TRANSIENCE AND MULTIFRACTAL ANALYSIS

GODOFREDO IOMMI, THOMAS JORDAN, AND MIKE TODD

ABSTRACT. We study dimension theory for dissipative dynamical systems, proving a conditional variational principle for the quotients of Birkhoff averages restricted to the recurrent part of the system. On the other hand, we show that when the whole system is considered (and not just its recurrent part) the conditional variational principle does not necessarily hold. Moreover, we exhibit the first example of a topologically transitive map having discontinuous Lyapunov spectrum. The mechanism producing all these pathological features on the multifractal spectra is transience, that is, the non-recurrent part of the dynamics.

1. INTRODUCTION

The dimension theory of dynamical systems has received a great deal of attention over the last fifteen years. Multifractal analysis is a sub-area of dimension theory devoted to study the complexity of level sets of invariant local quantities. Typical examples of these quantities are Birkhoff averages, Lyapunov exponents, local entropies and pointwise dimension. Usually, the geometry of the level sets is complicated and in order to quantify its size or complexity tools such as Hausdorff dimension or topological entropy are used. Thermodynamic formalism is, in most cases, the main technical device used in order to describe the various multifractal spectra. In this note we will be interested in multifractal analysis of Birkhoff averages and of quotients of Birkhoff averages. That is, given a dynamical system $T : X \to X$ and functions $\phi, \psi : X \to \mathbb{R}$, with $\psi(x) > 0$. We will be interested in the level sets determined by the quotient of Birkhoff averages of ϕ with ψ. Let

$$\alpha_m = \alpha_{m,\phi,\psi} := \inf \left\{ \lim_{n \to \infty} \frac{\sum_{i=0}^{n-1} \phi(T^i x)}{\sum_{i=0}^{n-1} \psi(T^i x)} : x \in X \right\} \text{ and} \qquad (1)$$

$$\alpha_M = \alpha_{M,\phi,\psi} := \sup \left\{ \lim_{n \to \infty} \frac{\sum_{i=0}^{n-1} \phi(T^i x)}{\sum_{i=0}^{n-1} \psi(T^i x)} : x \in X \right\}. \qquad (2)$$

For $\alpha \in [\alpha_m, \alpha_M]$ we define the level set of points having quotient of Birkhoff average equal to α by

$$J(\alpha) = J_{\phi,\psi}(\alpha) := \left\{ x \in X : \lim_{n \to \infty} \frac{\sum_{i=0}^{n-1} \phi(T^i x)}{\sum_{i=0}^{n-1} \psi(T^i x)} = \alpha \right\}. \tag{3}$$

Note that these sets induce the so called *multifractal decomposition* of the repeller,

$$X = \bigcup_{\alpha=\alpha_m}^{\alpha_M} J(\alpha) \ \cup J',$$

where J' is the *irregular set* defined by,

$$J' = J'_{\phi,\psi} := \left\{ x \in X : \text{ the limit } \lim_{n \to \infty} \frac{\sum_{i=0}^{n-1} \phi(T^i x)}{\sum_{i=0}^{n-1} \psi(T^i x)} \text{ does not exist} \right\}.$$

The *multifractal spectrum* is the function that encodes this decomposition and it is defined by

$$b(\alpha) = b_{\phi,\psi}(\alpha) := \dim_H(J_{\phi,\psi}(\alpha)),$$

where $\dim_H$ denotes the Hausdorff dimension (see Section 2.3 or [Fa] for more details). Note that if $\psi \equiv 1$ then $b_{\phi,1}$ gives a multifractal decomposition of Birkhoff averages. If the set X is a compact interval, the dynamical system is uniformly expanding with finitely many piecewise monotone branches and the potentials ϕ and ψ are Hölder, it turns out that the map $\alpha \mapsto b_{\phi,\psi}(\alpha)$ is very well behaved. Indeed, both $\alpha_{m,\phi,\psi}$ and $\alpha_{M,\phi,\psi}$ are finite and the map $\alpha \mapsto b_{\phi,\psi}(\alpha)$ is real analytic (see the work of Barreira and Saussol [BS]). Some of these results were extended by Iommi and Jordan [IJ2] to the case of expanding full-branched interval maps, with countably many branches. However in this situation it is not always the case that the spectrum is always real analytic. In the situation where the map is non-uniformly expanding, for example the Manneville-Pomeau map, again the multifractal spectrum is not as regular as in the uniformly hyperbolic setting. Indeed in this case there will be regions where the spectrum does vary analytically but the transitions between these regions may not be analytic or even continuous, see [JJOP] and in particular section 6 of [IJ2]. The lack of uniform hyperbolicity of the dynamical system being the reason for the irregular behaviour of the multifractal spectrum. The tool used in the proofs is thermodynamic formalism.

Another important result in the study of multifractal analysis are the so-called conditional variational principles. Indeed, it has been shown for a very large class of dynamical systems (not necessarily uniformly hyperbolic) and for a large class of potentials (not necessarily Hölder) that the following holds:

$$b_{\phi,\psi}(\alpha) = \sup \left\{ \frac{h(\mu)}{\int \log |F'| \ d\mu} : \frac{\int \phi \ d\mu}{\int \psi \ d\mu} = \alpha \text{ and } \mu \in \mathcal{M} \right\},$$

where $\mathcal{M}$ denotes the set of $T-$invariant probability measures. See [BS, Cl, FFW, FLW, H, IJ1, JJOP, O, PW] for works where this conditional variational principle has been obtained with different degrees of generality.

The aim of the present note is to study multifractal spectra of quotients of Birkhoff averages when the map is modelled by a topologically mixing countable Markov shift with no additional assumptions (e.g. the incidence matrix is not assumed to be finitely primitive). This allows us to study certain dissipative maps by which we

mean maps where the Hausdorff dimension of the set of recurrent points is smaller than the Hausdorff dimension of the repeller of the map (see Sections 2.2 and 2.3 for precise definitions). Note that in this situation we cannot use the techniques from [IJ1] and [IJ2] since both these papers are restricted to maps which can be modelled by a full shift and the techniques can not be applied without additional assumptions on the incidence matrix. The multifractal analysis for the local dimension of Gibbs measures in this setting has been studied in [I] but the technique of inducing used there does not work so well in the setting of Birkhoff averages and so we take a different approach.

