IDENTIFYING NEIGHBORS OF STABLE SURFACES

GIANCARLO URZÚA

ABSTRACT. We identify the stable surfaces around the stable limit of the examples of Y. Lee and J. Park [LP07], and H. Park, J. Park and D. Shin [PPS09] using the explicit 3-fold Mori theory in [HTU13]. These surfaces belong to the Kollár–Shepherd-Barron–Alexeev compactification of the moduli space of simply connected surfaces of general type with $p_g = 0$ and $K^2 = 1, 2, 3$.

CONTENTS

1. INTRODUCTION

A main application of [HTU13] is to have an explicit 3-fold Mori theory to find stable limits of $\mathbb{Q}$-Gorenstein one parameter degenerations of surfaces with only log terminal singularities. (We summarize some results of [HTU13] in Section 2.) The aim of this paper is to run [HTU13, §5] on the singular examples of Y. Lee and J. Park [LP07], and H. Park, J. Park and D. Shin [PPS09] to identify all the stable surfaces around them. These surfaces belong to the Kollár–Shepherd-Barron–Alexeev compactification of the moduli space of simply connected surfaces of general type with $p_g = 0$ and $K^2 = 1, 2, 3$. This moduli space has no explicit description for any K^2. It is not even known whether it is irreducible. Moreover, the only explicit surfaces with those invariants

Date: October 16, 2013.

are Barlow surfaces [BHPV04, VII.10] [1], where $K^2 = 1$, and for the rest we only know existence via the $\mathbb{Q}$-Gorenstein smoothing method pioneered in [LP07].

We work out one example for each K^2, and state results for the others. We find their stable models (i.e. their canonical models; see Lemma 3.1 for the general picture), and the smooth minimal models of the stable singular surfaces around them. Roughly speaking, these examples represent smooth points of the moduli space of stable surfaces (having dimension $10 - 2K^2$ there), and each of its Wahl singularities $\frac{1}{n^2}(1, na-1)$ defines a boundary divisor $\mathcal{D}\binom{n}{a}$. In this way, we will be identifying general points on these divisors.

This identification shows the presence of various special surfaces in the boundary. For example, there are singular stable surfaces whose smooth minimal models are $p_g = 0$ surfaces of general type with certain configurations of curves inside (see Sections 4 and 5). There are also stable surfaces coming from Dolgachev surfaces (i.e. simply connected elliptic fibrations with $p_g = 0$ and Kodaira dimension 1), and from special rational surfaces. In some cases, these rational examples are distinct from the type of examples in [LP07, PPS09] and related papers, where the construction depends on rational elliptic fibrations with certain singular fibers. Hence this brings new types of examples. We will discuss explicitness for them in a forthcoming article.

In the last section, we expose about elliptic surfaces with $p_g = 0$ through $\mathbb{Q}$-Gorenstein smoothings, to put them in perspective with the general type constructions, and to use them when describing boundary divisors of the moduli spaces in the previous sections. Dolgachev surfaces appear in Corollary 6.2.

We would like to remark that the techniques used here can be applied to surfaces with other invariants. The choice of invariants in this paper reflects the interest of the author.

Acknowledgements. I have benefited from many conversations with Paul Hacking and Jenia Tevelev. I was supported by the FONDECYT Inicio grant 11110047 funded by the Chilean Government.

2. Preliminaries

The following is a summary of some results from [HTU13] which will be used to do computations in the next sections. We first recall some common terminology and facts. The ground field is $\mathbb{C}$.

[1]Conjecturally we also have the Craighero-Gattazzo surfaces.

Let Y be a cyclic quotient singularity $\frac{1}{m}(1,q)$, i.e. a germ at the origin of the quotient of $\mathbb{C}^2$ by the action of μ_m given by $(x,y)\mapsto(\mu x,\mu^q y)$, where μ is a primitive m-th root of 1, and q is an integer with $0<q<m$ and $\gcd(q,m)=1$. Let $\sigma\colon X\to Y$ be the minimal resolution of Y. Figure 1 shows the exceptional curves $E_i=\mathbb{P}^1$ of σ, for $1\le i\le s$, and the proper transforms E_0 and E_{s+1} of $(y=0)$ and $(x=0)$ respectively.

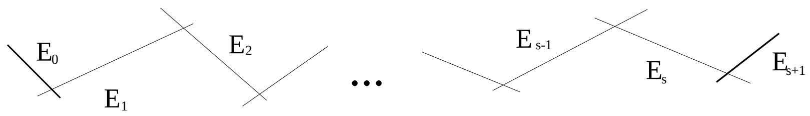

FIGURE 1. Exceptional divisors over $\frac{1}{m}(1,q)$, E_0 and E_{s+1}

The intersection numbers $E_i^2=-e_i$ are computed using the *Hirzebruch-Jung continued fraction*

$$\frac{m}{q}=e_1-\cfrac{1}{e_2-\cfrac{1}{\ddots-\frac{1}{e_s}}}=:[e_1,\ldots,e_s].$$

A configuration of curves $[e_1,\ldots,e_s]$ in a nonsingular surface will mean the corresponding exceptional divisor of the singularity $\frac{1}{m}(1,q)$.

The continued fraction $[e_1,\ldots,e_s]$ defines the sequence of integers

$$0=\beta_{s+1}<1=\beta_s<\ldots<q=\beta_1<m=\beta_0$$

where $\beta_{i+1}=e_i\beta_i-\beta_{i-1}$. In this way, $\frac{\beta_{i-1}}{\beta_i}=[e_i,\ldots,e_s]$. Partial fractions $\frac{\alpha_i}{\gamma_i}=[e_1,\ldots,e_{i-1}]$ are computed through the sequences

$$0=\alpha_0<1=\alpha_1<\ldots<q'=\alpha_s<m=\alpha_{s+1},$$

where $\alpha_{i+1}=e_i\alpha_i-\alpha_{i-1}$ (q' is the integer such that $0<q'<m$ and $qq'\equiv 1(\mathrm{mod}\ m)$), and $\gamma_0=-1$, $\gamma_1=0$, $\gamma_{i+1}=e_i\gamma_i-\gamma_{i-1}$. We have $\alpha_{i+1}\gamma_i-\alpha_i\gamma_{i+1}=-1$, $\beta_i=q\alpha_i-m\gamma_i$, and $\frac{m}{q'}=[e_s,\ldots,e_1]$. These numbers appear in the pull-back formulas

$$\sigma^*(E_0)=\sum_{i=0}^{s+1}\frac{\beta_i}{m}E_i,\quad\text{and}\quad\sigma^*(E_{s+1})=\sum_{i=0}^{s+1}\frac{\alpha_i}{m}E_i,$$

and $K_X\equiv\sigma^*(K_Y)-\sum_{i=1}^{s}(1-\frac{\beta_i+\alpha_i}{m})E_i$.

The following terminology and facts are from [KSB88].

Let Y be a normal surface with only quotient singularities, and let $\mathbb{D}$ be a smooth curve analytic germ. A deformation $(Y\subset\mathcal{Y})\to(0\in\mathbb{D})$ of Y is called a *smoothing* if its general fiber is smooth. It is $\mathbb{Q}$-*Gorenstein* if $K_{\mathcal{Y}}$ is $\mathbb{Q}$-Cartier. A germ of a normal surface Y

is called a T-singularity if it is a quotient singularity and admits a $\mathbb{Q}$-Gorenstein smoothing. Any T-singularity is either a du Val singularity or a cyclic quotient singularity of the form $\frac{1}{dn^2}(1, dna - 1)$ with $\gcd(n, a) = 1$ [KSB88, Prop.3.10]. A T-singularity with a one-dimensional $\mathbb{Q}$-Gorenstein versal deformation space is either a node A_1 or a *Wahl singularity* $\frac{1}{n^2}(1, na - 1)$.

Let $(Q \in Y)$ be a germ of a two dimensional quotient singularity. A proper birational map $f : X \to Y$ is called a *P-resolution* if f is an isomorphism away from Q, X has T-singularities only, and K_X is ample relative to f [KSB88, Def.3.8].

By [KSB88, 3.9], there is a natural bijection between P-resolutions $X^+ \to Y$ and irreducible components of the formal deformation space $\mathrm{Def}(Y)$. Namely, let $\mathrm{Def}^{\mathrm{QG}}(X^+)$ denote the versal $\mathbb{Q}$-Gorenstein deformation space of X^+. Recall that for any rational surface singularity Z and its partial resolution $X \to Z$, there is an induced map $\mathrm{Def}\, X \to \mathrm{Def}\, Z$ of formal deformation spaces [Wahl76, 1.4], which we refer to as *blowing down deformations*. In particular, we have a map $\mathrm{Def}^{\mathrm{QG}}(X^+) \to \mathrm{Def}(Y)$. The germ $\mathrm{Def}^{\mathrm{QG}}(X^+)$ is smooth, the map $\mathrm{Def}^{\mathrm{QG}}(X^+) \to \mathrm{Def}(Y)$ is a closed embedding, and it identifies $\mathrm{Def}^{\mathrm{QG}}(X^+)$ with an irreducible component of $\mathrm{Def}(Y)$. All irreducible components of $\mathrm{Def}(Y)$ arise in this fashion (in a unique way).

