THE SEMI-INFINITE q-BOSON SYSTEM WITH BOUNDARY INTERACTION

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Abstract. Upon introducing a one-parameter quadratic deformation of the q-boson algebra and a diagonal perturbation at the end point, we arrive at a semi-infinite q-boson system with a two-parameter boundary interaction. The eigenfunctions are shown to be given by Macdonald's hyperoctahedral Hall-Littlewood functions of type BC. It follows that the n-particle spectrum is bounded and absolutely continuous and that the corresponding scattering matrix factorizes as a product of two-particle bulk and one-particle boundary scattering matrices.

1. Introduction

The q-boson system [BIK] is a lattice discretization of the one-dimensional quantum nonlinear Schrödinger equation [G2, G, KBI, Mt, S] built of particle creation and annihilation operators representing the q-oscillator algebra [KS, Ch. 5]. Its n-particle eigenfunctions are given by Hall-Littlewood functions [T, K, DE]. In the present paper we study a system of q-bosons on the semi-infinite lattice with boundary interactions, in the spirit of previous works concerned with the quantum nonlinear Schrödinger equation on the half-line [G1, GLM, HL, CC, TW].

Specifically, by introducing at the end point creation and annihilation operators representing a quadratic deformation of the q-oscillator algebra together with a diagonal perturbation, we arrive at the hamiltonian of a q-boson system on the semi-infinite integer lattice endowed with a two-parameter boundary interaction. By means of an explicit formula for the action of the hamiltonian in the n-particle subspace, it is deduced that the Bethe Ansatz eigenfunctions are given by Macdonald's three-parameter Hall-Littlewood functions with hyperoctahedral symmetry associated with the BC-type root system [M, §10].

It follows that the q-boson system fits within a large class of discrete quantum models with bounded absolutely continous spectrum for which the scattering behaviour was determined in great detail by means of stationary phase techniques [R, D3]. In particular, the n-particle scattering matrix is seen to factorize as a product of explicitly computed two-particle bulk and one-particle boundary scattering matrices.

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2. Semi-infinite q-boson system

Let

$$\mathcal{F} := \bigoplus_{n\in\mathbb{N}} \mathcal{F}(\Lambda_n) \tag{2.1}$$

denote the algebraic Fock space consisting of finite linear combinations of $f_n \in \mathcal{F}(\Lambda_n)$, $n \in \mathbb{N} := \{0, 1, 2, \ldots\}$, where $\mathcal{F}(\Lambda_n)$ stands for the space of functions $f : \Lambda_n \to \mathbb{C}$ on the set of partitions of length at most n:

$$\Lambda_n := \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}, \tag{2.2}$$

with the additional convention that $\Lambda_0 := \{0\}$ and $\mathcal{F}(\Lambda_0) := \mathbb{C}$. For $l \in \mathbb{N}$, we introduce the following actions on $f \in \mathcal{F}(\Lambda_n) \subset \mathcal{F}$:

$$(\beta_l f)(\lambda) := f(\beta_l^* \lambda) \qquad\qquad (\lambda \in \Lambda_{n-1})$$

if $n > 0$ and $\beta_l f := 0$ if $n = 0$,

$$(\beta_l^* f)(\lambda) := \begin{cases} [m_l(\lambda)](1 - c\delta_l q^{m_0(\lambda)-1}) f(\beta_l \lambda) & \text{if } m_l(\lambda) > 0 \\ 0 & \text{otherwise} \end{cases} \qquad (\lambda \in \Lambda_{n+1}),$$

$$(q^{N_l+k} f)(\lambda) := q^{m_l(\lambda)+k} f(\lambda) \qquad (\lambda \in \Lambda_n),$$

with $q, c \in \mathbb{R}$ such that $|q| \neq 0, 1$ and $k \in \mathbb{Z}$. Here