Dissipative maps arise naturally in a wide range of contexts, but the study of their dimension properties is still at an early stage. For example, in the context of rational maps Avila and Lyubich [AL, Theorem D] have suggested the existence of a rational map with Julia set of positive area whose hyperbolic dimension (see the definition given in equation (10)) is strictly smaller than 2. In a different context, Urbański and Stratmann [SU] proved that there exists a Schottky group G with limit set $L(G)$ for which the critical exponent of the corresponding Poincaré series $\delta(G)$ satisfies $\delta(G) < \dim_H L(G)$. These results extend those obtained by Patterson [Pa]. In [I, Example 3.3] an explicit example of an interval Markov map with countably many branches for which the Hausdorff dimension of the recurrent set (see definition 2.2) is strictly smaller than the corresponding dimension of the repeller is constructed. In all the above mentioned works the dissipation of the system is somehow measured by the difference between the Hausodrff dimension of the repeller with that of the conservative part of the system.

In this note we exhibit some of the pathologies that can easily occur in the dimension theory of dissipative systems. We not only study the dimension of the conservative part of the system but also the multifractal decomposition of the whole repeller (see Section 4). The example to which we will devote more attention is a model for an induced map of a Fibonacci unimodal map (see Section 4) which has been studied by Stratmann and Vogt [SV] and by Bruin and Todd (see [BT1, BT2]).

We prove that the conditional variational principle for quotients of Birkhoff averages holds under certain assumptions when restricted to the recurrent set. Moreover, we exhibit a map for which the Birkhoff spectrum, $b(\alpha)$, is discontinuous. In this example the mechanism producing the discontinuity is *transience*. Note that the Birkhoff spectrum for this map does not satisfy the conditional variational principle for certain Hölder potentials. We stress that while recently [IJ2] examples of discontinuous Birkhoff spectra were found in the non-uniformly hyperbolic setting, the situation we treat here is of a completely different nature. The study of transience in dynamical systems has attracted some attention recently and its implications in thermodynamic formalism has been explored (see [C, CS, IT, Sa2]). In this note we study some of the consequences that transience has in dimension theory. Of particular interest is Proposition 4.4 where we exhibit a map having discontinuous Lyapunov spectrum. This particular case of Birkhoff spectrum has been thoroughly studied over the last years in a wide range of contexts. This is the first such example, as far as we are aware, that is discontinuous.

2. Notation and statement of our main result

This section is devoted to stating the conditional variational principle for the quotient of Birkhoff averages restricted to the recurrent set, followed by some preliminary results we will need to prove it. In order to do this, we will define the class of maps and potentials that we will consider as well as to recall some basic definitions from geometric measure theory.

2.1. Symbolic spaces.

Let (Σ, σ) be a one-sided Markov shift over the countable alphabet $\mathbb{N}$. This means that there exists a matrix $(t_{ij})_{\mathbb{N} \times \mathbb{N}}$ of zeros and ones (with no row and no column made entirely of zeros) such that

$$\Sigma := \left\{ (x_n)_{n \in \mathbb{N}} : t_{x_i x_{i+1}} = 1 \text{ for every } i \in \mathbb{N} \right\}.$$

The *shift map* $\sigma : \Sigma \to \Sigma$ is defined by $\sigma(x_1 x_2 x_2 \dots) = (x_2 x_2 \dots)$. We will always assume the system (Σ, σ) to be topologically mixing (see [Sa1] for a precise definition). The space Σ endowed with the topology generated by the cylinder sets

$$C_{i_1 i_2 \dots i_n} := \{(x_n) \in \Sigma : x_i = i_j \text{ for } j \in \{1, 2, 3 \dots n\}\},$$

is a non-compact space. We define the *n-th variation* of a function $\phi : \Sigma \to \mathbb{R}$ by

$$var_n(\phi) = \sup_{(i_1 \dots i_n) \in \mathbb{N}^n} \sup_{x,y \in C_{i_1 i_2 \dots i_n}} |\phi(x) - \phi(y)|.$$

A function $\phi : \Sigma \to \mathbb{R}$ is locally Hölder if there exists $0 < \gamma < 1$ and $C > 0$ such that for every $n \in \mathbb{N}$ we have $var_n(\phi) \leq C\gamma^n$.

2.2. The class of maps.

Given a compact interval $X \subset \mathbb{R}$, let $\{X_n\}_n \subset X$ be a countable collection closed subintervals such that their interiors are pairwise disjoint and let $T : \cup_n X_n \to X$ be a map. The *repeller* of the map T is defined by

$$X^\infty := \{x \in X : T^n(x) \text{ is defined for all } n \in \mathbb{N}\}.$$

We say that the map T is *Markov* if there exists a countable Markov shift (Σ, σ) and a continuous bijective map $\pi : \Sigma \to X^\infty$ such that $T \circ \pi = \pi \circ \sigma$. We will use the notation $[i_1, \dots, i_n] := \pi(C_{i_1 \dots i_n})$. Let $\mathcal{R}$ denote the set of potentials $\phi : \cup_n X_n \to \mathbb{R}$ such that $\phi \circ \pi$ is locally Hölder and let $\mathcal{R}_0$ denote the set of such potentials $\phi \in \mathcal{R}$ for which there exists $\varepsilon > 0$ such that $\phi \geq \varepsilon$.

Given $x \in X^\infty$, define the *lower pointwise Lyapunov exponent* of T at x by $\underline{\lambda}_T(x) := \liminf_n \frac{1}{n} \log |(T^n)'(x)|$. Denote by $\mathcal{M}$ the set of T−invariant probability measures. If $\mu \in \mathcal{M}$, we denote by $\lambda_T(\mu) := \int \log |T'| \, d\mu$ the *Lyapunov exponent* of T with respect to the measure μ.

Definition 2.1. *Given a compact interval $X \subset \mathbb{R}$, let $\{X_n\}_n$ be be a countable collection closed subintervals such that their interiors are pairwise disjoint. The map $T : \cup_n X_n \to X$ is called an EMV (Expanding Markov (summable) Variation) map if*

1. *it is C^1 on each X_n;*
2. *there exists $\xi > 1$ such that $\underline{\lambda}_T(x) > \log \xi$ for all $x \in \cup_n X_n$.*
3. *it is Markov and it can be coded by a topologically mixing countable Markov shift.*

4. with $\mathcal{R}$ defined by the shift structure above, $\log |T'| \in \mathcal{R}$

Observe that the second condition in Definition 2.1 means that for any $\mu \in \mathcal{M}$, $\int \log |T'| \, d\mu > \log \xi$, and in particular that for any periodic orbit $x, Tx, \ldots, T^{n-1}x$, we have $|(T^n)'(x)| > \xi^n$. The fact that the system can be coded by a topologically mixing Markov shift means that there is a dense orbit, so T is *topologically transitive*.

The following set will play an important part in the rest of the note.

