The absolute continuous spectrum of skew products of compact Lie groups

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Abstract

Let X and G be compact Lie groups, $F_1 : X \to X$ the time-one map of a C^∞ measure-preserving flow, $\phi : X \to G$ a continuous function and π a finite-dimensional irreducible unitary representation of G. Then, we prove that the skew products

$$T_\phi : X \times G \to X \times G, \quad (x, g) \mapsto \big(F_1(x), g\phi(x)\big),$$

have purely absolutely continuous spectrum in the subspace associated to π if $\pi \circ \phi$ has a Dini-continuous Lie derivative along the flow and if a matrix multiplication operator related to the topological degree of $\pi \circ \phi$ has nonzero determinant. This result provides a simple, but general, criterion for the presence of an absolutely continuous component in the spectrum of skew products of compact Lie groups. As an illustration, we consider the cases where F_1 is an ergodic translation on $\mathbb{T}^d$ and $X \times G = \mathbb{T}^d \times \mathbb{T}^{d'}$, $X \times G = \mathbb{T}^d \times \mathsf{SU}(2)$ and $X \times G = \mathbb{T}^d \times \mathsf{U}(2)$. Our proofs rely on recent results on positive commutator methods for unitary operators.

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1 Introduction

In his seminal work [5], H. Anzai has shown that the skew products

$$T_m : \mathbb{T} \times \mathbb{T} \to \mathbb{T} \times \mathbb{T}, \quad (x, g) \mapsto (x + y, g + mx), \quad \mathbb{T} \simeq \mathbb{R}/\mathbb{Z}, \ y \in \mathbb{R} \setminus \mathbb{Q}, \ m \in \mathbb{Z} \setminus \{0\},$$

have countable Lebesgue spectrum in the orthocomplement of functions depending only on the first variable. Since then, various generalisations of this result have been obtained (see [7, 8, 11, 12, 13, 16, 17, 20, 21, 22, 23, 25, 26] and see also [2, 9, 19, 31] for related results). In particular, A. Iwanik, M. Lemańzyk and D. Rudolph have shown in [23] that the skew products

$$T_\phi : \mathbb{T} \times \mathbb{T} \to \mathbb{T} \times \mathbb{T}, \quad (x, g) \mapsto \big(x + y, g + \phi(x)\big), \quad y \in \mathbb{R} \setminus \mathbb{Q},$$

have countable Lebesgue spectrum if the cocycle $\phi : \mathbb{T} \to \mathbb{T}$ is an absolutely continuous function with nonzero topological degree and with derivative of bounded variation. Also, A. Iwanik and K. Frączek have

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proved in [11, 12, 21] similar results for skew products of higher dimensional tori. Finally, K. Frączek has shown in [13] (see also [14]) that the skew products

$$T_\phi : \mathbb{T} \times \mathrm{SU}(2) \to \mathbb{T} \times \mathrm{SU}(2), \quad (x, g) \mapsto \big(x + y, g\phi(x)\big), \quad y \in \mathbb{R} \setminus \mathbb{Q},$$

have countable Lebesgue spectrum in an appropriate subspace of the orthocomplement of functions depending only on the first variable if the cocycle $\phi : \mathbb{T} \to \mathrm{SU}(2)$ is of class C^2, has nonzero topological degree and is cohomologous (in a suitable way) to a diagonal cocycle.

The purpose of this paper is to extend these types of spectral results to a general class of skew products of compact Lie groups. Our set-up is the following. We consider skew products

$$T_\phi : X \times G \to X \times G, \quad (x, g) \mapsto \big(F_1(x), g\phi(x)\big),$$

where X and G are compact Lie groups, $F_1 : X \to X$ the time-one map of a C^∞ measure-preserving flow and $\phi : X \to G$ a continuous function. We fix π a finite-dimensional irreducible unitary representation of G and we write $\mathscr{L}_Y$ for the Lie derivative associated to the flow. Then, our main result reads as follows. If the Lie derivative $\mathscr{L}_Y(\pi \circ \phi)$ exists and satisfies a Dini-type condition along the flow and if a matrix multiplication operator related to the topological degree of $\pi \circ \phi$ has nonzero determinant, then T_ϕ has purely absolutely continuous spectrum in the subspace associated to π (see Theorem 3.5 and Remark 3.6 for a precise satement). This result provides a simple, but general, criterion for the presence of an absolutely continuous component in the spectrum of skew products of compact Lie groups. Its proof relies on recent results [10] on positive commutator methods for unitary operators. As an illustration, we consider the cases where F_1 is an ergodic translation on $\mathbb{T}^d$ and $X \times G = \mathbb{T}^d \times \mathbb{T}^{d'}$, $X \times G = \mathbb{T}^d \times \mathrm{SU}(2)$ and $X \times G = \mathbb{T}^d \times \mathrm{U}(2)$. In the cases $X \times G = \mathbb{T}^d \times \mathbb{T}^{d'}$ and $X \times G = \mathbb{T}^d \times \mathrm{SU}(2)$, we obtain countable Lebesgue spectrum under conditions similar to the ones to be found in the literature; see Theorems 4.2 and 4.5 and the discussions that follow. In the case $X \times G = \mathbb{T}^d \times \mathrm{U}(2)$, our result (countable Lebesgue spectrum in an appropriate subspace) is new; see Theorem 4.9 and the discussion that follows.

Here is a brief description of the content of the paper. In Section 2, we recall the needed notations and results on positive commutator methods for unitary operators. In Section 3, we construct an appropriate conjugate operator (Lemma 3.2 and Formula (3.7)) and use it to prove our main theorem on the spectrum of skew products (Theorem 3.5). We also give an interpretation of our result in terms of the topological degree of $\pi \circ \phi$ (Remark 3.6). Finally, we present in Sections 4.1, 4.2 and 4.3 the examples $X \times G = \mathbb{T}^d \times \mathbb{T}^{d'}$, $X \times G = \mathbb{T}^d \times \mathrm{SU}(2)$ and $X \times G = \mathbb{T}^d \times \mathrm{U}(2)$.

To conclude, we mention two possible extensions of this paper worth studying: (i) We consider here skew products where the dynamics on the base space is given by the time-one map of a flow. This guarantees the existence of a distinguished Lie derivative which is used for the definition of the conjugate operator. It would be interesting to see if one can still construct a suitable conjugate operator even if the dynamics on the base space is not given by the time-one map of a flow. (ii) It would be interesting to see if the result of this paper could be extended to the case where X and G are noncompact Lie groups. In such a case, the main difficulty would be to deal with the unavailability of Peter-Weyl theorem, which is repeatedly used in this paper.

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2 Commutator methods for unitary operators

We briefly recall in this section some facts on commutator methods for unitary operators borrowed from [10]. We refer the reader to [4, 27] for standard references in the case of self-adjoint operators.

Let $\mathcal{H}$ be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ antilinear in the first argument, denote by $\mathscr{B}(\mathcal{H})$ the set of bounded linear operators on $\mathcal{H}$, and write $\| \cdot \|$ both for the norm on $\mathcal{H}$ and the norm on $\mathscr{B}(\mathcal{H})$.

Let A be a self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(A)$, and take $S \in \mathscr{B}(\mathcal{H})$. For any $k \in \mathbb{N}$, we say that S belongs to $C^k(A)$, with notation $S \in C^k(A)$, if the map

$$\mathbb{R} \ni t \mapsto \mathrm{e}^{-itA} S \, \mathrm{e}^{itA} \in \mathscr{B}(\mathcal{H}) \tag{2.1}$$

is strongly of class C^k. In the case $k = 1$, one has $S \in C^1(A)$ if and only if the quadratic form

$$\mathcal{D}(A) \ni \varphi \mapsto \langle \varphi, iSA\varphi \rangle - \langle A\varphi, iS\varphi \rangle \in \mathbb{C}$$

is continuous for the topology induced by $\mathcal{H}$ on $\mathcal{D}(A)$. We denote by $[iS, A]$ the bounded operator associated with the continuous extension of this form, or equivalently the strong derivative of the function (2.1) at $t = 0$.

A condition slightly stronger than the inclusion $S \in C^1(A)$ is provided by the following definition: S belongs to $C^{1+0}(A)$, with notation $S \in C^{1+0}(A)$, if $S \in C^1(A)$ and if $[A, S]$ satisfies the Dini-type condition

$$\int_0^1 \frac{\mathrm{d}t}{t} \left\| \mathrm{e}^{-itA}[A, S] \, \mathrm{e}^{itA} - [A, S] \right\| < \infty.$$

As banachisable topological vector spaces, the sets $C^2(A)$, $C^{1+0}(A)$, $C^1(A)$ and $C^0(A) \equiv \mathscr{B}(\mathcal{H})$ satisfy the continuous inclusions [4, Sec. 5.2.4]

$$C^2(A) \subset C^{1+0}(A) \subset C^1(A) \subset C^0(A) \equiv \mathscr{B}(\mathcal{H}).$$

Now, let $U \in C^1(A)$ be a unitary operator with (complex) spectral measure $E^U(\,\cdot\,)$ and spectrum $\sigma(U) \subset \mathbb{S}^1 := \{z \in \mathbb{C} \mid |z| = 1\}$. If there exist a Borel set $\Theta \subset \mathbb{S}^1$, a number $a > 0$ and a compact operator $K \in \mathscr{B}(\mathcal{H})$ such that

$$E^U(\Theta)\, U^*[A, U] E^U(\Theta) \geq a\, E^U(\Theta) + K, \tag{2.2}$$

then one says that U satisfies a Mourre estimate on Θ and that A is a conjugate operator for U on Θ. Also, one says that U satisfies a strict Mourre estimate on Θ if (2.2) holds with $K = 0$. The main consequence of a strict Mourre estimate is to imply a limiting absorption principle for (the Cayley transform of) U on Θ if U is also of class $C^{1+0}(A)$. This in turns implies that U has no singular spectrum in Θ. If U only satisfies a Mourre estimate on Θ, then the same holds up to the possible presence of a finite number of eigenvalues in Θ, each one of finite multiplicity. We recall here a version of these results (see [10, Thm. 2.7 & Rem. 2.8] for more details):

Theorem 2.1 (Mourre estimate for unitary operators)**.** *Let U and A be respectively a unitary and a self-ajoint operator in a Hilbert space $\mathcal{H}$, with $U \in C^{1+0}(A)$. Suppose there exist an open set $\Theta \subset \mathbb{S}^1$, a number $a > 0$ and a compact operator $K \in \mathscr{B}(\mathcal{H})$ such that*

$$E^U(\Theta)\, U^*[A, U] E^U(\Theta) \geq a\, E^U(\Theta) + K. \tag{2.3}$$

Then, U has at most finitely many eigenvalues in Θ, each one of finite multiplicity, and U has no singular continuous spectrum in Θ. Furthermore, if (2.3) holds with $K = 0$, then U has no singular spectrum in Θ.

