Γ-convergence approximation of fracture and cavitation in nonlinear elasticity

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Abstract

Our starting point is a variational model in nonlinear elasticity that allows for cavitation and fracture that was introduced by Henao and Mora-Corral (2010). The total energy to minimize is the sum of the elastic energy plus the energy produced by crack and surface formation. It is a free discontinuity problem, since the crack set and the set of new surface are unknowns of the problem. The expression of the functional involves a volume integral and two surface integrals, and this fact makes the problem numerically intractable. In this paper we propose an approximation (in the sense of Γ-convergence) by functionals involving only volume integrals, which makes a numerical approximation by finite elements feasible. This approximation has some similarities to the Modica–Mortola approximation of the perimeter and the Ambrosio–Tortorelli approximation of the Mumford–Shah functional, but with the added difficulties typical of nonlinear elasticity, in which the deformation is assumed to be one-to-one and orientation-preserving.

1 Introduction

Free-discontinuity problems have attracted a great amount of attention in the mathematical community in the last decades, because of their applications and of the mathematical challenges that they pose. We refer to the monograph [7] for an in-depth study. A common feature of these problems is the presence of an interaction between an n-dimensional volume energy and an $(n-1)$-dimensional surface energy. The latter involves a surface set, which is an unknown of the problem. A paradigmatic model is the Mumford & Shah [50] functional for image segmentation, which was recasted as a variational free-discontinuity problem by De Giorgi, Carriero and Leaci [29] as follows: for a given $f \in L^2(\Omega)$, minimize

$$\int_\Omega \left[|\nabla u|^2 + (u-f)^2 \right] \mathrm{d}\mathbf{x} + \mathcal{H}^{n-1}(J_u) \tag{1.1}$$

among $u \in SBV(\Omega)$. Here, Ω is a bounded open set of $\mathbb{R}^n$ and SBV is the space of functions of special bounded variation. In this case, the free discontinuity set is J_u, the *jump set* of u.

In elasticity theory, the paradigmatic free-discontinuity problem is that of fracture, which can be seen as a vectorial version of the Mumford–Shah functional. In its simplest form, the functional to minimize is

$$\int_\Omega |\nabla \mathbf{u}|^2 \, \mathrm{d}\mathbf{x} + \mathcal{H}^{n-1}(J_{\mathbf{u}}) \tag{1.2}$$

among $\mathbf{u} \in SBV(\Omega, \mathbb{R}^n)$. The first term of (1.2) is a handy substitute of the elastic energy, and the second term penalizes the crack formation, as stipulated by Griffith's [35] theory of fracture. It was Francfort & Marigo [32] who proposed a variational formulation of brittle fracture in a quasi-static setting.

Another phenomenon in elasticity theory that can be regarded as a free-discontinuity problem is that of cavitation, which is the process of formation and rapid expansion of voids in solids, typically under triaxial tension. The seminal paper of Ball [11] described this process as a singular ordinary differential equation, but in his work and in others following it, the location of the cavity points was prescribed. It was shown by Müller & Spector [48] that cavitation can be recasted as a free-discontinuity problem following the general scheme described above. In this case, the energy to minimize is

$$\int_\Omega W(D\mathbf{u})\,\mathrm{d}\mathbf{x} + \operatorname{Per}\mathbf{u}(\Omega) \tag{1.3}$$

among $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n)$ satisfying some invertibility conditions. The first term of (1.3) is the elastic energy of the deformation, while the second term represents the energy produced by the creation of new surface, and, hence, by the cavitation. The idea is that the image $\mathbf{u}(\Omega)$, properly defined, may create a hole which was not previously in Ω. The new surface created by the hole is detected by $\operatorname{Per}\mathbf{u}(\Omega)$, so in this case the free discontinuity set is the measure-theoretic boundary of $\mathbf{u}(\Omega)$, which lies in the deformed configuration.

Our free discontinuity problem to be approximated gathers the fracture functional with the cavitation functional. To be precise, Henao & Mora-Corral [37, 38, 39] showed that when the functional setting allows for cavitation and fracture, it was convenient to replace the term $\operatorname{Per}\mathbf{u}(\Omega)$ in (1.3) by the functional

$$\mathcal{E}(\mathbf{u}) := \sup\left\{\mathcal{E}(\mathbf{u},\mathbf{f}) :\ \mathbf{f} \in C_c^\infty(\Omega \times \mathbb{R}^n, \mathbb{R}^n),\ \|\mathbf{f}\|_\infty \leq 1\right\},$$

where

$$\mathcal{E}(\mathbf{u},\mathbf{f}) := \int_\Omega \left[\operatorname{cof}\nabla\mathbf{u}(\mathbf{x}) \cdot D_\mathbf{x}\mathbf{f}(\mathbf{x},\mathbf{u}(\mathbf{x})) + \det\nabla\mathbf{u}(\mathbf{x}) \operatorname{div}\mathbf{f}(\mathbf{x},\mathbf{u}(\mathbf{x}))\right] \mathrm{d}\mathbf{x}.$$

They proved that $\mathcal{E}(\mathbf{u})$ equals the $\mathcal{H}^{n-1}$-measure of the new surface created by $\mathbf{u}$, whether produced by cavitation, fracture or any other process of surface creation. They also proved the existence of minimizers of

$$\int_\Omega W(D\mathbf{u})\,\mathrm{d}\mathbf{x} + \mathcal{H}^{n-1}(J_\mathbf{u}) + \mathcal{E}(\mathbf{u}) \tag{1.4}$$

among $\mathbf{u} \in SBV(\Omega, \mathbb{R}^n)$ satisfying some invertibility conditions. We remark that in (1.3) and (1.4), the stored-energy function W is polyconvex and has the growth

$$W(\mathbf{F}) \to \infty \quad \text{as } \det\mathbf{F} \to 0. \tag{1.5}$$

In this paper, we define a slight variant of the functional $\mathcal{E}$, namely

$$\bar{\mathcal{E}}(\mathbf{u}) := \sup\left\{\mathcal{E}(\mathbf{u},\mathbf{f}) :\ \mathbf{f} \in C_c^\infty(\bar{\Omega} \times \mathbb{R}^n, \mathbb{R}^n),\ \|\mathbf{f}\|_\infty \leq 1\right\}.$$

The main difference of $\bar{\mathcal{E}}$ with respect to $\mathcal{E}$ is that, while $\mathcal{E}$ measures the surface created, $\bar{\mathcal{E}}$ also measures the stretching of the boundary $\partial\Omega$ by the deformation. In fact, it can be proved that, loosely speaking, the equality

$$\bar{\mathcal{E}}(\mathbf{u}) = \mathcal{E}(\mathbf{u}) + \mathcal{H}^{n-1}(\mathbf{u}(\partial\Omega))$$

holds.

A direct approach to numerical minimization of free-discontinuity functionals, as those described above, is unfeasible using standard methods. A fruitful procedure is the construction of an approximating sequence of elliptic functionals I_ε, possibly defined in a different functional space, that Γ-converge to the functional I to be approximated.

One of the first results in this direction was the example of Modica & Mortola [45], which was recasted by Modica [44] as an approximation of a model for phase transitions in liquids. They showed how the perimeter functional can be approximated by elliptic functionals via Γ-convergence. As a particular case, they showed the convergence of

$$3\int_\Omega \left[\varepsilon|Dw|^2 + \frac{w^2(1-w)^2}{\varepsilon}\right] \mathrm{d}\mathbf{x} \tag{1.6}$$

for functions $w \in W^{1,2}(\Omega)$ with prescribed mass $\int_\Omega w\,\mathrm{d}\mathbf{x}$, to the functional

$$\operatorname{Per} w^{-1}(0)$$

in the space $BV(\Omega, \{0,1\})$.

A landmark study was the approximation by Ambrosio & Tortorelli [8, 9] of the Mumford–Shah functional (1.1) by the functionals

$$\int_\Omega (v^2 + \eta_\varepsilon)\,|Du|^2\,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Omega \left[\varepsilon|Dv|^2 + \frac{(1-v)^2}{\varepsilon}\right]\mathrm{d}\mathbf{x}$$

for $u, v \in W^{1,2}(\Omega)$. Here v is an extra variable that converges a.e. to 1, and indicates healthy material when $v \simeq 1$ and damaged material when $v \simeq 0$. The infinitesimal η_ε goes to zero faster than ε.

The work of Ambrosio & Tortorelli [8] has given rise to many extensions (the reader is referred, in particular, to the monograph [18]), as well as actual numerical studies and experiments [15, 14, 22]. We ought to say that the numerical experiments of Bourdin, Francfort & Marigo [16] (see also the review paper [17]) were in fact a strong motivation for our work, and so was the analysis by Burke [21] of the Ambrosio–Tortorelli functional.

In our context of interest of fracture, we mention that Chambolle [23] was able to extend their result to approximate, instead of (1.2), the more realistic energy

$$\int_\Omega W(\nabla\mathbf{u})\,\mathrm{d}\mathbf{x} + \mathcal{H}^{n-1}(J_\mathbf{u}), \tag{1.7}$$

when W equals the quadratic functional corresponding to linear elasticity. Later, Braides, Chambolle & Solci [20] proved that the Γ-convergence still holds for a quasiconvex W with p-growth from above and below. As a by-product of our analysis, we cover the case where W is polyconvex and has the growth (1.5), as required in nonlinear elasticity. We believe that this is the first lower bound inequality proved for a stored energy function satisfying that growth condition.

This paper deals with the approximation of

$$\int_\Omega W(D\mathbf{u})\,\mathrm{d}\mathbf{x} + \mathcal{H}^{n-1}(J_\mathbf{u}) + \bar{\mathcal{E}}(\mathbf{u}) \tag{1.8}$$

which is, as mentioned above, a variant of (1.4), and, hence, a model for the energy of an elastic deformation that also exhibits cavitation and fracture. We chose the functional (1.8) instead of (1.4), that is to say, $\bar{\mathcal{E}}$ instead of $\mathcal{E}$, because the latter lends itself to an easier approximation. The study of a model that gathers cavitation and fracture was partially motivated by the role of cavitation in the initiation of fracture in rubber and ductile metals through void growth and coalescence (see [55, 52, 36, 34, 54, 33, 51]). In particular, the numerical experiments carried out using the method described in this work (see the companion paper [40]) aim to contribute to the understanding of void coalescence as a precursor of fracture.

In broad lines, the term $\mathcal{H}^{n-1}(J_\mathbf{u})$ of (1.8) can be treated as an Ambrosio–Tortorelli term, while the term $\bar{\mathcal{E}}(\mathbf{u})$ resembles a Modica–Mortola term, but it is subtler. The general scheme of the approximation of (1.8) proposed in this paper is as follows. We will use two phase-field functions: v for $\mathcal{H}^{n-1}(J_\mathbf{u})$ and w for $\bar{\mathcal{E}}(\mathbf{u})$. As in the Ambrosio–Tortorelli approximation, v lies in the reference configuration, and $v \simeq 1$ indicates healthy material, while $v \simeq 0$ represents damaged material. For technical reasons in our argument, we need v to be continuous, so instead of

$$\frac{1}{2}\int_\Omega \left[\varepsilon|Dv|^2 + \frac{(1-v)^2}{\varepsilon}\right]\mathrm{d}\mathbf{x},$$

we choose

$$\int_\Omega \left[\varepsilon^{q-1}\frac{|Dv|^q}{q} + \frac{(1-v)^{q'}}{q'\varepsilon}\right]\mathrm{d}\mathbf{x}$$

as an approximation of $\mathcal{H}^{n-1}(J_\mathbf{u})$, where $q > n$, and q' is the conjugate exponent of q. The Sobolev embedding guarantees that v is continuous. Thus, the approximation of the term $\mathcal{H}^{n-1}(J_\mathbf{u})$ of (1.8) follows the scheme of Braides, Chambolle & Solci [20].

The approximation of the term $\bar{\mathcal{E}}(\mathbf{u})$ is new and summarized as follows. As in the Modica–Mortola approximation, the phase-field function w is defined in the deformed configuration, and $w \simeq 1$ when there is matter, while $w \simeq 0$ when there is no matter. In other words, $w \simeq \chi_{\mathbf{u}(\Omega)}$. Naturally, there must be a relation between the phase-field variables, which is that w follows v but in the deformed configuration, so $w \simeq v \circ \mathbf{u}$. Imposing an exact equality $w = v \circ \mathbf{u}$ would make the construction of the recovery sequence too strict, and, in fact, is incompatible with the boundary condition for v and w. The exact way of expressing $w \simeq v \circ \mathbf{u}$ is that $w \leq v \circ \mathbf{u}$ and that w is close to $v \circ \mathbf{u}$ in L^1. Again for technical reasons, the function w is required to be continuous, so instead of (1.6), we choose

$$6 \int_Q \left[\varepsilon^{q-1} \frac{|Dw|^q}{q} + \frac{w^{q'}(1-w)^{q'}}{q'\varepsilon} \right] \mathrm{d}\mathbf{y}$$

to approximate $\bar{\mathcal{E}}(\mathbf{u})$. Here $Q \subset \mathbb{R}^n$ is a bounded open set containing a fixed compact set K, which in turn is assumed to contain the image of $\mathbf{u}$. A key result in this approximation is the representation formula

$$\bar{\mathcal{E}}(\mathbf{u}) = \operatorname{Per} \mathbf{u}(\Omega) + 2\, \mathcal{H}^{n-1}(J_{\mathbf{u}^{-1}}), \tag{1.9}$$

valid for deformations $\mathbf{u}$ that are one-to-one. Equality (1.9) is the analogue of the representation formula for $\mathcal{E}$ proved in [38, Th. 3]. We observe that the term $\operatorname{Per} \mathbf{u}(\Omega)$, explained above, appears together with the term $\mathcal{H}^{n-1}(J_{\mathbf{u}^{-1}})$, which measures the set of jumps of the inverse and accounts for a possible pathological phenomenom consisting in a sort of interpenetration of matter for deformations $\mathbf{u}$ that still are one-to-one. We refer to [38] for a discussion of this phenomenom, and just mention here that deformations $\mathbf{u}$ with $\mathcal{H}^{n-1}(J_{\mathbf{u}^{-1}}) > 0$ are, in general, not physical.

Given $\lambda_1, \lambda_2 > 0$, the main result of the paper is an approximation result of the functional

$$\begin{aligned} I_\varepsilon(\mathbf{u}, v, w) := \int_\Omega (v^2 + \eta_\varepsilon)\, W(D\mathbf{u})\, \mathrm{d}\mathbf{x} + \lambda_1 \int_\Omega \left[\varepsilon^{q-1} \frac{|Dv|^q}{q} + \frac{(1-v)^{q'}}{q'\varepsilon} \right] \mathrm{d}\mathbf{x} \\ + 6\lambda_2 \int_Q \left[\varepsilon^{q-1} \frac{|Dw|^q}{q} + \frac{w^{q'}(1-w)^{q'}}{q'\varepsilon} \right] \mathrm{d}\mathbf{y} \end{aligned} \tag{1.10}$$

to

$$I(\mathbf{u}) := \int_\Omega W(\nabla \mathbf{u})\, \mathrm{d}\mathbf{x} + \lambda_1 \left[\mathcal{H}^{n-1}(J_\mathbf{u}) + \mathcal{H}^{n-1}\left(\{\mathbf{x} \in \partial_D\Omega : \mathbf{u} \neq \mathbf{u}_0\}\right) + \frac{1}{2}\mathcal{H}^{n-1}(\partial_N\Omega) \right] + \lambda_2\, \bar{\mathcal{E}}(\mathbf{u}) \tag{1.11}$$

as $\varepsilon \to 0$, where $0 < \eta_\varepsilon \ll \varepsilon$. We explain the two terms in I that have not appeared so far. We impose to $\mathbf{u}$ a Dirichlet boundary condition $\mathbf{u}_0$ in the Dirichlet part $\partial_D\Omega$ of the boundary $\partial\Omega$, while the Neumann part $\partial_N\Omega$ is left free. The phase-field functions v and w are assumed to satisfy

$$v|_{\partial_D\Omega} = 1, \qquad v|_{\partial_N\Omega} = 0, \qquad w|_{Q \setminus \mathbf{u}(\Omega)} = 0.$$

The fact that v has to decrease to 0 at $\partial_N\Omega$ forces a transition from 1 to 0, whose energy is, approximately, $\frac{1}{2}\mathcal{H}^{n-1}(\partial_N\Omega)$. This term is a constant, and, hence, it does not affect the minimization problem. On the other hand, the term

$$\mathcal{H}^{n-1}\left(\{\mathbf{x} \in \partial_D\Omega : \mathbf{u}(\mathbf{x}) \neq \mathbf{u}_0(\mathbf{x})\}\right) \tag{1.12}$$

accounts for a possible fracture at the boundary. Indeed, it is well-known that the traces are not continuous with respect to the weak* convergence in BV (see, e.g., [7, Sect. 3.8]), so even though $\mathbf{u}_\varepsilon = \mathbf{u}_0$ on $\partial_D\Omega$ for a sequence of deformations $\mathbf{u}_\varepsilon$, it is possible that its weak* limit $\mathbf{u}$ in BV does not satisfy the boundary condition. This phenomenon is, nevertheless, penalized energetically by the term (1.12).

The admissible space for I_ε is the set of $(\mathbf{u}, v, w)$ such that $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n)$, $v \in W^{1,q}(\Omega)$, $w \in W^{1,q}(Q)$ satisfying the boundary conditions described above, and $\mathbf{u}$ is one-to-one a.e. Moreover, $\mathbf{u}$ is assumed to create no surface, which is expressed as $\mathcal{E}(\mathbf{u}) = 0$. The admissible space for I is the set of $\mathbf{u} \in SBV(\Omega, \mathbb{R}^n)$ such that $\mathbf{u}$ is one-to-one a.e.

The limit passage from I_ε to I is meant to be in the sense of Γ-convergence, but, unfortunately, in this paper we do not provide a full Γ-convergence result. The existence of minimizers, compactness and lower bound are indeed proved. To be precise, the functional I_ε has a minimizer for each ε. Moreover, if $(\mathbf{u}_\varepsilon, v_\varepsilon, w_\varepsilon)$ is a sequence of admissible maps with $\sup_\varepsilon I_\varepsilon(\mathbf{u}_\varepsilon, v_\varepsilon, w_\varepsilon) < \infty$ then, for a subsequence, there exists a one-to-one a.e. map $\mathbf{u} \in SBV(\Omega, \mathbb{R}^n)$ such that $\mathbf{u}_\varepsilon \to \mathbf{u}$, $v_\varepsilon \to 1$ and $w_\varepsilon \to \chi_{\mathbf{u}(\Omega)}$ a.e. In addition,

$$I(\mathbf{u}) \leq \liminf_{\varepsilon \to 0} I_\varepsilon(\mathbf{u}_\varepsilon, v_\varepsilon, w_\varepsilon).$$

Proving the upper bound, however, is out of reach at the moment, since it seems that the construction of the recovery sequence would require, in particular, a density result for invertible maps, whereas only partial results are known in this direction (see [46, 13, 42, 28, 47]). This is so because the usual approach to prove a *limsup* inequality consists in first proving it for a dense subset of smooth maps and then conclude by density. As mentioned above, in the presence of the constraint that $\mathbf{u}$ is one-to-one a.e., there are no known results of density of smooth functions that are useful for our analysis. There are, in fact, more difficulties that appear, such as to identify the set of limit functions $\mathbf{u}$. We only prove that this set is contained in the set of $\mathbf{u} \in SBV(\Omega, \mathbb{R}^n)$ such that $\mathbf{u}$ is one-to-one a.e., $\mathcal{H}^{n-1}(J_\mathbf{u}) < \infty$ and $\bar{\mathcal{E}}(\mathbf{u}) < \infty$. Once identified that set, another density result would be needed, this time of the style that piecewise smooth maps (for example, maps with finitely many smooth cavities and smooth cracks) are dense in this set to be identified; that result would be in the spirit of that of Cortesani [25] (see also [26]) stating that functions that are smooth away from a polyhedral crack are dense in SBV with respect to Mumford–Shah energy. Instead of a full upper bound inequality, what we perform is a series of examples of deformations $\mathbf{u}$ in dimension 2 that can be approximated by admissible maps $(\mathbf{u}_\varepsilon, v_\varepsilon, w_\varepsilon)$ satisfying

$$I(\mathbf{u}) = \lim_{\varepsilon \to 0} I_\varepsilon(\mathbf{u}_\varepsilon, v_\varepsilon, w_\varepsilon).$$

We have chosen the deformations $\mathbf{u}$ so that one creates a cavity, one creates an interior crack, one presents fracture at the boundary, and one exhibits *coalescence*, which is modelled as the creation of a crack joining two preexisting cavities. Those examples, as well as the numerical experiments of [40], allow us to believe that the stated functional I is indeed the Γ-limit of I_ε.

We now present the outline of this paper. In Section 2 we present the general notation as well as some results that will be used throughout the paper. In Section 3 we give a geometric meaning to $\bar{\mathcal{E}}$ by proving the equality

$$\bar{\mathcal{E}}(\mathbf{u}) = \operatorname{Per} \mathbf{u}(\Omega) + 2\, \mathcal{H}^{n-1}(J_{\mathbf{u}^{-1}}). \tag{1.13}$$

We also show a lower semicontinity property for this functional. In Section 4 we present the general assumptions for the stored energy functional W and for the deformations. We also define the admissible set for the functional I_ε. In Section 5 we prove the existence of minimizers for the functional I_ε. Section 6 proves the compactness and lower bound for the convergence $I_\varepsilon \to I$. Section 7 construct some examples for the upper bound.

2 Notation and preliminary results

In this section we set the general notation and concepts of the paper, and state some preliminary results.

2.1 General notation

We will work in dimension $n \geq 2$, and Ω is a bounded open set of $\mathbb{R}^n$ representing the body in its reference configuration. Vector-valued and matrix-valued quantities will be written in boldface. Coordinates in the reference configuration will generically be denoted by $\mathbf{x}$, while coordinates in the deformed configuration by $\mathbf{y}$.

The closure of a set A is denoted by $\bar{A}$, and its boundary by ∂A. Given two sets U, V of $\mathbb{R}^n$, we will write $U \subset\subset V$ if U is bounded and $\bar{U} \subset V$. The open ball of radius $r > 0$ centred at $\mathbf{x} \in \mathbb{R}^n$ is denoted by $B(\mathbf{x}, r)$, the closed ball by $\bar{B}(\mathbf{x}, r)$, while $\bar{B}(\bar{A}, r)$ is the set of $\mathbf{x}' \in \mathbb{R}^n$ such that $\operatorname{dist}(\mathbf{x}', \bar{A}) \leq r$. The function dist indicates the distance from a point to a set. Unless otherwise stated, a *ball* will always be an open ball.

Given a square matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, its transpose is denoted by $\mathbf{A}^T$, and its determinant by $\det \mathbf{A}$. The cofactor matrix of $\mathbf{A}$, denoted by $\operatorname{cof} \mathbf{A}$, is the matrix that satisfies $(\det \mathbf{A})\mathbf{1} = \mathbf{A}^T \operatorname{cof} \mathbf{A}$, where $\mathbf{1}$ denotes the identity matrix. If $\mathbf{A}$ is invertible, its inverse is denoted by $\mathbf{A}^{-1}$. The inner (dot) product of vectors and of matrices will be denoted by $\cdot$. The Euclidean norm of a vector $\mathbf{x}$ is denoted by $|\mathbf{x}|$, and the associated matrix norm is also denoted by $|\cdot|$. Given $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, the tensor product $\mathbf{a} \otimes \mathbf{b}$ is the $n \times n$ matrix whose component (i, j) is $a_i\, b_j$.

Unless otherwise stated, expressions like *measurable* or *a.e.* (for *almost everywhere* or *almost every*) refer to the Lebesgue measure in $\mathbb{R}^n$, which is denoted by $\mathcal{L}^n$. The $(n-1)$-dimensional Hausdorff measure will be indicated by $\mathcal{H}^{n-1}$. The measure $\mathcal{H}^0$ is simply the counting measure.

The Lebesgue L^p and Sobolev $W^{1,p}$ spaces are defined in the usual way. So are the sets of class C^k, for an integer $k \geq 0$ or infinity, and their versions C^k_c of compact support. Note that we do not identify functions that coincide a.e. We will always indicate the domain and target space, as in, for example, $L^p(\Omega, \mathbb{R}^n)$, except if the target space is $\mathbb{R}$, in which case we will simply write $L^p(\Omega)$. If $K \subset \mathbb{R}^n$, we indicate by $L^p(\Omega, K)$ the set of $\mathbf{u} \in L^p(\Omega, \mathbb{R}^n)$ such that $\mathbf{u}(\mathbf{x}) \in K$ for a.e. $\mathbf{x} \in \Omega$, and analogously for other function spaces. The space $L^p_{\text{loc}}(\Omega)$ indicates the set of $f : \Omega \to \mathbb{R}$ such that $f|_A \in L^p(A)$ for all open $A \subset\subset \Omega$, and analogously for other function spaces.