Definition 2.2. *Let T be an EMV map. The recurrent set of T is defined by*

$$X_R := \{x \in X : \exists X_n \text{ and } n_k \to \infty \text{ with } T^{n_k}(x) \in X_n \text{ for all } k \in \mathbb{N}\}.$$

We let $\phi \in \mathcal{R}$ and $\psi \in \mathcal{R}_0$ and let α_m, α_m be as in equation (1) and $J(\alpha)$ be as in equation (3). We will consider the restriction of the level set $J(\alpha)$ to the recurrent set for T,

$$J_R(\alpha) = J_{R,\phi,\psi} := J_{\phi,\psi}(\alpha) \cap X_R.$$

2.3. **Hausdorff dimension.** We briefly recall the definition of the Hausdorff measure (see [Ba, Fa] for further details). Let $F \subset \mathbb{R}^d$ and $s, \delta \in \mathbb{R}^+$,

$$H_\delta^s(F) := \inf \left\{ \sum_{i=1}^\infty |U_i|^s : \{U_i\}_i \text{ is a } \delta\text{-cover of } F \right\}.$$

The *s-Hausdorff measure* of the set F is defined by

$$H^s(F) := \lim_{\delta \to 0} H_\delta^s(F)$$

and the *Hausdorff dimension* by

$$\dim_H F := \inf\{s : H^s(F) = 0\} = \sup\{s : H^s(F) = \infty\}.$$

We call a measure μ on X *dissipative* if $\mu(X_R) < \mu(X^\infty)$. In the same spirit, we call the system *dissipative* if $\dim_H(X_R) < \dim_H(X^\infty)$.

2.4. **Main results.** Our main result establishes the conditional variational principle for the sets $J_R(\alpha)$. In the final section of the note we will give an example to show that it is not always true for the sets $J(\alpha)$.

Theorem 2.3. *Let $T : \cup_n X_n \to X$ be a EMV map and $\phi, \psi : \cup_n X_n \to \mathbb{R}$ be such that $\phi \in \mathcal{R}$ and $\psi \in \mathcal{R}_0$. Let $\alpha \in (\alpha_m, \alpha_M)$. If there exists $K > 0$ such that for every $x \in J_R(\alpha)$ we have that*

$$\limsup_{n \to \infty} \frac{S_n \psi(x)}{n} < K, \tag{4}$$

then

$$\dim_H(J_R(\alpha)) = \sup \left\{ \frac{h(\mu)}{\lambda_T(\mu)} : \frac{\int \phi \, d\mu}{\int \psi \, d\mu} = \alpha, \max\left\{ \lambda_T(\mu), \int \psi \, d\mu \right\} < \infty, \mu \in \mathcal{M} \right\}.$$

By taking ψ to be the constant function 1 we obtain the following corollary.

Corollary 2.4 (Birkhoff spectrum)**.** *Let $T : \cup_n X_n \to X$ be a EMV map and $\phi : \cup_n X_n \to \mathbb{R}$ be such that $\phi \in \mathcal{R}$. Let $\alpha \in (\alpha_m, \alpha_M)$ then*

$$\dim_H(J_R(\alpha)) = \sup\left\{\frac{h(\mu)}{\lambda_T(\mu)} : \int \phi \; \mathrm{d}\mu = \alpha, \lambda_T(\mu) < \infty, \mu \in \mathcal{M}\right\}.$$

Remark 2.5. *It is a direct consequence of results by Barreira and Schmeling* [BSc] *(see also* [BS, Theorem 11]*) that if $\alpha_m \neq \alpha_M$ then*

$$\dim_H X_R = \dim_H \left(J' \cap X_R\right).$$

2.5. **Thermodynamic formalism.** The proof of Theorem 2.3 uses tools from thermodynamic formalism. We briefly recall the basic notions and results that will be used. The *Gurevich Pressure* of a locally Hölder potential $\phi : \cup_n X_n \to \mathbb{R}$ was introduced by Sarig in [Sa1]. It is defined by letting

$$Z_n(\phi) = \left(\sum_{T^n x = x} \exp\left(\sum_{j=0}^{n-1} \phi(T^j(x))\right) \mathbb{1}_{X_i}(x)\right),$$

where $\mathbb{1}_{X_i}(x)$ denotes the characteristic function of the cylinder X_i, and

$$P(\phi) := \lim_{n \to \infty} \frac{\log(Z_n(\phi))}{n}.$$

The limit always exists and its value does not depend on the cylinder X_i considered. This notion of pressure satisfies the following variational principle: if ϕ is a locally Hölder potential then

$$P(\phi) = \sup\left\{h_\sigma(\mu) + \int \phi \; \mathrm{d}\mu : \mu \in \mathcal{M} \text{ and } -\int \min\{\phi, 0\} \; \mathrm{d}\mu < \infty\right\}.$$

In this generality, this result is [IJT, Theorem 2.10]. Since the form of this statement is classical, in this note we refer to this as the Variational Principle. A measure attaining the supremum above will be called *equilibrium measure* for ϕ. An important property of the Gurevich pressure is that it can be approximated by considering functions restricted to certain compact invariant sets. Let

$\mathcal{K} := \{M \subset X : M \neq \varnothing \text{ is compact, } T\text{-invariant and } T|M \text{ is Markov and mixing}\}.$

Lemma 2.6. *There exists an increasing sequence, $\{M_n\}_{n \in \mathbb{N}}$ of sets in $\mathcal{K}$ such that*

1. *for any $\psi \in \mathcal{R}$ we have that $P(\psi) = \lim_{n \to \infty} P_{M_n}(\psi)$;*
2. *for any $M \in \mathcal{K}$ there exists $n \in \mathbb{N}$ such that $M \subset M_n$.*

Proof. The proof of [Sa1, Theorem 2] gives this lemma. $\qquad\square$

3. Proof of Theorem 2.3

In this section we give the proof of the main result of this note, Theorem 2.3. The proof is similar to the one developed in [H] to study multifractal spectra for interval maps. It will be convenient to consider invariant measures supported on compact sets. Thus we define

$$\mathcal{M}_\mathcal{K} := \{\mu \in \mathcal{M} : \text{there exists } M \in \mathcal{K} \text{ such that } \mu(X \setminus M) = 0\}.$$

The following quantities will be crucial in our proof.

