ZERO TEMPERATURE LIMITS OF GIBBS STATES FOR ALMOST-ADDITIVE POTENTIALS

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Abstract. This paper is devoted to study ergodic optimisation problems for almost-additive sequences of functions (rather than a fixed potential) defined over countable Markov shifts (that is a non-compact space). Under certain assumptions we prove that any accumulation point of a family of Gibbs equilibrium measures is a maximising measure. Applications are given in the study of the joint spectral radius and to multifractal analysis of Lyapunov exponent of non-conformal maps.

1. Introduction

In statistical mechanics a very important problem is that of describing how do Gibbs states varies as the temperature changes. Of particular importance is the case when the temperature decreases to zero. Indeed, this case case is related to ground states, that is, measures supported on configurations of minimal energy [EFS, Appendix B.2]. It turns out that materials at low temperature tend to be highly ordered, they might even reach crystal or quasi crystal configurations. Grounds states are the measures that account for this phenomena (see [BLL, Chapter 3] for details). A similar problem, in the context of dynamical systems, has been the subject of great interest over the last years. Indeed, given a dynamical system (Σ, σ) and an observable $\phi : \Sigma \to \mathbb{R}$ we say that a σ-invariant measure μ is a *maximising* measure for ϕ if

$$\int \phi \, d\mu = \sup \left\{ \int \phi \, d\nu : \nu \in \mathcal{M} \right\},$$

where $\mathcal{M}$ denotes the set of $\sigma-$invariant probability measures. In certain cases, some maximising measures can be described as the limit of Gibbs states as the temperature goes to zero. Indeed, assume that (Σ, σ) is a transitive sub-shift of finite type defined over a finite alphabet and that ϕ is a Hölder potential. It is well known that for every $t \in \mathbb{R}$ there exists a unique Gibbs state μ_t (which is also an equilibrium measure) for the potential $t\phi$ (see [Bow1, Theorem 1.2 and 1.22]). It turns out that if μ is any weak-star accumulation point of $\{\mu_t\}_{t>0}$ then μ is a maximsing measure (see, for example, [J1, Section 4]). Note that the value t can be thought of as the inverse of the temperature, hence as the temperature decreases to

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zero the value of t tends to infinity. The theory that studies maximising measures is usually called *ergodic optimisation*. See [Bo, J1, J2] for more details.

The purpose of the present paper is to study a similar problem in the context of countable Markov shifts and for sequences of potentials. To be more precise, let (Σ, σ) be a countable Markov shift and let $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ an almost-additive sequence of continuous functions $\log f_n : \Sigma \to \mathbb{R}$ (see Section 2 for precise definitions). We say that a σ-invariant measure μ is a *maximising* measure for $\mathcal{F}$ if

$$\lim_{n\to\infty} \frac{1}{n} \int \log f_n \, d\mu = \sup \left\{ \lim_{n\to\infty} \frac{1}{n} \int \log f_n \, d\nu : \nu \in \mathcal{M} \right\}.$$

In our main result, Theorem 2.2, we prove that under certain assumptions there exists a sequence of Gibbs states $\{\mu_t\}_{t>0}$ corresponding to $t\mathcal{F}$ and that this sequences has an accumulation point μ. Moreover, the measure μ is maximising for $\mathcal{F}$. We stress that since the space Σ is not compact the space $\mathcal{M}$ is not compact either. Therefore, the existence of an accumulation point is far from trivial. A similar problem, in the case of a single function instead of a sequence $\mathcal{F}$, was first studied by Jenkinson, Mauldin and Urbański [JMU1] (see also [BGa, BF, I]).

Note that even though we prove the existence of an accumulation point for the sequence $\{\mu_t\}_{t>0}$, this does not imply that the limit $\lim_{t\to\infty} \mu_t$ exists. Actually, in the simpler setting of compact sub-shifts of finite type and Hölder potentials, Hochman and Chazottes [ChH] constructed an example where there is no convergence. However, under certain finite range assumptions convergence has been proved in [Br, L, ChGU].

The thermodynamic formalism needed in the context of almost-additive sequences for countable Markov shifts was developed in [IY]. In particular, the existence of Gibbs states was established in [IY, Theorem 4.1]. New results in this direction are obtained in Section 3, where we establish conditions that ensure that certain Gibbs states are actually equilibrium measures (recall that in this non-compact setting this is not always the case, see [S2, p.1757]).

We stress that our formalism allows us to deal with products of matrices and it is well suited for applications. In particular, in Section 5, applications of our results are given to the study of the joint spectral radius of a set of matrices. We construct an invariant measure that realises the joint spectral radius. Moreover, we construct a sequence of Gibbs states that can be used to approximate the value of the joint spectral radius. It should be pointed out that these results are new even in the case of finitely many matrices. In the same spirit, Section 6 is devoted to another application of our results in the setting of multifractal analysis. For non-conformal dynamical systems defined on the plane, we obtain the upper bound for the Lyapunov spectrum. Moreover, we construct a measure supported on the maximal level set.

2. Preliminaries

In this section, we give a brief overview of recent results of thermodynamic formalism for almost-additive sequences on countable Markov shifts. We collect results mostly from [IY].

Let (Σ, σ) be a one-sided Markov shift over a countable alphabet S. This means that there exists a matrix $(t_{ij})_{S\times S}$ of zeros and ones (with no row and no column made entirely of zeros) such that

$$\Sigma = \left\{ x \in S^{\mathbb{N}} : t_{x_i x_{i+1}} = 1 \text{ for every } i \in \mathbb{N} \right\}.$$

The *shift map* $\sigma : \Sigma \to \Sigma$ is defined by $\sigma(x_1 x_2 \dots) = (x_2 x_3 \dots)$. Sometimes we simply say that (Σ, σ) is a *countable Markov shift.* The set

$$C_{i_1 \cdots i_n} = \{ x \in \Sigma : x_j = i_j \text{ for } 1 \leq j \leq n \}.$$

is called *cylinder* of length n. The space Σ endowed with the topology generated by cylinder sets is a non-compact space. We denote by $\mathcal{M}$ the set of σ-invariant Borel probability measures on Σ. We will always assume (Σ, σ) to be topologically mixing, that is, for every $a, b \in S$ there exists $N_{ab} \in \mathbb{N}$ such that for every $n > N_{ab}$ we have $C_a \cap \sigma^{-n} C_b \neq \emptyset$.

Definition 2.1. Let (Σ, σ) be a one-sided countable state Markov shift. For each $n \in \mathbb{N}$, let $f_n : \Sigma \to \mathbb{R}^+$ be a continuous function. A sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ on Σ is called *almost-additive* if there exists a constant $C \geq 0$ such that for every $n, m \in \mathbb{N}, x \in \Sigma$, we have

$$f_n(x) f_m(\sigma^n x) e^{-C} \leq f_{n+m}(x), \tag{1}$$

and

$$f_{n+m}(x) \leq f_n(x) f_m(\sigma^n x) e^{C}. \tag{2}$$

Throughout this paper, we will assume the sequence $\mathcal{F}$ to be almost-additive. We also assume the following regularity condition.

Definition 2.2. Let (Σ, σ) be a one-sided countable Markov shift. For each $n \in \mathbb{N}$, let $f_n : \Sigma \to \mathbb{R}^+$ be continuous. A sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ on Σ is called a *Bowen* sequence if there exists $M \in \mathbb{R}^+$ such that

$$\sup\{A_n : n \in \mathbb{N}\} \leq M, \tag{3}$$

where

$$A_n = \sup \left\{ \frac{f_n(x)}{f_n(y)} : x, y \in \Sigma, x_i = y_i \text{ for } 1 \leq i \leq n \right\}.$$

In [IY] thermodynamic formalism was developed for almost-additive Bowen sequences. The following definition of pressure is a generalisation of the one given by Sarig [S1] to the case of almost-additive sequences.

Definition 2.3. Let $\mathcal{F} = \{\log f_n\}_{n \in \mathbb{N}}$ be an almost-additive Bowen sequence, the *Gurevich pressure* of $\mathcal{F}$, denoted by $P(\mathcal{F})$, is defined by

$$P(\mathcal{F}) = \lim_{n \to \infty} \frac{1}{n} \log \left(\sum_{\sigma^n x = x} f_n(x) \chi_{C_a}(x) \right),$$

where $\chi_{C_a}(x)$ is the characteristic function of the cylinder C_a.

Let us stress that the limit always exists and does not depend on the choice of the cylinder. Note that if $f : \Sigma \to \mathbb{R}$ is a continuous function, the sequence of Birkhoff sums of f form an almost additive sequence (in this case the constant C in definition 2.1 is equal to zero). For every $n \in \mathbb{N}, x \in \Sigma$, define $f_n : \Sigma \to \mathbb{R}^+$ by $f_n(x) = e^{f(x) + f(\sigma x) + \cdots + f(\sigma^{n-1} x)}$. Then the sequence $\{\log f_n\}_{n=1}^{\infty}$ is additive. This remark is the link that ties up the thermodynamic formalism for a continuous function with that of sequences of continuous functions. Therefore, the definition of pressure given in definition 2.3 generalises that of Gurevich pressure given by Sarig [S1]. Also, we note that $\lim_{n\to\infty} \frac{1}{n}\int \log f_n \, d\mu = \int f \, d\mu$ for any $\mu \in \mathcal{M}$. Therefore, the next theorem is a generalisation of the variational principle for continuous functions to the setting of almost-additive sequence of continuous functions. It was proved in [IY, Theorem 3.1]. In order to state it we need the following definition, given $f : \Sigma \to \mathbb{R}$ a continuous function, the *transfer operator* L_f applied to function $g : \Sigma \to \mathbb{R}$ is formally defined by

$$(L_f g)(x) := \sum_{\sigma z = x} f(z) g(z) \quad \text{ for every } x \in \Sigma.$$

Theorem 2.1. *Let (Σ, σ) be a topologically mixing countable state Markov shift and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be an almost-additive Bowen sequence on Σ with $||L_{f_1} 1||_{\infty} < \infty$. Then $-\infty < P(\mathcal{F}) < \infty$ and*

$$\begin{aligned} P(\mathcal{F}) &= \sup\left\{ h(\mu) + \lim_{n\to\infty} \frac{1}{n} \int \log f_n \, d\mu : \mu \in \mathcal{M} \text{ and } \lim_{n\to\infty} \frac{1}{n} \int \log f_n \, d\mu \neq -\infty \right\} \\ &= \sup\left\{ h(\mu) + \int \lim_{n\to\infty} \frac{1}{n} \log f_n \, d\mu : \mu \in \mathcal{M} \text{ and } \int \lim_{n\to\infty} \frac{1}{n} \log f_n \, d\mu \neq -\infty \right\}. \end{aligned}$$

A measure $\mu \in \mathcal{M}$ is said to be an *equilibrium measure* for $\mathcal{F}$ if

$$P(\mathcal{F}) = h(\mu) + \lim_{n\to\infty} \frac{1}{n} \int \log f_n \, d\mu.$$

In [B1, M], the notion of Gibbs state for continuous functions was extended to the case of almost-additive sequences.

