]$. [BT2, Theorem 1] applies to a class of non-Collet Eckmann multimodal maps with t in a left-sided neighbourhood of 1, again proving analyticity of pressure in that range. For rational maps, a result of this kind has been obtained by Przytycki and Rivera-Letelier [PR], in which they consider the full range of t and obtain analyticity of pressure in that range. A more detailed study of the behaviour of the pressure function at the phase transition t^+, inspired by [MS], was carried out in [CR].

As we have seen, the expanding properties of the systems coded by the Lyapunov exponents are of fundamental importance in order to describe the dynamical properties of the system. It is, therefore, natural to try to characterise the sets

$$J(\lambda) := \left\{ x \in I : \lambda(x) = \lambda \right\}.$$

for $\lambda \in [\lambda_{\inf}, \lambda_{\sup}]$. Defining

$$J' := \left\{ x \in I : \lim_{n \to \infty} \frac{1}{n} \log |Df^n(x)| \text{ does not exist} \right\},$$

we can write

$$I = J' \cup \bigcup_{\lambda \in [\lambda_{\inf}, \lambda_{\sup}]} J(\lambda),$$

which is the *multifractal decomposition of I by Lyapunov exponents*. In particular, we are interested in the Hausdorff dimension of these sets:

$$L(\lambda) := \dim_H(J(\lambda)).$$

This function, usually called *Lyapunov spectrum*, can be computed by means of thermodynamic formalism, in particular using the pressure and its derivative.

To state our second main theorem, we define

$$A_f := \begin{cases} [\lambda_{\inf}, -D^- p(t^+)) & \text{if } \lambda_m(f) > 0, \\ \{0\} & \text{if } \lambda_m(f) = 0. \end{cases}$$

The following Theorem was obtained in [IT2]. A further theorem regarding the multifractal spectrum of local dimension was also proved there, but we omit an exposition of that for brevity.

Theorem B. *Let $f \in F_{acip}$ and $\lambda \in \mathbb{R} \setminus A$. Then*

$$L(\lambda) = \frac{1}{\lambda} \inf_{t \in \mathbb{R}} (p(t) + t\lambda).$$

If $\lambda \in (-D^- p(t^+), \lambda_M(f))$ then

$$L(\lambda) = \frac{1}{\lambda}(p(t_\lambda) + t_\lambda \lambda) = \frac{h(\mu_{t_\lambda})}{\lambda}.$$

If $\lambda_m(f) > 0$ and $\lambda \in A$ then

$$L(\lambda) \geqslant \frac{1}{\lambda} \inf_{t \in \mathbb{R}} (p(t) + t\lambda).$$

Moreover, $\dim_H(J') = 1$.

While the rigorous study of Lyapunov spectrum began with the work of Weiss [We], previous studies which built up the theory include [CLP, EP, HJK, Ra]. Since then the relation established in Theorem B between the pressure and the Lyapunov spectrum has been proved in a wide range of different settings. For example, for maps with parabolic fixed points results have been obtained in [BI, GR, N, PolW]. For rational maps a good description of the Lyapunov spectrum has been given in [GPR, GPRR]; we note that here, as well as in [GR], very different methods were used to those presented here. For a study of this kind of equilibrium state and of Lyapunov exponents in classes of non-uniformly hyperbolic systems in dimension two see [LR, LOR, ST1, ST2]. For an example of Theorems A and B in the case of a flow, see [PT], where they were proved for contracting Lorenz-like maps, via the results described here applied to the class of cusp maps given below.

1.3. **Extension to other classes of interval maps.** The theory outlined in this paper applies to various classes of piecewise smooth interval/circle maps. This includes Manneville-Pomeau maps and Lorenz-like maps, but our main focus is on smooth interval maps with critical points, the class $\mathcal{F}$. However, we briefly describe a generalisation of this class, so-called *cusp maps*, studied by Ledrappier [Le] and later extended by Dobbs [D]. This class includes the class of contracting Lorenz-like maps, see for example [Ro]. The extension of our main theorems to this class is described in [IT2].

Definition 1.1. *$f : \cup_j I_j \to I$ is a* cusp map *if there exist constants $C, \alpha > 1$ and a set $\{I_j\}_j$ is a finite collection of disjoint open subintervals of I such that*

(1) $f_j := f|_{I_j}$ is $C^{1+\alpha}$ on each $I_j =: (a_j, b_j)$ and $|Df_j| \in (0, \infty)$.

(2) $D^+ f(a_j)$, $D^- f(b_j)$ *exist and are equal to 0 or* $\pm\infty$.
(3) *For all* $x, y \in \overline{I_j}$ *such that* $0 < |Df_j(x)|, |Df_j(y)| \leqslant 2$ *we have* $|Df_j(x) - Df_j(y)| < C|x - y|^\alpha$.
(4) *For all* $x, y \in \overline{I_j}$ *such that* $|Df_j(x)|, |Df_j(y)| \geqslant 2$, *we have* $|Df_j^{-1}(x) - Df_j^{-1}(y)| < C|x - y|^\alpha$.

Matching the notation above, denote the set of points a_j, b_j *by* $\mathcal{C}r$.

Remark 1.2. *Notice that if for some* j, $b_j = a_{j+1}$, *i.e.* $I_j \cap I_{j+1}$ *intersect, then* f *may not continuously extend to a well defined function at the intersection point* b_j, *since the definition above would then allow* f *to take either one or two values there. So in the definition above, the value of* $f_j(a_j)$ *is taken to be* $\lim_{x \searrow a_j} f_j(x)$ *and* $f_j(b_j) = \lim_{x \nearrow b_j} f_j(x)$, *so for each* j, f_j *is well defined on* $\overline{I_j}$.

Remark 1.3. *In contrast to the class of smooth maps* $\mathcal{F}$ *considered previously in this paper, for cusp maps we can have* $\lambda_M = \infty$ *and/or* $\lambda_m = -\infty$. *The first possibility follows since we allow singularities (points where the one-sided derivative is* ∞). *The second possibility follows from the presence of critical points (although is avoided for smooth multimodal maps with non-flat critical points by* [Pr]). *Examples of both of these possibilities can be found in* [D, Section 11].

1.4. **Outline of the paper.** The layout of the paper is as follows. In Section 2 we give a detailed exposition of thermodynamic formalism in the context of the full-shift on a countable alphabet; examples of some of the irregular behaviour that can occur in this setting are given. The importance of that section relies on the fact that the inducing schemes we consider are conjugated to the full-shift on a countable alphabet. In Section 3 a brief summary of classical definitions and results on dimension theory that will be used in the rest of the paper is given. Section 4 introduces and describes the induced systems. This is our main technical device and it allows for the proof of all the results presented here. Several important results are given in this section, for example it is explained how the entropy of a measure in the induced system relates to the entropy of the measure on the original system (Abramov's formula) and also how the integral of a potential relates to the integral of its induced version (Kac's formula). Moreover, a fundamental result stating that we only need to consider a countable collection of inducing schemes in order to 'see' all the ergodic theory of the original system is given. In Section 5 we explain how to implement the inducing approach in order to describe the thermodynamic formalism of the original system: we outline the proof of the relevant result, Theorem 5.1. In Section 6 we explain the proof of Theorem B where we describe the Lyapunov spectrum in the context of multimodal maps. We also include appendices, containing both explanatory comments and new proofs of some results. For instance in Appendix A we describe and explain the Hofbauer extension technique which is implicit in our results. In Appendix B we give a new proof of a result that relates conformal measures for the induced system with conformal measures in the original one. We give conditions that ensure that we can project a conformal measure from the induced system into the original one. Finally, in Appendix C we correct an error in a result we previously used from [PolW, Proposition 3], where the pointwise dimension is related to the Markov dimension. This result is used in our proof of Theorem B.

2. Thermodynamic formalism for countable Markov shifts

This section is devoted to study thermodynamic formalism for countable Markov shifts. While the theory is well developed for topologically mixing systems (see [S1]), we will concentrate on a particular case, namely the full-shift, which corresponds to the symbolic representation of the *inducing schemes* that we will consider. In this setting the theory was mostly developed by Mauldin and Urbański [MU1]. We will see that the thermodynamic formalism in this setting is similar to that of finite state Markov shifts [Ru].

The *full-shift* on a countable alphabet (Σ, σ) is the set

$$\Sigma := \left\{ (x_n)_{n \in \mathbb{N}_0} : x_n \in \mathbb{N}_0 \text{ for every } n \in \mathbb{N}_0 \right\},$$

together with the shift map $\sigma : \Sigma \to \Sigma$ defined by $\sigma(x_0, x_1, \dots) = (x_1, x_2, \dots)$. The set $C_{i_0 \dots i_{n-1}} := \{ (x_n)_n \in \Sigma : x_0 = i_0, \dots, x_{n-1} = i_{n-1} \}$ is called a *cylinder* of length n. The space Σ endowed with the topology generated by the cylinder sets is a non-compact space. This fact is one of the main difficulties that need to be addressed to develop the theory. Indeed, the classical definition of pressure using (n, ϵ)-separated sets (see [W, Chapter 9]) depends upon the metric and, if the space is not compact, can be different even for two equivalent metrics. The regularity assumptions on the potentials are fundamental when it comes to describing and developing the thermodynamic formalism. The *n-th variation* of $\varphi : \Sigma \to \mathbb{R}$ is defined by

$$V_n := \sup\{ |\varphi(x) - \varphi(y)| : x, y \in \Sigma, x_i = y_i, 0 \le i \le n - 1 \}.$$

Note that φ is continuous if and only if $V_n \to 0$. We say that φ is of *summable variations* if $\sum_{n=2}^{\infty} V_n(\varphi) < \infty$. We say that it is *weakly Hölder* (with parameter θ) if there exists $\theta \in (0,1)$ such that for $n \ge 2$ we have $V_n(\varphi) = O(\theta^n)$. Clearly, if φ is of summable variations then it is continuous and if φ is weakly Hölder then it is of summable variations. The following definition was given by Mauldin and Urbański [MU1].