Now some definitions from [KM92]. An *extremal neighborhood*

$$(C^- \subset \mathcal{X}^-) \to (Q \in \mathcal{Y})$$

is a proper birational morphism between normal 3-folds $\mathcal{X}^- \to \mathcal{Y}$ such that

(1) The canonical class $K_{\mathcal{X}^-}$ is $\mathbb{Q}$-Cartier and $\mathcal{X}^-$ has only terminal singularities.
(2) There is a distinguished point $Q \in \mathcal{Y}$ such that ${F^-}^{-1}(Q)$ consists of an irreducible curve $C^- \subset \mathcal{X}^-$.
(3) $K_{\mathcal{X}^-} \cdot C^- < 0$.

There are two types of extremal nbds according to the dimension of the exceptional loci $\mathrm{Exc}(F^-)$ of F^-. An extremal nbd is *flipping* if $\mathrm{Exc}(F^-) = C^-$. Otherwise, $\mathrm{Exc}(F^-)$ is two dimensional and F^- is called *divisorial*.

In the flipping case, $K_{\mathcal{Y}}$ is not $\mathbb{Q}$-Cartier. Then one attempts another type of surgery. A *flip* of a flipping extremal nbd

$$F^- \colon (C^- \subset \mathcal{X}^-) \to (Q \in \mathcal{Y})$$

is a proper birational morphism

$$F^+ \colon (C^+ \subset \mathcal{X}^+) \to (Q \in \mathcal{Y})$$

where $\mathcal{X}^+$ is normal with terminal singularities, $\mathrm{Exc}(F^+) = C^+$ is a curve, and $K_{\mathcal{X}^+}$ is $\mathbb{Q}$-Cartier and F^+-ample. A flip induces a birational map $\mathcal{X}^- \dashrightarrow \mathcal{X}^+$ to which we also refer as flip. When a flip exists then it is unique (cf. [KM98]). Mori [Mori88] proves that (3-fold) flips always exist.

In [HTU13], we focus in two particular types of extremal nbds which appear naturally when working on the Kollár–Shepherd-Barron–Alexeev compactification of the moduli of surfaces of general type [KSB88].

Definition 2.1. Let $(Q \in Y)$ be a two dimensional cyclic quotient singularity germ. Assume there is a partial resolution $f^- \colon X^- \to Y$ of Y such that $f^{-1}(Q)$ is a smooth rational curve C^- with one (two) Wahl singularity(ies) on it. Suppose $K_{X^-} \cdot C^- < 0$. Let $(X^- \subset \mathcal{X}^-) \to (0 \in \mathbb{D})$ be a $\mathbb{Q}$-Gorenstein smoothing of X^- over a smooth curve germ $\mathbb{D}$. Let $(Y \subset \mathcal{Y}) \to (0 \in \mathbb{D})$ be the corresponding blowing down deformation of Y. The induced birational morphism $(C^- \subset \mathcal{X}^-) \to (Q \in \mathcal{Y})$ is called *extremal neighborhood of type mk1A (mk2A)*; we denote it by mk1A (mk2A).

These extremal neighborhoods are of type k1A and k2A (cf. [KM92, Mori02]), and minimal with respect to the second betti number $(= 1)$) of the Milnor fiber of $(Y \subset \mathcal{Y}) \to (0 \in \mathbb{D})$ (see [HTU13, Prop.2.1] for more discussion on this).

Definition 2.2. A P-resolution $f^+ \colon X^+ \to Y$ of a two dimensional cyclic quotient singularity germ $(Q \in Y)$ is called *extremal P-resolution* if ${f^+}^{-1}(Q)$ is a smooth rational curve C^+, and X^+ has only Wahl singularities (at most two).

Proposition 2.3. *If a mk1A or mk2A is flipping, then the flip $(C^+ \subset \mathcal{X}^+) \to (Q \in \mathcal{Y})$ is induced by the blowing down deformation of a $\mathbb{Q}$-Gorenstein smoothing of a surface X^+ of an extremal P-resolution $(C^+ \subset X^+) \to (Q \in Y)$. The commutative diagram of morphisms is*

$$\begin{tikzcd}
C^- \subset X^- \subset \mathcal{X}^- \arrow[rr, dashed, "flip"] \arrow[dr] \arrow[ddr] & & C^+ \subset X^+ \subset \mathcal{X}^+ \arrow[dl] \arrow[ddl] \\
& Q \in Y \subset \mathcal{Y} \arrow[d] & \\
& 0 \in \mathbb{D} &
\end{tikzcd}$$

Proof. [KM92, Sect.11 and Thm.13.5]. (See [Mori02, HTU13] for explicit equations.) □

Proposition 2.4. *If a mk1A or mk2A is divisorial, then $(Q \in Y)$ is a Wahl singularity. The divisorial contraction $\mathcal{X}^- \to \mathcal{Y}$ induces the blowing down of a (-1)-curve between the smooth fibers of $\mathcal{X} \to \mathbb{D}$ and $\mathcal{Y} \to \mathbb{D}$.*

Proof. See [HTU13, §3.3]. □

The following is the numerical description of the X^- in a mk1A or in a mk2A, and of the X^+ in an extremal P-resolution.

Let us fix a mk1A with Wahl singularity $\frac{1}{m^2}(1, ma-1)$. Let $\frac{m^2}{ma-1} = [e_1, \ldots, e_s]$ be its continued fraction. Let $E_1, \ldots, E_s$ be the exceptional curves of the minimal resolution $\widetilde{X}^-$ of X^- with $E_j^2 = -e_j$ for all j. Notice that $K_{X^-} \cdot C^- < 0$ and $C^- \cdot C^- < 0$ imply that the proper transform of C^- in $\widetilde{X}^-$ is a (-1)-curve intersecting only one component E_i transversally at one point. This data will be written as $[e_1, \ldots, \overline{e_i}, \ldots, e_s]$ so that

$$\frac{\Delta}{\Omega} = [e_1, \ldots, e_i - 1, \ldots, e_s]$$

[2] where $0 < \Omega < \Delta$, and $(Q \in Y)$ is $\frac{1}{\Delta}(1, \Omega)$. Let $\beta_i, \alpha_i, \gamma_i$ be the numbers defined above for $[e_1, \ldots, e_s]$. Then

$$\Delta = m^2 - \beta_i \alpha_i \qquad \Omega = ma - 1 - \gamma_i \beta_i$$

and, if $\delta := \frac{\beta_i + \alpha_i}{m}$, we have $K_{X^-} \cdot C^- = \frac{-\delta}{m} < 0$ and $C^- \cdot C^- = \frac{-\Delta}{m^2} < 0$; See [HTU13, §2.2].

Similarly, for a mk2A with Wahl singularities $\frac{1}{m_j^2}(1, m_j a_j - 1)$ $(j = 1, 2)$, let $E_1, \ldots, E_{s_1}$ and $F_1, \ldots, F_{s_2}$ be the exceptional divisors over $\frac{1}{m_1^2}(1, m_1 a_1 - 1)$ and $\frac{1}{m_2^2}(1, m_2 a_2 - 1)$ respectively, such that $\frac{m_1^2}{m_1 a_1 - 1} = [e_1, \ldots, e_{s_1}]$ and $\frac{m_2^2}{m_2 a_2 - 1} = [f_1, \ldots, f_{s_2}]$ with $E_i^2 = -e_i$ and $F_j^2 = -f_j$. We know that the proper transform of C^- in the minimal resolution $\widetilde{X}^-$ of X^- is a (-1)-curve intersecting only one E_i and one F_j transversally at one point, and these two exceptional curves are at the ends of these exceptional chains. The data for mk2A will be written as $[f_{s_2}, \ldots, f_1] - [e_1, \ldots, e_{s_1}]$ so that the (-1)-curve intersects F_1 and E_1, and

$$\frac{\Delta}{\Omega} = [f_{s_2}, \ldots, f_1, 1, e_1, \ldots, e_{s_1}]$$

where $0 < \Omega < \Delta$ and $(Q \in Y)$ is $\frac{1}{\Delta}(1, \Omega)$.

We define $\delta := m_1 a_2 + m_2 a_1 - m_1 m_2$, and so

$$\Delta = m_1^2 + m_2^2 - \delta m_1 m_2, \quad \Omega = (m_2 - \delta m_1)(m_2 - a_2) + m_1 a_1 - 1.$$

[2]We use same notation for continued fractions even when some entries are 1.

We have $K_{X^-} \cdot C^- = \frac{-\delta}{m_1 m_2} < 0$ and $C^- \cdot C^- = \frac{-\Delta}{m_1^2 m_2^2} < 0$.

In analogy to a mk2A, an extremal P-resolution has data $[f_{s_2}, \ldots, f_1]-c-[e_1, \ldots, e_{s_1}]$, so that

$$\frac{\Delta}{\Omega} = [f_{s_2}, \ldots, f_1, c, e_1, \ldots, e_{s_1}]$$

where $-c$ is the self-intersection of the proper transform of C^+ in the minimal resolution of X^+, $0 < \Omega < \Delta$, and $(Q \in Y)$ is $\frac{1}{\Delta}(1, \Omega)$. As above, here $\frac{m_1^2}{m_1 a_1 - 1} = [e_1, \ldots, e_{s_1}]$ and $\frac{m_2^2}{m_2 a_2 - 1} = [f_1, \ldots, f_{s_2}]$. If a Wahl singularity (or both) is (are) actually smooth, then we set $m_i = a_i = 1$. We define

$$\delta = c m_1 m_2 - m_1 a_2 - m_2 a_1,$$

and so $\Delta = m_1^2 + m_2^2 + \delta m_1 m_2$ and, when both $m_i \neq 1$,

$$\Omega = -m_1^2(c-1) + (m_2 + \delta m_1)(m_2 - a_2) + m_1 a_1 - 1.$$

One easily computes Ω when one or both $m_i = 1$. We have

$$K_{X^+} \cdot C^+ = \frac{\delta}{m_1 m_2} > 0 \quad \text{and} \quad C^+ \cdot C^+ = \frac{-\Delta}{m_1^2 m_2^2} < 0.$$

In [HTU13, §2.3] it is proved that a given exceptional nbhd of type mk1A degenerates to two mk2A sharing the following numerics.