Definition 2.4. Let (Σ, σ) be a topologically mixing countable state Markov shift and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be an almost-additive sequence on Σ. A measure $\mu \in \mathcal{M}$ is said to be a *Gibbs* state for $\mathcal{F}$ if there exist constants $C_0 > 0$ and $P \in \mathbb{R}$ such that for every $n \in \mathbb{N}$ and every $x \in C_{i_1 \dots i_n}$ we have

$$\frac{1}{C_0} \leq \frac{\mu(C_{i_1 \dots i_n})}{\exp(-nP) f_n(x)} \leq C_0. \tag{4}$$

In the case of a single continuous function defined over a countable Markov shift, there exists a combinatorial obstruction on Σ that prevent the existence of Gibbs states (see [S2]). This, of course, is also the case in the setting of almost additive sequences. The combinatorial condition on Σ is the following

Definition 2.5. A countable Markov shift (Σ, σ) is said to satisfy the *big images and preimages property (BIP property)* if there exists $\{b_1, b_2, \dots, b_n\}$ in the alphabet S such that

$$\forall a \in S \ \exists i, j \text{ such that } t_{b_i a} t_{a b_j} = 1.$$

In [IY, Theorem 4.1], the existence of Gibbs states for an almost-additive sequence of continuous functions defined over a BIP shift Σ was established. Moreover, under a finite entropy assumption it was shown that this Gibbs state is also an equilibrium measure.

Theorem 2.2. *Let (Σ, σ) be a topologically mixing countable state Markov shift with the BIP property. Let $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be an almost-additive Bowen sequence defined on Σ. Assume that $\sum_{a\in S} \sup f_1|_{C_a} < \infty$. Then there exits a unique invariant Gibbs state μ for $\mathcal{F}$ and it is mixing. Moreover, If $h(\mu) < \infty$, then it is the unique equilibrium measure for $\mathcal{F}$.*

Remark 2.1. Note that $\sum_{a\in S}\sup f_1|_{C_a}<\infty$ implies that $-\infty<P(\mathcal{F})<\infty$.

3. Existence of Gibbs equilibrium states

In Theorem 2.2 we established conditions under which the existence of a Gibbs state μ for an almost-additive sequence $\mathcal{F}=\{\log f_n\}_{n=1}^{\infty}$ is guaranteed. It might happen that $h(\mu)=\infty$ and that $\lim_{n\to\infty}\frac{1}{n}\int\log f_n\ d\mu=-\infty$. In this case, since the sum of these two quantities is meaningless, we don't say that the measure μ is an equilibrium measure. However, if $h(\mu)<\infty$ then μ is indeed an equilibrium measure. The purpose of the following section is to establish other conditions that would also imply that the Gibbs state μ is an equilibrium measure. Our result could be compared with those in [MU1], where a single function (instead of an almost-additive sequence) is studied.

Proposition 3.1. *Let (Σ,σ) be a topologically mixing countable state Markov shift with the BIP property. Let $\mathcal{F}=\{\log f_n\}_{n=1}^{\infty}$ be an almost-additive Bowen defined on Σ satisfying $\sum_{i\in\mathbb{N}}\sup f_1|_{C_i}<\infty$ and $\mu_{\mathcal{F}}$ be the unique invariant Gibbs state for $\mathcal{F}$. The followings statements are equivalent:*

(1) $h_{\mu_{\mathcal{F}}}(\sigma)<\infty$,
(2) $\lim_{n\to\infty}\frac{1}{n}\int\log f_n d\mu_{\mathcal{F}}>-\infty$,
(3) $\int\log f_1 d\mu_{\mathcal{F}}>-\infty$,
(4) $\sum_{i=1}^{\infty}\sup\{\log f_1(x):x\in C_i\}\sup\{f_1(x):x\in C_i\}>-\infty$.

Therefore, if one of the above is satisfied, then $\mu_{\mathcal{F}}$ is the unique Gibbs equilibrium state for $\mathcal{F}$.

Proof. Suppose we have (1), it follows from Theorem 2.2 that $\mu_{\mathcal{F}}$ is the unique Gibbs equilibrium state for $\mathcal{F}$, thus (1) implies (2). Assume now (2). Since $\sum_{i\in\mathbb{N}}\sup f_1|_{C_i}<\infty$ implies $-\infty<P(\mathcal{F})<\infty$, it is a direct consequence of the variational principle that $h_{\mu_{\mathcal{F}}}(\sigma)<\infty$. Thus (2) implies (1). Next we show that (2) if and only if (3). It directly follows form Definition 2.1 (equations (1) and (2)), that

$$-\frac{n-1}{n}C+\int\log f_1(x)d\mu_{\mathcal{F}}\leq\frac{1}{n}\int\log f_n d\mu_{\mathcal{F}}\leq\frac{n-1}{n}C+\int\log f_1 d\mu_{\mathcal{F}}.$$

Letting $n\to\infty$, we obtain the result. Now we show that (3) if and only if (4). Fix $i\in\mathbb{N}$. Since $\mathcal{F}$ is a Bowen sequence on Σ, there exists M such that

$$\sup\left\{\frac{f_1(x)}{f_1(y)}:x_1=y_1=i\right\}\leq M.$$

This implies that $-\log M\leq\log f_1(x)-\log f_1(y)\leq\log M$ for any $x,y\in C_i$. Therefore,

$$\sup\left\{\log\frac{f_1(x)}{M}:x\in C_i\right\}\leq\inf\left\{\log f_1(x):x\in C_i\right\}.$$

Let x_i be an arbitrary point in C_i , then

$$\begin{aligned}\int \log f_1 d\mu_{\mathcal{F}} &= \sum_{i=1}^{\infty} \int_{C_i} \log f_1 d\mu_{\mathcal{F}} \\ &\geq \sum_{i=1}^{\infty} \inf \{\log f_1(x) : x \in C_i\} \, e^{-P(\mathcal{F})} \frac{f_1(x_i)}{C_0} \\ &\geq \sum_{i=1}^{\infty} \sup \left\{\log \frac{f_1(x)}{M} : x \in C_i\right\} e^{-P(\mathcal{F})} \frac{f_1(x_i)}{C_0},\end{aligned}$$

where in the second inequality we used the definition of Gibbs sate. Taking the supremun over $x_i \in C_i$, we have

$$\int \log f_1 d\mu_{\mathcal{F}} \geq \frac{e^{-P(\mathcal{F})}}{C_0} \sum_{i=1}^{\infty} \sup \left\{\log \frac{f_1(x)}{M} : x \in C_i\right\} \sup\{f_1(x) : x \in C_i\}.$$

Therefore, (4) implies (3). To see that (3) implies (4), consider

$$\begin{aligned}\int \log f_1 d\mu_{\mathcal{F}} &= \sum_{i=1}^{\infty} \int_{C_i} \log f_1 d\mu_{\mathcal{F}} \\ &\leq e^{-P(\mathcal{F})} C_0 \sum_{i=1} \sup\{\log f_1(x) : x \in C_i\} f_1(x_i) \\ &\leq e^{-P(\mathcal{F})} C_0 \sum_{i=1} \sup\{\log f_1(x) : x \in C_i\} \sup\{f_1(x) : x \in C_i\}.\end{aligned}$$

Therefore, the desired result is obtained. □

The rest of the section is devoted to study the relation between the Gibbs equilibrium state for $\log f_1$ on Σ and that for $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ on Σ. We begin establishing conditions on a continuous function $\log f_1$ so that it has a Gibbs equilibrium state (see [MU1, S1, S2] for related work).

Proposition 3.2. *Let (Σ, σ) be a topologically mixing countable Markov shift with the BIP property. Suppose that $\log f_1$ is a continuous function on Σ satisfying:*

1. $\sup_{n\in\mathbb{N}} \left\{ \frac{f_1(x)f_1(\sigma x)\dots f_1(\sigma^{n-1}x)}{f_1(y)f_1(\sigma x)\dots f_1(\sigma^{n-1}y)} : x_i = y_i, 1 \leq i \leq n \right\} < \infty,$
2. $\sum_{i\in\mathbb{N}} e^{\sup \log f_1|_{C_i}} < \infty.$

Then there exists a unique invariant Gibbs measure $\mu_{\log f_1}$ for $\log f_1$. Moreover,

1. *If $\int \log f_1 d\mu_{\log f_1} > -\infty$, then $\mu_{\log f_1}$ is the unique Gibbs equilibrium state for $\log f_1$.*
2. *$\int \log f_1 d\mu_{\log f_1} > -\infty$ if and only if $\sum_{i=1}^{\infty} \sup\{\log f_1(x) : x \in C_i\} \sup\{f_1(x) : x \in C_i\} > -\infty$.*

Proof. Let $g(x) = \log f_1(x)$ and $g_n(x) = e^{g(x)+g(\sigma x)+\dots+g(\sigma^{n-1}x)}$. Define $\Phi = \{\log g_n\}_{n=1}^{\infty}$. We have that Φ is an additive Bowen sequence. Indeed, we only need to prove that Φ is a Bowen sequence. In order to do so, we note that for each $n \in \mathbb{N}$ we have

$$\sup \left\{ \frac{g_n(x)}{g_n(y)} : x_i = y_i, 1 \leq i \leq n \right\} =$$

$$\sup \left\{ \frac{f_1(x)f_1(\sigma x)\dots f_1(\sigma^{n-1}x)}{f_1(y)f_1(\sigma x)\dots f_1(\sigma^{n-1}y)} : x_i = y_i, 1 \leq i \leq n \right\}.$$