$$\delta_l := \begin{cases} 1 & \text{for } l = 0, \\ 0 & \text{otherwise} \end{cases}, \qquad [m] := \frac{1 - q^m}{1 - q} = \begin{cases} 0 & \text{for } m = 0 \\ 1 + q + \cdots + q^{m-1} & \text{for } m > 0 \end{cases},$$

and the multiplicity $m_l(\lambda)$ counts the number of parts λ_j, $1 \leq j \leq n$ of size $\lambda_j = l$ (so $m_0(\lambda)$, $\lambda \in \Lambda_n$ is equal to n minus the number of nonzero parts), while $\beta_l^* \lambda \in \Lambda_{n+1}$ and $\beta_l \lambda \in \Lambda_{n-1}$ stand for the partitions obtained from $\lambda \in \Lambda_n$ by inserting/deleting a part of size l, respectively (where it is assumed in the latter situation that $m_l(\lambda) > 0$). It is clear from these definitions that β_l, β_l^* and q^{N_l+k} map $\mathcal{F}(\Lambda_n)$ into $\mathcal{F}(\Lambda_{n-1})$, $\mathcal{F}(\Lambda_{n+1})$ and $\mathcal{F}(\Lambda_n)$, respectively (with the convention that $\mathcal{F}(\Lambda_{-1})$ is the null space).

The operators in question represent a quadratic deformation of the q-boson field algebra at the boundary site $l = 0$ parametrized by the constant c:

$$\begin{gathered} \beta_l q^{N_l} = q^{N_l+1} \beta_l, \;\; \beta_l^* q^{N_l} = q^{N_l-1} \beta_l^*, \\ \beta_l \beta_l^* = [N_l + 1](1 - c\delta_l q^{N_0}), \;\; [\beta_l, \beta_l^*]_q = 1 - c\delta_l q^{2N_0} \end{gathered} \tag{2.3a}$$

and preserving the ultralocality:

$$[\beta_l, \beta_k] = [\beta_l^*, \beta_k^*] = [N_l, N_k] = [N_l, \beta_k] = [N_l, \beta_k^*] = [\beta_l, \beta_k^*] = 0 \tag{2.3b}$$

for $l \neq k$ (where $[A, B] := AB - BA$, $[A, B]_q := AB - qBA$, and $[N_l + r] := (1 - q^{N_l+r})/(1 - q)$).

When interpreting the characteristic function $|\lambda\rangle \in \mathcal{F}(\Lambda_n)$ supported on $\lambda \in \Lambda_n$ as a state representing a configuration of n particles on $\mathbb{N}$ such that $m_l(\lambda)$ particles are siting on the site $l \in \mathbb{N}$, it is clear that the operators β_l and β_l^* act as particle annihilation and creation operators:

$$\beta_l |\lambda\rangle = \begin{cases} |\beta_l \lambda\rangle & \text{if } m_l(\lambda) > 0 \\ 0 & \text{otherwise} \end{cases}, \quad \beta_l^* |\lambda\rangle = [m_l(\lambda) + 1](1 - c\delta_l q^{m_0(\lambda)}) |\beta_l^* \lambda\rangle,$$

while q^{N_l} counts the number of particles at the site l (as a power of q):

$$q^{N_l}|\lambda\rangle = q^{m_l(\lambda)}|\lambda\rangle.$$

The dynamics of our q-boson system is governed by a hamiltonian built of left and right hopping operators together with a diagonal boundary term:

$$\mathrm{H}_q = a[N_0] + \sum_{l\in\mathbb{N}} (\beta_{l+1}\beta_l^* + \beta_{l+1}^*\beta_l), \tag{2.4}$$

$a \in \mathbb{R}$. This hamiltonian constitutes a well-defined operator on $\mathcal{F}$ (2.1) as for any $f \in \mathcal{F}(\Lambda_n)$ and $\lambda \in \Lambda_n$ the infinite sum $(\mathrm{H}_q f)(\lambda)$ contains only a finite number of nonvanishing terms.

3. The n-Particle hamiltonian and its eigenfunctions

By construction H_q (2.4) preserves the n-particle subspace $\mathcal{F}(\Lambda_n)$. The following proposition describes the action of the hamiltonian in this subspace explicitly.