Proposition 2.5. *Let $[e_1, \ldots, \overline{e_i}, \ldots, e_s]$ be the data of a mk1A with $\frac{m^2}{ma-1} = [e_1, \ldots, e_s]$. Let δ, Δ, Ω be its numbers. Let $\frac{m_2}{m_2 - a_2} = [e_1, \ldots, e_{i-1}]$ and $\frac{m_1}{m_1 - a_1} = [e_s, \ldots, e_{i+1}]$, if possible (this is, for the first $i > 1$, for the second $i < s$). Then, there are mk2A with data*

$$[f_{s_2}, \ldots, f_1] - [e_1, \ldots, e_s] \quad \text{and} \quad [e_1, \ldots, e_s] - [g_1, \ldots, g_{s_1}],$$

where $\frac{m_2^2}{m_2 a_2 - 1} = [f_1, \ldots, f_{s_2}]$, $\frac{m_1^2}{m_1 a_1 - 1} = [g_1, \ldots, g_{s_1}]$, such that the corresponding cyclic quotient singularity $\frac{1}{\Delta}(1, \Omega)$ and δ are the same for the mk1A and the mk2A. Moreover, each of the mk2A deforms (over a smooth curve germ) to the mk1A by $\mathbb{Q}$-Gorenstein smoothing up $\frac{1}{m_i^2}(1, m_i a_i - 1)$ while keeping $\frac{1}{m^2}(1, ma-1)$, and there are two possibilities: either these three extremal nbds are

1. *flipping, with the same extremal P-resolution for the flip, or*
2. *divisorial, with $\Delta = \delta^2 > 1$.*

Proposition 2.5 allows us to compute the flip or the divisorial contraction for any mk1A through the Mori algorithm [Mori02] for extremal neighborhoods of type k2A as follows.

Consider a mk2A with Wahl singularities defined by m_2, a_2 and m_1, a_1, and numbers δ, Δ and Ω, such that $\delta m_1 - m_2 \leq 0$. We call it *initial mk2A*. We also allow the mk1A special case $m_1 = a_1 = 1$.

For $i \geq 2$, we have the Mori recursions

$$d(1) = m_1, \quad d(2) = m_2, \quad d(i-1) + d(i+1) = \delta d(i)$$

and $c(1) = a_1$, $c(2) = m_2 - a_2$, $c(i-1) + c(i+1) = \delta c(i)$.

When $\delta > 1$, we have for each i a mk2A with data $m_2 = d(i+1), a_2 = d(i+1) - c(i+1)$ and $m_1 = d(i), a_1 = c(i)$, with same δ, Δ and Ω. For two consecutive i's, we actually have the two mk2A of Proposition 2.5. We call this sequence of mk2A's a *Mori sequence*. If $\delta = 1$, then the initial mk2A must be flipping, and Mori's recursion gives only one more mk2A with data $m_2 = d(2) - d(1), a_2 = d(2) - d(1) + c(1) - c(2)$ and $m_1 = d(2), a_1 = c(2)$.

In [Mori02], Mori proves that any mk2A belongs to a unique Mori sequence (including the case $\delta = 1$), and that it is flipping if and only if $\delta d(1) - d(2) < 0$.

Below we do computations for divisorial and flipping nbhds using the initial mk2A of a Mori sequence with $\delta m_1 - m_2 \leq 0$.

- ($= 0$) For divisorial type, we have that $m_1 = \delta$, $m_2 = \delta^2 = \Delta$, $\Omega = \delta a_1 - 1$, and $a_2 = \delta^2 - \Omega$.
- (< 0) For flipping type, we have that the corresponding extremal P-resolution has $m_2 = d(1)$, $a_2 = d(1) - c(1)$, and

$$m_1 = d(2) - \delta d(1), \qquad a_1 \equiv c(2) - \delta c(1) \pmod{d(2) - \delta d(1)}.$$

  (The self-intersection of the flipping curve can be found using the formula for the δ of an extremal P-resolution.)

Conversely, for a given Wahl singularity $\frac{1}{\delta^2}(1, \delta a - 1)$ we have one Mori sequence of divisorial type following the previous recipe. For a given extremal P-resolution we have at most two Mori sequences, corresponding to each of its Wahl singularities, following the recipe above.

Example 2.6. Consider the Wahl singularity $(Q \in Y) = \frac{1}{4}(1, 1)$. Then the numerical data of any mk1A and any mk2A of divisorial type associated to $(Q \in Y)$ can be read from

$$[4] - [2, \bar{2}, 6] - [2, 2, 2, \bar{2}, 8] - [2, 2, 2, 2, 2, \bar{2}, 10] - \cdots$$

Notice that $\delta = 2$.

Example 2.7. Let $\frac{1}{11}(1, 3)$ be the cyclic quotient singularity $(Q \in Y)$. Consider the extremal P-resolution $[4] - 3$. Here $\delta = 3$, and the "middle" curve is a (-3)-curve. Then the numerical data of any mk1A and any mk2A associated to X^+ can be read from

$$[\bar{2}, 5, 3] - [2, 3, \bar{2}, 2, 7, 3] - [2, 3, 2, 2, 2, \bar{2}, 5, 7, 3] - \cdots$$

and

$$[4] - [2, \bar{2}, 5, 4] - [2, 2, 3, \bar{2}, 2, 7, 4] - [2, 2, 3, 2, 2, 2, \bar{2}, 5, 7, 4] - \cdots$$

These two Mori sequences are the numerical data of the universal antiflip [HTU13, §3] of $[4] - 3$.

A flip which shows up very often in calculations is the following

Proposition 2.8. *Let $[e_1, \ldots, e_{s-1}, \overline{e_s}]$ be a flipping mk1A. Let $e_i \geq 3$ be such that $e_j = 2$ for all $j > i$.*
Then the data for X^+ is $e_1 - [e_2, \ldots, e_i - 1]$.

A corollary is the useful fact

Proposition 2.9. [HP10, p.188] *Let $\widetilde{Y}$ be a smooth surface with a chain of rational smooth curves $E_1, \ldots, E_s$, which is the exceptional divisor of a Wahl singularity. Let C_1, C_2 be (-1)-curves in $\widetilde{Y}$ such that $C_1 \cdot C_2 = 0$, $C_1 \cdot E_1 = 1$, and $C_2 \cdot E_s = 1$, and C_1, C_2 do not intersect any other E_i's. Let $\sigma \colon \widetilde{Y} \to Y$ be the contraction of the chain $E_1, \ldots, E_s$ (to a Wahl singularity), and let $C_0 = \sigma(C_1) \cup \sigma(C_2)$. Assume there is a $\mathbb{Q}$-Gorenstein smoothing $(Y \subset \mathcal{Y}) \to (0 \in \mathbb{D})$.*
Then there is a (-1)-curve C_t in the smooth fiber over $t \in \mathbb{D} \setminus \{0\}$ which degenerates to C_0.

Proof. Notice that $C^- := \sigma(C_2)$ defines a mk1A of flipping type as in Proposition 2.8. After be perform the flip, we obtain a surface Y^+ (from the corresponding extremal P-resolution) and the proper transform of $\sigma(C_1)$ in Y^+ does not pass through the singularity. Therefore the $\mathbb{Q}$-Gorenstein smoothing of Y^+, which gives the flip, would have a (-1)-curve C_t in the general fiber that deforms to $\sigma(C_2)$. It is clear that in $(Y \subset \mathcal{Y}) \to (0 \in \mathbb{D})$ this (-1)-curve degenerates to C_0. $\square$

We point out that the previous proposition can be generalized for other "positions" of the (-1)-curves C_1, C_2 according to the extremal P-resolution of the flipping mk1A coming from $\sigma(C_2)$.

Finally some notation. We write the same letter to denote a curve and its proper transform under a birational map. We use Kodaira's notation for singular fibers of elliptic fibrations. Throughout this paper we use the dotted diagrams of [HTU13, §5] to perform the birational operations associated to mk1A and mk2A.

3. $K^2 = 1$

We begin with the example corresponding to [LP07, Fig.5]. Consider the pencil of curves in $\mathbb{P}^2_{x_0,x_1,x_2}$

$$\alpha x_0^3 + \beta x_1(x_0^2 + x_1^2 - x_2^2) = 0$$

with $(\alpha : \beta) \in \mathbb{P}^1_{\alpha,\beta}$. We have base points $p = (0 : 1 : 1)$, $q = (0 : 1 : -1)$, and $r = (0 : 0 : 1)$ which we blow-up three times each to obtain an elliptic fibration $g \colon Y \to \mathbb{P}^1$ with a configuration of singular fibers $IV^*, 2I_1, I_2$. Let $A = \{x_0 = 0\}$, $B = \{x_1 = 0\}$, and $C = \{x_0^2+x_1^2 = x_2^2\}$. Let P and Q be the last exceptional divisors over p and q. This and more notation is shown in Figure 2.