The claim now follows from assumption (5). It is a direct consequence of assumption (7) that $-\infty < P(\Phi) < \infty$. Therefore, by Theorem 2.2, there exists a unique invariant Gibbs state μ_Φ for Φ. Since $P(\Phi) = P(\log f_1)$ we have that μ_Φ is the unique invariant Gibbs state for $\log f_1$. Moreover, since for any $\mu \in \mathcal{M}$ we have that $\lim_{n\to\infty} \frac{1}{n} \int \log g_n d\mu = \int \log f_1 d\mu$, then if $\int \log f_1 d\mu_\Phi > -\infty$, it follows that μ_Φ is the unique Gibbs equilibrium state for $\log f_1$. This proves the first part of the Proposition. The second claim is proved in the exact same way as the corresponding one in Proposition 3.1. $\square$

Corollary 3.1. *Let (Σ, σ) be a topologically mixing countable Markov shift with the BIP property. Suppose that the function $\log f_1 : \Sigma \to \mathbb{R}$ is of summable variation and satisfies $\sum_{i\in\mathbb{N}} e^{\sup \log f_1|_{C_i}} < \infty$. Then,*

$$\sup_{n\in\mathbb{N}} \left\{ \frac{f_1(x)f_1(\sigma x)\dots f_1(\sigma^{n-1}x)}{f_1(y)f_1(\sigma x)\dots f_1(\sigma^{n-1}y)} : x_i = y_i, 1 \le i \le n \right\} < \infty$$

and

1. *If $\int \log f_1 d\mu_{\log f_1} > -\infty$, then $\mu_{\log f_1}$ is the unique Gibbs equilibrium state for $\log f_1$.*
2. *$\int \log f_1 d\mu_{\log f_1} > -\infty$ if and only if $\sum_{i=1}^{\infty} \sup\{\log f_1(x) : x \in C_i\} \sup\{f_1(x) : x \in C_i\} > -\infty$.*

Proof. Since $\log f_1$ is summable variation, there exists $N \in \mathbb{R}$ such that

$$\sum_{n=1}^{\infty} \sup \left\{ \left| \log \frac{f_1(x)}{f_1(y)} \right| : x_i = y_i, 1 \le i \le n \right\} \le N.$$

Therefore, for $x, y \in \Sigma$ such that $x_i = y_i$, $1 \le i \le n$ we have that

$$\begin{aligned}
\log \frac{f_1(x)f_1(\sigma x)\dots f_1(\sigma^{n-1}x)}{f_1(y)f_1(\sigma y)\dots f_1(\sigma^{n-1}y)} &\le \\
\sup \left\{ \log \frac{f_1(x)}{f_1(y)} : x_i = y_i, 1 \le i \le n \right\} &+ \\
\sup \left\{ \log \frac{f_1(\sigma x)}{f_1(\sigma y)} : (\sigma x)_i = (\sigma y)_i, 1 \le i \le n-1 \right\} + &\dots \\
+ \dots + \sup \left\{ \log \frac{f_1(\sigma^{n-1}x)}{f_1(\sigma^{n-1}y)} : (\sigma^{n-1}x)_i = (\sigma^{n-1}y)_i, i = 1 \right\} &\le N.
\end{aligned}$$

Then

$$\sup_{n\in\mathbb{N}} \left\{ \frac{f_1(x)f_1(\sigma x)\dots f_1(\sigma^{n-1}x)}{f_1(y)f_1(\sigma y)\dots f_1(\sigma^{n-1}y)} : x_i = y_i, 1 \le i \le n \right\} \le e^N,$$

and the result follows. $\square$

The next theorem characterises the existence of a Gibbs equilibrium state for $\mathcal{F}$ in terms of the existence of the Gibbs equilibrium state for $\log f_1$.

Theorem 3.1. *Let Σ be a topologically mixing countable Markov shift with the BIP property. Let $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be an almost-additive Bowen sequence on Σ, with $\sum_{i\in\mathbb{N}} \sup f_1|_{C_i} < \infty$ and*

$$\sup_{n\in\mathbb{N}} \left\{ \frac{f_1(x)f_1(\sigma x)\dots f_1(\sigma^{n-1}x)}{f_1(y)f_1(\sigma x)\dots f_1(\sigma^{n-1}y)} : x_i = y_i, 1 \le i \le n \right\} < \infty.$$

Then $\mathcal{F}$ has a unique Gibbs equilibrium state $\mu_{\mathcal{F}}$ if and only if $\log f_1$ has a unique Gibbs equilibrium state $\mu_{\log f_1}$.

Proof. To begin with, note that $P(\log f_1) < \infty$ and that the assumptions made on f_1 together with Proposition 3.2 imply that there exists a Gibbs state $\mu_{\log f_1}$ for $\log f_1$.

Let us first assume that there exists a unique Gibbs equilibrium state, $\mu_{\mathcal{F}}$, for $\mathcal{F}$ and prove that $\mu_{\log f_1}$ is an equilibrium measure for $\log f_1$. Since $\mu_{\mathcal{F}}$ is a Gibbs equilibrium state we have that $\lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu_{\mathcal{F}} > -\infty$. Proposition 3.1 then implies that

$$\sum_{i=1}^{\infty} \sup\{\log f_1(x) : x \in C_i\} \sup\{f_1(x) : x \in C_i\} > -\infty.$$

Thus, there exists constants $C_0', M' \in \mathbb{R}^+$ such that

$$\int \log f_1 d\mu_{\log f_1} \geq \frac{e^{-P(\log f_1)}}{C_0'} \sum_{i=1}^{\infty} \sup\left\{\log \frac{f_1(x)}{M'} : x \in C_i\right\} \sup\{f_1(x) : x \in C_i\}.$$

Therefore, $\int \log f_1 d\mu_{\log f_1} > -\infty$ and we can conclude that $\mu_{\log f_1}$ is the unique Gibbs equilibrium state for $\log f_1$.

Conversely, suppose $\log f_1$ has a unique Gibbs equilibrium state $\mu_{\log f_1}$. Then $\int \log f_1 d\mu_{\log f_1} > -\infty$ and Proposition 3.2 implies that

$$\sum_{i=1}^{\infty} \sup\{\log f_1(x) : x \in C_i\} \sup\{f_1(x) : x \in C_i\} > -\infty.$$

Therefore, applying Theorem 2.2 and Proposition 3.1 we obtain that $\mu_{\mathcal{F}}$ is the unique Gibbs equilibrium state for $\mathcal{F}$. □

Remark 3.1. A similar result to that in Corollary 3.1 but under different regularity assumptions was proved in [MU1].

Corollary 3.2. *Let Σ be a topologically mixing countable Markov shift with the BIP property. Let $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be an almost-additive Bowen sequence on X, where $\log f_1$ is summable variation that satisfies $\sum_{i\in\mathbb{N}} \sup f_1|_{C_i} < \infty$. Then $\mathcal{F}$ has a unique Gibbs equilibrium state $\mu_{\mathcal{F}}$ if and only if $\log f_1$ has a unique Gibbs equilibrium state $\mu_{\log f_1}$.*

Proof. The result immediately follows from Proposition 3.2 and Theorem3.1. □

4. Zero temperature limits of Gibbs equilibrium states

This section is devoted to state and prove our main result. We prove that for a certain class of almost-additive potentials we can associate a family of Gibbs equilibrium measures and that this sequence has, at least, one accumulation point. It turns out that this measure is a maximising one. This result generalise the zero temperature limit theorems obtained for a single function in the compact (see [J1, Section 4]) and in the non-compact settings (see [JMU1, BGa, BF, I]). The major difficulty we have to face in this context is that the space of invariant probability measures is not compact, hence the existence of an accumulation point is far from trivial. The techniques we use in the proof are inspired in results by Jenkinson, Mauldin and Urbański [JMU1]. We begin by defining the class of sequence of potentials that we will be interested in.

Definition 4.1. Let (Σ, σ) be a countable Markov shift satisfying the BIP property. A sequence of continuous functions $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$, with $f_i : \Sigma \to \mathbb{R}$, belongs the class $\mathcal{R}$ if it satisfies the following properties:

(1) The sequence $\mathcal{F}$ is almost-additive, Bowen and $\sum_{i\in\mathbb{N}} \sup f_1|_{C_i} < \infty$.
(2) $\sum_{i=1}^{\infty} \sup\{\log f_1(x) : x \in C_i\} \sup\{f_1(x) : x \in C_i\} > -\infty$.

Remark 4.1. Note that it follows from Theorem 2.2 that if $\mathcal{F} \in \mathcal{R}$ then there exists a unique invariant Gibbs measure $\mu_{\mathcal{F}}$ for $\mathcal{F}$. In particular, using Proposition 3.1 and (2) in Definition 4.1, for every $t \in \mathbb{R}$, there exists a unique invariant Gibbs measure $\mu_{t\mathcal{F}}$ for $t\mathcal{F}$.

We begin proving the upper semi-continuity of the limit of the integrals. This is an essential result and it holds under weaker assumptions than those considered in the definition of the class $\mathcal{R}$.

Lemma 4.1. *Let (Σ, σ) be a countable Markov shift and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ an almost-additive sequence of continuous functions on Σ with $\sup f_1 < \infty$. Then the map $m : \mathcal{M} \to \mathbb{R}$ defined by $m(\mu) = \lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu$ is upper semi continuous.*

Proof. We use an argument similar to that in the proof of [Y, Proposition 3.7]. Let $\{\mu_i\}_{n=1}^{\infty}$ be a sequence of measures in $\mathcal{M}$ which converge to the measure $\mu \in \mathcal{M}$ in the weak* topology.