Strong or a.e. convergence is denoted with $\to$, while weak convergence is denoted with $\rightharpoonup$.

With $\langle\cdot,\cdot\rangle$ we will indicate the duality product between a distribution and a smooth function. The identity function in $\mathbb{R}^n$ is denoted by $\mathbf{id}$.

We will use the divergence operator in the deformed configuration, which will be called div.

If μ is a measure on a set U, and V is a μ-measurable subset of U, then the restriction of μ to V is the measure on U, denoted by $\mu \llcorner V$, that satisfies $\mu \llcorner V(A) = \mu(A \cap V)$ for all μ-measurable sets A. The measure $|\mu|$ denotes the total variation of μ.

Given two sets A, B of $\mathbb{R}^n$, we write $A = B$ a.e. if $\mathcal{L}^n(A \setminus B) = \mathcal{L}^n(B \setminus A) = 0$, and analogously when we write that $A = B$ holds $\mathcal{H}^{n-1}$-a.e. In particular, the expression $A \subset B$ $\mathcal{H}^{n-1}$-a.e. means $\mathcal{H}^{n-1}(A \setminus B) = 0$.

2.2 Boundary and perimeter

Given a measurable set $A \subset \Omega$, its characteristic function will be denoted by χ_A. Its *perimeter*, written $\operatorname{Per} A$ or $\operatorname{Per}(A)$, is defined as

$$\operatorname{Per} A := \sup \left\{ \int_A \operatorname{div} \mathbf{g}(\mathbf{y})\, \mathrm{d}\mathbf{y} : \ \mathbf{g} \in C^\infty_c(\mathbb{R}^n, \mathbb{R}^n),\ \|\mathbf{g}\|_\infty \leq 1 \right\},$$

while its perimeter in Ω is defined as

$$\operatorname{Per}(A, \Omega) := \sup \left\{ \int_A \operatorname{div} \mathbf{g}(\mathbf{y})\, \mathrm{d}\mathbf{y} : \ \mathbf{g} \in C^\infty_c(\Omega, \mathbb{R}^n),\ \|\mathbf{g}\|_\infty \leq 1 \right\}.$$

Half-spaces are denoted by

$$H^+(\mathbf{a}, \boldsymbol{\nu}) := \{\mathbf{x} \in \mathbb{R}^n : (\mathbf{x} - \mathbf{a}) \cdot \boldsymbol{\nu} \geq 0\}, \qquad H^-(\mathbf{a}, \boldsymbol{\nu}) := H^+(\mathbf{a}, -\boldsymbol{\nu}),$$

for a given $\mathbf{a} \in \mathbb{R}^n$ and a nonzero vector $\boldsymbol{\nu} \in \mathbb{R}^n$. The set of unit vectors in $\mathbb{R}^n$ is denoted by $\mathbb{S}^{n-1}$.

Given a measurable set $A \subset \mathbb{R}^n$ and a point $\mathbf{x} \in \mathbb{R}^n$, the *density* of A at $\mathbf{x}$ is defined as

$$D(A, \mathbf{x}) := \lim_{r \searrow 0} \frac{\mathcal{L}^n(B(\mathbf{x}, r) \cap A)}{\mathcal{L}^n(B(\mathbf{x}, r))}.$$

Definition 2.1. *Let A be a measurable set of $\mathbb{R}^n$. We define the reduced boundary of A, and denote it by $\partial^* A$, as the set of points $\mathbf{y} \in \mathbb{R}^n$ for which a unit vector $\boldsymbol{\nu}_A(\mathbf{y})$ exists such that*

$$D(A \cap H^-(\mathbf{y}, \boldsymbol{\nu}_A(\mathbf{y})), \mathbf{y}) = \frac{1}{2} \quad \text{and} \quad D(A \cap H^+(\mathbf{y}, \boldsymbol{\nu}_A(\mathbf{y})), \mathbf{y}) = 0.$$

This $\boldsymbol{\nu}_A(\mathbf{y})$ is uniquely determined and is called the unit outward normal to A.

This definition of boundary may differ from other usual definitions, but thanks to Federer's [31] theorem (see also [7, Th. 3.61] or [56, Sect. 5.6]) they coincide $\mathcal{H}^{n-1}$-a.e. with all other usual definitions of reduced (or *essential* or *measure-theoretic*) boundary for sets of finite perimeter. In particular, if $\operatorname{Per}(A, \Omega) < \infty$ then $\operatorname{Per}(A, \Omega) = \mathcal{H}^{n-1}(\partial^* A \cap \Omega)$.

2.3 Approximate differentiability and functions of bounded variation

We assume the reader some familiarity with the set BV of functions of bounded variation, and of special bounded variation SBV; see [7], if necessary, for the definitions. This section is meant primarily to set some notation.

The *total variation* of $\mathbf{u} \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is defined as

$$V(\mathbf{u}, \Omega) := \sup \left\{ \int_\Omega \mathbf{u}(\mathbf{x}) \cdot \operatorname{Div} \boldsymbol{\varphi}(\mathbf{x}) \, \mathrm{d}\mathbf{x} : \, \boldsymbol{\varphi} \in C^1_c(\Omega, \mathbb{R}^{n \times n}), \, |\boldsymbol{\varphi}| \leq 1 \right\},$$

where $\operatorname{Div} \boldsymbol{\varphi}$ is the divergence of the rows of $\boldsymbol{\varphi}$.

The following notions are essentially due to Federer [31].

Definition 2.2. *Let A be a measurable set in $\mathbb{R}^n$, and $\mathbf{u} : A \to \mathbb{R}^n$ a measurable function. Let $\mathbf{x}_0 \in \mathbb{R}^n$ satisfy $D(A, \mathbf{x}_0) = 1$, and let $\mathbf{y}_0 \in \mathbb{R}^n$.*

(a) We will say that $\mathbf{x}_0$ is an approximate jump point of $\mathbf{u}$ if there exist $\mathbf{a}^+, \mathbf{a}^- \in \mathbb{R}^n$ and $\boldsymbol{\nu} \in \mathbb{S}^{n-1}$ such that $\mathbf{a}^+ \neq \mathbf{a}^-$ and

$$D\left(\left\{\mathbf{x} \in A \cap H^\pm(\mathbf{x}_0, \boldsymbol{\nu}) : \left|\mathbf{u}(\mathbf{x}) - \mathbf{a}^\pm\right| \geq \delta\right\}, \mathbf{x}_0\right) = 0$$

for all $\delta > 0$. The unit vector $\boldsymbol{\nu}$ is uniquely determined up to a sign. When a choice of $\boldsymbol{\nu}$ has been done, it is denoted by $\boldsymbol{\nu}_{\mathbf{u}}(\mathbf{x}_0)$. The points $\mathbf{a}^+$ and $\mathbf{a}^-$ are called the lateral traces of $\mathbf{u}$ at $\mathbf{x}_0$ with respect to the $\boldsymbol{\nu}_{\mathbf{u}}(\mathbf{x}_0)$, and are denoted by $\mathbf{u}^+(\mathbf{x}_0)$ and $\mathbf{u}^-(\mathbf{x}_0)$, respectively. The set of approximate jump points of $\mathbf{u}$ is called the jump set of $\mathbf{u}$, and is denoted by $J_{\mathbf{u}}$.

(b) We will say that $\mathbf{u}$ is approximately differentiable at $\mathbf{x}_0 \in A$ if there exists $\mathbf{L} \in \mathbb{R}^{n \times n}$ such that

$$D\left(\left\{\mathbf{x} \in A \setminus \{\mathbf{x}_0\} : \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}_0) - \mathbf{L}(\mathbf{x} - \mathbf{x}_0)|}{|\mathbf{x} - \mathbf{x}_0|} \geq \delta\right\}, \mathbf{x}_0\right) = 0$$

for all $\delta > 0$. In this case, $\mathbf{L}$ (which is uniquely determined) is called the approximate differential of $\mathbf{u}$ at $\mathbf{x}_0$, and will be denoted by $\nabla \mathbf{u}(\mathbf{x}_0)$.

We will say that a map $\mathbf{u} : \Omega \to \mathbb{R}^n$ is *approximately differentiable a.e.* when it is measurable and approximately differentiable at almost each point of Ω.

If $\mathbf{u} : \Omega \to \mathbb{R}^n$ is a function of locally bounded variation, $D\mathbf{u}$ denotes the distributional derivative of $\mathbf{u}$, which is a Radon measure in Ω. The Calderón–Zygmund theorem asserts that if $\mathbf{u}$ is locally of bounded variation then it is approximately differentiable a.e. and $\nabla \mathbf{u}$ coincides a.e. with the absolutely continuous part of $D\mathbf{u}$.

Lemma 2.3. *Let $\mathbf{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable a.e., and let $E \subset \Omega$ be measurable. Then $\chi_E \mathbf{u}$ is approximately differentiable a.e., and $\nabla(\chi_E \mathbf{u}) = \chi_E \nabla \mathbf{u}$ a.e.*

Proof. As E is measurable, by Lebesgue's theorem, almost every point in E has density 1 in E, and almost every point in $\Omega \setminus E$ has density 1 in $\Omega \setminus E$. It is immediate to check that if $\mathbf{x} \in E$ satisfies $D(E, \mathbf{x}) = 1$ and $\mathbf{u}$ is approximately differentiable at $\mathbf{x}$ then $\chi_E \mathbf{u}$ is approximately differentiable at $\mathbf{x}$ with $\nabla(\chi_E \mathbf{u})(\mathbf{x}) = \nabla \mathbf{u}(\mathbf{x})$, while if $\mathbf{x} \in \Omega \setminus E$ satisfies $D(\Omega \setminus E, \mathbf{x}) = 1$ then $\chi_E \mathbf{u}$ is approximately differentiable at $\mathbf{x}$ with $\nabla(\chi_E \mathbf{u})(\mathbf{x}) = \mathbf{0}$. □

The following is a known result in the theory of BV functions; it is in fact a particular case of [7, Th. 3.84].

Lemma 2.4. *Let* $\mathbf{u} \in SBV(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ *and let* E *be a measurable subset of* Ω *with* $\operatorname{Per}(E, \Omega) < \infty$. *Then* $\chi_E \mathbf{u} \in SBV(\Omega, \mathbb{R}^n)$ *and* $J_{\chi_E \mathbf{u}} \subset (J_\mathbf{u} \cap E) \cup (\partial^* E \cap \Omega)$ $\mathcal{H}^{n-1}$*-a.e.*

2.4 Area formula and geometric image

We recall the *area formula* of Federer [31]. The formulation is taken from [48, Prop. 2.6].

Proposition 2.5. *Let* $\mathbf{u} : \Omega \to \mathbb{R}^n$ *be approximately differentiable a.e., and denote the set of approximate differentiability points of* $\mathbf{u}$ *by* Ω_d. *Then, for any measurable set* $A \subset \Omega$ *and any measurable function* $\varphi : \mathbb{R}^n \to \mathbb{R}$,

$$\int_A \varphi(\mathbf{u}(\mathbf{x})) \, |\det \nabla \mathbf{u}(\mathbf{x})| \, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n} \varphi(\mathbf{y}) \, \mathcal{H}^0(\{\mathbf{x} \in \Omega_d \cap A : \mathbf{u}(\mathbf{x}) = \mathbf{y}\}) \, \mathrm{d}\mathbf{y},$$

whenever either integral exists. Moreover, if $\psi : A \to \mathbb{R}$ *is measurable and* $\bar{\psi} : \mathbf{u}(\Omega_d \cap A) \to \mathbb{R}$ *is given by*

$$\bar{\psi}(\mathbf{y}) := \sum_{\substack{\mathbf{x} \in \Omega_d \cap A \\ \mathbf{u}(\mathbf{x}) = \mathbf{y}}} \psi(\mathbf{x}),$$

then $\bar{\psi}$ *is measurable and*

$$\int_A \psi(\mathbf{x}) \, \varphi(\mathbf{u}(\mathbf{x})) \, |\det \nabla \mathbf{u}(\mathbf{x})| \, \mathrm{d}\mathbf{x} = \int_{\mathbf{u}(\Omega_d \cap A)} \bar{\psi}(\mathbf{y}) \, \varphi(\mathbf{y}) \, \mathrm{d}\mathbf{y}, \tag{2.1}$$

whenever the integral on the left-hand side of (2.1) *exists.*

An immediate consequence of Proposition 2.5 is that if $\mathbf{u} \in L^\infty(\Omega, \mathbb{R}^n)$ is one-to-one a.e. and approximately differentiable a.e. then $\det \nabla \mathbf{u} \in L^1(\Omega)$.

The area formula of Proposition 2.5 has given rise to the notion of the *geometric image* (or *measure-theoretic image*, using the expression in [48]) of a measurable set $A \subset \Omega$ under an approximately differentiable map $\mathbf{u} : \Omega \to \mathbb{R}^n$. This was defined as $\mathbf{u}(A \cap \Omega_d)$ by Müller and Spector [48]; for technical convenience, however, we use the following definition, which is an adaptation of that of Conti and De Lellis [24].

Definition 2.6. *Let* $\mathbf{u} : \Omega \to \mathbb{R}^n$ *be approximately differentiable a.e. and suppose that* $\det \nabla \mathbf{u}(\mathbf{x}) \neq 0$ *for a.e.* $\mathbf{x} \in \Omega$. *Define* Ω_0 *as the set of* $\mathbf{x} \in \Omega$ *such that* $\mathbf{u}$ *is approximately differentiable at* $\mathbf{x}$ *with* $\det \nabla \mathbf{u}(\mathbf{x}) \neq 0$, *and there exist* $\mathbf{w} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ *and a compact set* $K \subset \Omega$ *of density* 1 *at* $\mathbf{x}$ *such that* $\mathbf{u}|_K = \mathbf{w}|_K$ *and* $\nabla \mathbf{u}|_K = D\mathbf{w}|_K$. *For any measurable set* A *of* Ω, *we define the geometric image of* A *under* $\mathbf{u}$ *as* $\mathbf{u}(A \cap \Omega_0)$, *and denote it by* $\operatorname{im_G}(\mathbf{u}, A)$.

Standard arguments, essentially due to Federer [31, Thms. 3.1.8 and 3.1.16] (see also [48, Prop. 2.4] and [24, Rk. 2.5]), show that the set Ω_0 in Definition 2.6 is of full measure in Ω.

2.5 Notation about sequences

When computing the Γ-limit of I_ε in (1.10), we will fix a sequence of positive numbers tending to zero, and denote it by $\{\varepsilon\}_\varepsilon$. The letter ε is reserved for a member of the fixed sequence, so expressions like "for every ε" mean "for every member ε of the sequence", and $\{\mathbf{u}_\varepsilon\}_\varepsilon$ denotes the sequence of $\mathbf{u}_\varepsilon$ labelled by the sequence of ε. We will repeatedly take subsequences, which will not be relabelled. All convergences involving ε are understood as the sequence $\{\varepsilon\}_\varepsilon$ goes to zero, abbreviated to $\varepsilon \to 0$. For example, in the expression $\mathbf{u}_\varepsilon \to \mathbf{u}$ it is understood that the convergence holds as $\varepsilon \to 0$.

Given two sequences $\{a_\varepsilon\}_\varepsilon$ and $\{b_\varepsilon\}_\varepsilon$ of positive numbers, we write

$$\begin{aligned}
&a_\varepsilon \lessapprox b_\varepsilon && \text{when } \limsup_{\varepsilon\to 0} \frac{a_\varepsilon}{b_\varepsilon} < \infty, \\
&a_\varepsilon \ll b_\varepsilon && \text{when } \lim_{\varepsilon\to 0} \frac{a_\varepsilon}{b_\varepsilon} = 0, \\
&a_\varepsilon \simeq b_\varepsilon && \text{when } \lim_{\varepsilon\to 0} \frac{a_\varepsilon}{b_\varepsilon} = 1, \\
&a_\varepsilon \approx b_\varepsilon && \text{when } a_\varepsilon \lessapprox b_\varepsilon \text{ and } b_\varepsilon \lessapprox a_\varepsilon.
\end{aligned}$$

Sometimes, the sequences $\{a_\varepsilon\}_\varepsilon$ and $\{b_\varepsilon\}_\varepsilon$ will be positive functions. In this case, and when a domain A of definition is clear from the context, the notation $a_\varepsilon \lessapprox b_\varepsilon$ means

$$\limsup_{\varepsilon\to 0} \sup_{x\in A} \frac{a_\varepsilon(\mathbf{x})}{b_\varepsilon(\mathbf{x})} < \infty,$$

and analogously for the other notation.

2.6 Inverses of one-to-one a.e. maps

A function is *one-to-one a.e.* when its restriction to a set of full measure is one-to-one. The following result was proved in [38, Lemma 3].

Lemma 2.7. *Let $\mathbf{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable a.e., one-to-one a.e., and suppose that $\det \nabla \mathbf{u}(\mathbf{x}) \neq 0$ for a.e. $\mathbf{x} \in \Omega$. Let Ω_0 be as in Definition 2.6. Then $\mathbf{u}|_{\Omega_0}$ is one-to-one.*

Definition 2.8. *Let $\mathbf{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable a.e., one-to-one a.e., and suppose that $\det \nabla \mathbf{u}(\mathbf{x}) \neq 0$ for a.e. $\mathbf{x} \in \Omega$. Let Ω_0 be as in Definition 2.6. The inverse $\mathbf{u}^{-1} : \operatorname{im}_{\mathrm{G}}(\mathbf{u}, \Omega) \to \mathbb{R}^n$ of $\mathbf{u}$ is defined as the function that sends every $\mathbf{y} \in \operatorname{im}_{\mathrm{G}}(\mathbf{u}, \Omega)$ to the only $\mathbf{x} \in \Omega_0$ such that $\mathbf{u}(\mathbf{x}) = \mathbf{y}$. Analogously, given any measurable subset A of Ω, we define $\mathbf{u}_A^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ as*

$$\mathbf{u}_A^{-1}(\mathbf{y}) := \begin{cases} \mathbf{u}^{-1}(\mathbf{y}) & \text{if } \mathbf{y} \in \operatorname{im}_{\mathrm{G}}(\mathbf{u}, A), \\ \mathbf{0} & \text{if } \mathbf{y} \in \mathbb{R}^n \setminus \operatorname{im}_{\mathrm{G}}(\mathbf{u}, A). \end{cases}$$

By Proposition 2.5, the maps $\mathbf{u}^{-1}$ and $\mathbf{u}_A^{-1}$ are measurable. The following result was proved in [38, Th. 2].

Lemma 2.9. *Let $\mathbf{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable a.e., one-to-one a.e. and satisfy $\det \nabla \mathbf{u}(\mathbf{x}) \neq 0$ for a.e. $\mathbf{x} \in \Omega$. Then $\mathbf{u}^{-1}$ is approximately differentiable in $\operatorname{im}_{\mathrm{G}}(\mathbf{u}, \Omega)$ and $\nabla \mathbf{u}^{-1}(\mathbf{u}(\mathbf{x})) = (\nabla \mathbf{u}(\mathbf{x}))^{-1}$ for all $\mathbf{x} \in \Omega_0$.*

Putting together Lemmas 2.3 and 2.9, we arrive at the following result.

Lemma 2.10. *Let $\mathbf{u} : \Omega \to \mathbb{R}^n$ be approximately differentiable a.e., one-to-one a.e. and satisfy $\det \nabla \mathbf{u}(\mathbf{x}) \neq 0$ for a.e. $\mathbf{x} \in \Omega$. Let A be a measurable subset of Ω. Then $\mathbf{u}_A^{-1}$ is approximately differentiable a.e. and*

$$\nabla \mathbf{u}_A^{-1}(\mathbf{y}) = \begin{cases} \nabla \mathbf{u}^{-1}(\mathbf{y}) & \text{for a.e. } \mathbf{y} \in \operatorname{im}_{\mathrm{G}}(\mathbf{u}, A), \\ \mathbf{0} & \text{for a.e. } \mathbf{y} \in \mathbb{R}^n \setminus \operatorname{im}_{\mathrm{G}}(\mathbf{u}, A). \end{cases}$$

2.7 Weak convergence of products and minors

We will frequently use the following convergence result, whose proof can be found, e.g., in [53, Lemma 6.7].

Lemma 2.11. *For each $j \in \mathbb{N}$, let $f_j, f \in L^\infty(\Omega)$ and $g_j, g \in L^1(\Omega)$ satisfy*

$$f_j \to f \text{ a.e.} \quad \text{and} \quad g_j \rightharpoonup g \text{ in } L^1(\Omega) \qquad \text{as } j \to \infty.$$

Assume that $\sup_{j\in\mathbb{N}} \|f_j\|_{L^\infty(\Omega)} < \infty$. *Then*

$$f_j\, g_j \rightharpoonup f\, g \quad \text{in} \quad L^1(\Omega) \quad \text{as } j \to \infty.$$

We denote by $\mathbb{R}^{n\times n}_+$ the set of $\mathbf{F} \in \mathbb{R}^{n\times n}$ such that $\det \mathbf{F} > 0$. Let $\tau = \tau(n)$ be the number of minors (subdeterminants) of a matrix in $\mathbb{R}^{n\times n}$. Given $\mathbf{F} \in \mathbb{R}^{n\times n}$, let $\boldsymbol{\mu}_0(\mathbf{F}) \in \mathbb{R}^{\tau-1}$ be the vector composed by all minors of $\mathbf{F}$ except the determinant, and $\boldsymbol{\mu}(\mathbf{F}) \in \mathbb{R}^\tau$ is defined as $\boldsymbol{\mu}(\mathbf{F}) := (\boldsymbol{\mu}_0(\mathbf{F}), \det \mathbf{F})$. We denote by $\mathbb{R}^\tau_+$ the set of vectors in $\mathbb{R}^\tau$ whose last component is positive.

The following result on the weak continuity of minors is well known and can be proved as in Ambrosio [5, Cor. 4.9] (see also [7, Cor. 5.31]).

Lemma 2.12. *For each $j \in \mathbb{N}$, let $\mathbf{u}_j, \mathbf{u} \in SBV(\Omega, \mathbb{R}^n)$ satisfy that the sequences $\{\|\nabla \mathbf{u}_j\|_{L^{n-1}(\Omega,\mathbb{R}^{n\times n})}\}_{j\in\mathbb{N}}$ and $\{\mathcal{H}^{n-1}(J_{\mathbf{u}_j})\}_{j\in\mathbb{N}}$ are bounded. Assume that $\mathbf{u}_j \to \mathbf{u}$ in $L^1(\Omega, \mathbb{R}^n)$ as $j \to \infty$, and the sequence $\{\operatorname{cof} \nabla \mathbf{u}_j\}_{j\in\mathbb{N}}$ is equiintegrable. Then*

$$\boldsymbol{\mu}_0(\nabla \mathbf{u}_j) \rightharpoonup \boldsymbol{\mu}_0(\nabla \mathbf{u}) \qquad \text{in} \quad L^1(\Omega, \mathbb{R}^{\tau-1}) \quad \text{as} \quad j \to \infty.$$

2.8 Slicing

We will use the following slicing notation.