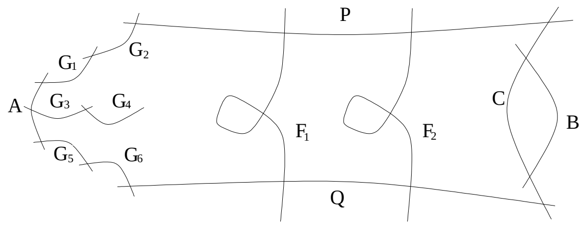

FIGURE 2. Elliptic fibration with $IV^*, 2I_1, I_2$

We now blow-up Y 11 times as in Figure 5 of [LP07] (see Figure 3). Let Y' be the corresponding surface, and let X' be the singular normal projective surface obtained by contracting the configurations of curves $[2, 2, 2, 7]$, $[4]$, $[6, 2, 2]$, and $[2, 6, 2, 3]$. One can check that $K_{X'}$ is not nef: the intersection of the image of G (see Figure 3) in X' with $K_{X'}$ is $-\frac{1}{12}$. However, a $\mathbb{Q}$-Gorenstein smoothing of these 4 singularities has the claimed properties in [LP07]. To see this, we perform a flip of type mk2A (Definition 2.1) on a $\mathbb{Q}$-Gorenstein one parameter smoothing of X'. We flip the curve $G^- := G$ in X'. It passes through the singularities corresponding to $[4]$ and $[2, 6, 2, 3]$. The flip of G^- produces a surface X'^+, and a curve G^+ (the flip of G^-) which passes through two Wahl singularities. The diagram of this operation is shown in Figure 4. For the computation of any flip we refer to [HTU13].

After this flip, the minimal resolution $\tilde{Y}$ of X'^+ is the blow-up of Y 10 times. The new configuration of relevant curves is in Figure 5. Hence $X := X'^+$ is the contraction of the configurations $[4]$ (C), $[2, 2, 6]$ $(E_4+E_3+F_1)$, $[2, 2, 2, 7]$ $(A+G_5+G_6+Q)$, and $[2, 5, 3]$ (E_7+F_2+P). It is easy to check there are no local-to-global obstructions to deform X via an argument as in [LP07]. (This is actually inherited from X'.)

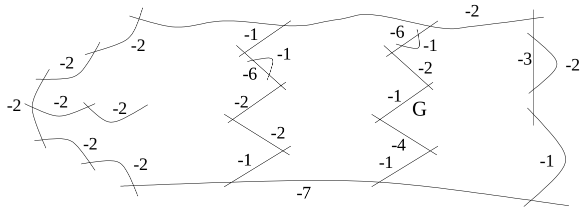

FIGURE 3. The blow-up Y' of Y 11 times

FIGURE 4. A flip

We have the $\mathbb{Q}$-numerical equivalence

$$K_{\tilde{Y}} \equiv -\frac{1}{2}F_1 - \frac{1}{2}F_2 + \frac{1}{2}E_2 + \frac{1}{2}E_3 + \frac{3}{2}E_4 + \frac{5}{2}E_5 + \frac{1}{2}E_7 + \frac{3}{2}E_8 + E_9 + E_{10},$$

and so, by adding the discrepancies of the singularities of X, we verify that the pull-back of K_X is $\mathbb{Q}$-numerically effective and nef. Therefore, the general fiber of the $\mathbb{Q}$-Gorenstein smoothing is a smooth minimal projective surface of general type with $K^2 = 1$, $p_g = 0$, and trivial π_1.

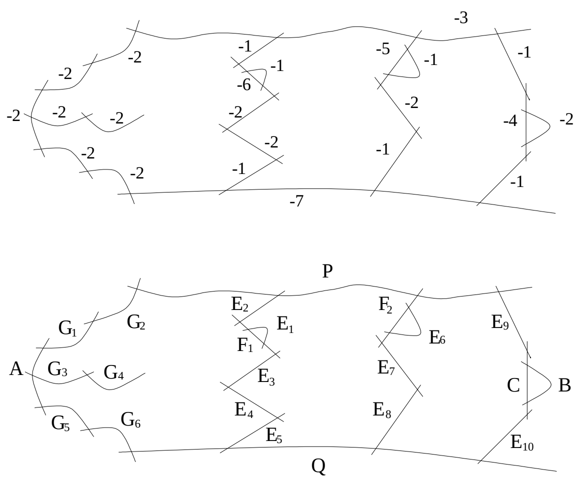

FIGURE 5. The blow-up $\tilde{Y}$ of Y 10 times

We are going to find the canonical model $X_{\text{can}} \subset \mathcal{X}_{\text{can}}$ of $X \subset \mathcal{X}$. The following lemma says what type of singularities we can have in X_{can} in general.

Lemma 3.1. *Assume we have a $\mathbb{Q}$-Gorenstein smoothing $(X \subset \mathcal{X}) \to (0 \in \mathbb{D})$ of a projective surface X with only Wahl singularities and K_X nef. Suppose that $K_X^2 > 0$. Then, its canonical model $(X_{can} \subset \mathcal{X}_{can}) \to (0 \in \mathbb{D})$ has X_{can} projective surface with only T-singularities, this is, it has du Val singularities or cyclic quotient singularities $\frac{1}{dn^2}(1, dna-1)$ with $gcd(n,a) = 1$.*

Proof. We know there is $(X_{\text{can}} \subset \mathcal{X}_{\text{can}}) \to (0 \in \mathbb{D})$; cf. [KM98]. We have a birational morphism $\mathcal{X} \to \mathcal{X}_{\text{can}}$ over $\mathbb{D}$ such that $K_{\mathcal{X}_{\text{can}}}$ is $\mathbb{Q}$-Cartier and ample. Notice that X_{can} has log terminal singularities because X does [KM98, pp.102–103]. The singularities of X_{can} must be T-singularities by [KSB88, §5.2]. $\square$

Notice first that X_{can} is not X since $G_4 \cdot K_X = 0$. Let $\pi \colon \tilde{Y} \to X$ be the minimal resolution. The strategy to find X_{can} will be to identify all curves Γ in $\tilde{Y}$ not contracted by π, such that $\Gamma \cdot \pi^*(K_X) = 0$. In his case we have $\Gamma \cdot K_{\tilde{Y}} = 0$, because of the curves in the $\mathbb{Q}$-numerical effective support of $\pi^*(K_X)$. Also, since $\Gamma \cdot E_i \neq 0$ may only happen for $i = 1$ and $i = 6$, we have that $\Gamma \cdot K_Z = 0$ for Z equal to the blow-up of Y at the nodes of F_1 and F_2. Notice that Γ does not intersects P and Q as well.

The following turns out to be a useful lemma at this point.

Lemma 3.2. *Let $Y \to \mathbb{P}^1$ be a rational elliptic fibration. Assume it has two fibers F_1, F_2 of type I_1, and two sections P, Q. Let Y' be the surface obtained by blowing up the nodes of both F_1 and F_2 in Y, and blowing down P and Q. Then, Y' is a Halphen surface* [CD12, §2] *of index 2, i.e., Y' has an elliptic fibration with a unique multiple fiber of multiplicity 2. The curve $F_1 + F_2$ in Y' is a non-multiple fiber of type I_2.*

Proof. Let $\pi \colon Y \to \mathbb{P}^2$ be a blow-down to $\mathbb{P}^2$ starting with the sections P, Q (see proof of [CD12, Prop. 2.2] for example). Then, the elliptic fibration $Y \to \mathbb{P}^1$ comes from the pencil of cubics

$$\{af_1 + bf_2 \colon (a : b) \in \mathbb{P}^1\},$$

where f_1, f_2 are the cubic polynomials of the images of F_1, F_2 under π. Notice that the node of F_i is not in F_j for $i \neq j$. Hence there exists a unique cubic Λ passing through the node of F_1, the node of F_2, and the 7 base points of the pencil above not including the ones corresponding to P and Q. This gives the existence of the Halphen pencil of index 2

$$\{cf_1f_2 + d\lambda^2 \colon (c : d) \in \mathbb{P}^1\}$$

where $\lambda = 0$ is the equation of Λ. The associated Halphen surface is the Y' described in the statement of this lemma. $\square$

Now contract P and Q to obtain a Halphen surface Z' of index 2 as in Lemma 3.2. In Z' we have $\Gamma \cdot K_{Z'} = 0$. But this means that Γ does not intersect a general fiber, and so it is contained in a singular fiber. In this way, Γ must be a smooth rational curve with self-intersection (-2). The elliptic fibration on Z' has three singular fibers: one I_2^* and two I_2. The two I_2 are F_1+F_2 and $B+D$, where $D = \{x_0^2+3x_1^2 = 3x_2^2\}$. The two conics $M = \{x_0^2+3x_1^2 = 3x_1x_2\}$ and $N = \{x_0^2+3x_1^2 = -3x_1x_2\}$ are part of I_2^*, together with G_4, G_3, A, G_1, and G_5. In this way, we conclude that Γ can only be G_4, and the canonical model X_{can} of X is the contraction of G_4.

The surface X_{can} belongs to the Kollár–Shepherd-Barron–Alexeev compactification of the moduli space of surfaces of general type with $K^2 = 1$ and $p_g = 0$. The versal $\mathbb{Q}$-Gorenstein deformation space of X_{can}, denoted by $\text{Def}^{\text{QG}}(X_{\text{can}})$, is smooth and 8 dimensional; cf. [H11].