Let $M_1 \in \mathbb{R}$ be such that $\sup f_1 \leq M_1$ and $C \in \mathbb{R}$ the almost-additive constant that appear in equations (1) and (2) in Definition 2.1. Let $g_n(x) = f_n(x)e^C$ and note that $\{\log g_n\}_{n=1}^{\infty}$ is a sub-additive sequence. Moreover, $(\log g_1(x))^+ \leq \log M_1 + \log e^C$ and so $(\log g_1)^+ \in L_1(\mu_i)$ for each $i \in \mathbb{N}$. Also, for each fixed $n \in \mathbb{N}$ we have that $(\log g_n(x))^+ \leq n \log M_1 + nC < \infty$ which implies that for each $i \in \mathbb{N}$ we have $(\log g_n)^+ \in L_1(\mu_i)$. The sub-additive ergodic theorem (see [W, Theorem 10.1]) implies that for each fixed $\mu_i \in \mathcal{M}$ and $k \in \mathbb{N}$ we have,

$$\lim_{n\to\infty} \frac{1}{n} \int \log g_n d\mu_i \leq \frac{1}{k} \int \log g_k d\mu_i.$$

Thus,

$$\lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu_i \leq \frac{1}{k} \int \log f_k d\mu_i + \frac{C}{k}.$$

Therefore,

$$\limsup_{i\to\infty} \lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu_i \leq \frac{1}{k} \int \log f_k d\mu + \frac{C}{k}.$$

Letting $k \to \infty$, we obtain

$$\limsup_{i\to\infty} \lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu_i \leq \lim_{k\to\infty} \frac{1}{k} \int \log f_k d\mu,$$

which shows that $\limsup_{i\to\infty} m(\mu_i) \leq m(\mu)$, concluding the proof. □

In our next Lemma we we consider a one parameter family of Gibbs measures μ_t corresponding to a family $t\mathcal{F}$. We show how do the constants in the Gibbs property depend upon the parameter t. The proof of the Lemma strongly uses the combinatorial assumptions we made on the system and an approximation argument we now describe. Since (Σ, σ) is topologically mixing and has the BIP property, there exist $k \in \mathbb{N}$ and a finite collection W of admissible words of length k such that for any $a, b \in S$, there exists $w \in W$ such that awb is admissible (see [S2, p.1752] and [MU1]). Denote by A the transition matrix for Σ. It is known (see [IY, S1]) that rearranging the set $\mathbb{N}$, there is an increasing sequence $\{l_n\}_{n=1}^{\infty}$ such that the matrix $A|_{\{1,\dots,l_n\}\times\{1,\dots,l_n\}}$ is primitive (for a definition of primitive see [MU2, p.5]). Let Y_{l_n} be the topologically mixing finite state Markov shift with the transition matrix $A|_{\{1,\dots,l_n\}\times\{1,\dots,l_n\}}$. Then there exists $p \in \mathbb{N}$ such that for all $n \geq p$, the set Y_{l_n} contains all admissible words in W. We denote by $B_n(Y_l)$ the set of admissible words of length n in Y_l. Given $w \in W$ we let $N_w = \sup\{f_k(z) : z \in C_w\}$ and $\bar{N} = \min\{N_w : w \in W\}$. The proof of the next Lemma makes use of ideas from [IY, Claim 4.1].

Lemma 4.2. *Let (Σ, σ) be a countable Markov shift with the BIP property and $\mathcal{F} \in \mathcal{R}$. Let $t \in \mathbb{R}$ and $\mu_{t\mathcal{F}}$ be the unique invariant Gibbs measure for $t\mathcal{F}$. Then, for every $n \in \mathbb{N}$ and for all $x \in C_{i_1\dots i_n}$, we have*

$$\frac{\mu_{t\mathcal{F}}(C_{i_1\dots i_n})}{e^{-nP(t\mathcal{F})}f_n^t(x)} \leq \left(\frac{Me^{6C}}{D^5}\right)^t,$$

where $D = \bar{N}e^{-3C}/(M^3 e^{(k-1)C}(\sum_{i\in\mathbb{N}} f_1|C_i)^k)$.

Proof. Fix $t \in \mathbb{N}$ and Y_{l_m} with $m \geq p$. In order to simplify the notation we denote Y_{l_m} by Y. Recall that Y is a compact set. Define $\alpha_{n,t}^Y := \sum_{i_1\cdots i_n \in B_n(Y)} \sup\{f_n^t|_Y(y) : y \in C_{i_1\dots i_n}\}$. For $l \in \mathbb{N}$, let $\nu_{l,t}$ be the Borel probability measure on Y defined by

$$\nu_{l,t}(C_{i_1\dots i_l}) = \frac{\sup\{f_l^t|_Y(y) : y \in C_{i_1\dots i_l}\}}{\alpha_{l,t}^Y}.$$

Let $a_{i_1\dots i_l} := \sup\{f_l^t|_Y(y) : y \in C_{i_1\dots i_l}\}$. Let $n \in \mathbb{N}$ and $l \geq n+k$. Using the same arguments used to prove equations (15), (16), (17) of [IY, Claim 4.1], for each fixed $i_1 \dots i_n \in B_n(Y)$, we have

$$\sum_{t_1\dots t_l} \sup\{f_{n+l}^t|_Y(y) : y \in C_{i_1\dots i_n t_1\dots t_l}\} \geq \frac{\bar{N}^t e^{-2Ct}}{M^{3t}} a_{i_1\dots i_n}\alpha_{l-k,t}^Y. \tag{5}$$

It is easy to see that

$$\alpha_{l,t}^Y \leq e^{Ct}\alpha_{n,t}^Y\alpha_{l-n,t}^Y. \tag{6}$$

Thus we obtain

$$\alpha_{n+l,t}^Y \geq \frac{\bar{N}^t e^{-3Ct}\alpha_{n,t}^Y\alpha_{l,t}^Y}{M^{3t}\alpha_{k,t}^Y}. \tag{7}$$

Also, from the proof of [IY, Claim 4.1], we have that

$$\alpha_{k,t}^Y \leq e^{(k-1)Ct}\left(\sum_{i\in\mathbb{N}} f_1|C_i\right)^{tk}. \tag{8}$$

Using (5), (7) and (8), we obtain

$$\alpha_{n+l,t}^Y \geq D^t\alpha_{n,t}^Y\alpha_{l,t}^Y. \tag{9}$$

Therefore, again using the arguments in the proof of [IY, Claim 4.1], we obtain

$$D^t\alpha_{n,t}^Y \leq e^{nP(t\mathcal{F}|_Y)} \leq e^{Ct}\alpha_{n,t}^Y. \tag{10}$$

Now, for each fixed $i_1 \dots i_n \in B_n(Y)$, we have

$$
\begin{aligned}
&\nu_{l,t}(C_{i_1\dots i_n})\\
&\leq \sum_{j_1\dots j_{l-n}} \frac{\sup\{f_l^t|_Y(y) : y \in C_{i_1\dots i_n j_1\dots j_{l-n}}\}}{\alpha_{l,t}^Y}\\
&\leq \sum_{j_1\dots j_{l-n}} \frac{e^{Ct}\sup\{f_n^t|_Y(y) f_{l-n}^t|_Y(\sigma^n y) : y \in C_{i_1\dots i_n j_1\dots j_{l-n}}\}}{\alpha_{l,t}^Y}\\
&\leq e^{Ct} \sum_{j_1\dots j_{l-n}} \frac{\sup\{f_n^t|_Y(y) : y \in C_{i_1\dots i_n}\}\sup\{f_{l-n}^t|_Y(\sigma^n y) : y \in C_{i_1\dots i_n j_1\dots j_{l-n}}\}}{\alpha_{l,t}^Y}\\
&\leq \frac{e^{Ct}}{\alpha_{l,t}^Y}\sup\{f_n^t|_Y(y) : y \in C_{i_1\dots i_n}\} \sum_{j_1\dots j_{l-n}} \sup\{f_{l-n}^t|_Y(y) : y \in C_{j_1\dots\dots j_{l-n}}\}\\
&\leq \frac{e^{Ct}}{D^t\alpha_{n,t}^Y}\sup\{f_n^t|_Y(y) : y \in C_{i_1\dots i_n}\} \text{ (by (9))}\\
&\leq \frac{e^{2Ct}e^{-nP(t\mathcal{F}|_Y)}}{D^t}\sup\{f_n^t|_Y(y) : y \in C_{i_1\dots i_n}\} \text{ (by (10))}.
\end{aligned}
$$

Therefore, we obtain

$$
\frac{\nu_{l,t}(C_{i_1\dots i_n})}{a_{i_1\dots i_n}e^{-nP(t\mathcal{F}|_Y)}} \leq \frac{e^{2Ct}}{D^t}. \tag{11}
$$

Using the property of bounded variation, for all $y \in C_{i_1\dots i_n}$, we have

$$
\frac{\nu_{l,t}(C_{i_1\dots i_n})}{f_n^t|_Y(y)e^{-nP(t\mathcal{F}|_Y)}} \leq \frac{e^{2Ct}M^t}{D^t}. \tag{12}
$$

Also,

$$
\begin{aligned}
&\nu_{l,t}(C_{i_1\dots i_n}) \geq \frac{\bar{N}^t e^{-2Ct}}{\alpha_{l,t}^Y M^{3t}} a_{i_1\dots i_n}\alpha_{l-n-k,t}^Y \text{ (by (5))} \geq \frac{\bar{N}^t e^{-2Ct} a_{i_1\dots i_n}\alpha_{l-n,t}^Y}{\alpha_{l,t}^Y\alpha_{k,t}^Y M^{3t}e^{Ct}} \text{ (by (6))}\\
&\geq \frac{D^t a_{i_1\dots i_n}}{e^{Ct}\alpha_{n,t}^Y} \text{ (by (6) and (8))} \geq \frac{D^{2t}}{e^{Ct}} a_{i_1\dots i_n} e^{-nP(t\mathcal{F}|_Y)} \text{ (by (10))}.
\end{aligned}
$$

Thus

$$
\frac{\nu_{l,t}(C_{i_1\dots i_n})}{a_{i_1\dots i_n}e^{-nP(t\mathcal{F}|_Y)}} \geq \frac{D^{2t}}{e^{Ct}}. \tag{13}
$$