3 Spectrum of skew products of compact Lie groups

Let X be a compact Lie group with normalised Haar measure μ_X and neutral element e_X, and let $\{F_t\}_{t\in\mathbb{R}}$ be a C^∞ measure-preserving flow on (X, μ_X). Then, the family of operators $\{V_t\}_{t\in\mathbb{R}}$ given by

$$V_t \varphi := \varphi \circ F_t, \quad \varphi \in \mathrm{L}^2(X, \mu_X),$$

defines a strongly continuous one-parameter unitary group satisfying $V_t C^\infty(X) \subset C^\infty(X)$ for each $t \in \mathbb{R}$ [1, Prop. 2.6.14]. It follows from Nelson's theorem [3, Prop. 5.3] that the generator H of the group $\{V_t\}_{t\in\mathbb{R}}$

$$H\varphi := \text{s-}\lim_{t\to 0} it^{-1}(V_t - 1)\varphi, \quad \varphi \in \mathcal{D}(H) := \left\{\varphi \in \mathsf{L}^2(X,\mu_X) \mid \lim_{t\to 0} |t|^{-1}\big\|(V_t - 1)\varphi\big\| < \infty\right\},$$

is essentially self-adjoint on $C^\infty(X)$, and one has

$$H\varphi := -i\mathscr{L}_Y\varphi, \quad \varphi \in C^\infty(X),$$

with Y the divergence-free vector field associated to $\{F_t\}_{t\in\mathbb{R}}$ and $\mathscr{L}_Y$ the corresponding Lie derivative.

Let G be a second compact Lie group with normalised Haar measure μ_G and neutral element e_G. Then, each $\phi \in C(X;G)$ induces a cocycle $X \times \mathbb{Z} \ni (x,n) \mapsto \phi^{(n)}(x) \in G$ over the diffeomorphism F_1 given by

$$\phi^{(n)}(x) := \begin{cases} \phi(x)(\phi\circ F_1)(x)\cdots(\phi\circ F_{n-1})(x) & \text{if } n \geq 1 \\ e_G & \text{if } n = 0 \\ \big\{(\phi\circ F_n)(x)(\phi\circ F_{n+1})(x)\cdots(\phi\circ F_{-1})(x)\big\}^{-1} & \text{if } n \leq -1. \end{cases}$$

We thus call cocycle any function belonging to $C(X;G)$. Two cocycles $\phi,\xi \in C(X;G)$ are C^0-cohomologous if there exists a function $\zeta \in C(X;G)$, called transfer function, such that

$$\phi(x) = \zeta(x)^{-1}\xi(x)\zeta\big(F_1(x)\big), \quad x \in X.$$

In such a case, the map $X \times G \ni (x,g) \mapsto \big(x,\zeta(x)g\big)$ establishes a C^0-conjugation of T_ϕ and T_ξ. The skew product T_ϕ associated to ϕ,

$$T_\phi : (X \times G, \mu_X \otimes \mu_G) \to (X \times G, \mu_X \otimes \mu_G), \quad (x,g) \mapsto \big(F_1(x), g\phi(x)\big),$$

is an automorphism of the measure space $(X \times G, \mu_X \otimes \mu_G)$ which satisfies

$$T_\phi^n(x,g) = \big(F_n(x), g\phi^{(n)}(x)\big), \quad x \in X,\ g \in G,\ n \in \mathbb{Z}. \tag{3.1}$$

Since T_ϕ is invertible, the corresponding Koopman operator

$$U_\phi\psi := \psi \circ T_\phi, \quad \psi \in \mathcal{H} := \mathsf{L}^2(X \times G, \mu_X \otimes \mu_G),$$

is a unitary operator in $\mathcal{H}$.

Let $\widehat{G}$ be the set of all (equivalence classes of) finite-dimensional irreducible unitary representations (IUR) of G. Then, each representation $\pi \in \widehat{G}$ is a C^∞ group homomorphism from G to the unitary group $\mathsf{U}(d_\pi)$ of degree $d_\pi := \dim(\pi) < \infty$, and the Peter-Weyl theorem implies that the set of all matrix elements $\{\pi_{jk}\}_{j,k=1}^{d_\pi}$ of all representations $\pi \in \widehat{G}$ form an orthogonal basis of $\mathsf{L}^2(G,\mu_G)$. Accordingly, one has the orthogonal decomposition

$$\mathcal{H} = \bigoplus_{\pi\in\widehat{G}}\bigoplus_{j=1}^{d_\pi}\mathcal{H}_j^{(\pi)} \quad \text{with} \quad \mathcal{H}_j^{(\pi)} := \left\{\sum_{k=1}^{d_\pi}\varphi_k \otimes \pi_{jk} \mid \varphi_k \in \mathsf{L}^2(X,\mu_X),\ k = 1,\ldots,d_\pi\right\}, \tag{3.2}$$

and one has a natural isomorphism

$$\mathcal{H}_j^{(\pi)} \simeq \bigoplus_{k=1}^{d_\pi}\mathsf{L}^2(X,\mu_X), \tag{3.3}$$

due to the orthogonality of the matrix elements $\{\pi_{jk}\}_{j,k=1}^{d_\pi}$.

A direct calculation shows that the operator U_ϕ is reduced by the decomposition (3.2) and that the restriction $U_{\pi,j} := U_\phi|_{\mathcal{H}_j^{(\pi)}}$ is given by

$$U_{\pi,j} \sum_{k=1}^{d_\pi} \varphi_k \otimes \pi_{jk} = \sum_{k,\ell=1}^{d_\pi} (V_1 \varphi_k)(\pi_{\ell k} \circ \phi) \otimes \pi_{j\ell}, \quad \varphi_k \in \mathsf{L}^2(X, \mu_X).$$

This, together with (3.1), implies that

$$(U_{\pi,j})^n \sum_{k=1}^{d_\pi} \varphi_k \otimes \pi_{jk} = \sum_{k,\ell=1}^{d_\pi} (V_n \varphi_k)\big(\pi_{\ell k} \circ \phi^{(n)}\big) \otimes \pi_{j\ell}, \quad n \in \mathbb{Z}, \ \varphi_k \in \mathsf{L}^2(X, \mu_X).$$

Assumption 3.1 (Cocycle)**.** *For each $k, \ell \in \{1, \dots, d_\pi\}$, the function $\pi_{k\ell} \circ \phi \in C(X; \mathbb{C})$ has a Lie derivative $\mathscr{L}_Y(\pi_{k\ell} \circ \phi)$ which satisfies the following Dini-type condition along the flow $\{F_t\}_{t \in \mathbb{R}}$:*

$$\int_0^1 \frac{\mathrm{d}t}{t} \left\| \mathscr{L}_Y(\pi_{k\ell} \circ \phi) \circ F_t - \mathscr{L}_Y(\pi_{k\ell} \circ \phi) \right\|_{\mathsf{L}^\infty(X)} < \infty.$$

Assumption 3.1 implies that $-i\mathscr{L}_Y(\pi \circ \phi) \cdot (\pi^* \circ \phi)$ is a continuous hermitian matrix-valued function. In particular, the matrix multiplication operator M given by

$$M \sum_{k=1}^{d_\pi} \varphi_k \otimes \pi_{jk} := -i \sum_{k,\ell=1}^{d_\pi} a_k \big\{\mathscr{L}_Y(\pi \circ \phi) \cdot (\pi^* \circ \phi)\big\}_{k\ell} \varphi_\ell \otimes \pi_{jk}, \quad a_k \in \mathbb{R}, \ \varphi_\ell \in \mathsf{L}^2(X, \mu_X),$$

is bounded in $\mathcal{H}_j^{(\pi)}$. We use the notation

$$M_{k\ell} := -i a_k \big\{\mathscr{L}_Y(\pi \circ \phi) \cdot (\pi^* \circ \phi)\big\}_{k\ell}, \quad k, \ell \in \{1, \dots, d_\pi\}, \tag{3.4}$$

for the matrix elements of M.

Lemma 3.2 (Conjugate operator for $U_{\pi,j}$)**.** *Let ϕ satisfy Assumption 3.1 and suppose that $(a_k - a_\ell)(\pi_{\ell k} \circ \phi) \equiv 0$ for all $k, \ell \in \{1, \dots, d_\pi\}$. Then,*

(a) The operator A given by

$$A \sum_{k=1}^{d_\pi} \varphi_k \otimes \pi_{jk} := \sum_{k=1}^{d_\pi} a_k H \varphi_k \otimes \pi_{jk}, \quad a_k \in \mathbb{R}, \ \varphi_k \in C^\infty(X),$$

is essentially self-adjoint in $\mathcal{H}_j^{(\pi)}$, and its closure (which we denote by the same symbol) has domain

$$\mathcal{D}(A) = \left\{ \sum_{k=1}^{d_\pi} \varphi_k \otimes \pi_{jk} \mid \varphi_k \in \mathcal{D}(H), \ k = 1, \dots, d_\pi \right\}.$$

Furthermore, one has

$$\mathrm{e}^{itA} \sum_{k=1}^{d_\pi} \varphi_k \otimes \pi_{jk} = \sum_{k=1}^{d_\pi} \mathrm{e}^{ita_k H} \varphi_k \otimes \pi_{jk}, \quad t \in \mathbb{R}, \ \varphi_k \in \mathsf{L}^2(X, \mu_X). \tag{3.5}$$

(b) For all $k, \ell \in \{1, \dots, d_\pi\}$ and all $t \in \mathbb{R}$, one has

$$\left\| \mathrm{e}^{-ita_k H} M_{k\ell} \, \mathrm{e}^{ita_\ell H} - M_{k\ell} \right\|_{\mathscr{B}(\mathsf{L}^2(X, \mu_X))} = \left\| \mathrm{e}^{-ita_k H} M_{k\ell} \, \mathrm{e}^{ita_k H} - M_{k\ell} \right\|_{\mathscr{B}(\mathsf{L}^2(X, \mu_X))}.$$

(c) $U_{\pi,j} \in C^{1+0}(A)$ *with* $[A, U_{\pi,j}] = MU_{\pi,j}$.

Remark 3.3. *(i) The commutation assumption* $(a_k - a_\ell)(\pi_{\ell k} \circ \phi) \equiv 0$ *for all* $k, \ell \in \{1, \dots, d_\pi\}$ *implies that the matrix* $M(x)$ *is hermitian for each* $x \in X$, *and thus that the operator* M *is self-adjoint. This assumption is satisfied if all the* a_k*'s are equal or if the matrix-valued function* $\pi \circ \phi$ *is diagonal (which occurs for instance when* G *is abelian). (ii) Instead of the diagonal operator* A, *one could use the more general, non-diagonal, self-adjoint operator*

$$A' \sum_{k=1}^{d_\pi} \varphi_k \otimes \pi_{jk} := \sum_{k=1}^{d_\pi} a_{k\ell} H\varphi_\ell \otimes \pi_{jk}, \quad \varphi_k \in C^\infty(X), \ a_{k\ell} = \overline{a_{\ell k}} \in \mathbb{C}.$$

However, doing this, one ends up with the same commutation relation in Lemma 3.2(c) (the scalars a_k *appearing in the matrix* M *are just replaced by the column sums* $\sum_{\ell=1}^{d_\pi} a_{k\ell}$*). So, there is no gain in using the operator* A' *instead of the simpler operator* A. *(iii) The commutation relation in Lemma 3.2(c) is a matrix version of the commutation relation put into evidence in [30, Sec. 2].*

Proof of Lemma 3.2. (a) The image of operator A under the isomorphism (3.3) is the operator $\bigoplus_{k=1}^{d_\pi} a_k H$. So, all the claims follow from standard results on direct sums of self-adjoint operators (see for instance [28, p. 268]).

