Resonances and Spectral Shift Function Singularities for Magnetic Quantum Hamiltonians

Jean-François Bony, Vincent Bruneau, Georgi Raikov

Abstract. In this survey article we consider the operator pair (H, H_0) where H_0 is the shifted 3D Schrödinger operator with constant magnetic field, $H := H_0 + V$, and V is a short-range electric potential of a fixed sign. We describe the asymptotic behavior of the Krein spectral shift function (SSF) $\xi(E; H, H_0)$ as the energy E approaches the Landau levels $2bq$, $q \in \mathbb{Z}_+$, which play the role of thresholds in the spectrum of H_0. The main asymptotic term of $\xi(E; H, H_0)$ as $E \to 2bq$ with a fixed $q \in \mathbb{Z}_+$ is written in the terms of appropriate compact Berezin–Toeplitz operators. Further, we investigate the relation between the threshold singularities of the SSF and the accumulation of resonances at the Landau levels. We establish the existence of resonance free sectors adjoining any given Landau level and prove that the number of the resonances in the complementary sectors is infinite. Finally, we obtain the main asymptotic term of the local resonance counting function near an arbitrary fixed Landau level; this main asymptotic term is again expressed via the Berezin–Toeplitz operators which govern the asymptotics of the SSF at the Landau levels.

Keywords: magnetic Schrödinger operators, resonances, spectral shift function

2000 AMS Mathematics Subject Classification: 35P25, 35J10, 47F05, 81Q10

1 Introduction

This article is a survey of our results on two closely related topics:

- The threshold singularities of the Krein spectral shift function (SSF) for the operator pair (H, H_0) where H_0 is the shifted 3D Schrödinger operator with constant magnetic field, and $H := H_0 + V$ with an appropriate short-range potential V of a fixed, positive or negative, sign;
- The accumulation of the resonances of the operator H at its spectral thresholds.

The self-adjoint unperturbed operator

$$H_0 = H_0(b) := (-i\nabla - A)^2 - b \tag{1.1}$$

is defined initially on $C_0^\infty(\mathbb{R}^3)$ and then is closed in $L^2(\mathbb{R}^3)$. Here $A = \left(-\frac{bx_2}{2}, \frac{bx_1}{2}, 0\right)$ is a magnetic potential generating the magnetic field $B = \text{curl}\, A = (0, 0, b)$ where $b > 0$ is the constant scalar intensity of B.
It is well known that the spectrum $\sigma(H_0)$ of the operator H_0 is absolutely continuous and coincides with $[0, \infty)$ (see e.g. [1]). Moreover, the so called *Landau levels* $2bq$, $q \in \mathbb{Z}_+ :=$

$\{0, 1, 2, \ldots\}$, play the role of thresholds in $\sigma(H_0)$ (see below Subsection 2.2).
The perturbation of H_0 is the electric potential $V : \mathbb{R}^3 \to \mathbb{R}$. Assume that V is Lebesgue measurable, and bounded. On the domain of H_0 define the perturbed operator

$$H := H_0 + V \tag{1.2}$$

which obviously is self-adjoint in $L^2(\mathbb{R}^3)$.
In order to describe the assumptions on the decay of V we need the following notations. For $x \in \mathbb{R}^3$ we will occasionally write $x = (X_\perp, x_3)$ where $X_\perp = (x_1, x_2) \in \mathbb{R}^2$ are the variables in the plane perpendicular to the magnetic field, and $x_3 \in \mathbb{R}$ is the variable along the magnetic field. We will suppose that V satisfies one of the following estimates:

- D (anisotropic decay): $V(x) = O(\langle X_\perp \rangle^{-m_\perp} \langle x_3 \rangle^{-m_3})$ with $m_\perp > 2$, $m_3 > 1$;
- D_0 (isotropic decay): $V(x) = O(\langle x \rangle^{-m_0})$ with $m_0 > 3$;
- $\mathrm{D}_{\mathrm{exp}}$ (fast decay with respect to x_3): $V(x) = O(\langle X_\perp \rangle^{-m_\perp} \exp(-N|x_3|))$ with some $m_\perp > 0$ and any $N > 0$.

Note that assumption D_0 implies D. Moreover, evidently, assumption $\mathrm{D}_{\mathrm{exp}}$ with $m_\perp > 2$ again implies D.
The article is organized as follows. In Section 2 we discuss the behavior of the SSF for the operator pair (H, H_0) near the Landau levels $2bq$, $q \in \mathbb{Z}_+$. Its main asymptotic term (see Theorem 2.1) is given in terms of auxiliary Berezin–Toeplitz operators discussed in detail in Subsection 2.3. As an example of the possible applications of Theorem 2.1, a generalized Levinson formula is deduced from it (see Corollary 2.1). In Subsection 2.6 we describe briefly the extensions of Theorem 2.1 to Pauli and Dirac operators with non constant magnetic field.
Further, Section 3 is dedicated to the resonances and the embedded eigenvalues of H. In Subsection 3.1 we state Theorem 3.1 which implies that under very general assumptions about the decay of V the operator H has infinitely many eigenvalues embedded in its essential spectrum, provided that V is axisymmetric and non positive. Next, in Subsection 3.2 we define the resonances of H as the poles of an appropriate meromorphic continuation of the resolvent $(H - z)^{-1}$, $\mathrm{Im}\, z > 0$, to an appropriate infinitely sheeted Riemann surface. Further, we establish the existence of resonance-free regions and of regions with infinitely many resonances in a vicinity of each Landau levels (see Theorem 3.3). Finally, for every $q \in \mathbb{Z}_+$ fixed, we obtain in Theorem 3.4 the main asymptotic term of the number of the resonances on an annulus centered at the Landau level $2bq$ as the inner radius of the annulus tends to zero.
We have been working on the problems discussed in the article for almost a decade. Some of the results obtained have already been surveyed (see e.g [4, 34]). We decided to include them again in the present opus since we wanted to tell here our story from the very beginning, referring the reader when necessary to the original works but preferably not to other surveys.

2 Singularities of the spectral shift function at the Landau levels

2.1 The spectral shift function $\xi(E; H, H_0)$

Let V satisfy D. Then the diamagnetic inequality easily implies that the operator $V(H_0 + 1)^{-1/2}$ is Hilbert–Schmidt, and hence the resolvent difference $(H - i)^{-1} - (H_0 - i)^{-1}$ is a trace-class operator. Therefore, there exists a unique

$$\xi = \xi(\cdot; H, H_0) \in L^1(\mathbb{R}; (1 + E^2)^{-1} dE)$$

such that the Lifshits-Krein trace formula

$$\mathrm{Tr}\,(f(H) - f(H_0)) = \int_{\mathbb{R}} \xi(E; H, H_0) f'(E) dE$$

holds for each $f \in C_0^\infty(\mathbb{R})$ and the normalization condition $\xi(E; H, H_0) = 0$ is fulfilled for each $E \in (-\infty, \inf \sigma(H))$ (see the original works [29, 25] or [48, Chapter 8]). Then $\xi(\cdot; H, H_0)$ is called *the spectral shift function* (SSF) for the operator pair (H, H_0).
By the *Birman–Krein formula*, for almost every $E > 0 = \inf \sigma_{\rm ac}(H)$, the SSF $\xi(E; H, H_0)$ coincides with *the scattering phase* for the operator pair (H, H_0) (see the original work [2] or the monograph [48]).
Further, for almost every $E < 0$ we have

$$-\xi(E; H, H_0) = \mathrm{Tr}\,\mathbf{1}_{(-\infty, E)}(H),$$

where $\mathrm{Tr}\,\mathbf{1}_{(-\infty,E)}(H)$ is just the number of the eigenvalues of H less than E, counted with their multiplicities.
The above properties follow directly from the general abstract theory of the SSF (see e.g. [48, Chapter 8]). By [8, Proposition 2.5], the SSF for the operator pair defined in (1.1) and (1.2) possesses the following more particular features:

- $\xi(\cdot; H, H_0)$ is bounded on every compact subset of $\mathbb{R} \setminus 2b\mathbb{Z}_+$;
- $\xi(\cdot; H, H_0)$ is continuous on $\mathbb{R} \setminus (2b\mathbb{Z}_+ \cup \sigma_{\rm pp}(H))$ where $\sigma_{\rm pp}(H)$ is the set of the eigenvalues of H.

Our first goal is to describe the asymptotic behavior of the SSF $\xi(E; H, H_0)$ as $E \to 2bq$, $q \in \mathbb{Z}_+$. This behavior will be described in terms of auxiliary Berezin–Toeplitz operators studied in more detail in Subsection 2.3. The next subsection deals with the well-known properties of the Landau Hamiltonian, i.e. the 2D Schrödinger operator with constant magnetic fields.