Definition 2.13. *For every $\boldsymbol{\xi} \in \mathbb{S}^{n-1}$ let $\Pi_{\boldsymbol{\xi}}$ be the linear subspace of $\mathbb{R}^n$ orthogonal to $\boldsymbol{\xi}$. For $B \subset \mathbb{R}^n$, let $B^{\boldsymbol{\xi}}$ be the orthogonal projection of B on $\Pi_{\boldsymbol{\xi}}$. For every $\mathbf{x}' \in \Pi_{\boldsymbol{\xi}}$ define $B^{\boldsymbol{\xi},\mathbf{x}'} := \{t \in \mathbb{R} : \mathbf{x}' + t\boldsymbol{\xi} \in B\}$. If $f : B \to \mathbb{R}$ and $\mathbf{x}' \in B^{\boldsymbol{\xi}}$, let $f^{\boldsymbol{\xi},\mathbf{x}'} : B^{\boldsymbol{\xi},\mathbf{x}'} \to \mathbb{R}$ be defined by $f^{\boldsymbol{\xi},\mathbf{x}'}(t) := f(\mathbf{x}' + t\boldsymbol{\xi})$.*

Proposition 2.14. *Suppose that $u \in L^\infty(\Omega)$ satisfies that for all $\boldsymbol{\xi} \in \mathbb{S}^{n-1}$,*

i) $u^{\boldsymbol{\xi},\mathbf{x}'} \in SBV(\Omega^{\boldsymbol{\xi},\mathbf{x}'})$ *for a.e.* $\mathbf{x}' \in \Omega^{\boldsymbol{\xi}}$, *and*

ii) $\displaystyle\int_{\Omega^{\boldsymbol{\xi}}} \left[\int_{\Omega^{\boldsymbol{\xi},\mathbf{x}'}} |\nabla u^{\boldsymbol{\xi},\mathbf{x}'}| \,\mathrm{d}t + \mathcal{H}^0(J_{u^{\boldsymbol{\xi},\mathbf{x}'}}) \right] \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}') < \infty.$

Then $u \in SBV(\Omega)$, $\mathcal{H}^{n-1}(J_u) < \infty$, and for all $\boldsymbol{\xi} \in \mathbb{S}^{n-1}$, the following assertions hold:

a) $\nabla u(\mathbf{x}' + t\boldsymbol{\xi}) \cdot \boldsymbol{\xi} = \nabla u^{\boldsymbol{\xi},\mathbf{x}'}(t)$, *for $\mathcal{H}^{n-1}$-a.e. $\mathbf{x}' \in \Omega^{\boldsymbol{\xi}}$ and a.e. $t \in \Omega^{\boldsymbol{\xi},\mathbf{x}'}$.*

b) The normal $\boldsymbol{\nu}_u : J_u \to \mathbb{S}^{n-1}$ satisfies

$$\int_{J_u} |\boldsymbol{\nu}_u \cdot \boldsymbol{\xi}| \,\mathrm{d}\mathcal{H}^{n-1} = \int_{\Omega^{\boldsymbol{\xi}}} \mathcal{H}^0(J_{u^{\boldsymbol{\xi},\mathbf{x}'}}) \,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}').$$

c) For any $\mathcal{H}^{n-1}$-rectifiable subset A of $\partial\Omega$,

$$\int_A |\boldsymbol{\nu} \cdot \boldsymbol{\xi}| \,\mathrm{d}\mathcal{H}^{n-1} = \int_{A^{\boldsymbol{\xi}}} \mathcal{H}^0(A^{\boldsymbol{\xi},\mathbf{x}'}) \,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}').$$

d) For any $p \geq 1$, any $v \in C(\bar{\Omega})$ with $v \geq 0$ and any measurable set $A \subset \Omega$,

$$\int_{\Omega^{\boldsymbol{\xi}}} \int_{A^{\boldsymbol{\xi},\mathbf{x}'}} v^{\boldsymbol{\xi},\mathbf{x}'} \, |\nabla u^{\boldsymbol{\xi},\mathbf{x}'}|^p \,\mathrm{d}t \,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}') \leq \int_A v \,|\nabla u|^p \,\mathrm{d}\mathbf{x} \quad \text{and} \quad \int_{\Omega^{\boldsymbol{\xi}}} \int_{A^{\boldsymbol{\xi},\mathbf{x}'}} v^{\boldsymbol{\xi},\mathbf{x}'} \,\mathrm{d}t \,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}') = \int_A v \,\mathrm{d}\mathbf{x}.$$

e) For any set $E \subset \Omega$ *with* $\mathrm{Per}(E, \Omega) < \infty$,

$$\int_{\Omega^{\boldsymbol{\xi}}} \mathcal{H}^0(\partial^* E^{\boldsymbol{\xi},\mathbf{x}'} \cap \Omega^{\boldsymbol{\xi},\mathbf{x}'}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}') \le \mathcal{H}^{n-1}(\partial^* E \cap \Omega).$$

Proof. Part *c*) is proved in [31, Th. 3.2.22]. Part *d)* is a consequence of *a)* and Fubini's theorem, and part *e)* is a consequence of *c)*. The remaining parts are proved, e.g., in [3, Th. 3.3] or in [4, Sect. 3] or in [7, Sect. 3.11] (in particular Remark 3.104 and Thm. 3.108). □

2.9 Coarea formula

We will use the coarea formula in the following two versions. They are known to experts, but we have not found an exact reference.

Proposition 2.15. *a) Let* $u : \Omega \to \mathbb{R}$ *be Lipschitz and let* $f \in L^\infty(\mathbb{R})$ *be Borel measurable. Then*

$$\int_\Omega f(u(\mathbf{x})) \, |Du(\mathbf{x})| \, \mathrm{d}\mathbf{x} = \int_{-\infty}^{\infty} f(t) \, \mathcal{H}^{n-1}(\{\mathbf{x} \in \Omega : u(\mathbf{x}) = t\}) \, \mathrm{d}t. \tag{2.2}$$

b) Let $u \in W^{1,1}(\Omega)$ *be continuous and let* $f \in L^\infty(\mathbb{R})$ *be Borel measurable. Then*

$$\int_\Omega f(u(\mathbf{x})) \, |Du(\mathbf{x})| \, \mathrm{d}\mathbf{x} = \int_{-\infty}^{\infty} f(t) \, \mathrm{Per}(\{\mathbf{x} \in \Omega : u(\mathbf{x}) < t\}, \Omega) \, \mathrm{d}t. \tag{2.3}$$

and

$$\int_\Omega f(u(\mathbf{x})) \, |Du(\mathbf{x})| \, \mathrm{d}\mathbf{x} = \int_{-\infty}^{\infty} f(t) \, \mathrm{Per}(\{\mathbf{x} \in \Omega : u(\mathbf{x}) > t\}, \Omega) \, \mathrm{d}t. \tag{2.4}$$

Proof. In this proof, we will use the notation $\{u < t\}$ as a shorthand for $\{\mathbf{x} \in \Omega : u(\mathbf{x}) < t\}$, and similarly for other sets.

A standard approximation procedure shows that it is enough to prove formulas (2.2), (2.3) and (2.4) for the case when f is the characteristic function of an open interval (a, b). We will assume so throughout the proof.

Under the assumptions of *a)*, we have that, thanks to the usual coarea formula for Lipschitz functions (see, e.g., [7, Th. 2.93]),

$$\begin{aligned}
\int_\Omega f(u(\mathbf{x})) \, |Du(\mathbf{x})| \, \mathrm{d}\mathbf{x} &= \int_{\{a<u<b\}} |Du(\mathbf{x})| \, \mathrm{d}\mathbf{x} \\
&= \int_{-\infty}^{\infty} \mathcal{H}^{n-1}\left(\{a < u < b, \ u = t\}\right) \mathrm{d}t \\
&= \int_a^b \mathcal{H}^{n-1}\left(\{u = t\}\right) \mathrm{d}t \\
&= \int_{-\infty}^{\infty} f(t) \, \mathcal{H}^{n-1}(\{u = t\}) \, \mathrm{d}t,
\end{aligned}$$

which proves (2.2).

Under the assumptions of *b)*, we have instead, thanks to the usual coarea formula for *BV* functions (see, e.g., [7, Th. 2.93]) and the continuity of u, that

$$\int_\Omega f(u(\mathbf{x})) \, |Du(\mathbf{x})| \, \mathrm{d}\mathbf{x} = \int_{\{a<u<b\}} |Du(\mathbf{x})| \, \mathrm{d}\mathbf{x} = \int_{-\infty}^{\infty} \mathrm{Per}\left(\{u < t\}, \{a < u < b\}\right) \mathrm{d}t. \tag{2.5}$$

Let $t \in \mathbb{R}$. One always has

$$\mathrm{Per}\left(\{u < t\}, \{a < u < b\}\right) \le \mathrm{Per}\left(\{u < t\}, \Omega\right). \tag{2.6}$$

Moreover, if $\mathbf{g} \in C_c^1(\{a < u < b\}, \mathbb{R}^n)$ and $t < a$ then obviously

$$\int_{\{u>t\}} \operatorname{div} \mathbf{g} \, \mathrm{d}\mathbf{x} = 0, \tag{2.7}$$

whereas if $t > b$ then

$$\int_{\{u<t\}} \operatorname{div} \mathbf{g} \, \mathrm{d}\mathbf{x} = \int_{\{a<u<b\}} \operatorname{div} \mathbf{g} \, \mathrm{d}\mathbf{x} = 0, \tag{2.8}$$

because of the divergence theorem. Thanks to (2.6), (2.7) and (2.8) we conclude that

$$\int_{-\infty}^{\infty} \operatorname{Per}(\{u < t\}, \{a < u < b\}) \, \mathrm{d}t \le \int_a^b \operatorname{Per}(\{u < t\}, \Omega) \, \mathrm{d}t = \int_{-\infty}^{\infty} f(t) \operatorname{Per}(\{u < t\}, \Omega) \, \mathrm{d}t. \tag{2.9}$$

Equations (2.5) and (2.9) show that

$$\int_\Omega f(u(\mathbf{x})) \, |Du(\mathbf{x})| \, \mathrm{d}\mathbf{x} \le \int_{-\infty}^{\infty} f(t) \operatorname{Per}(\{u < t\}, \Omega) \, \mathrm{d}t. \tag{2.10}$$

As $f \circ u \in L^\infty(\Omega)$, we can apply the coarea formula for Sobolev maps (see [43, (1.1)]) to show that

$$\int_\Omega f(u(\mathbf{x})) \, |Du(\mathbf{x})| \, \mathrm{d}\mathbf{x} = \int_{-\infty}^{\infty} \int_{\{u=t\}} f(u(\mathbf{x})) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}) \, \mathrm{d}t = \int_{-\infty}^{\infty} f(t) \, \mathcal{H}^{n-1}(\{u = t\}) \, \mathrm{d}t. \tag{2.11}$$

For a.e. $t \in \mathbb{R}$, the set $\{u < t\}$ has finite perimeter in Ω, hence

$$\operatorname{Per}(\{u < t\}, \Omega) = \mathcal{H}^{n-1}\left(\partial^* \{u < t\} \cap \Omega\right). \tag{2.12}$$

The continuity of u shows that

$$\partial^* \{u < t\} \cap \Omega \subset \partial \{u < t\} \cap \Omega \subset \{u = t\}. \tag{2.13}$$

Equations (2.12) and (2.13) show that

$$\operatorname{Per}(\{u < t\}, \Omega) \le \mathcal{H}^{n-1}\left(\{u = t\}\right), \tag{2.14}$$

whereas equations (2.11) and (2.14) show that

$$\int_\Omega f(u(\mathbf{x})) \, |Du(\mathbf{x})| \, \mathrm{d}\mathbf{x} \ge \int_{-\infty}^{\infty} f(t) \operatorname{Per}(\{u < t\}, \Omega) \, \mathrm{d}t. \tag{2.15}$$

Inequalities (2.10) and (2.15) conclude (2.3). The proof of formula (2.4) is analogous. □

3 Representation of the surface energy functional

In this section we prove the representation formula (1.13) and a lower semicontinuity result for $\bar{\mathcal{E}}$. We start with the definitions of $\mathcal{E}$ and $\bar{\mathcal{E}}$.

Definition 3.1. *Let* $\mathbf{u} : \Omega \to \mathbb{R}^n$ *be approximately differentiable a.e. Suppose that* $\det \nabla \mathbf{u} \in L^1(\Omega)$ *and* $\operatorname{cof} \nabla \mathbf{u} \in L^1(\Omega, \mathbb{R}^{n \times n})$*. For every* $\mathbf{f} \in C_c^\infty(\bar{\Omega} \times \mathbb{R}^n, \mathbb{R}^n)$*, define*

$$\mathcal{E}(\mathbf{u}, \mathbf{f}) := \int_\Omega \left[\operatorname{cof} \nabla \mathbf{u}(\mathbf{x}) \cdot D_\mathbf{x} \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x})) + \det \nabla \mathbf{u}(\mathbf{x}) \operatorname{div} \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x}))\right] \mathrm{d}\mathbf{x} \tag{3.1}$$

and

$$\mathcal{E}(\mathbf{u}) := \sup\left\{\mathcal{E}(\mathbf{u}, \mathbf{f}) : \ \mathbf{f} \in C_c^\infty(\Omega \times \mathbb{R}^n, \mathbb{R}^n), \ \|\mathbf{f}\|_\infty \le 1\right\},$$
$$\bar{\mathcal{E}}(\mathbf{u}) := \sup\left\{\mathcal{E}(\mathbf{u}, \mathbf{f}) : \ \mathbf{f} \in C_c^\infty(\bar{\Omega} \times \mathbb{R}^n, \mathbb{R}^n), \ \|\mathbf{f}\|_\infty \le 1\right\}.$$

In equation (3.1), $D_{\mathbf{x}}\mathbf{f}(\mathbf{x},\mathbf{y})$ denotes the derivative of $\mathbf{f}(\cdot,\mathbf{y})$ evaluated at $\mathbf{x}$, while div always denotes the divergence operator in the deformed configuration, so $\operatorname{div}\mathbf{f}(\mathbf{x},\mathbf{y})$ is the divergence of $\mathbf{f}(\mathbf{x},\cdot)$ evaluated at $\mathbf{y}$.

The functional $\mathcal{E}$ was introduced in [37] to measure the creation of new surface of a deformation. The functional $\bar{\mathcal{E}}$ is new, and its difference with respect to $\mathcal{E}$ is that $\bar{\mathcal{E}}$ also takes into account what happens on $\partial\Omega$, and, in particular, it also measures the stretching of $\partial\Omega$ by $\mathbf{u}$.

It was shown in [38, Th. 2] that the inequality $\mathcal{E}(\mathbf{u})<\infty$ implies that suitable truncations of $\mathbf{u}^{-1}$ (see Definition 2.8) are in SBV. The adaptation of that result is as follows.

Proposition 3.2. *Let $\mathbf{u}\in L^\infty(\Omega,\mathbb{R}^n)$ be approximately differentiable a.e., one-to-one a.e., and such that* $\det\nabla\mathbf{u}>0$ *a.e.,* $\operatorname{cof}\nabla\mathbf{u}\in L^1(\Omega,\mathbb{R}^{n\times n})$ *and $\bar{\mathcal{E}}(\mathbf{u})<\infty$. Then $\mathbf{u}_\Omega^{-1}\in SBV(\mathbb{R}^n,\mathbb{R}^n)$.*

Proof. As a consequence of Proposition 2.5, we have that $\det\nabla\mathbf{u}\in L^1(\Omega)$.

In order to calculate the total variation of $\mathbf{u}_\Omega^{-1}$, fix $\alpha\in\{1,\dots,n\}$, denote by v_α the α-th component of $\mathbf{u}_\Omega^{-1}$, and notice that $v_\alpha\in L^\infty(\mathbb{R}^n)$. For each $\boldsymbol{\varphi}\in C_c^\infty(\mathbb{R}^n,\mathbb{R}^n)$ with $\|\boldsymbol{\varphi}\|_\infty\le 1$ we have, thanks to Proposition 2.5,

$$\int_{\mathbb{R}^n} v_\alpha(\mathbf{y})\operatorname{div}\boldsymbol{\varphi}(\mathbf{y})\,\mathrm{d}\mathbf{y}=\int_\Omega x_\alpha\operatorname{div}\boldsymbol{\varphi}(\mathbf{u}(\mathbf{x}))\det\nabla\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{3.2}$$

Let $\mathbf{e}_\alpha$ denote the α-th vector of the canonical basis of $\mathbb{R}^n$. When we define $\mathbf{f}_\alpha\in C_c^\infty(\bar\Omega\times\mathbb{R}^n,\mathbb{R}^n)$ as

$$\mathbf{f}_\alpha(\mathbf{x},\mathbf{y}):=x_\alpha\,\boldsymbol{\varphi}(\mathbf{y}),$$

we have that

$$\mathcal{E}(\mathbf{u},\mathbf{f}_\alpha)=\int_\Omega[\operatorname{cof}\nabla\mathbf{u}(\mathbf{x})\cdot(\boldsymbol{\varphi}(\mathbf{u}(\mathbf{x}))\otimes\mathbf{e}_\alpha)+x_\alpha\operatorname{div}\boldsymbol{\varphi}(\mathbf{u}(\mathbf{x}))\det\nabla\mathbf{u}(\mathbf{x})]\,\mathrm{d}\mathbf{x},$$

hence, by (3.2) we find that

$$\left|\int_{\mathbb{R}^n} v_\alpha(\mathbf{y})\operatorname{div}\boldsymbol{\varphi}(\mathbf{y})\,\mathrm{d}\mathbf{y}\right|\le\bar{\mathcal{E}}(\mathbf{u})\,\|\mathbf{id}\|_{L^\infty(\Omega,\mathbb{R}^n)}+\|\operatorname{cof}\nabla\mathbf{u}\|_{L^1(\Omega,\mathbb{R}^{n\times n})}.$$

This shows that v_α has finite total variation, and, hence $\mathbf{u}_\Omega^{-1}\in BV(\mathbb{R}^n,\mathbb{R}^n)$.

Fix a bounded open set Q such that $\operatorname{im_G}(\mathbf{u},\Omega)\subset\subset Q$. Let $\mathbf{g}\in C_c^\infty(\mathbb{R}^n)$ have support in Q and satisfy $\|\mathbf{g}\|_\infty\le 1$, consider $\psi\in C^1(\mathbb{R})\cap W^{1,\infty}(\mathbb{R})$ and fix $\alpha\in\{1,\dots,n\}$.

When we define $\mathbf{f}\in C_c^\infty(\bar\Omega\times\mathbb{R}^n,\mathbb{R}^n)$ as

$$\mathbf{f}(\mathbf{x},\mathbf{y}):=(\psi(x_\alpha)-\psi(0))\,\mathbf{g}(\mathbf{y}),$$

we have that, thanks to Lemma 2.9, for a.e. $\mathbf{x}\in\Omega$ and all $\mathbf{y}\in\mathbb{R}^n$,

$$\begin{aligned}D_{\mathbf{x}}\mathbf{f}(\mathbf{x},\mathbf{y})\cdot\operatorname{cof}\nabla\mathbf{u}(\mathbf{x})&=(\mathbf{g}(\mathbf{y})\otimes\psi'(x_\alpha)\,\mathbf{e}_\alpha)\cdot\operatorname{cof}\nabla\mathbf{u}(\mathbf{x})=\psi'(x_\alpha)\,(\operatorname{cof}\nabla\mathbf{u}(\mathbf{x})\,\mathbf{e}_\alpha)\cdot\mathbf{g}(\mathbf{y})\\&=\det\nabla\mathbf{u}(\mathbf{x})\,\psi'(x_\alpha)\left((\nabla\mathbf{u}^{-1}(\mathbf{u}(\mathbf{x})))^T\mathbf{e}_\alpha\right)\cdot\mathbf{g}(\mathbf{y})=\det\nabla\mathbf{u}(\mathbf{x})\,\psi'(x_\alpha)\,\nabla v_\alpha(\mathbf{u}(\mathbf{x}))\cdot\mathbf{g}(\mathbf{y})\end{aligned}$$

and

$$\operatorname{div}\mathbf{f}(\mathbf{x},\mathbf{y})=(\psi(x_\alpha)-\psi(0))\operatorname{div}\mathbf{g}(\mathbf{y}),$$

so, thanks to Proposition 2.5,

$$\begin{aligned}\mathcal{E}(\mathbf{u},\mathbf{f})&=\int_\Omega\det\nabla\mathbf{u}(\mathbf{x})\,[\psi'(x_\alpha)\,\nabla v_\alpha(\mathbf{u}(\mathbf{x}))\cdot\mathbf{g}(\mathbf{u}(\mathbf{x}))+(\psi(x_\alpha)-\psi(0))\operatorname{div}\mathbf{g}(\mathbf{u}(\mathbf{x}))]\,\mathrm{d}\mathbf{x}\\&=\int_{\operatorname{im_G}(\mathbf{u},\Omega)}[\psi'(v_\alpha(\mathbf{y}))\nabla v_\alpha(\mathbf{y})\cdot\mathbf{g}(\mathbf{y})+\psi(v_\alpha(\mathbf{y}))\operatorname{div}\mathbf{g}(\mathbf{y})]\,\mathrm{d}\mathbf{y}-\psi(0)\int_{\operatorname{im_G}(\mathbf{u},\Omega)}\operatorname{div}\mathbf{g}(\mathbf{y})\,\mathrm{d}\mathbf{y}.\end{aligned}$$

On the other hand, using Lemma 2.3,

$$\langle D(\psi \circ v_\alpha|_Q) - \psi' \circ v_\alpha \, \nabla v_\alpha \, \mathcal{L}^n \llcorner Q, \mathbf{g}|_Q \rangle = - \int_Q [\psi(v_\alpha(\mathbf{y})) \operatorname{div} \mathbf{g}(\mathbf{y}) + \psi'(v_\alpha(\mathbf{y})) \, \nabla v_\alpha(\mathbf{y}) \cdot \mathbf{g}(\mathbf{y})] \, \mathrm{d}\mathbf{y}$$

$$= - \int_{\operatorname{im}_{\mathrm{G}}(\mathbf{u},\Omega)} [\psi(v_\alpha(\mathbf{y})) \operatorname{div} \mathbf{g}(\mathbf{y}) + \psi'(v_\alpha(\mathbf{y})) \, \nabla v_\alpha(\mathbf{y}) \cdot \mathbf{g}(\mathbf{y})] \, \mathrm{d}\mathbf{y} - \psi(0) \int_{Q \setminus \operatorname{im}_{\mathrm{G}}(\mathbf{u},\Omega)} \operatorname{div} \mathbf{g}(\mathbf{y}) \, \mathrm{d}\mathbf{y}.$$

Summing the last two expressions and using the divergence theorem, we obtain that

$$\mathcal{E}(\mathbf{u}, \mathbf{f}) + \langle D(\psi \circ v_\alpha|_Q) - \psi' \circ v_\alpha \, \nabla v_\alpha \, \mathcal{L}^n \llcorner Q, \mathbf{g}|_Q \rangle = -\psi(0) \int_Q \operatorname{div} \mathbf{g}(\mathbf{y}) \, \mathrm{d}\mathbf{y} = 0.$$

Therefore,

$$\begin{aligned} |\langle D(\psi \circ v_\alpha|_Q) - \psi' \circ v_\alpha \, \nabla v_\alpha \, \mathcal{L}^n \llcorner Q, \mathbf{g}|_Q \rangle| &\leq \bar{\mathcal{E}}(\mathbf{u}) \, \|\mathbf{f}\|_{L^\infty(\bar{\Omega} \times \mathbb{R}^n, \mathbb{R}^n)} \\ &\leq \bar{\mathcal{E}}(\mathbf{u}) \sup_{\mathbf{x} \in \bar{\Omega}} |\psi(x_\alpha) - \psi(0)| \leq \bar{\mathcal{E}}(\mathbf{u}) \sup_{t,s \in \mathbb{R}} |\psi(t) - \psi(s)| \, . \end{aligned}$$

By the characterization of SBV given in [7, Prop. 4.12], this implies that $v_\alpha|_Q \in SBV(Q)$. As v_α is zero outside Q and in a neigbourhood of ∂Q, we have that $v_\alpha \in SBV(\mathbb{R}^n)$, and, hence $\mathbf{u}_\Omega^{-1} \in SBV(\mathbb{R}^n, \mathbb{R}^n)$. □

The following is a representation result for $\bar{\mathcal{E}}$. We follow the proof of [38, Th. 3], which showed an analogous statement for the surface energy $\mathcal{E}$.

Theorem 3.3. *Let Ω be a bounded Lipschitz domain satisfying $\mathbf{0} \notin \bar{\Omega}$. Let $\mathbf{u} \in L^\infty(\Omega, \mathbb{R}^n)$ be approximately differentiable a.e. with $\operatorname{cof} \nabla \mathbf{u} \in L^1(\Omega, \mathbb{R}^{n \times n})$. Suppose that there exists a measurable subset A of Ω such that*

a) $\mathbf{u}|_{\Omega \setminus A} = \mathbf{0}$.

b) $\mathbf{u}|_A$ *is one-to-one a.e.*

c) $\det \nabla \mathbf{u} > 0$ *a.e. in A.*

d) $\mathbf{u}_A^{-1} \in SBV(\mathbb{R}^n, \mathbb{R}^n)$.