(b) One has

$$\big\| e^{-ita_k H} M_{k\ell} e^{ita_\ell H} - M_{k\ell} \big\|_{\mathscr{B}(L^2(X,\mu_X))} = \big\| e^{-ita_k H} M_{k\ell} e^{ita_k H} - M_{k\ell} + M_{k\ell}\big(1 - e^{it(a_k - a_\ell)H}\big) \big\|_{\mathscr{B}(L^2(X,\mu_X))}.$$

Therefore, it is sufficient to show that $M_{k\ell}\big(1 - e^{it(a_k-a_\ell)H}\big) = 0$, which is equivalent to the condition

$$\big\langle \varphi, M_{k\ell}\big(1 - e^{it(a_k-a_\ell)H}\big)\varphi \big\rangle_{L^2(X,\mu_X)} = 0 \quad \text{for all } \varphi \in C^\infty(X),$$

due to the density of $C^\infty(X)$ in $L^2(X, \mu_X)$. So, take $\varphi \in C^\infty(X)$, set

$$F_{k\ell}(t) := \big\langle \varphi, M_{k\ell}\big(1 - e^{it(a_k-a_\ell)H}\big)\varphi \big\rangle_{L^2(X,\mu_X)},$$

and note that $M_{k\ell}(a_k - a_\ell) = 0$ due to the assumption $(a_k - a_\ell)(\pi_{\ell k} \circ \phi) \equiv 0$ for all $k, \ell \in \{1, \dots, d_\pi\}$. Then, one has

$$\frac{d}{dt} F_{k\ell}(t) := -i \big\langle \varphi, M_{k\ell}(a_k - a_\ell) e^{it(a_k-a_\ell)H} H\varphi \big\rangle_{L^2(X,\mu_X)} = 0.$$

It follows that $F_{k\ell}(t) = F_{k\ell}(0) = 0$ for all $t \in \mathbb{R}$, which proves the claim.

(c) Using first that $\mathscr{L}_Y$ and V_1 commute and then that $(a_k - a_\ell)(\pi_{\ell k} \circ \phi) \equiv 0$ for all $k, \ell \in \{1, \dots, d_\pi\}$, one gets for $\varphi_k \in C^\infty(X)$ that

$$\begin{aligned}
&\big(AU_{\pi,j} - U_{\pi,j}A\big) \sum_{k=1}^{d_\pi} \varphi_k \otimes \pi_{jk} \\
&= -i \sum_{k,\ell=1}^{d_\pi} a_\ell \big(\mathscr{L}_Y(V_1\varphi_k)\big)(\pi_{\ell k} \circ \phi) \otimes \pi_{j\ell} - i \sum_{k,\ell=1}^{d_\pi} a_\ell (V_1\varphi_k)\big(\mathscr{L}_Y(\pi_{\ell k} \circ \phi)\big) \otimes \pi_{j\ell} \\
&\quad + i \sum_{k,\ell=1}^{d_\pi} a_k \big(V_1\big(\mathscr{L}_Y\varphi_k\big)\big)(\pi_{\ell k} \circ \phi) \otimes \pi_{j\ell} \\
&= i \sum_{k,\ell=1}^{d_\pi} (a_k - a_\ell)\big(V_1\big(\mathscr{L}_Y\varphi_k\big)\big)(\pi_{\ell k} \circ \phi) \otimes \pi_{j\ell} - i \sum_{k,\ell=1}^{d_\pi} a_\ell (V_1\varphi_k)\big(\mathscr{L}_Y(\pi_{\ell k} \circ \phi)\big) \otimes \pi_{j\ell} \\
&= -i \sum_{k,\ell=1}^{d_\pi} a_\ell (V_1\varphi_k)\big(\mathscr{L}_Y(\pi_{\ell k} \circ \phi)\big) \otimes \pi_{j\ell}.
\end{aligned}$$

This, together with the fact that $\mathscr{L}_Y(\pi_{\ell k}\circ\phi)\in \mathsf{L}^\infty(X)$ for all $k,\ell\in\{1,\dots,d_\pi\}$ and the density of the vectors $\sum_{k=1}^{d_\pi}\varphi_k\otimes\pi_{jk}$ in $\mathcal{H}_j^{(\pi)}$, implies that $U_{\pi,j}\in C^1(A)$ with

$$[A,U_{\pi,j}]\sum_{k=1}^{d_\pi}\varphi_k\otimes\pi_{jk}=-i\sum_{k,\ell=1}^{d_\pi}a_\ell(V_1\varphi_k)\big(\mathscr{L}_Y(\pi_{\ell k}\circ\phi)\big)\otimes\pi_{j\ell},\quad \varphi_k\in\mathsf{L}^2(X,\mu_X).$$

Then, one obtains

$$\begin{aligned}
[A,U_{\pi,j}](U_{\pi,j})^*\sum_{k=1}^{d_\pi}\varphi_k\otimes\pi_{jk}&=-i\sum_{k,\ell,m=1}^{d_\pi}a_m\big\{V_1\big((V_{-1}\varphi_k)(\pi_{\ell k}\circ\phi^{(-1)})\big)\big\}\big(\mathscr{L}_Y(\pi_{m\ell}\circ\phi)\big)\otimes\pi_{jm}\\
&=-i\sum_{k,\ell,m=1}^{d_\pi}a_m\,\varphi_k(\pi^*\circ\phi)_{\ell k}\big(\mathscr{L}_Y(\pi\circ\phi)\big)_{m\ell}\otimes\pi_{jm}\\
&=M\sum_{k=1}^{d_\pi}\varphi_k\otimes\pi_{jk},
\end{aligned}$$

which shows the equality $[A,U_{\pi,j}]=MU_{\pi,j}$.

To prove that $U_{\pi,j}\in C^{1+0}(A)$, one has to check that

$$\int_0^1\frac{\mathrm{d}t}{t}\big\|\,\mathrm{e}^{-itA}[A,U_{\pi,j}]\,\mathrm{e}^{itA}-[A,U_{\pi,j}]\big\|_{\mathscr{B}(\mathcal{H}_j^{(\pi)})}<\infty.$$

But since $[A,U_{\pi,j}]=MU_{\pi,j}$ with $U_{\pi,j}\in C^1(A)$, it is sufficient to show that

$$\int_0^1\frac{\mathrm{d}t}{t}\big\|\,\mathrm{e}^{-itA}\,M\,\mathrm{e}^{itA}-M\big\|_{\mathscr{B}(\mathcal{H}_j^{(\pi)})}<\infty.$$

Now, Formula (3.5) implies that

$$\big(\mathrm{e}^{-itA}\,M\,\mathrm{e}^{itA}-M\big)\sum_{k=1}^{d_\pi}\varphi_k\otimes\pi_{jk}=\sum_{k,\ell=1}^{d_\pi}\big(\mathrm{e}^{-ita_kH}\,M_{k\ell}\,\mathrm{e}^{ita_\ell H}-M_{k\ell}\big)\varphi_\ell\otimes\pi_{jk},\quad \varphi_k\in\mathsf{L}^2(X,\mu_X).$$

It follows that

$$\begin{aligned}
&\int_0^1\frac{\mathrm{d}t}{t}\big\|\,\mathrm{e}^{-itA}\,M\,\mathrm{e}^{itA}-M\big\|_{\mathscr{B}(\mathcal{H}_j^{(\pi)})}\\
&\le\sum_{k,\ell=1}^{d_\pi}\int_0^1\frac{\mathrm{d}t}{t}\big\|\,\mathrm{e}^{-ita_kH}\,M_{k\ell}\,\mathrm{e}^{ita_\ell H}-M_{k\ell}\big\|_{\mathscr{B}(\mathsf{L}^2(X,\mu_X))}\\
&=\sum_{k,\ell=1}^{d_\pi}\int_0^1\frac{\mathrm{d}t}{t}\big\|\,\mathrm{e}^{-ita_kH}\,M_{k\ell}\,\mathrm{e}^{ita_kH}-M_{k\ell}\big\|_{\mathscr{B}(\mathsf{L}^2(X,\mu_X))}\\
&=\sum_{k,\ell=1}^{d_\pi}\int_0^{a_k}\frac{\mathrm{d}s}{s}\big\|V_s\,M_{k\ell}V_{-s}-M_{k\ell}\big\|_{\mathscr{B}(\mathsf{L}^2(X,\mu_X))}\\
&\le\mathrm{Const.}\sum_{k,\ell=1}^{d_\pi}\int_0^{a_k}\frac{\mathrm{d}s}{s}\,\big\|\big\{\mathscr{L}_Y(\pi\circ\phi)\cdot(\pi^*\circ\phi)\big\}_{k\ell}\circ F_s-\big\{\mathscr{L}_Y(\pi\circ\phi)\cdot(\pi^*\circ\phi)\big\}_{k\ell}\big\|_{\mathsf{L}^\infty(X)}\\
&<\infty,
\end{aligned}$$

due to point (b) and the Dini-type condition satisfied by the functions $\mathscr{L}_Y(\pi_{k\ell}\circ\phi)$. $\square$

In the following theorem, we present a first set of conditions implying a strict Mourre estimate for $U_{\pi,j}$ on all of $\mathbb{S}^1$ and thus the absolute continuity of the spectrum of $U_{\pi,j}$. For each $x \in X$, we write $\lambda_k\big(M(x)\big)$, $k \in \{1,\dots,d_\pi\}$, for the eigenvalues of the hermitian matrix $M(x)$, and we use the notation

$$\lambda_* := \inf_{k\in\{1,\dots,d_\pi\},\, x\in X} \lambda_k\big(M(x)\big). \tag{3.6}$$

Theorem 3.4 (First Mourre estimate for $U_{\pi,j}$)**.** *Let ϕ satisfy Assumption 3.1, suppose that $(a_k - a_\ell)(\pi_{\ell k}\circ\phi) \equiv 0$ for all $k,\ell \in \{1,\dots,d_\pi\}$, and assume that $\lambda_* > 0$. Then, $U_{\pi,j}$ satisfies the strict Mourre estimate*

$$(U_{\pi,j})^*[A, U_{\pi,j}] \geq \lambda_*,$$

and $U_{\pi,j}$ has purely absolutely continuous spectrum.