Hence we obtain for each $y \in C_{i_1\dots i_n}$

$$
\frac{D^{2t}}{e^{Ct}} \leq \frac{\nu_{l,t}(C_{i_1\dots i_n})}{f_n^t|_Y(y)e^{-nP(t\mathcal{F}|_Y)}} \leq \frac{e^{2Ct}M^t}{D^t}. \tag{14}
$$

Consider now a convergent subsequence $\{\nu_{l_k,t}\}_{k=1}^{\infty}$ of $\{\nu_{l,t}\}_{l=1}^{\infty}$ and let ν_t be the corresponding limit point. Then ν_t also satisfies (11),(12), (13) and (14). We know by [B1, Lemma 2] that ν_t is ergodic. Using the arguments in [B1], we construct σ-invariant ergodic Gibbs measure $\mu_{t\mathcal{F}|_Y}$ for $t\mathcal{F}|_Y$. A limit point of the sequence $\{\frac{1}{n}\sum_{l=1}^{n-1}\nu_t \circ \sigma^{-l}\}_{n=1}^{\infty}$ is the unique equilibrium state for $t\mathcal{F}|_Y$ which is also Gibbs (see [IY]). Let $i_1 \dots i_n \in B_n(Y)$ be fixed, then

$$
\begin{aligned}
&\nu_t(\sigma^{-l}(C_{i_1\dots i_n}))\\
&\leq \frac{e^{2Ct}}{D^t} \sum_{j_1\dots j_l i_1 \dots i_n} \sup\{f^t_{l+n}|_Y(y) : y \in C_{j_1\dots j_l i_1\dots i_n}\} e^{-(l+n)P(t\mathcal{F}|_Y)} \text{ (replacing } \nu_{l,t} \text{ by } \nu_t \text{ in (11))}\\
&\leq \frac{e^{3Ct}}{D^t} \sum_{j_1\dots j_l} \sup\{f^t_l(y) f^t_n(\sigma^l y) : y \in C_{j_1\dots j_l i_1\dots i_n}\} e^{-(l+n)P(t\mathcal{F}|_Y)}\\
&\leq \frac{e^{3Ct}}{D^t} \sup\{f^t_n(y) : y \in C_{i_1\dots i_n}\} \alpha^Y_{l,t} e^{-(l+n)P(t\mathcal{F}|_Y)}\\
&\leq \frac{e^{3Ct}}{D^{2t}} \sup\{f^t_n(y) : y \in C_{i_1\dots i_n}\} e^{-nP(tF|_Y)} \text{ (by (10)).}
\end{aligned}
$$

Therefore, using (13) (replacing $\nu_{l,t}$ by ν_t), we have

$$\nu_t(\sigma^{-l}(C_{i_1\dots i_n})) \leq \frac{e^{4Ct}}{D^{4t}} \nu_t(C_{i_1\dots i_n}).$$

Using (14) (replacing $\nu_{l,t}$ by ν_t), for all $y \in C_{i_1\dots i_n}$,

$$\frac{1}{n} \sum_{i=1}^{n} \nu_t(\sigma^{-1}(C_{i_1\dots i_n})) \leq \frac{e^{6Ct} M^t f^t_n|_Y(y) e^{-nP(t\mathcal{F}|_Y)}}{D^{5t}},$$

and hence for all $y \in C_{i_1\dots i_n}$,

$$\mu_{t\mathcal{F}|_Y}(C_{i_1\dots i_n}) \leq \frac{e^{6Ct} M^t f^t_n|_Y(y) e^{-nP(t\mathcal{F}|_Y)}}{D^{5t}}. \tag{15}$$

Therefore, for each fixed $l_m, m \geq p$, $t\mathcal{F}|_{Y_{l_m}}$ has a unique equilibrium state $\mu_{t\mathcal{F}|_{Y_{l_m}}}$ which is Gibbs and satisfies (15) (replacing $\mu_{t\mathcal{F}|_Y}$ by $\mu_{t\mathcal{F}|_{Y_{l_m}}}$). The proof of [IY, Theorem 4.1] shows that (15) holds when we replace $\mu_{t\mathcal{F}|_Y}$ by the unique Gibbs equilibrium state μ for $\mathcal{F}$. This proves the lemma. □

Lemma 4.3. *The family of Gibbs equilibrium states $\{\mu_{t\mathcal{F}}\}_{t=1}^{\infty}$ is tight, i.e., for all $\epsilon > 0$, there exists a compact set $K \subset \Sigma$ such that for all $t \geq 1$ we have $\mu_{t\mathcal{F}}(K) > 1 - \epsilon$.*

Proof. The proof is based on [JMU1, Lemma 2]. Let $\epsilon > 0$. We construct an increasing sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ such that the compact set

$$K = \{x \in \Sigma : 1 \ \leq x_k \leq n_k, \text{ for all } k \in \mathbb{N}\}$$

satisfies $\mu_{t\mathcal{F}}(K) > 1 - \epsilon$ for all $t \geq 1$. Let $\pi_k : \Sigma \to \mathbb{N}$ be the projection map into the k-th coordinate. Note that

$$
\begin{aligned}
\mu_{t\mathcal{F}(K)} =& \mu_{t\mathcal{F}}\left(\Sigma \cap (\cup_{k=1}^{\infty}\{x \in \Sigma : x_k > n_k\})^c\right)\\
&\geq 1 - \sum_{k=1}^{\infty} \mu_{t\mathcal{F}}(\{x \in \Sigma : x_k > n_k\})\\
&= 1 - \sum_{k=1}^{\infty} \sum_{i=n_k+1}^{\infty} \mu_{t\mathcal{F}}(\pi_k^{-1}(i))\\
&= 1 - \sum_{k=1}^{\infty} \sum_{i=n_k+1}^{\infty} \mu_{t\mathcal{F}}[C_i].
\end{aligned}
$$

Therefore, in order to show that $\{\mu_{t\mathcal{F}}\}_{t=1}^{\infty}$ is tight, it is enough to find $\{n_k\}_{k=1}^{\infty}$ such that

$$\sum_{i=n_k+1}^{\infty} \mu_{t\mathcal{F}}[C_i] < \frac{\epsilon}{2^k}, \text{ for all } k \in \mathbb{N}, t \geq 1. \tag{16}$$

Now, let $N = Me^{6C}/D^5$ in Lemma 4.2. If $n = 1$, we have

$$\mu_{t\mathcal{F}}[C_i] \leq N^t e^{-P(t\mathcal{F})} \sup\{f_1^t(x) : x \in C_i\} \text{ for all } t \geq 1.$$

Now, let m be any σ-invariant Borel probability measure for which the limit

$$I = \lim_{n \to \infty} \frac{1}{n} \int \log f_n dm$$

is finite. Then

$$\begin{aligned} P(t\mathcal{F}) - tI &= \sup\left\{h_\mu(\sigma) + t \lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu : \mu \in \mathcal{M}\right\} - tI \\ &= P(t\mathcal{F} - tI) \geq h_m(\sigma) \geq 0. \end{aligned}$$

Therefore, for $t \geq 1$,

$$\mu_{t\mathcal{F}}[C_i] \leq N^t e^{-P(t(\mathcal{F}-I))} e^{-tI} (\sup\{f_1(x) : x \in C_i\})^t \tag{17}$$

$$\leq (N^t e^{-tI} (\sup\{f_1(x) : x \in C_i\})^t \tag{18}$$

$$= \left(Ne^{-I}\right)^t (\sup\{f_1(x) : x \in C_i\})^t \tag{19}$$

Note that Definition 4.1 (1) implies that, given $\epsilon > 0$, we can find $J \in \mathbb{N}$ such that

$$\sum_{i>J} \sup\{f_1(x) : x \in C_i\} < \frac{\epsilon}{Ne^{-I}} \frac{1}{2^k}. \tag{20}$$

Now we show equation (16). Using (17) and (20), we obtain

$$\begin{aligned} \sum_{i>J} \mu_{t\mathcal{F}}[C_i] &\leq \left(Ne^{-I}\right)^t \sum_{i>J} (\sup\{f_1(x) : x \in C_i\})^t \\ &= \left(\frac{\epsilon}{2^k}\right)^t \leq \frac{\epsilon}{2^k}. \end{aligned}$$

Thus we obtain (16). □

Remark 4.2. Lemma 4.3 implies that the family of Gibbs equilibrium states $\{\mu_{t\mathcal{F}}\}_{n=1}^{\infty}$ has a subsequence that converges weakly to a σ-invariant Borel probability measure μ.

Before stating our main result, we prove a Lemma of thermodynamic nature. We show that under certain assumptions the pressure function $t \mapsto P(t\mathcal{F})$ is differentiable and we give an explicit formula for its derivative. This result could be compared with the case of a single potential (see [PP, Chapter 4]).

Lemma 4.4. *Let (Σ, σ) be a countable Markov shift with the BIP property and let $\mathcal{F} \in \mathcal{R}$, then the function $t \to P(t\mathcal{F})$, when finite, is differentiable. Moreover,*

$$\frac{d}{dt} P(t\mathcal{F})\Big|_{t=s} = \lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu_{s\mathcal{F}}.$$

The proof of this Lemma closely follows the arguments developed in [B3, Theorem 10.4.1] and in [BD, K].