This is the argument. The smoothness of $\text{Def}^{\text{QG}}(X_{\text{can}})$ follows from $H^2(T_{X_{\text{can}}}) = 0$ and [H11, Sect.3]. To compute the dimension, we observe that if $\mathcal{X}_{\text{can}} \to \Delta$ is a $\mathbb{Q}$-Gorenstein smoothing of $\mathcal{X}_{\text{can},0} = X_{\text{can}}$ and $\mathcal{T}_{\mathcal{X}_{\text{can}}|\Delta}$ is the dual of $\Omega^1_{\mathcal{X}_{\text{can}}|\Delta}$, then $\mathcal{T}_{\mathcal{X}_{\text{can}}|\Delta}$ restricts to $\mathcal{X}_{\text{can},t}$ as $T_{\mathcal{X}_{\text{can},t}}$ (tangent bundle of $\mathcal{X}_{\text{can},t}$) when $t \neq 0$, and $\mathcal{T}_{\mathcal{X}_{\text{can}}|\Delta}|_{\mathcal{X}_{\text{can},0}} \subset T_{\mathcal{X}_{\text{can},0}}$ with cokernel supported at the singular points of $\mathcal{X}_{\text{can},0}$; cf. [Wahl81]. Then the flatness of $\mathcal{T}_{\mathcal{X}_{\text{can}}|\Delta}$ and semicontinuity in cohomology plus the fact that $H^2(\mathcal{T}_{X_{\text{can}}}) = 0$ gives $H^2(\mathcal{X}_{\text{can},t}, T_{\mathcal{X}_{\text{can},t}}) = 0$ for any t. But then, since $\mathcal{X}_{\text{can},t}$ is of general type, the Hirzebruch-Riemann-Roch Theorem says

$$H^1(\mathcal{X}_{\text{can},t}, T_{\mathcal{X}_{\text{can},t}}) = 10\chi(\mathcal{X}_{\text{can},t}, \mathcal{O}_{\mathcal{X}_{\text{can},t}}) - 2K^2_{\mathcal{X}_{\text{can},t}} = 10 - 2 = 8.$$

The space $\text{Def}^{\text{QG}}(X_{can})$ has 5 divisors whose general point represents a singular normal surface with one singularity. These general points are obtained by $\mathbb{Q}$-Gorenstein smoothing up four of the five singularities of X_{can}. The singularities are $\frac{1}{2}(1,1)$, $\frac{1}{4}(1,1)$, $\frac{1}{16}(1,11)$, $\frac{1}{25}(1,19)$, and $\frac{1}{25}(1,9)$. We denote the corresponding divisors by $\mathcal{D}(A_1)$, $\mathcal{D}\binom{2}{1}$, $\mathcal{D}\binom{4}{1}$, $\mathcal{D}\binom{5}{1}$, and $\mathcal{D}\binom{5}{2}$. It is well-known that for du Val singularities we have simultaneous resolutions, and so there is no question for $\mathcal{D}(A_1)$. The goal now is to identify the smooth minimal model of the surface represented by a general point of $\mathcal{D}\binom{n}{a}$.

The general point of $\mathcal{D}\binom{2}{1}$. Since there are no local-to-global obstructions to deform X, we consider a one parameter $\mathbb{Q}$-Gorenstein smoothing of all singularities of X except $\frac{1}{4}(1,1)$. In this family, we simultaneously resolve the singularity $\frac{1}{4}(1,1)$, obtaining a $\mathbb{Q}$-Gorenstein smoothing $\mathcal{X}_t \to \mathbb{D}$ of X_0 (= X with $\frac{1}{4}(1,1)$ resolved), where $\mathbb{D}$ is a smooth curve germ with parameter t. The general fiber is the minimal

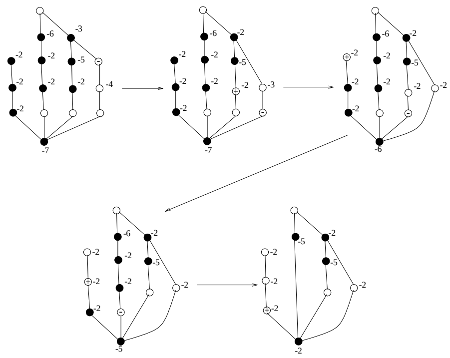

FIGURE 6. Flips for $\mathcal{D}\binom{2}{1}$

resolution of the general fiber of the old deformation. The minimal resolution of X_0 is still $\tilde{Y}$. On this surface $\tilde{Y}$, we considerer the relevant curves to perform flips of type mk1A and mk2A on $\mathcal{X}_t \to \mathbb{D}$. The flips are shown in Figure 6. We use 4 flips total.

Let $\mathcal{X}_{t,5} \to \mathbb{D}$ be the final deformation. The minimal resolution $\widetilde{X}_5$ of $\mathcal{X}_{0,5} = X_5$ is the blow-up of Y at four points: the nodes of F_1 and F_2, the intersection of P and F_1, and the intersection between Q and F_2. The surface X_5 is obtained by contracting $P + F_2$ and $Q + F_1$ in $\widetilde{X}_5$. By Lemma 3.2, we can see $\widetilde{X}_5$ as the blow-up at four points of a Halphen surface of index 2, and then Lemma 6.2 with a configuration $[3, 3]$ (coming from $[2, 5] - 1 - [2, 5]$), we obtain that a $\mathbb{Q}$-Gorenstein smoothing of X_5 is a Dolgachev surface of type $(2, 3)$ (cf. [BHPV04, p.383]).

Proposition 3.3. *The minimal resolution of a surface representing the general point in* $\mathcal{D}\binom{2}{1}$ *is a Dolgachev surface of type* $(2, 3)$*. It contains a smooth rational curve with self-intersection* (-4)*.*

Because of the simplicity of the singularity $\frac{1}{4}(1, 1)$, the previous proposition can also be proved in the following way. Let Y be a smooth projective surface containing a (-4)-curve Γ and $K_Y^2 = 0$. Let $f : Y \to X$ be the contraction of Γ. If K_X is nef, then Y is not rational. Indeed, if Y is rational, then by Riemann-Roch $h^0(Y, -K_Y) \geq 1$ and so $-K_Y \sim E \geq 0$. Since $K_Y \cdot \Gamma = 2$, we have $\Gamma \subset E$. We know that $f^*(2K_X) \sim -2E + \Gamma$. But $E \neq \Gamma$, and so $f^*(2K_X)$ cannot be nef. In this way, in Proposition 3.3 we cannot have that the resolution of $\frac{1}{4}(1, 1)$ is rational. Also, the Kodaira dimension cannot be 0 because of Γ and cannot be 2 because of Proposition 3.9. Therefore it is 1, and so it has an elliptic fibration. Since it is simply connected, then it has exactly two multiple fibers of multiplicities a and b. But now it is easy to check using the canonical class formula and Γ that the only possibility is $a = 2$ and $b = 3$: a Dolgachev surface of type $(2, 3)$.

The general point of $\mathcal{D}\binom{4}{1}$. As in the previous case, we do the same but now with the singularity $\frac{1}{16}(1, 11)$. We perform 7 flips as shown in Figure 7. Let X_7 be the central singular fiber of the corresponding deformation after the 7th flip. It has only a $\frac{1}{4}(1, 1)$ singularity. The minimal resolution of X_7 is the blow up of Y at two points, which are disjoint from the (-4)-curve. This situation is as in Theorem 6.1 part (-1). The general fiber of the $\mathbb{Q}$-Gorenstein smoothing is rational.

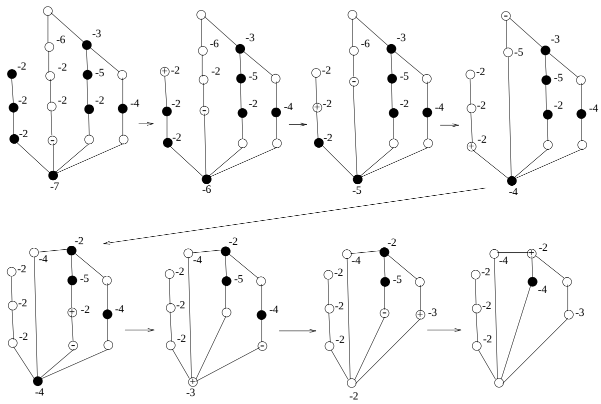

FIGURE 7. Flips for $\mathcal{D}\binom{4}{1}$

Proposition 3.4. *The minimal resolution of a surface representing the general point in $\mathcal{D}\binom{4}{1}$ is a rational surface with $K^2 = -2$. It contains the configuration of rational smooth curves $[6, 2, 2]$ and a (-1)-curve intersecting the (-6)-curve transversally at two points.*

The (-1)-curve intersecting the (-6)-curve transversally at two points comes from the (-1)-curve having the same property in the special fiber. This curve does not contain any singularity of the special fiber and so it lifts in any deformation.

The general point of $\mathcal{D}\binom{5}{1}$. Following the same recipe for the singularity $\frac{1}{25}(1, 19)$, we perform the sequence of 3 flips shown in Figure 8. Notice that the situation after the last flip is very similar to the previous case.

Proposition 3.5. *The minimal resolution of a surface representing the general point in $\mathcal{D}\binom{5}{1}$ is a rational surface with $K^2 = -3$. It contains the configuration of rational smooth curves $[7, 2, 2, 2]$ and two*

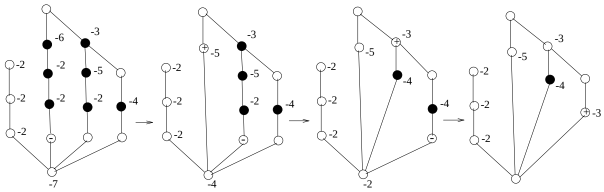

Figure 8. Flips for $\mathcal{D}\binom{5}{1}$

disjoint (-1)*-curves intersecting the* (-7)*-curve transversally at two points each.*

The existence of the (-1)-curves intersecting the (-7)-curve is an application of Proposition 2.9. It is applied several times via partial smoothings. We start with X_0, which is X with $\frac{1}{5^2}(1,4)$ resolved, and $\mathbb{Q}$-Gorenstein smooth up $\frac{1}{4^2}(1,3)$. Then the curves E_2 and E_5 produce a (-1)-curve E_t in the general fiber, intersecting the (-7)-curve at one point, using Proposition 2.9. We now $\mathbb{Q}$-Gorenstein smooth up $\frac{1}{4}(1,1)$, and by the same proposition we obtain a (-1)-curve E'_t in the general fiber from E_{10} and E_9. Finally we $\mathbb{Q}$-Gorenstein smooth up $\frac{1}{5^2}(1,9)$ to get the two claimed (-1)-curves, each from E_t, E_8, and E'_t, E_8, applying again Proposition 2.9. The intersection properties can be easily checked.