Proof. Since M is hermitian matrix-valued, there exists a function $U : X \to \mathrm{U}(d_\phi)$ such that $M = U^*DU$ with D diagonal:

$$D(x) := \begin{pmatrix} \lambda_1\big(M(x)\big) & & 0 \\ & \ddots & \\ 0 & & \lambda_{d_\pi}\big(M(x)\big) \end{pmatrix}, \quad x \in X.$$

Furthermore, one has the orthogonality relation $\big\langle \pi_{jm}, \pi_{jk}\big\rangle_{\mathsf{L}^2(G,\mu_G)} = \delta_{mk}(d_\pi)^{-1}$. Therefore, one obtains for $\varphi_k \in \mathsf{L}^2(X,\mu_X)$ that

$$\begin{aligned}
\left\langle \sum_{m=1}^{d_\pi} \varphi_m \otimes \pi_{jm}, M \sum_{k=1}^{d_\pi} \varphi_k \otimes \pi_{jk} \right\rangle_{\mathcal{H}_j^{(\pi)}} &= \sum_{k,\ell=1}^{d_\pi} \big\langle \varphi_k, M_{k\ell}\varphi_\ell\big\rangle_{\mathsf{L}^2(X,\mu_X)} (d_\pi)^{-1} \\
&= \sum_{m=1}^{d_\pi} \left\langle \left(\sum_{k=1}^{d_\pi} U_{mk}\varphi_k\right), \lambda_m\big(M(\cdot)\big) \left(\sum_{\ell=1}^{d_\pi} U_{m\ell}\varphi_\ell\right)\right\rangle_{\mathsf{L}^2(X,\mu_X)} (d_\pi)^{-1} \\
&\geq \lambda_* \sum_{m=1}^{d_\pi} \left\langle \left(\sum_{k=1}^{d_\pi} U_{mk}\varphi_k\right), \left(\sum_{\ell=1}^{d_\pi} U_{m\ell}\varphi_\ell\right)\right\rangle_{\mathsf{L}^2(X,\mu_X)} (d_\pi)^{-1} \\
&= \lambda_* \left\langle \sum_{m=1}^{d_\pi} \varphi_m \otimes \pi_{jm}, \sum_{k=1}^{d_\pi} \varphi_k \otimes \pi_{jk} \right\rangle_{\mathcal{H}_j^{(\pi)}},
\end{aligned}$$

which is equivalent to the inequality $M \geq \lambda_*$. We thus infer from Lemma 3.2(c) that $U_{\pi,j} \in C^{1+0}(A)$ with

$$(U_{\pi,j})^*[A, U_{\pi,j}] = (U_{\pi,j})^* M U_{\pi,j} \geq \lambda_*.$$

It follows from Theorem 2.1 that $U_{\pi,j}$ has purely absolutely continuous spectrum. $\square$

Sometimes (as when F_1 is uniquely ergodic), it is advantageous to replace the positivity condition $\lambda_* > 0$ of Theorem 3.4 by an averaged positivity condition more likely to be satisfied. For this, we have to modify the conjugate operator A. Following the approach of [10, Sec. 4] and [30, Sec. 2], we use the operator A_N obtained by averaging the operator A along the flow generated by $U_{\pi,j}$:

$$A_N\varphi := \frac{1}{N}\sum_{n=0}^{N-1}(U_{\pi,j})^n A (U_{\pi,j})^{-n}\varphi, \quad N \in \mathbb{N}_{\geq 1},\ \varphi \in \mathcal{D}(A_N) := \mathcal{D}(A), \tag{3.7}$$

(the operator A_N is self-adjoint on $\mathcal{D}(A_N) = \mathcal{D}(A)$ because $(U_{\pi,j})^n \in C^1(A)$ for each $n \in \mathbb{Z}$, see [10, Sec. 4]). In such a case, the averages

$$M_N := \frac{1}{N}\sum_{n=0}^{N-1} \big(\pi\circ\phi^{(n)}\big)\big(M\circ F_n\big)\big(\pi^*\circ\phi^{(n)}\big) \tag{3.8}$$

of the operator M appear, and we thus use the notation

$$\lambda_{*,N} := \inf_{k\in\{1,\dots,d_\pi\},\, x\in X} \lambda_k\big(M_N(x)\big). \tag{3.9}$$

Note that if the matrix-valued function $\pi\circ\phi$ is diagonal, then $\big(\pi\circ\phi^{(n)}\big)$, $(M\circ F_n)$ and $\big(\pi^*\circ\phi^{(n)}\big)$ are also diagonal and M_N reduces to the Birkhoff sum

$$M_N = \frac{1}{N}\sum_{n=0}^{N-1} M\circ F_n.$$

Theorem 3.5 (Second Mourre estimate for $U_{\pi,j}$)**.** *Let ϕ satisfy Assumption 3.1, suppose that $(a_k - a_\ell)(\pi_{\ell k}\circ\phi)\equiv 0$ for all $k,\ell\in\{1,\dots,d_\pi\}$, and assume that $\lambda_{*,N}>0$ for some $N\in\mathbb{N}_{\geq 1}$. Then, $U_{\pi,j}$ satisfies the strict Mourre estimate*

$$(U_{\pi,j})^*[A_N, U_{\pi,j}] \geq \lambda_{*,N},$$

and $U_{\pi,j}$ has purely absolutely continuous spectrum.

Proof. We know from Lemma 3.2(c) that $U_{\pi,j}\in C^{1+0}(A)$. So, it follows from the abstract result [10, Lemma 4.1] that $U_{\pi,j}\in C^{1+0}(A_N)$ with $[A_N, U_{\pi,j}] = \frac{1}{N}\sum_{n=0}^{N-1}(U_{\pi,j})^n[A,U_{\pi,j}](U_{\pi,j})^{-n}$. Using the equality $[A,U_{\pi,j}] = MU_{\pi,j}$, one thus obtains that

$$[A_N, U_{\pi,j}] = \left(\frac{1}{N}\sum_{n=0}^{N-1}(U_{\pi,j})^n M (U_{\pi,j})^{-n}\right) U_{\pi,j}$$

with

$$\begin{aligned}
(U_{\pi,j})^n M (U_{\pi,j})^{-n}\sum_{k=1}^{d_\pi}\varphi_k\otimes\pi_{jk} &= (U_{\pi,j})^n \sum_{k,\ell,m=1}^{d_\pi} M_{\ell m}(\varphi_k\circ F_{-n})\big(\pi_{mk}\circ\phi^{(-n)}\big)\otimes\pi_{j\ell}\\
&= \sum_{k,\ell,m,p=1}^{d_\pi}(M_{\ell m}\circ F_n)\big(\pi_{mk}\circ\phi^{(-n)}\circ F_n\big)\big(\pi_{p\ell}\circ\phi^{(n)}\big)\varphi_k\otimes\pi_{jp}\\
&= \sum_{k,\ell,m,p=1}^{d_\pi}(M_{\ell m}\circ F_n)\big(\pi^*_{mk}\circ\phi^{(n)}\big)\big(\pi_{p\ell}\circ\phi^{(n)}\big)\varphi_k\otimes\pi_{jp}\\
&= \sum_{k,p=1}^{d_\pi}\big\{\big(\pi\circ\phi^{(n)}\big)(M\circ F_n)\big(\pi^*\circ\phi^{(n)}\big)\big\}_{pk}\varphi_k\otimes\pi_{jp}
\end{aligned}$$

for $\varphi_k\in \mathsf{L}^2(X,\mu_X)$. Thus, $U_{\pi,j}\in C^{1+0}(A_N)$ with $[A_N,U_{\pi,j}] = M_N\, U_{\pi,j}$. Since $M_N(x)$ is a hermitian matrix for each $x\in X$, one can then conclude using the same argument as in the proof of Theorem 3.4. $\square$

Remark 3.6 (Relation with the topological degree)**.** *The matrix-valued function M_N, which came out from a commutator calculation, is related to the notion of topological degree of the cocycle ϕ in the representation π. Indeed, the assumption $(a_k-a_\ell)(\pi_{\ell k}\circ\phi)\equiv 0$ for all $k,\ell\in\{1,\dots,d_\pi\}$ implies that the matrices $\pi\circ\phi^{(n)}$ and $D_a := \mathrm{diag}(a_1,\dots,a_{d_\pi})$ commute. Thus, one has the equalities*

$$\begin{aligned}
M_N &= D_a\frac{1}{N}\sum_{n=0}^{N-1}\big(\pi\circ\phi^{(n)}\big)\mathscr{L}_Y(\pi\circ\phi\circ F_n)(\pi^*\circ\phi\circ F_n)\big(\pi^*\circ\phi^{(n)}\big)\\
&= D_a\frac{1}{N}\sum_{n=0}^{N-1}(\pi\circ\phi)\cdots(\pi\circ\phi\circ F_{n-1})\mathscr{L}_Y(\pi\circ\phi\circ F_n)(\pi^*\circ\phi\circ F_n)(\pi^*\circ\phi\circ F_{n-1})\cdots(\pi^*\circ\phi)\\
&= D_a\frac{1}{N}\mathscr{L}_Y\big((\pi\circ\phi)^{(N)}\big)\big((\pi\circ\phi)^{(N)}\big)^*,
\end{aligned}$$

and M_N is the product of $D_a \frac{1}{N}$ times the matrix-valued function $\mathscr{L}_Y\big((\pi\circ\phi)^{(N)}\big)\big((\pi\circ\phi)^{(N)}\big)^*$, which can be associated to the winding number of the curve $(\pi\circ\phi)^{(N)}$ in the unitary group $\mathrm{U}(d_\pi)\equiv\mathrm{codomain}(\pi)$ (the Lie derivative $\mathscr{L}_Y$ replaces the complex derivative of the scalar case). It follows that the limit $\lim_{N\to\infty} M_N$ (if it exists, in some topology to be specified) can be interpreted as the matrix topological degree of $\pi\circ\phi$, up to multiplication by the constant matrix D_a.

This furnishes an alternative interpretation to the result of Theorem 3.5: If N is large enough, then M_N is close to D_a times the topological degree of $\pi\circ\phi$. So, the condition $\lambda_{*,N}>0$ means that the topological degree of $\pi\circ\phi$ has nonzero determinant, and Theorem 3.5 tells us that in this case U_ϕ has purely absolutely continuous spectrum in the subspace associated to π. This is nothing else but a local version, in each representation π, of the result (already known in various cases, see [5, 12, 13, 15, 23, 31]) that the continuous component of spectrum of skew products is purely absolutely continuous if ϕ is regular enough and has nonzero topological degree. The main novelty here is that X and G are general compact Lie groups.