Proposition 3.1 (n-Particle hamiltonian)**.** *For any $f \in \mathcal{F}(\Lambda_n)$ and $\lambda \in \Lambda_n$, one has that*

$$(H_q f)(\lambda) = a[m_0(\lambda)]f(\lambda) + \sum_{\substack{1\le j\le n\\ \lambda+e_j\in\Lambda_n}} (1 - c\delta_{\lambda_j} q^{m_0(\lambda)-1})[m_{\lambda_j}(\lambda)]f(\lambda+e_j) + \sum_{\substack{1\le j\le n\\ \lambda-e_j\in\Lambda_n}} [m_{\lambda_j}(\lambda)]f(\lambda-e_j),$$

where $e_1, \ldots, e_n$ refer to the unit vectors comprising the standard basis of $\mathbb{Z}^n$.

Proof. It is clear from the definitions that $([N_0]f)(\lambda) = [m_0(\lambda)]f(\lambda)$, and that for any $l \in \mathbb{N}$:

$$(\beta_{l+1}\beta_l^* f)(\lambda) = \begin{cases} [m_l(\lambda)](1 - c\delta_l q^{m_0(\lambda)-1})f(\beta_{l+1}^*\beta_l\lambda) & \text{if } m_l(\lambda) > 0,\\ 0 & \text{otherwise,}\end{cases}$$

where $\beta_{l+1}^*\beta_l\lambda = \lambda + e_j$ with $j = \min\{k \mid \lambda_k = l\}$ (so $l = \lambda_j$), and

$$(\beta_{l+1}^*\beta_l f)(\lambda) = \begin{cases} [m_{l+1}(\lambda)]f(\beta_{l+1}\beta_l^*\lambda) & \text{if } m_{l+1}(\lambda) > 0,\\ 0 & \text{otherwise,}\end{cases}$$

where $\beta_{l+1}\beta_l^*\lambda = \lambda - e_j$ with $j = \max\{k \mid \lambda_k = l+1\}$ (so $l = \lambda_j - 1$). □

The n-particle hamiltonian has Bethe Ansatz eigenfunctions given by the following plane wave expansion

$$\phi_\xi(\lambda) := \sum_{\substack{\sigma\in S_n\\ \epsilon\in\{\pm1\}^n}} C(\epsilon\xi_\sigma)e^{i\langle\lambda,\epsilon\xi_\sigma\rangle}, \tag{3.1a}$$

with expansion coefficients of the form

$$C(\xi) := \prod_{1\le j\le n} \frac{1 - ae^{-i\xi_j} + ce^{-2i\xi_j}}{1 - e^{-2i\xi_j}} \times \prod_{1\le j<k\le n} \Bigl(\frac{1 - qe^{-i(\xi_j-\xi_k)}}{1 - e^{-i(\xi_j-\xi_k)}}\Bigr)\Bigl(\frac{1 - qe^{-i(\xi_j+\xi_k)}}{1 - e^{-i(\xi_j+\xi_k)}}\Bigr). \tag{3.1b}$$

Here $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $\mathbb{R}^n$, $\epsilon\xi_\sigma := (\epsilon_1\xi_{\sigma_1}, \epsilon_2\xi_{\sigma_2}, \ldots, \epsilon_n\xi_{\sigma_n})$, and the summation is meant over all permutations σ in the symmetric group S_n

and all sign configurations $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{1, -1\}^n$. Viewed as a function of the spectral parameter $\xi = (\xi_1, \ldots, \xi_n)$ in the fundamental alcove

$$A := \{(\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n \mid \pi > \xi_1 > \xi_2 > \cdots > \xi_n > 0\}, \tag{3.2}$$

the expression $\phi_\xi(\lambda)$, $\lambda \in \Lambda_n$ amounts to Macdonald's three-parameter Hall-Littlewood polynomial with hyperoctahedral symmetry associated with the root system BC_n [M, §10].