The general point of $\mathcal{D}\binom{5}{2}$. In this case we perform the flips shown in Figure 9. At the end, the special fiber is not singular anymore, and so we know that the general fiber of the deformation is a rational surface.

Proposition 3.6. *The minimal resolution of a surface representing the general point in* $\mathcal{D}\binom{5}{2}$ *is a rational surface with* $K^2=-2$*. It contains the configuration of rational smooth curves* $[2,5,3]$ *and a* (-1)*-curve intersecting the* (-5)*-curve transversally at two points.*

The (-1)-curve comes from the (-1)-curve E_6 intersecting the (-5)-curve transversally at two points.

This finishes the description of the stable neighbors of X_{can}.

Remark 3.7. The construction of a surface Z with same Wahl singularities as X can be done over an elliptic rational surface with singular fibers $I_4+6I_1+I_2$. This elliptic fibration has moduli dimension 4. From the 4 Wahl singularities $\frac{1}{4}(1,1)$, $\frac{1}{16}(1,3)$, $\frac{1}{25}(1,4)$, and $\frac{1}{25}(1,9)$ we get the other 4 dimensions to complete the 8 dimensions we have around the stable surface in the moduli space.

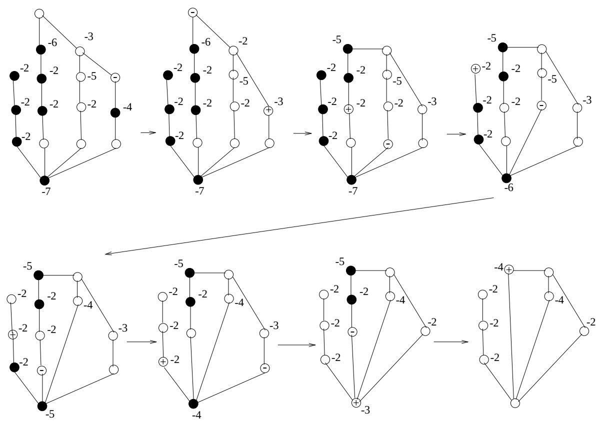

FIGURE 9. Flips for $\mathcal{D}\binom{5}{2}$

Remark 3.8. For the other example with $K^2 = 1$ in [LP07, Fig.6] we have a surface with Wahl singularities and canonical class nef. This example is related to the previous in the following way. Take a (-1)-curve from [LP07, Fig.6] between the configuration $[2,2,6]$ and $[4]$ (there are two choices). The configuration $[2,2,6]-1-[4]$ represents the data of an extremal P-resolution of $\frac{1}{36}(1,13)$. But this singularity admits another extremal P-resolution: $[3,5,2]-2$. (We recall that in [HTU13, §4] we have a section devoted to this type of singularities.) Now consider the corresponding "dual" deformation. The canonical class of the central fiber is now not nef, because there is a (-1)-curve intersecting the (-8)-curve at one point. So we perform one flip of type mk1A and obtain the previous example. Therefore, we have a sort of dual families. This is a common phenomena in these sort of examples, coming from a cyclic quotient singularity having two extremal P-resolutions.

The analog results for partial smoothings of the Wahl singularities in the example [LP07, Fig. 6] are: for both $[4]$ Dolgachev surfaces of type $(2,3)$ (for $[3,3]$ we also have Dolgachev surfaces of the same type), for the other singularities we obtain rational surfaces.

One may wonder at this point what sort of surfaces with only Wahl singularities one can expect in the boundary in general. The following proposition, due to Kawamata [K92], says that at least there is a hierarchy with respect to K^2 and the Kodaira dimension.

Proposition 3.9. *Let $f\colon \mathcal{X} \to \mathbb{D}$ be a $\mathbb{Q}$-Gorenstein smoothing of a normal singular projective surface X_0 with only Wahl singularities over a smooth curve germ $\mathbb{D}$. Let Y_0 be the minimal resolution of X_0, and let Z_0 be the smooth minimal model of Y_0. Assume that $K_{\mathcal{X}}$ is nef. If*

Z_0 is of general type, then the general fiber X_t is of general type and $K^2_{X_t} > K^2_{Z_0}$.

Proof. By Kawamata [K92, Lemma 2.4], there exist positive integers m_1 and m_2 such that the inequalities of m-plurigenera $P_m(X_t) > P_m(Z_0)$ hold for positive integers m with m_1 dividing m and $m_2 < m$. This implies that X_t is of general type. Moreover, this inequality becomes [BHPV04, VII Cor(5.4)]

$$\frac{m(m-1)}{2}K^2_{X_t} + \chi(X_t) > \frac{m(m-1)}{2}K^2_{Z_0} + \chi(Z_0)$$

for those m, and so we have the claim. $\square$

This implies that the stable boundary appearing in this way for $K^2 = 1$ consists of surfaces whose minimal resolution is not of general type. This is not the case for $K^2 > 1$, as we will see in the next sections.

4. $K^2 = 2$

We take the example [LP07, Fig.2]. It uses the same elliptic fibration of Section 3. The corresponding surface X with only Wahl singularities has K_X nef. One can use Lemma 3.2 to show that K_X is ample in this case, so X is a stable surface. The 5 Wahl singularities define 5 boundary divisors. We label them as before: $\mathcal{D}\binom{2}{1}$ for $[4]$, $\mathcal{D}\binom{3}{1}$ for $[2,5]$, $\mathcal{D}\binom{5}{1}$ for $[7,2,2,2]$, $\mathcal{D}\binom{9}{4}$ for $[2,7,2,2,3]$, and $\mathcal{D}\binom{15}{7}$ for $[2,10,2,2,2,2,2,2,3]$.

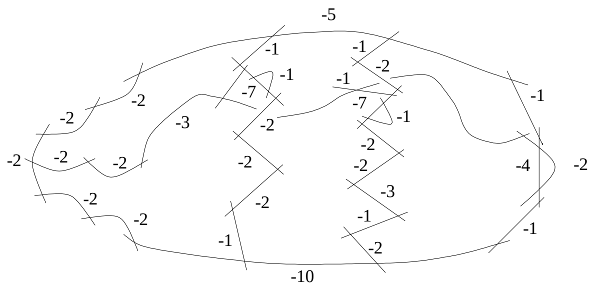

FIGURE 10. The example [LP07, Fig. 2]

The general point of $\mathcal{D}\binom{2}{1}$. We proceed as in Section 3. We perform 4 flips as in Figure 11: the first two are mk1A flips, the last two are mk2A flips. If X_4 is the last singular surface, then it has 5 Wahl singularities and K_{X_4} is nef. Notice that $K^2_{X_4} = 1$.

Proposition 4.1. *The minimal resolution of a surface representing the general point in $\mathcal{D}\binom{2}{1}$ is a simply connected surface of general type with $p_g = 0$ and $K^2 = 1$. It contains a (-4)-curve.*

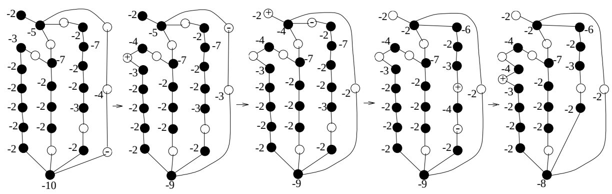

FIGURE 11. Flips for $\mathcal{D}\binom{2}{1}$

Notice that this flipping procedure gives in this case new examples for $K^2 = 1$ from X_4: its minimal resolution has T-configurations [4], [4], $[2, 6, 2, 3]$, $[7, 2, 2, 2]$, and $[3, 2, 2, 2, 8, 2]$.

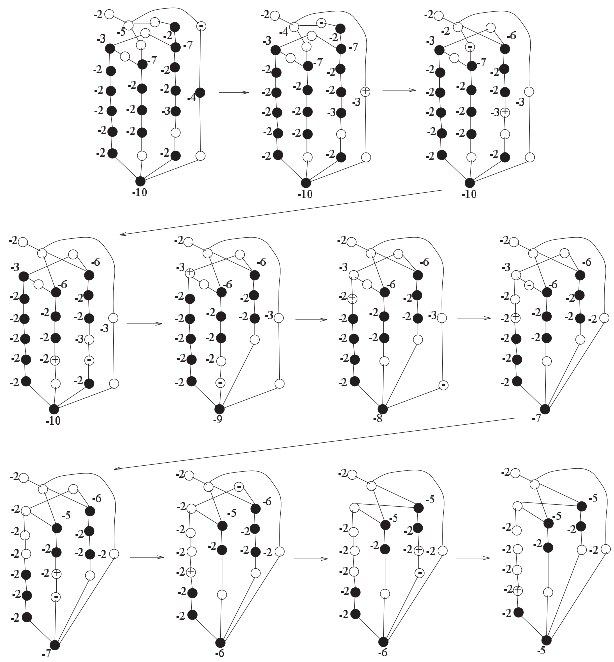

FIGURE 12. Flips for $\mathcal{D}\binom{3}{1}$

The general point of $\mathcal{D}\binom{3}{1}$. Here we perform the 11 flips shown in Figure 12. One can check that X_{11}, the last surface, has $K^2_{X_{11}} = 0$ and $K_{X_{11}}$ nef. Therefore, the general fiber of the $\mathbb{Q}$-Gorenstein smoothing is a Dolgachev surface of some type (n_1, n_2), since we know it is also simply connected. To find n_1, n_2, one can argue that a $\mathbb{Q}$-Gorenstein smoothing of X_{11} was used in the second example with $K^2 = 1$ of the previous section. There we knew that the Dolgachev surface contained a (-4)-curve, and so one obtains $n_1 = 2$, $n_2 = 3$. So we have same multiplicities for our current example (although we do not know if there is a (-4)-curve inside).