We conclude the section by noting that in the particular case where $\{F_t\}_{t\in\mathbb{R}}$ is a translation flow on a torus $X=\mathbb{T}^d\simeq\mathbb{R}^d/\mathbb{Z}^d$, $d\geq 1$, with F_1 ergodic along one coordinate, the spectrum of $U_{\pi,j}$ is Lebesgue if it is purely absolutely continuous. Indeed, assume that

$$F_t(x):=x+ty \ (\mathrm{mod}\ \mathbb{Z}^d),\quad t\in\mathbb{R},\ x\in\mathbb{T}^d,$$

for some $y:=(y_1,\dots,y_d)\in\mathbb{R}^d$ with $y_{k_0}\in\mathbb{R}\setminus\mathbb{Q}$ for some $k_0\in\{1,\dots,d\}$. Let $Q_{k_0}\in\mathscr{B}\big(\mathcal{H}_j^{(\pi)}\big)$ be the unitary operator given by

$$\left(Q_{k_0}\sum_{k=1}^{d_\pi}\varphi_k\otimes\pi_{jk}\right)(x,g):=\mathrm{e}^{2\pi i x_{k_0}}\sum_{k=1}^{d_\pi}\varphi_k(x)\,\pi_{jk}(g),\quad \varphi_k\in\mathrm{L}^2(X,\mu_X),\ (x,g)\in X\times G,$$

and let $T_0:\mathbb{T}\to\mathbb{T}$ be the ergodic translation given by $T_0(z):=z+y_{k_0}$. Finally, denote by σ_ψ the spectral measure of $U_{\pi,j}$ associated to a vector $\psi\in\mathcal{H}_j^{(\pi)}$; that is, the Borel measure on $\mathbb{T}$ defined by the equalities

$$(\mathscr{F}\sigma_\psi)(-m)=\int_{\mathbb{T}}\mathrm{e}^{2\pi i m z}\,\mathrm{d}\sigma_\psi(z)=\big\langle (U_{\pi,j})^m\psi,\psi\big\rangle,\quad m\in\mathbb{Z},$$

with $\mathscr{F}$ the Fourier transform. Then, we have the identities

$$(U_{\pi,j})^m Q_{k_0}=\mathrm{e}^{2\pi i m y_{k_0}}\,Q_{k_0}(U_{\pi,j})^m$$

and

$$\int_{\mathbb{T}}\mathrm{e}^{2\pi i m z}\,\mathrm{d}\sigma_{Q_{k_0}\psi}(z)=\big\langle (U_{\pi,j})^m Q_{k_0}\psi,Q_{k_0}\psi\big\rangle_{\mathcal{H}_j^{(\pi)}}=\mathrm{e}^{-2\pi i m y_{k_0}}\,\big\langle (U_{\pi,j})^m\psi,\psi\big\rangle_{\mathcal{H}_j^{(\pi)}}=\int_{\mathbb{T}}\mathrm{e}^{2\pi i m z}\,\mathrm{d}(T_0^*\sigma_\psi)(z)$$

for all $m\in\mathbb{Z}$ and $\psi\in\mathcal{H}_j^{(\pi)}$. Thus, $\sigma_{Q_{k_0}\psi}=T_0^*\sigma_\psi$, and one has the following result:

Lemma 3.7. *Assume that $X=\mathbb{T}^d$, $d\geq 1$, and let $\{F_t\}_{t\in\mathbb{R}}$ be given by*

$$F_t(x):=x+ty \ (\mathrm{mod}\ \mathbb{Z}^d),\quad t\in\mathbb{R},\ x\in\mathbb{T}^d,$$

for some $y:=(y_1,\dots,y_d)\in\mathbb{R}^d$ with $y_{k_0}\in\mathbb{R}\setminus\mathbb{Q}$ for some $k_0\in\{1,\dots,d\}$. Then, the spectrum of $U_{\pi,j}$ is Lebesgue if it is purely absolutely continuous.

Proof. The claim follows from the identity $\sigma_{Q_{k_0}\psi}=T_0^*\sigma_\psi$ and the ergodicity of T_0 (see the proof of [23, Lemma 3] or [13, Lemma 3.1] for details). □

4 Examples

4.1 The abelian case $X = \mathbb{T}^d$ and $G = \mathbb{T}^{d'}$

Suppose that $X = \mathbb{T}^d$ and $G = \mathbb{T}^{d'}$ for some $d, d' \geq 1$, set $\mathcal{H} := \mathsf{L}^2\big(\mathbb{T}^d \times \mathbb{T}^{d'}, \mu_{\mathbb{T}^d} \otimes \mu_{\mathbb{T}^{d'}}\big)$, and let $\{F_t\}_{t\in\mathbb{R}}$ be the translation flow on $\mathbb{T}^d$ given by

$$F_t(x) := x + ty \ (\mathrm{mod}\ \mathbb{Z}^d), \quad t \in \mathbb{R},\ x \in \mathbb{T}^d,$$

for some $y := (y_1, y_2, \dots, y_d) \in \mathbb{R}^d$. Then, each element $\chi_q \in \widehat{\mathbb{T}^{d'}}$ is a 1-dimensional IUR (character) of $\mathbb{T}^{d'}$ given by $\chi_q(z) := \mathrm{e}^{2\pi i q\cdot z}$ for some $q \in \mathbb{Z}^{d'}$. One has

$$\mathcal{H}_1^{(\chi_q)} = \big\{\varphi \otimes \chi_q \mid \varphi \in \mathsf{L}^2\big(\mathbb{T}^d, \mu_{\mathbb{T}^d}\big)\big\},$$

the Lie derivative $\mathscr{L}_Y$ is given by $\mathscr{L}_Y = y \cdot \nabla_x$, and F_1 is uniquely ergodic if and only if the numbers $y_1, y_2, \dots, y_d, 1$ are rationally independent.

Given $q \in \mathbb{Z}^{d'}$, we choose the function $\phi \in C(\mathbb{T}^d; \mathbb{T}^{d'})$ as follows:

Assumption 4.1. *The function $\phi \in C(\mathbb{T}^d; \mathbb{T}^{d'})$ satisfies $\phi = \xi + \eta$, where*

(i) $\xi : \mathbb{T}^d \to \mathbb{T}^{d'}$ is a Lie group homomorphism; that is, ξ is given by $\xi(x) := Bx$ (mod $\mathbb{Z}^{d'}$) for some $d' \times d$ matrix B with integer entries,

(ii) $\eta \in C(\mathbb{T}^d; \mathbb{T}^{d'})$ is such that $\mathscr{L}_Y(q \cdot \eta)$ exists and satisfies the Dini-type condition

$$\int_0^1 \frac{\mathrm{d}t}{t} \left\|\mathscr{L}_Y(q\cdot\eta) \circ F_t - \mathscr{L}_Y(q\cdot\eta)\right\|_{\mathsf{L}^\infty(\mathbb{T}^d)} < \infty. \tag{4.1}$$

Then, the function ϕ satisfies Assumption 3.1, the skew product T_ϕ is given by

$$T_\phi(x, g) = \big(x + y, g + \phi(x)\big), \quad (x, g) \in \mathbb{T}^d \times \mathbb{T}^{d'},$$

and the matrix-valued function M defined in (3.4) reduces to the scalar function

$$M = -i a_1 \mathscr{L}_Y(\chi_q \circ \phi) \cdot (\overline{\chi_q} \circ \phi) = 2\pi a_1 \big(\mathscr{L}_Y(q\cdot\xi) + \mathscr{L}_Y(q\cdot\eta)\big) = 2\pi a_1 \big(y \cdot (B^\mathsf{T} q) + \mathscr{L}_Y(q\cdot\eta)\big).$$

Accordingly, the matrix-valued function M_N defined in (3.8) reduces to the scalar function

$$M_N = \frac{1}{N}\sum_{n=0}^{N-1} M \circ F_n = 2\pi a_1 \left(y \cdot (B^\mathsf{T} q) + \frac{1}{N}\sum_{n=0}^{N-1} \mathscr{L}_Y(q\cdot\eta) \circ F_n\right).$$

So, if $B^\mathsf{T} q \neq 0$ and if the numbers $y_1, y_2, \dots, y_d, 1$ are rationally independent, one has $y \cdot (B^\mathsf{T} q) \neq 0$. Thus, one can set $a_1 := \big(2\pi y \cdot (B^\mathsf{T} q)\big)^{-1}$, so that M_N takes the form

$$M_N = 1 + \big(y \cdot (B^\mathsf{T} q)\big)^{-1} \left(\frac{1}{N}\sum_{n=0}^{N-1} \mathscr{L}_Y(q\cdot\eta) \circ F_n\right). \tag{4.2}$$

Collecting what precedes, one obtains the following result on the spectrum of the operators $U_{\chi_q,1}$ and U_ϕ associated to the skew product T_ϕ.

Theorem 4.2. *Let ϕ satisfy Assumption 4.1, suppose that $B^\mathsf{T} q \neq 0$, and assume that $y_1, y_2, \dots, y_d, 1$ are rationally independent. Then, $U_{\chi_q,1}$ has purely Lebesgue spectrum. In particular, if ϕ satisfies Assumption 4.1 for each $q \in \mathbb{Z}^{d'}$, then the restriction of U_ϕ to the subspace $\bigoplus_{q\in\mathbb{Z}^{d'},\, B^\mathsf{T} q\neq 0} \mathcal{H}_1^{(\chi_q)} \subset \mathcal{H}$ has countable Lebesgue spectrum.*

Proof. We know that ϕ satisfies Assumption 3.1, and it is obvious that $(a_1 - a_1)(\chi_q \circ \phi) \equiv 0$. Furthermore, due to the unique ergodicity of F_1, we infer from (4.2) that

$$\begin{aligned}
\lim_{N\to\infty} M_N &= 1 + \big(y \cdot (B^{\mathsf{T}} q)\big)^{-1} \lim_{N\to\infty} \left(\frac{1}{N}\sum_{n=0}^{N-1} \mathscr{L}_Y(q\cdot\eta)\circ F_n\right) \\
&= 1 + \big(y \cdot (B^{\mathsf{T}} q)\big)^{-1} \int_{\mathbb{T}^d} \mathrm{d}\mu_{\mathbb{T}^d}\, \mathscr{L}_Y(q\cdot\eta) \\
&= 1
\end{aligned}$$

uniformly on $\mathbb{T}^d$. Therefore,

$$\lambda_{*,N} = \inf_{x\in\mathbb{T}^d} \lambda_1\big(M_N(x)\big) = \inf_{x\in\mathbb{T}^d} M_N(x) > 0$$

if N is large enough. So, it follows from Theorem 3.5 and Lemma 3.7 that $U_{\chi_q,1}$ has purely Lebesgue spectrum. The claim on U_ϕ follows from what precedes if one takes into account the separability of the Hilbert space $\mathcal{H} \equiv \mathsf{L}^2\big(\mathbb{T}^d \times \mathbb{T}^{d'}, \mu_{\mathbb{T}^d} \otimes \mu_{\mathbb{T}^{d'}}\big)$. □

Theorem 4.2 is consistent with Corollary 4.5 of [30], where the same spectral result is obtained using a less general framework. We refer to the discussion after [30, Cor. 4.5] for a comparison with prior results on the spectral analysis of skew products of tori.