2.2 The Landau Hamiltonian

We have

$$H_0 = H_\perp \otimes I_\parallel + I_\perp \otimes H_\parallel \tag{2.1}$$

where $I_\perp$ and $I_\|$ are the identities in $L^2(\mathbb{R}^2_{X_\perp})$ and $L^2(\mathbb{R}_{x_3})$ respectively,

$$H_\perp := \left(-i\frac{\partial}{\partial x_1} + \frac{bx_2}{2}\right)^2 + \left(-i\frac{\partial}{\partial x_2} - \frac{bx_1}{2}\right)^2 - b$$

is the (shifted) *Landau Hamiltonian*, self-adjoint in $L^2(\mathbb{R}^2_{X_\perp})$, and

$$H_\| := -\frac{d^2}{dx_3^2}$$

is the 1D free Hamiltonian, self-adjoint in $L^2(\mathbb{R}_{x_3})$. Note that $H_\perp = a^* a$ where

$$a := -2ie^{-b|z|^2/4}\frac{\partial}{\partial \bar{z}}e^{b|z|^2/4}, \;\; \bar{z} = x_1 - ix_2,$$

is *the magnetic annihilation operator*, and

$$a^* := -2ie^{b|z|^2/4}\frac{\partial}{\partial z}e^{-b|z|^2/4}, \;\; z = x_1 + ix_2,$$

is *the magnetic creation operator*, adjoint to a in $L^2(\mathbb{R}^2)$. Moreover, $[a, a^*] = 2b$. Therefore, $\sigma(H_\perp) = \cup_{q=0}^\infty \{2bq\}$. Furthermore,

$$\mathrm{Ker}\, H_\perp = \mathrm{Ker}\, a = \left\{f \in L^2(\mathbb{R}^2) | f = ge^{-b|z|^2/4}, \; \frac{\partial g}{\partial \bar{z}} = 0\right\}$$

is the classical *Fock–Segal–Bargmann* space (see e.g. [21]), and

$$\mathrm{Ker}\,(H_\perp - 2bq) = (a^*)^q \mathrm{Ker}\, H_\perp, \quad q \geq 1.$$

Evidently,

$$\dim \mathrm{Ker}\,(H_\perp - 2bq) = \infty$$

for each $q \in \mathbb{Z}_+$.

Representation (2.1) shows that the operator H_0 has *a waveguide structure* since the transversal operator $H_\perp$ has a purely point spectrum and the set of its eigenvalues is a discrete subset of the real axis, while the longitudinal operator $H_\|$ has a purely absolutely continuous spectrum. This waveguide structure of H_0 explains the qualification of the Landau levels, i.e. the eigenvalues of the transversal operator $H_\perp$, as *thresholds* in the spectrum of the "total" 3D operator H_0. The important difference with the usual waveguides (see e.g. [20, 10]) is that the eigenvalues $2bq$, $q \in \mathbb{Z}_+$, of the transversal operator $H_\|$ are of infinite multiplicity. We would like to underline here that most of the phenomena discussed in the present article are due to the infinite degeneracy of the Landau levels regarded as eigenvalues of $H_\perp$.

2.3 Berezin–Toeplitz operators

Fix $q \in \mathbb{Z}_+$. Denote by p_q the orthogonal projection onto $\mathrm{Ker}\,(H_\perp - 2bq)$. As discussed in the previous subsection, we have $\mathrm{rank}\, p_q = \infty$.

Let $U : \mathbb{R}^2 \to \mathbb{C}$ be a Lebesgue measurable function. Introduce *the Berezin–Toeplitz operator*

$$p_q U p_q : p_q L^2(\mathbb{R}^2) \to p_q L^2(\mathbb{R}^2).$$

We will call U *the symbol* of the operator $p_q U p_q$. Evidently, if $U \in L^\infty(\mathbb{R}^2)$ then $p_q U p_q$ is bounded, and $\|p_q U p_q\| \le \|U\|_{L^\infty(\mathbb{R}^2)}$. Moreover, if $U \in L^p(\mathbb{R}^2)$, $p \in [1,\infty)$, then by [33, Lemma 5.1] or [13, Lemma 3.1], we have $p_q U p_q \in S_p$, the pth Schatten–von Neumann class, and

$$\|p_q U p_q\|_{S_p}^p \le \frac{b}{2\pi} \|U\|_{L^p(\mathbb{R}^2)}^p.$$

As a corollary, if $U \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ and $U(x) \to 0$ as $|x| \to \infty$, then $p_q U p_q$ is compact.
Further, $p_0 U p_0$ with domain $p_0 L^2(\mathbb{R}^2)$ is unitarily equivalent to the ΨDO : $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with anti-Wick symbol

$$\omega(y,\eta) := U(b^{-1/2}\eta, b^{-1/2}y), \ (y,\eta) \in T^*\mathbb{R},$$

while by [8, Lemma 9.2] the operator $p_q U p_q$ with any $q \in \mathbb{Z}_+$ is unitarily equivalent to

$$p_0 \left(\sum_{s=0}^{q} \frac{q!}{(2b)^s (s!)^2 (q-s)!} \Delta^s U \right) p_0 \tag{2.2}$$

which is quite useful when we want to reduce the analysis at the higher Landau levels to analysis at the first Landau level. Note that the differential operation occurring in (2.2) can be written as $\mathrm{L}_q\left(-\frac{\Delta}{2b}\right)$ where

$$\mathrm{L}_q(t) := \sum_{s=0}^{q} \frac{q!}{(s!)^2 (q-s)!} (-t)^s, \quad t \in \mathbb{R},$$

is the qth Laguerre polynomial. A more abstract point of view concerning the unitary equivalence between $p_q U p_q$ and the operator defined in (2.2) could be found in [16].
The following three lemmas deal with the spectral asymptotics for compact Berezin–Toeplitz operators whose symbols U admit respectively a power-like decay, an exponential decay, or have a compact support. More precisely, we discuss the asymptotics as $s \downarrow 0$ of the eigenvalue counting function $\mathrm{Tr}\, \mathbf{1}_{(s,\infty)}(p_q U p_q)$.

Lemma 2.1. [33, Theorem 2.6] *Let $0 \le U \in C^1(\mathbb{R}^2)$, and*

$$U(X_\perp) = u_0(X_\perp/|X_\perp|)|X_\perp|^{-\alpha}(1+o(1)),$$

$$|\nabla U(X_\perp)| = O(|X_\perp|^{-\alpha-1}),$$

as $|X_\perp| \to \infty$, with $\alpha > 0$, and $0 < u_0 \in C(\mathbb{S}^1)$. Fix $q \in \mathbb{Z}_+$. Then

$$\mathrm{Tr}\, \mathbf{1}_{(s,\infty)}(p_q U p_q) = \frac{b}{2\pi} \left| \left\{ X_\perp \in \mathbb{R}^2 | U(X_\perp) > s \right\} \right| (1+o(1)) = \psi_\alpha(s)(1+o(1)), \quad s \downarrow 0, \tag{2.3}$$

where $|\cdot|$ denotes the Lebesgue measure, and

$$\psi_\alpha(s) := s^{-2/\alpha} \frac{b}{4\pi} \int_{\mathbb{S}^1} u_0(t)^{2/\alpha} dt. \tag{2.4}$$

Lemma 2.2. [36, Theorem 2.1, Proposition 4.1] *Let* $0 \le U \in L^\infty(\mathbb{R}^2)$ *and*

$$\ln U(X_\perp) = -\mu|X_\perp|^{2\beta}(1+o(1)), \quad |X_\perp| \to \infty,$$

with $\beta \in (0,\infty)$*,* $\mu \in (0,\infty)$*. Fix* $q \in \mathbb{Z}_+$*. Then*

$$\mathrm{Tr}\, \mathbf{1}_{(s,\infty)}(p_q U p_q) = \varphi_\beta(s)(1+o(1)), \quad s \downarrow 0, \tag{2.5}$$

where

$$\varphi_\beta(s) := \begin{cases} \frac{b}{2\mu^{1/\beta}}|\ln s|^{1/\beta} \text{ if } 0 < \beta < 1, \\ \frac{1}{\ln(1+2\mu/b)}|\ln s| \text{ if } \beta = 1, \\ \frac{\beta}{\beta-1}(\ln|\ln s|)^{-1}|\ln s| \text{ if } 1 < \beta < \infty. \end{cases} \tag{2.6}$$

Lemma 2.3. [36, Theorem 2.2, Proposition 4.1] *Let* $0 \le U \in L^\infty(\mathbb{R}^2)$*,* $\mathrm{supp}\, U$ *be compact, and* $U \ge C > 0$ *on an open non-empty subset of* $\mathbb{R}^2$*. Fix* $q \in \mathbb{Z}_+$*. Then*

$$\mathrm{Tr}\, \mathbf{1}_{(s,\infty)}(p_q U p_q) = \varphi_\infty(s)(1+o(1)), \quad s \downarrow 0, \tag{2.7}$$

where

$$\varphi_\infty(s) := (\ln|\ln s|)^{-1}|\ln s|. \tag{2.8}$$

Asymptotic relation (2.3) is of semiclassical nature in the sense that it is written in terms of the measure of that part of "the phase space" $\mathbb{R}^2$ where the symbol U of the operator $p_q U p_q$ is greater than $s > 0$. Similarly, asymptotic relation (2.5) with $\beta \in (0,1)$ is of semiclassical nature. Asymptotic relation (2.5) with $\beta = 1$ is the border-line one: the order is semiclassical but the coefficient $\frac{1}{\ln(1+2\mu/b)}$ is not. Note that the main asymptotic term of $\frac{1}{\ln(1+2\mu/b)}$ as $b \to \infty$ coincides with the semiclassical coefficient $\frac{b}{2\mu}$. Finally, asymptotic relation (2.5) with $\beta \in (1,\infty)$ as well as asymptotic relation (2.7) are not of semiclassical nature.

Lemmas 2.1, 2.2, 2.3 have been cited in a similar form in several works of the present authors (see e.g. [13, 5, 6]). We have chosen this form since, in our opinion, it contains a reasonable scale of the possible types of decay of V, which, in particular, reveals clearly enough the passage from semiclassical to non semiclassical asymptotic behavior of the eigenvalue counting function for $p_q U p_q$. Of course, these three lemmas do not cover all possible symbols U for which the main asymptotic term of $\mathrm{Tr}\, \mathbf{1}_{(s,\infty)}(p_q U p_q)$ as $s \downarrow 0$ can be found explicitly. For example, if $U \ge C > 0$ on an open non-empty subset of $\mathbb{R}^2$, and

$$\lim_{|X_\perp| \to \infty} \frac{\ln(-\ln U(X_\perp))}{\ln|X_\perp|} = \infty,$$

then (2.5) and (2.7) easily imply that

$$\frac{\beta}{\beta-1} \le \liminf_{s \downarrow 0} \frac{\mathrm{Tr}\, \mathbf{1}_{(s,\infty)}(p_q U p_q)}{(\ln|\ln s|)^{-1}|\ln s|} \le \limsup_{s \downarrow 0} \frac{\mathrm{Tr}\, \mathbf{1}_{(s,\infty)}(p_q U p_q)}{(\ln|\ln s|)^{-1}|\ln s|} \le 1$$

for any $\beta \gg 1$. Letting $\beta \to \infty$, we find that (2.7) again holds true.