Definition 2.1. *Let (Σ, σ) be the full-shift on a countable alphabet and let $\varphi : \Sigma \to \mathbb{R}$ a potential of summable variations. The* pressure *of φ is given by*

$$P(\varphi) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \exp \left(\sum_{i=0}^{n-1} \varphi(\sigma^i x) \right).$$

The limit always exists but it can be infinity. Indeed $P(0) = \infty$, which by the Variational Principle, which we'll see later, means that the entropy of the full-shift is infinity.

Remark 2.1. *Sarig* [S1], *building up on work by Gurevich* [Gu1, Gu2], *extended this definition to arbitrary topologically mixing countable Markov shifts (Σ, σ). He defined the* Gurevich pressure *of a potential $\varphi : \Sigma \to \mathbb{R}$ of summable variations by*

$$P_G(\varphi) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \exp \left(\sum_{i=0}^{n-1} \varphi(\sigma^i x) \right) \chi_{C_{i_0}}(x),$$

where $\chi_{C_{i_0}}(x)$ is the characteristic function of the cylinder C_{i_0}. It can be proved that the limit always exists (it can be infinity) and that it is independent of i_0.

Moreover, if (Σ, σ) is the full-shift then the Gurevich pressure coincides with the pressure defined by Mauldin and Urbański [S4], i.e., $P_G(\varphi) = P(\varphi)$.

If the potential $\varphi : \Sigma \to \mathbb{R}$ depends only on the first coordinate, that is for every $i \in \mathbb{N}$ there exists $\lambda_i > 0$ such that $\varphi(x_i x_1 \dots) = \log \lambda_i$, then the pressure can be explicitly computed:

$$
\begin{aligned}
P(\varphi) &= \lim_{n \to \infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \exp \left(\sum_{i=0}^{n-1} \varphi(\sigma^i x) \right) \\
&= \lim_{n \to \infty} \frac{1}{n} \log \sum_{(j_0 \dots j_{n-1}) \in \mathbb{N}^n} (\lambda_{j_0} \lambda_{j_1} \cdots \lambda_{j_{n-1}}) \\
&= \lim_{n \to \infty} \frac{1}{n} \log \left(\sum_{i \in \mathbb{N}_0} \lambda_i \right)^n = \log \sum_{i=0}^{\infty} \lambda_i.
\end{aligned}
$$

The pressure satisfies the following approximation property (see [MU1, S1]).

Theorem 2.1. *Let (Σ, σ) be the full-shift and $\varphi : \Sigma \to \mathbb{R}$ a potential of summable variations. If $\mathcal{K} := \{K \subset \Sigma : K \text{ compact and } \sigma\text{-invariant}, K \neq \emptyset\}$ then*

$$
P(\varphi) = \sup\{P(\varphi|K) : K \in \mathcal{K}\},
$$

where $P(\varphi|K)$ is the topological pressure of φ restricted to the compact set K (for definition and properties see [W, Chapter 9]).

This notion of pressure also satisfies the Variational Principle (see [MU1, S1]):

Theorem 2.2 (Variational Principle)**.** *Let (Σ, σ) be the full-shift and $\varphi : \Sigma \to \mathbb{R}$ a potential of summable variations then*

$$
P(\varphi) = \sup \left\{ h(\mu) + \int \varphi \, d\mu : \mu \in \mathcal{M}_\sigma \text{ and } - \int \varphi \, d\mu < \infty \right\}
$$

where $\mathcal{M}_\sigma$ denotes the set of $\sigma-$invariant probability measures and $h(\mu)$ denotes the entropy of the measure μ (for a precise definition see [W, Chapter 4]).

It is worth pointing out the remarkable fact that this theorem relates quantities of a different nature. While the pressure is defined in topological terms, the left hand side of the Variational Principle only depends on the Borel structure. Also note that since the system has infinite entropy, it is possible for a measure $\mu \in \mathcal{M}_\sigma$ to have $h(\mu) = \infty$ and $\int \varphi \, d\mu = -\infty$. In that case, the sum of both quantities is meaningless, which is why the condition $- \int \varphi \, d\mu < \infty$ is imposed.

A measure $\mu \in \mathcal{M}_\sigma$ attaining the supremum in the Variational Principle, that is

$$
P(\varphi) = h(\mu) + \int \varphi \, d\mu,
$$

is called an *equilibrium state* for φ. Buzzi and Sarig [BuS] proved that a potential of summable variations has at most one equilibrium state. There are examples of locally constant potentials of finite pressure that don't have equilibrium states (see Example 2.1).

Given a potential $\varphi : \Sigma \to \mathbb{R}$, a measure μ on Σ is called a *Gibbs measure* for the potential φ if there exist constants $C > 0$ and $P \in \mathbb{R}$ such that for every cylinder $C_{i_0 i_1 \ldots i_{n-1}}$ and every $x \in C_{i_0 i_1 \ldots i_{n-1}}$ we have

$$\frac{1}{C} \leq \frac{\mu(C_{i_0 i_1 \ldots i_{n-1}})}{\exp(-nP + \sum_{i=0}^{n-1} \varphi(\sigma^i x))} \leq C.$$

That is, the measure of a cylinder is comparable to the Birkhoff sum of the potential, normalised by some P, the *Gibbs constant*. This a useful property that enables us to estimate the measure of a set. In the settings described here, if μ is an equilibrium state for φ, then it is a Gibbs measure with Gibbs constant $P = P(\varphi)$. The following result was obtained (in a slightly more general setting) by Sarig in [S4]. The sufficient part of the theorem was proved by Mauldin and Urbański [MU2].

Theorem 2.3. *Let (Σ, σ) be the full-shift and $\varphi : \Sigma \to \mathbb{R}$ a potential such that $\sum_{i=1}^{\infty} V_i(\varphi) < \infty$ then φ has an invariant Gibbs measure if and only if $P(\varphi) < \infty$.*

Example 2.1. *The following example was given by Sarig in [S4]. Let (Σ, σ) the the full-shift and for $n \in \mathbb{N}$, let $a_n := 1/(2n(\log 2n)^2)$, and set $a_0 \in \mathbb{R}$ such that $\sum_{n=0}^{\infty} a_n = 1$. Then define the potential $\varphi : \Sigma \to \mathbb{R}$ by $\varphi(x_0 x_1 \ldots) = \log a_{x_0}$. Note that $P(\varphi) = 0$. The potential φ does not have an equilibrium state. Indeed, the corresponding Gibbs measure is the Bernoulli measure $\mu(C_{i_0 \ldots i_n}) = \prod_{j=0}^{n} a_{i_j}$. We have that $h(\mu) = \infty$ and $\int \varphi \, d\mu = -\infty$. In particular this example shows that, opposite to what happens for finite state Markov shifts, a Gibbs measure might not be an equilibrium state.*

In [S2] Sarig generalises the Ruelle Perron Frobenius Theorem to countable Markov shifts. For φ a potential of summable variations define the Ruelle operator defined formally in some space of functions by:

$$L_\varphi g(x) := \sum_{\sigma y = x} \exp(\varphi(y)) g(y).$$

Theorem 2.4. *Let (Σ, σ) be the full-shift and $\varphi : \Sigma \to \mathbb{R}$ a potential such that $\sum_{i=1}^{\infty} V_i(\varphi) < \infty$ and $P(\varphi) = \log \lambda$. Then there exists a conservative conformal measure m and a continuous function h such that $L_\varphi^* m = \lambda m$, $L_\varphi h = \lambda h$ and $\int h \, dm < \infty$. The measure $\mu = hm$ is an equilibrium state for φ if $\int \varphi \, d\mu < \infty$.*

2.1. **Possible pressure behaviours.** Given a potential $\varphi : \Sigma \to \mathbb{R}$ we will be particularly interested in the pressure function $t \mapsto P(t\varphi)$. It follows directly from the approximation property of the pressure (Theorem 2.1) and the fact that pressure for finite state Markov shifts is convex that the pressure function is convex. Moreover, if the potential φ is non-positive then the pressure function is non-increasing. Actually, if φ is non-positive there exists $t_0 \in [0, \infty]$ such that

$$P(t\varphi) = \begin{cases} \infty & \text{if } t < t_0; \\ \text{finite} & \text{if } t > t_0. \end{cases}$$

The regularity of the pressure function is well understood. The following result was proved by Sarig and by Cyr and Sarig [S3, CS].