Proposition 3.2 (Bethe Ansatz eigenfunctions)**.** *The n-particle Bethe Ansatz wave function ϕ_ξ, $\xi \in A$ solves the eigenvalue equation*

$$H_q \phi_\xi = E_n(\xi)\phi_\xi, \qquad E_n(\xi) := 2\sum_{j=1}^{n} \cos(\xi_j). \tag{3.3}$$

Proof. It follows from Proposition 3.1 that the stated eigenvalue equation boils down to the following identity

$$a[m_0(\lambda)]\phi_\xi(\lambda) \; + \sum_{\substack{1\le j\le n \\ \lambda+e_j\in\Lambda_n}} (1 - c\delta_{\lambda_j} q^{m_0(\lambda)-1})[m_{\lambda_j}(\lambda)]\phi_\xi(\lambda + e_j)$$
$$+ \sum_{\substack{1\le j\le n \\ \lambda-e_j\in\Lambda_n}} [m_{\lambda_j}(\lambda)]\phi_\xi(\lambda - e_j) = 2\phi_\xi(\lambda)\sum_{j=1}^{n}\cos(\xi_j),$$

which is in turn equivalent to the Pieri formula for the hyperoctahedral Hall-Littlewood function in Eq. (A.3) of Appendix A. □

4. Diagonalization

From now on it will be assumed unless stated otherwise that $0 < |q| < 1$ and that the boundary parameters a and c are chosen such that the roots r_1, r_2 of the quadratic polynomial $r^2 - ar + c$ belong to the interval $(-1, 1)$:

$$a = r_1 + r_2 \text{ and } c = r_1 r_2 \text{ with } r_1, r_2 \in (-1, 1). \tag{4.1}$$

Let $L^2(A, \Delta\mathrm{d}\xi)$ be the Hilbert space of functions $\hat{f} : A \to \mathbb{C}$ characterized by the inner product

$$\langle \hat{f}, \hat{g}\rangle_\Delta = \frac{1}{(2\pi)^n}\int_A \hat{f}(\xi)\overline{\hat{g}(\xi)}\Delta(\xi)\mathrm{d}\xi, \quad \text{where} \quad \Delta(\xi) := \frac{1}{|C(\xi)|^2} \tag{4.2}$$

with $C(\xi)$ given by Eq. (3.1b). It is well-known that for the parameter regime in question Macdonald's hyperoctahedral Hall-Littlewood functions form an orthogonal basis of $L^2(A, \Delta\mathrm{d}\xi)$ [M, §10]:

$$\langle \phi(\lambda), \phi(\mu)\rangle_\Delta = \begin{cases} \mathcal{N}(\lambda) & \text{if } \lambda = \mu, \\ 0 & \text{otherwise,} \end{cases} \tag{4.3a}$$

where

$$\mathcal{N}(\lambda) := (c; q)_{m_0(\lambda)} \prod_{\ell\in\mathbb{N}} [m_\ell(\lambda)]! \tag{4.3b}$$

with $(c; q)_m := (1 - c)(1 - cq)\cdots(1 - cq^{m-1})$ (and the convention that $(c; q)_0 := 1$) and $[m]! := (q; q)_m/(q; q)_1^m = [m][m - 1]\cdots[2][1]$. By combining the orthogonality in Eqs. (4.3a), (4.3b) with Proposition 3.2, the spectral decomposition of H_q in the n-particle Hilbert space $\ell^2(\Lambda_n, \mathcal{N}^{-1}) \subset \mathcal{F}(\Lambda_n)$ characterized by the inner product

$$\langle f, g\rangle_n := \sum_{\lambda\in\Lambda_n} f(\lambda)\overline{g(\lambda)}\mathcal{N}^{-1}(\lambda) \tag{4.4}$$

becomes immediate.