Then $\operatorname{im}_{\mathrm{G}}(\mathbf{u}, A)$ has finite perimeter, for any $\mathbf{f} \in C_c^\infty(\bar{\Omega} \times \mathbb{R}^n, \mathbb{R}^n)$ we have that

$$\begin{aligned} \mathcal{E}(\mathbf{u}, \mathbf{f}) = \int_{J_{(\mathbf{u}|_A)^{-1}}} & \left[\mathbf{f}\big(((\mathbf{u}|_A)^{-1})^-(\mathbf{y}), \mathbf{y} \big) - \mathbf{f}\big(((\mathbf{u}|_A)^{-1})^+(\mathbf{y}), \mathbf{y} \big) \right] \cdot \boldsymbol{\nu}_{(\mathbf{u}|_A)^{-1}}(\mathbf{y}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}) \\ & + \int_{\partial^* \operatorname{im}_{\mathrm{G}}(\mathbf{u},A)} \mathbf{f}\big(((\mathbf{u}|_A)^{-1})^-(\mathbf{y}), \mathbf{y} \big) \cdot \boldsymbol{\nu}_{\operatorname{im}_{\mathrm{G}}(\mathbf{u},A)}(\mathbf{y}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}), \end{aligned} \tag{3.3}$$

and

$$\bar{\mathcal{E}}(\mathbf{u}) = \operatorname{Per} \operatorname{im}_{\mathrm{G}}(\mathbf{u}, A) + 2 \, \mathcal{H}^{n-1}(J_{(\mathbf{u}|_A)^{-1}}). \tag{3.4}$$

Proof. As in Proposition 3.2, the assumptions imply that $\det \nabla \mathbf{u} \in L^1(\Omega)$.

Assumption *d)* and the chain rule in BV (see [6, Prop. 1.2] or [7, Th. 3.96]) show that $|\mathbf{u}_A^{-1}| \in BV(\mathbb{R}^n)$, so, as a particular case of the coarea formula for BV functions (see, e.g., [7, Th. 3.40]), almost all superlevel sets of $|\mathbf{u}_A^{-1}|$ have finite perimeter. Since for each $0 \leq t < \inf_{\mathbf{x} \in \Omega} |\mathbf{x}|$ we have

$$\left\{ \mathbf{y} \in \mathbb{R}^n : |\mathbf{u}_A^{-1}(\mathbf{y})| > t \right\} = \operatorname{im}_{\mathrm{G}}(\mathbf{u}, A),$$

we conclude that

$$\operatorname{Per} \operatorname{im}_{\mathrm{G}}(\mathbf{u}, A) < \infty. \tag{3.5}$$

In this proof, given $B \subset \mathbb{R}^n$ and a function $\mathbf{h} : B \to \mathbb{R}^n$, we define the function

$$\mathbf{h} \bowtie \mathbf{id} : B \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n, \qquad (\mathbf{h} \bowtie \mathbf{id})(\mathbf{y}_1, \mathbf{y}_2) := (\mathbf{h}(\mathbf{y}_1), \mathbf{y}_2).$$

Let $\mathbf{f} \in C_c^\infty((\bar{\Omega} \cup \{\mathbf{0}\}) \times \mathbb{R}^n, \mathbb{R}^n)$. As the image of $\mathbf{u}_A^{-1}$ is contained in $\Omega \cup \{\mathbf{0}\}$, the function $\mathbf{f} \circ (\mathbf{u}_A^{-1} \bowtie \mathbf{id})$ is well defined; moreover, thanks to assumption *d)* and the chain rule in BV, it belongs to $SBV(\mathbb{R}^n, \mathbb{R}^n)$, and

$$\begin{aligned} \nabla \left(\mathbf{f} \circ (\mathbf{u}_A^{-1} \bowtie \mathbf{id})\right) &= D_{\mathbf{x}} \mathbf{f} \circ \left(\mathbf{u}_A^{-1} \bowtie \mathbf{id}\right) \nabla \mathbf{u}_A^{-1} + D_{\mathbf{y}} \mathbf{f} \circ \left(\mathbf{u}_A^{-1} \bowtie \mathbf{id}\right), \\ D^j \left(\mathbf{f} \circ (\mathbf{u}_A^{-1} \bowtie \mathbf{id})\right) &= \left[\mathbf{f} \circ \left((\mathbf{u}_A^{-1})^+ \bowtie \mathbf{id}\right) - \mathbf{f} \circ \left((\mathbf{u}_A^{-1})^- \bowtie \mathbf{id}\right)\right] \otimes \boldsymbol{\nu}_{\mathbf{u}_A^{-1}} \, \mathcal{H}^{n-1} \llcorner J_{\mathbf{u}_A^{-1}}, \end{aligned} \tag{3.6}$$

where we have used the trivial identities

$$J_{\mathbf{u}_A^{-1} \bowtie \mathbf{id}} = J_{\mathbf{u}_A^{-1}}, \qquad \boldsymbol{\nu}_{\mathbf{u}_A^{-1} \bowtie \mathbf{id}} = \boldsymbol{\nu}_{\mathbf{u}_A^{-1}}, \qquad \left(\mathbf{u}_A^{-1} \bowtie \mathbf{id}\right)^\pm = \left(\mathbf{u}_A^{-1}\right)^\pm \bowtie \mathbf{id}$$

and the notation D^j represents the jump part of the derivative (see, e.g., [7, Def. 3.91]). It is easy to check through the definitions and property (3.5) that the following equalities hold up to $\mathcal{H}^{n-1}$-null sets:

$$\begin{aligned} & J_{\mathbf{u}_A^{-1}} = J_{(\mathbf{u}|_A)^{-1}} \cup \partial^* \operatorname{im_G}(\mathbf{u}, A), \qquad J_{(\mathbf{u}|_A)^{-1}} \cap \partial^* \operatorname{im_G}(\mathbf{u}, A) = \varnothing, \\ & \boldsymbol{\nu}_{\mathbf{u}_A^{-1}} = \begin{cases} \boldsymbol{\nu}_{\mathbf{u}_A^{-1}} & \text{in } J_{(\mathbf{u}|_A)^{-1}}, \\ \boldsymbol{\nu}_{\operatorname{im_G}(\mathbf{u}, A)} & \text{in } \partial^* \operatorname{im_G}(\mathbf{u}, A), \end{cases} \qquad (\mathbf{u}_A^{-1})^+ = \begin{cases} ((\mathbf{u}|_A)^{-1})^+ & \text{in } J_{(\mathbf{u}|_A)^{-1}}, \\ \mathbf{0} & \text{in } \partial^* \operatorname{im_G}(\mathbf{u}, A), \end{cases} \\ & (\mathbf{u}_A^{-1})^- = ((\mathbf{u}|_A)^{-1})^-. \end{aligned} \tag{3.7}$$

Let $\eta \in C_c^\infty(\mathbb{R}^n)$. On the one hand, we have that

$$\langle D\big(\mathbf{f} \circ (\mathbf{u}_A^{-1} \bowtie \mathbf{id})\big), \eta \, \mathbf{1} \rangle = -\int_{\mathbb{R}^n} \left(\mathbf{f} \circ (\mathbf{u}_A^{-1} \bowtie \mathbf{id})\right) \cdot \operatorname{div}(\eta \mathbf{1}) \, \mathrm{d}\mathbf{y} = -\int_{\mathbb{R}^n} \mathbf{f}(\mathbf{u}_A^{-1}(\mathbf{y}), \mathbf{y}) \cdot D\eta(\mathbf{y}) \, \mathrm{d}\mathbf{y}, \tag{3.8}$$

whereas using (3.6) we find that

$$\begin{aligned} \langle D\big(\mathbf{f} \circ (\mathbf{u}_A^{-1} \bowtie \mathbf{id})\big), \eta \, \mathbf{1} \rangle = & \int_{\mathbb{R}^n} \left[\nabla \mathbf{u}_A^{-1}(\mathbf{y})^T \cdot D_{\mathbf{x}} \mathbf{f}(\mathbf{u}_A^{-1}(\mathbf{y}), \mathbf{y}) + \operatorname{div} \mathbf{f}(\mathbf{u}_A^{-1}(\mathbf{y}), \mathbf{y})\right] \eta(\mathbf{y}) \, \mathrm{d}\mathbf{y} \\ & + \int_{J_{\mathbf{u}_A^{-1}}} \left[\mathbf{f}\big((\mathbf{u}_A^{-1})^+(\mathbf{y}), \mathbf{y}\big) - \mathbf{f}\big((\mathbf{u}_A^{-1})^-(\mathbf{y}), \mathbf{y}\big)\right] \cdot \boldsymbol{\nu}_{\mathbf{u}_A^{-1}}(\mathbf{y}) \, \eta(\mathbf{y}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}). \end{aligned} \tag{3.9}$$

Recall that div denotes the divergence operator in the deformed configuration, that is, with respect to the $\mathbf{y}$ variables. If η is chosen so that $\eta = 1$ in a neigbourhood of $\operatorname{im_G}(\mathbf{u}, A)$, equalities (3.8) and (3.9) read, respectively, as

$$\langle D\big(\mathbf{f} \circ (\mathbf{u}_A^{-1} \bowtie \mathbf{id})\big), \eta \, \mathbf{1} \rangle = -\int_{\mathbb{R}^n \setminus \operatorname{im_G}(\mathbf{u}, A)} \mathbf{f}(\mathbf{0}, \mathbf{y}) \cdot D\eta(\mathbf{y}) \, \mathrm{d}\mathbf{y}, \tag{3.10}$$

and

$$\begin{aligned} \langle D\big(\mathbf{f} \circ (\mathbf{u}_A^{-1} \bowtie \mathbf{id})\big), \eta \, \mathbf{1} \rangle = & \int_{\mathbb{R}^n \setminus \operatorname{im_G}(\mathbf{u}, A)} \operatorname{div} \mathbf{f}(\mathbf{0}, \mathbf{y}) \, \eta(\mathbf{y}) \, \mathrm{d}\mathbf{y} \\ & + \int_{\operatorname{im_G}(\mathbf{u}, A)} \left[\nabla \mathbf{u}_A^{-1}(\mathbf{y})^T \cdot D_{\mathbf{x}} \mathbf{f}(\mathbf{u}_A^{-1}(\mathbf{y}), \mathbf{y}) + \operatorname{div} \mathbf{f}(\mathbf{u}_A^{-1}(\mathbf{y}), \mathbf{y})\right] \mathrm{d}\mathbf{y} \\ & + \int_{J_{\mathbf{u}_A^{-1}}} \left[\mathbf{f}\big((\mathbf{u}_A^{-1})^+(\mathbf{y}), \mathbf{y}\big) - \mathbf{f}\big((\mathbf{u}_A^{-1})^-(\mathbf{y}), \mathbf{y}\big)\right] \cdot \boldsymbol{\nu}_{\mathbf{u}_A^{-1}}(\mathbf{y}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}), \end{aligned} \tag{3.11}$$

where we have used that $J_{\mathbf{u}_A^{-1}} \subset \overline{\operatorname{im_G}(\mathbf{u}, A)}$ as well as Lemma 2.10. Now, the divergence theorem for sets of finite perimeter shows that

$$\int_{\mathbb{R}^n \setminus \operatorname{im_G}(\mathbf{u}, A)} \left[\mathbf{f}(\mathbf{0}, \mathbf{y}) \cdot D\eta(\mathbf{y}) + \operatorname{div} \mathbf{f}(\mathbf{0}, \mathbf{y}) \, \eta(\mathbf{y})\right] \mathrm{d}\mathbf{y} = -\int_{\partial^* \operatorname{im_G}(\mathbf{u}, A)} \mathbf{f}(\mathbf{0}, \mathbf{y}) \cdot \boldsymbol{\nu}_{\operatorname{im_G}(\mathbf{u}, A)}(\mathbf{y}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}). \tag{3.12}$$

Comparing (3.10), (3.11) and (3.12), we find that

$$\begin{aligned}
&\int_{\partial^* \operatorname{im_G}(\mathbf{u},A)} \mathbf{f}(\mathbf{0},\mathbf{y}) \cdot \boldsymbol{\nu}_{\operatorname{im_G}(\mathbf{u},A)}(\mathbf{y}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}) \\
&= \int_{\operatorname{im_G}(\mathbf{u},A)} \big[\nabla \mathbf{u}_A^{-1}(\mathbf{y})^T \cdot D_{\mathbf{x}} \mathbf{f}(\mathbf{u}_A^{-1}(\mathbf{y}),\mathbf{y}) + \operatorname{div} \mathbf{f}(\mathbf{u}_A^{-1}(\mathbf{y}),\mathbf{y})\big] \, \mathrm{d}\mathbf{y} \\
&\quad + \int_{J_{\mathbf{u}_A^{-1}}} \big[\mathbf{f}\big((\mathbf{u}_A^{-1})^+(\mathbf{y}),\mathbf{y}\big) - \mathbf{f}\big((\mathbf{u}_A^{-1})^-(\mathbf{y}),\mathbf{y}\big)\big] \cdot \boldsymbol{\nu}_{\mathbf{u}_A^{-1}}(\mathbf{y}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}),
\end{aligned} \tag{3.13}$$

Using identities (3.7) we obtain that, in fact,

$$\begin{aligned}
&\int_{J_{\mathbf{u}_A^{-1}}} \big[\mathbf{f}\big((\mathbf{u}_A^{-1})^-(\mathbf{y}),\mathbf{y}\big) - \mathbf{f}\big((\mathbf{u}_A^{-1})^+(\mathbf{y}),\mathbf{y}\big)\big] \cdot \boldsymbol{\nu}_{\mathbf{u}_A^{-1}}(\mathbf{y}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}) \\
&= \int_{J_{(\mathbf{u}|_A)^{-1}}} \big[\mathbf{f}\big(((\mathbf{u}|_A)^{-1})^-(\mathbf{y}),\mathbf{y}\big) - \mathbf{f}\big(((\mathbf{u}|_A)^{-1})^+(\mathbf{y}),\mathbf{y}\big)\big] \cdot \boldsymbol{\nu}_{(\mathbf{u}|_A)^{-1}}(\mathbf{y}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}) \\
&\quad + \int_{\partial^* \operatorname{im_G}(\mathbf{u},A)} \big[\mathbf{f}\big(((\mathbf{u}|_A)^{-1})^-(\mathbf{y}),\mathbf{y}\big) - \mathbf{f}\big(\mathbf{0},\mathbf{y}\big)\big] \cdot \boldsymbol{\nu}_{\operatorname{im_G}(\mathbf{u},A)}(\mathbf{y}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}).
\end{aligned} \tag{3.14}$$

Equalities (3.13) and (3.14), together with Lemmas 2.3 and 2.10, thus yield

$$\begin{aligned}
&\int_{\operatorname{im_G}(\mathbf{u},A)} \big[\nabla (\mathbf{u}|_A)^{-1}(\mathbf{y})^T \cdot D_{\mathbf{x}} \mathbf{f}((\mathbf{u}|_A)^{-1}(\mathbf{y}),\mathbf{y}) + \operatorname{div} \mathbf{f}((\mathbf{u}|_A)^{-1}(\mathbf{y}),\mathbf{y})\big] \, \mathrm{d}\mathbf{y} \\
&= \int_{J_{(\mathbf{u}|_A)^{-1}}} \big[\mathbf{f}\big(((\mathbf{u}|_A)^{-1})^-(\mathbf{y}),\mathbf{y}\big) - \mathbf{f}\big(((\mathbf{u}|_A)^{-1})^+(\mathbf{y}),\mathbf{y}\big)\big] \cdot \boldsymbol{\nu}_{(\mathbf{u}|_A)^{-1}}(\mathbf{y}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}) \\
&\quad + \int_{\partial^* \operatorname{im_G}(\mathbf{u},A)} \mathbf{f}\big(((\mathbf{u}|_A)^{-1})^-(\mathbf{y}),\mathbf{y}\big) \cdot \boldsymbol{\nu}_{\operatorname{im_G}(\mathbf{u},A)}(\mathbf{y}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}).
\end{aligned} \tag{3.15}$$

Now we use assumption *a)*, Proposition 2.5 and equality (3.15) to find that

$$\begin{aligned}
&\int_{\Omega} [\operatorname{cof} \nabla \mathbf{u}(\mathbf{x}) \cdot D_{\mathbf{x}} \mathbf{f}(\mathbf{x},\mathbf{u}(\mathbf{x})) + \det \nabla \mathbf{u}(\mathbf{x}) \operatorname{div} \mathbf{f}(\mathbf{x},\mathbf{u}(\mathbf{x}))] \, \mathrm{d}\mathbf{x} \\
&= \int_{A} [\operatorname{cof} \nabla \mathbf{u}(\mathbf{x}) \cdot D_{\mathbf{x}} \mathbf{f}(\mathbf{x},\mathbf{u}(\mathbf{x})) + \det \nabla \mathbf{u}(\mathbf{x}) \operatorname{div} \mathbf{f}(\mathbf{x},\mathbf{u}(\mathbf{x}))] \, \mathrm{d}\mathbf{x} \\
&= \int_{\operatorname{im_G}(\mathbf{u},A)} \big[\nabla (\mathbf{u}|_A)^{-1}(\mathbf{y})^T \cdot D_{\mathbf{x}} \mathbf{f}((\mathbf{u}|_A)^{-1}(\mathbf{y}),\mathbf{y}) + \operatorname{div} \mathbf{f}((\mathbf{u}|_A)^{-1}(\mathbf{y}),\mathbf{y})\big] \, \mathrm{d}\mathbf{y} \\
&= \int_{J_{(\mathbf{u}|_A)^{-1}}} \big[\mathbf{f}\big(((\mathbf{u}|_A)^{-1})^-(\mathbf{y}),\mathbf{y}\big) - \mathbf{f}\big(((\mathbf{u}|_A)^{-1})^+(\mathbf{y}),\mathbf{y}\big)\big] \cdot \boldsymbol{\nu}_{(\mathbf{u}|_A)^{-1}}(\mathbf{y}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}) \\
&\quad + \int_{\partial^* \operatorname{im_G}(\mathbf{u},A)} \mathbf{f}\big(((\mathbf{u}|_A)^{-1})^-(\mathbf{y}),\mathbf{y}\big) \cdot \boldsymbol{\nu}_{\operatorname{im_G}(\mathbf{u},A)}(\mathbf{y}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}).
\end{aligned} \tag{3.16}$$

Expression (3.16) is independent of the value of $\mathbf{f}$ at $\mathbf{0}$. Therefore, for any $\mathbf{f} \in C_c^\infty(\bar{\Omega} \times \mathbb{R}^n, \mathbb{R}^n)$, equality (3.3) holds. Consequently,

$$\bar{\mathcal{E}}(\mathbf{u}) \leq \operatorname{Per} \operatorname{im_G}(\mathbf{u},A) + 2\,\mathcal{H}^{n-1}(J_{(\mathbf{u}|_A)^{-1}}). \tag{3.17}$$

In particular, equation (3.4) holds if $\bar{\mathcal{E}}(\mathbf{u}) = \infty$. Suppose, then, that $\bar{\mathcal{E}}(\mathbf{u}) < \infty$. By Riesz' representation theorem, there exists an $\mathbb{R}^n$-valued Borel measure $\boldsymbol{\Lambda}$ in $\bar{\Omega} \times \mathbb{R}^n$ such that

$$|\boldsymbol{\Lambda}|(\bar{\Omega} \times \mathbb{R}^n) = \mathcal{E}(\mathbf{u}) \tag{3.18}$$

and

$$\mathcal{E}(\mathbf{u},\mathbf{f}) = \int_{\bar\Omega\times\mathbb{R}^n} \mathbf{f}(\mathbf{x},\mathbf{y})\cdot \mathrm{d}\boldsymbol{\Lambda}(\mathbf{x},\mathbf{y}), \qquad \mathbf{f}\in C_c^\infty(\bar\Omega\times\mathbb{R}^n,\mathbb{R}^n). \tag{3.19}$$

Assumption *d)* implies that the set $J_{\mathbf{u}_A^{-1}}$ is σ-finite with respect to $\mathcal{H}^{n-1}$. Let $F\subset J_{\mathbf{u}_A^{-1}}$ be a Borel set such that $\mathcal{H}^{n-1}(F)<\infty$, and consider the $\mathbb{R}^n$-valued measure

$$\begin{aligned}\boldsymbol{\lambda}_F :=& \left(((\mathbf{u}|_A)^{-1})^- \bowtie \mathbf{id}\right)_\sharp \left(\boldsymbol{\nu}_{\mathrm{im_G}(\mathbf{u},A)}\mathcal{H}^{n-1}\llcorner(\partial^*\mathrm{im_G}(\mathbf{u},A)\cap F)\right)\\ &+\left[\left(((\mathbf{u}|_A)^{-1})^- \bowtie \mathbf{id}\right)_\sharp - \left(((\mathbf{u}|_A)^{-1})^+ \bowtie \mathbf{id}\right)_\sharp\right]\left(\boldsymbol{\nu}_{(\mathbf{u}|_A)^{-1}}\mathcal{H}^{n-1}\llcorner(J_{(\mathbf{u}|_A)^{-1}}\cap F)\right).\end{aligned} \tag{3.20}$$

Here, the operator $\sharp$ denotes the push-forward of a measure (see, e.g., [7, Def. 1.70]). By definition of lateral traces,

$$\left(((\mathbf{u}|_A)^{-1})^- \bowtie \mathbf{id}\right)(\mathrm{im_G}(\mathbf{u},A)) \cap \left(((\mathbf{u}|_A)^{-1})^+ \bowtie \mathbf{id}\right)(\mathrm{im_G}(\mathbf{u},A)) = \varnothing, \tag{3.21}$$

whereas the definition of jump set yields that any point in $J_{(\mathbf{u}|_A)^{-1}}$ has density one in $\mathrm{im_G}(\mathbf{u},A)$, hence

$$\mathcal{H}^{n-1}\left(J_{(\mathbf{u}|_A)^{-1}}\cap\partial^*\mathrm{im_G}(\mathbf{u},A)\right)=0. \tag{3.22}$$

Using (3.21) and (3.22), it is easy to check, by the definition of total variation of a measure (see, e.g., [7, Def. 1.4]), that

$$\begin{aligned}|\boldsymbol{\lambda}_F| =& \left|\left(((\mathbf{u}|_A)^{-1})^- \bowtie \mathbf{id}\right)_\sharp \left(\boldsymbol{\nu}_{\mathrm{im_G}(\mathbf{u},A)}\mathcal{H}^{n-1}\llcorner(\partial^*\mathrm{im_G}(\mathbf{u},A)\cap F)\right)\right|\\ &+\left|\left(((\mathbf{u}|_A)^{-1})^- \bowtie \mathbf{id}\right)_\sharp \left(\boldsymbol{\nu}_{(\mathbf{u}|_A)^{-1}}\mathcal{H}^{n-1}\llcorner(J_{(\mathbf{u}|_A)^{-1}}\cap F)\right)\right|\\ &+\left|\left(((\mathbf{u}|_A)^{-1})^+ \bowtie \mathbf{id}\right)_\sharp \left(\boldsymbol{\nu}_{(\mathbf{u}|_A)^{-1}}\mathcal{H}^{n-1}\llcorner(J_{(\mathbf{u}|_A)^{-1}}\cap F)\right)\right|.\end{aligned}$$

In fact, by [6, Lemma 1.3] and [7, Prop. 1.23],

$$\begin{aligned}|\boldsymbol{\lambda}_F| =& \left(((\mathbf{u}|_A)^{-1})^- \bowtie \mathbf{id}\right)_\sharp \left(\mathcal{H}^{n-1}\llcorner(\partial^*\mathrm{im_G}(\mathbf{u},A)\cap F)\right)\\ &+\left(((\mathbf{u}|_A)^{-1})^- \bowtie \mathbf{id}\right)_\sharp \left(\mathcal{H}^{n-1}\llcorner(J_{(\mathbf{u}|_A)^{-1}}\cap F)\right)\\ &+\left(((\mathbf{u}|_A)^{-1})^+ \bowtie \mathbf{id}\right)_\sharp \left(\mathcal{H}^{n-1}\llcorner(J_{(\mathbf{u}|_A)^{-1}}\cap F)\right).\end{aligned}$$

Thus, on the one hand,

$$\begin{aligned}|\boldsymbol{\lambda}_F|\left(\bar\Omega\times\mathbb{R}^n\right) =& \mathcal{H}^{n-1}\left(\left\{\mathbf{y}\in\partial^*\mathrm{im_G}(\mathbf{u},A)\cap F:\ ((\mathbf{u}|_A)^{-1})^-(\mathbf{y})\in\bar\Omega\right\}\right)\\ &+\mathcal{H}^{n-1}\left(\left\{\mathbf{y}\in J_{(\mathbf{u}|_A)^{-1}}\cap F:\ ((\mathbf{u}|_A)^{-1})^-(\mathbf{y})\in\bar\Omega\right\}\right)\\ &+\mathcal{H}^{n-1}\left(\left\{\mathbf{y}\in J_{(\mathbf{u}|_A)^{-1}}\cap F:\ ((\mathbf{u}|_A)^{-1})^+(\mathbf{y})\in\bar\Omega\right\}\right)\\ =& \mathcal{H}^{n-1}\left(\partial^*\mathrm{im_G}(\mathbf{u},A)\cap F\right)+2\,\mathcal{H}^{n-1}\left(J_{(\mathbf{u}|_A)^{-1}}\cap F\right).\end{aligned} \tag{3.23}$$