Definition 3.1. *For $\alpha \in (\alpha_m, \alpha_M)$ let*

$$V(\alpha) := \sup \left\{ \frac{h(\mu)}{\lambda_T(\mu)} : \frac{\int \phi \, \mathrm{d}\mu}{\int \psi \, \mathrm{d}\mu} = \alpha, \max \left\{ \lambda_T(\mu), \int \psi \, \mathrm{d}\mu \right\} < \infty \text{ and } \mu \in \mathcal{M} \right\}$$

and

$$\mathcal{E}(\alpha) := \sup \left\{ \frac{h(\mu)}{\lambda_T(\mu)} : \frac{\int \phi \, \mathrm{d}\mu}{\int \psi \, \mathrm{d}\mu} = \alpha, \text{ and } \mu \in \mathcal{M}_{\mathcal{K}} \text{ is ergodic} \right\}.$$

To start the proof we first relate the quantity $V(\alpha)$ to the pressure function. To do this we need the following preparatory lemma.

Lemma 3.2. *If $\alpha \in (\alpha_m, \alpha_M)$, $\delta > 0$ and $\inf\{P(q(\phi - \alpha\psi) - \delta \log |T'|) : q \in \mathbb{R}\} > 0$ then there exists $M \in \mathcal{K}$ such that:*

1. *$P_M(q(\phi - \alpha\psi) - \delta \log |T'|) > 0$ for every $q \in \mathbb{R}$,*
2. *the following equality holds*

$$\lim_{q \to \infty} P_M(q(\phi - \alpha\psi) - \delta \log |T'|) = \lim_{q \to -\infty} P_M(q(\phi - \alpha\psi) - \delta \log |T'|) = \infty.$$

Proof. We start with the second part. Since $\alpha \in (\alpha_m, \alpha_M)$ we can find $K_1, K_2 \in \mathcal{K}$, $\mu_1 \in M_{K_1}$ and $\mu_2 \in \mathcal{M}_{K_2}$ such that

$$\frac{\int \phi \, \mathrm{d}\mu_1}{\int \psi \, \mathrm{d}\mu_1} < \alpha < \frac{\int \phi \, \mathrm{d}\mu_2}{\int \psi \, \mathrm{d}\mu_2}. \tag{5}$$

We will use Lemma 2.6 and the Variational Principle to show that there exists $K_3 \in \mathcal{K}$ such that

$$\lim_{q \to \infty} P_{K_3}(q(\phi - \alpha\psi) - \delta \log |T'|) = \infty = \lim_{q \to -\infty} P_M(q(\phi - \alpha\psi) - \delta \log |T'|). \tag{6}$$

Indeed, by equation (5),

$$\int (\phi - \alpha\psi) \, \mathrm{d}\mu_2 > 0.$$

So by the Variational Principle,

$$P_{K_3}(q(\phi - \alpha\psi) - \delta \log |T'|) \geq \left(h(\mu_2) - \delta \int \log |T'| \, \mathrm{d}\mu_2 \right) + q \int (\phi - \alpha\psi) \, \mathrm{d}\mu_2.$$

Hence the first inequality in (6) follows since

$$\lim_{q \to \infty} q \int (\phi - \alpha\psi) \, \mathrm{d}\mu_2 = \infty.$$

An analogous argument using μ_1 yields the second equality in (6).

Now let $\gamma := \inf\{P(q(\phi - \alpha\psi) - \delta \log |T'|) : q \in \mathbb{R}\} > 0$ and $I = \{q \in \mathbb{R} : P_{K_3}(q(\phi - \alpha\psi) - \delta \log |T'|) \leq \gamma\}$. If $I = \varnothing$ then the proof is complete. If $I \neq \varnothing$ then by the convexity of pressure it is a compact set.

By Lemma 2.6 there exists an increasing sequence of sets $\{M_n\}_n \subset \mathcal{K}$ where $K_3 \subset M_1$ such that

$$P(q(\phi - \alpha\psi) - \delta \log |T'|) = \lim_{n \to \infty} P_{M_n}(q(\phi - \alpha\psi) - \delta \log |T'|).$$

Therefore, for each $q \in I$ we have that $\lim_{n\to\infty} P_{M_n}(q(\phi - \alpha\psi) - \delta \log |T'|) \geq \gamma$. Now suppose that for each $n \in \mathbb{N}$ there exists $q_n \in I$ such that $P_{M_n}(q_n(\phi - \alpha\psi) - \delta \log |T'|) \leq \gamma/2$ then since I is compact we can assume, passing to a subsequence if necessary, that there exists $q_* = \lim_{n\to\infty} q_n$. By the continuity of the pressure, for any fixed $n \in \mathbb{N}$ we have that

$$P_{M_n}(q_*(\phi - \alpha\psi) - \delta \log |T'|) = \lim_{k\to\infty} P_{M_n}(q_k(\phi - \alpha\psi) - \delta \log |T'|). \qquad (7)$$

On the other hand, since for every $k \geq n$ we have that $M_n \subset M_k$, we obtain

$$P_{M_n}((q_k(\phi - \alpha\psi) - \delta \log |T'|) \leq P_{M_k}((q_k(\phi - \alpha\psi) - \delta \log |T'|) \leq \frac{\gamma}{2}. \qquad (8)$$

Combining equations (7) with (8) we obtain that for every $n \in \mathbb{N}$ we have

$$\lim_{n\to\infty} P_{M_n}(q_*(\phi - \alpha\psi) - \delta \log |T'|) \leq \frac{\gamma}{2}.$$

Thus $P(q_*(\phi - \alpha\psi) - \delta \log |T'|) \leq \gamma/2$ which is a contradiction. Therefore we can conclude that there exists $M \in K$ such that $P_M(q(\phi - \alpha\psi) - \delta \log |T'|) > 0$ for all $q \in \mathbb{R}$ and

$$\lim_{q\to\infty} P_M(q(\phi - \alpha\psi) - \delta \log |T'|) = \lim_{q\to-\infty} P_M(q(\phi - \alpha\psi) - \delta \log |T'|) = \infty.$$

$\square$

We can now relate $V(\alpha)$ to the pressure function in the following lemma, which is the main engine of the proof of Theorem 2.3.