Theorem 2.5. *Let (Σ, σ) be the full-shift and $\varphi : \Sigma \to \mathbb{R}$ a weakly Hölder potential of finite pressure then, when finite, the pressure function is real analytic and*

$$\frac{d}{dt}P(t\varphi) = \int \varphi \ d\mu_t,$$

where μ_t is the equilibrium state corresponding to $t\varphi$.

Summarising we see that for regular potentials the thermodynamic formalism in the full-shift behaves in similar way as the finite state Markov shifts. A potential does have a corresponding invariant Gibbs measure, and if this measure has finite entropy then it is the unique equilibrium state. The pressure function, when finite, is real analytic. Exactly the same properties are satisfied when considering topologically mixing finite state Markov shifts and Hölder potentials (see [Bo, Chapter 2] and [PP, Chapters 3 and 4]).

Example 2.2. *There exist potentials for which the pressure function is either infinite or strictly negative. The class of potentials with that property was called* irregular *by Mauldin and Urbański [MU1]. Let $N \in \mathbb{N}$ be such that the sequence $a_n = 1/(2(n + N)(\log 2(n + N))^2)$ satisfies $\sum_{n=1}^{\infty} a_n < 1$. Then the potential $\varphi(x_0 x_1 \dots) = \log a_{x_0}$ is such that*

$$P(t\varphi) = \begin{cases} \text{infinite} & \text{if } t < 1; \\ \text{negative} & \text{if } t \geq 1. \end{cases}$$

The following example was given by Mauldin and Urbański [MU1]. For each $n \in \mathbb{N}$ consider $2^{n^2-1} + 1$ cylinders of length one (partition the space in cylinder of length one according to this criteria). Define a locally constant potential $\psi(x) = 2^{-(n^2+n)}$, if x belongs to one of the above cylinders. Then

$$P(t\psi) = \begin{cases} \text{infinite} & \text{if } t < 1; \\ \text{negative} & \text{if } t \geq 1. \end{cases}$$

Example 2.3. *The following is an example of a potential which is not irregular. Let $a_n = n(n+1)$ and consider $\varphi : \Sigma \to \mathbb{R}$ to be the locally constant potential defined by $\varphi(x)\big|_{C_n} := -\log a_n$. Then*

$$P(t\varphi) = \log \sum_{n=1}^{\infty} \left(\frac{1}{n(n+1)}\right)^t.$$

Therefore if $t \leqslant 1/2$ then $P(t\psi) = \infty$ Moreover, $t \mapsto P(t\psi)$ is real analytic on $(1/2, \infty)$. Furthermore,

$$\lim_{t \to 1/2} P(t\varphi) = \infty.$$

As we have seen, for a non-positive potential φ there exists a critical value $t_0 \geq 0$ such that for $t < t_0$ the pressure function is infinite and for $t > t_0$ is finite. If $P(t_0\varphi) < \infty$ it is possible for

$$\lim_{t \to t_0^+} \frac{d}{dt}P(t\varphi)$$

to be either finite or infinite.

Example 2.4. *The following examples are due to Kesseböhmer, Munday and Strat-mann* [KMS]. *Let* $a_n := n^{-2}(\log(n+5))^{-12}/C$, *where* $C := \sum_{n=1}^{\infty} n^{-2}(\log(n+5))^{-12}$. *Consider* $\varphi : \Sigma \to \mathbb{R}$ *to be the locally constant potential defined by* $\varphi(x)\big|_{C_n} := \log a_n$. *In this case* $t_0 = 1/2$ *and* $P((1/2)\varphi) < 1$. *Moreover,*

$$\lim_{t \to 1/2^+} \frac{d}{dt} P(t\varphi) < \infty.$$

Consider now $b_n := n^{-2}(\log(n+5))^{-4}/C$, *where* $C := \sum_{n=1}^{\infty} n^{-2}(\log(n+5))^{-4}$. *Consider* $\psi : \Sigma \to \mathbb{R}$ *to be the locally constant potential defined by* $\psi(x)\big|_{C_n} := \log a_n$. *In this case* $t_0 = 1/2$ *and* $P((1/2)\psi) < 1$ *and*

$$\lim_{t \to 1/2^+} \frac{d}{dt} P(t\psi) = \infty.$$

3. Dimension theory

This section is devoted to recalling basic definitions and results from dimension theory that will be used in what follows (see [P] and [PrU] for details). A countable collection of sets $\{U_i\}_{i \in \mathbb{N}}$ is called a δ-cover of $F \subset \mathbb{R}$ if $F \subset \bigcup_{i \in \mathbb{N}} U_i$, and U_i has diameter $|U_i|$ at most δ for every $i \in \mathbb{N}$. Letting $s > 0$, we define

$$\mathcal{H}^s(J) := \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\}_i \text{ is a } \delta\text{-cover of } J \right\}.$$

The *Hausdorff dimension* of the set J is defined by

$$\dim_H(J) := \inf \left\{ s > 0 : \mathcal{H}^s(J) = 0 \right\}.$$

Hausdorff dimension is invariant under bi-Lipschitz transformations

Proposition 3.1. *Let* $\pi : J \subset \mathbb{R}^n \to \mathbb{R}^n$ *be a bi-Lipschitz transformation then*

$$\dim_H(J) = \dim_H(\pi(J)).$$

Given a finite Borel measure μ in F, the *lower pointwise dimension* and *upper pointwise dimension* of μ at the point x are defined by

$$\underline{d}_\mu(x) := \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \quad \text{and} \quad \overline{d}_\mu(x) := \limsup_{r \to 0} \frac{\log \mu(B(x,r))}{\log r},$$

respectively, where $B(x,r)$ is the ball at x of radius r. Whenever these limits are equal, we denote their common limit by $d_\mu(x)$, the *pointwise dimension* of μ at x. This function describes the power law behaviour of $\mu(B(x,r))$ as $r \to 0$, that is

$$\mu(B(x,r)) \sim r^{d_\mu(x)}.$$

The pointwise dimension quantifies how concentrated a measure is around a point: the larger it is the less concentrated the measure is around that point. Note that if μ is an atomic measure supported at the point x_0 then $d_\mu(x_0) = 0$ and if $x_1 \neq x_0$ then $d_\mu(x_1) = \infty$.

The following propositions relating the pointwise dimension with the Hausdorff dimension can be found in [P, Section 7].

Proposition 3.2. *Given a finite Borel measure μ, if $\underline{d}_\mu(x) \leqslant d$ for every $x \in F$, then $\dim_H(F) \leqslant d$.*

The *Hausdorff dimension* of the measure μ is defined by

$$\dim_H(\mu) := \inf \{\dim_H(Z) : \mu(Z) = 1\}.$$

Proposition 3.3. *Given a finite Borel measure μ, if $d_\mu(x) = d$ for μ-almost every $x \in F$, then $\dim_H(\mu) = d$.*

We finish this section by defining another notion of dimension, which applies whenever we have a dynamical system $f : X \to X$ and a partition $\mathcal{P}$, which thus gives rise to the notion of k-cylinders. Denote the k-cylinder at $x \in X$ by $I_k(x)$. Note that x could be in two k-cylinders simultaneously, but in this case we make an arbitrary choice: in fact these points will not be important here due to their small dimension and measure. In the case of a shift, this issue does not arise, and $\mathcal{P}$ can be taken as the partition into 1-cylinders.

Definition 3.1. *Given a system as above with a probability measure μ on X, we define the* lower Markov pointwise dimension *and* upper Markov pointwise dimension *of μ at the point x as*

$$\underline{\delta}_\mu(x) := \liminf_{k \to \infty} \frac{\log \mu(I_k(x))}{\log |I_k(x)|} \quad and \quad \overline{\delta}_\mu(x) := \limsup_{k \to \infty} \frac{\log \mu(I_k(x))}{\log |I_k(x)|},$$

respectively. Here, $|\cdot|$ denotes the Euclidean length. If these values coincide, then their common value, the Markov pointwise dimension *of μ at x, is denoted $\delta_\mu(x)$.*

4. INDUCING SCHEMES

In this section we introduce our main technical tool, namely *inducing schemes*. We show how its ergodic theory is related to that of the original system and how can it be understood using thermodynamic formalism for countable Markov shifts. A natural way of thinking of inducing schemes is as a generalisation of the first return time map.

We say that $(X, \{X_i\}_i, F, \tau) = (X, F, \tau)$ is an *inducing scheme* for (I, f) if

- X is an interval containing a finite or countable collection of disjoint intervals $\{X_i\}_i$ such that F is expanding and maps each X_i diffeomorphically onto X, with bounded distortion (i.e. there exist $C > 0$ and $\lambda > 1$ such that $|DF^n| \geqslant C\lambda^n$, and $K > 0$ so that for all i and $x, y \in X_i$, $|DF(x)/DF(y) - 1| \leqslant K|x-y|$);
- $\tau|_{X_i} = \tau_i$ for $\tau_i \in \mathbb{N}$ and $F|_{X_i} = f^{\tau_i}$. If $x \notin \cup_i X_i$ then $\tau(x) = \infty$.