2.4 Asymptotics of $\xi(E; H, H_0)$ as $E \to 2bq$

Let V satisfy D. For $X_\perp \in \mathbb{R}^2$, $\lambda \geq 0$, set

$$W(X_\perp) := \int_{\mathbb{R}} |V(X_\perp, x_3)| dx_3, \tag{2.9}$$

$$\mathcal{W}_\lambda = \mathcal{W}_\lambda(X_\perp) := \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix},$$

where

$$w_{11} := \int_{\mathbb{R}} |V(X_\perp, x_3)| \cos^2(\sqrt{\lambda} x_3) dx_3, \quad w_{22} := \int_{\mathbb{R}} |V(X_\perp, x_3)| \sin^2(\sqrt{\lambda} x_3) dx_3,$$

$$w_{12} = w_{21} := \int_{\mathbb{R}} |V(X_\perp, x_3)| \cos(\sqrt{\lambda} x_3) \sin(\sqrt{\lambda} x_3) dx_3.$$

Unless $V = 0$ almost everywhere, we have

$$\operatorname{rank} p_q W p_q = \infty, \quad \operatorname{rank} p_q \mathcal{W}_\lambda p_q = \infty, \ \lambda \geq 0.$$

If $F_j(V; \lambda)$, $j = 1, 2$, are two real non decreasing functionals of V, depending on $\lambda > 0$, we write

$$F_1(V; \lambda) \sim F_2(V; \lambda), \quad \lambda \downarrow 0,$$

if for each $\varepsilon \in (0, 1)$ we have

$$F_2((1-\varepsilon)V; \lambda) + O_\varepsilon(1) \leq F_1(V; \lambda) \leq F_2((1+\varepsilon)V; \lambda) + O_\varepsilon(1).$$

We also use analogous notations for non increasing functionals $F_j(V; \lambda)$ of V.

Theorem 2.1. [13, Theorems 3.1, 3.2] *Let V satisfy* D$_0$*, and $V \geq 0$ or $V \leq 0$. Fix $q \in \mathbb{Z}_+$. Then we have*

$$\xi(2bq - \lambda; H, H_0) = O(1), \quad \lambda \downarrow 0, \tag{2.10}$$

if $V \geq 0$, and

$$\xi(2bq - \lambda; H, H_0) \sim -\operatorname{Tr} \mathbf{1}_{(2\sqrt{\lambda}, \infty)} (p_q W p_q), \ \lambda \downarrow 0, \tag{2.11}$$

if $V \leq 0$. Moreover,

$$\xi(2bq + \lambda; H, H_0) \sim \frac{1}{\pi} \operatorname{Tr} \arctan\left(\frac{p_q \mathcal{W}_\lambda p_q}{2\sqrt{\lambda}}\right), \ \lambda \downarrow 0, \tag{2.12}$$

if $V \geq 0$, and

$$\xi(2bq + \lambda; H, H_0) \sim -\frac{1}{\pi} \operatorname{Tr} \arctan\left(\frac{p_q \mathcal{W}_\lambda p_q}{2\sqrt{\lambda}}\right), \ \lambda \downarrow 0, \tag{2.13}$$

if $V \leq 0$.

Note that in the case $q = 0$ asymptotic relation (2.11) concerns the distribution of the discrete eigenvalues of the operator H with $V \leq 0$ near the origin which coincides with the infimum of its essential spectrum. Results of this type have been known for a long time, and could be found in:

- [42, 43, 45, 33, 22] in the case of *a power-like decay of* V;
- [36] in the case of *an exponential decay of* V;
- [36, 30] in the case of *compactly supported potentials* V.

Inserting the results of Lemmas 2.1, 2.2, or 2.3 into (2.11), (2.12), and (2.13), we could obtain the main asymptotic term of the SSF as $E \to 2bq$. We omit here these explicit formulae referring the reader to the original work (see [13, Corollary 3.1]), and prefer to state here only the following intriguing

Corollary 2.1. [34] *Let* V *satisfy* D_0, *and* $V \leq 0$. *Fix* $q \in \mathbb{Z}_+$. *Then*

$$\lim_{\lambda \downarrow 0} \frac{\xi(2bq+\lambda; H, H_0)}{\xi(2bq-\lambda; H, H_0)} = \frac{1}{2\cos\frac{\pi}{\alpha}} \tag{2.14}$$

if W *satisfies the assumptions of Lemma 2.1, i.e. if* W *admits a power-like decay with decay rate* $\alpha > 2$, *or*

$$\lim_{\lambda \downarrow 0} \frac{\xi(2bq+\lambda; H, H_0)}{\xi(2bq-\lambda; H, H_0)} = \frac{1}{2} \tag{2.15}$$

if W *satisfies the assumptions of Lemma 2.2 or Lemma 2.3, i.e. if* W *decays exponentially*[1] *or has a compact support.*

Relations (2.14)–(2.15) could be interpreted as *generalized Levinson formulae*. We recall that the classical Levinson formula relates the number of the negative eigenvalues of $-\Delta + V$ with V which decays sufficiently fast at infinity, and $\lim_{E \downarrow 0} \xi(E; -\Delta + V, -\Delta)$ (see the original work [28] or the survey article [38]).

2.5 Sketch of the proof of Theorem 2.1

We start with a representation of the SSF due to A. Pushnitski [32, 8]. Assume that V satisfies D. Then the norm limit

$$T(E) := \lim_{\delta \downarrow 0} |V|^{1/2}(H_0 - E - i\delta)^{-1}|V|^{1/2}$$

exists for every $E \in \mathbb{R} \setminus 2b\mathbb{Z}_+$. Moreover, $T(E)$ is compact, and $0 \leq \mathrm{Im}\, T(E) \in S_1$ (see [8, Lemma 4.2]). Assume in addition that $\pm V \geq 0$. Then for $E \in \mathbb{R} \setminus 2b\mathbb{Z}_+$ we have

$$\xi(E; H, H_0) = \pm\frac{1}{\pi}\int_{\mathbb{R}} \mathrm{Tr}\mathbf{1}_{(1,\infty)}(\mp(\mathrm{Re}\, T(E) + t\mathrm{Im}\, T(E)))\frac{dt}{1+t^2}$$

(see [32, Theorem 1.2], [8, Subsection 3.3]).
The first important step in the proof of Theorem 2.1 is the estimate

$$\pm\xi(E; H, H_0) \sim \frac{1}{\pi}\int_{\mathbb{R}} \mathrm{Tr}\mathbf{1}_{(1,\infty)}(\mp(\mathrm{Re}\, T_q(E) + t\mathrm{Im}\, T_q(E)))\frac{dt}{1+t^2}, \quad E \to 2bq \tag{2.16}$$

[1] In the case of exponential decay of W we should also suppose that V satisfies D with $m_\perp > 2$ and $m_3 > 2$.

where

$$\begin{aligned} T_q(E) &:= \lim_{\delta\downarrow 0} |V|^{1/2}(p_q \otimes I_{\|})(H_0 - E - i\delta)^{-1}|V|^{1/2} \\ &= \lim_{\delta\downarrow 0} |V|^{1/2}(p_q \otimes (H_{\|} + 2bq - E - i\delta)^{-1})|V|^{1/2} \qquad E \neq 2bq. \end{aligned}$$

If $E = 2bq - \lambda$ with $\lambda > 0$, then $T_q(E) = T_q(E)^*$, and (2.16) implies

$$\pm\xi(E; H, H_0) \sim \mathrm{Tr}\mathbf{1}_{(1,\infty)}(\mp T_q(E)), \quad E \to 2bq. \tag{2.17}$$

Moreover, we have $T_q(E) \geq 0$, i.e. $\mathrm{Tr}\,\mathbf{1}_{(1,\infty)}(-T_q(E)) = 0$. Then (2.17) with the upper sign implies

$$\xi(E; H, H_0) = O(1), \quad E \uparrow 2bq,$$

provided that $V \geq 0$, i.e. we obtain (2.10).
Assume now that $V \leq 0$. The second important step in the proof of Theorem 2.1 is the estimate

$$\mathrm{Tr}\mathbf{1}_{(1,\infty)}(T_q(2bq - \lambda)) \sim \mathrm{Tr}\mathbf{1}_{(1,\infty)}\left(|V|^{1/2}\left(p_q \otimes S_-(\lambda)\right)|V|^{1/2}\right), \quad \lambda \downarrow 0, \tag{2.18}$$

where $S_-(\lambda)$ denotes the operator with constant integral kernel $\frac{1}{2\sqrt{\lambda}}$. Note that $\frac{1}{2\sqrt{\lambda}}$ could be interpreted as the divergent part as $\lambda \downarrow 0$ of the integral kernel

$$\frac{e^{-\sqrt{\lambda}|x_3 - x_3'|}}{2\sqrt{\lambda}}, \quad x_3, x_3' \in \mathbb{R}, \tag{2.19}$$

of the resolvent $(H_{\|} + \lambda)^{-1}$. Our next step requires the following abstract