Theorem 4.1 (Diagonalization)**.** *For $0 < |q| < 1$ and values of the boundary parameters a and c in the orthogonality domain (4.1), the q-boson Hamiltonian H_q (2.4) restricts to a bounded self-adjoint operator in $\ell^2(\Lambda_n, \mathcal{N}^{-1})$ with purely absolutely continuous spectrum. More specifically, its spectral decomposition reads explicitly*

$$H_q = \boldsymbol{F_q}^{-1} \circ \hat{E} \circ \boldsymbol{F_q}, \tag{4.5}$$

where $\boldsymbol{F_q} : \ell^2(\Lambda_n, \mathcal{N}^{-1}) \to L^2(A, \Delta d\xi)$ denotes the unitary Fourier transform associated with the hyperoctahedral Macdonald-Hall-Littlewood basis:

$$(\boldsymbol{F_q} f)(\xi) := \langle f, \phi_\xi\rangle_n = \sum_{\lambda\in\Lambda_n} f(\lambda)\overline{\phi_\xi(\lambda)}\mathcal{N}^{-1}(\lambda) \tag{4.6a}$$

($f \in \ell^2(\Lambda_n, \mathcal{N}^{-1})$) with the inversion formula given by

$$(\boldsymbol{F_q}^{-1}\hat{f})(\lambda) = \langle \hat{f}, \overline{\phi(\lambda)}\rangle_\Delta = \frac{1}{(2\pi)^n}\int_A \hat{f}(\xi)\phi_\xi(\lambda)\Delta(\xi)\,d\xi \tag{4.6b}$$

($\hat{f} \in L^2(A, \Delta d\xi)$), and $(\hat{E}\hat{f})(\xi) := E_n(\xi)\hat{f}(\xi)$ stands for the bounded real multiplication operator in $L^2(A, \Delta d\xi)$ associated with the n-particle eigenvalue $E_n(\xi)$ (3.3).

In the Fock space $\mathcal{H} := \bigoplus_{n\geq 0} \ell^2(\Lambda_n, \mathcal{N}^{-1})$, built of all linear combinations $\sum_{n\geq 0} c_n f_n$ with $c_n \in \mathbb{C}$ and $f_n \in \ell^2(\Lambda_n, \mathcal{N}^{-1})$ such that $\sum_{n\geq 0} |c_n|^2\langle f_n, f_n\rangle_n < \infty$, the q-boson hamiltonian H_q (2.4) constitutes an unbounded operator that is essentially self-adjoint on the dense domain $\mathcal{D} := \mathcal{F} \cap \mathcal{H}$ (because for $z \in \mathbb{C}\setminus\mathbb{R}$ the range $(\mathrm{H}_q - z)\mathcal{D}$ is dense in $\mathcal{H}$ and $\lim_{n\to\infty} \sup_{\xi\in A} |E_n(\xi)| = \infty$). The representation of the deformed q-boson field algebra in Section 2 on the other hand gives rise to a bounded representation on $\mathcal{H}$:

$$\begin{aligned}
\langle \beta_l f, \beta_l f\rangle_{n-1} &\leq \frac{1 + |c|\delta_l}{1-q}\langle f, f\rangle_n, \\
\langle \beta_l^* f, \beta_l^* f\rangle_{n+1} &\leq \frac{1 + |c|\delta_l}{1-q}\langle f, f\rangle_n, \\
\langle q^{N_l} f, q^{N_l} f\rangle_n &\leq \langle f, f\rangle_n,
\end{aligned}$$

preserving the $*$-structure:

$$\langle \beta_l^* f, g\rangle_{n+1} = \langle f, \beta_l g\rangle_n \quad \text{and} \quad \langle q^{N_l} f, g\rangle_n = \langle f, q^{N_l} g\rangle_n.$$

Remark 4.2. Upon rescaling the lattice Λ_n (2.2) and performing an appropriate continuum limit [D2, Sec. 5], Macdonald's hyperoctahedral Hall-Littlewood functions tend to the eigenfunctions of the quantum nonlinear Schrödinger equation on the half-line with a boundary interaction [G1, GLM, HL, CC, TW]. In particular, it follows from [D2, Sec. 5.3] that for $a = 0$ (which corresponds to a reduction from type BC to type C root systems) a renormalized version of the q-boson hamiltonian