For our main theorems to hold, we don't need such a strong bounded distortion condition, but for other applications we often do, so we leave the condition in here. The function $\tau : \cup_i X_i \to \mathbb{N}$ is called the *inducing time*. It may happen that $\tau(x)$ is the first return time of x to X, but that is certainly not the general case. We denote the set of points $x \in I$ for which there exists $k \in \mathbb{N}$ such that $\tau(F^n(f^k(x))) < \infty$ for all $n \in \mathbb{N}$ by $(X, F, \tau)^\infty$.

The space of F-invariant measures is related to the space of f-invariant measures. Indeed, given an f-invariant measure μ, if there is an F-invariant measure μ_F such that for a subset $A \subset I$,

$$\mu(A) = \frac{1}{\int \tau \, d\mu_F} \sum_i \sum_{k=0}^{\tau_i - 1} \mu_F \left(f^{-k}(A) \cap X_i \right) \qquad (2)$$

where $\frac{1}{\int \tau \, d\mu_F} < \infty$, we call μ_F the *lift* of μ and say that μ is a *liftable* measure. Conversely, given a measure μ_F that is F-invariant we say that μ_F *projects* to μ if (2) holds. We say that an f-invariant probability measure μ is *compatible* with the inducing scheme (X, F, τ) if

- $\mu(X) > 0$ and $\mu(X \setminus (X, F)^\infty) = 0$, and
- there exists a F-invariant measure μ_F which projects to μ.

For a potential $\varphi : I \to \mathbb{R}$, $x \in I$ and $k \in \mathbb{N}$, we define

$$S_k \varphi(x) := \varphi(x) + \varphi \circ f(x) + \cdots + \varphi \circ f^{k-1}(x).$$

Given an inducing scheme (X, F, τ), we define the induced potential $\Phi : X \to \mathbb{R}$ by $\Phi(x) = S_{\tau(x)} \varphi(x)$ for $x \in X$ whenever $\tau(x) < \infty$, and $-\infty$ otherwise.

Let μ be a liftable measure and be ν be its lift. A classical result by Abramov [A] (see also [PS, Z]) allows us to relate the entropy of both measures. Further results obtained in [PS, Z] allow us to do the same with the integral of a given potential $\varphi : I \to \mathbb{R}$. Indeed, for the induced potential Φ we have that

$$h(\mu) = \frac{h(\nu)}{\int \tau \, d\nu} \text{ and } \int \varphi \, d\mu = \frac{\int \Phi \, d\nu}{\int \tau \, d\nu}.$$

The following result appeared in [IT2, Theorem 3.3] (it can be proved using [T]). It provides us with a countable family of relevant inducing schemes.

Theorem 4.1. *Let* $f \in \mathcal{F}$. *There exist a countable collection* $\{(X^n, F_n)\}_n$ *of inducing schemes such that:*

(a) *any ergodic invariant probability measure* μ *with* $\lambda(\mu) > 0$ *is compatible with one of the inducing schemes* (X^n, F_n). *In particular there exists an ergodic* F_n-*invariant probability measure* μ_{F_n} *which projects to* μ;
(b) *any equilibrium state for* $-t \log |Df|$ *where* $t \in \mathbb{R}$ *with* $\lambda(\mu) > 0$, *or for a Hölder continuous potential* $\varphi : I \to \mathbb{R}$ *with* $\sup \varphi < P(\varphi)$, *is compatible with all inducing schemes* (X^n, F_n).
(c) *if* $f \in \mathcal{F}_{acip}$ *then*

$$dim_H \left(I \setminus (\cup_{n=1}^\infty (X^n, F_n)^\infty) \right) = 0.$$

To give a trivial example of part of the importance of this theorem, suppose that a point $x_0 \in I$ is a repelling periodic point for f: there exists $p \geqslant 1$ such that $f^p(x_0) = x_0$ and $|Df^p(x_0)| > 1$. Then for a small interval $X \ni x_0$ there is a subset $X_1 \ni p$ such that $f^p(X_1) = X$. So we have produced an inducing scheme (X, F) with one branch X_1 and $\tau|_{X_1} = p$. The measure $\mu := \frac{1}{p} (\delta_{x_0} + \delta_{f(x_0)} + \cdots + \delta_{f^{p-1}(x_0)})$ lifts to (X, F), but is the unique measure which does so. Therefore, if we were only

interested in μ, then (X, F) is sufficient for our purposes, but if we wanted to consider other measures, for example with positive entropy, then (X, F) is useless.

4.1. **Young towers.** Observe that an inducing scheme (X, F, τ) for (I, f) can be used to build a tower, which in the context described here is often referred to as a Young tower (see eg [Y]). Even though we don't use these ideas here directly, we give a short description to clarify its relationship with the inducing scheme. The tower is defined as the disjoint union

$$\Delta = \bigsqcup_i \bigsqcup_{j=0}^{\tau_i - 1} (X_i, j),$$

with dynamics

$$f_\Delta(x, j) = \begin{cases} (x, j+1) & \text{if } x \in X_i, j < \tau_i - 1; \\ (F(x), 0) & \text{if } x \in X_i, j = \tau_i - 1. \end{cases}$$

Given i and $0 \leqslant j < \tau_i$, let $\Delta_{i,j} := \{(x, j) : x \in X_i\}$ and $\Delta_l := \bigcup_i \Delta_{i,l}$ is called the l-th floor. Define the natural projection $\pi_\Delta : \Delta \to X$ by $\pi_\Delta(x, j) = f^j(x)$, and $\pi_X : \Delta \to X$ by $\pi_X(x, j) = x$. Note that (Δ, f_Δ) is a Markov system, and the first return map of f_Δ to the *base* Δ_0 is isomorphic to (X, F, τ).

Also, given $\psi : I \to \mathbb{R}$, let $\psi_\Delta : \Delta \to \mathbb{R}$ be defined by $\psi_\Delta(x, j) = \psi(f^j(x))$. Then the induced potential of ψ_Δ to the first return map to Δ_0 is exactly the same as the induced potential of ψ to the inducing scheme (X, F, τ).

4.2. **Coding.** One of the main tools in the proof of the results presented here is that an inducing scheme as described here can be coded by the full-shift on at most countably many symbols. That is, given a point $x \in (X, F)^\infty$, this point is given the code $\pi x = (x_0, x_1, \ldots)$ where $x_k = i$ if $F^k(x) \in X_i$. Notice that π is bijective due to the hyperbolicity of F. Thus there is a conjugacy

$$\begin{array}{ccc} (X, F)^\infty & \xrightarrow{\;F\;} & (X, F)^\infty \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \Sigma & \xrightarrow{\;\sigma\;} & \Sigma \end{array}$$

Moreover, the potentials $\Phi : (X, F) \to [-\infty, \infty]$ have symbolic versions $\Phi \circ \pi^{-1} : \Sigma \to [-\infty, \infty]$. We abuse notation by not explicitly distinguishing between the symbolic and standard version of potentials. Observe that if $\Phi : \cup_i X_i \to [-\infty, \infty]$ is Hölder, then the hyperbolicity and distortion conditions on the inducing scheme imply that the symbolic version is also weakly Hölder continuous. Moreover the properties of our inducing schemes imply that the potential $-\log |DF|$ is also Hölder continuous and its symbolic version is weakly Hölder continuous.

4.3. **Induced level sets.** Given an inducing scheme (X, F, τ), and $\lambda \in \mathbb{R}$, let

$$J_F(\lambda) := \left\{ x \in X : \lim_{k \to \infty} \frac{\sum_{j=0}^{k-1} \log |DF(F^j(x))|}{\sum_{j=0}^{k-1} \tau(F^j(x))} = \lambda \right\}.$$

A key observation here is that if $x \in (X, F)^\infty$, then $x \in J(\lambda)$ if and only if $x \in J_F(\lambda)$. Moreover, the same holds if $f^k(x) \in (X, F)^\infty$ for some $k \in \mathbb{N}$. Therefore the Lyapunov spectrum of (I, f) can be completely described via inducing schemes so long as inducing schemes cover a sufficient amount of points in $J(\lambda)$: that is, so long as there is no subset of $J(\lambda)$ of larger Hausdorff dimension than $J_F(\lambda)$ for all our inducing schemes (X, F, τ).

5. Implementing the inducing approach

In this section we take the potential $\varphi_t - p_f(t) = -t \log |Df| - p_f(t)$ and consider its induced version $\Psi_t = -t \log |DF| - \tau p_f(t)$. Furthermore, for $i \in \mathbb{N}$, let $\Psi_{i,t} := \sup_{x \in X_i} \Psi_t(x)$. The principal result guaranteeing that we can study our measures and multifractal properties using inducing schemes is the following.

Theorem 5.1. *Given $f \in \mathcal{F}_T$, there exists an inducing scheme (X, F, τ) such that for each $t \in (-\infty, t^+)$,*

  (a) $P(\Psi_t) = 0$;
  (b) $\sum_i \tau_i e^{\Psi_{i,t}} < \infty$.

We sketch the proof of this below, but first state and prove a corollary.

Corollary 5.1. *Under the conditions of Theorem 5.1,*
  (i) there exists an equilibrium state μ_t for φ_t;
  (ii) $t \mapsto P(\varphi_t)$ is C^1 in $(-\infty, t^+)$;
 (iii) there exists a ψ_t-conformal measure m_t, and $\mu_t \ll m_t$.