Lemma 2.4. [3, Theorem 8.1.4] *Let L be a linear compact operator acting between two, possible different, Hilbert spaces. Then for each $s > 0$ we have*

$$\mathrm{Tr}\,\mathbf{1}_{(s,\infty)}(L^*L) = \mathrm{Tr}\,\mathbf{1}_{(s,\infty)}(LL^*).$$

Applying Lemma 2.4 with appropriate L, we immediately find that

$$\mathrm{Tr}\mathbf{1}_{(1,\infty)}\left(|V|^{1/2}\left(p_q \otimes S_-(\lambda)\right)|V|^{1/2}\right) = \mathrm{Tr}\mathbf{1}_{(1,\infty)}\left(\frac{p_q W p_q}{2\sqrt{\lambda}}\right) = \mathrm{Tr}\mathbf{1}_{(2\sqrt{\lambda},\infty)}\left(p_q W p_q\right). \tag{2.20}$$

Putting together (2.17), (2.18), and (2.20), we obtain (2.11).
Let now $E = 2bq + \lambda$ with $\lambda \downarrow 0$. Then the next important step is the estimate

$$\begin{aligned} \frac{1}{\pi}\int_{\mathbb{R}} \mathrm{Tr}\mathbf{1}_{(1,\infty)}(\mp(\mathrm{Re}\,T_q(E) + t\mathrm{Im}\,T_q(E)))\frac{dt}{1+t^2} &\sim \frac{1}{\pi}\int_{\mathbb{R}} \mathrm{Tr}\mathbf{1}_{(1,\infty)}(\mp t\mathrm{Im}\,T_q(E))\frac{dt}{1+t^2} \\ = \frac{1}{\pi}\mathrm{Tr}\arctan\left(\mathrm{Im}\,T_q(E)\right) &= \frac{1}{\pi}\mathrm{Tr}\arctan\left(|V|^{1/2}\left(p_q \otimes S_+(\lambda)\right)|V|^{1/2}\right) \end{aligned} \tag{2.21}$$

where $S_+(\lambda)$ is the operator with integral kernel $\frac{\cos\sqrt{\lambda}(x_3 - x_3')}{2\sqrt{\lambda}}$, $x_3, x_3' \in \mathbb{R}$. Applying Lemma 2.4 with appropriate L, we get

$$\frac{1}{\pi}\mathrm{Tr}\arctan\left(|V|^{1/2}\left(p_q \otimes S_+(\lambda)\right)|V|^{1/2}\right) = \frac{1}{\pi}\mathrm{Tr}\arctan\left(\frac{p_q \mathcal{W}_\lambda p_q}{2\sqrt{\lambda}}\right). \tag{2.22}$$

Now the combination of (2.16), (2.21), and (2.22), yields (2.12)–(2.13).

2.6 Extensions of Theorem 2.1 to Pauli and Dirac operators

Theorem 2.1 admits extensions to Pauli and Dirac operators with non constant magnetic fields $(0,0,b)$ of constant direction. Here

$$b = b_0 + \tilde{b},$$

$b_0 \neq 0$ is a constant, and the function $\tilde{b} : \mathbb{R}^2 \to \mathbb{R}$ is such that the Poisson equation

$$\Delta\tilde{\varphi} = \tilde{b}$$

has a solution $\tilde{\varphi} \in C_{\rm b}^2(\mathbb{R}^2)$. In particular, b may belong to a fairly large class of periodic or almost periodic functions of non zero mean value.
In the case of the Pauli operator, the role of the Landau levels is played by the origin. The analogue of Theorem 2.1 could be found in [35]. Related results for negative energies (when the SSF is proportional to the eigenvalue counting function) are contained in [23].
In the case of the Dirac operator, the role of the Landau levels is played by the points $\pm m$ where $m > 0$ is the mass of the relativistic quantum particle. The analogue of Theorem 2.1 could be found in [46].

3 Resonances near the Landau levels

3.1 Embedded eigenvalues of H

The singularity of the SSF as $E \uparrow 0$ in the case $V \leq 0$ has a simple explanation: the existence of infinitely many negative discrete eigenvalues of H accumulating at the origin which coincides with the infimum of the essential spectrum of H. The explanation of the singularities of the SSF at the higher Landau levels is much less transparent. There is no evidence that in the general case these singularities are due (only) to the accumulation at the Landau levels of embedded eigenvalues of H. That is why the natural conjecture is that the singularities of the SSF at the higher Landau levels are related to the accumulation of resonances of H at these levels; we discuss this possible accumulation in the several following subsections.
Our next theorem however stresses the fact that the magnetic Hamiltonians in the presence of an appropriate symmetry are much apter to have embedded eigenvalues than the non magnetic ones.

Theorem 3.1. *Let the operator $V(H_0+1)^{-1}$ be compact in $L^2(\mathbb{R}^3)$. Assume moreover, that V is axisymmetric, i.e. it depends only on $\rho := |X_\perp|$ and x_3.*
(i) Suppose that V satisfies

$$-2b < V(x) \leq -C\mathbf{1}_K(x), \quad x \in \mathbb{R}^3, \tag{3.1}$$

where $C > 0$, and $K \subset \mathbb{R}^3$ is an open non empty set. Then each interval

$$(2b(q-1), 2bq), \quad q \in \mathbb{N},$$

contains at least one (embedded) eigenvalue of H.
(ii) Suppose now that V satisfies

$$-2b < V(x) \leq -C\mathbf{1}_{\tilde{K}}(X_\perp)\langle x_3\rangle^{-m_3}, \quad x = (X_\perp, x_3) \in \mathbb{R}^3, \tag{3.2}$$

where $C > 0$, $m_3 \in (0,2)$, and $\tilde{K} \subset \mathbb{R}^2$ is an open non empty set. Then each interval

$$(2b(q-1), 2bq), \quad q \in \mathbb{N},$$

contains a sequence of (embedded) eigenvalues of H which converges to $2bq$.

The first part of the theorem is contained in [1, Theorem 5.1], and the simple modifications needed for the second part are briefly outlined in [34, Subsection 3.1]. Nonetheless, due to some imprecise statements of the results of Theorem 3.1 which appeared in [13, p. 385] and [9, p. 3457], and were already commented in [34, Subsection 3.1], we include in the Appendix a detailed sketch of the proof of Theorem 3.1.

On the contrary, it is expected that H has no (embedded) eigenvalues in the case $V \geq 0$. The fact that the SSF is bounded above each Landau level (see (2.10)) confirms this conjecture. It has been proved for small V. More precisely, we have

Theorem 3.2. [5, Proposition 7] *Let $V \geq 0$ satisfy* D *with $m_\perp > 0$ and $m_3 > 2$. There exists $\kappa_0 > 0$ such that, for any $0 \leq \kappa \leq \kappa_0$, $H_0 + \kappa V$ has no (embedded) eigenvalues in $\mathbb{R} \setminus 2b\mathbb{Z}_+$.*

Moreover, without the smallness assumption, [6, Corollary 6.7] states that the (embedded) eigenvalues of H form a discrete set for generic potentials $V \geq 0$ satisfying $\mathrm{D}_{\mathrm{exp}}$. Nevertheless, the absence of eigenvalues for general non-negative V remains an open problem.

3.2 Meromorphic continuation of the resolvent of H and definition of resonances

As mentioned in the previous subsection, it is expected that in the generic case the singularities of the SSF described in Theorem 2.1 are related to the accumulation of the resonances of H at the Landau levels. The first step in this investigation is, of course, the definition of the resonances themselves. As is generally accepted nowadays, we will define these resonances as the poles of a meromorphic extension of the resolvent $(H-z)^{-1}$ to an appropriate Riemann surface $\mathcal{M}$. For $z \in \mathbb{C}_+ := \{\zeta \in \mathbb{C} \,|\, \mathrm{Im}\, \zeta > 0\}$ we have

$$(H_0 - z)^{-1} = \sum_{q=0}^{\infty} p_q \otimes \left(H_\parallel + 2bq - z\right)^{-1}.$$

Recall that the resolvent $\left(H_\parallel - z\right)^{-1}$ with $z \in \mathbb{C}_+$ admits the integral kernel

$$-\frac{e^{i\sqrt{z}|x_3 - x_3'|}}{2i\sqrt{z}}, \quad x_3, x_3' \in \mathbb{R}, \quad \mathrm{Im}\,\sqrt{z} > 0,$$

(cf. (2.19)). Hence, for any $q \in \mathbb{Z}_+$ the operator $p_q \otimes (H_\parallel + 2bq - z)^{-1}$ admits a standard analytic extension to the two-sheeted Riemann surface of the square root $\sqrt{z - 2bq}$ which however depends on q. Therefore, we define $\mathcal{M}$ as the infinite-sheeted Riemann surface of the countable family

$$\left\{\sqrt{z - 2bq}\right\}_{q \in \mathbb{Z}_+}. \tag{3.3}$$

Let $\mathcal{P}_G : \mathcal{M} \to \mathbb{C} \setminus 2b\mathbb{Z}_+$ be the corresponding covering.
The properties of the Riemann surface $\mathcal{M}$ have been studied in detail in [5, Section 2]. Similar infinite-sheeted Riemann surfaces appearing in the spectral and resonance theory for perturbed waveguides have been introduced e.g. in [20, 12, 10]. In this case the family analogous to (3.3) is

$$\left\{\sqrt{z-\mu_q}\right\}_{q\in\mathbb{Z}_+}$$

where μ_q, $q \in \mathbb{Z}_+$, are the distinct eigenvalues of the transversal operator which in the case of a waveguide is lower bounded, and has a discrete spectrum.
The global structure of the Riemann surface $\mathcal{M}$ is quite complicated and may make difficult the analysis of the resonances of H. The investigation of their asymptotic distribution near a fixed Landau level $2bq$, $q \in \mathbb{Z}_+$, however is facilitated by the fact that in this case we are concerned with the local properties of $\mathcal{M}$, and in a domain diffeomorphic to a vicinity of $2bq$ the surface $\mathcal{M}$ resembles the two-sheeted Riemann surface of the square root $\sqrt{z-2bq}$. Namely, if we put

$$D(\lambda_0,\varepsilon) := \{\lambda \in \mathbb{C} \,|\, |\lambda-\lambda_0| < \varepsilon\}, \quad D(\lambda_0,\varepsilon)^* := \{\lambda \in \mathbb{C} \,|\, 0 < |\lambda-\lambda_0| < \varepsilon\},$$

for $\lambda_0 \in \mathbb{C}$ and $\varepsilon > 0$, then there exists a domain $D_q^* \subset \mathcal{M}$, and a an analytic bijection

$$D(0,\sqrt{2b})^* \ni k \mapsto z_q(k) \in D_q^* \subset \mathcal{M}, \tag{3.4}$$

such that $\mathcal{P}_G(z_q(k)) = 2bq + k^2$.
For $N > 0$ denote by $\mathcal{M}_N$ the part of $\mathcal{M}$ where $\operatorname{Im}\sqrt{z-2bq} > -N$ for all $q \in \mathbb{Z}_+$. Then, $\cup_{N>0}\mathcal{M}_N = \mathcal{M}$.