H_q (2.4) then converges in the n-particle subspace in the strong resolvent sense to a hamiltonian that can be written formally as:

$$-\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}+g\sum_{1\le j<k\le n}\big(\delta(x_j-x_k)+\delta(x_j+x_k)\big)+g_0\sum_{1\le j\le n}\delta(x_j)$$

with $g,g_0>0$ (where $\delta(\cdot)$ stands for the 'delta potential').

5. Factorized scattering

The similarity transformation

$$H:=\mathcal{N}^{-1/2}\,\mathrm{H}_q\,\mathcal{N}^{1/2} \tag{5.1}$$

turns the n-particle q-boson hamiltonian in Proposition 3.1 into a self-adjoint operator in $\ell^2(\Lambda_n)$ diagonalized by the normalized wave function

$$\begin{aligned}\Psi_\xi(\lambda)&:=e^{\frac{\pi i}{2}n^2}|C(\xi)|^{-1}\mathcal{N}(\lambda)^{-1/2}\phi_\xi(\lambda)\\&=\mathcal{N}(\lambda)^{-1/2}\sum_{\substack{\sigma\in S_n\\ \epsilon\in\{\pm1\}^n}}\mathrm{sign}(\epsilon\sigma)\hat{\mathcal{S}}(\epsilon\xi_\sigma)^{1/2}e^{i\langle\rho+\lambda,\epsilon\xi_\sigma\rangle},\end{aligned} \tag{5.2a}$$

with $\xi\in A$ (3.2), $\mathrm{sign}(\epsilon\sigma):=\epsilon_1\cdots\epsilon_n\mathrm{sign}(\sigma)$, $\rho:=(n-1,n-2,\ldots,2,1,0)$, and

$$\hat{\mathcal{S}}(\xi):=\prod_{1\le j<k\le n}s(\xi_j-\xi_k)s(\xi_j+\xi_k)\prod_{1\le j\le n}s_0(\xi_j), \tag{5.2b}$$

where

$$s(x):=\frac{1-qe^{-ix}}{1-qe^{ix}}\quad\text{with}\quad s(x)^{1/2}=\frac{1-qe^{-ix}}{|1-qe^{ix}|} \tag{5.2c}$$

and

$$s_0(x):=\frac{1-ae^{-ix}+ce^{-2ix}}{1-ae^{ix}+ce^{2ix}}\quad\text{with}\quad s_0(x)^{1/2}=\frac{1-ae^{-ix}+ce^{-2ix}}{|1-ae^{ix}+ce^{2ix}|}. \tag{5.2d}$$

Specifically, one has that $H=\boldsymbol{F}^{-1}\circ\hat{E}\circ\boldsymbol{F}$ where $\boldsymbol{F}:\ell^2(\Lambda_n)\to L^2(A,\mathrm{d}\xi)$ denotes the unitary Fourier transformation determined by the kernel $\Psi_\xi(\lambda)$ (and $\hat{E}$ is now interpreted as a bounded multiplication operator in $L^2(A,\mathrm{d}\xi)$). For $q,a,c\to0$ the n-particle q-boson hamiltonian H (5.1) simplifies to a hamiltonian modeling impenetrable bosons on $\mathbb{N}$:

$$(H_0f)(\lambda)=\sum_{\substack{1\le j\le n\\ \lambda+e_j\in\Lambda_n}}f(\lambda+e_j)+\sum_{\substack{1\le j\le n\\ \lambda-e_j\in\Lambda_n}}f(\lambda-e_j)$$

($f\in\ell^2(\Lambda_n)$), which is diagonalized by the conventional Fourier transform $\boldsymbol{F_0}$: $\ell^2(\Lambda_n)\to L^2(A,\mathrm{d}\xi)$ obtained from $\boldsymbol{F}$ by setting $\hat{\mathcal{S}}(\xi)\equiv1$, $\mathcal{N}(\lambda)\equiv1$.