Proof. The existence of an equilibrium state μ_{Ψ_t} for Ψ_t follows from Theorem 2.3 and Theorem 5.1(a). To prove that μ_Φ projects to an equilibrium state for the original system, we need to check that $\int \tau \, d\mu_{\Psi_t} < \infty$. Since μ_{Ψ_t} is a Gibbs measure, this integral is comparable to the sum $\sum_i \tau_i e^{\Psi_{i,t}}$, which is finite by Theorem 5.1(b), proving (i).

The fact that $s \mapsto \int \tau \, d\mu_{\Psi_s}$ and $s \mapsto \int \log |DF| \, d\mu_{\Psi_s}$ are continuous for $s \in (t - \varepsilon, t + \varepsilon)$ can be easily deduced, so long as μ_{Ψ_s} all lift to the same inducing scheme. Therefore $s \mapsto \lambda(\mu_s)$ is continuous. The proof of (ii) then follows from an analogue of Theorem 2.5, which is standard.

The proof of the existence of a Ψ_t-conformal measure, which is equivalent to μ_{Ψ_t} follows from Theorem 5.1(a) and Theorem 2.4. The fact that it projects to a ψ_t-conformal measure follows as in Theorem B.1. $\square$

The main steps in the proof of Theorem 5.1 are contained in Sections 4-6 of [IT1].
(1) $P(\Psi_t) = 0$:
    (i) $P(\Psi_t) \leqslant 0$ is straightforward, see [IT1, Lemma 4.1]. The idea is to assume the contrary. Then Theorem 2.1 implies that there is an approximation of (X, F) by a subsystem with N branches with $h(\nu_N) + \int \Psi_t > 0$. Projecting ν_N to μ_N, Abramov's formula implies that $h(\mu_N) + \int \psi_t \, d\mu_N > 0$, contradicting the Variational Principle.

(ii) Proving $P(\Psi_t) \geqslant 0$ is much harder. The strategy is to find a sequence $(\mu_n)_n$ such that $h(\mu_n) + \int \psi_t \, d\mu_n \to 0$, but crucially to show that $\int \tau \, d\mu_n$ is uniformly bounded. The idea for this is that by our choice of $t \in (-\infty, t^+)$, there is a constant $K > 0$ such that for all large n, $h(\mu_n) \geqslant K$. This means that there is a finite set of domains in the Hofbauer extension (see Appendix A, particularly Theorem A.1) such that each μ_n gives definite mass to these domains. By a compactness argument, this gives a domain $\hat{X}$ such that a subsequence of these lifted measures give $\hat{X}$ mass $\geqslant \varepsilon > 0$. The inducing scheme is a first return map in the Hofbauer extension, so Kac's Lemma ultimately gives $\int \tau \, d\nu_{n_k} \leqslant \frac{1}{\varepsilon}$ where $(\nu_{n_k})_k$ are the lifted versions of these measures, hence giving a uniform bound. Moreover, Abramov's formula implies that $h(\nu_{n_k}) + \int \Psi_t \, d\nu_{n_k} \to 0$, so by the Variational Principle $P(\Psi_t) \geqslant 0$.

(2) The existence of the equilibrium state μ_{Ψ_t} then follows from Theorem 2.3. To project this, we need $\int \tau \, d\mu_{\Psi_t} < \infty$. This follows since we can show that the measures ν_{n_k} above converge to μ_{Ψ_t}.

(3) The fact that the inducing scheme works for all $t \in (-\infty, t^+)$ follows from an argument on the Hofbauer extension which we sketch briefly here. Suppose that μ_{ψ_t} lifts to (X, F) and $\mu_{\psi_{t'}}$ lifts to (X', F'), with lifted measures ν_{Ψ_t} and $\nu_{\Psi_{t'}}$ respectively. Branches of these different inducing schemes are connected by the dynamics at the level of the Hofbauer extension (orbits of domains of one pass through those of the other). It can be shown that if $\mu_{\psi_{t'}}$ doesn't lift to (X, F) then $\nu_{\Psi_{t'}}$ doesn't give mass to those branches of (X', F') which are connected to (X, F). This is impossible since our equilibrium states give all branches positive mass.

6. Multifractal analysis: The Lyapunov spectrum

This Section is devoted to explaining and commenting on the proof of Theorem B.

We will first prove that if $\lambda \in \mathbb{R} \setminus A$ then $L(\lambda) = \frac{1}{\lambda} \inf_{t \in \mathbb{R}} (p(t) + t\lambda)$. In this context our proof relies on the fact that we can an construct equilibrium state μ_λ such that $\lambda(\mu_\lambda) = \lambda$.

Proof of the lower bound. Let $\lambda \in (-D^- p(t^+), \lambda_M)$. This part of the proof goes very much along the lines of the usual hyperbolic theory (see [P, Chapter 7]). Consider the equilibrium measure μ_λ corresponding to $-t_\lambda \log |Df|$ such that $\lambda(\mu_\lambda) = \lambda$. We have

(1) $\mu_\lambda(I \setminus J(\lambda)) = 0$;
(2) the measure μ_λ is ergodic;
(3) by [H], the pointwise dimension is μ_λ-almost everywhere equal to

$$\lim_{r \to 0} \frac{\log \mu_\lambda(B(x, r))}{\log r} = \frac{h(\mu_\lambda)}{\lambda}.$$

Therefore, Proposition 3.3 implies

$$\dim_H(J(\lambda)) \geqslant \frac{h(\mu_\lambda)}{\lambda} = \frac{1}{\lambda} \inf_{t \in \mathbb{R}} (p(t) + t\lambda),$$

where the final equality follows from the fact that $t \mapsto p(t)$ is C^1 and strictly convex in the relevant domain.

Proof of the upper bound. Again, let $\lambda \in (-D^- p(t^+), \lambda_M)$. The situation in this case is a bit more subtle, we will make use of the induced systems and of its Markov structure. Initially, rather than considering pointwise dimension , we consider a local dimension that is adapted to the Markov structure, namely the Markov dimension defined at the end of Section 3. In Appendix C we show that under suitable conditions, these two local dimensions coincide. This will allow us to obtain an upper bound for the induced level set. In order to prove that this same bound also holds for the original level sets we will make use of the bi-Lipschitz property of the projection map.

Let (X^n, F_n, τ) be an induced system as constructed in Theorem 4.1 and $J_{F_n}(\lambda)$ be the induced level set defined in Section 4.3. Note that if μ_t is the equilibrium measure for $-t \log |Df|$ and $\lambda(\mu_t) = \lambda$ then the lifted measure $\mu_{F_n,t}$ has

$$\mu_{F_n,t}\left(I \setminus J_{F_n}(\lambda) \right) = 0.$$

Denote by $I_k^n(x)$ the cylinder (with respect to the Markov dynamical system (X^n, F_n)) of length k that contains the point $x \in X$, and by $|I_k^n(x)|$ its Euclidean length. By definition there exists a positive constant $K \geqslant 1$ such that for every $x \in X$ and every $k \in \mathbb{N}$ we have

$$\frac{1}{K} \leqslant \frac{|I_k^n(x)|}{|DF_n^k(x)|} \leqslant K.$$

A simple calculation (see [IT2, Lemma 4.3]) then shows that for every $x \in J_{F_n}(\lambda)$, $\delta_{\mu_{F_n,t}}(x) = \frac{h(\mu_t)}{\lambda}$. As proved in Appendix C (see also [PolW, Proposition 3]), if $\delta_{\mu_{F_n,t}}(x)$ and $\lambda(x)$ exist then

$$d_{\mu_{F_n,t}}(x) = \delta_{\mu_{F_n,t}}(x).$$

Therefore Proposition 3.2 implies that

$$\dim_H(J_{F_n}(\lambda)) \leqslant \frac{h(\mu_t)}{\lambda}.$$

The definition of $J(\lambda)$ and Theorem 4.1(c) implies that up to a set of Hausdorff dimension zero, any $x \in J(\lambda)$ has $k, n \in \mathbb{N}$ such that $f^k(x) \in J_{F_n}(\lambda)$, and since moreover the projection map $\pi : X_n \to I$ is bi-Lipschitz,

$$\dim_H(J(\lambda)) \leqslant \dim_H \left(\cup_n \cup_{k \geqslant 0} f^{-k} \left(\pi_n(J_{F_n}(\lambda)) \right) \right) =$$

$$\sup_n \left\{ \dim_H \left(\pi_n(J_{F_n}(\lambda)) \right) \right\} = \frac{h(\mu_t)}{\lambda},$$

thus completing the proof of the upper bound in the case $\lambda \in (-D^- p(t^+), \lambda_M)$.

The situation can be much more complicated if $\lambda \in (0, -D^- p(t^+))$ due to the lack of natural equilibrium states (note that the case $\lambda = -D^- p(t^+)$ follows as above, provided $-D^- p(t^+) > 0$). However, we are able to prove the following.