On the other hand, equalities (3.3) and (3.19) together with a standard approximation argument based on Lusin's theorem, show that the equality

$$\begin{aligned}\int_{\bar\Omega\times\mathbb{R}^n}\phi(\mathbf{x})\,\mathbf{g}(\mathbf{y})\cdot\mathrm{d}\boldsymbol{\Lambda}(\mathbf{x},\mathbf{y}) =& \int_{\partial^*\mathrm{im_G}(\mathbf{u},A)}\phi(((\mathbf{u}|_A)^{-1})^-(\mathbf{y}))\,\mathbf{g}(\mathbf{y})\cdot\boldsymbol{\nu}_{\mathrm{im_G}(\mathbf{u},A)}(\mathbf{y})\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{y})\\ &+\int_{J_{(\mathbf{u}|_A)^{-1}}}\left[\phi(((\mathbf{u}|_A)^{-1})^-(\mathbf{y}))-\phi(((\mathbf{u}|_A)^{-1})^+(\mathbf{y}))\right]\mathbf{g}(\mathbf{y})\cdot\boldsymbol{\nu}_{(\mathbf{u}|_A)^{-1}}(\mathbf{y})\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{y})\end{aligned} \tag{3.24}$$

is valid for any $\phi\in C^\infty(\bar\Omega)$ and any bounded Borel function $\mathbf{g}:\mathbb{R}^n\to\mathbb{R}^n$. Let now $\phi\in C^\infty(\bar\Omega)$ and $\mathbf{g}\in C_c(\mathbb{R}^n)$, and apply (3.24) to ϕ and $\mathbf{g}\chi_F$ so as to obtain

$$\begin{aligned}\int_{\bar\Omega\times F}\phi(\mathbf{x})\,\mathbf{g}(\mathbf{y})\cdot\mathrm{d}\boldsymbol{\Lambda}(\mathbf{x},\mathbf{y}) =& \int_{\partial^*\mathrm{im_G}(\mathbf{u},A)\cap F}\phi(((\mathbf{u}|_A)^{-1})^-(\mathbf{y}))\,\mathbf{g}(\mathbf{y})\cdot\boldsymbol{\nu}_{\mathrm{im_G}(\mathbf{u},A)}(\mathbf{y})\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{y})\\ &+\int_{J_{(\mathbf{u}|_A)^{-1}}\cap F}\left[\phi(((\mathbf{u}|_A)^{-1})^-(\mathbf{y}))-\phi(((\mathbf{u}|_A)^{-1})^+(\mathbf{y}))\right]\mathbf{g}(\mathbf{y})\cdot\boldsymbol{\nu}_{(\mathbf{u}|_A)^{-1}}(\mathbf{y})\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}),\end{aligned}$$

which, together with (3.20), yields

$$\int_{\bar{\Omega}\times F} \phi(\mathbf{x})\, \mathbf{g}(\mathbf{y}) \cdot \mathrm{d}\mathbf{\Lambda}(\mathbf{x},\mathbf{y}) = \int_{\bar{\Omega}\times \mathbb{R}^n} \phi(\mathbf{x})\, \mathbf{g}(\mathbf{y}) \cdot \mathrm{d}\boldsymbol{\lambda}_F(\mathbf{x},\mathbf{y}). \tag{3.25}$$

Using that the set of sums of functions the form

$$\phi(\mathbf{x})\, \mathbf{g}(\mathbf{y}) \text{ with } \phi \in C^\infty(\bar{\Omega}) \text{ and } \mathbf{g} \in C_c(\mathbb{R}^n)$$

is dense in $C_c(\bar{\Omega}\times\mathbb{R}^n, \mathbb{R}^n)$, we conclude from (3.25) that

$$\int_{\bar{\Omega}\times F} \mathbf{f}(\mathbf{x},\mathbf{y}) \cdot \mathrm{d}\mathbf{\Lambda}(\mathbf{x},\mathbf{y}) = \int_{\bar{\Omega}\times \mathbb{R}^n} \mathbf{f}(\mathbf{x},\mathbf{y}) \cdot \mathrm{d}\boldsymbol{\lambda}_F(\mathbf{x},\mathbf{y})$$

holds true for all $\mathbf{f} \in C_c(\bar{\Omega}\times\mathbb{R}^n, \mathbb{R}^n)$. By Riesz' representation theorem, this shows that $\mathbf{\Lambda} \llcorner (\bar{\Omega}\times F) = \boldsymbol{\lambda}_F$. By virtue of (3.23), we obtain that

$$|\mathbf{\Lambda}|\,(\bar{\Omega}\times F) = \mathcal{H}^{n-1}\big(\partial^* \operatorname{im_G}(\mathbf{u}, A) \cap F\big) + 2\,\mathcal{H}^{n-1}\big(J_{(\mathbf{u}|_A)^{-1}} \cap F\big),$$

so, in particular,

$$|\mathbf{\Lambda}|\,(\bar{\Omega}\times \mathbb{R}^n) \geq \mathcal{H}^{n-1}\big(\partial^* \operatorname{im_G}(\mathbf{u}, A) \cap F\big) + 2\,\mathcal{H}^{n-1}\big(J_{(\mathbf{u}|_A)^{-1}} \cap F\big).$$

As $J_{\mathbf{u}_A^{-1}}$ is σ-finite with respect to $\mathcal{H}^{n-1}$, we conclude that

$$|\mathbf{\Lambda}|\,(\bar{\Omega}\times \mathbb{R}^n) \geq \mathcal{H}^{n-1}\big(\partial^* \operatorname{im_G}(\mathbf{u}, A)\big) + 2\,\mathcal{H}^{n-1}\big(J_{(\mathbf{u}|_A)^{-1}}\big),$$

but equations (3.17) and (3.18) show that, in fact, equality (3.4) holds. □

The following is a lower semicontinuity result for $\bar{\mathcal{E}}$ and will represent a key step in the proof of the compactness and lower bound result for the Γ-convergence of I_ε (see (1.10)) to be proved in Section 6. Its proof is an adaptation of those of [37, Thms. 2 and 3].

Theorem 3.4. *Let Ω be a bounded Lipschitz domain satisfying $\mathbf{0} \notin \bar{\Omega}$. For each ε, let $\mathbf{u}_\varepsilon : \Omega \to \mathbb{R}^n$ be approximately differentiable a.e., and let A_ε be a measurable subset of Ω such that*

a) $\operatorname{cof} \nabla\mathbf{u}_\varepsilon \in L^1(A_\varepsilon, \mathbb{R}^{n\times n})$ *and* $\det \nabla\mathbf{u}_\varepsilon \in L^1(A_\varepsilon)$.

b) $\mathcal{L}^n(A_\varepsilon) \to \mathcal{L}^n(\Omega)$.

c) $\mathbf{u}_\varepsilon|_{A_\varepsilon}$ *is one-to-one a.e.*

d) $\det \nabla\mathbf{u}_\varepsilon > 0$ *a.e. in* A_ε.

e) $\mathbf{u}_{\varepsilon,A_\varepsilon}^{-1} \in SBV(\mathbb{R}^n, \mathbb{R}^n)$.

f) $\sup_\varepsilon \left[\operatorname{Per} \operatorname{im_G}(\mathbf{u}_\varepsilon, A_\varepsilon) + \mathcal{H}^{n-1}(J_{(\mathbf{u}_\varepsilon|_{A_\varepsilon})^{-1}})\right] < \infty$.

g) There exists $\theta \in L^1(\Omega)$ *with* $\theta > 0$ *a.e. such that* $\chi_{A_\varepsilon} \det \nabla\mathbf{u}_\varepsilon \rightharpoonup \theta$ *in* $L^1(\Omega)$.

h) $\{\mathbf{u}_\varepsilon\}_\varepsilon$ *is equiintegrable.*

i) There exists a map $\mathbf{u} : \Omega \to \mathbb{R}^n$ *approximately differentiable a.e. such that* $\mathbf{u}_\varepsilon \to \mathbf{u}$ *a.e.*

j) $\chi_{A_\varepsilon} \operatorname{cof} \nabla\mathbf{u}_\varepsilon \rightharpoonup \operatorname{cof} \nabla\mathbf{u}$ *in* $L^1(\Omega, \mathbb{R}^{n\times n})$.

Then

i) $\theta = \det \nabla\mathbf{u}$ *a.e.*

ii) $\mathbf{u}$ *is one-to-one a.e.*

iii) $\chi_{\mathrm{im_G}(\mathbf{u}_\varepsilon, A_\varepsilon)} \to \chi_{\mathrm{im_G}(\mathbf{u},\Omega)}$ *in* $L^1(\mathbb{R}^n)$.

iv) $\operatorname{Per} \mathrm{im_G}(\mathbf{u}, \Omega) + 2\,\mathcal{H}^{n-1}(J_{\mathbf{u}^{-1}}) \leq \liminf_{\varepsilon\to 0} \left[\operatorname{Per} \mathrm{im_G}(\mathbf{u}_\varepsilon, A_\varepsilon) + 2\,\mathcal{H}^{n-1}(J_{(\mathbf{u}_\varepsilon|_{A_\varepsilon})^{-1}})\right]$.

Proof. As $\sup_\varepsilon \operatorname{Per} \mathrm{im_G}(\mathbf{u}_\varepsilon, A_\varepsilon) < \infty$, there exists a measurable set $V \subset \mathbb{R}^n$ such that, for a subsequence, $\mathrm{im_G}(\mathbf{u}_\varepsilon, A_\varepsilon) \to V$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We will see that, in fact, there is no need of taking a subsequence.

Let $\varphi \in C_c(\mathbb{R}^n)$. By Proposition 2.5, for all ε,

$$\int_{\mathrm{im_G}(\mathbf{u}_\varepsilon, A_\varepsilon)} \varphi(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_{A_\varepsilon} \varphi(\mathbf{u}_\varepsilon(\mathbf{x})) \det \nabla \mathbf{u}_\varepsilon(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Letting $\varepsilon \to 0$ and using assumption *g)* and Lemma 2.11, we obtain

$$\int_{\mathbb{R}^n} \varphi(\mathbf{y})\,\chi_V(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_\Omega \varphi(\mathbf{u}(\mathbf{x}))\,\theta(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{3.26}$$

A standard approximation procedure using Lusin's theorem shows that (3.26) holds true for any bounded Borel function $\varphi : \mathbb{R}^n \to \mathbb{R}$.

Now we show that $\det \nabla \mathbf{u}(\mathbf{x}) \neq 0$ for a.e. $\mathbf{x} \in \Omega$. Let Ω_d be the set of approximate differentiability points of $\mathbf{u}$, and let Z be the set of $\mathbf{x} \in \Omega_d$ such that $\det \nabla \mathbf{u}(\mathbf{x}) = 0$. As a consequence of Proposition 2.5, we find that $\mathcal{L}^n(\mathbf{u}(Z)) = 0$. Thus, there exists a Borel set U containing $\mathbf{u}(Z)$ such that $\mathcal{L}^n(U) = 0$. Applying (3.26) with $\varphi = \chi_U$, we obtain that

$$0 \leq \int_Z \theta\,\mathrm{d}\mathbf{x} \leq \int_\Omega \chi_U(\mathbf{u}(\mathbf{x}))\,\theta(\mathbf{x})\,\mathrm{d}\mathbf{x} = \mathcal{L}^n(U \cap V) \leq \mathcal{L}^n(U) = 0,$$

and, since $\theta > 0$ a.e., we conclude that $\mathcal{L}^n(Z) = 0$.

Define Ω_1 as the set of $\mathbf{x} \in \Omega_d$ such that $\det \nabla \mathbf{u}(\mathbf{x}) \neq 0$ and $\theta(\mathbf{x}) > 0$. We have just shown that Ω_1 has full measure in Ω. The function $\tilde{\psi} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\tilde{\psi}(\mathbf{y}) := \sum_{\substack{\mathbf{x}\in\Omega_1\\ \mathbf{u}(\mathbf{x})=\mathbf{y}}} \frac{\theta(\mathbf{x})}{|\det \nabla \mathbf{u}(\mathbf{x})|}, \qquad \mathbf{y} \in \mathbb{R}^n$$

satisfies that $\tilde{\psi} > 0$ in $\mathbf{u}(\Omega_1)$, $\tilde{\psi} = 0$ in $\mathbb{R}^n \setminus \mathbf{u}(\Omega_1)$ and, thanks to Proposition 2.5, for any bounded Borel function $\varphi : \mathbb{R}^n \to \mathbb{R}$,

$$\int_\Omega \varphi(\mathbf{u}(\mathbf{x}))\,\theta(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n} \varphi(\mathbf{y})\,\tilde{\psi}(\mathbf{y})\,\chi_{\mathrm{im_G}(\mathbf{u},\Omega)}(\mathbf{y})\,\mathrm{d}\mathbf{y}. \tag{3.27}$$

Equalities (3.26) and (3.27) show that $\chi_V = \tilde{\psi}\chi_{\mathrm{im_G}(\mathbf{u},\Omega)}$ a.e. Since $\tilde{\psi} > 0$ in $\mathbf{u}(\Omega_1)$, necessarily $V = \mathrm{im_G}(\mathbf{u}, \Omega)$ a.e. and $\tilde{\psi} = \chi_{\mathrm{im_G}(\mathbf{u},\Omega)}$ a.e. Moreover, $\mathrm{im_G}(\mathbf{u}_\varepsilon, A_\varepsilon) \to \mathrm{im_G}(\mathbf{u}, \Omega)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ for the whole sequence ε.

Define $\tilde{\mathbf{u}}_\varepsilon := \chi_{A_\varepsilon}\mathbf{u}_\varepsilon$. Assumptions *b)* and *h)* show that $\tilde{\mathbf{u}}_\varepsilon - \mathbf{u}_\varepsilon \to \mathbf{0}$ in $L^1(\Omega, \mathbb{R}^n)$, and, hence, for a subsequence, the convergence also holds a.e., so, thanks to assumption *i)*, $\tilde{\mathbf{u}}_\varepsilon \to \mathbf{u}$ a.e. For each $\mathbf{f} \in C^\infty_c(\bar{\Omega} \times \mathbb{R}^n, \mathbb{R}^n)$, thanks to assumptions *g)* and *j)*, and Lemma 2.11, one has

$$\lim_{\varepsilon\to 0} \mathcal{E}(\tilde{\mathbf{u}}_\varepsilon, \mathbf{f}) = \int_\Omega [\operatorname{cof} \nabla \mathbf{u}(\mathbf{x}) \cdot D_{\mathbf{x}}\mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x})) + \theta(\mathbf{x}) \operatorname{div} \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x}))]\,\mathrm{d}\mathbf{x}.$$

Since $\mathcal{E}(\tilde{\mathbf{u}}_\varepsilon, \mathbf{f}) \leq \bar{\mathcal{E}}(\tilde{\mathbf{u}}_\varepsilon)\|\mathbf{f}\|_\infty$ for each ε, thanks to Theorem 3.3 and assumption *f)*, the linear functional $\Lambda : C^\infty_c(\bar{\Omega} \times \mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}$ given by

$$\Lambda(\mathbf{f}) := \int_\Omega [\operatorname{cof} \nabla \mathbf{u}(\mathbf{x}) \cdot D_{\mathbf{x}}\mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x})) + \theta(\mathbf{x}) \operatorname{div} \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x}))]\,\mathrm{d}\mathbf{x}$$

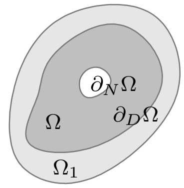


Figure 1: Ω is coloured in grey, and Ω_1 is the union of the grey and light-grey parts.

satisfies

$$|\Lambda(\mathbf{f})| \leq \liminf_{\varepsilon\to 0} \bar{\mathcal{E}}(\tilde{\mathbf{u}}_\varepsilon)\, \|\mathbf{f}\|_\infty\,, \qquad \mathbf{f} \in C_c^\infty(\bar{\Omega}\times\mathbb{R}^n, \mathbb{R}^n).$$

By Riesz' representation theorem, we obtain that Λ can be identified with an $\mathbb{R}^n$-valued measure in $\bar{\Omega}\times\mathbb{R}^n$. At this point, one can repeat the proof of [37, Th. 3] and conclude that $\theta = \det \nabla\mathbf{u}$ a.e. In particular, for each $\mathbf{f} \in C_c^\infty(\bar{\Omega}\times\mathbb{R}^n, \mathbb{R}^n)$, we have that $\mathcal{E}(\tilde{\mathbf{u}}_\varepsilon, \mathbf{f}) \to \mathcal{E}(\mathbf{u}, \mathbf{f})$, so taking suprema we obtain that $\bar{\mathcal{E}}(\mathbf{u}) \leq \liminf_{\varepsilon\to 0} \bar{\mathcal{E}}(\tilde{\mathbf{u}}_\varepsilon)$, and we conclude assertion *iv)* thanks to Theorem 3.3 and Proposition 3.2.

The fact that $\theta = \det\nabla\mathbf{u}$ a.e. shows that $\tilde{\psi}(\mathbf{y}) = \mathcal{H}^0(\{\mathbf{x}\in\Omega_1 : \mathbf{u}(\mathbf{x}) = \mathbf{y}\})$ for a.e. $\mathbf{y}\in\mathbb{R}^n$. Using now that $\tilde{\psi} = \chi_{\mathrm{im_G}(\mathbf{u},\Omega)}$ a.e., we infer that $\mathbf{u}$ is one-to-one a.e. $\square$

The list of assumptions of Theorem 3.4 may look artificial, but we will see in Section 6 that they are naturally satisfied for a truncation of the maps $\mathbf{u}_\varepsilon$ generating a minimizing sequence for the functional I_ε of (1.10).

4 General assumptions for the approximated energy

In this section we present the admissible set for the functional I_ε of (1.10). We also list the general assumptions for the stored energy function W.

The reference configuration of the body is represented by a bounded domain Ω of $\mathbb{R}^n$. We distinguish the Dirichlet part $\partial_D\Omega$ of the boundary $\partial\Omega$, where the deformation is prescribed, and the Neumann part $\partial_N\Omega := \partial\Omega\setminus\partial_D\Omega$. We impose that both $\partial_D\Omega$ and $\partial_N\Omega$ are closed. We assume that $\partial_D\Omega$ is non-empty and Lipschitz; in particular, $\mathcal{H}^{n-1}(\partial_D\Omega) > 0$. Moreover, we suppose that there exists an open set $\Omega_1\subset\mathbb{R}^n$ such that $\Omega\cup\partial_D\Omega\subset\Omega_1$ and $\partial_N\Omega\subset\partial\Omega_1$. A typical configuration is shown in Figure 1. We will also need sets $K\subset Q\subset\mathbb{R}^n$ in the deformed configuration such that Q is open and K is compact.

Recall the notation for minors from Section 2.7. The assumptions for the function $W:\Omega\times K\times\mathbb{R}^{n\times n}_+\to\mathbb{R}$ are the following:

(W1) There exists $\tilde{W}:\Omega\times K\times\mathbb{R}^\tau_+\to\mathbb{R}$ such that the function $\tilde{W}(\cdot,\mathbf{y},\boldsymbol{\xi})$ is measurable for every $(\mathbf{y},\boldsymbol{\xi})\in K\times\mathbb{R}^\tau_+$, the function $\tilde{W}(\mathbf{x},\cdot,\cdot)$ is continuous for a.e. $\mathbf{x}\in\Omega$, the function $\tilde{W}(\mathbf{x},\mathbf{y},\cdot)$ is convex for a.e. $\mathbf{x}\in\Omega$ and every $\mathbf{y}\in K$, and

$$W(\mathbf{x},\mathbf{y},\mathbf{F}) = \tilde{W}(\mathbf{x},\mathbf{y},\boldsymbol{\mu}(\mathbf{F})) \quad \text{for a.e. } \mathbf{x}\in\Omega \text{ and all } (\mathbf{y},\mathbf{F})\in K\times\mathbb{R}^{n\times n}_+.$$

(W2) There exist a constant $c>0$, an exponent $p\geq n-1$, an increasing function $h_1:(0,\infty)\to[0,\infty)$ and a convex function $h_2:(0,\infty)\to[0,\infty)$ such that

$$\lim_{t\to\infty}\frac{h_1(t)}{t} = \lim_{t\to\infty}\frac{h_2(t)}{t} = \lim_{t\to 0^+} h_2(t) = \infty$$

and

$$W(\mathbf{x},\mathbf{y},\mathbf{F}) \geq c\,|\mathbf{F}|^p + h_1(|\operatorname{cof}\mathbf{F}|) + h_2(\det\mathbf{F})$$

for a.e. $\mathbf{x}\in\Omega$, all $\mathbf{y}\in K$ and all $\mathbf{F}\in\mathbb{R}^{n\times n}_+$.

Assumptions (W1)–(W2) are the usual ones in nonlinear elasticity (see, e.g., [10, 49]), in which W is assumed to be polyconvex and blows up when the determinant of the deformation gradients goes to zero. However, the growth conditions are slow enough to allow for cavitation (see, e.g., [48, 53, 37, 39]): this is why p is only required to be greater than or equal to $n-1$, and h_1 is only required to be superlinear at infinity. We also remark that the dependence of W on $\mathbf{y}$ is not physical, but we have included it for the sake of generality, since it does not affect the mathematical analysis.

Given parameters $\lambda_1, \lambda_2, \varepsilon, \eta, b > 0$, an exponent $q > n$ and functions $\mathbf{u} \in W^{1,p}(\Omega, K)$, $v \in W^{1,q}(\Omega, [0,1])$, $w \in W^{1,q}(Q, [0,1])$, we define the approximated energy as

$$
\begin{aligned}
I(\mathbf{u}, v, w) := & \int_\Omega (v(\mathbf{x})^2 + \eta)\, W(\mathbf{x}, \mathbf{u}(\mathbf{x}), D\mathbf{u}(\mathbf{x}))\, \mathrm{d}\mathbf{x} + \lambda_1 \int_\Omega \left[\varepsilon^{q-1} \frac{|Dv(\mathbf{x})|^q}{q} + \frac{(1-v(\mathbf{x}))^{q'}}{q'\varepsilon} \right] \mathrm{d}\mathbf{x} \\
& + 6\, \lambda_2 \int_Q \left[\varepsilon^{q-1} \frac{|Dw(\mathbf{y})|^q}{q} + \frac{w(\mathbf{y})^{q'}(1-w(\mathbf{y}))^{q'}}{q'\varepsilon} \right] \mathrm{d}\mathbf{y}.
\end{aligned} \tag{4.1}
$$

We assume the existence of a bi-Lipschitz homeomorphism $\mathbf{u}_0 : \Omega_1 \to K$ such that $\det D\mathbf{u}_0 > 0$ a.e. and

$$
\int_\Omega W(\mathbf{x}, \mathbf{u}_0(\mathbf{x}), D\mathbf{u}_0(\mathbf{x}))\, \mathrm{d}\mathbf{x} < \infty. \tag{4.2}
$$

Note that $\operatorname{im}_{\mathrm{G}}(\mathbf{u}_0, \Omega)$ is open, as it coincides with $\mathbf{u}_0(\Omega)$. Moreover, $\mathcal{E}(\mathbf{u}_0) = 0$ (see, e.g., [37, Sect. 4]).

We define $\mathcal{A}^E$ as the set of $\mathbf{u} \in W^{1,p}(\Omega, K)$ such that

$$
\mathbf{u} = \mathbf{u}_0 \text{ on } \partial_D \Omega, \tag{4.3}
$$

in the sense of traces, and that, defining

$$
\bar{\mathbf{u}} := \begin{cases} \mathbf{u} & \text{in } \Omega, \\ \mathbf{u}_0 & \text{in } \Omega_1 \setminus \Omega, \end{cases} \tag{4.4}
$$

we have that $\bar{\mathbf{u}}$ is one-to-one a.e., $\det D\bar{\mathbf{u}} > 0$ a.e. and

$$
\mathcal{E}(\bar{\mathbf{u}}) = 0. \tag{4.5}
$$

Note that the following properties are automatically satisfied: $\bar{\mathbf{u}} \in W^{1,p}(\Omega_1, K)$,

$$
\operatorname{im}_{\mathrm{G}}(\mathbf{u}, \Omega) \subset K \quad \text{a.e.} \tag{4.6}
$$

and

$$
\mathcal{L}^n \left(\operatorname{im}_{\mathrm{G}}(\bar{\mathbf{u}}, \Omega_1 \setminus \Omega) \cap \operatorname{im}_{\mathrm{G}}(\mathbf{u}, \Omega) \right) = 0. \tag{4.7}
$$

Moreover, $\mathbf{u}_0 \in \mathcal{A}^E$.