Proposition 3.1. [5, Propositions 1,2] (i) *For each $N > 0$ the operator-valued function*

$$(H_0 - z)^{-1} : e^{-N\langle x_3\rangle}L^2(\mathbb{R}^3) \to e^{N\langle x_3\rangle}L^2(\mathbb{R}^3)$$

has an analytic extension from $\mathbb{C}_+$ to $\mathcal{M}_N$.
(ii) *Suppose that V satisfies* $\mathrm{D}_{\exp}$ *with $m_\perp > 0$. Then for each $N > 0$ the operator-valued function*

$$(H - z)^{-1} : e^{-N\langle x_3\rangle}L^2(\mathbb{R}^3) \to e^{N\langle x_3\rangle}L^2(\mathbb{R}^3),$$

has a meromorphic extension from $\mathbb{C}_+$ to $\mathcal{M}_N$ whose poles and residue ranks do not depend on N.

We define the resonances of H as the poles of the meromorphic extension of the resolvent $(H-z)^{-1}$, and denote their set by $\operatorname{Res}(H)$. For $z_0 \in \operatorname{Res}(H)$ define its multiplicity by

$$\operatorname{mult}(z_0) := \operatorname{rank} \frac{1}{2i\pi}\int_\gamma (H-z)^{-1}dz,$$

where γ is a circle centered at z_0 and run over in the clockwise direction, such that $\overline{\operatorname{Int}\gamma}$ contains no elements of $\operatorname{Res}(H) \setminus \{z_0\}$.

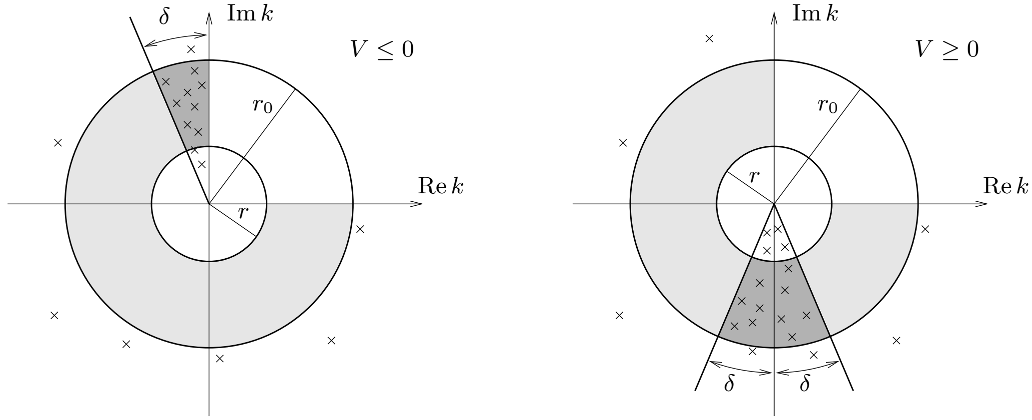

Figure 1: Resonances near a Landau level for V of definite sign, concentrated near the semi-axis $k = -i(\mathrm{sgn} V)(0, +\infty)$.

3.3 Resonance-free regions and regions with infinitely many resonances

One of the main technical achievements of our article [5] was the identification of the resonances (together with their multiplicities) as the zeroes of an appropriate 2-determinant. We recall that for a Hilbert–Schmidt operator T the 2-determinant is defined as

$$\det_2(I+T) = \det(I+T)e^{-T}.$$

Proposition 3.2. [5, Proposition 3] *Suppose that* V *satisfies* $\mathrm{D}_{\mathrm{exp}}$ *with* $m_\perp > 2$. *Introduce* $\mathcal{T}_V(z)$, *the analytic extension from* $\mathbb{C}_+$ *to* $\mathcal{M}_N$, *of*

$$\mathcal{T}_V(z) := \operatorname{sign} V\, |V|^{1/2}(H_0 - z)^{-1}|V|^{1/2}. \tag{3.5}$$

Then $z_0 \in \mathcal{M}$ *is a resonance of* H *if and only if* -1 *is an eigenvalue of* $\mathcal{T}_V(z_0)$. *Moreover,*

$$\det_2\big((H-z)(H_0-z)^{-1}\big) = \det_2\big(I + \mathcal{T}_V(z)\big) \tag{3.6}$$

has an analytic continuation from $\mathbb{C}_+$ *to* $\mathcal{M}$ *whose zeroes are the resonances of* H, *and if* z_0 *is a resonance, then there exists a holomorphic function* $f(z)$, *for* z *close to* z_0, *such that* $f(z_0) \neq 0$ *and*

$$\det_2\big(I + \mathcal{T}_V(z)\big) = (z - z_0)^{\mathrm{mult}\,(z_0)} f(z).$$

In the case of a trace-class perturbation $H - H_0$ the determinant $\det\big((H-z)(H_0-z)^{-1}\big)$, $z \in \mathbb{C}_+$, coincides with the classical perturbation determinant introduced my M. G. Krein in [26] (see also [17, Section IV.3]). In the case of Hilbert-Schmidt perturbations $H - H_0 \in S_2$ (or relatively Hilbert–Schmidt perturbations $(H - H_0)(H_0 - z)^{-1} \in S_2$) the (generalized) perturbation 2-determinants were introduced by L. S. Koplienko in [24] where he considered as well the whole Schatten–von Neumann scale $H - H_0 \in S_r$ (or $(H - H_0)(H_0 - z)^{-1} \in S_r$), $r \geq 1$.

The identification of the resonances of H as zeroes of the 2-determinant (3.6) allowed us to obtain in [5] various local estimates of the number of resonances near any fixed Landau level $2bq$, $q \in \mathbb{Z}_+$. As a typical and important example, we state below Theorem 3.3. It shows that if V is small, and its sign is fixed, then the possible resonances of H near $2bq$, parametrized according to (3.4), are concentrated in sectors centered at $2bq$, adjoining the imaginary axis, and depending on the sign of V, as illustrated in Figure 1. Moreover, for rapidly decaying V, the number of the resonances of H in each of these sectors is infinite.

Theorem 3.3. [5, Theorem 2] *Let $0 < r_0 < \sqrt{2b}$ and $q \in \mathbb{Z}_+$. Assume V satisfies* $\mathrm{D}_{\mathrm{exp}}$ *with $m_\perp > 2$, and is of definite sign J. Then for any $\delta > 0$ there exists $\varkappa_0 > 0$ such that:*
(i) *$H_0 + \varkappa V$ has no resonances in*

$$\{z = z_q(k) \,|\, 0 < |k| < r_0,\ -J\mathrm{Im}\, k \leq \frac{1}{\delta}|\mathrm{Re}\, k|\}$$

for any $0 \leq \varkappa \leq \varkappa_0$.
(ii) *If the function W defined in* (2.9)*, satisfies $\ln W(X_\perp) \leq -C\langle X_\perp\rangle^2$, then for any $0 \leq \varkappa \leq \varkappa_0$, the operator $H_0 + \varkappa V$ has an infinite number of resonances in*

$$\{z = z_q(k) \,|\, 0 < |k| < r_0,\ -J\mathrm{Im}\, k > \frac{1}{\delta}|\mathrm{Re}\, k|\}.$$

In the above result, the assumption $m_\perp > 2$ is not necessary. It was made in [5] in order to define the 2-determinant, but using the notion of index (see Subsection 3.5), Theorem 3.3 can be proved for $m_\perp > 0$.

3.4 Asymptotics of the resonance counting function

In spite of the undisputable usefulness of the methods developed and applied in [5], they led to a loss of sharpness in some of the crucial estimates which hindered us to obtain the main asymptotic term of the number of the resonances of H lying on an annulus centered at the landau level $2bq$, $q \in \mathbb{Z}_+$, as the inner radius of the annulus tends to zero. This result central in the theory of the resonances of H, was obtained recently in [6] using methods different from those of [5], and is contained below in Theorem 3.4.
For $q \in \mathbb{Z}_+$ and $z \in D(0, \sqrt{2b})$ set

$$\mathcal{A}_q(z) := J|V|^{1/2}\Big(\frac{1}{2}\Big(p_q \otimes e^{z|x_3 - x_3'|}\Big) - z\sum_{j \neq q}\Big(p_j \otimes \Big(H_\| + 2b(j-q) + z^2\Big)^{-1}\Big)\Big)|V|^{1/2}.$$

Note that for $q \in \mathbb{N}$ and $k \in D(0, \sqrt{2b})^*$ we have

$$I + \mathcal{T}_V(z_q(k)) = I - \frac{\mathcal{A}_q(ik)}{ik}, \tag{3.7}$$

the operator-valued function $\mathcal{T}_V(z)$ being defined in (3.5). Let Π_q be the orthogonal projection onto $\mathrm{Ker}\,\mathcal{A}_q(0)$.