Lemma 6.1. *Given $f \in F_{acip}$, for any $\lambda \in (\lambda_m, -D^- p(t^+))$ and $\varepsilon > 0$ there exists an ergodic measure $\mu \in \mathcal{M}$ with $\lambda(\mu) = \lambda$ and $\dim_H(\mu) \geqslant t^+ + \frac{p_f(t^+)}{\lambda} - \varepsilon$.*

The proof is obtained by approximating (I, f) by hyperbolic sets on which we have equilibrium states with relatively small Lyapunov exponent and large Hausdorff dimension. The hyperbolic sets are invariant sets for truncated inducing schemes. For the particular case of $\lambda_m(f) > 0$ this covers the statement of Theorem B for $\lambda \in A$. For the particular case that $\lambda_m(f) = 0$, since $t^+ = 1$ and $p_f(t^+) = 0$, such a choice of λ gives a measure with $\dim_H(\mu) \geqslant 1$.

The same approximation argument together with the result of Barreira and Schmeling [BaS] allows us to prove that the irregular set J' has full Hausdorff dimension.

Acknowledgements

Appendix A. Hofbauer extension

In this section we describe the Hofbauer extension, the construction of which underlies many of the results presented here. We will not explain our use of its properties in great detail for the sake of brevity. The setup we present here can be applied to general dynamical systems, since it only uses the structure of dynamically defined cylinders. An alternative way of thinking of the Hofbauer extension specifically for the case of multimodal interval maps, which explicitly makes use of the critical set, is presented in [BB].

We let $C_n[x]$ denote the member of $\mathcal{P}_n$, defined as above, containing x. If $x \in \cup_{n \geqslant 0} f^{-n}(\mathcal{C}r)$ there may be more than one such interval, but this ambiguity will not cause us any problems here.

The *Hofbauer extension* is defined as

$$\hat{I} := \bigsqcup_{k \geqslant 0} \bigsqcup_{C_k \in \mathcal{P}_k} f^k(C_k)/ \sim$$

where $f^k(C_k) \sim f^{k'}(C_{k'})$ as components of the disjoint union $\hat{I}$ if $f^k(C_k) = f^{k'}(C_{k'})$ as subsets in I. Let $\mathcal{D}$ be the collection of domains of $\hat{I}$ and $\hat{\pi} : \hat{I} \to I$ be the natural inclusion map. A point $\hat{x} \in \hat{I}$ can be represented by (x, D) where $\hat{x} \in D$ for $D \in \mathcal{D}$ and $x = \hat{\pi}(\hat{x})$. Given $\hat{x} \in \hat{I}$, we can denote the domain $D \in \mathcal{D}$ it belongs to by $D_{\hat{x}}$.

The map $\hat{f} : \hat{I} \to \hat{I}$ is defined by

$$\hat{f}(\hat{x}) = \hat{f}(x, D) = (f(x), D')$$

if there are cylinder sets $C_k \supset C_{k+1}$ such that $x \in f^k(C_{k+1}) \subset f^k(C_k) = D$ and $D' = f^{k+1}(C_{k+1})$. In this case, we write $D \to D'$, giving $(\mathcal{D}, \to)$ the structure of a directed graph. Therefore, the map $\hat{\pi}$ acts as a semiconjugacy between $\hat{f}$ and f:

$$\hat{\pi} \circ \hat{f} = f \circ \hat{\pi}.$$

We denote the 'base' of $\hat{I}$, the copy of I in $\hat{I}$, by D_0. For $D \in \mathcal{D}$, we define $\text{lev}(D)$ to be the length of the shortest path $D_0 \to \cdots \to D$ starting at the base D_0. For

each $R \in \mathbb{N}$, let $\hat{I}_R$ be the compact part of the Hofbauer extension defined by

$$\hat{I}_R := \sqcup\{D \in \mathcal{D} : \text{lev}(D) \leqslant R\}.$$

For maps in $\mathcal{F}$, we can say more about the graph structure of $(\mathcal{D}, \rightarrow)$ since Lemma 1 of [BT2] implies that if $f \in \mathcal{F}$ then there is a closed primitive subgraph $\mathcal{D}_\mathcal{T}$ of $\mathcal{D}$. That is, for any $D, D' \in \mathcal{D}_\mathcal{T}$ there is a path $D \rightarrow \cdots \rightarrow D'$; and for any $D \in \mathcal{D}_\mathcal{T}$, if there is a path $D \rightarrow D'$ then $D' \in \mathcal{D}_\mathcal{T}$ too. We can denote the disjoint union of these domains by $\hat{I}_\mathcal{T}$. The same lemma says that if $f \in \mathcal{F}$ then $\hat{\pi}(\hat{I}_\mathcal{T}) = \Omega$, the non-wandering set and $\hat{f}$ is transitive on $\hat{I}_\mathcal{T}$. Theorem A.1 below gives these properties for transitive cusp maps.

Given an ergodic measure $\mu \in \mathcal{M}_f$, we say that μ *lifts to* $\hat{I}$ if there exists an ergodic $\hat{f}$-invariant probability measure $\hat{\mu}$ on $\hat{I}$ such that $\hat{\mu} \circ \hat{\pi}^{-1} = \mu$. For $f \in \mathcal{F}$, if $\mu \in \mathcal{M}_f$ is ergodic and $\lambda(\mu) > 0$ then μ lifts to $\hat{I}$, see [K2, BK].

For use later, we say that (†) holds if for any $\hat{x}, \hat{y} \notin \partial \hat{I}$ with $\hat{\pi}(x) = \hat{\pi}(y)$ there exists n such that $\hat{f}^n(\hat{x}) = \hat{f}^n(\hat{y})$. This property holds for cusp maps by the construction of $\hat{I}$ using the partition $\{I_i\}_i$ given in Definition 1.1.

We only use the following result in the context of equilibrium states for cusp maps with no singularities. However, for interest we state the theorem in greater generality.

Theorem A.1. *Suppose that $f : I \rightarrow I$ is a transitive cusp map with topological entropy $h_{top}(f) > 0$. Then:*

(1) *there is a transitive part $\hat{I}_\mathcal{T}$ of the extension such that $\hat{\pi}(\hat{I}_\mathcal{T}) = I$;*
(2) *any measure $\mu \in \mathcal{M}_f$ with $0 < \lambda(\mu) < \infty$ lifts to $\hat{\mu}$ with $\mu = \hat{\mu} \circ \hat{\pi}^{-1}$;*
(3) *for each $\varepsilon > 0$ there exists $\eta > 0$ and a compact set $\hat{K} \subset \hat{I}_\mathcal{T} \setminus \partial \hat{I}$ such that any measure $\mu \in \mathcal{M}_f$ with $h(\mu) > \varepsilon$ and $0 < \lambda(\mu) < \infty$ has $\hat{\mu}(\hat{K}) > \eta$.*

APPENDIX B. PROJECTING CONFORMAL MEASURES

In Section 4 we outlined the relation between invariant measures for inducing schemes and for the original system. Here we are concerned with proving analogous results for conformal measures. We show how, given a suitable inducing scheme, the conformal measure for the induced system can be projected to the original system (and vice versa). One way of doing this was described in [T] where the structure of the Hofbauer extension was used. Here we use a method inspired by that construction, but not directly appealing to it. Note that one could also write the proofs here with notation from the Young tower. The conformal measure so-constructed can by used to prove Theorem B via Proposition B.2 in [IT2].

Lemma B.1. *A φ-conformal measure m_φ for (I, f) is also a Φ-conformal measure for (X, F) if $m_\varphi(\cup_i X_i) = m_\varphi(X)$.*

The lemma says that m_φ 'lifts' to (X, F). The proof is elementary since for $x \in X_i$,

$$dm_\varphi(F(x)) = dm_\varphi(f^{\tau_i}(x)) = e^{-S_{\tau_i}\varphi(x)}dm_\varphi(x) = e^{-\Phi(x)}dm_\varphi(x),$$

and the complement of $\cup_i X_i$ has zero measure. The point here is to prove the other direction: that for certain inducing schemes (X, F, Φ), any Φ-conformal measure, if it exists, projects to a φ-conformal measure. To do this we will work with a certain type of inducing schemes:

Condition $(*)$**:** An inducing scheme (X, F, τ) satisfies condition $(*)$ if for any $x \in (X, F)^\infty$, if $y, y' \in (X, F)^\infty$ have $f^k(y) = f^{k'}(y') = x$ for $k, k' \in \mathbb{N}_0$, then there exists $n \in \mathbb{N}$ such that $k + n$ and $k' + n$ are inducing times for y and y' respectively.

It will turn out that this condition is satisfied by a natural class of inducing schemes, one way of obtaining which is the Hofbauer extension. In fact $(\dagger)$ in Appendix A implies that inducing schemes obtained from the Hofbauer tower, as in Theorem 4.1, satisfy condition $(*)$.

Theorem B.1. *Suppose that $f \in \mathcal{F}$ and (X, F, τ) is an inducing scheme satisfying condition $(*)$. If $\varphi : I \to [-\infty, \infty]$ has an induced version $\Phi : X \to [-\infty, \infty)$ with a Φ-conformal probability measure m_Φ, then m_Φ projects to a σ-finite φ-conformal measure m_φ, where $\mu_\varphi \ll m_\varphi$.*

Moreover, m_φ is a finite measure if φ is bounded below: the measure is also finite if for each $x \in \{\varphi = -\infty\}$ there exists $n \in \mathbb{N}_0$ such that $f^n(x) \in X$ and for any y in neighbourhood of x, $\varphi(f^k(y)) < \infty$ for $1 \leqslant k \leqslant n - 1$.