Proposition 4.2. *The minimal resolution of a surface representing the general point in* $\mathcal{D}\binom{3}{1}$ *is a Dolgachev surface of type* $(2,3)$ *which contains a configuration* $[2,5]$.

For the other 3 divisors we perform some flips to deduce that its general point is rational and

$\mathcal{D}\binom{5}{1}$: $K^2 = -2$ with a configuration $[2,2,2,7]$ inside.
$\mathcal{D}\binom{9}{4}$: $K^2 = -3$ with a configuration $[3,2,2,7,2]$ inside.
$\mathcal{D}\binom{15}{7}$: $K^2 = -6$ and a configuration $[3,2,2,2,2,2,10,2]$ inside.

For the other example [LP07, Fig.4], we find: for each of the [4] a simply connected surface of general type with $K^2 = 1$ and $p_g = 0$, when keeping both singularities $\frac{1}{4}(1,1)$ a Dolgachev surface $(2,3)$ with two disjoint (-4)-curves, and finally for each of the other Wahl singularities we obtain rational surfaces.

5. $K^2 = 3$

In [PPS09] there are five examples producing simply connected surfaces of general type with $p_g = 0$ and $K^2 = 3$. We take the one in [PPS09, Fig.8] because, as explained in [PPS09e], it contains a negative curve which makes the canonical divisor of the singular surface not nef. This curve gives the data of a flipping mk2A. The flip is shown in Figure 13.

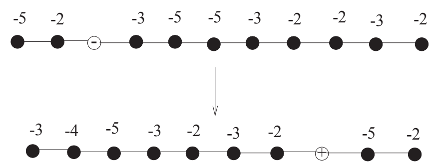

FIGURE 13. Flip for [PPS09, Fig.8]

We can show that after this flip, the resulting surface has nef canonical divisor. Therefore, the example has the claimed properties in [PPS09]. The minimal resolution $\widetilde{X}$ of the singular resulting surface X is in Figure 14. Let F be the general fiber of the induced elliptic fibration on $\widetilde{X}$. Then, following the notation in Figure 14, we have

$$K_{\widetilde{X}} \sim \sum_{i=1}^{15} E_i + E_7 + 2E_8 + E_{11} + E_{13} + E_{15} - F$$

and so $K_{\widetilde{X}} \equiv -\frac{1}{2}F_1 - \frac{1}{2}F_2 + E_1 + E_2 + \frac{1}{2}E_4 + \frac{1}{2}E_5 + \frac{1}{2}E_7 + E_8 + \frac{1}{2}E_9 + E_{10} + 2E_{11} + E_{12} + 2E_{13} + E_{14} + 2E_{15}$. We add the discrepancies and get an effective $\mathbb{Q}$-divisor for $\sigma^*(K_X)$. One checks that it is indeed nef.

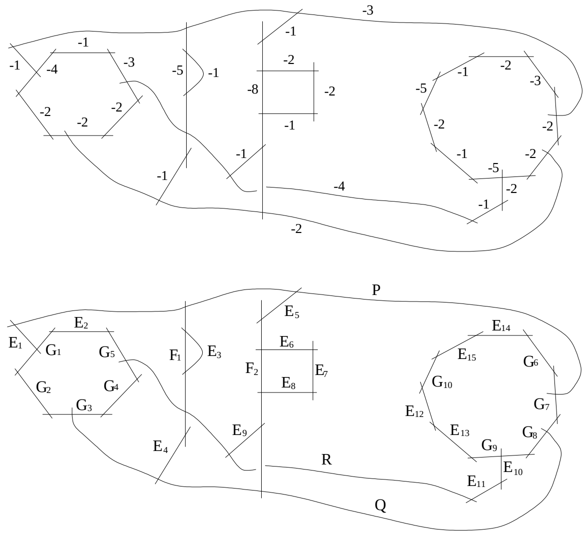

Figure 14. $\widetilde{X}$ and relevant curves

Moreover, its support contains E_5, F_2, E_6, E_7, E_8, and E_9 which is the support of a fiber. This implies that the only curves which could have intersection 0 with K_X are components of fibers. Then, the only one is E_{13}. Let X_{can} be the contraction of E_{13} in X, so $K_{X_{can}}$ is ample and X_{can} is stable. As we did in Section 3, the corresponding stable point in the moduli space is smooth of dimension 4. The singularities of X_{can} are $\frac{1}{30^2}(1, 30 \cdot 11 - 1)$, $\frac{1}{2 \cdot 3^2}(1, 2 \cdot 3 \cdot 1 - 1)$, and $\frac{1}{16^2}(1, 16 \cdot 11 - 1)$. Their $\mathbb{Q}$-Gorenstein deformation spaces give precisely the dimension $4 = 1 + 2 + 1$. In that sense, this is a “maximal degeneration”.

The loci in the moduli space defined by keeping the singularity $\frac{1}{2 \cdot 3^2}(1, 2 \cdot 3 \cdot 1 - 1)$ has codimension 2. We have that the (minimal model of a minimal resolution of) general point of it is a simply connected surface of general type with $K^2 = 1$ (and $p_g = 0$), and they have a configuration $[4, 3, 2]$ inside. The singularity $\frac{1}{2 \cdot 3^2}(1, 2 \cdot 3 \cdot 1 - 1)$ also $\mathbb{Q}$-Gorenstein deforms to $\frac{1}{9}(1, 2)$, and if we smooth up the other singularities, we obtain a surface of general type with $K^2 = 1$ as well. Finally, for each of the other two singularities we have divisors parametrizing rational surfaces.

Remark 5.1. With the example [PPS09, Fig.9] we can show that there are $K^2 = 2$ surfaces of general type with $p_g = 0$ in the boundary of the moduli space for $K^2 = 3$. We keep in a $\mathbb{Q}$-Gorenstein deformation the singularity $\frac{1}{4}(1,1)$ and smooth up the other two. After few flips we get a singular surface with 4 Wahl singularities whose exceptional configurations are $[2,3,2,3,5,4,3]$, $[2,5]$, $[2,5]$, and $[6,2,2]$. Its canonical class is nef and $K^2 = 2$.

6. Elliptic surfaces via $\mathbb{Q}$-Gorenstein smoothings

Notice that the exceptional divisor of any T-singularity

$$\frac{1}{dn^2}(1, dna - 1)$$

can be obtained from an I_d elliptic singular fiber by blowing-up over a node. We blow up a node of I_d and subsequent nodes coming from the new (-1)-curves. The exceptional divisor appears as the chain of curves of the total transform of I_d which does not contain the (last) (-1)-curve. We call this construction a *T-blow-up* of I_d. If $g\colon Y \to B$ is the elliptic fibration with the singular fiber I_d, then we denote by $\sigma\colon Y' \to Y$ the composition of blow-ups. This way of looking at T-singularities is in Kawamata's paper [K92].

The following is a useful list of cases of $\mathbb{Q}$-Gorenstein smoothings from fibers of rational elliptic fibrations. We used it a bit to identify smooth models of surfaces around stable surfaces with only T-singularities.

Theorem 6.1. *Let $g\colon Y \to \mathbb{P}^1$ be a minimal rational elliptic fibration with a section.*

(-1): *Assume g has a fiber of type I_d. Consider a T-blow-up of I_d with the notation above. Let $\{E_1, \ldots, E_s\}$ be the corresponding T-configuration where $\frac{1}{dn^2}(1, dna-1) = [e_1, \ldots, e_s]$ and $E_i^2 = -e_i$. Write $\sigma^*(I_d) = \sum_{i=1}^{s+1} \nu_i E_i$, where E_{s+1} is the (-1)-curve. Then there are $\mathbb{Q}$-Gorenstein smoothings X of X', and any such X is rational. We have $n = \nu_{s+1}$, $a = \nu_{s+1} - \nu_s$, and the discrepancy at E_i is $-1 + \frac{\nu_i}{\nu_{s+1}}$ for any $i = 1, \ldots, s$.*

(0): *Assume g has two fibers I_{d_1} and I_{d_2}. Let Y' be the blow-up of Y at one node of I_{d_1} and at one node of I_{d_2}. Hence we have two T-configurations of type $\frac{1}{4d_i}(1, 2d_i - 1)$. Let X' be the contraction of these configurations. Then there are $\mathbb{Q}$-Gorenstein smoothings X of X', and any such X is an Enriques surface.*

(1): *Assume it has two fibers I_{d_1} and I_{d_2}. We apply T-blow-ups to each of them. Assume that for one of them we blew-up at least twice. Let X' be the contraction of both T-configurations, where $\frac{1}{d_i n_i^2}(1, d_i n_i a_i -$*

1) *are the T-singularities. Then there are $\mathbb{Q}$-Gorenstein smoothings X of X', and any such X has Kodaira dimension* 1.