Theorem 3.4. [6, Theorem 6.5] *Let V satisfy* $\mathrm{D}_{\mathrm{exp}}$ *with $m_\perp > 0$ and have a definite sign $J = \pm 1$. Let the function W defined in* (2.9) *satisfy the assumptions of Lemma 2.1, 2.2, or 2.3. Fix $q \in \mathbb{Z}_+$, and assume that $I - \mathcal{A}_q'(0)\Pi_q$ is invertible. Then the conclusions of Theorem 3.3 hold for r_0 small enough and we have*

$$\sum_{z_q(k) \in \mathrm{Res}(H): r < |k| < r_0} \mathrm{mult}\,(z_q(k)) = \mathrm{Tr}\,\mathbf{1}_{(2r,\infty)}(p_q W p_q)(1 + o(1))$$

as $r \downarrow 0$.

A sketch of the proof of Theorem 3.4 can be found in the next subsection. Here we make some brief comments on its hypotheses as well on some related results.
First, the assumption that $I - \mathcal{A}_q'(0)\Pi_q$ is invertible is essential and does not have a purely technical character. As shown in [6, Section 7], abstract results close by spirit to Theorem 3.4 may cease to be valid if this assumption is canceled. On the other hand, it holds true for generic potentials V because it is satisfied for gV provided that $g \in \mathbb{R}$ is not in a discrete set ($1/g$ has to be distinct from the eigenvalues of the compact operator $\mathcal{A}_q'(0)\Pi_q$). A simple sufficient condition is that the norm $\|V\|_{L^\infty(\mathbb{R}^3)}$ is small enough.
As explained at the end of Subsection 2.3 the assumptions of Lemma 2.1, 2.2, or 2.3 could be replaced by more general ones. We stick to these assumptions of the sake of a coherent exposition of the article.
Theorem 3.4 has been extended recently by Sambou in [39] to the setting of Pauli and Dirac Hamiltonians with non constant magnetic fields. In the case of the Pauli (resp., Dirac) operator Sambou obtained the main asymptotic term of the resonance counting function for an annulus centered at the origin (resp., for annuli centered at $\pm m$). The class of the magnetic fields considered in [39] is quite close to the one described in Subsection 2.5 above.
In a more general context, Theorem 3.4 belongs to a large group of results concerning the asymptotic behavior of various resonance counting functions. Among the best known problems of this type, notorious for its hardness, is the problem of finding the first asymptotic term as $r \to \infty$ of the number $N(r)$ of the resonances lying on the disk $\{z \in \mathbb{C} \,|\, |z| < r\}$ for the operator $-\Delta + V$ with, say, compactly supported V, self-adjoint in $L^2(\mathbb{R}^n)$, $n \geq 1$. Actually, the first asymptotic term as $r \to \infty$ of $N(r)$ is known in the general case only if $n = 1$. Then we have

$$N(r) = \frac{2a}{\pi} r(1 + o(1)), \quad r \to \infty, \tag{3.8}$$

where a is the diameter (not the Lebesgue measure!) of the support of V (see [49, 15]). Hence, asymptotic relation (3.8) is not of Weyl type. In the case $n \geq 3$, n odd, only sharp upper bounds of $N(r)$ are known in the general case (see e.g. [50, 47, 44]). The main asymptotic term of $N(r)$ is known only in the exceptional cases when V is radial and satisfies some additional properties (see [50, 44]); again this main term is not of Weyl type. Note that lower bounds for generic perturbations are also known (see for instance [11]) and recently Sjöstrand [41] obtained probabilistic Weyl law for random perturbations.
For more information on the asymptotics of $N(r)$ as $r \to \infty$ we refer the reader to the evolving lecture notes [51].

3.5 Sketch of the proof of Theorem 3.4

In order to outline the proof of Theorem 3.4, we need the following abstract results. Let $\mathcal{D}$ be a domain of $\mathbb{C}$ containing 0, and let $\mathcal{H}$ be a separable Hilbert space. Consider the analytic function

$$A : \mathcal{D} \longrightarrow S_\infty(\mathcal{H}).$$

Let $\Pi(A)$ be the orthogonal projection onto $\mathrm{Ker}\, A(0)$.
In the sequel we will suppose that the following assumptions are fulfilled:

- $\mathcal{C}_1$: The operator $A(0)$ is self-adjoint;
- $\mathcal{C}_2$: The operator $I - A'(0)\Pi(A)$ is invertible.

Let $\Omega \subset \mathcal{D}\setminus\{0\}$. Define the characteristic values of $I - A(z)/z$ on Ω as the points $z \in \Omega$ for which the operator $I - A(z)/z$ is not invertible. We will denote the characteristic values of $I - A(z)/z$ on Ω by $\mathcal{Z}_A(\Omega)$. By $\mathcal{C}_1$ and $\mathcal{C}_2$ the set $\mathcal{Z}_A(\Omega)$ is discrete. For $z_0 \in \mathcal{Z}_A(\Omega)$ define its multiplicity by

$$\mathrm{Mult}(z_0) := \frac{1}{2\pi i}\mathrm{Tr}\int_\gamma \Big(I - \frac{A(z)}{z}\Big)'\Big(I - \frac{A(z)}{z}\Big)^{-1} dz$$

where γ is an appropriate circle centered at z_0.

Proposition 3.3. [6, Corollary 3.4] *Assume $\mathcal{C}_1$ and $\mathcal{C}_2$. Suppose that the origin is an accumulation point of $\mathcal{Z}_A(\mathcal{D}\setminus\{0\})$. Then we have*

$$|\mathrm{Im}\, z_0| = o(|z_0|), \quad z_0 \in \mathcal{Z}_A(\mathcal{D}\setminus\{0\}),$$

as $z_0 \to 0$. If, moreover, $\pm A(0) \geq 0$, then $\pm\mathrm{Re}\, z_0 \geq 0$ for $z_0 \in \mathcal{Z}_A(\mathcal{D}\setminus\{0\})$ with $|z_0|$ small enough.

Set

$$\mathcal{N}_A(\Omega) := \sum_{z_0 \in \mathcal{Z}_A(\Omega)} \mathrm{Mult}(z_0).$$

If $\partial\Omega$ is sufficiently regular, and $\mathcal{Z}_A(\Omega) \cap \partial\Omega = \emptyset$, then we have

$$\mathcal{N}_A(\Omega) = \mathrm{ind}_{\partial\Omega}\Big(I - \frac{A(z)}{z}\Big) := \frac{1}{2\pi i}\mathrm{Tr}\int_{\partial\Omega} \Big(I - \frac{A(z)}{z}\Big)'\Big(I - \frac{A(z)}{z}\Big)^{-1} dz.$$

The index $\mathrm{ind}_{\partial\Omega}\Big(I - \frac{A(z)}{z}\Big)$ plays a central role in the proof of Theorem 3.4. More information about its properties could be found in [19], [18, Section 4], and [6, Section 2] (see also [41] where the notion of index allows to define generalized determinants).
For $0 < a < b < \infty$ and $\theta > 0$ define the domain

$$C_\theta(a,b) := \{x + iy \in \mathbb{C} \,|\, a < x < b, \quad |y| < \theta x\}. \tag{3.9}$$

Proposition 3.4. [6, Corollary 3.11] *Assume $\mathcal{C}_1$ and $\mathcal{C}_2$. Suppose moreover that*

$$\mathrm{Tr}\, \mathbf{1}_{(r,\infty)}(A(0)) = \Phi(r)(1 + o(1)), \quad r \downarrow 0,$$

where Φ satisfies $\Phi(r) \to \infty$ as $r \downarrow 0$, and

$$\Phi(r(1 \pm \delta)) = \Phi(r)(1 + o(1) + O(\delta)), \quad r \downarrow 0, \tag{3.10}$$

for each sufficiently small $\delta > 0$. Then we have

$$\mathcal{N}_A(C_\theta(r,1)) = \Phi(r)(1 + o(1)), \quad r \downarrow 0,$$

for any $\theta > 0$.

It is easy to check that the functions $\Phi(r) = Cr^{-\gamma}$, $\Phi(r) = C|\ln r|^{\gamma}$, or $\Phi(r) = C\frac{|\ln r|}{\ln|\ln r|}$, with some $\gamma, C > 0$, satisfy asymptotic relation (3.10). Hence the functions ψ_α, φ_β, and φ_∞ defined respectively in (2.4), (2.6), and (2.8), satisfy it as well.
Now we are in position to prove Theorem 3.4. By (3.7) we have that $z_q(k) \in \operatorname{Res}(H)$ if and only if ik is a characteristic value of $I - \mathcal{A}_q(z)/z$. Moreover,

$$\operatorname{mult}(z_q(k)) = \operatorname{Mult}(ik).$$

By Proposition 3.3 with $A = \mathcal{A}_q$,

$$\{z_q(k) \in \operatorname{Res}(H) \,|\, r < |k| < r_0\} = \{z_q(k) \in \operatorname{Res}(H) \,|\, \pm ik \in C_\theta(r, r_0)\} + O(1), \quad r \downarrow 0.$$

Now the claim of Theorem 3.4 follows from Proposition 3.4 with $A = \mathcal{A}_q$ combined with Lemmas 2.1, 2.2, and 2.3, since, by Lemma 2.4 with appropriate L, we have $\operatorname{Tr} \mathbf{1}_{(r,\infty)}(\mathcal{A}_q(0)) = \operatorname{Tr} \mathbf{1}_{(2r,\infty)}(p_q W p_q)$.