Note that we allow Φ to be $-\infty$, but not ∞: indeed for each i, and $x \in X_i$ we allow $S_j\varphi(x) = -\infty$ for any $1 \leqslant j \leqslant \tau_i - 1$, but not $S_j\varphi(x) = +\infty$.

Proof. First note that $\Phi < \infty$ implies that for any set $A \subset X$, $m_\Phi(A) > 0$ implies $m_\Phi(F(A)) > 0$. This means that no set of positive measure can leave $\cup_i X_i$ under iteration of F, i.e., for all $k \in \mathbb{N}$, $m_\Phi(F^{-k}(\cup_i X_i)) = 1$ and thus $m_\Phi(X \cap (X, F)^\infty) = 1$. We will spread the measure m_Φ onto $(X, F)^\infty$ as follows.

Suppose that $x \in (X, F)^\infty$ is contained in some set $f^k(X_i)$ for $0 \leqslant k \leqslant \tau_i - 1$. There may be many such pairs (i, k), but we pick one arbitrarily and then later show that we could have chosen any and obtained the same result. There exists a unique $y \in X_i$ such that $f^k(y) = x$. We define ν_φ so that for any $j \in \mathbb{N}_0$,

$$d\nu_\varphi(f^j(x)) = e^{-S_{k+j}(y)}dm_\Phi(y).$$

Clearly this gives a conformal measure locally since for $j \in \mathbb{N}$,

$$d\nu_\varphi(f^j(x)) = e^{-S_{k+j}\varphi(y)}dm_\Phi(y) = e^{-S_j\varphi(x)}e^{-S_k\varphi(y)}dm_\Phi(y) = e^{-S_j\varphi(x)}d\nu_\varphi(x).$$

Note that if $k + j = \tau_i$ then $f^{k+j}(x) = F(y)$ and

$$d\nu_\varphi(F(y)) = d\nu_\varphi(f^{k+j}(x)) = e^{-S_{k+j}\varphi(y)}dm_\Phi(y) = e^{-\Phi(y)}dm_\Phi(y) = dm_\Phi(F(y)),$$

by the Φ-conformality of m_Φ. This also extends to the case when $k + j = \tau^p(y)$ for $p \in \mathbb{N}$, we obtain

$$d\nu_\varphi(f^j(x)) = d\nu_\varphi(f^{k+j}(y)) = e^{-S_{k+j}\varphi(y)}dm_\Phi(y)$$
$$= e^{-S_p\Phi(y)}dm_\Phi(y) = dm_\Phi(F^p(y)).$$

To prove that the procedure given above is well-defined, we need to check that the same measure is assigned at x when there is $i' \neq i$ and $1 \leqslant k' \leqslant \tau_{i'} - 1$ such that x

is also contained in $f^{i'}(X_{i'})$ and $\tau_{i'} - k' = \tau_i - k$. If we let ν'_φ be the measure at x obtained analogously to ν_φ but with i' and k' in place of i and k, and some point $y' \in X_{i'}$ in place of $y \in X_i$, we must show that $\nu'_\varphi = \nu_\varphi$.

By condition $(*)$ there exists n such that $k + n$ and $k' + n$ are inducing times for y and y' respectively. Therefore, as in (**??**),

$$d\nu'_\varphi(x) = e^{-S_n\varphi(x)}d\nu'_\varphi(f^n(x)) = e^{-S_n\varphi(x)}dm_\Phi(f^n(x))$$
$$= e^{-S_n\varphi(x)}d\nu_\varphi(f^n(x)) = d\nu_\varphi(x),$$

so $\nu'_\varphi = \nu_\varphi$, as required.

For σ-finiteness of ν_φ, notice that for any $x \in \cup_{i \geqslant 1} \cup_{k=0}^{\tau_i - 1} f^k(X_i)$, we can choose a single element $f^k(X_i)$ as above containing x to apportion measure at x. Since the resulting measure is finite, ν_φ is σ-finite.

To prove that ν_φ is actually finite when $\varphi > -\infty$, we use a topological property of f: that for any open set $U \subset I$, there exists $n \in \mathbb{N}$ such that $I = f^n(U)$. We take $U = X$, and note that $\nu_\varphi(X) = m_\Phi(X) = 1$. Then by the conformality of ν_φ and the fact that φ is bounded below, we have

$$\nu_\varphi(I) = \nu_\varphi(f^n(X)) \leqslant \int_X e^{-S_n\varphi}\,d\nu_\varphi \leqslant e^{-n\inf\varphi}m_\Phi(X) < \infty.$$

The fact that $\nu_\varphi \ll m_\varphi$ follows from the construction.

By this argument, the only possible obstacle to ν_φ being finite in the general case is if there is a point $x \in I$ such that $\varphi(x) = -\infty$: in this case, any neighbourhood of x may have infinite measure. We consider this possibility as in the final part of the statement of the theorem: we take $x \in \{\varphi = -\infty\}$, and $n \in \mathbb{N}_0$ such that $f^n(x) \in X$ and $\varphi(f^k(x)) < \infty$ for $1 \leqslant k \leqslant n - 1$. Then there exists an interval $U \ni x$ such that $f^n(U) \subset X$ and $S_k\varphi(y) < \infty$ for all $y \in U$. It suffices to prove that $\nu_\varphi(U) < \infty$. This follows since

$$1 = \nu_\varphi(X) \geqslant \nu_\varphi(U) = \int_U e^{-S_n\varphi}\,d\nu_\varphi \geqslant e^{-\sup_{y \in U} S_n\varphi(y)}\nu_\varphi(U).$$

We finish by setting $m_\varphi = \nu_\varphi/\nu_\varphi(I)$. $\square$

One of the main applications of this theorem is to maps $f \in \mathcal{F}$ with potential $-t\log|Df|$. In particular we are interested in conformal measures for the potential $\psi_t : x \mapsto -t\log|Df(x)| - P(-t\log|Df|)$. By Theorem B.1, in order to obtain a ψ_t-conformal measure it is sufficient to find an inducing scheme (X, F, τ) satisfying condition $(*)$ with a Ψ_t-conformal measure m_{Ψ_t} (where $\Psi_t = -t\log|DF| - \tau P(-t\log|DF|)$). The existence of such schemes is studied in [BK], [PS], [BT2] and [IT1]. We address this in the next section.

B.1. **Decent returns and natural conformal measures.** In this section we introduce a useful kind of inducing scheme, which in particular satisfies condition $(*)$.

Suppose that $A \subset I$ is an interval and $\delta > 0$. We define $(1+2\delta)A$ to be the interval concentric with A, but with length $(1 + 2\delta)|A|$ (we will always implicitly assume that $(1 + 2\delta)A$ is still contained in I). For a set $B \subset I$, we say that A is δ-*well inside* B if $(1 + 2\delta)A \subset B$.

The following is simply an application of the Koebe lemma, see [MeSt, Chapter IV], which applies in our case since $f \in \mathcal{F}$ has negative Schwarzian derivative.

Lemma B.2. *Suppose that $f \in \mathcal{F}$ and $A \subset I$ is such that $f^n : (1 + 2\delta)A \to f^n((1+2\delta)A)$ is a diffeomorphism. Then for all $x, y \in A$,*

$$\frac{\delta^2}{1+\delta^2} \leqslant \frac{Df^n(x)}{Df^n(y)} \leqslant \frac{1+\delta^2}{\delta^2}.$$

Given $f \in \mathcal{F}$, an interval $X \in I$ and $\delta > 0$, we say that a point $x \in X$ makes a δ-*decent return to X at time $n \in \mathbb{N}$* if there exists an interval $A'(\delta, x)$ containing x such that $f^n : A'(\delta, x) \to (1 + 2\delta)X$ is a diffeomorphism. Let $A(\delta, x) \subset A'(\delta, x)$ be such that $f^n : A(\delta, x) \to X$ is a diffeomorphism. The smallest such $n \in \mathbb{N}$ is called the *first δ-decent return to X*.

The idea is that this will give us a very natural inducing scheme. To ensure this, we need one more idea: we say that our interval X is δ-*nice* for any $x, y \in X$, either $A(\delta, x) = A(\delta, y)$ or $A(\delta, x)$ and $A(\delta, y)$ intersect in at most one point. One way to ensure this is to use the usual notion of 'nice interval': an interval X is *nice* if $f^n(\partial X) \cap int(X) = \emptyset$ for all $n \in \mathbb{N}$. This in turn can, for example, be guaranteed if we let p be a periodic point and let X be an open interval bounded by points in the orbit of p, but not containing any points of that orbit. By Lemma B.2 below, this process produces a 'decent' inducing scheme.

It is proved in [B] that decent inducing schemes can be produced by the Hofbauer extension for f, and thus that such an inducing scheme (X, F, τ) satisfies condition $(*)$.