4.2 The case $X = \mathbb{T}^d$ and $G = \mathsf{SU}(2)$

Suppose that $X = \mathbb{T}^d$ for some $d \geq 1$, let $G = \mathsf{SU}(2)$, set $\mathcal{H} := \mathsf{L}^2\big(\mathbb{T}^d \times \mathsf{SU}(2), \mu_{\mathbb{T}^d} \otimes \mu_{\mathsf{SU}(2)}\big)$, let $\{F_t\}_{t\in\mathbb{R}}$ be the translation flow on $\mathbb{T}^d$ given by

$$F_t(x) := x + ty \ (\mathrm{mod}\ \mathbb{Z}^d), \quad t \in \mathbb{R},\ x \in \mathbb{T}^d,$$

for some $y := (y_1, y_2, \dots, y_d) \in \mathbb{R}^d$, and let $\xi : \mathbb{T}^d \to \mathsf{SU}(2)$ be a Lie group homomorphism. Then, one has $\mathscr{L}_Y = y \cdot \nabla_x$, the function ξ satisfies Assumption 3.1, and the skew product T_ξ is given by

$$T_\xi(x, g) = \big(x + y, g\xi(x)\big), \quad (x, g) \in \mathbb{T}^d \times \mathsf{SU}(2). \tag{4.3}$$

Since $\mathbb{T}^d$ is abelian, the range of ξ is contained in a maximal torus of $\mathsf{SU}(2)$. But, all of these are mutually conjugate to the subgroup $\big\{\big(\begin{smallmatrix} z & 0 \\ 0 & \overline{z}\end{smallmatrix}\big)\big\}_{z\in\mathbb{S}^1}$ (see [6, Thm. IV.1.6 & Prop. IV.3.1]). So, we can suppose without loss of generality that

$$\xi(x) = h \begin{pmatrix} \mathrm{e}^{2\pi i(b\cdot x)} & 0 \\ 0 & \mathrm{e}^{-2\pi i(b\cdot x)} \end{pmatrix} h^*, \quad x \in \mathbb{T}^d, \tag{4.4}$$

for some vector $b \in \mathbb{Z}^d$ and some element $h \in \mathsf{SU}(2)$, and thus that

$$\big(\pi \circ \xi\big)(x) = \pi(h)\,\pi \begin{pmatrix} \mathrm{e}^{2\pi i(b\cdot x)} & 0 \\ 0 & \mathrm{e}^{-2\pi i(b\cdot x)} \end{pmatrix} \big(\pi(h)\big)^* \tag{4.5}$$

for each π, finite-dimensional IUR of $\mathsf{SU}(2)$.

The set $\widehat{\mathsf{SU}(2)}$ of all (equivalence classes of) finite-dimensional IUR's of $\mathsf{SU}(2)$ can be described as follows (see [29, Chap. II]). For each $n \in \mathbb{N}$, let V_n be the $(n+1)$-dimensional vector space of homogeneous polinomials of degree n in the variables $z_1, z_2 \in \mathbb{C}$. Endow V_n with the basis

$$p_k(z_1, z_2) := z_1^k z_2^{n-k}, \quad k \in \{0, \dots, n\},$$

and the scalar product $\langle\cdot,\cdot\rangle_{V_n} : V_n \times V_n \to \mathbb{C}$ defined by

$$\left\langle \sum_{k=0}^n \alpha_k z_1^k z_2^{n-k}, \sum_{\ell=0}^n \beta_\ell z_1^\ell z_2^{n-\ell} \right\rangle_{V_n} := \sum_{k=0}^n k!(n-k)!\,\alpha_k \overline{\beta_k}, \quad \alpha_k, \beta_\ell \in \mathbb{C}.$$

Then, the function $\pi^{(n)}:\mathrm{SU}(2)\to \mathrm{U}(V_n)\simeq \mathrm{U}(n+1)$ given by

$$\big(\pi^{(n)}(g)p\big)(z_1,z_2):=p\big(g_{11}z_1+g_{21}z_2,g_{12}z_1+g_{22}z_2\big),\quad g=\begin{pmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}\in \mathrm{SU}(2),\ p\in V_n,$$

defines a $(n+1)$-dimensional IUR of SU(2) on V_n, and each finite-dimensional IUR of SU(2) is unitarily equivalent to an element of the family $\{\pi^{(n)}\}_{n\in\mathbb{N}}$ [29, Prop. II.1.1 & Thm. II.4.1]. A calculation using the binomial theorem shows that the matrix elements $\pi^{(n)}_{jk}$ of $\pi^{(n)}$ with respect to the basis $\{p_k\}_{k=0}^n$ satisfy

$$\pi^{(n)}_{jk}(g):=\big\langle p_j,\pi^{(n)}(g)p_k\big\rangle_{V_n}=j!(n-j)!\sum_{\ell=0}^{k}\binom{k}{\ell}\binom{n-k}{j-\ell}g_{11}^{\ell}g_{12}^{j-\ell}g_{21}^{k-\ell}g_{22}^{n+\ell-k-j},\quad j,k\in\{0,\dots,n\},$$

with $\binom{\cdot}{\cdot}$ the binomial coefficients. In the particular case of diagonal elements $g=\left(\begin{smallmatrix} g_{11} & 0\\ 0 & \overline{g_{11}}\end{smallmatrix}\right)\in \mathrm{SU}(2)$, we thus get that

$$\pi^{(n)}_{jk}\begin{pmatrix} g_{11} & 0\\ 0 & \overline{g_{11}}\end{pmatrix}=j!(n-j)!\,g_{11}^{2j-n}\delta_{jk},\quad \delta_{jk}:=\begin{cases}1 & \text{if } j=k\\ 0 & \text{if } j\neq k.\end{cases}\tag{4.6}$$

Therefore, by replacing $\pi^{(n)}(\,\cdot\,)$ by the unitarily equivalent representation $\big(\pi^{(n)}(h)\big)^*\pi^{(n)}(\cdot)\pi^{(n)}(h)$, we infer from (4.5) that

$$\big(\pi^{(n)}_{jk}\circ\xi\big)(x)=j!(n-j)!\,\mathrm{e}^{2\pi i(2j-n)(b\cdot x)}\,\delta_{jk}.$$

Then, putting this expression in the formula (3.4) for the matrix-valued function M, one obtains that

$$\begin{aligned}
M&=-i\begin{pmatrix} a_0 & & 0\\ & \ddots & \\ 0 & & a_n\end{pmatrix}\big(\mathscr{L}_Y(\pi^{(n)}\circ\xi)\big)\cdot\big((\pi^{(n)})^*\circ\xi\big)\\
&=-i\begin{pmatrix} a_0 & & 0\\ & \ddots & \\ 0 & & a_n\end{pmatrix}\big(\pi^{(n)}\circ\xi\big)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\big(\pi^{(n)}\circ\xi\big)(ty)\Big|_{t=0}\cdot\big((\pi^{(n)})^*\circ\xi\big)\\
&=2\pi(y\cdot b)\begin{pmatrix} a_0\,0!\,(n-0)!\,(2\cdot 0-n) & & 0\\ & \ddots & \\ 0 & & a_n\,n!\,(n-n)!\,(2\cdot n-n)\end{pmatrix}.
\end{aligned}\tag{4.7}$$

In the case $y\cdot b\neq 0$, we can set $a_j:=(2j-n)\big(2\pi(y\cdot b)j!(n-j)!\big)^{-1}$, and thus obtain that

$$M_{jk}=(2j-n)^2\,\delta_{jk},$$

which (in view of Equation (3.6)) implies that

$$\lambda_*=\inf_{k\in\{0,\dots,n\}}(2k-n)^2=\begin{cases}0 & \text{if } n\in 2\mathbb{N}\\ 1 & \text{if } n\in 2\mathbb{N}+1.\end{cases}$$

Collecting what precedes, one ends up with the following result on the spectrum of the operators $U_{\pi^{(n)},j}$ and U_ξ associated to the skew product T_ξ.

Lemma 4.3. *Let ξ satisfy (4.4) with $y\cdot b\neq 0$, and take $n\in 2\mathbb{N}+1$ and $j\in\{0,\dots,n\}$. Then, $U_{\pi^{(n)},j}$ has purely absolutely continuous spectrum. In particular, the restriction of U_ξ to the subspace $\bigoplus_{n\in 2\mathbb{N}+1}\bigoplus_{j=0}^n \mathcal{H}_j^{(\pi^{(n)})}\subset\mathcal{H}$ has purely absolutely continuous spectrum.*

Proof. We know that ξ satisfies Assumption 3.1, that $(a_k-a_\ell)\big(\pi^{(n)}_{\ell k}\circ\xi\big)\equiv 0$ for all $k,\ell\in\{0,\dots,n\}$, and that $\lambda_*=1$. So, the claim is a direct consequence of Theorem 3.4. $\square$

As in Section 4.1, we can treat more general cocycles ϕ (namely, perturbations of group homomorphisms) if F_1 is uniquely ergodic. So, from now on, we assume that $y_1, y_2, \dots, y_d, 1$ are rationally independent (so that F_1 is uniquely ergodic) and we suppose that $\phi : \mathbb{T}^d \to \mathsf{SU}(2)$ is a perturbation of ξ in the sense that

$$\phi(x) := h \begin{pmatrix} \mathrm{e}^{2\pi i(b\cdot x+\eta(x))} & 0 \\ 0 & \mathrm{e}^{-2\pi i(b\cdot x+\eta(x))} \end{pmatrix} h^*, \quad x \in \mathbb{T}^d, \tag{4.8}$$

with $b \in \mathbb{Z}^d \setminus \{0\}$ and with $\eta \in C(\mathbb{T}^d; \mathbb{R})$ satisfying the following:

Assumption 4.4. $\eta \in C(\mathbb{T}^d; \mathbb{R})$ *is such that* $\mathscr{L}_Y \eta$ *exists and satisfies the Dini-type condition*

$$\int_0^1 \frac{\mathrm{d}t}{t} \left\| \mathscr{L}_Y \eta \circ F_t - \mathscr{L}_Y \eta \right\|_{\mathrm{L}^\infty(\mathbb{T}^d)} < \infty.$$

So, we have

$$\big(\pi^{(n)} \circ \phi\big)(x) = \pi^{(n)}(h)\,\pi^{(n)} \begin{pmatrix} \mathrm{e}^{2\pi i(b\cdot x+\eta(x))} & 0 \\ 0 & \mathrm{e}^{-2\pi i(b\cdot x+\eta(x))} \end{pmatrix} \big(\pi^{(n)}(h)\big)^*,$$

and, replacing $\pi^{(n)}(\,\cdot\,)$ by the unitarily equivalent representation $\big(\pi^{(n)}(h)\big)^* \pi^{(n)}(\,\cdot\,)\,\pi^{(n)}(h)$, we infer from (4.6) that

$$\big(\pi^{(n)}_{jk} \circ \phi\big)(x) = j\,!(n-j)!\,\mathrm{e}^{2\pi i(2j-n)(b\cdot x+\eta(x))}\,\delta_{jk}.$$

Therefore, a calculation similar to that of (4.7) gives

$$M_{jk} = 2\pi a_j\big((y \cdot b) + \mathscr{L}_Y \eta\big)\,j!\,(n-j)!\,(2j-n)\delta_{jk}.$$

But now we know that $y \cdot b \neq 0$, since $y_1, y_2, \dots, y_d, 1$ are rationally independent. So, we can set $a_j := (2j-n)\big(2\pi(y\cdot b)\,j!(n-j)!\big)^{-1}$, and thus obtain that

$$M_{jk} = \big(1 + (y\cdot b)^{-1}\mathscr{L}_Y\eta\big)(2j-n)^2\delta_{jk}.$$

Accordingly, the matrix-valued function M_N given in (3.8) reduces to

$$M_N = \frac{1}{N}\sum_{m=0}^{N-1} M \circ F_m = \left(1 + (y\cdot b)^{-1}\frac{1}{N}\sum_{m=0}^{N-1}\mathscr{L}_Y\eta \circ F_m\right) \begin{pmatrix} (2\cdot 0 - n)^2 & & 0 \\ & \ddots & \\ 0 & & (2\cdot n - n)^2 \end{pmatrix}. \tag{4.9}$$

The following theorem on the spectrum of the operators $U_{\pi^{(n)},j}$ and U_ϕ associated to the skew product T_ϕ complements Lemma 4.3.