Proof. We will prove that if the almost-additive families $\mathcal{F}$ and $\mathcal{G}$ belong to $\mathcal{R}$. In particular $\mathcal{F}+t\mathcal{G} \in \mathcal{R}$. Denote by μ_t and $\mu_{\mathcal{F}}$ the Gibbs measures corresponding to $\mathcal{F}+t\mathcal{G}$ and $\mathcal{F}$ respectively. It is a direct consequence of the variational principle that

$$P(\mathcal{F}+t\mathcal{G}) - P(\mathcal{F}) \geq h(\mu_{\mathcal{F}}) + \lim_{n\to\infty} \frac{1}{n} \int (f_n + tg_n) d\mu_F - P(\mathcal{F}) \quad = \\ t \lim_{n\to\infty} \frac{1}{n} \int g_n d\mu_F.$$

Moreover,

$$P(\mathcal{F}+t\mathcal{G}) - P(\mathcal{F}) = P(\mathcal{F}+t\mathcal{G}) - P(\mathcal{F}+t\mathcal{G}-t\mathcal{G}) \leq \\ P(\mathcal{F}+t\mathcal{G}) - h(\mu_t) - \lim_{n\to\infty} \frac{1}{n} \int (f_n + tg_n - tg_n) d\mu_t = \\ t \lim_{n\to\infty} \frac{1}{n} \int g_n d\mu_t.$$

We therefore have for $t > 0$ that

$$\lim_{n\to\infty} \frac{1}{n} \int g_n d\mu_F \leq \frac{P(\mathcal{F}+t\mathcal{G}) - P(\mathcal{F})}{t} \leq \lim_{n\to\infty} \frac{1}{n} \int g_n d\mu_t.$$

On the other hand, if $t < 0$ we obtain

$$\lim_{n\to\infty} \frac{1}{n} \int g_n d\mu_F \geq \frac{P(\mathcal{F}+t\mathcal{G}) - P(\mathcal{F})}{t} \geq \lim_{n\to\infty} \frac{1}{n} \int g_n d\mu_t.$$

We conclude the proof noticing that since there exists a unique equilibrium measure for $\mathcal{F}$ we obtain that

$$\lim_{t\to 0} \left(\lim_{n\to\infty} \frac{1}{n} \int g_n d\mu_t \right) = \lim_{n\to\infty} \frac{1}{n} \int g_n d\mu_F.$$

□

We now state and prove our main result.

Theorem 4.1. *Let (Σ, σ) be a countable Markov shift with the BIP property and let $\mathcal{F} \in \mathcal{R}$. Denote by $\mu \in \mathcal{M}$ any accumulation point of the sequence of Gibbs equilibrium measures $\{\mu_{t\mathcal{F}}\}_{t=1}^{\infty}$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu = \lim_{t\to\infty} \lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu_{t\mathcal{F}}, \tag{21}$$

and μ is a maximising measure for $\mathcal{F}$.

Proof. Assume that $\{\mu_{t_k\mathcal{F}}\}_{k=1}^{\infty}$ converges weakly to $\mu \in \mathcal{M}$. Note that if

$$\lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu = -\infty,$$

then the result is immediate by Lemma 4.1. Therefore, in what follows we assume that

$$-\infty < \lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu.$$

We consider the sub-additive sequence defined by $\{\log f_n e^C\}_{n=1}^{\infty}$. Note that since $\sum_{i=1}^{\infty} \sup f_1|_{C_i} < \infty$, there exists $M_1 \in \mathbb{R}$ such that $\sup f_1 \leq M_1$. Therefore, for each $t \in \mathbb{R}$ we have that $(\log f_1 e^C)^+ \in L^1(\mu_{t\mathcal{F}})$. Applying the sub-additive ergodic theorem (see [W, Theorem 10.1]), we obtain that for $m \in \mathbb{N}$ we have

$$\lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu_{t\mathcal{F}} = \lim_{n\to\infty} \int \frac{1}{n} \log f_n e^C d\mu_{t\mathcal{F}} \tag{22}$$

$$\leq \frac{1}{m} \int \log f_m e^C d\mu_{t\mathcal{F}} \leq \frac{1}{m} \int \log f_m d\mu_{t\mathcal{F}} + \frac{C}{m}. \tag{23}$$

Similarly, we consider the sub-additive sequence of continuous functions $\{\log \frac{e^C}{f_n}\}_{n=1}^{\infty}$. We claim that for each $t \in \mathbb{R}$ we have $(\log \frac{e^C}{f_1})^+ \in L^1(\mu_{t\mathcal{F}})$. In order to prove this claim first note that since for each $t \in \mathbb{R}$ the measure $\mu_{t\mathcal{F}}$ is a Gibbs equilibrium state for $t\mathcal{F}$; we have (see Theorem 3.1)

$$-\infty < \lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu_{t\mathcal{F}}.$$

Therefore, for every $m \in \mathbb{N}$ we have

$$-\infty < \lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu_{t\mathcal{F}} \leq \frac{1}{m} \int \log f_m d\mu_{t\mathcal{F}} + \frac{C}{m}.$$

Thus, there exists a constant $M_2 \in \mathbb{R}$, where M_2 depends on t, such that

$$M_2 \leq \int \log f_1 d\mu_{t\mathcal{F}} \leq M_1.$$

Note that

$$\int \log f_1 d\mu_{t\mathcal{F}} = \int_{\{x: f_1(x) > 1\}} \log f_1 d\mu_{t\mathcal{F}} + \int_{\{x: f_1(x) \leq 1\}} \log f_1 d\mu_{t\mathcal{F}},$$

and $0 \leq \int_{\{x: f_1(x)>1\}} \log f_1 d\mu_{t\mathcal{F}} \leq M_1$. We thus obtain

$$-\int_{\{x: f_1(x) \leq 1\}} \log f_1 d\mu_{t\mathcal{F}} = \int_{\{x: f_1(x) > 1\}} \log f_1 d\mu_{t\mathcal{F}} - \int \log f_1 d\mu_{t\mathcal{F}}.$$

Therefore,

$$-M_1 \leq \int_{\{x: f_1(x) \leq 1\}} \log \frac{1}{f_1} d\mu_{t\mathcal{F}} \leq M_1 - M_2,$$

which implies that $(\log \frac{e^C}{f_1})^+ \in L^1(\mu_{t\mathcal{F}})$. Applying the sub-additive ergodic theorem to the sequence $\{\log \frac{e^C}{f_n}\}_{n=1}^{\infty}$, for each $t \in \mathbb{R}$ and $m \in \mathbb{N}$ we obtain

$$\lim_{n\to\infty} \frac{1}{n} \int \log \frac{1}{f_n} d\mu_{t\mathcal{F}} \leq \frac{1}{m} \int \log \frac{e^C}{f_m} d\mu_{t\mathcal{F}}.$$

Hence,

$$\frac{1}{m} \int \log f_m d\mu_{t\mathcal{F}} - \frac{C}{m} \leq \lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu_{t\mathcal{F}}. \tag{24}$$

Replacing $\mu_{t\mathcal{F}}$ by $\mu_{t_k\mathcal{F}}$ in equations (22) and (24), we obtain

$$\lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu_{t_k\mathcal{F}} \leq \frac{1}{m} \int \log f_m d\mu_{t_k\mathcal{F}} + \frac{C}{m} \tag{25}$$

and

$$\frac{1}{m} \int \log f_m d\mu_{t_k\mathcal{F}} - \frac{C}{m} \leq \lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu_{t_k\mathcal{F}}. \tag{26}$$

Letting $k \to \infty$ in (25) we obtain

$$\limsup_{k\to\infty} \lim_{n\to\infty} \frac{1}{n}\int \log f_n d\mu_{t_k\mathcal{F}} \le \frac{1}{m}\int \log f_m d\mu + \frac{C}{m} \tag{27}$$

and

$$\liminf_{k\to\infty} \lim_{n\to\infty} \frac{1}{n}\int \log f_n d\mu_{t_k\mathcal{F}} \le \frac{1}{m}\int \log f_m d\mu + \frac{C}{m}. \tag{28}$$

On the other hand, letting $k \to \infty$ in (26), we obtain

$$\frac{1}{m}\int \log f_m d\mu - \frac{C}{m} \le \liminf_{k\to\infty} \lim_{n\to\infty} \frac{1}{n}\int \log f_n d\mu_{t_k\mathcal{F}} \tag{29}$$

and

$$\frac{1}{m}\int \log f_m d\mu - \frac{C}{m} \le \limsup_{k\to\infty} \lim_{n\to\infty} \frac{1}{n}\int \log f_n d\mu_{t_k\mathcal{F}}. \tag{30}$$

Letting $m \to \infty$ in equations (27), (28), (29) and (30), we have

$$\liminf_{k\to\infty} \lim_{n\to\infty} \frac{1}{n}\int \log f_n d\mu_{t_k\mathcal{F}} = \limsup_{k\to\infty} \lim_{n\to\infty} \frac{1}{n}\int \log f_n d\mu_{t_k\mathcal{F}} \tag{31}$$

$$= \lim_{n\to\infty} \frac{1}{n}\int \log f_n d\mu < \infty. \tag{32}$$

The last inequality follows from the fact that for any $\mu \in \mathcal{M}$ we have

$$\lim_{n\to\infty} \frac{1}{n}\int \log f_n d\mu \le \int \log f_1 d\mu + C \le \sum_{i=1}^{\infty} \sup f_1|_{C_i} + C < \infty.$$

Let us consider the pressure function map $p : \mathbb{R} \to \mathbb{R}$ defined by $p(t) = P(t\mathcal{F})$. This is a convex function (see [IY, Corollary 3.2]) and, when finite, is differentiable (see Lemma 4.4). Thus, its derivative, which is given by

$$p'(t) = \lim_{n\to\infty} \frac{1}{n}\int \log f_n d\mu_{t\mathcal{F}},$$

is non-decreasing. Moreover, since $\lim_{n\to\infty} \frac{1}{n}\int \log f_n d\mu_{t\mathcal{F}} \le \int \log f_1 d\mu_{t\mathcal{F}} + C < \infty$, we have that the limit $\lim_{t\to\infty} p'(t)$, exists. This fact together with the equality (31) proves the equality (21) of Theorem4.1.

$$\lim_{n\to\infty} \frac{1}{n}\int \log f_n d\mu = \lim_{t\to\infty} \lim_{n\to\infty} \frac{1}{n}\int \log f_n d\mu_{t\mathcal{F}}.$$

Finally, we show that the accumulation point $\mu \in \mathcal{M}$ is a maximising measure, i.e., for all $\nu \in \mathcal{M}$, $\lim_{n\to\infty} \frac{1}{n}\int \log f_n d\mu \ge \lim_{n\to\infty} \frac{1}{n}\int \log f_n d\nu$. In order to prove this we will make use of Lemma 4.4 together with standard arguments in ergodic optimisation (see for example [JMU1, Theorem 1]) Assume by way of contradiction that μ is not a maximising measure. Then, there exists $\nu \in \mathcal{M}$ and $\epsilon > 0$ satisfying $\lim_{n\to\infty} \frac{1}{n}\int \log f_n d\nu - \lim_{n\to\infty} \frac{1}{n}\int \log f_n d\nu = \epsilon > 0$. It is clear that $\lim_{n\to\infty} \frac{1}{n} \log f_n d\nu < \infty$. Since $-\infty < P(\mathcal{F}) < \infty$, we have $h_\nu(\sigma) < \infty$. Now define the map $l_\nu : \mathbb{R} \to \mathbb{R}$ by $l_\nu(t) = h_\nu(\sigma) + t \lim \frac{1}{n}\int \log f_n d\nu$. Since the map $t \to p'(t)$ is non-decreasing and

$$\lim_{t\to\infty} p'(t) = \lim_{n\to\infty} \frac{1}{n}\int \log f_n d\mu,$$

for every large enough $t \in \mathbb{R}$ we have

$$\lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu \geq \lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu_{t\mathcal{F}} = p'(t).$$

Thus

$$l'_\nu(t) = \lim_{n\to\infty} \frac{1}{n} \int \log f_n d\nu = \lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu + \epsilon$$

for all $t \geq 1$. This will give us a contradiction to Theorem 2.1. Therefore, μ is $\mathcal{F}$-maximising.