Theorem B.2. *Given $f \in \mathcal{F}$, for each $t < t^+$ there exists an inducing scheme (X, F, τ) satisfying condition $(*)$ and a Ψ_t-conformal measure m_{Ψ_t}.*

This is essentially proved putting together the main theorem of [IT1] and the decent inducing schemes produced in [B].

An immediate corollary of Theorems B.1 and B.2 is:

Corollary B.1. *Given $f \in \mathcal{F}$, for each $t < t^+$ there exists a ψ_t-conformal measure m_{ψ_t}.*

On can also apply this result to any Hölder potential $\varphi : I \to \mathbb{R}$ satisfying $\sup \varphi < P(\varphi)$, see [BT1] and [IT1]. However, such conformal measures were already known, see [K1]. Actually, it has been recently shown by Li and Rivera-Letelier [LiR] that for maps $f \in \mathcal{F}$ such that for every critical value c we have $\lim_{n \to \infty} |Df^n(f(c))| = \infty$ every Hölder potential φ satisfies $\frac{1}{n} \sup S_n \varphi < P(\varphi)$ and that this is enough to guarantee the existence and uniqueness of equilibrium states, conformal measures

and real analyticity of the pressure function. This remarkable result allows for the improvement of Theorem B in [IT2].

Appendix C. Equality of pointwise and Markov dimensions

In [OP], work was done to explain a case of [PolW, Proposition 3] which relates pointwise dimension and Markov dimension for a so-called EMR (Expanding Markov Rényi) transformation (I_0, T): for a precise definition of an EMR see [PolW], but for our purposes here, just suppose that it satisfies all the properties of an inducing scheme described in Section 4, but with $\tau \equiv 1$. The key issue is the part where they show, under the assumption that the Markov dimension $\delta_\mu(x)$ exists at x, that $\underline{d}_\mu(x) \geqslant \delta_\mu(x)$. In this section we alter the definition of an EMR transformation in order to obtain this result.

For $I_0 \subset I$, the EMR is a map $T : I_0' \to I_0$ where $I_0' := \cup_{i \in \mathbb{N}} I_{1,i}$ and for each $i \in \mathbb{N}$, $T : I_{1,i} \to I_0$ is a diffeomorphism, and $\inf\{|DT(x)| : x \in I_0'\} > 1$. For $x \in I_0'$, if $T^n(x)$ is defined, let $I_n(x)$ denote the n-cylinder at x; recall that this is the maximal set around x such that $T^n : I_n(x) \to I_0$ is a diffeomorphism. We can list these intervals as $\{I_{n,1}, I_{n,2}, \ldots\}$. Let I_0'' be the set of points for which T^n is defined for all $n \in \mathbb{N}_0$, i.e., $I_0'' = \{x \in I_0' : T^n(x) \in I_0' \text{ for all } n \geqslant 1\}$.

Given that we have a Markov structure for our system, and thus a natural partition into 1-cylinders, we can define the Markov dimension as usual for points in I_0''.

By definition of an EMR, or an inducing scheme, there exists $K_1 \geqslant 1$ such that

$$\frac{1}{K_1} \leqslant \frac{|I_n(x)|}{|DT^n(x)|^{-1}} \leqslant K_1. \tag{3}$$

Now recall that if μ is an equilibrium state (a Gibbs measure) for Φ then there exists $C \geqslant 1$ such that

$$\frac{1}{C} \leqslant \frac{\mu(I_n(x))}{e^{-nP(\Phi) + \sum_{j=0}^{n-1} \Phi(T^j(y))}} \leqslant C, \tag{4}$$

for all $y \in I_n(x)$. Let $\overline{\Phi}(x)$ denote the Birkhoff average at x, if it exists.

The following is essentially proved in [PolW], but with some minor errors, so we give the proof again here.

Lemma C.1. *Suppose that for $x \in I_0''$, $\delta_\mu(x)$ and $\overline{\Phi}(x)$ exist. Then $\overline{d}_\mu(x) \leqslant \delta_\mu(x)$.*

Proof. Given $r > 0$ there exists a unique $n = n(r)$ such that $K_1|DT^n(x)|^{-1} < r \leqslant K_1|DT^{n-1}(x)|^{-1}$. So $r > |I_n(x)|$ by (3) and $I_n(x) \subset B(x, |I_n(x)|) \subset B(x, r)$. Then

$$\frac{\log \mu(B(x,r))}{\log r} \leqslant \frac{\log(\mu(I_n(x)))}{\log(K_1|DT^{n-1}(x)|^{-1})} \leqslant \frac{\log \mu(I_{n-1}(x))}{\log(K_1^2|I_{n-1}(x)|)} \frac{\log \mu(I_n(x))}{\log \mu(I_{n-1}(x))}.$$

By the Gibbs property (4),

$$\frac{\log \mu(I_n(x))}{\log \mu(I_{n-1}(x))} \leqslant \frac{-nP(\Phi) + \sum_{j=0}^{n-1} \Phi(T^j(x)) + \log C}{-(n-1)P(\Phi) + \sum_{j=0}^{n-2} \Phi(T^j(x)) - \log C}.$$

Suppose that $\overline{\Phi}(x)$ exists. Then we can divide the numerator and denominator of the ratio on the right above by n to show that $\lim_{n\to\infty} \frac{\log \mu(I_n(x))}{\log \mu(I_{n-1}(x))} \leqslant 1$. This follows since both $\frac{1}{n}\sum_{j=0}^{n-1} \Phi(T^j(x))$ and $\frac{1}{n}\sum_{j=0}^{n-2} \Phi(T^j(x)) = \left(\frac{n-1}{n}\right)\frac{1}{n-1}\sum_{j=0}^{n-2}\Phi(T^j(x))$ tend to $\overline{\Phi}(x)$ as $n\to\infty$. Therefore, taking limits in r we prove the lemma.   □

In order to prove a reverse inequality to that in the previous lemma, we change the definition of an EMR. Observing that this new condition can be satisfied by the inducing schemes used in, for example, [IT2].

We extend the system as follows. In the interval I (which is the ambient space containing I_0), let I_0^ℓ, I_0^r be intervals adjacent to I_0 and set $\hat{I}_0 := I_0^\ell \cup I_0 \cup I_0^r$. We will assume that for any $n \in \mathbb{N}$ and $i \in \mathbb{N}$, the map $T^n : I_{n,i} \to I_0$ has an extension $\hat{T}^n$ so that for an interval $\hat{I}_0 \supset \hat{I}_{n,i} \supset I_{n,i}$, the map $\hat{T}^n|_{\hat{I}_{n,i}} : \hat{I}_{n,i} \to \hat{I}_0$ is a diffeomorphism. By 'extension', we mean that $\hat{T}^n|_{\hat{I}_{n,i}} = T^n|_{\hat{I}_{n,i}}$.

We further assume that for there exists $K_2 \geqslant 1$ such that

$$\frac{1}{K_2} \leqslant \frac{|I_n^\ell(x)|}{|I_n(x)|}, \frac{|I_n^r(x)|}{|I_n(x)|} \leqslant K_2. \tag{5}$$

Also assume that the Gibbs property extends to $\hat{T}$, that is (4) holds as well as the following: there exists $K_3 \geqslant 1$ such that

$$\frac{1}{K_3} \leqslant \frac{\mu(\hat{I}_n(x))}{\mu(I_n(x))} \leqslant K_3.$$

Note that in the setting of [IT2], $\frac{\mu(\hat{I}_n(x))}{\mu(I_n(x))}$ is simply $\frac{m_\varphi(\hat{I}_0)}{m_\varphi(I_0)}$ multiplied by some distortion constant (here m_φ is φ-conformal measure on I).

Lemma C.2. *Suppose that for $x \in I_0''$, $\delta_\mu(x)$ and $\overline{\Phi}(x)$ exist. Then $\underline{d}_\mu(x) \geqslant \delta_\mu(x)$.*

Therefore, combining Lemmas C.1 and C.2, if $\delta_\mu(x)$ and $\overline{\Phi}(x)$ exist, then $\overline{d}_\mu(x) = \delta_\mu(x)$.

Proof. Given $r > 0$ there exists a unique $n = n(r)$ such that $\frac{1}{K_1 K_2}|DT^n(x)|^{-1} < r \leqslant \frac{1}{K_1 K_2}|DT^{n-1}(x)|^{-1}$. So $r \leqslant \frac{|I_{n-1}(x)|}{K_2} \leqslant |I_{n-1}^\ell(x)|, |I_{n-1}^r(x)|$ by (3) and (5), which implies that $B(x,r) \subset \hat{I}_{n-1}(x)$. Then

$$\frac{\log \mu(B(x,r))}{\log r} \geqslant \frac{\log(\mu(\hat{I}_{n-1}(x)))}{\log\left(\frac{1}{K_1 K_2}|DT^n(x)|^{-1}\right)} \geqslant \frac{\log(K_3 \mu(I_n(x)))}{\log\left(\frac{1}{K_1^2 K_2}|I_n(x)|\right)} \frac{\log(K_3\mu(I_{n-1}(x)))}{\log(K_3\mu(I_n(x)))}.$$

As in Lemma C.1, taking limits in r proves the lemma.   □

The rest of [PolW, Proposition 3] follows similarly.

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