Theorem 4.5. *Let* ϕ *satisfy* (4.8) *with* $b \in \mathbb{Z}^d \setminus \{0\}$ *and Assumption 4.4. Suppose that* $y_1, y_2, \dots, y_d, 1$ *are rationally independent, and take* $n \in 2\mathbb{N}+1$ *and* $j \in \{0, \dots, n\}$*. Then,* $U_{\pi^{(n)},j}$ *has purely Lebesgue spectrum. In particular, the restriction of* U_ϕ *to the subspace* $\bigoplus_{n\in 2\mathbb{N}+1}\bigoplus_{j=0}^n \mathcal{H}_j^{(\pi^{(n)})} \subset \mathcal{H}$ *has countable Lebesgue spectrum.*

Proof. We know that ϕ satisfies Assumption 3.1 and that $(a_k - a_\ell)\big(\pi^{(n)}_{\ell k} \circ \phi\big) \equiv 0$ for all $k, \ell \in \{0, \dots, n\}$.

Furthermore, due to the unique ergodicity of F_1, we deduce from (4.9) that

$$\begin{aligned}\lim_{N\to\infty} M_N &= \left(1+(y\cdot b)^{-1}\lim_{N\to\infty}\frac{1}{N}\sum_{m=0}^{N-1}\mathscr{L}_Y\eta\circ F_m\right)\begin{pmatrix}(2\cdot 0-n)^2 & & 0\\ & \ddots & \\ 0 & & (2\cdot n-n)^2\end{pmatrix}\\ &= \left(1+(y\cdot b)^{-1}\int_{\mathbb{T}^d}\mathrm{d}\mu_{\mathbb{T}^d}\,\mathscr{L}_Y\eta\right)\begin{pmatrix}(2\cdot 0-n)^2 & & 0\\ & \ddots & \\ 0 & & (2\cdot n-n)^2\end{pmatrix}\\ &= \begin{pmatrix}(2\cdot 0-n)^2 & & 0\\ & \ddots & \\ 0 & & (2\cdot n-n)^2\end{pmatrix}\end{aligned}$$

uniformly on $\mathbb{T}^d$. Therefore, since $n\in 2\mathbb{N}+1$, one has that

$$\lim_{N\to\infty}\lambda_{*,N} = \inf_{k\in\{0,\dots,n\},\,x\in\mathbb{T}^d}\lambda_k\left(\lim_{N\to\infty}M_N(x)\right) = \inf_{k\in\{0,\dots,n\}}\lambda_k\begin{pmatrix}(2\cdot 0-n)^2 & & 0\\ & \ddots & \\ 0 & & (2\cdot n-n)^2\end{pmatrix} = 1,$$

and thus $\lambda_{*,N}>0$ if N is large enough. So, it follows from Theorem 3.5 and Lemma 3.7 that $U_{\pi^{(n)},j}$ has purely Lebesgue spectrum. The claim on U_ϕ follows from what precedes if one takes into account the separability of the Hilbert space $\mathcal{H}\equiv\mathsf{L}^2\big(\mathbb{T}^d\times\mathsf{SU}(2),\mu_{\mathbb{T}^d}\otimes\mu_{\mathsf{SU}(2)}\big)$. $\square$

Theorem 4.5 should be compared with prior results for skew products on $\mathbb{T}^d\times\mathsf{SU}(2)$ obtained by K. Frączek (but see also [18, 24]). When F_1 is an ergodic translation on $\mathbb{T}^d$ ($d=1,2$), K. Frączek exhibits in [13, Thm. 6.1 & Thm. 8.2] conditions on the cocycle ϕ guaranteeing that the restriction of U_ϕ to the subspace $\bigoplus_{n\in2\mathbb{N}+1}\bigoplus_{j=0}^{n}\mathcal{H}_j^{(\pi^{(n)})}$ has countable Lebesgue spectrum. In dimension $d=1$, these conditions are verified if $\phi\in C^2\big(\mathbb{T};\mathsf{SU}(2)\big)$, if ϕ has nonzero topological degree, and if ϕ is cohomologous to a diagonal cocyle with a transfer function $\zeta:\mathbb{T}\to\mathsf{SU}(2)$ of bounded variation and with $\zeta'\zeta^{-1}\in\mathsf{L}^2\big(\mathbb{T};\mathfrak{su}(2)\big)$ (see [13, Cor. 6.5]). This is similar (but not completely equivalent) to the conditions satisfied by ϕ when $d=1$ in Theorem 4.5 (in Theorem 4.5, ϕ has Dini-continuous derivative along the flow $\{F_t\}_{t\in\mathbb{R}}$, it has topological degree $b\neq0$, and it is cohomologous to a diagonal cocyle with a constant transfer function). K. Frączek also shows various properties of the topological degree of cocycles $\phi\in C^2\big(\mathbb{T};\mathsf{SU}(2)\big)$ such as the fact that it takes values in $\mathbb{Z}$ or the fact that it is invariant under the relation of measurable cohomology (see [14, Thm. 2.7 & Thm. 2.10]).

4.3 The case $X=\mathbb{T}^d$ and $G=\mathsf{U}(2)$

Suppose that $X=\mathbb{T}^d$ for some $d\geq1$, let $G=\mathsf{U}(2)$, set $\mathcal{H}:=\mathsf{L}^2\big(\mathbb{T}^d\times\mathsf{U}(2),\mu_{\mathbb{T}^d}\otimes\mu_{\mathsf{U}(2)}\big)$, let $\{F_t\}_{t\in\mathbb{R}}$ be the translation flow on $\mathbb{T}^d$ given by

$$F_t(x):=x+ty\ (\mathrm{mod}\ \mathbb{Z}^d),\quad t\in\mathbb{R},\ x\in\mathbb{T}^d,$$

for some $y:=(y_1,y_2,\dots,y_d)\in\mathbb{R}^d$, take $\xi:\mathbb{T}^d\to\mathsf{U}(2)$ a Lie group homomorphism, and let

$$T_\xi(x,g)=\big(x+y,g\xi(x)\big),\quad (x,g)\in\mathbb{T}^d\times\mathsf{U}(2).$$

Since $\mathbb{T}^d$ is abelian, the range of ξ is contained in a maximal torus of $\mathsf{U}(2)$ of the form $\big\{h\left(\begin{smallmatrix}z_1&0\\0&z_2\end{smallmatrix}\right)h^*\big\}_{z_1,z_2\in\mathbb{S}^1}$ for some $h\in\mathsf{U}(2)$ (see [6, Thm. IV.1.6 & Prop. IV.3.1]). So, we can suppose without loss of generality that

$$\xi(x)=h\begin{pmatrix}\mathrm{e}^{2\pi i(b_1\cdot x)} & 0\\ 0 & \mathrm{e}^{2\pi i(b_2\cdot x)}\end{pmatrix}h^*,\quad x\in\mathbb{T}^d, \tag{4.10}$$

for some vectors $b_1, b_2 \in \mathbb{Z}^d$, and thus that

$$(\pi \circ \xi)(x) = \pi(h)\pi \begin{pmatrix} e^{2\pi i(b_1\cdot x)} & 0 \\ 0 & e^{2\pi i(b_2\cdot x)} \end{pmatrix} (\pi(h))^* \tag{4.11}$$

for each π, finite-dimensional IUR of $\mathrm{U}(2)$.

Now, the fact that the map $\mathbb{S}^1 \times \mathrm{SU}(2) \ni (z,g) \mapsto zg \in \mathrm{U}(2)$ is an epimorphism with kernel $\{(1, e_{\mathrm{SU}(2)}), (-1, -e_{\mathrm{SU}(2)})\}$ implies that the set $\widehat{\mathrm{U}(2)}$ of all equivalence classes of finite-dimensional IUR's of $\mathrm{U}(2)$ coincides (up to unitary equivalence) with the set of tensors products $\{\rho_{2m-n} \otimes \pi^{(n)}\}_{m\in\mathbb{Z},n\in\mathbb{N}}$, with $\pi^{(n)}$ as in Section 4.2 and $\rho_{2m-n} : \mathbb{S}^1 \to \mathrm{U}(1) \equiv \mathbb{S}^1$ given by $\rho_{2m-n}(z) := z^{2m-n}$ (see [6, Sec. II.5]). Therefore, if one uses (4.6), (4.11) and the factorisation

$$\begin{pmatrix} e^{2\pi i(b_1\cdot x)} & 0 \\ 0 & e^{2\pi i(b_2\cdot x)} \end{pmatrix} = e^{\pi i(b_+\cdot x)} \begin{pmatrix} e^{\pi i(b_-\cdot x)} & 0 \\ 0 & e^{-\pi i(b_-\cdot x)} \end{pmatrix}, \quad b_\pm := b_1 \pm b_2,$$

and if one replaces $(\rho_{2m-n} \otimes \pi^{(n)})(\,\cdot\,)$ by the unitarily equivalent representation

$$\big((\rho_{2m-n} \otimes \pi^{(n)})(h)\big)^* (\rho_{2m-n} \otimes \pi^{(n)})(\,\cdot\,)(\rho_{2m-n} \otimes \pi^{(n)})(h),$$

one obtains that

$$\big((\rho_{2m-n} \otimes \pi^{(n)})_{jk} \circ \xi\big)(x) = j!(n-j)!\, e^{\pi i((2m-n)(b_+\cdot x)+(2j-n)(b_-\cdot x))}\, \delta_{jk}, \quad j,k \in \{0,\dots,n\}.$$

Then, putting this expression in the formula (3.4) for the matrix-valued function M, one obtains that

$$M_{jk} = \pi a_j j!(n-j)!\big((2m-n)(b_+\cdot y) + (2j-n)(b_-\cdot y)\big)\,\delta_{jk}.$$

Setting $a_j := \big((2m-n)(b_+\cdot y) + (2j-n)(b_-\cdot y)\big)\big(\pi j!(n-j)!\big)^{-1}$, one thus obtains that

$$M_{jk} = \big((2m-n)(b_+\cdot y) + (2j-n)(b_-\cdot y)\big)^2\,\delta_{jk}.$$

As a consequence, we obtain the following result on the spectrum of the operators $U_{\rho_{2m-n}\otimes\pi^{(n)},j}$ and U_ξ associated to the skew product T_ξ.