As a very special case of the results in [D3, Sec. 4], it now follows that the wave- and scattering operators comparing the q-boson dynamics

$$(e^{itH}f)(\lambda)=\frac{1}{(2\pi)^n}\int_A e^{itE_n(\xi)}\hat{f}(\xi)\Psi_\xi(\lambda)\mathrm{d}\xi\qquad\hat{f}=\boldsymbol{F}f \tag{5.3}$$

with the corresponding impenetrable boson dynamics generated by H_0 are governed by a unitary S-matrix $\hat{\mathcal{S}}:L^2(A,\mathrm{d}\xi)\to L^2(A,\mathrm{d}\xi)$ of the form

$$(\hat{\mathcal{S}}\hat{f})(\xi):=\hat{\mathcal{S}}(\epsilon_\xi\xi_{\sigma_\xi})\hat{f}(\xi)\qquad(\hat{f}\in C_0(A_r). \tag{5.4}$$

Here $C_0(A_r)$ denotes the dense subspace of $L^2(A, \mathrm{d}\xi)$ consisting of smooth test functions with compact support in the open dense subset $A_r \subset A$ for which the components of $\nabla E_n(\xi) = (-2\sin(\xi_1), \ldots, -2\sin(\xi_n))$ do not vanish and are all distinct in absolute value, and the sign-configuration ϵ_ξ and the permutation σ_ξ are such that the components of $\nabla E_n(\epsilon_\xi \xi_{\sigma_\xi})$ are all positive and ordered from large to small. Specifically, by comparing the large-time asymptotics of oscillatory integrals of the form in Eq. (5.3) for the dynamics generated by H and H_0 one concludes that [D3, Thm. 4.2 and Cor. 4.3]:

Theorem 5.1 (Wave and scattering operators)**.** *The operator limits*

$$\Omega^{\pm} := s - \lim_{t\to\pm\infty} e^{itH} e^{-itH_0} \tag{5.5a}$$

converge in the strong $\ell^2(\Lambda_n)$-norm topology and the corresponding wave operators $\Omega_r^{\pm}$ are given by unitary operators in $\ell^2(\Lambda_n)$ of the form

$$\Omega_r^{\pm} = \boldsymbol{F}^{-1} \circ \hat{\mathcal{S}}^{\mp 1/2} \circ \boldsymbol{F_0}. \tag{5.5b}$$

Hence, the scattering operator comparing the dynamics of H and H_0 is given by the unitary operator

$$\mathcal{S} := (\Omega_r^{+})^{-1}\Omega_r^{-} = \boldsymbol{F_0}^{-1} \circ \hat{\mathcal{S}} \circ \boldsymbol{F_0}. \tag{5.5c}$$

Appendix A. Pieri formula for Macdonald's hyperoctahedral Hall-Littlewood function

Let $x := (x_1, \ldots, x_n) = (e^{i\xi_1}, \ldots, e^{i\xi_n})$ and $\tau := (\tau_1, \ldots, \tau_n)$, where $\tau_j = rq^{n-j}$ ($j = 1, \ldots, n$) with $r = \frac{a}{2} + \sqrt{(\frac{a}{2})^2 - c}$ (cf. Eq. (4.1)). Upon setting

$$P_\lambda(x) := \frac{\tau_1^{\lambda_1} \cdots \tau_n^{\lambda_n}}{\mathcal{N}(0)} \phi_\xi(\lambda) \qquad (\lambda \in \Lambda_n), \tag{A.1}$$

where $\mathcal{N}(0)$ is given by Eq. (4.3b) with $\lambda = 0$, the hyperoctahedral Hall-Littlewood function is renormalized to have unital principal specialization values: $P_\lambda(\tau) = 1$ ($\forall \lambda \in \Lambda_n$) [M, §10]. With this normalization, the following Pieri formula holds:

$$\begin{aligned} P_\lambda(x) \sum_{j=1}^{n} (x_j + x_j^{-1} - \tau_j - \tau_j^{-1}) = \\ \sum_{\substack{1\leq j\leq n \\ \lambda+e_j\in\Lambda_n}} V_j^{+}(\lambda) \left(P_{\lambda+e_j}(x) - P_\lambda(x)\right) + \sum_{\substack{1\leq j\leq n \\ \lambda-e_j\in\Lambda_n}} V_j^{-}(\lambda) \left(P_{\lambda-e_j}(x) - P_\lambda(x)\right), \end{aligned} \tag{A.2}$$

where

$$V_j^{+}(\lambda) = \tau_j^{-1} \Bigl(\frac{1 - c^2\delta_{\lambda_j} q^{2(n-j)}}{1 + c\delta_{\lambda_j} q^{2(n-j)}}\Bigr) \prod_{\substack{j<k\leq n \\ \lambda_k=\lambda_j}} \Bigl(\frac{1 - q^{1+k-j}}{1 - q^{k-j}}\Bigr)\Bigl(\frac{1 + c\delta_{\lambda_j} q^{1+2n-k-j}}{1 + c\delta_{\lambda_j} q^{2n-k-j}}\Bigr),$$

$$V_j^{-}(\lambda) = \tau_j \prod_{\substack{1\leq k<j \\ \lambda_k=\lambda_j}} \Bigl(\frac{1 - q^{1+j-k}}{1 - q^{j-k}}\Bigr).$$

The formula in question is readily obtained through degeneration from an analogous Pieri formula for a BC_n-type Macdonald function that arises as a special case of the Pieri formulas in [D1, Sec. 6.1]. Specifically, by substituting $t_2 = q^{1/2}$,

$t_3 = -q^{1/2}$ (which amounts to a reduction from the Macdonald-Koornwinder function to the BC_n-type Macdonald function) in the Pieri formula of [D1, Eqs. (6.4), (6.5)] with coefficients taken from [D1, Eqs. (6.12), (6.13)], the relation in Eq. (A.2) is retrieved for $q \to 0$ (which corresponds to a transition from Macdonald type functions to Hall-Littlewood type functions). Notice in this connection that the parameters q, a, c (and r) of the present paper are related to the parameters t, t_0, t_1 of Ref. [D1] via $q = t$, $a = t_0 + t_1$, $c = t_0 t_1$ (and $r = t_0$).

Since

$$V_j^+(\lambda) = \tau_j^{-1}(1 - c\delta_{\lambda_j} q^{m_0(\lambda)-1})[m_{\lambda_j}(\lambda)], \qquad V_j^-(\lambda) = \tau_j [m_{\lambda_j}(\lambda)],$$

and

$$\sum_{j=1}^{n} (\tau_j + \tau_j^{-1}) - \sum_{\substack{1\le j\le n \\ \lambda - e_j \in \Lambda_n}} \tau_j [m_{\lambda_j}(\lambda)] - \sum_{\substack{1\le j\le n \\ \lambda + e_j \in \Lambda_n}} \tau_j^{-1} [m_{\lambda_j}(\lambda)] = r[m_0(\lambda)],$$

the Pieri formula (A.2) can be condensed into the more compact form

$$\begin{aligned} P_\lambda(x) \sum_{j=1}^{n} (x_j + x_j^{-1}) = a[m_0(\lambda)] &+ \sum_{\substack{1\le j\le n \\ \lambda - e_j \in \Lambda_n}} \tau_j [m_{\lambda_j}(\lambda)] P_{\lambda - e_j}(x) \\ &+ \sum_{\substack{1\le j\le n \\ \lambda + e_j \in \Lambda_n}} \tau_j^{-1} (1 - c\delta_{\lambda_j} q^{m_0(\lambda)-1}) [m_{\lambda_j}(\lambda)] P_{\lambda + e_j}(x). \end{aligned} \tag{A.3}$$

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