Lemma 3.3. *For any $\alpha \in (\alpha_m, \alpha_M)$,*

$$\mathcal{E}(\alpha) = V(\alpha) = \sup \left\{ \delta \in \mathbb{R} : \inf\{P(q(\phi - \alpha\psi) - \delta \log |T'|) : q \in \mathbb{R}\} > 0 \right\}.$$

Proof. Let $\varepsilon > 0$. By the definition of $V(\alpha)$, we can find $\mu \in \mathcal{M}$ such that $\frac{h(\mu)}{\int \log |T'| \, d\mu} > V(\alpha) - \varepsilon$ and $\frac{\int \phi \, d\mu}{\int \psi \, d\mu} = \alpha$. Then it is a consequence of the Variational Principle that

$$P\big(q(\phi - \alpha\psi) - (V(\alpha) - \varepsilon) \log |T'|\big)$$

$$\geq h(\mu) + \int q(\phi - \alpha\psi) \, d\mu - (V(\alpha) - \varepsilon) \int \log |T'| \, d\mu$$

$$= h(\mu) - (V(\alpha) - \varepsilon) \int \log |T'| \, d\mu > 0.$$

Therefore, $\sup \left\{ \delta \in \mathbb{R} : P(q(\phi - \alpha\psi) - \delta \log |T'|) > 0 \right\} \geq V(\alpha) - \varepsilon$ for all $\varepsilon > 0$, so $V(\alpha)$ and hence $\mathcal{E}(\alpha)$ are lower bounds.

For the upper bound suppose that $s \in \mathbb{R}$ satisfies that

$$P(q(\phi - \alpha\psi) - s \log |T'|) > 0$$

for all $q \in \mathbb{R}$. By Lemma 3.2 we can find $M \in \mathcal{K}$ such that

$$P_M(q(\phi - \alpha\psi) - s \log |T'|) > 0$$

for all $q \in \mathbb{R}$ and such that

$$\lim_{q\to\infty} P_M(q(\phi - \alpha\psi) - s \log |T'|) = \lim_{q\to\infty} P_M(q(\phi - \alpha\psi) - s \log |T'|) = \infty. \qquad (9)$$

Since the function $q \mapsto P_M(q(\phi - \alpha\psi) - s\log|T'|)$ is real analytic (see [BS]), it is a consequence of (9) that there exists $q_0 \in \mathbb{R}$ such that

$$\frac{\partial}{\partial q} P_M(q(\phi - \alpha\psi) - s\log|T'|)\Big|_{q=q_0} = 0.$$

Therefore, using Ruelle's formula for the derivative of pressure (see [PU, Lemma 5.6.4]), we obtain that

$$\int (\phi - \alpha\psi) \, \mathrm{d}\mu_0 = 0,$$

where μ_0 denotes the equilibrium measure for the potential $q(\phi - \alpha\psi) - s\log|T'|$ and the dynamical system T restricted to M. Thus, we have that

$$\frac{\int \phi \, \mathrm{d}\mu_0}{\int \psi \, \mathrm{d}\mu_0} = \alpha.$$

But it also follows from the Variational Principle that

$$h(\mu_0) + \int (\phi - \alpha\psi) \, \mathrm{d}\mu_0 - s\int \log|T'| \, \mathrm{d}\mu_0 > 0.$$

That is,

$$\frac{h(\mu_0)}{\int \log|T'| \, \mathrm{d}\mu_0} > s.$$

Therefore, since μ_0 is ergodic we obtain that $\mathcal{E}(\alpha) \geq V(\alpha) \geq s$ and the result follows. $\qquad\square$

It is now straightforward to prove the lower bound.

Lemma 3.4. *For all $\alpha \in (\alpha_m, \alpha_M)$ we have that $\dim_H(J_R(\alpha)) \geq V(\alpha)$.*

Proof. Let $\epsilon > 0$. Since Lemma 3.3 implies that $V(\alpha) = \mathcal{E}(\alpha)$, there exists a compactly supported invariant ergodic measure μ such that $\frac{\int \phi \mathrm{d}\mu}{\int \psi \mathrm{d}\mu} = \alpha$ and $\frac{h(\mu)}{\lambda_T(\mu)} > V(\alpha) - \epsilon$. Thus since $\mu(J_{\phi,\psi}(\alpha) \cap X_R) = 1$, the well known formula for the dimension of μ (see for example [HR, M]) implies that $\dim_H(J_{\phi,\psi}(\alpha) \cap X_R) \geq V(\alpha)$. $\qquad\square$

In order to prove the upper bound we will use a covering argument. To start with we set

$$\tilde{J}(\alpha, j) = \tilde{J}_{\phi,\psi}(\alpha, j) := \{x \in X : x \in J_{\phi,\psi}(\alpha) \text{ and } \#\{n \in \mathbb{N} : T^n(x) \in X_j\} = \infty\}$$

and

$$J(\alpha, j) = J_{\phi,\psi}(\alpha, j) := \tilde{J}_{\phi,\psi}(\alpha, j) \cap X_j.$$

The following lemma can be immediately deduced from the definition.

Lemma 3.5. *For all $j \in \mathbb{N}$ we have that*

$$\dim_H \tilde{J}(\alpha, j) = \dim_H J(\alpha, j)$$

and thus

$$\dim_H J_R(\alpha) = \sup_{j \in \mathbb{N}} \dim_H J(\alpha, j).$$

The next lemma is the main step in the proof of the upper bound.

Lemma 3.6. *Let $0 < \delta < 1$, if there exists $q \in \mathbb{R}$ such that*

$$P(q(\phi - \alpha\psi) - \delta \log |T'|) \leq 0$$

then $\dim_H J(\alpha, j) \leq \delta$ for all $j \in \mathbb{N}$.

Proof. Let $\epsilon > 0$ be fixed. Note that since for every $x \in X$ we have $\underline{\lambda}_T(x) > \log \xi > 0$ and $P(q(\phi - \alpha\psi) - \delta \log |T'|) \leq 0$ we can conclude that

$$P(q(\phi - \alpha\psi) - (\delta + \epsilon) \log |T'|) < 0.$$

Denote by $B(x, r)$ the ball of centre x and radius r. Let $j, n \in \mathbb{N}$, we define

$$G(\alpha, n, \epsilon) := \left\{ x \in X_j : T^n(x) \in X_j, \frac{S_n\phi(x)}{S_n\psi(x)} \in B\left(\alpha, \frac{\epsilon \log \xi}{q2K}\right) \right\}.$$

Observe that $J(\alpha, j) \subset \bigcap_{r=1}^\infty \bigcup_{n=r}^\infty G(\alpha, n, \epsilon)$. Consider now the set of cylinders that intersect $G(\alpha, n, \epsilon)$,

$$C(\alpha, n, \epsilon) := \{[i_1, \ldots, i_n] : [i_1, \ldots, i_n] \cap G(\alpha, n, \epsilon) \neq \varnothing\}.$$

We can choose N such that for all $n \geq N$ if $[i_1, \ldots, i_n] \in C(a, n, \epsilon)$ then for any $x \in [i_1, \ldots, i_n]$ we have

$$\log |[i_1, \ldots, i_n]| \leq - S_n(\log |T'|)(x) + \sum_{n=1}^n var_n(\log |T'|)$$

and such that for any $x \in [i_1, \ldots, i_n]$

$$-n\epsilon \log \xi \leq S_n(q(\phi - \alpha\psi))(x) \leq n\epsilon \log \xi.$$