□

We stress that the result obtained in Theorem 4.1 is new even in the compact setting, where convergence of Gibbs states directly follows as a consequence of the fact that the space of invariant measures is compact. Also note that in this compact setting, since the system has finite topological entropy, every Gibbs measure is an equilibrium measure.

Corollary 4.1. *Let (Σ, σ) be a transitive sub-shift of finite type defined over a finite alphabet and let $\mathcal{F}$ be an almost-additive Bowen sequence on Σ. Denote by $\mu \in \mathcal{M}$ any accumulation point of the sequence of Gibbs equilibrium measures $\{\mu_{t\mathcal{F}}\}_{t=1}^{\infty}$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu = \lim_{t\to\infty} \lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu_{t\mathcal{F}},$$

and μ is a maximising measure for $\mathcal{F}$.

Proof. Since (Σ, σ) is a sub-shift of finite type over a finite alphabet, it is clear that $\mathcal{F}$ satisfies (1) and (2) of Definition 4.1. Now we apply Theorem 4.1. □

5. The Joint spectral radius

In this section we will show that the techniques developed in this paper have interesting applications in functional analysis. Even in the compact (finite alphabet) setting, Theorem 4.1 can be used to obtain results in spectral theory. We begin recalling some basic definitions. Let A be a $d \times d$ real matrix, the *spectral radius* of A, is defined by

$$\rho(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}.$$

It is well known that if $\| \cdot \|$ is any sub-multiplicative matrix norm the following relation (sometimes called Gelfand property) holds

$$\rho(A) = \lim_{n\to\infty} \|A\|^{1/n}.$$

Let $\mathcal{A} := \{A_1, A_2, \dots, A_k\}$ be a a set of $d \times d$ real matrices and $\| \cdot \|$ a sub-multiplicative matrix norm. The *joint spectral radius* $\varrho(\mathcal{A})$ is defined by

$$\varrho(\mathcal{A}) := \lim_{n\to\infty} \max\left\{ \|A_{i_n} \cdots A_{i_1}\|^{1/n} : i_j \in \{1, 2, \dots, k\} \right\}.$$

This notion, that generalises the notion of spectral radius to a set of matrices, was introduced by G.-C. Rota and W.G. Strang in 1960 [RS]. Interest on it was strongly renewed by its applications in the study of wavelets discovered by Daubechies and Lagarias [DL1, DL2]. The value of $\varrho(\mathcal{A})$ is independent of the choice of the norm since all of them are equivalent. There exists a wide range of applications of the joint spectral radius in different topics including not only wavelets [P], but for example, combinatorics [DST]. Lagarias and Wang [LW] conjectured that for any finite set of matrices $\mathcal{A}$ there exists integers $\{i_1, \dots, i_n\}$ such that the periodic product $A_{i_1} \cdots A_{i_n}$ satisfies

$$\varrho(\mathcal{A}) = \rho(A_{i_n} \cdots A_{i_1})^{1/n}.$$

This conjecture was proven to be false by Bousch and Mariesse [BoM] and explicit counterexamples were first obtained by Hare, Morris, Sidorov and Theys [HMSY].

It is possible to restate the definition of joint spectral radius in terms of dynamical systems. Indeed, let (Σ_k, σ) be the full-shift on k symbols and consider the family of maps, $\phi_n : \Sigma_k \to \mathbb{R}$ defined by

$$\phi_n(x) := \|A_{i_n} \cdots A_{i_1}\|.$$

The family $\mathcal{F} := \{\log \phi_n\}_{n=1}^{\infty}$ is sub-additive on Σ_k. Denote by $\mathcal{M}$ the set of $\sigma-$invariant probability measures. If $\nu \in \mathcal{M}$ is ergodic then Kingman's sub-additive theorem implies that ν-almost everywhere the following equality holds

$$\lim_{n\to\infty} \frac{1}{n} \int \log \phi_n \; d\nu = \lim_{n\to\infty} \frac{1}{n} \log \phi_n(x).$$

We have that

Lemma 5.1. *Let $\mathcal{A} := \{A_1, A_2, \dots, A_k\}$ be a a set of $d \times d$ real matrices then there exits a measure $\mu \in \mathcal{M}$ such that*

$$\varrho(\mathcal{A}) = \exp\left(\sup\left\{\lim_{n\to\infty} \frac{1}{n} \int \log \phi_n \; d\nu : \nu \in \mathcal{M}\right\}\right) = \exp\left(\lim_{n\to\infty} \frac{1}{n} \int \log \phi_n d\mu\right).$$

Proof. Note that since the space Σ_k is compact, for every $n \in \mathbb{N}$ there exists a point $x_n \in \Sigma_k$ such that

$$\sup\left\{\frac{1}{n}\phi_n(x) : x \in \Sigma_k\right\} = \frac{1}{n}\phi_n(x_n).$$

Let δ_x be the atomic measure supported at the point $\{x\}$. Consider the probability measure $\mu_n := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{\sigma^i x_n}$. We have that

$$\sup\left\{\frac{1}{n}\phi_n(x)\right\} = \frac{1}{n}\phi_n(x_n) = \frac{1}{n}\int \log \phi_n d\mu_n.$$

The space of probability measures defined on Σ_k is compact, therefore there exists an accumulation point μ. It turns out that $\mu \in \mathcal{M}$ (see [W, Section 6.2]). Moreover, since

$$\exp\left(\lim_{n\to\infty} \frac{1}{n}\phi_n(x_n)\right) = \varrho(\mathcal{A}),$$

we have that

$$\varrho(\mathcal{A}) = \exp\left(\lim_{n\to\infty} \frac{1}{n} \int \log \phi_n d\mu\right).$$

□

The above Lemma was first obtained, with a different method though, in [DHX]. It is worth stressing that the definition of joint spectral radius and Lemma 5.1 hold in a broader context. Indeed, we can consider (Σ, σ) to be any mixing sub-shift of finite type defined on a finite alphabet and define the corresponding joint spectral radius by

$$\varrho_\Sigma(\mathcal{A}) := \exp\left(\lim_{n\to\infty} \max\left\{\|A_{i_n} \cdots A_{i_1}\|^{1/n} : (i_1 i_2 \dots i_n) \text{ is an admissible word }\right\}\right).$$

Under certain cone conditions for the set $\mathcal{A}$, we will show that there exists a one parameter family of dynamically relevant (Gibbs states) invariant measures $\{\mu_t\}_{t=1}^{\infty}$ such that any weak start accumulation point of it μ, satisfies

$$\varrho(\mathcal{A}) = \exp\left(\lim_{n\to\infty}\frac{1}{n}\int \log\phi_n \, d\mu\right). \tag{33}$$

Theorem 5.1 (Compact case)**.** *Let $\mathcal{A} := \{A_1, A_2, \ldots, A_k\}$ be a a set of $d \times d$ and (Σ, σ) mixing sub-shift of finite type defined on the alphabet $\{1, 2 \ldots, k\}$. Let $\phi_n : \Sigma_k \to \mathbb{R}$ be defined by*

$$\phi_n(w) = \phi_n((i_1, i_2, \ldots)) = \|A_{i_n} \cdots A_{i_1}\|.$$

If the family $\mathcal{F} = \{\log\phi_n\}_{n=1}^{\infty}$ is almost-additive then for every $t \in \mathbb{R}$ there exists a unique Gibbs state μ_t corresponding to $t\mathcal{F}$ and there exists a weak start accumulation point μ for $\{\mu_t\}_{t=1}^{\infty}$. The measure μ is such that

$$\varrho(\mathcal{A}) = \exp\left(\lim_{n\to\infty}\frac{1}{n}\int \log\phi_n d\mu\right) = \exp\left(\lim_{t\to\infty}\left(\lim_{n\to\infty}\frac{1}{n}\int \log\phi_n d\mu_t\right)\right).$$

Proof. Since the family $\mathcal{F}$ is almost-additive and is a Bowen sequence then it is a consequence of Theorem 2.2 that there exists a unique Gibbs state μ_t corresponding to $t\mathcal{F}$ for every $t \in \mathbb{R}$ which is also an equilibrium measure. Since the space $\mathcal{M}$ is compact there exists a weak star accumulation point μ for the sequence $\{\mu_t\}_{t=1}^{\infty}$. The result now follows from Theorem 4.1 or Corollary 4.1. □

Corollary 5.1. *Let $\mathcal{B} = \{A_1, A_2, \ldots, A_n\}$ be a finite set of positive matrices then the family $\mathcal{F} = \{\log\phi_n\}_{n=1}^{\infty}$ is almost-additive and therefore the sequence of Gibbs measures $\{\mu_t\}_{t=1}^{\infty}$ for $t\mathcal{F}$ has an accumulation point $\mu \in \mathcal{M}$ and*

$$\varrho(\mathcal{A}) = \exp\left(\lim_{n\to\infty}\frac{1}{n}\int \log\phi_n d\mu\right) = \exp\left(\lim_{t\to\infty}\left(\lim_{n\to\infty}\frac{1}{n}\int \log\phi_n d\mu_t\right)\right).$$

Proof. It was proved in [Fe, Lemma 2.1] that the set $\mathcal{B}$ is almost-additive. The result follows from Theorem 5.2. □

Let us consider now the non-compact case. Let $\mathcal{A} := \{A_1, A_2, \ldots\}$ be a countable set of $d \times d$ real matrices. We can again consider the joint spectral radius of them. However, the conclusion of Lemma 5.1 might not be true. Even for the case of one potential $\psi : \Sigma \to \mathbb{R}$, there are examples of non-compact dynamical systems for which

$$\sup\left\{\int \psi d\mu : \mu \in \mathcal{M}\right\} < \limsup_{n\to\infty}\frac{1}{n}\sup\left\{\sum_{i=0}^{n-1}\psi(\sigma^i x) : x \in \Sigma\right\}.$$

See for instance [JMU2, Example 4]. So we consider a slightly different situation.