It was shown in [39, Th. 4.6] that condition (4.5) prevents the creation of cavities of $\bar{\mathbf{u}}$ in Ω_1. In particular, it prevents the creation of cavities in Ω and at $\partial_D\Omega$ (as in [53]). Moreover, (4.5) is automatically satisfied if $p \geq n$ (see [37, Sect. 4]), or if $\bar{\mathbf{u}}$ satisfies condition INV and $\operatorname{Det} D\bar{\mathbf{u}} = \det D\bar{\mathbf{u}}$ (see [39, Lemma 5.3] and also [48] for the definition of condition INV and of the distributional determinant Det).

We define $\mathcal{A}$ as the set of triples $(\mathbf{u}, v, w)$ such that $\mathbf{u} \in \mathcal{A}^E$, $v \in W^{1,q}(\Omega, [0,1])$, $w \in W^{1,q}(Q, [0,1])$ and

$$
v = 1 \text{ on } \partial_D \Omega, \tag{4.8}
$$

$$
v = 0 \text{ on } \partial_N \Omega, \tag{4.9}
$$

$$
w = 0 \text{ in } Q \setminus \operatorname{im}_{\mathrm{G}}(\mathbf{u}, \Omega), \tag{4.10}
$$

$$
v(\mathbf{x}) \geq w(\mathbf{u}(\mathbf{x})) \quad \text{a.e. } \mathbf{x} \in \Omega, \tag{4.11}
$$

$$
\int_\Omega [v(\mathbf{x}) - w(\mathbf{u}(\mathbf{x}))]\, \mathrm{d}\mathbf{x} \leq b. \tag{4.12}
$$

The functional I of (4.1) will be defined on the set $\mathcal{A}$. We explain the choice of conditions (4.8)–(4.12). The functions v and w are phase-field variables: v in the reference configuration, and w in the deformed configuration. A value of v close to 1 indicates healthy material, while if it is close to zero, it indicates a region with a crack. The function w indicates where there is matter, so $w \simeq \chi_{\operatorname{im_G}(\mathbf{u},\Omega)}$. Except close to the boundary, the function w follows v in the deformed configuration, so $w \simeq v \circ \mathbf{u}$: this is expressed by inequalities (4.11)–(4.12). The fact that $w \simeq \chi_{\operatorname{im_G}(\mathbf{u},\Omega)}$ agrees with the boundary condition (4.10). Condition (4.8) is also natural since the trace equality (4.3) and the existence (4.4) of an extension $\bar{\mathbf{u}}$ in $W^{1,p}(\Omega_1, \mathbb{R}^n)$ prevent a fracture at $\partial_D\Omega$. Condition (4.9) is somewhat artificial and comes from a technical part of the proof. As $\partial_N\Omega$ is the free part of the boundary, there is no information about whether $\mathbf{u}$ presents fracture at $\partial_N\Omega$. Condition (4.9) allows for it but it does not impose it. At some point of the proof of the lower bound inequality (see Proposition 6.8, and, in particular, relation (6.59)), we need to distinguish $\partial_N\Omega$ from $\partial_D\Omega$ with the mere information of v, and we are only able to do it with (4.9). Naturally, condition (4.9) has an effect on the limit energy, since it forces a transition from 1 to 0 close to $\partial_N\Omega$, whose cost is approximately $\frac{1}{2}\mathcal{H}^{n-1}(\partial_N\Omega)$. This term is a constant, hence it does not affect the minimization problem, and explains its appearance in the limit energy (1.11).

5 Existence for the approximated functional

In this section we prove that the functional (4.1) has a minimizer in $\mathcal{A}$, so the approximated problem is well posed.

Theorem 5.1. *Let $\lambda_1, \lambda_2, \varepsilon, \eta, b > 0$, $p \geq n-1$ and $q > n$. Let I be as in* (4.1)*. Then there exists a minimizer of I in $\mathcal{A}$.*

Proof. We show first that the set $\mathcal{A}$ is not empty and that I is not identically infinity in $\mathcal{A}$. As $\partial_D\Omega$ and $\partial_N\Omega$ are disjoint compact sets, there exists a Lipschitz function $v_0 : \bar{\Omega} \to [0,1]$ such that $v_0 = 1$ on $\partial_D\Omega$ and $v_0 = 0$ on $\partial_N\Omega$.

Let $\mathbf{u}_0$ be as in Section 4. By the regularity of the Lebesgue measure, there exists a compact $E \subset \mathbf{u}_0(\Omega)$ such that

$$\mathcal{L}^n(\mathbf{u}_0(\Omega) \setminus E) \leq \frac{b}{L^n}, \tag{5.1}$$

where L is the Lipschitz constant of $\mathbf{u}_0^{-1}$ in $\mathbf{u}_0(\Omega)$. As $\mathbf{u}_0(\Omega)$ is open, there exists a Lipschitz function $w_1 : Q \to [0,1]$ such that $w_1 = 1$ in a neighbourhood of E, and $w_1 = 0$ in $Q \setminus \mathbf{u}_0(\Omega)$. Define $w_0 : Q \to [0,1]$ as

$$w_0 := \begin{cases} v_0 \circ \mathbf{u}_0^{-1} & \text{in } E, \\ \min\{w_1, v_0 \circ \mathbf{u}_0^{-1}\} & \text{in } \mathbf{u}_0(\Omega) \setminus E, \\ 0 & \text{in } Q \setminus \mathbf{u}_0(\Omega). \end{cases}$$

It is easy to check that w_0 is Lipschitz, and that $v_0 \geq w_0 \circ \mathbf{u}_0$ a.e. in Ω. Moreover, thanks to (5.1) we find that

$$\int_\Omega [v_0 - w_0 \circ \mathbf{u}_0] \, \mathrm{d}\mathbf{x} = \int_{\Omega \setminus \mathbf{u}_0^{-1}(E)} [v_0 - w_0 \circ \mathbf{u}_0] \, \mathrm{d}\mathbf{x} \leq \mathcal{L}^n \left(\Omega \setminus \mathbf{u}_0^{-1}(E)\right) \leq b.$$

Thus, conditions (4.8)–(4.12) hold for the triple $(\mathbf{u}, v, w) = (\mathbf{u}_0, v_0, w_0)$. Consequently, $(\mathbf{u}_0, v_0, w_0) \in \mathcal{A}$. In addition,

$$\int_\Omega \left[|Dv_0|^q + (1-v_0)^{q'}\right] \mathrm{d}\mathbf{x} < \infty \quad \text{and} \quad \int_Q \left[|Dw_0|^q + w_0^{q'}(1-w_0)^{q'}\right] \mathrm{d}\mathbf{y} < \infty. \tag{5.2}$$

Using (4.2) and (5.2), we find that $I(\mathbf{u}_0, v_0, w_0) < \infty$. Furthermore, assumption (W2) shows that $I \geq 0$. Therefore, there exists a minimizing sequence $\{(\mathbf{u}_j, v_j, w_j)\}_{j\in\mathbb{N}}$ of I in $\mathcal{A}$. Again assumption (W2) implies the bound

$$\sup_{j\in\mathbb{N}} \left[\|D\mathbf{u}_j\|_{L^p(\Omega,\mathbb{R}^{n\times n})} + \|h_1(|\operatorname{cof} D\mathbf{u}_j|)\|_{L^1(\Omega)} + \|h_2(\det D\mathbf{u}_j)\|_{L^1(\Omega)}\right] < \infty.$$

Moreover, calling $\bar{\mathbf{u}}_j$ the extension of $\mathbf{u}_j$ as in (4.4), and using De la Vallée–Poussin criterion, we find that the sequence $\{D\bar{\mathbf{u}}_j\}_{j\in\mathbb{N}}$ is bounded in $L^p(\Omega_1, \mathbb{R}^{n\times n})$, while the sequences $\{\operatorname{cof} D\bar{\mathbf{u}}_j\}_{j\in\mathbb{N}}$ and $\{\det D\bar{\mathbf{u}}_j\}_{j\in\mathbb{N}}$ are equiintegrable. As, in addition, $\det D\bar{\mathbf{u}}_j > 0$ a.e., $\bar{\mathbf{u}}_j$ is one-to-one a.e. and $\mathcal{E}(\bar{\mathbf{u}}_j) = 0$ for all $j \in \mathbb{N}$, the same proof of [37, Th. 4] shows that there exists $\bar{\mathbf{u}} \in W^{1,p}(\Omega_1, K)$ such that $\bar{\mathbf{u}}$ is one-to-one a.e., $\det D\bar{\mathbf{u}} > 0$ a.e., $\mathcal{E}(\bar{\mathbf{u}}) = 0$ and that, for a subsequence,

$$\bar{\mathbf{u}}_j \to \bar{\mathbf{u}} \text{ a.e. in } \Omega_1, \qquad \bar{\mathbf{u}}_j \rightharpoonup \bar{\mathbf{u}} \text{ in } W^{1,p}(\Omega_1, \mathbb{R}^n), \qquad \det D\bar{\mathbf{u}}_j \rightharpoonup \det D\bar{\mathbf{u}} \text{ in } L^1(\Omega_1) \tag{5.3}$$

as $j \to \infty$. Moreover, a standard result on the continuity of minors (see, e.g., [27, Th. 8.20], which is fact is a particular case of Lemma 2.12) shows that $\boldsymbol{\mu}_0(D\mathbf{u}_j) \rightharpoonup \boldsymbol{\mu}_0(D\mathbf{u})$ in $L^1(\Omega, \mathbb{R}^{\tau-1})$ as $j \to \infty$, where we are using the notation for minors explained in Section 2.7. With (5.3) we obtain

$$\boldsymbol{\mu}(D\mathbf{u}_j) \rightharpoonup \boldsymbol{\mu}(D\mathbf{u}) \text{ in } L^1(\Omega, \mathbb{R}^\tau) \text{ as } j \to \infty. \tag{5.4}$$

In addition, $\bar{\mathbf{u}} = \mathbf{u}_0$ in $\Omega_1 \setminus \Omega$, so, calling $\mathbf{u} := \bar{\mathbf{u}}|_\Omega$ we have that condition (4.3) is satisfied and, hence, $\mathbf{u} \in \mathcal{A}^E$.

Using that $q > n$, the Sobolev embedding theorem, the estimate

$$\sup_{j\in\mathbb{N}} \left[\|Dv_j\|_{L^q(\Omega,\mathbb{R}^n)} + \|Dw_j\|_{L^q(Q,\mathbb{R}^n)}\right] < \infty,$$

and the inclusions $v_j(\Omega), w_j(Q) \subset [0,1]$ for all $j \in \mathbb{N}$, we find that there exist $v \in W^{1,q}(\Omega, [0,1])$ and $w \in W^{1,q}(Q, [0,1])$ such that, for a subsequence,

$$v_j \to v \text{ in } C^{0,\alpha}(\bar{\Omega}), \qquad v_j \rightharpoonup v \text{ in } W^{1,q}(\Omega), \qquad w_j \to w \text{ in } C^{0,\alpha}(\bar{Q}), \qquad w_j \rightharpoonup w \text{ in } W^{1,q}(Q), \tag{5.5}$$

for some $\alpha > 0$. Now, for all $j \in \mathbb{N}$ and a.e. $\mathbf{x} \in \Omega$,

$$\begin{aligned} |w_j(\mathbf{u}_j(\mathbf{x})) - w(\mathbf{u}(\mathbf{x}))| &\le |w_j(\mathbf{u}_j(\mathbf{x})) - w_j(\mathbf{u}(\mathbf{x}))| + |w_j(\mathbf{u}(\mathbf{x})) - w(\mathbf{u}(\mathbf{x}))| \\ &\le \|w_j\|_{C^{0,\alpha}(\bar{Q})} |\mathbf{u}_j(\mathbf{x}) - \mathbf{u}(\mathbf{x})|^\alpha + \|w_j - w\|_{L^\infty(Q)}, \end{aligned}$$

so, thanks to the convergences (5.3) and (5.5), we infer that

$$w_j \circ \mathbf{u}_j \to w \circ \mathbf{u} \text{ a.e. as } j \to \infty. \tag{5.6}$$

Thanks to (5.5), (5.6) and dominated convergence, we have that inequalities (4.11)–(4.12) are satisfied, as well as the boundary conditions (4.8)–(4.9). We show next that condition (4.10) is also satisfied. For this, we first prove that

$$\chi_{\operatorname{im_G}(\mathbf{u}_j,\Omega)} \to \chi_{\operatorname{im_G}(\mathbf{u},\Omega)} \text{ as } j \to \infty \tag{5.7}$$

in $L^1(\mathbb{R}^n)$. Thanks to [37, Th. 2], there exists an increasing sequence $\{V_k\}_{k\in\mathbb{N}}$ of open sets such that $\Omega = \bigcup_{k\in\mathbb{N}} V_k$ and, for each $k \in \mathbb{N}$,

$$\chi_{\operatorname{im_G}(\mathbf{u}_j,V_k)} \to \chi_{\operatorname{im_G}(\mathbf{u},V_k)} \qquad \text{as } j \to \infty \tag{5.8}$$

in $L^1_{\text{loc}}(\mathbb{R}^n)$, up to a subsequence. In fact, as $\chi_{\operatorname{im_G}(\mathbf{u}_j,\Omega)} \le \chi_K$ a.e. for all $j \in \mathbb{N}$, we have that the convergence (5.8) is in $L^1(\mathbb{R}^n)$. For all $j, k \in \mathbb{N}$ we have that

$$\begin{aligned} \left\|\chi_{\operatorname{im_G}(\mathbf{u}_j,\Omega)} - \chi_{\operatorname{im_G}(\mathbf{u},\Omega)}\right\|_{L^1(\mathbb{R}^n)} &\le \left\|\chi_{\operatorname{im_G}(\mathbf{u}_j,\Omega)} - \chi_{\operatorname{im_G}(\mathbf{u}_j,V_k)}\right\|_{L^1(\mathbb{R}^n)} \\ &+ \left\|\chi_{\operatorname{im_G}(\mathbf{u}_j,V_k)} - \chi_{\operatorname{im_G}(\mathbf{u},V_k)}\right\|_{L^1(\mathbb{R}^n)} + \left\|\chi_{\operatorname{im_G}(\mathbf{u},V_k)} - \chi_{\operatorname{im_G}(\mathbf{u},\Omega)}\right\|_{L^1(\mathbb{R}^n)}. \end{aligned} \tag{5.9}$$

Thanks to Proposition 2.5,

$$\left\|\chi_{\operatorname{im_G}(\mathbf{u}_j,\Omega)} - \chi_{\operatorname{im_G}(\mathbf{u}_j,V_k)}\right\|_{L^1(\mathbb{R}^n)} = \left\|\chi_{\operatorname{im_G}(\mathbf{u}_j,\Omega\setminus V_k)}\right\|_{L^1(\mathbb{R}^n)} = \int_{\Omega\setminus V_k} \det D\mathbf{u}_j(\mathbf{x})\,\mathrm{d}\mathbf{x} \tag{5.10}$$

and

$$\left\|\chi_{\mathrm{im_G}(\mathbf{u},V_k)} - \chi_{\mathrm{im_G}(\mathbf{u},\Omega)}\right\|_{L^1(\mathbb{R}^n)} = \int_{\Omega\setminus V_k} \det D\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{5.11}$$

Let $\bar{\varepsilon} > 0$. By the equiintegrability of the sequence $\{\det D\mathbf{u}_j\}_{j\in\mathbb{N}}$ given by (5.3), there exists $k \in \mathbb{N}$ such that for all $j \in \mathbb{N}$,

$$\int_{\Omega\setminus V_k} \det D\mathbf{u}_j(\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{\Omega\setminus V_k} \det D\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x} \leq \bar{\varepsilon}. \tag{5.12}$$

Using the $L^1(\mathbb{R}^n)$ convergence of (5.8), for such $k \in \mathbb{N}$ there exists $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$,

$$\left\|\chi_{\mathrm{im_G}(\mathbf{u}_j,V_k)} - \chi_{\mathrm{im_G}(\mathbf{u},V_k)}\right\|_{L^1(\mathbb{R}^n)} \leq \bar{\varepsilon}. \tag{5.13}$$

Thus, the $L^1(\mathbb{R}^n)$ convergence (5.7) follows from (5.9)–(5.13). For a subsequence, it also holds a.e. To conclude the argument, we let $\mathbf{y} \in Q \setminus \mathrm{im_G}(\mathbf{u},\Omega)$. By the a.e. convergence of (5.7), there exists $j_0 \in \mathbb{N}$ such that $\mathbf{y} \notin \mathrm{im_G}(\mathbf{u}_j,\Omega)$ for all $j \geq j_0$, and, by (4.10), $w_j(\mathbf{y}) = 0$. Passing to the limit using (5.5) shows that $w(\mathbf{y}) = 0$. Therefore, condition (4.10) holds and we conclude that $(\mathbf{u}, v, w) \in \mathcal{A}$.

On the other hand, convergences (5.5) show that

$$\int_\Omega (1-v)^{q'}\mathrm{d}\mathbf{x} = \lim_{j\to\infty} \int_\Omega (1-v_j)^{q'}\mathrm{d}\mathbf{x}, \qquad \int_\Omega |Dv|^q\,\mathrm{d}\mathbf{x} \leq \liminf_{j\to\infty} \int_\Omega |Dv_j|^q\,\mathrm{d}\mathbf{x} \tag{5.14}$$

and

$$\int_Q w^{q'}(1-w)^{q'}\mathrm{d}\mathbf{y} = \lim_{j\to\infty} \int_Q w_j^{q'}(1-w_j)^{q'}\mathrm{d}\mathbf{y}, \qquad \int_Q |Dw|^q\,\mathrm{d}\mathbf{y} \leq \liminf_{j\to\infty} \int_Q |Dw_j|^q\,\mathrm{d}\mathbf{y}. \tag{5.15}$$

In addition, we can apply the lower semicontinuity result of [12, Th. 5.4], according to which, thanks to the polyconvexity of W given by (W1) and to convergences (5.3), (5.4) and (5.5), we have that

$$\int_\Omega (v(\mathbf{x})^2+\eta)\,W(\mathbf{x},\mathbf{u}(\mathbf{x}),D\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x} \leq \liminf_{j\to\infty} \int_\Omega (v_j(\mathbf{x})^2+\eta)\,W(\mathbf{x},\mathbf{u}_j(\mathbf{x}),D\mathbf{u}_j(\mathbf{x}))\,\mathrm{d}\mathbf{x}. \tag{5.16}$$

Inequalities (5.14), (5.15) and (5.16) show that $(\mathbf{u}, v, w)$ is a minimizer of I in $\mathcal{A}$. □

6 Compactness and lower bound

For the rest of the paper, we fix a sequence $\{\varepsilon\}_\varepsilon$ of positive numbers going to zero. As in Section 4, we fix parameters $\lambda_1, \lambda_2 > 0$, exponents $p \geq n-1$ and $q > n$ and sequences $\{\eta_\varepsilon\}_\varepsilon$ and $\{b_\varepsilon\}_\varepsilon$ of positive numbers such that

$$\sup_\varepsilon \eta_\varepsilon < \infty \tag{6.1}$$

and

$$b_\varepsilon \to 0. \tag{6.2}$$

The functional I of (4.1) corresponding to the parameters $\lambda_1, \lambda_2, \varepsilon, \eta_\varepsilon, p, q$ will be called I_ε, and the admissible set $\mathcal{A}$ of Section 4 corresponding to $b = b_\varepsilon$ in the restriction (4.12) will be called $\mathcal{A}_\varepsilon$.

Given ε, measurable sets $A \subset \Omega$ and $B \subset Q$, and $(\mathbf{u}, v, w) \in \mathcal{A}_\varepsilon$, define

$$\begin{gathered} I_\varepsilon^E(\mathbf{u},v;A) := \int_A (v(\mathbf{x})^2+\eta_\varepsilon)\,W(\mathbf{x},\mathbf{u}(\mathbf{x}),D\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}, \qquad I_\varepsilon^V(v;A) := \int_A \left[\varepsilon^{q-1}\frac{|Dv(\mathbf{x})|^q}{q} + \frac{(1-v(\mathbf{x}))^{q'}}{q'\varepsilon}\right]\mathrm{d}\mathbf{x} \\ I_\varepsilon^W(w;B) := \int_B \left[\varepsilon^{q-1}\frac{|Dw(\mathbf{y})|^q}{q} + \frac{w(\mathbf{y})^{q'}(1-w(\mathbf{y}))^{q'}}{q'\varepsilon}\right]\mathrm{d}\mathbf{y}. \end{gathered} \tag{6.3}$$

Define also

$$I_\varepsilon^E(\mathbf{u}, v) := I_\varepsilon^E(\mathbf{u}, v; \Omega), \qquad I_\varepsilon^V(v) := I_\varepsilon^V(v; \Omega) \quad \text{and} \quad I_\varepsilon^W(w) := I_\varepsilon^W(w; Q),$$

so that

$$I_\varepsilon(\mathbf{u}, v, w) = I_\varepsilon^E(\mathbf{u}, v) + \lambda_1 \, I_\varepsilon^V(v) + 6\lambda_2 \, I_\varepsilon^W(w).$$

This section is devoted to the proof of the following theorem.

Theorem 6.1. *For each ε, let $(\mathbf{u}_\varepsilon, v_\varepsilon, w_\varepsilon) \in \mathcal{A}_\varepsilon$ satisfy*

$$\sup_\varepsilon I_\varepsilon(\mathbf{u}_\varepsilon, v_\varepsilon, w_\varepsilon) < \infty. \tag{6.4}$$

Then there exists $\mathbf{u} \in SBV(\Omega, K)$ such that $\mathbf{u}$ is one-to-one a.e., $\det D\mathbf{u} > 0$ a.e. and, for a subsequence,

$$\mathbf{u}_\varepsilon \to \mathbf{u} \ \text{a.e.}, \qquad v_\varepsilon \to 1 \ \text{a.e.} \qquad \text{and} \qquad w_\varepsilon \to \chi_{\operatorname{im_G}(\mathbf{u}, \Omega)} \ \text{a.e.} \tag{6.5}$$

Moreover, for any such $\mathbf{u}$, we have that

$$\int_\Omega W(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla \mathbf{u}(\mathbf{x})) \, \mathrm{d}\mathbf{x} + \lambda_1 \left[\mathcal{H}^{n-1}(J_\mathbf{u}) + \mathcal{H}^{n-1} \left(\{ \mathbf{x} \in \partial_D \Omega : \mathbf{u}(\mathbf{x}) \neq \mathbf{u}_0(\mathbf{x}) \} \right) + \frac{1}{2} \mathcal{H}^{n-1}(\partial_N \Omega) \right]$$
$$+ \lambda_2 \left[\operatorname{Per} \operatorname{im_G}(\mathbf{u}, \Omega) + 2 \, \mathcal{H}^{n-1}(J_{\mathbf{u}^{-1}}) \right] \leq \liminf_{\varepsilon \to 0} I_\varepsilon(\mathbf{u}_\varepsilon, v_\varepsilon, w_\varepsilon).$$

In the inequality above, the value of $\mathbf{u}$ on $\partial\Omega$ is understood in the sense of traces (see, e.g., [7, Th. 3.87]). Theorem 6.1 constitutes the usual *compactness* and *lower bound* parts of a Γ-convergence result. Its proof spans the next subsections, and will be divided into partial results.

6.1 A first compactness result

For the sake of brevity, for each ε we define $W_\varepsilon : \Omega \to [0, \infty]$ through

$$W_\varepsilon(\mathbf{x}) := W(\mathbf{x}, \mathbf{u}_\varepsilon(\mathbf{x}), D\mathbf{u}_\varepsilon(\mathbf{x})). \tag{6.6}$$

The following is a preliminary compactness result for the sequence $\{(\mathbf{u}_\varepsilon, v_\varepsilon)\}_\varepsilon$.