Since $S_n(q(\phi - \alpha\psi))(x) \geq -n\epsilon \log \xi \geq -\epsilon S_n(\log |T'|)(x)$, for $x \in G(\alpha, n, \epsilon)$ and N large enough that the derivative sufficiently dominates the sum of the variations (indeed we require $N \cdot \inf_x\{\underline{\lambda}_T(x)\} > \sum_n var_n(\log |T'|)$),

$$H_{\xi_n}^{\delta+4\epsilon}\left(\cup_{n \geq N} G(\alpha, n, \epsilon)\right) \leq \sum_{n \geq N} \sum_{C(\alpha, n, \epsilon)} |i_1, \ldots, i_n|^{\delta+4\epsilon}$$

$$\leq \sum_{n \geq N} \sum_{x \in G(\alpha, n, \epsilon): T^n(x)=x} e^{-(\delta+3\epsilon)(S_n \log |T'|)(x)}$$

$$\leq \sum_{n \geq N} \sum_{x \in G(\alpha, n, \epsilon): T^n(x)=x} e^{q(S_n\phi(x)-\alpha S_n\psi(x))-(\delta+2\epsilon)(S_n \log |T'|)(x)}$$

$$\leq \sum_{n \geq N} \sum_{x \in X_j: T^n(x)=x} e^{q(S_n\phi(x)-\alpha S_n\psi(x))-(\delta+2\epsilon)(S_n \log |T'|)(x)}$$

$$\leq \sum_{n \geq N} e^{nP(q(\phi-\alpha\psi)-(\delta+\epsilon) \log |T'|)} < \infty$$

where for the penultimate inequality, we assume N is so large that $Z_n(q(\phi - \alpha\psi) - (2\delta + \epsilon) \log |T'|, X_j) \leq e^{nP(q(\phi-\alpha\psi)-(\delta+\epsilon) \log |T'|)}$ for $n \geq N$. By letting $N \to \infty$ and then $\epsilon \to 0$ we have that $\dim_H J(\alpha, j) \leq \delta$. $\qquad\square$

We can now prove the upper bound.

Lemma 3.7. *For all $\alpha \in (\alpha_m, \alpha_M)$ we have that $\dim_H(J_{\phi,\psi}(\alpha) \cap X_R) \leq V(\alpha)$.*

Proof. Let $\alpha \in (\alpha_m, \alpha_M)$ and $\epsilon > 0$ and $s \geq V(\alpha) + \epsilon$. By Lemma 3.3 we can conclude that there exists q such that

$$P(q(\phi - \alpha\psi) - s \log |T'|) \leq 0.$$

Therefore by Lemmas 3.5 and 3.6 it follows that $\dim_H(J_{\phi,\psi}(\alpha) \cap X_R) \leq V(\alpha)$. $\square$

This completes the proof of Theorem 2.3.

4. DISCONTINUOUS BIRKHOFF SPECTRA

This section is devoted to exhibiting pathologies and new phenomena that occur when studying dimension theory of a specific dissipative map. We consider a piecewise linear, uniformly expanding map which is Markov over a countable partition and that has been studied in detail by Bruin and Todd (see [BT1, BT2]). This map was proposed by van Strien to Stratmann as a model for an induced map of a Fibonacci unimodal map. Stratmann and Vogt [SV] computed the Hausdorff dimension of points that converge to zero under iteration of it. The map we consider is the following: let $\lambda \in (1/2, 1)$ and consider the partition of the interval $(0, 1]$ given by $\{X_n\}_{n\geq 1}$, where $X_n = (\lambda^n, \lambda^{n-1})$. The map $F_\lambda : (0, 1] \to (0, 1]$ is defined by

$$F_\lambda(x) := \begin{cases} \frac{x-\lambda}{1-\lambda} & \text{if } x \in X_1; \\ \frac{x-\lambda^n}{\lambda(1-\lambda)} & \text{if } x \in X_n, \ n \geq 2. \end{cases}$$

We stress that the phase space is non-compact. Bruin and Todd [BT1] studied the thermodynamic formalism for this map. They showed that even though the map F_λ is expanding and transitive there is dissipation in the system and they where able to quantify it. It is a direct consequence of Theorem 2.3 that the conditional variational principle for quotients of Birkhoff averages holds when restricted to the recurrent set:

Theorem 4.1. *Let $\phi \in \mathcal{R}$ and $\psi \in \mathcal{R}_0$. Then*

$$\dim_H(J_{R,\phi,\psi}(\alpha)) = \sup \left\{ \frac{h(\mu)}{\lambda_{F_\lambda}(\mu)} : \frac{\int \phi \, \mathrm{d}\mu}{\int \psi \, \mathrm{d}\mu} = \alpha \ and \ \mu \in \mathcal{M} \right\}.$$

However, if we consider the whole repeller the situation is more complicated as the following theorem shows,

Theorem 4.2. *Let $\phi : (0, 1] \to \mathbb{R}$ be a Hölder potential such that $\lim_{x\to 0} \phi(x) = a$. The Birkhoff spectrum of ϕ with respect to the dynamical system F_λ satisfies*

1. *If $\alpha = a$ then $\dim_H J_{\phi,1}(\alpha) = 1$.*
2. *If $\alpha \neq a$ then $\dim_H J_{\phi,1}(\alpha) \leq -\frac{\log 4}{\log(\lambda(1-\lambda))}$*

In particular the function $b_{\phi,1}$ is discontinuous at $\alpha = a$. Moreover, the multifractal spectrum $b_{\phi,1}$ in the set $[\alpha_m, \alpha_M] \setminus \{a\}$ satisfies the following conditional variational principle

$$b_{\phi,1}(\alpha) = \sup \left\{ \frac{h(\mu)}{\lambda_{F_\lambda}(\mu)} : \int \phi \, \mathrm{d}\mu = \alpha \ and \ \mu \in \mathcal{M} \right\}.$$

For $\alpha = a$ the function $b_{\phi,1}(\alpha)$ does not satisfy the conditional variational principle.

We therefore exhibit a map for which the Birkhoff spectrum is discontinuous and does not satisfy the conditional variational principle in one point, $\alpha = a$. However it does satisfy it in the complement of the point $\alpha = a$.