Proof. For the proof, we assume g has the singular fibers I_{d_1} and I_{d_2}. This situation adjusts to prove all cases simultaneously. Let $\sigma\colon Y' \to Y$ be the composition of blow ups for both T-blow-ups, so that Y' contains the T-configurations $\{E_1, \ldots, E_s\}$ and $\{F_1, \ldots, F_r\}$ of types $\frac{1}{d_1 n_1^2}(1, d_1 n_1 a_1 - 1) = [e_1, \ldots, e_s]$ and $\frac{1}{d_2 n_2^2}(1, d_2 n_2 a_2 - 1) = [f_1, \ldots, f_r]$, where $E_i^2 = -e_i$ and $F_i^2 = -f_i$. We also have the (-1)-curves E_{s+1} and F_{r+1}, so that $\sigma^*(I_{d_1}) = \sum_{i=1}^{s+1} \nu_i E_i$, and $\sigma^*(I_{d_2}) = \sum_{i=1}^{r+1} \mu_i F_i$. Let $h\colon Y' \to X'$ be the contraction of both T-configurations.

Through simple arguments as in [LP07], we know that there are no local-to-global obstructions to deform X' because

$$H^2(Y', T_{Y'}(-\log(E_1 + \ldots + E_s + F_1 + \ldots + F_r))) = 0.$$

Let C be the general fiber of g. Then,

$$K_{Y'} \sim -\sigma^* C + \sum_{i=1}^{s+1} (\nu_i - 1) E_i + \sum_{i=1}^{r+1} (\mu_i - 1) F_i$$

and $K_{Y'} \equiv h^* K_{X'} - \sum_{i=1}^{s} \mathrm{discr}(E_i) E_i - \sum_{i=1}^{r} \mathrm{discr}(F_i) F_i$, where discr stands for minus the discrepancy. Then, by intersecting with all E_i's and F_i's, we get a linear system of equations on the ν_i's and μ_i's, which is uniquely solved by our numerical claims: $\mathrm{discr}(E_i) = 1 - \frac{\nu_i}{n_1}$ and $\mathrm{discr}(F_i) = 1 - \frac{\mu_i}{n_2}$. In this way, we have

$$h^*(K_{X'}) \equiv -\frac{1}{n_1} \sum_{i=1}^{s+1} \nu_i E_i \equiv -\frac{1}{n_1} \sigma^* C$$

for the case **(-1)**, and

$$h^*(K_{X'}) \equiv \frac{n_1 - 2}{2n_1} \sum_{i=1}^{s+1} \nu_i E_i + \frac{n_2 - 2}{2n_2} \sum_{i=1}^{r+1} \mu_i F_i \equiv \Big(1 - \frac{1}{n_1} - \frac{1}{n_2}\Big) \sigma^* C$$

for cases **(0)** and **(1)**. Then, in case **(-1)** we have that $-K_{X'}$ is nef and not $\equiv 0$, and so X is a rational surface. We recall that in any case, $K_X^2 = 0$, $q(X) = p_g(X) = 0$. For the case **(0)** we see that $K_{X'} \equiv 0$ and so for K_X. It follows that X is an Enriques surface. For the last case **(1)**, $K_{X'}$ is nef and not trivial, and so X is a minimal surface with Kodaira dimension 1.

□

We recall that a Dolgachev surface of type X_{n_1, n_2} is a simply connected elliptic fibration with exactly two multiple fibers of multiplicities n_1 and n_2; cf. [BHPV04, p.383].

Corollary 6.2. *If in case **(1)** we have $gdc(n_1, n_2) = 1$, then a smooth fiber of any $\mathbb{Q}$-Gorenstein smoothing is a Dolgachev surface of type X_{n_1,n_2}.*

Proof. First, we recall that given a Hirzebruch-Jung continued fraction $\frac{m}{q} = [e_1, \ldots, e_s]$ has associated sequences $\alpha_i, \beta_i, \gamma_i$ as in Section 2. The corresponding E_i divisor of the minimal resolution of $\frac{1}{m}(1, q)$ has discrepancy $-1 + \frac{\beta_i+\alpha_i}{m}$. Also, the fundamental group of a neighborhood of the complement of the exceptional divisor is cyclic of order m and it is generated by a loop ξ around E_1 (or E_s). The loops ξ_i around E_i are conjugate to ξ^{α_i} (or ξ^{β_i}). (Cf. [Mum61].) For $m = n^2$ and $q = na - 1$ with $\gcd(n, a) = 1$, we have $\beta_i + \alpha_i = \nu_i n$ by Theorem 6.1, and so $\nu_i = a\alpha_i - n\gamma_i$.

We compute the fundamental group π_1 of a smooth fiber of a $\mathbb{Q}$-Gorenstein smoothing following the strategy of [LP07, p.493]. The computation is done on the minimal resolution $\widetilde{X'}$ of the singular fiber X'. It is enough to show that $\pi_1(\widetilde{X'} \setminus E)$ is trivial, where E is the exceptional divisor. We consider two small loops ξ and ρ around the two components of E which intersect a section of the elliptic fibration. By above, we notice that for those component the $\nu_{j_i,i} = 1$ $(i = 1, 2)$, and so $\gcd(\beta_{j_i,i}, n_i) = 1$ and these loops generate the fundamental groups of the nbhds of the complements of each exceptional component. This section, which is a $\mathbb{P}^1$, gives the relation $\xi \sim t\rho t^{-1}$ for some path t. We now use that $\gcd(n_1, n_2) = 1$ to conclude that ξ and ρ become trivial in $\pi_1(\widetilde{X'} \setminus E)$. This implies that $\pi_1(\widetilde{X'} \setminus E) = 1$.

Therefore, the smooth fiber X_t is a simply connected elliptic fibration with exactly two multiple fibers. If A_t and B_t are the reduced curves of the multiple fibers of the elliptic fibration $X_t \to \mathbb{P}^1$, and C_t is its general fiber, then C_t becomes the general fiber of the elliptic fibration $X' \to \mathbb{P}^1$, and M_t, N_t become multiples of the reduced curves of the multiple fibers of $X' \to \mathbb{P}^1$. By means of the formula for the canonical class for the fibrations $X' \to \mathbb{P}^1$ and $X_t \to \mathbb{P}^1$, and because the multiplicities for both fibrations are coprime, we conclude that the multiplicities must match. □

References

BHPV04. W. P. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven. *Compact complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., second edition, vol. 4, Springer-Verlag, Berlin, 2004.

CD12. S. Cantat, and I. Dolgachev. *Rational surfaces with a large group of automorphisms*, J. Amer. Math. Soc., vol 25 (2012), no. 3, 863–905.

H11. P. Hacking. *Compact moduli spaces of surfaces of general type*, Compact moduli spaces and vector bundles, 1-18, Contemp. Math., 564, Amer. Math. Soc., Providence, RI, 2012.

HTU13. P. Hacking, J. Tevelev and G. Urzúa. *Flipping surfaces*, arXiv:1310.1580 [math.AG], preprint 2013.

HP10. P. Hacking and Y. Prokhorov. *Smoothable del Pezzo surfaces with quotient singularities*, Compositio Math. 146(2010), 169–192.

K92. Y. Kawamata. *Moderate degenerations of algebraic surfaces*, Complex algebraic varieties (Bayreuth, 1990), 113132, Lecture Notes in Math., 1507, Springer, Berlin, 1992.

KSB88. J. Kollár and N. I. Shepherd-Barron. *Threefolds and deformations of surface singularities*, Invent. math. 91, 299–338 (1988).

KM92. J. Kollár and S. Mori. *Classification of three-dimensional flips*, J. Amer. Math. Soc. v.5, number 3, July 1992, 533–703.

KM98. J. Kollár and S. Mori. *Birational geometry of algebraic varieties*, CTM 134 (1998).

LP07. Y. Lee and J. Park. *A simply connected surface of general type with $p_g = 0$ and $K^2 = 2$*, Invent. Math. 170, 483–505 (2007).

Mori88. S. Mori. *Flip theorem and the existence of minimal models for 3-folds*, J. Amer. Math. Soc. 1(1988), 117–253.

Mori02. S. Mori. *On semistable extremal neighborhoods*, Higher dimensional birational geometry (Kyoto 1997), Adv. Stud. Pure Math. 35, Math. Soc. Japan, Tokyo, 157–184 (2002).

Mum61. D. Mumford. *The topology of normal singularities of an algebraic surface and a criterion for simplicity*, Publ. Math., Inst. Hautes Étud. Sci. 36, 229–246 (1961).

PPS09. H. Park, J. Park and D. Shin. *A simply connected surface of general type with $p_g = 0$ and $K^2 = 3$*, Geom. Topol. 13 (2009), 743–767.

PPS09e. H. Park, J. Park and D. Shin. *Erratum to the article A simply connected surface of general type with $p_g = 0$ and $K^2 = 3$*, Geom. Topol. 15 (2011), no. 1, 499-500.

Wahl76. J. Wahl, *Equisingular deformations of normal surface singularities*, Ann. Math, 104 (1976), 325–356.

Wahl81. J. Wahl. *Smoothings of normal surface singularities*, Topology 20(1981), no. 3, 219-246.

Facultad de Matemáticas,
Pontificia Universidad Católica de Chile,
Santiago, Chile.
E-mail address: `urzua@mat.puc.cl`