Proposition 6.2. *For each ε, let $(\mathbf{u}_\varepsilon, v_\varepsilon) \in \mathcal{A}^E \times W^{1,q}(\Omega, [0,1])$ satisfy*

$$\sup_\varepsilon \left[I_\varepsilon^E(\mathbf{u}_\varepsilon, v_\varepsilon) + I_\varepsilon^V(v_\varepsilon) \right] < \infty. \tag{6.7}$$

Then, for a subsequence,

$$v_\varepsilon \to 1 \ \text{ in } L^1(\Omega), \ \text{a.e. and in measure,} \tag{6.8}$$

and there exists $\mathbf{u} \in BV(\Omega, K)$ such that

$$\mathbf{u}_\varepsilon \to \mathbf{u} \ \text{ a.e. and in } L^1(\Omega, \mathbb{R}^n). \tag{6.9}$$

Proof. For each ε, we use the equality

$$D\left(\left(3v_\varepsilon^2 - 2v_\varepsilon^3 \right) \mathbf{u}_\varepsilon \right) = 6 v_\varepsilon (1 - v_\varepsilon) \mathbf{u}_\varepsilon \otimes D v_\varepsilon + v_\varepsilon^2 \left(3 - 2 v_\varepsilon \right) D\mathbf{u}_\varepsilon,$$

the bound $0 \leq v_\varepsilon \leq 1$ and the L^∞ a priori bound for $\mathbf{u}_\varepsilon$ given by K to find that

$$\left| D\left(\left(3v_\varepsilon^2 - 2v_\varepsilon^3 \right) \mathbf{u}_\varepsilon \right) \right| \lessapprox (1 - v_\varepsilon) \left| \mathbf{u}_\varepsilon \otimes D v_\varepsilon \right| + v_\varepsilon^2 \left| D\mathbf{u}_\varepsilon \right| \lessapprox (1 - v_\varepsilon) \left| D v_\varepsilon \right| + v_\varepsilon^{\frac{2}{p}} \left| D\mathbf{u}_\varepsilon \right|,$$

so by Hölder's inequality, Young's inequality and assumption (W2) we obtain that

$$\int_\Omega \left|D\left(\left(3v_\varepsilon^2 - 2v_\varepsilon^3\right)\mathbf{u}_\varepsilon\right)\right| \mathrm{d}\mathbf{x} \lessapprox \int_\Omega (1-v_\varepsilon)\,|Dv_\varepsilon|\,\mathrm{d}\mathbf{x} + \left(\int_\Omega v_\varepsilon^2\,|D\mathbf{u}_\varepsilon|^p\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{p}}$$
$$\lessapprox I_\varepsilon^V(v_\varepsilon) + \left(\int_\Omega v_\varepsilon^2\,W_\varepsilon\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{p}} \le I_\varepsilon^V(v_\varepsilon) + I_\varepsilon^E(\mathbf{u}_\varepsilon, v_\varepsilon)^{\frac{1}{p}} \lessapprox 1.$$

Therefore, there exists $\mathbf{u} \in BV(\Omega, K)$ such that, for a subsequence, $(3v_\varepsilon^2 - 2v_\varepsilon^3)\mathbf{u}_\varepsilon \to \mathbf{u}$ a.e.

On the other hand,

$$\int_\Omega (1-v_\varepsilon)^{q'}\,\mathrm{d}\mathbf{x} \le q'\varepsilon\, I_\varepsilon^V(v_\varepsilon) \lessapprox \varepsilon,$$

so, taking a subsequence, the convergences (6.8) hold and, hence,

$$\mathbf{u}_\varepsilon = \frac{\left(3v_\varepsilon^2 - 2v_\varepsilon^3\right)\mathbf{u}_\varepsilon}{(3v_\varepsilon^2 - 2v_\varepsilon^3)} \to \mathbf{u} \quad \text{a.e.}$$

By dominated convergence, $\mathbf{u}_\varepsilon \to \mathbf{u}$ in $L^1(\Omega, \mathbb{R}^n)$ as well. □

6.2 Fracture energy term

In this section we study the term I_ε^V. Its analysis is essentially due to Ambrosio & Tortorelli [8, 9], who proved it in the scalar case when W is the Dirichlet energy. In this section, we take many ideas from the exposition of [19, Sect. 10.2] and [20, Sect. 5.2], who extended the result to the vectorial case for a quasiconvex W. Some adaptations are to be made, though, because of the boundary conditions (4.3), (4.8) and (4.9), so that inequality (6.11) of Proposition 6.5 below is stronger than the usual lower bound inequality for I_ε^V. In addition, our W is polyconvex, is allowed to have a slow growth at infinity and blows up when the determinant of the deformation gradient goes to zero, all of which add further difficulties in the analysis.

We first present a version of the intermediate value theorem for measurable functions, which will be used several times in the sequel. Although the result is well known for experts, we have not found a precise reference.

Lemma 6.3. *Let $I \subset \mathbb{R}$ be a measurable set with $\mathcal{L}^1(I) > 0$. Let $f, g : I \to [0, \infty]$ be two measurable functions such that $f \in L^1(I)$. Then the set of $s_0 \in I$ such that*

$$\int_I f(s)\,g(s)\,\mathrm{d}s \ge \int_I f(s)\,\mathrm{d}s\,g(s_0)$$

has positive measure.

Proof. Let J be the set of $s \in I$ such that $f(s) > 0$. The result is immediate if $\mathcal{L}^1(J) = 0$, so assume that $\mathcal{L}^1(J) > 0$. The result is also trivial if g is constant a.e. in J, so assume that this is not the case. Then

$$\frac{\int_J f(s)\,g(s)\,\mathrm{d}s}{\int_J f(s)\,\mathrm{d}s} > \operatorname*{ess\,inf}_J g.$$

By definition of essential infimum, we have that

$$\mathcal{L}^1\left(\{s_0 \in J : g(s_0) \le \frac{\int_J f(s)\,g(s)\,\mathrm{d}s}{\int_J f(s)\,\mathrm{d}s}\}\right) > 0. \tag{6.10}$$

Assume the conclusion of the lemma to be false. Then, together with (6.10) we would infer that there exists $s_0 \in J$ such that

$$\int_J f(s)\,g(s)\,\mathrm{d}s < \int_J f(s)\,\mathrm{d}s\,g(s_0) \quad \text{and} \quad g(s_0) \le \frac{\int_J f(s)\,g(s)\,\mathrm{d}s}{\int_J f(s)\,\mathrm{d}s},$$

which is a contradiction. □

The following lemma is a restatement of the well-known fact that Lipschitz domains satisfy both the interior and exterior cone conditions (see, e.g., [2, Prop. 3.7]).

Lemma 6.4. *Let Ω be a Lipschitz domain. Then there exist $\delta > 0$ and $\gamma_0 \in (0,1)$ such that for $\mathcal{H}^{n-1}$-a.e. $\mathbf{x} \in \partial\Omega$ and every $\boldsymbol{\xi} \in \mathbb{S}^{n-1}$ such that $\boldsymbol{\xi} \cdot \boldsymbol{\nu}_\Omega(\mathbf{x}) > \gamma_0$,*

$$\{t \in (-\delta, \delta) : \mathbf{x} + t\boldsymbol{\xi} \in \Omega\} = (-\delta, 0).$$

The compactness result of Proposition 6.2 is complemented by the following one, in which we also prove the lower bound inequality for the term I_ε^V.

Proposition 6.5. *For each ε, let $(\mathbf{u}_\varepsilon, v_\varepsilon) \in \mathcal{A}^E \times W^{1,q}(\Omega, [0,1])$ satisfy (6.7). Let $\mathbf{u} \in BV(\Omega, K)$ satisfy (6.9). Then $\mathbf{u} \in SBV(\Omega, K)$ and*

$$\mathcal{H}^{n-1}(J_\mathbf{u}) + \mathcal{H}^{n-1}\left(\{\mathbf{x} \in \partial_D\Omega : \mathbf{u}(\mathbf{x}) \neq \mathbf{u}_0(\mathbf{x})\}\right) + \frac{1}{2}\mathcal{H}^{n-1}(\partial_N\Omega) \leq \liminf_{\varepsilon\to 0} I_\varepsilon^V(v_\varepsilon). \tag{6.11}$$

Proof. Fix $0 < \delta < \frac{1}{2}$. We perform a slicing argument, for which we will use the notation of Definition 2.13. By Fatou's lemma, Proposition 2.14 and (W2), we have that for every $\boldsymbol{\xi} \in \mathbb{S}^{n-1}$,

$$\begin{aligned}\int_{\Omega^{\boldsymbol{\xi}}} \liminf_{\varepsilon\to 0} \int_{\Omega^{\boldsymbol{\xi},\mathbf{x}'}} (v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'})^2\, |D\mathbf{u}_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}|^p \,\mathrm{d}t\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}') &\leq \liminf_{\varepsilon\to 0} \int_{\Omega^{\boldsymbol{\xi}}} \int_{\Omega^{\boldsymbol{\xi},\mathbf{x}'}} (v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'})^2\, |D\mathbf{u}_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}|^p \,\mathrm{d}t\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}') \\ &\leq \liminf_{\varepsilon\to 0} \int_\Omega v_\varepsilon^2\, |D\mathbf{u}_\varepsilon|^p \,\mathrm{d}\mathbf{x} \lessapprox \liminf_{\varepsilon\to 0} I_\varepsilon^E(\mathbf{u}_\varepsilon, v_\varepsilon)\end{aligned} \tag{6.12}$$

and

$$\begin{aligned}&\int_{\Omega^{\boldsymbol{\xi}}} \liminf_{\varepsilon\to 0} \int_{\Omega^{\boldsymbol{\xi},\mathbf{x}'}} \left[\varepsilon^{q-1} \frac{|Dv_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}|^q}{q} + \frac{(1 - v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'})^{q'}}{q'\varepsilon}\right] \mathrm{d}t\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}') \\ &\qquad \leq \liminf_{\varepsilon\to 0} \int_{\Omega^{\boldsymbol{\xi}}} \int_{\Omega^{\boldsymbol{\xi},\mathbf{x}'}} \left[\varepsilon^{q-1} \frac{|Dv_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}|^q}{q} + \frac{(1 - v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'})^{q'}}{q'\varepsilon}\right] \mathrm{d}t\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}') \leq \liminf_{\varepsilon\to 0} I_\varepsilon^V(v_\varepsilon).\end{aligned} \tag{6.13}$$

Inequalities (6.12)–(6.13) and the energy bound (6.7) imply that for $\mathcal{H}^{n-1}$-a.e. $\mathbf{x}' \in \Omega^{\boldsymbol{\xi}}$,

$$\liminf_{\varepsilon\to 0} \int_{\Omega^{\boldsymbol{\xi},\mathbf{x}'}} (v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'})^2\, |D\mathbf{u}_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}|^p \,\mathrm{d}t < \infty \quad \text{and} \quad \liminf_{\varepsilon\to 0} \int_{\Omega^{\boldsymbol{\xi},\mathbf{x}'}} \left[\varepsilon^{q-1} \frac{|Dv_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}|^q}{q} + \frac{(1 - v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'})^{q'}}{q'\varepsilon}\right] \mathrm{d}t < \infty. \tag{6.14}$$

By (6.8)–(6.9), using slicing theory and passing to a subsequence (which may depend on $\mathbf{x}'$), we also have that, for $\mathcal{H}^{n-1}$-a.e. $\mathbf{x}' \in \Omega^{\boldsymbol{\xi}}$,

$$\mathcal{L}^1\left(\{t \in \Omega^{\boldsymbol{\xi},\mathbf{x}'} : v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}(t) < 1 - \delta\}\right) \to 0 \qquad \text{and} \qquad \mathbf{u}_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'} \to \mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'} \text{ in } L^1(\Omega^{\boldsymbol{\xi},\mathbf{x}'}, \mathbb{R}^n). \tag{6.15}$$

Fix any $\mathbf{x}' \in \Omega^{\boldsymbol{\xi}}$ for which equations (6.14)–(6.15) hold, and let U be a non-empty open subset of Ω. Then $U^{\boldsymbol{\xi},\mathbf{x}'}$ is also open, hence it is the union of a disjoint countable family $\{I_k\}_{k\in\mathbb{N}}$ of open intervals. Note that each I_k depends also on U, $\mathbf{x}'$ and $\boldsymbol{\xi}$, but this dependence will not be emphasized in the notation. Also for simplicity, we use the notation $\{I_k\}_{k\in\mathbb{N}}$, even though the family of intervals may be finite.

By Young's inequality, the coarea formula (2.3) and Lemma 6.3, for each $k \in \mathbb{N}$ and each ε there exists $s_{\varepsilon,k} \in (\delta, 1 - \delta)$ such that, when we define

$$a_\delta := \int_\delta^{1-\delta} (1 - s)\,\mathrm{d}s, \qquad E_{\varepsilon,k} := \{t \in I_k : v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}(t) < s_{\varepsilon,k}\}, \tag{6.16}$$

we have

$$\begin{aligned}\int_{I_k}\left[\varepsilon^{q-1}\frac{|Dv_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}|^q}{q}+\frac{(1-v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'})^{q'}}{q'\varepsilon}\right]\mathrm{d}t &\geq \int_{I_k}(1-v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'})\,|Dv_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}|\,\mathrm{d}t\\ &\geq \int_\delta^{1-\delta}(1-s)\,\mathcal{H}^0\big(\partial^*\{t\in I_k: v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}(t)<s\}\cap I_k\big)\,\mathrm{d}s \geq a_\delta\,\mathcal{H}^0(\partial^* E_{\varepsilon,k}\cap I_k).\end{aligned}\tag{6.17}$$

The function $v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}$ is absolutely continuous, hence differentiable a.e. In addition, by a version of Sard's theorem for Sobolev maps (see, e.g., [30, Sect. 5]), we have that

$$\mathcal{L}^1\left(v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}\left(\{t\in\Omega^{\boldsymbol{\xi},\mathbf{x}'}: v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'} \text{ is differentiable at } t \text{ and } (v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'})'(t)=0\}\right)\right)=0.$$

On the other hand, it is easy to see that for any $s_0\in\mathbb{R}$ with the property that

$$\text{all } t_0\in(v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'})^{-1}(s_0) \text{ is such that } v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'} \text{ is differentiable at } t_0 \text{ and } (v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'})'(t_0)\neq 0,$$

one has

$$\partial^*\{t\in\Omega^{\boldsymbol{\xi},\mathbf{x}'}: v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}(t)<s_0\}=\partial\{t\in\Omega^{\boldsymbol{\xi},\mathbf{x}'}: v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}(t)<s_0\}.$$

Moreover, since $v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}$ is continuous, $E_{\varepsilon,k}$ is an open set. These facts together with Lemma 6.3 allow us to assume that the number $s_{\varepsilon,k}$ in (6.16) was chosen so that not only (6.17) holds, but also $\partial^*E_{\varepsilon,k}=\partial E_{\varepsilon,k}$. Thus,

$$\begin{aligned}&\frac{1}{\delta^2}\liminf_{\varepsilon\to0}\int_{U^{\boldsymbol{\xi},\mathbf{x}'}}(v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'})^2\,|D\mathbf{u}_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}|^p\,\mathrm{d}t\geq\sum_{k\in\mathbb{N}}\liminf_{\varepsilon\to0}\int_{I_k\setminus E_{\varepsilon,k}}|D\mathbf{u}_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}|^p\,\mathrm{d}t,\\ &\liminf_{\varepsilon\to0}\int_{U^{\boldsymbol{\xi},\mathbf{x}'}}\left[\varepsilon^{q-1}\frac{|Dv_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}|^q}{q}+\frac{(1-v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'})^{q'}}{q'\varepsilon}\right]\mathrm{d}t\geq a_\delta\liminf_{\varepsilon\to0}\mathcal{H}^0(\partial E_{\varepsilon,k}\cap I_k).\end{aligned}\tag{6.18}$$

Fix $k\in\mathbb{N}$. From (6.14) and (6.18), we infer that $\liminf_{\varepsilon\to0}\mathcal{H}^0(\partial E_{\varepsilon,k}\cap I_k)<\infty$, and, hence, for a subsequence, $E_{\varepsilon,k}$ has a uniformly bounded number of connected components. Let F_k be the Hausdorff limit of a subsequence of $\{\overline{E_{\varepsilon,k}}\}_\varepsilon$, i.e., F_k is characterized by the facts that it is compact, contained in $\overline{I_k}$ and for each $\eta>0$ there exists ε_η such that if $\varepsilon<\varepsilon_\eta$ then

$$E_{\varepsilon,k}\subset\bar{B}(F_k,\eta)\quad\text{and}\quad F_k\subset\bar{B}\left(\overline{E_{\varepsilon,k}},\eta\right).\tag{6.19}$$

Moreover, F_k can be found by taking the limit of the sequences of endpoints of the connected components of $E_{\varepsilon,k}$. Call

$$G_{k,0}:=\{t\in F_k\cap\partial I_k:\lim_{\varepsilon\to0}v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}(t)=0\},\qquad G_{k,1}:=\{t\in F_k\cap\partial I_k:\lim_{\varepsilon\to0}v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}(t)=1\},$$

where the value of $v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}$ in ∂I_k is understood in the sense of traces, and it always exists because $v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}$ is uniformly continuous. By (6.15) and (6.16) we have that $\mathcal{L}^1(E_{\varepsilon,k})\to0$, hence F_k necessarily consists of a finite number of points. Using this and that each $E_{\varepsilon,k}$ is a union of a uniformly bounded number of open intervals, the following argument allows us to conclude that

$$\mathcal{H}^0(F_k\cap I_k)+\mathcal{H}^0(G_{k,1})+\frac12\,\mathcal{H}^0(G_{k,0})\leq\liminf_{\varepsilon\to0}\frac12\,\mathcal{H}^0(\partial E_{\varepsilon,k}\cap I_k).\tag{6.20}$$

Indeed, we first observe that for each $t\in F_k$ there exist sequences $\{\underline{\tau}_\varepsilon\}_\varepsilon$ and $\{\overline{\tau}_\varepsilon\}_\varepsilon$ tending to t such that

$$\underline{\tau}_\varepsilon<\overline{\tau}_\varepsilon,\qquad \underline{\tau}_\varepsilon,\overline{\tau}_\varepsilon\in\partial E_{\varepsilon,k}\quad\text{and}\quad(\underline{\tau}_\varepsilon,\overline{\tau}_\varepsilon)\subset E_{\varepsilon,k}\qquad\text{for all }\varepsilon.$$

Consider the following two cases.

a) If $t \in I_k$, then $\underline{\tau}_\varepsilon, \overline{\tau}_\varepsilon \in I_k$ for every ε sufficiently small. Therefore, to t there correspond two points in $\partial E_{\varepsilon,k} \cap I_k$: $\underline{\tau}_\varepsilon$ and $\overline{\tau}_\varepsilon$.

b) If $t \in \partial I_k$, assume, for definiteness, that $t = \inf I_k$. Then $t \le \underline{\tau}_\varepsilon$ for all ε sufficiently small. If $\lim_{\varepsilon\to 0} v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}(t) = 1$, then, by (6.16) we have that $t \neq \underline{\tau}_\varepsilon$, and, hence $\underline{\tau}_\varepsilon, \overline{\tau}_\varepsilon \in I_k$. Therefore, to t there correspond two points in $\partial E_{\varepsilon,k} \cap I_k$: $\underline{\tau}_\varepsilon$ and $\overline{\tau}_\varepsilon$. If, instead, $\lim_{\varepsilon\to 0} v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}(t) = 0$ then still $\overline{\tau}_\varepsilon \in I_k$, but it may happen that $\underline{\tau}_\varepsilon = t$ for all ε sufficiently small, so we cannot guarantee that $\underline{\tau}_\varepsilon \in I_k$. Hence we only conclude that to t there corresponds at least one point in $\partial E_{\varepsilon_k} \cap I_k$: $\overline{\tau}_\varepsilon$.

This discussion completes the proof of (6.20).

Now, for each $\eta > 0$ there exists ε_η such that if $\varepsilon < \varepsilon_\eta$, the inclusions (6.19) hold. Thus, by (6.14) and (6.18),

$$\infty > \liminf_{\varepsilon\to 0} \int_{I_k \setminus E_{\varepsilon,k}} |D\mathbf{u}_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}|^p \,\mathrm{d}t \ge \liminf_{\varepsilon\to 0} \int_{I_k \setminus \bar{B}(F_k,\eta)} |D\mathbf{u}_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}|^p \,\mathrm{d}t. \tag{6.21}$$

From (6.15) and (6.21) we obtain that $\mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'} \in W^{1,p}(I_k \setminus \bar{B}(F_k,\eta), \mathbb{R}^n)$ and

$$\int_{I_k \setminus \bar{B}(F_k,\eta)} |D\mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'}|^p \,\mathrm{d}t \le \liminf_{\varepsilon\to 0} \int_{I_k \setminus E_{\varepsilon,k}} |D\mathbf{u}_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}|^p \,\mathrm{d}t. \tag{6.22}$$

Since the right-hand side of (6.22) is independent of η, we conclude that $\mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'} \in W^{1,p}(I_k \setminus F_k, \mathbb{R}^n)$ and

$$\int_{I_k} |\nabla \mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'}|^p \,\mathrm{d}t \le \liminf_{\varepsilon\to 0} \int_{I_k \setminus E_{\varepsilon,k}} |D\mathbf{u}_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}|^p \,\mathrm{d}t. \tag{6.23}$$

A standard result in the theory of SBV functions (see, e.g., [7, Prop. 4.4]) shows then that $\mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'} \in SBV(I_k, \mathbb{R}^n)$ and

$$J_{\mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'}} \cap I_k \subset F_k \cap I_k. \tag{6.24}$$

In particular, $\mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'} \in SBV_{\mathrm{loc}}(U^{\boldsymbol{\xi},\mathbf{x}'}, \mathbb{R}^n)$ and, by (6.24), (6.20) and (6.18),

$$\mathcal{H}^0(J_{\mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'}} \cap U^{\boldsymbol{\xi},\mathbf{x}'}) + \sum_{k\in\mathbb{N}} \left[\mathcal{H}^0(G_{k,1}) + \frac{1}{2}\,\mathcal{H}^0(G_{k,0})\right] \le \frac{1}{2a_\delta} \liminf_{\varepsilon\to 0} \int_{U^{\boldsymbol{\xi},\mathbf{x}'}} \left[\varepsilon^{q-1} \frac{|Dv_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}|^q}{q} + \frac{(1-v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'})^{q'}}{q'\varepsilon}\right] \mathrm{d}t. \tag{6.25}$$

The analysis above is true for any non-empty open $U \subset \Omega$. In the rest of the paragraph, we take U to be Ω. We have

$$V\left(\mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'}, \Omega^{\boldsymbol{\xi},\mathbf{x}'}\right) = \sum_{k\in\mathbb{N}} V\left(\mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'}, I_k\right) = \sum_{k\in\mathbb{N}} \left[\int_{I_k} \left|\nabla \mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'}\right| \mathrm{d}t + \sum_{t\in J_{\mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'}} \cap I_k} \left|\mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'}(t^+) - \mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'}(t^-)\right|\right]. \tag{6.26}$$

Both equalities of (6.26) are standard: see, e.g., [56, Rk. 5.1.2] for the first and [7, Cor. 3.33] for the second In (6.26), $\mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'}(t^+)$ denotes the limit at t of the precise representative of $\mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'}$ from the right, and $\mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'}(t^-)$ from the left. On the one hand, we have, due to (6.25) and (6.14),

$$\sum_{k\in\mathbb{N}} \sum_{t\in J_{\mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'}} \cap I_k} \left|\mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'}(t^+) - \mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'}(t^-)\right| \le 2 \sup_{\mathbf{y}\in K} |\mathbf{y}|\, \mathcal{H}^0(J_{\mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'}}) < \infty \tag{6.27}$$

and, on the other hand, using (6.23), (6.18), (6.14) and Fatou's lemma,

$$\sum_{k\in\mathbb{N}} \int_{I_k} |\nabla \mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'}|^p \,\mathrm{d}t \le \liminf_{\varepsilon\to 0} \sum_{k\in\mathbb{N}} \int_{I_k \setminus E_{\varepsilon,k}} |D\mathbf{u}_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}|^p \,\mathrm{d}t \le \frac{1}{\delta^2} \liminf_{\varepsilon\to 0} \int_{\Omega^{\boldsymbol{\xi},\mathbf{x}'}} (v_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'})^2\, |D\mathbf{u}_\varepsilon^{\boldsymbol{\xi},\mathbf{x}'}|^p \,\mathrm{d}t < \infty. \tag{6.28}$$

Thus, equations (6.26), (6.27) and (6.28) show that $\mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'} \in SBV(\Omega^{\boldsymbol{\xi},\mathbf{x}'}, \mathbb{R}^n)$. In addition, by (6.25) and (6.13),

$$\int_{\Omega^{\boldsymbol{\xi}}} \mathcal{H}^0(J_{\mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'}})\, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}') \leq \frac{1}{2a_\delta} \liminf_{\varepsilon\to 0} I_\varepsilon^V(v_\varepsilon), \tag{6.29}$$

whereas, by (6.28) and (6.12),

$$\int_{\Omega^{\boldsymbol{\xi}}} \int_{\Omega^{\boldsymbol{\xi},\mathbf{x}'}} |\nabla \mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'}|^p \,\mathrm{d}t\, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}') = \int_{\Omega^{\boldsymbol{\xi}}} \sum_{k\in\mathbb{N}} \int_{I_k} |\nabla \mathbf{u}^{\boldsymbol{\xi},\mathbf{x}'}|^p \,\mathrm{d}t\, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}') \lessapprox \liminf_{\varepsilon\to 0} I_\varepsilon^E(\mathbf{u}_\varepsilon, v_\varepsilon). \tag{6.30}$$

Proposition 2.14 and equations (6.29), (6.30), and (6.7) conclude that $\mathbf{u} \in SBV(\Omega, \mathbb{R}^n)$ and $\mathcal{H}^{n-1}(J_\mathbf{u}) < \infty$.