In order to prove Theorem 4.2 we first recall the thermodynamic and dimension theoretic description that Bruin and Todd have made of the map F_λ. The *escaping set* of the map F_λ is defined by

$$\Omega_\lambda := \left\{ x \in (0,1] : \lim_{n \to \infty} F_\lambda^n(x) = 0 \right\},$$

and the *hyperbolic dimension* is defined by

$$\dim_{hyp}(F_\lambda) := \sup\{\dim_H \Lambda : \Lambda \subset (0,1] \text{ compact, non-empty and } F_\lambda - \text{invariant}\}. \tag{10}$$

It was proved in [BT1, Theorems A and C] that

Theorem 4.3 (Bruin-Todd). *If $\lambda \in (1/2, 1)$ the for the map F_λ we have*

1. *The Lebesgue measure is dissipative.*
2. *The Hausdorff dimension of the escaping set is given by $\dim_H \Omega_\lambda = 1$.*
3. *The Hausdorff dimension of the recurrent set is given by*

$$\dim_{hyp}(F_\lambda) = \frac{-\log 4}{\log(\lambda(1-\lambda))} < 1.$$

We can now prove Theorem 4.2.

Proof of Theorem 4.2. If $x \in \Omega_\lambda$ then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^\infty \phi(F_\lambda^n x) = a.$$

By Theorem 4.3, $\dim_H \Omega_\lambda = 1$, so $b(a) = 1$. On the other hand, for every $\alpha \neq a$ we have that $J(\alpha) \subset (0,1] \setminus \Omega_\lambda$. A direct consequence of Theorem 4.3 yields

$$b(\alpha) = \dim_H J(\alpha) \leq -\frac{\log 4}{\log(\lambda(1-\lambda))} < 1.$$

Therefore, the multifractal spectrum, $b(\alpha)$, is discontinuous at $\alpha = a$.

Since for every $\mu \in \mathcal{M}$ we have that

$$\dim_H \mu \leq -\frac{\log 4}{\log(\lambda(1-\lambda))} < 1$$

it is clear that the conditional variational principle does no hold for $\alpha = a$. The fact that it does hold in the recurrent set follows from Theorem 4.1. □

4.1. **Lyapunov spectrum.** Perhaps the most important potential to consider is $\phi(x) = \log|F'_\lambda(x)|$. In this context the Birkhoff spectrum is called the *Lyapunov spectrum*. In the example we are considering we can describe in great detail the spectrum. Indeed, we can show that it varies analytically in a half open interval and that it is discontinuous in one point. This is the first example where a discontinuous Lyapunov spectrum for a topologically transitive map has been explicitly calculated that we are aware of. Note that this phenomenon is likely to occur in situations where the hyperbolic dimension is different from the Hausdorff dimension of the repeller, see [SU].

Note that in this case we have that

$$\alpha_m = -\log(1-\lambda) \text{ and } \alpha_M = -\log\lambda(1-\lambda) := a.$$

We also have an explicit form for the pressure of $-t\phi$ given in [BT1] which in particular says that

$$P(-t\phi) = t\log(1-\lambda) - \log(1-\lambda^t) \text{ for } t \geq \frac{-\log 2}{\log\lambda}.$$

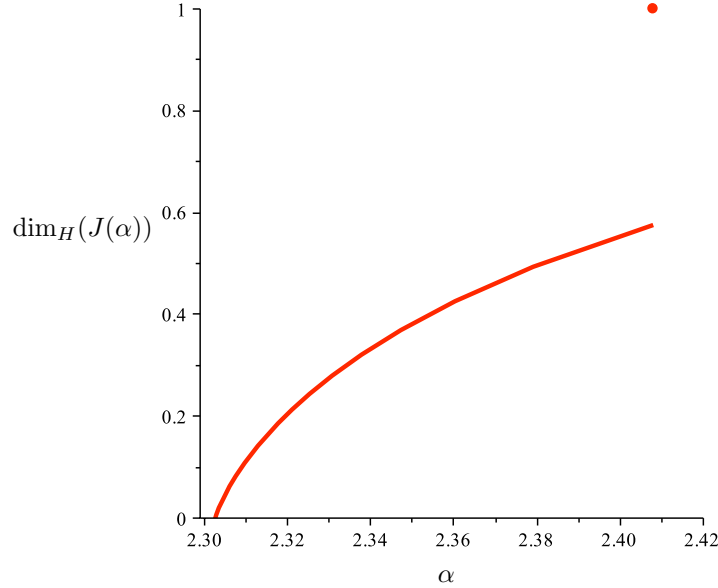This allows us to deduce the following result, see Figure 4.1.

FIGURE 1. Lyapunov spectrum for $\lambda = 0.9$

Proposition 4.4. *Consider the map F_λ for $\lambda \in (\frac{1}{2}, 1)$. Then for any $t > \frac{-\log 2}{\log\lambda}$,*

$$\dim_H J\left(-\log(1-\lambda) - \frac{\lambda^t \log\lambda}{1-\lambda^t}\right) = \frac{t\log(1-\lambda) - \log(1-\lambda^t)}{-\log(1-\lambda) - \frac{\lambda^t \log\lambda}{1-\lambda^t}} + t. \qquad (11)$$

and $dim_H(J(-\log\lambda(1-\lambda))) = 1$. In particular the function $\alpha \to \dim_H J(\alpha)$ is analytic in (α_m, α_M) but discontinuous at α_M.

Proof. Given $t > \frac{-\log 2}{\log \lambda}$, set $\alpha_t := \left(-\log(1-\lambda) - \frac{\lambda^t \log \lambda}{1-\lambda^t} \right)$. Then defining $g : (-\log 2/\log \lambda, \infty) \to \mathbb{R}$ by $g(t) = P(-t\phi)$, we obtain $g'(t) = -\alpha_t$. Moreover by the results in [BT1] it follows that for t in our specified range, the potential $-t\phi$ has an unique equilibrium state μ_t with $\lambda(\mu_t) = \alpha_t$ and $\frac{h(\mu_t)}{\lambda(\mu_t)} = g(t)/\alpha_t + t$. If we let μ be an F_λ invariant measure such that $\lambda(\mu) = \alpha$ then by the Variational Principle, $h(\mu) \leq h(\mu_t)$. Therefore $\frac{h(\mu)}{\lambda(\mu)} \leq g(t)/\alpha_t + t$ and thus $\dim_H(J_R(\alpha)) = V(\alpha) = g(t)/\alpha_t + t$. We next check the range of values of α for which equation (11) holds. Clearly, $\lim_{t \searrow \frac{-\log 2}{\log \lambda}} \alpha_t = \alpha_M$ and $\lim_{t \to \infty} \alpha_t = \alpha_m$, so we have analyticity of $\alpha \mapsto \dim_H J(\alpha)$ on (α_m, α_M). Since $\lambda \neq \frac{1}{2}$ we have

$$\lim_{\alpha \to a} \dim_H J(\alpha) = \left(\frac{\log 2}{\log \lambda} \right) \left(\frac{\log \left(\frac{\lambda}{1-\lambda} \right)}{-\log(\lambda(1-\lambda))} - 1 \right) < 1 = \dim_H J(\alpha_M),$$

so there is a discontinuity at α_M, as claimed. $\square$

REFERENCES

[AL] A. Avila and M. Lyubich, *Hausdorff dimension and conformal measures of Feigenbaum Julia sets.* J. Amer. Math. Soc. 21 (2008), no. 2, 305–363. 1