3.6 Link between the SSF and the resonances

The spectral shift function and the resonances are usually connected by the Breit–Wigner formula. This formula represents the derivative of the SSF as a sum of a harmonic measure associated to the resonances, and the imaginary part of a holomorphic function. When the resonances are close to the real axis, the Breit–Wigner approximation can be exploited to obtain asymptotic expansions of the SSF (see e.g. [7]). On the contrary, it can be used to localize resonances as in [31, Appendix]. Eventually, it can imply local trace formulas in the spirit of [40].

Theorem 3.5. [5, Theorem 3] *Let V satisfies $\mathrm{D}_{\exp}$ with $m_\perp > 2$. Then for $q \in \mathbb{Z}_+$ and $\varepsilon, \theta > 0$, there exist $r_0 > 0$ and functions $g_\pm(\cdot, r)$ holomorphic in $\pm C_\theta(1,2)$, such that, for $\lambda \in r[1+\varepsilon, 2-\varepsilon]$, we have*

$$\xi'(2bq \pm \lambda; H, H_0) = \sum_{\substack{2bq \pm w \in \operatorname{Res}(H) \\ w \in rC_\theta(1,2)\setminus\mathbb{R}}} \frac{\operatorname{Im} w}{\pi|\lambda - w|^2} - \sum_{\substack{2bq \pm w \in \operatorname{Res}(H) \\ w \in r[1,2]}} \delta(\lambda - w) + \frac{1}{r} \operatorname{Im} g'_\pm\Big(\frac{\lambda}{r}, r\Big),$$

where $g_\pm(z, r)$ satisfies the estimate

$$g_\pm(z, r) = O\big(|\ln r|\, r^{-\frac{1}{m_\perp}}\big),$$

uniformly with respect to $0 < r < r_0$ and $z \in C_\theta(1+\varepsilon, 2-\varepsilon)$.

A more general statement and some applications of this formula can be found in [5]. Note that this Breit–Wigner approximation implies that the SSF is analytic outside of the resonances (including the embedded eigenvalues) and their complex conjugate.
We close this subsection with the remark that we are not able yet to make the link between Theorem 2.1 and Theorem 3.4 using Theorem 3.5 for V of definite sign. We believe that such a result would shed additional light on the relation between the SSF threshold singularities and the resonances of H, and hope to obtain it in a future work. Nevertheless,

we can formally deduce the localization of the resonances from the asymptotic of the SSF. Indeed, for $V \leq 0$, the singularities of the main term in the asymptotic (2.11) are the numbers $2bq + k^2$ with $2k \in i\sigma(p_q W p_q)$, which is in agreement with Theorem 3.4. For $V \geq 0$, the uniform bound in (2.10) should be explained by some cancelations in the Breit–Wigner formula due to the particular form of the Riemann surface $\mathcal{M}$ and the localization of the resonances. Finally, the leading terms of (2.12) and (2.13) have singularities at $2bq + k^2$ with $2k \in \pm i\sigma(p_q W p_q)$ which can be explained by Theorem 3.4 and the symmetry of the singularities of the SSF with respect to the real axis.

4 Appendix: Sketch of the proof of Theorem 3.1

Passing to cylindrical coordinates (ρ, φ, x_3), and decomposing $u \in \mathrm{Dom}\, H_0$ into a Fourier series with respect to φ, we find that the operator H_0 is unitarily equivalent to the orthogonal sum $\sum_{m\in\mathbb{Z}} \oplus H_0^{(m)}$ where

$$H_0^{(m)} := -\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \left(b\rho + \frac{m}{\rho}\right)^2 - \frac{\partial^2}{\partial x_3^2} - b$$

is self-adjoint in $L^2(\mathbb{R}_+ \times \mathbb{R}; \rho d\rho dx_3)$, while H is unitarily equivalent to $\sum_{m\in\mathbb{Z}} \oplus H^{(m)}$ with $H^{(m)} := H_0^{(m)} + V(\rho, x_3)$. We have

$$\sigma(H_0^{(m)}) = [2bm_+, \infty), \quad m \in \mathbb{Z}_+,$$

where $m_+ := \max\{m, 0\}$ is the positive part of m. Fix $q \in \mathbb{N}$. Since the operator $V(H_0 + 1)^{-1}$ is compact in $L^2(\mathbb{R}^3)$, the operator $V(H_0^{(q)} + 1)^{-1}$ is compact in $L^2(\mathbb{R}_+ \times \mathbb{R}; \rho d\rho dx_3)$. Therefore, the eigenvalues of the operator $H^{(q)}$ lying on the interval $(2b(q-1), 2bq)$ are discrete, and at the same time they are embedded eigenvalues of the "total" operator H since $\sigma_{\mathrm{ess}}(H) = [0, \infty)$. Let us estimate from below the quantity $\mathrm{Tr}\, \mathbf{1}_{(2b(q-1),2bq)}(H^{(q)})$. By the lower bounds in (3.1) – (3.2) we have

$$\mathrm{Tr}\, \mathbf{1}_{(2b(q-1),2bq)}(H^{(q)}) = \mathrm{Tr}\, \mathbf{1}_{(-\infty,2bq)}(H^{(q)}).$$

Evidently, it suffices to show that

$$\mathrm{Tr}\, \mathbf{1}_{(-\infty,2bq)}(H^{(q)}) \geq 1 \tag{4.1}$$

in order to prove the first part of Theorem 3.1. Moreover, since the spectrum of $H^{(q)}$ on $(-\infty, 2bq)$ is discrete and can accumulate only at $2bq$, it suffices to show that

$$\mathrm{Tr}\, \mathbf{1}_{(-\infty,2bq)}(H^{(q)}) = \infty \tag{4.2}$$

in order to prove the second part of this theorem. Let $\varphi_q : \mathbb{R}_+ \to \mathbb{R}$ be an eigenfunction of the transversal part of the operator $H^{(q)}$ satisfying

$$-\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\varphi_q}{d\rho}\right) + \left(b\rho + \frac{m}{\rho}\right)^2 \varphi_q - b\varphi_q = 2bq\varphi_q,$$

and $\int_0^\infty \varphi_q^2 \rho d\rho = 1$. On $\mathrm{H}^2(\mathbb{R}_{x_3})$ introduce the 1D Schrödinger operator $h_q := -\frac{d^2}{dx_3^2} + v_q$ where

$$v_q(x_3) := \int_0^\infty V(\rho, x_3)\varphi_q(\rho)^2 \rho d\rho, \quad x_3 \in \mathbb{R}.$$

Restricting the quadratic form of the operator $H^{(q)}$ onto the subspace

$$\left\{u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C} \,|\, u(\rho, x_3) = \varphi_q(\rho)\, w(x_3),\ w \in \mathrm{H}^1(\mathbb{R})\right\},$$

we find that the mini-max principle implies

$$\operatorname{Tr} \mathbf{1}_{(-\infty, 2bq)}(H^{(q)}) \geq \operatorname{Tr} \mathbf{1}_{(-\infty,0)}(h_q). \tag{4.3}$$

Further, we could assume without loss of generality that the set K in (3.1) is bounded and axisymmetric, while the set $\tilde{K}$ in (3.2) is radially symmetric. By the upper bounds in (3.1)–(3.2) and the mini-max principle, we have

$$\operatorname{Tr} \mathbf{1}_{(-\infty,0)}(h_q) \geq \operatorname{Tr} \mathbf{1}_{(-\infty,0)}(\widetilde{h_q}) \tag{4.4}$$

where $\widetilde{h_q} := -\frac{d^2}{dx_3^2} + \widetilde{v_q}$, and

$$\widetilde{v_q}(x_3) := -C \left\{ \begin{array}{ll} \int_0^\infty \mathbf{1}_K(\rho, x_3)\varphi_q(\rho)^2 \rho d\rho & \text{if (3.1) holds true,} \\ \langle x_3 \rangle^{-m_3} \int_0^\infty \mathbf{1}_{\tilde{K}}(\rho)\varphi_q(\rho)^2 \rho d\rho & \text{if (3.2) holds true,} \end{array} \right. \quad x_3 \in \mathbb{R}.$$

Now note that the operator $\widetilde{h_q}$ has at least one negative eigenvalue if (3.1) holds true (see e.g. [1, Lemma 5.2]), and it has an infinite sequence of negative discrete eigenvalues accumulating at the origin if (3.2) holds true (see e.g. [37, Theorem XIII.82]). Therefore, estimates (4.1)–(4.2) now follow from (4.3)–(4.4).

Acknowledgments. G. Raikov thanks Fumio Hiroshima for the opportunity to give a talk on the results of the present article at the Conference Spectral and Scattering Theory and Related Topics, RIMS, Kyoto, Japan, 14 - 17 December 2011.
J.-F. Bony and V. Bruneau were partially supported by ANR-08-BLAN-0228. G. Raikov was partially supported by the Chilean Science Foundation *Fondecyt* under Grant 1090467, and by *Núcleo Científico ICM* P07-027-F "*Mathematical Theory of Quantum and Classical Magnetic Systems*".

References

[1] J. Avron, I. Herbst, B. Simon, *Schrödinger operators with magnetic fields. I. General interactions*, Duke Math. J. **45** (1978), 847–883.

[2] M. Š. Birman, M. G. Kreĭn, *On the theory of wave operators and scattering operators*, Dokl. Akad. Nauk SSSR **144** (1962), 475–478 (Russian); English translation in Soviet Math. Doklady **3** (1962).