Lemma 4.6. *Let ξ satisfy* (4.10)*, set*

$$R := \left\{(m,n) \in \mathbb{Z}\times\mathbb{N} \mid \inf_{k\in\{0,\dots,n\}} \big((2m-n)(b_+\cdot y) + (2k-n)(b_-\cdot y)\big)^2 > 0\right\}, \tag{4.12}$$

and take $(m,n) \in R$ and $j \in \{0,\dots,n\}$. Then, $U_{\rho_{2m-n}\otimes\pi^{(n)},j}$ has purely absolutely continuous spectrum. In particular, the restriction of U_ξ to the subspace $\bigoplus_{(m,n)\in R}\bigoplus_{j=0}^n \mathcal{H}_j^{(\rho_{2m-n}\otimes\pi^{(n)})} \subset \mathcal{H}$ has purely absolutely continuous spectrum.

Proof. We know that ξ satisfies Assumption 3.1, that $(a_k - a_\ell)\big(\pi^{(n)}_{\ell k} \circ \xi\big) \equiv 0$ for all $k,\ell \in \{0,\dots,n\}$, and that

$$\lambda_* = \inf_{k\in\{0,\dots,n\}} \big((2m-n)(b_+\cdot y) + (2k-n)(b_-\cdot y)\big)^2 > 0.$$

So, the claim is a direct consequence of Theorem 3.4. $\square$

Remark 4.7. *Some particular cases of Lemma 4.6 are worth mentioning. First, if $b_1 = b_2$, then $\lambda_* = 4(2m-n)^2(b_1\cdot y)^2$. Thus, $U_{\rho_{2m-n}\otimes\pi^{(n)},j}$ has purely absolutely continuous spectrum if $2m \neq n$ and $b_1\cdot y \neq 0$. Second, if $b_1 = -b_2$, then*

$$\lambda_* = \inf_{k\in\{0,\dots,n\}} 4(2k-n)^2(b_1\cdot y)^2 = \begin{cases} 0 & \text{if } n \in 2\mathbb{N} \\ 4(b_1\cdot y)^2 & \text{if } n \in 2\mathbb{N}+1. \end{cases}$$

Thus, $U_{\rho_{2m-n}\otimes\pi^{(n)},j}$ has purely absolutely continuous spectrum if $n \in 2\mathbb{N}+1$ and $b_1 \cdot y \neq 0$ (we recover the result of Lemma 4.3, since ξ takes values in $\mathsf{SU}(2)$). Finally, if $b_1 = 0$ (or if $b_2 = 0$; this is similar), then

$$\lambda_* = \inf_{k\in\{0,\dots,n\}} 4(m-k)^2(b_2\cdot y)^2 = \begin{cases} 0 & \text{if } m \in \{0,\dots,n\} \\ 4(b_2\cdot y)^2 & \text{if } m \in \mathbb{Z}\setminus\{0,\dots,n\}. \end{cases}$$

Thus, $U_{\rho_{2m-n}\otimes\pi^{(n)},j}$ has purely absolutely continuous spectrum if $m \in \mathbb{Z}\setminus\{0,\dots,n\}$ and $b_2 \cdot y \neq 0$.

As in the previous sections, we can treat more general cocycles ϕ if F_1 is uniquely ergodic. So, from now on we assume that $y_1, y_2, \dots, y_d, 1$ are rationally independent and we suppose that $\phi : \mathbb{T}^d \to \mathsf{U}(2)$ is a perturbation of ξ in the sense that

$$\phi(x) = h \begin{pmatrix} \mathrm{e}^{2\pi i(b_1\cdot x+\eta_1(x))} & 0 \\ 0 & \mathrm{e}^{2\pi i(b_2\cdot x+\eta_2(x))} \end{pmatrix} h^*, \quad x \in \mathbb{T}^d, \tag{4.13}$$

with $\eta_1, \eta_2 \in C(\mathbb{T}^d;\mathbb{R})$ satisfying the following:

Assumption 4.8. *For $k = 1,2$, the function $\eta_k \in C(\mathbb{T}^d;\mathbb{R})$ is such that $\mathscr{L}_Y\eta_k$ exists and satisfies the Dini-type condition*

$$\int_0^1 \frac{\mathrm{d}t}{t} \left\|\mathscr{L}_Y\eta_k \circ F_t - \mathscr{L}_Y\eta_k\right\|_{\mathsf{L}^\infty(\mathbb{T}^d)} < \infty.$$

If we proceed as before, we obtain that

$$\big((\rho_{2m-n}\otimes\pi^{(n)})_{jk}\circ\xi\big)(x) = j!(n-j)!\,\mathrm{e}^{\pi i\{(2m-n)(b_+\cdot x+\eta_+(x))+(2j-n)(b_-\cdot x+\eta_-(x))\}}\,\delta_{jk}, \quad \eta_\pm := \eta_1 \pm \eta_2.$$

Then, calculations similar to those of the previous section lead to the following result on the spectrum of the operators $U_{\rho_{2m-n}\otimes\pi^{(n)},j}$ and U_ϕ associated to the skew product T_ϕ (review (4.12) for the definition of R).

Theorem 4.9. *Let ϕ satisfy (4.13) and Assumption 4.8, suppose that $y_1, y_2, \dots, y_d, 1$ are rationally independent, and take $(m,n) \in R$ and $j \in \{0,\dots,n\}$. Then, $U_{\rho_{2m-n}\otimes\pi^{(n)},j}$ has purely Lebesgue spectrum. In particular, the restriction of U_ϕ to the subspace $\bigoplus_{(m,n)\in R}\bigoplus_{j=0}^n \mathcal{H}_j^{(\rho_{2m-n}\otimes\pi^{(n)})} \subset \mathcal{H}$ has countable Lebesgue spectrum.*

As far as we know, the result of Theorem 4.9 is new. Besides, it would be possible to build on and apply the method of Section 3 to cocycles $\phi : X \to G$ taking values in other (higher dimensional) Lie groups than $\mathsf{SU}(2)$ or $\mathsf{U}(2)$. However, doing this leads one to consider more and more involved (combinations of tensor products of) families of IUR's in order to prove spectral results similar to Theorems 4.5 and 4.9. Thus, we curbed our enthusiasm, hoping that the transition from $G = \mathsf{SU}(2)$ in Section 4.2 to $G = \mathsf{U}(2)$ in this section already illustrates the type of procedure one has to follow.

References

[1] R. Abraham and J. E. Marsden. *Foundations of mechanics*. Benjamin/Cummings Publishing Co. Inc. Advanced Book Program, Reading, Mass., 1978. Second edition, revised and enlarged, With the assistance of Tudor Rațiu and Richard Cushman.

[2] Y. Ahn and G. H. Choe. Spectral types of skewed Bernoulli shift. *Proc. Amer. Math. Soc.*, 128(2):503–510, 2000.

[3] W. O. Amrein. *Hilbert space methods in quantum mechanics*. Fundamental Sciences. EPFL Press, Lausanne, 2009.

[4] W. O. Amrein, A. Boutet de Monvel, and V. Georgescu. *C_0-groups, commutator methods and spectral theory of N-body Hamiltonians*, volume 135 of *Progress in Math.* Birkhäuser, Basel, 1996.

[5] H. Anzai. Ergodic skew product transformations on the torus. *Osaka Math. J.*, 3:83–99, 1951.

[6] T. Bröcker and T. tom Dieck. *Representations of compact Lie groups*, volume 98 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1985.

[7] G. H. Choe. *SPECTRAL PROPERTIES OF COCYCLES*. ProQuest LLC, Ann Arbor, MI, 1987. Thesis (Ph.D.)–University of California, Berkeley.

[8] G. H. Choe. Spectral types of uniform distribution. *Proc. Amer. Math. Soc.*, 120(3):715–722, 1994.

[9] B. R. Fayad. Skew products over translations on $\mathbf{T}^d$, $d \geq 2$. *Proc. Amer. Math. Soc.*, 130(1):103–109 (electronic), 2002.

[10] C. Fernández, S. Richard, and R. Tiedra de Aldecoa. Commutator methods for unitary operators. to appear in J. Spectr. Theory.

[11] K. Frączek. Spectral properties of cocycles over rotations. Master's thesis, Nicolaus Copernicus University, Toruń, 1995. preprint on `http://www-users.mat.umk.pl/~fraczek/SPECPROP.pdf`.

[12] K. Frączek. Circle extensions of $\mathbf{Z}^d$-rotations on the d-dimensional torus. *J. London Math. Soc. (2)*, 61(1):139–162, 2000.

[13] K. Frączek. On cocycles with values in the group SU(2). *Monatsh. Math.*, 131(4):279–307, 2000.

[14] K. Frączek. On the degree of cocycles with values in the group SU(2). *Israel J. Math.*, 139:293–317, 2004.

[15] P. Gabriel, M. Lemańczyk, and P. Liardet. Ensemble d'invariants pour les produits croisés de Anzai. *Mém. Soc. Math. France (N.S.)*, (47):102, 1991.

[16] G. R. Goodson. A survey of recent results in the spectral theory of ergodic dynamical systems. *J. Dynam. Control Systems*, 5(2):173–226, 1999.

[17] H. Helson and W. Parry. Cocycles and spectra. *Ark. Mat.*, 16(2):195–206, 1978.

[18] X. Hou. Ergodicity, minimality and reducibility of cocycles on some compact groups. *Taiwanese J. Math.*, 15(3):1247–1259, 2011.

[19] A. Iwanik. Generic smooth cocycles of degree zero over irrational rotations. *Studia Math.*, 115(3):241–250, 1995.

[20] A. Iwanik. Anzai skew products with Lebesgue component of infinite multiplicity. *Bull. London Math. Soc.*, 29(2):195–199, 1997.

[21] A. Iwanik. Spectral properties of skew-product diffeomorphisms of tori. *Colloq. Math.*, 72(2):223–235, 1997.

[22] A. Iwanik, M. Lemańczyk, and C. Mauduit. Piecewise absolutely continuous cocycles over irrational rotations. *J. London Math. Soc. (2)*, 59(1):171–187, 1999.

[23] A. Iwanik, M. Lemańczyk, and D. Rudolph. Absolutely continuous cocycles over irrational rotations. *Israel J. Math.*, 83(1-2):73–95, 1993.

[24] R. Krikorian. Global density of reducible quasi-periodic cocycles on $\mathbf{T}^1 \times \mathrm{SU}(2)$. *Ann. of Math. (2)*, 154(2):269–326, 2001.

[25] A. G. Kushnirenko. Spectral properties of certain dynamical systems with polynomial dispersal. *Moscow Univ. Math. Bull.*, 29(1):82–87, 1974.

[26] H. A. Medina. Spectral types of unitary operators arising from irrational rotations on the circle group. *Michigan Math. J.*, 41(1):39–49, 1994.

[27] É. Mourre. Absence of singular continuous spectrum for certain selfadjoint operators. *Comm. Math. Phys.*, 78(3):391–408, 1980/81.

[28] M. Reed and B. Simon. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978.

[29] M. Sugiura. *Unitary representations and harmonic analysis*, volume 44 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1990. An introduction.

[30] R. Tiedra de Aldecoa. Commutator methods for the spectral analysis of uniquely ergodic dynamical systems. to appear in Ergodic Theory Dynam. Systems.

[31] M. Wysokińska. A class of real cocycles over an irrational rotation for which Rokhlin cocycle extensions have Lebesgue component in the spectrum. *Topol. Methods Nonlinear Anal.*, 24(2):387–407, 2004.