[Ba] L. Barreira, *Dimension and recurrence in hyperbolic dynamics.* Progress in Mathematics, 272. Birkhauser Verlag, Basel, 2008. 2.3

[BS] L. Barreira and B. Saussol, *Variational principles and mixed multifractal spectra.* Trans. Amer. Math. Soc. 353 (2001), no. 10, 3919–3944. 1, 2.5, 3

[BSc] L. Barreira and J. Schmeling, *Sets of "non-typical" points have full topological entropy and full Hausdorff dimension.* Israel J. Math. **116** (2000), 29–70. 2.5

[BT1] H. Bruin and M. Todd, *Transience and thermodynamic formalism for infinitely branched interval maps.* J. London Math. Soc. **86** (2012), 171–194. 1, 4, 4, 4.1, 4.1

[BT2] H. Bruin and M. Todd, *Wild attractors and thermodynamic formalism.* Preprint, arXiv:1202.1770. 1, 4

[Cl] V. Climenhaga, *The thermodynamic approach to multifractal analysis.* Preprint, arXiv:1102.5427. 1

[C] V. Cyr, *Countable Markov shifts with Transient Potentials.* Proc. London Math. Soc. **103** (2011) 923–949. 1

[CS] V. Cyr and O. Sarig, *Spectral Gap and Transience for Ruelle Operators on Countable Markov Shifts.* Comm. Math. Phys. **292**, 637–666 (2009). 1

[Fa] K. Falconer, *Fractal geometry. Mathematical foundations and applications.* Second edition. John Wiley & Sons, Inc., Hoboken, NJ, 2003. 1, 2.3

[FFW] A. Fan, D. Feng and J. Wu, *Recurrence, dimension and entropy.* J. London Math. Soc. (2) 64 (2001), no. 1, 229–244. 1

[FLW] D. Feng, K-S. Lau and J. Wu, *Ergodic limits on the conformal repellers.* Adv. Math. **169** (2002), no. 1, 58–91. 1

[H] F. Hofbauer, *M*ultifractal spectra of Birkhoff averages for a piecewise monotone interval map. Fund. Math. **208** (2010), no. 2, 95–121. 1, 3

[HR] F. Hofbauer and P. Raith, *The Hausdorff dimension of an ergodic invariant measure for a piecewise monotonic map of the interval.* Canad. Math. Bull. **35** (1992), no. 1, 84–98. 3

[I] G. Iommi, *Multifractal analysis for countable Markov shifts.* Ergodic Theory Dynam. Systems **25** (2005) 1881–1907. 1

[IJ1] G. Iommi and T. Jordan *Multifractal analysis of Birkhoff averages for countable Markov maps.* Preprint, arXiv:1003.2979. 1

[IJ2] G. Iommi and T. Jordan *Multifractal analysis of quotients of Birkhoff sums for countable Markov maps.* Preprint, arXiv:1303.2596. 1

[IJT] G. Iommi, T. Jordan and M. Todd, *Recurrence and transience for suspension flows.* Preprint, arXiv:1209.6542. 2.5

[IT] G. Iommi and M. Todd, *Transience in Dynamical Systems.* 'FirstView' article at Ergodic Theory Dynam. Systems. 1

[JJOP] A. Johansson, T. Jordan, A. Oberg and M. Pollicott, *Multifractal analysis of non-uniformly hyperbolic systems.* Israel J. Math. **177** (2010), 125–144. 1

[M] A. Manning, *A relation between Lyapunov exponents, Hausdorff dimension and entropy.* Ergodic Theory Dynamical Systems **1** (1981), no. 4, 451–459 (1982). 3

[O] L. Olsen, *Multifractal analysis of divergence points of deformed measure theoretical Birkhoff averages.*, J. Math. Pures Appl. (9) **82** (2003), no. 12, 1591–1649. 1

[Pa] S.J. Patterson, *Further remarks on the exponent of convergence of Poincaré series.* Tohoku Math. J. (2) 35 (1983), no. 3, 357–373. 1

[PW] Y. Pesin and H. Weiss, *The multifractal analysis of Birkhoff averages and large deviations.* Global analysis of dynamical systems, 419–431, Inst. Phys., Bristol, 2001. 1

[PU] F. Przytycki, M. Urbański, *Conformal Fractals: Ergodic Theory Methods,* Cambridge University Press 2010. 3

[Sa1] O. Sarig, *Thermodynamic formalism for countable Markov shifts.* Ergodic Theory Dynam. Systems **19** (1999), 1565–1593. 2.1, 2.5, 2.5

[Sa2] O. Sarig, *Phase transitions for countable Markov shifts.* Comm. Math. Phys. **217** (2001), no. 3, 555–577. 1

[SV] B. Stratmann and R. Vogt, *Fractal dimension of dissipative sets.* Nonlinearity **10** (1997) 565–577. 1, 4

[SU] B. Stratmann and M. Urbański, *Pseudo-Markov systems and infinitely generated Schottky groups.* Amer. J. Math. 129 (2007), no. 4, 1019–1062. 1, 4.1

FACULTAD DE MATEMÁTICAS, PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE (PUC), AVENIDA VICUÑA MACKENNA 4860, SANTIAGO, CHILE

E-mail address: giommi@mat.puc.cl

URL: http://www.mat.puc.cl/~giommi/

THE SCHOOL OF MATHEMATICS, THE UNIVERSITY OF BRISTOL, UNIVERSITY WALK, CLIFTON, BRISTOL, BS8 1TW, UK

E-mail address: Thomas.Jordan@bristol.ac.uk

URL: http://www.maths.bris.ac.uk/~matmj/

MIKE TODD, MATHEMATICAL INSTITUTE, UNIVERSITY OF ST ANDREWS, NORTH HAUGH, ST ANDREWS, KY16 9SS, SCOTLAND

E-mail address: mjt20@st-andrews.ac.uk

URL: http://www.mcs.st-and.ac.uk/~miket/