[3] M.Š.Birman, M.Z.Solomjak, *Spectral Theory of Self-Adjoint Operators in Hilbert Space*, D. Reidel Publishing Company, Dordrecht, 1987.

[4] J.-F. Bony, Ph. Briet, V. Bruneau, G. D. Raikov, *Resonances and SSF singularities for magnetic Schrdinger operators*, Cubo **11** (2009), 2338.

[5] J.-F. Bony, V. Bruneau, G. D. Raikov, *Resonances and spectral shift function near Landau levels*, Ann. Inst. Fourier **57** (2007), 629–671.

[6] J.-F. Bony, V. Bruneau, G. D. Raikov, *Counting function of characteristic values and magnetic resonances*, ArXiv Preprint: 1109.3985 (2011).

[7] V. Bruneau, V. Petkov, *Meromorphic continuation of the spectral shift function*, Duke Math. J. **116** (2003), 389-430.

[8] V. Bruneau, A. Pushnitski, G. D. Raikov, *Spectral shift function in strong magnetic fields*, Algebra i Analiz **16** (2004), 207–238; see also St. Petersburg Math. J. **16** (2005), 181–209.

[9] V. Bruneau, G. D. Raikov, *High energy asymptotics of the magnetic spectral shift function*, J. Math. Phys. **45** (2004), 3453-3461.

[10] T. Christiansen, *Some upper bounds on the number of resonances for manifolds with infinite cylindrical ends*, Ann. Henri Poincaré **3** (2002), 895-920.

[11] T. Christiansen, *Several complex variables and the distribution of resonances in potential scattering*, Comm. Math. Phys. **259** (2005), no. 3, 711–728

[12] J. Edward, *On the resonances of the Laplacian on waveguides*, J. Math. Anal. Appl. **272** (2002), 89116.

[13] C. Fernández, G. D. Raikov, *On the singularities of the magnetic spectral shift function at the Landau levels*, Ann. Henri Poincaré **5** (2004), 381 – 403.

[14] V. Fock, *Bemerkung zur Quantelung des harmonischen Oszillators im Magnetfeld*, Z. Physik **47** (1928), 446–448.

[15] R. Froese, *Asymptotic distribution of resonances in one dimension*, J. Differential Equations **137** (1997), 251-272.

[16] M. Goffeng, *Index formulas and charge deficiencies on the Landau levels*, J. Math. Phys. **51** (2010), 023509, 18 pp.

[17] I. C. Gohberg, M. G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*. Translations of Mathematical Monographs, **18** American Mathematical Society, Providence, R.I. 1969.

[18] I. Gohberg, J. Leiterer, *Holomorphic Operator Functions of One Variable and Applications*, Methods from complex analysis in several variables. Operator Theory: Advances and Applications, **192**, Birkhäuser Verlag, Basel, 2009.

[19] I. C. Gohberg, E. I. Sigal, *An operator generalization of the logarithmic residue theorem and Rouché's theorem*, Mat. Sb. (N.S.) **84** (1971), 607-629 (Russian); English translation in: Math. USSR-Sb. **13** (1971), 603–625.

[20] L. Guillopé, *Théorie spectrale de quelques variétés à bouts*, Ann. Sci. Ecole Norm. Sup. **22** (1989), 137-160.

[21] B. C. Hall, *Holomorphic methods in analysis and mathematical physics*, In: First Summer School in Analysis and Mathematical Physics, Cuernavaca Morelos, 1998, 1–59, Contemp.Math. **260**, AMS, Providence, RI, 2000.

[22] V. Ivrii, *Microlocal Analysis and Precise Spectral Asymptotics*, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[23] A. Iwatsuka, H. Tamura, *Asymptotic distribution of eigenvalues for Pauli operators with nonconstant magnetic fields*, Duke Math. J. **93** (1998), 535-574.

[24] L. S. Koplienko, *The trace formula for perturbations of nonnuclear type*, (Russian) Sibirsk. Mat. Zh. **25** (1984), 62-71; English translation: Siberian Math. J. **25** (1984), 735743.

[25] M. G. Krein, *On the trace formula in perturbation theory*, Mat. Sb. **33** (1953), 597–626 (Russian).

[26] M. G. Krein, *On perturbation determinants and a trace formula for unitary and self-adjoint operators*, (Russian) Dokl. Akad. Nauk SSSR **144** (1962), 268271.

[27] L. Landau, *Diamagnetismus der Metalle*, Z. Physik **64** (1930), 629–637.

[28] N. Levinson, *On the uniqueness of the potential in a Schrödinger equation for a given asymptotic phase*, Danske Vid. Selsk. Mat.-Fys. Medd. **25**, (1949), no. 9, 29 pp.

[29] I. M. Lifshits, *On a problem in perturbation theory*, Uspekhi Mat. Nauk **7** (1952), 171–180 (Russian).

[30] M. Melgaard, G. Rozenblum, *Eigenvalue asymptotics for weakly perturbed Dirac and Schrödinger operators with constant magnetic fields of full rank*, Comm. Partial Differential Equations **28** (2003), 697-736.

[31] S. Nakamura, P. Stefanov, M. Zworski, *Resonance expansions of propagators in the presence of potential barriers*, J. Funct. Anal. **205** (2003), 180–205.

[32] A. Pushnitskiĭ, *A representation for the spectral shift function in the case of perturbations of fixed sign*, Algebra i Analiz **9** (1997), 197–213 (Russian); English translation in St. Petersburg Math. J. **9** (1998), 1181–1194.

[33] G. D. Raikov, *Eigenvalue asymptotics for the Schrödinger operator with homogeneous magnetic potential and decreasing electric potential. I. Behaviour near the essential spectrum tips*, Comm. PDE **15** (1990), 407–434; Errata: Comm. PDE **18** (1993), 1977–1979.

[34] G. D. Raikov, *Spectral shift function for magnetic Schrödinger operators*, Mathematical Physics of Quantum Mechanics, Lecture Notes in Physics, **690** (2006), 451–465.

[35] G. D. Raikov, *Low energy asymptotics of the spectral shift function for Pauli operators with nonconstant magnetic fields*, Publ. Res. Inst. Math. Sci. **46** (2010), 565-590.

[36] G. D. Raikov, S. Warzel, *Quasi-classical versus non-classical spectral asymptotics for magnetic Schrödinger operators with decreasing electric potentials*, Rev. Math. Phys. **14** (2002), 1051–1072.

[37] M. Reed, B. Simon, *Methods of Modern Mathematical Physics. IV. Analysis of Operators*, Academic Press, New York, 1979.

[38] D. Robert, *Semiclassical asymptotics for the spectral shift function*, In: Differential Operators and Spectral theory, AMS Translations Ser. 2 **189**, 187–203, AMS, Providence, RI, 1999.

[39] D. Sambou, *Résonances près de seuils d'opérateurs magnétiques de Pauli et de Dirac*, ArXiv Preprint: 1201.6552 (2012).

[40] J. Sjöstrand, *A trace formula and review of some estimates for resonances*, In: *Microlocal analysis and spectral theory*, Lucca 1996, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **490**, Dordrecht, Kluwer Acad. Publ. 1997, 377–437.

[41] J. Sjöstrand, *Weyl law for semi-classical resonances with randomly perturbed potentials*, Preprint ArXiv:1111.3549 (2011)

[42] A. V. Sobolev, *Asymptotic behavior of energy levels of a quantum particle in a homogeneous magnetic field perturbed by an attenuating electric field, I*, Probl. Mat. Anal., **9**, 67-84, Leningrad. Univ., Leningrad, 1984 (Russian).

[43] A. V. Sobolev, *Asymptotic behavior of energy levels of a quantum particle in a homogeneous magnetic field perturbed by an attenuating electric field. II*, Probl. Mat. Fiz., **11**, 232-248, 278, Leningrad. Univ., Leningrad, 1986 (Russian).

[44] P. Stefanov, *Sharp upper bounds on the number of the scattering poles*, J. Funct. Anal. **231** (2006), 111142.

[45] H. Tamura, *Asymptotic distribution of eigenvalues for Schrödinger operators with homogeneous magnetic fields*, Osaka J. Math. **25** (1988), 633-647.

[46] R. Tiedra de Aldecoa, *Asymptotics near $\pm m$ of the spectral shift function for Dirac operators with non-constant magnetic fields*, Comm. Partial Differential Equations **36** (2011), 10-41.

[47] G. Vodev, *Sharp polynomial bounds on the number of scattering poles for perturbations of the Laplacian*, Comm. Math. Phys. **146** (1992) 205-216.

[48] D. R. Yafaev, *Mathematical Scattering Theory. General Theory.* Translations of Mathematical Monographs, **105** AMS, Providence, RI, 1992.

[49] M. Zworski, *Distribution of poles for scattering on the real line*, J. Funct. Anal. **73** (1987), 277-296.

[50] M. Zworski, *Sharp polynomial bounds on the number of scattering poles*, Duke Math. J. **59** (1989), 311323.

[51] M. Zworski, *Lecture Notes on Resonances*, available at http://math.berkeley.edu/ zworski/.

Jean-François Bony
Université Bordeaux I, Institut de Mathématiques de Bordeaux,
UMR CNRS 5251, 351, Cours de la Libération, 33405 Talence, France
E-mail: bony@math.u-bordeaux1.fr

Vincent Bruneau
Université Bordeaux I, Institut de Mathématiques de Bordeaux,
UMR CNRS 5251, 351, Cours de la Libération, 33405 Talence, France
E-mail: vbruneau@math.u-bordeaux1.fr

Georgi Raikov
Departamento de Matemática, Facultad de Matemáticas,
Pontificia Universidad Católica de Chile, Vicuña Mackenna 4860, Santiago de Chile
E-mail: graikov@mat.puc.cl