Theorem 5.2. *Let $\mathcal{A} = \{A_1, A_2, \ldots\}$ be a countable set of matrices and (Σ, σ) a topologically mixing countable Markov shift satisfying the BIP property. Let*

$$\phi_n(w) = \|A_{i_n} \cdots A_{i_1}\|.$$

If the family $\mathcal{F} = \{\log\phi_n\}_{n=1}^{\infty}$ is almost-additive then for every $t \in \mathbb{R}$ there exists a unique Gibbs measure μ_t corresponding to $t\mathcal{F}$ and there exists a weak star *accumulation point μ for $\{\mu_t\}_{t=1}^{\infty}$. The measure μ is such that*

$$\sup\left\{\lim_{n\to\infty}\frac{1}{n}\int\log\phi_n d\nu : \nu\in\mathcal{M}\right\} = \lim_{n\to\infty}\frac{1}{n}\int\log\phi_n d\mu = \lim_{t\to\infty}\lim_{n\to\infty}\frac{1}{n}\int\log\phi_n d\mu_t.$$

The proof of this results follows from the zero temperature limit theorems obtained in the previous sections.

Proof. Since the family $\mathcal{F}$ is almost-additive and is a Bowen sequence then it is a consequence of Theorem 2.2 that there exists a unique Gibbs state μ_t corresponding to $t\mathcal{F}$ for every $t\in\mathbb{R}$ such that $P(t\mathcal{F})<\infty$. Lemma 4.3 implies that there exists a weak star accumulation point μ for the sequence $\{\mu_t\}_{t=1}^{\infty}$. The result now follows from Theorem 4.1. $\square$

Corollary 5.2. *Let $\mathcal{A}=\{A_1,A_2,\dots\}$ be a sequence of matrices having strictly positive entries and such that there exists a constant $C>0$ with the property that for every $k\in\mathbb{N}$ the following holds*

$$\frac{\min_{i,j}(A_k)_{i,j}}{\max_{i,j}(A_k)_{i,j}}\geq C$$

then for every sufficiently large $t\in\mathbb{R}$ there exists a Gibbs state μ_t for $t\mathcal{F}$ and the sequence $\{\mu_t\}_{t=1}^{\infty}$ has an accumulation point $\mu\in\mathcal{M}$. Moreover,

$$\sup\left\{\lim_{n\to\infty}\frac{1}{n}\int\log\phi_n d\nu : \nu\in\mathcal{M}\right\} = \lim_{n\to\infty}\frac{1}{n}\int\log\phi_n d\mu = \lim_{t\to\infty}\lim_{n\to\infty}\frac{1}{n}\int\log\phi_n d\mu_t.$$

Proof. Under the assumptions of the theorem the family $\mathcal{F}=\{\log\phi_n\}_{n=1}^{\infty}$ is almost-additive (see [IY, Lemma 7.1]) on Σ and therefore the results directly follows from Theorem 5.2. $\square$

Let us stress that the space of invariant measures is not compact, so the existence of such an invariant measure is non trivial.

6. Maximising the singular value function

Ever since the pioneering work of Bowen [Bow2] the relation between thermodynamic formalism and the dimension theory of dynamical systems has been thoroughly studied and exploited (see for example [B2, B3, Pe]). Multifractal analysis is a sub-area of dimension theory where the results obtained out of this relation has been particularly successful. The main goal in multifractal analysis is to study the complexity of level sets of invariant local quantities. Examples of these quantities are Birkhoff averages, Lyapunov exponents, local entropies and pointwise dimension. In general the structure of these level sets is very complicated so tools such as Hausdorff dimension or topological entropy are used to describe them. In dimension two (or higher) where a typical dynamical system is non-conformal computing the exact value of the Hausdorff dimension of the level sets is an extremely complicated task and at this point there are no techniques available to deal with such problem.

In this section we show how the results obtained in Section 4 can be used in the study of multifractal analysis of Lyapunov exponents for certain non-conformal repellers. Indeed, we will combine our results on ergodic optimisation with those of Barreira and Gelfert [BG] (which in turn uses ideas of Falconer [F]) to construct a measure supported on the extreme level sets.

Let $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a C^1 map and let $\Lambda \subset \mathbb{R}^2$ be a repeller of f. That is, the set Λ is compact, f-invariant, and the map f is expanding on Λ, i.e., there exist $c > 0$ and $\beta > 1$ such that

$$\|d_x f^n(v)\| \geq c\beta^n \|v\|,$$

for every $x \in \Lambda$, $n \in \mathbb{N}$ and $v \in T_x\mathbb{R}^2$. We will also assume that there exists an open set $U \subset \mathbb{R}^2$ such that $\Lambda \subset U$ and $\Lambda = \cap_{n\in\mathbb{N}} f^n(U)$ and that f restricted to Λ is topologically mixing. A pair (Λ, f) satisfying the above assumptions will be called *expanding repeller*. All the above assumptions are standard and there is a large literature describing the dynamics of expanding maps (see for example [BDV]). It is important to stress that the system (Λ, f) can be coded with a finite state transitive Markov shift. For each $x \in \mathbb{R}^2$ and $v \in T_x R^2$ we define the *Lyapunov exponent* of (x, v) by

$$\lambda(x, v) := \limsup_{n\to\infty} \frac{1}{n} \log \|d_x f^n v\|.$$

For each $x \in \mathbb{R}^2$ there exists a positive integer $s(x) \leq 2$, numbers $\lambda_1(x) \geq \lambda_2(x)$, and linear subspaces

$$\{0\} = E_{s(x)+1}(x) \subset E_{s(x)}(x) \subset E_1(x) = T_x R^2,$$

such that

$$E_i(x) = \left\{v \in T_x\mathbb{R}^2 : \lambda(x, v) = \lambda_i(x)\right\}$$

and $\lambda(x, v) = \lambda_i(x)$ if $v \in E_i(x) \setminus E_{i+1}(x)$.

In order to study study multifractal analysis of Lyapunov exponents in this context, Barreira and Gelfert [BG] used a construction originally made by Falconer [F] that we know recall. The singular values $s_1(A), s_2(A)$ of a 2×2 matrix A are the eigenvalues, counted with multiplicities, of the matrix $(A^*A)^{1/2}$, where A^* denotes the transpose of A. The singular values can be interpreted as the length of the semi-axes of the ellipse which is the image of the unit ball under A. The functions, $\phi_{i,n} : \Lambda \to \mathbb{R}$ be defined by

$$\phi_{i,n}(x) = \log s_i(d_x f^n)$$

and called *singular value functions*. Falconer [F] studied them with the purpose of estimating the Hausdorff dimension of Λ and have become one of the major tools in the dimension theory for non-conformal systems. It directly follows from Oseledets' multiplicative ergodic theorem [BP, Chapter 3] that for each finite $f-$invariant measure μ there exists a set $X \subset \mathbb{R}^2$ of full μ measure such that

$$\lim_{n\to\infty} \frac{\phi_{i,n}(x)}{n} = \lim_{n\to\infty} \frac{1}{n} \log s_i(d_x f^n) = \lambda_i(x). \tag{34}$$

Given $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ define the corresponding level set by

$$L(\alpha) := \{x \in \Lambda : \lambda_1(x) = \alpha_1 \text{ and } \lambda_2(x) = \alpha_2\}.$$

Barreira and Gelfert [BG] described the entropy spectrum of the Lyapunov exponents of f, that is they studied the function $\alpha \to h_{top}(f|L(\alpha))$, where h_{top} denotes the topological entropy of the set $L(\alpha)$. Their study exploited the relation established in equation (34), where it is shown that the level sets for the Lyapunov exponent correspond to level sets of the ergodic averages of the sub-additive sequences defined by $S_1 = \{\phi_{1,n}\}_{n=1}^{\infty}$ and $S_2 = \{\phi_{2,n}\}_{n=1}^{\infty}$.

The following result, which is a direct consequence of the theorems obtained in Section 4, allows us to describe the maximal Lyapunov exponent of the map f.

Proposition 6.1. *Let (Λ, f) be an expanding repeller such that the singular value function are almost-additive then for every $t > 0$ there exists a unique Gibbs measure μ_t corresponding to tS_1. Moreover, the sequence $\{\mu_t\}_{t=1}^{\infty}$ has an accumulation point μ and*

$$\sup\left\{\lim_{n\to\infty}\frac{1}{n}\phi_{1,n}(x)\right\} = \lim_{n\to\infty}\frac{1}{n}\int\phi_{1,n}(x)d\mu.$$

In particular, we obtain invariant measure supported on the set of points for which the Lyapunov exponent is maximised.

Conditions on the dynamical system f so that the sequences S_1 and S_2 are almost-additive can be found, for example, in [BG, Proposition 4] where it is proved that

Lemma 6.1. *Let (Λ, f) be an expanding repeller. If*

(1) *for every $x \in \Lambda$ the derivative $d_x f$ is represented by a positive 2×2 matrix, or*
(2) *if Λ posses a dominated splitting (see* [B3, p.234] *or* [BDV] *for a precise definition).*

Then the sequences S_1 and S_2 are almost-additive.

Actually, a cone condition of the type discussed in Corollary 5.1 is enough to obtain almost-additivity. This is discussed also in [BG].

We have considered maps defined in $\mathbb{R}^2$, similar results can be obtained in any finite dimension. Let us stress that we have only used a compact version of the results of Section 4, namely Corollary 4.1, which hold true in the countable (non-compact) setting.

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