We pass to prove (6.11). Fix a dense countable set $\{\boldsymbol{\xi}_j\}_{j\in\mathbb{N}}$ in $\mathbb{S}^{n-1}$ and $\gamma \in [\gamma_0, 1)$, where γ_0 is the number appearing in Lemma 6.4. Define the sets

$$\begin{aligned} S &:= \{\mathbf{x} \in \partial_D\Omega : \mathbf{u}(\mathbf{x}) \neq \mathbf{u}_0(\mathbf{x})\}, \\ S_j &:= \{\mathbf{x} \in \partial\Omega : \text{there exists } \sigma > 0 \text{ such that } \mathbf{x} - (0,\sigma)\boldsymbol{\xi}_j \subset \Omega \text{ and } \mathbf{x} + (0,\sigma)\boldsymbol{\xi}_j \subset \mathbb{R}^n \setminus \Omega\}, \\ A_j &:= \{\mathbf{x} \in J_\mathbf{u} \cup S \cup \partial_N\Omega : \boldsymbol{\nu}(\mathbf{x}) \cdot \boldsymbol{\xi}_j > \gamma \text{ and } \boldsymbol{\nu}(\mathbf{x}) \cdot \boldsymbol{\xi}_i \leq \gamma \text{ for all } i < j\}, \end{aligned}$$

where $\boldsymbol{\nu}(\mathbf{x})$ in the definition of A_j denotes either $\boldsymbol{\nu}_\mathbf{u}(\mathbf{x})$ if $\mathbf{x} \in J_\mathbf{u}$, or $\boldsymbol{\nu}_\Omega(\mathbf{x})$ if $\mathbf{x} \in S \cup \partial_N\Omega$. For convenience, the Borel maps $\boldsymbol{\nu}_\mathbf{u} : J_\mathbf{u} \to \mathbb{S}^{n-1}$ and $\boldsymbol{\nu}_\Omega : \partial\Omega \to \mathbb{S}^{n-1}$ are defined everywhere, even at those points where $J_\mathbf{u}$ or $\partial\Omega$ do not admit an approximate tangent space; for those points $\mathbf{x}$ (which form an $\mathcal{H}^{n-1}$-null set), $\boldsymbol{\nu}_\mathbf{u}(\mathbf{x})$ and $\boldsymbol{\nu}_\Omega(\mathbf{x})$ are defined arbitrarily so that the resulting maps $\boldsymbol{\nu}_\mathbf{u}$ and $\boldsymbol{\nu}_\Omega$ are Borel. Note that $\{A_j\}_{j\in\mathbb{N}}$ is a disjoint family whose union is $J_\mathbf{u} \cup S \cup \partial_N\Omega$. Indeed, for each $\mathbf{x} \in J_\mathbf{u} \cup S \cup \partial_N\Omega$ there exists $j \in \mathbb{N}$ such that $|\boldsymbol{\nu}(\mathbf{x}) \cdot \boldsymbol{\xi}_j| > \gamma$, since $\{\boldsymbol{\xi}_j\}_{j\in\mathbb{N}}$ is dense in $\mathbb{S}^{n-1}$. If $j_0 \in \mathbb{N}$ is the first such j, then $\mathbf{x} \in A_{j_0}$. Notice, in addition, that

$$S_j^{\boldsymbol{\xi}_j} \subset \Omega^{\boldsymbol{\xi}_j}. \tag{6.31}$$

Indeed, let $\pi_{\boldsymbol{\xi}_j}$ be the linear projection onto Π_{ξ_j} (see Definition 2.13). If $\mathbf{x}_0 \in S_j^{\boldsymbol{\xi}_j}$ then there exists $\mathbf{x} \in S_j$ such that $\mathbf{x}_0 = \pi_{\boldsymbol{\xi}_j}(\mathbf{x})$. By definition of S_j, there exists $t > 0$ such that $\mathbf{x} - t\boldsymbol{\xi}_j \in \Omega$, so $\pi_{\boldsymbol{\xi}_j}(\mathbf{x} - t\boldsymbol{\xi}_j) \in \Omega^{\boldsymbol{\xi}_j}$, but $\pi_{\boldsymbol{\xi}_j}(\mathbf{x} - t\boldsymbol{\xi}_j) = \pi_{\boldsymbol{\xi}_j}(\mathbf{x}) = \mathbf{x}_0$. This shows (6.31). Now, Lemma 6.4 implies that if $\gamma \geq \gamma_0$ then

$$A_j \cap \partial\Omega \cap S_j = A_j \cap \partial\Omega \quad \mathcal{H}^{n-1}\text{-a.e.} \tag{6.32}$$

From now on, take such a γ.

Use the regularity of the finite Radon measure $\mathcal{H}^{n-1} \llcorner (J_\mathbf{u} \cup S \cup \partial_N\Omega)$ to find, for each $j \in \mathbb{N}$, an open set U_j such that $A_j \subset U_j$ and

$$\mathcal{H}^{n-1}\left((J_\mathbf{u} \cup S \cup \partial_N\Omega) \cap U_j \setminus A_j\right) \leq 2^{-j}(1-\gamma). \tag{6.33}$$

For each $\mathbf{x} \in J_\mathbf{u} \cup S \cup \partial_N\Omega$, let $j \in \mathbb{N}$ satisfy $\mathbf{x} \in A_j$, and define $\mathcal{F}_\mathbf{x}$ as the family of all closed balls B centred at $\mathbf{x}$ such that $B \subset U_j$ and

$$\mathcal{H}^{n-1}\left((J_\mathbf{u} \cup S \cup \partial_N\Omega) \cap \partial B\right) = 0. \tag{6.34}$$

Then the family

$$\mathcal{F} := \{B : B \in \mathcal{F}_\mathbf{x} \text{ for some } \mathbf{x} \in J_\mathbf{u} \cup S \cup \partial_N\Omega\}$$

forms a fine cover of $J_\mathbf{u} \cup S \cup \partial_N\Omega$. Apply Besicovitch's theorem (see, e.g., [7, Th. 2.19]) to obtain a disjoint subfamily $\mathcal{G}$ of $\mathcal{F}$ such that $\mathcal{H}^{n-1}((J_\mathbf{u} \cup S \cup \partial_N\Omega) \setminus \bigcup \mathcal{G}) = 0$. For each $j \in \mathbb{N}$, call V_j the union of the interiors of all the balls in $\mathcal{G}$ that are centred at a point in A_j. Each V_j is open and contained in U_j, the family $\{V_j\}_{j\in\mathbb{N}}$ is disjoint, and

$$\mathcal{H}^{n-1}\left((J_\mathbf{u} \cup S \cup \partial_N\Omega) \setminus \bigcup_{j\in\mathbb{N}} V_j\right) = 0, \tag{6.35}$$

because of condition (6.34).

Fix $j \in \mathbb{N}$ and $\mathbf{x}' \in \Omega^{\boldsymbol{\xi}_j}$ such that equations (6.14)–(6.15) hold for $\boldsymbol{\xi} = \boldsymbol{\xi}_j$. As each V_j is open, we can apply (6.25) to $U = \Omega \cap V_j$ so as to obtain

$$\begin{aligned} \mathcal{H}^0(J_{\mathbf{u}^{\boldsymbol{\xi}_j,\mathbf{x}'}} \cap (\Omega \cap V_j)^{\boldsymbol{\xi}_j,\mathbf{x}'}) + \sum_{k\in\mathbb{N}} \left[\mathcal{H}^0(G^{j,\mathbf{x}'}_{k,1}) + \frac{1}{2}\,\mathcal{H}^0(G^{j,\mathbf{x}'}_{k,0})\right] \\ \leq \frac{1}{2a_\delta} \liminf_{\varepsilon\to 0} \int_{(\Omega\cap V_j)^{\boldsymbol{\xi}_j,\mathbf{x}'}} \left[\varepsilon^{q-1}\frac{|Dv^{\boldsymbol{\xi}_j,\mathbf{x}'}_\varepsilon|^q}{q} + \frac{(1-v^{\boldsymbol{\xi}_j,\mathbf{x}'}_\varepsilon)^{q'}}{q'\varepsilon}\right] \mathrm{d}t, \end{aligned} \tag{6.36}$$

where the family $\{I_k\}_{k\in\mathbb{N}}$ of intervals this time corresponds to $(\Omega \cap V_j)^{\boldsymbol{\xi}_j,\mathbf{x}'}$, and the dependence of $G_{k,0}$ and $G_{k,1}$ on V_j, $\boldsymbol{\xi}_j$, and $\mathbf{x}'$ has been made explicit in the notation. Now we analyze the last two terms of the left-hand side of (6.36). We discuss the following two cases.

a) Let $t_0 \in (\partial_N\Omega \cap S_j \cap V_j)^{\boldsymbol{\xi}_j,\mathbf{x}'}$. Thus, there exist $\mathbf{x} \in \partial_N\Omega \cap S_j \cap V_j$ and $\mathbf{x}' \in (\partial_N\Omega \cap S_j \cap V_j)^{\boldsymbol{\xi}_j}$ such that $\mathbf{x} = \mathbf{x}' + t_0\boldsymbol{\xi}_j$. Then $t_0 \in \partial I_k$ for some $k \in \mathbb{N}$, by definition of S_j. By (4.9) we have that $v^{\boldsymbol{\xi}_j,\mathbf{x}'}_\varepsilon(t_0) = 0$ for all ε, so by the continuity of $v^{\boldsymbol{\xi}_j,\mathbf{x}'}_\varepsilon$, we infer that $t \in E_{\varepsilon,k}$ for all $t \in \Omega^{\boldsymbol{\xi}_j,\mathbf{x}'}$ with $t \simeq t_0$. Since $\mathbf{x} \in S_j$, this implies that $t_0 \in \overline{E_{\varepsilon,k}}$. From the definition of F_k we conclude that $t_0 \in F_k$. This shows that

$$(\partial_N\Omega \cap S_j \cap V_j)^{\boldsymbol{\xi}_j,\mathbf{x}'} \subset \bigcup_{k\in\mathbb{N}} G^{j,\mathbf{x}'}_{k,0}. \tag{6.37}$$

b) Note now that $\mathcal{H}^{n-1}$-a.e. $\mathbf{x} \in \partial_D\Omega$ satisfies $\mathbf{u}_\varepsilon(\mathbf{x}) = \mathbf{u}_0(\mathbf{x})$, thanks to (4.3). Take such an $\mathbf{x}$ that in addition belongs to $S \cap S_j \cap V_j$. As in the previous case, let $\mathbf{x}' \in (S \cap S_j \cap V_j)^{\boldsymbol{\xi}_j}$ and $t_0 \in (S \cap S_j \cap V_j)^{\boldsymbol{\xi}_j,\mathbf{x}'}$ be such that $\mathbf{x} = \mathbf{x}' + t_0\boldsymbol{\xi}_j$, so $t_0 = \sup I_k$ for some $k \in \mathbb{N}$. By (4.8), $v^{\boldsymbol{\xi}_j,\mathbf{x}'}_\varepsilon(t_0) = 1$ for all ε, while we have just seen that

$$\mathbf{u}^{\boldsymbol{\xi}_j,\mathbf{x}'}_\varepsilon(t_0) = \mathbf{u}_0(\mathbf{x}). \tag{6.38}$$

On the other hand, t_0 must belong to F_k, since otherwise, having in mind equation (6.19) and the fact that F_k is compact, there would exist $\eta > 0$ such that $(t_0 - \eta, t_0) \subset I_k \setminus E_{\varepsilon,k}$ for all ε sufficiently small. By (6.14), (6.15), (6.38) and trace theory for maps in $W^{1,p}((t_0-\eta,t_0),\mathbb{R}^n)$, we would conclude that $\mathbf{u}^{\boldsymbol{\xi}_j,\mathbf{x}'}(t_0) = \mathbf{u}_0(\mathbf{x})$, which contradicts the fact that $\mathbf{x} \in S$. This shows that for $\mathcal{H}^{n-1}$-a.e. $\mathbf{x}' \in (S \cap S_j \cap V_j)^{\boldsymbol{\xi}_j}$,

$$(S \cap S_j \cap V_j)^{\boldsymbol{\xi}_j,\mathbf{x}'} \subset \bigcup_{k\in\mathbb{N}} G^{j,\mathbf{x}'}_{k,1}. \tag{6.39}$$

Inclusions (6.37) and (6.39) imply that

$$\begin{aligned} \int_{(\partial_N\Omega\cap S_j\cap V_j)^{\boldsymbol{\xi}_j}} \mathcal{H}^0\Big((\partial_N\Omega \cap S_j \cap V_j)^{\boldsymbol{\xi}_j,\mathbf{x}'}\Big)\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}') &\leq \sum_{k\in\mathbb{N}} \int_{(\partial_N\Omega\cap S_j\cap V_j)^{\boldsymbol{\xi}_j}} \mathcal{H}^0(G^{j,\mathbf{x}'}_{k,0})\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}'), \\ \int_{(S\cap S_j\cap V_j)^{\boldsymbol{\xi}_j}} \mathcal{H}^0\Big((S \cap S_j \cap V_j)^{\boldsymbol{\xi}_j,\mathbf{x}'}\Big)\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}') &\leq \sum_{k\in\mathbb{N}} \int_{(S\cap S_j\cap V_j)^{\boldsymbol{\xi}_j}} \mathcal{H}^0(G^{j,\mathbf{x}'}_{k,1})\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}'). \end{aligned} \tag{6.40}$$

Now recall from (6.31) that

$$(\partial_N\Omega \cap S_j \cap V_j)^{\boldsymbol{\xi}_j} \subset (\Omega \cap V_j)^{\boldsymbol{\xi}_j} \quad \text{and} \quad (S \cap S_j \cap V_j)^{\boldsymbol{\xi}_j} \subset (\Omega \cap V_j)^{\boldsymbol{\xi}_j}. \tag{6.41}$$

Thus, combining (6.40), (6.41), (6.36), Fatou's lemma and Proposition 2.14, we find that

$$\begin{aligned} &\int_{(\Omega\cap V_j)^{\boldsymbol{\xi}_j}} \mathcal{H}^0\Big(J_{\mathbf{u}^{\boldsymbol{\xi}_j,\mathbf{x}'}} \cap (\Omega \cap V_j)^{\boldsymbol{\xi}_j,\mathbf{x}'}\Big)\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}') + \int_{(S\cap S_j\cap V_j)^{\boldsymbol{\xi}_j}} \mathcal{H}^0\Big((S \cap S_j \cap V_j)^{\boldsymbol{\xi}_j,\mathbf{x}'}\Big)\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}') \\ &+ \frac{1}{2}\int_{(\partial_N\Omega\cap S_j\cap V_j)^{\boldsymbol{\xi}_j}} \mathcal{H}^0\Big((\partial_N\Omega \cap S_j \cap V_j)^{\boldsymbol{\xi}_j,\mathbf{x}'}\Big)\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}') \leq \frac{1}{2a_\delta} \liminf_{\varepsilon\to 0} I^V_\varepsilon(v_\varepsilon; \Omega \cap V_j). \end{aligned} \tag{6.42}$$

By Proposition 2.14,

$$
\begin{aligned}
\int_{(\Omega\cap V_j)^{\boldsymbol{\xi}_j}} \mathcal{H}^0(J_{\mathbf{u}^{\boldsymbol{\xi}_j,\mathbf{x}'}} \cap (\Omega\cap V_j)^{\boldsymbol{\xi}_j,\mathbf{x}'}) \,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}') &= \int_{V_j\cap J_{\mathbf{u}}} \left|\boldsymbol{\nu}_{\mathbf{u}}\cdot\boldsymbol{\xi}_j\right| \mathrm{d}\mathcal{H}^{n-1}, \\
\int_{(S\cap S_j\cap V_j)^{\boldsymbol{\xi}_j}} \mathcal{H}^0((S\cap S_j\cap V_j)^{\boldsymbol{\xi}_j,\mathbf{x}'}) \,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}') &= \int_{S\cap S_j\cap V_j} \left|\boldsymbol{\nu}_{\Omega}\cdot\boldsymbol{\xi}_j\right| \mathrm{d}\mathcal{H}^{n-1}, \\
\int_{(\partial_N\Omega\cap S_j\cap V_j)^{\boldsymbol{\xi}_j}} \mathcal{H}^0((\partial_N\Omega\cap S_j\cap V_j)^{\boldsymbol{\xi}_j,\mathbf{x}'}) \,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}') &= \int_{\partial_N\Omega\cap S_j\cap V_j} \left|\boldsymbol{\nu}_{\Omega}\cdot\boldsymbol{\xi}_j\right| \mathrm{d}\mathcal{H}^{n-1}.
\end{aligned}
\tag{6.43}
$$

Using the definition of A_j, we find that

$$
\begin{aligned}
&\int_{V_j\cap J_{\mathbf{u}}\cap A_j} \left|\boldsymbol{\nu}_{\mathbf{u}}\cdot\boldsymbol{\xi}_j\right| \mathrm{d}\mathcal{H}^{n-1} + \int_{V_j\cap S\cap A_j} \left|\boldsymbol{\nu}_{\Omega}\cdot\boldsymbol{\xi}_j\right| \mathrm{d}\mathcal{H}^{n-1} + \frac{1}{2}\int_{V_j\cap\partial_N\Omega\cap A_j} \left|\boldsymbol{\nu}_{\Omega}\cdot\boldsymbol{\xi}_j\right| \mathrm{d}\mathcal{H}^{n-1} \\
&\geq \gamma\left[\mathcal{H}^{n-1}(V_j\cap J_{\mathbf{u}}\cap A_j) + \mathcal{H}^{n-1}(V_j\cap S\cap A_j) + \frac{1}{2}\,\mathcal{H}^{n-1}(V_j\cap\partial_N\Omega\cap A_j)\right].
\end{aligned}
\tag{6.44}
$$

On the other hand, using the inclusion $V_j\subset U_j$ and (6.33), we find that

$$
\begin{aligned}
&\mathcal{H}^{n-1}(V_j\cap J_{\mathbf{u}}) + \mathcal{H}^{n-1}(V_j\cap S) + \frac{1}{2}\,\mathcal{H}^{n-1}(V_j\cap\partial_N\Omega) \\
&\leq \mathcal{H}^{n-1}(V_j\cap J_{\mathbf{u}}\cap A_j) + \mathcal{H}^{n-1}(V_j\cap S\cap A_j) + \frac{1}{2}\,\mathcal{H}^{n-1}(V_j\cap\partial_N\Omega\cap A_j) + 2^{-j}(1-\gamma).
\end{aligned}
\tag{6.45}
$$

Applying (6.32), we obtain that

$$
\begin{aligned}
&\int_{V_j\cap J_{\mathbf{u}}} \left|\boldsymbol{\nu}_{\mathbf{u}}\cdot\boldsymbol{\xi}_j\right| \mathrm{d}\mathcal{H}^{n-1} + \int_{S_j\cap S\cap V_j} \left|\boldsymbol{\nu}_{\Omega}\cdot\boldsymbol{\xi}_j\right| \mathrm{d}\mathcal{H}^{n-1} + \frac{1}{2}\int_{V_j\cap\partial_N\Omega\cap S_j} \left|\boldsymbol{\nu}_{\Omega}\cdot\boldsymbol{\xi}_j\right| \mathrm{d}\mathcal{H}^{n-1} \\
&\geq \int_{V_j\cap J_{\mathbf{u}}\cap A_j} \left|\boldsymbol{\nu}_{\mathbf{u}}\cdot\boldsymbol{\xi}_j\right| \mathrm{d}\mathcal{H}^{n-1} + \int_{A_j\cap S\cap V_j} \left|\boldsymbol{\nu}_{\Omega}\cdot\boldsymbol{\xi}_j\right| \mathrm{d}\mathcal{H}^{n-1} + \frac{1}{2}\int_{A_j\cap\partial_N\Omega\cap V_j} \left|\boldsymbol{\nu}_{\Omega}\cdot\boldsymbol{\xi}_j\right| \mathrm{d}\mathcal{H}^{n-1}.
\end{aligned}
\tag{6.46}
$$

By (6.35) and (6.45), we have that

$$
\begin{aligned}
&\mathcal{H}^{n-1}(J_{\mathbf{u}}) + \mathcal{H}^{n-1}(S) + \frac{1}{2}\,\mathcal{H}^{n-1}(\partial_N\Omega) \\
&\leq \sum_{j\in\mathbb{N}}\left[\mathcal{H}^{n-1}(J_{\mathbf{u}}\cap V_j\cap A_j) + \mathcal{H}^{n-1}(A_j\cap S\cap V_j) + \frac{1}{2}\,\mathcal{H}^{n-1}(A_j\cap\partial_N\Omega\cap V_j)\right] + 1-\gamma.
\end{aligned}
\tag{6.47}
$$

Putting together succesively inequalities (6.47), (6.44), (6.46), (6.43), (6.42), we obtain

$$
\mathcal{H}^{n-1}(J_{\mathbf{u}}) + \mathcal{H}^{n-1}(S) + \frac{1}{2}\,\mathcal{H}^{n-1}(\partial_N\Omega) \leq \frac{1}{2a_\delta\gamma}\liminf_{\varepsilon\to 0} I_\varepsilon(v_\varepsilon) + 1-\gamma.
$$

Letting $\gamma\to 1$ and $\delta\to 0$, we conclude (6.11). □

6.3 Surface and elastic energy terms

In this section we study the terms $I_\varepsilon^E(\mathbf{u}_\varepsilon, v_\varepsilon)$ and $I_\varepsilon^W(w_\varepsilon)$. The analysis of $I_\varepsilon^E(\mathbf{u}_\varepsilon, v_\varepsilon)$ is initially based on Braides, Chambolle and Solci [20, Sect. 3], who proved a Γ-convergence result for a quasiconvex stored energy function W with p-growth. The term $I_\varepsilon^W(w_\varepsilon)$ resembles a Modica–Mortola [45] functional, but for its analysis we also need the convergence result of Theorem 3.4. In fact, in order to deal with a polyconvex W that grows as in (W2) and with the invertibility constraint for the deformation, we need to apply the techniques of [37].

The following auxiliary results will be used several times. Recall from Section 2.7 the notation for minors.

Lemma 6.6. *For each ε, let $(\mathbf{u}_\varepsilon, v_\varepsilon) \in \mathcal{A}^E \times W^{1,q}(\Omega, [0,1])$ satisfy* (6.7)*. Let $\{A_\varepsilon\}_\varepsilon$ be a sequence of measurable subsets of Ω such that $\inf_\varepsilon \inf_{A_\varepsilon} v_\varepsilon > 0$. Then, the sequence $\{\nabla(\chi_{A_\varepsilon} \mathbf{u}_\varepsilon)\}_\varepsilon$ is bounded in $L^p(\Omega, \mathbb{R}^{n\times n})$, and the sequence $\{\boldsymbol{\mu}(\nabla(\chi_{A_\varepsilon} \mathbf{u}_\varepsilon))\}_\varepsilon$ is equiintegrable.*

Proof. Call $\delta := \inf_\varepsilon \inf_{A_\varepsilon} v_\varepsilon$. Using Lemma 2.3 and (W2), we find that

$$\int_\Omega |\nabla(\chi_{A_\varepsilon} \mathbf{u}_\varepsilon)|^p \, \mathrm{d}\mathbf{x} \leq \frac{1}{\delta^2} \int_{A_\varepsilon} v_\varepsilon^2 \, |D\mathbf{u}_\varepsilon|^p \, \mathrm{d}\mathbf{x} \lessapprox \int_{A_\varepsilon} v_\varepsilon^2 \, W_\varepsilon \, \mathrm{d}\mathbf{x} \leq I_\varepsilon^E(\mathbf{u}_\varepsilon, v_\varepsilon) \lessapprox 1.$$

Let h_1 and h_2 be the functions of (W2). For $i \in \{1,2\}$, define $\bar{h}_i : [0,\infty) \to [0,\infty)$ as $\bar{h}_i(t) := h_i(\max\{1,t\})$. Then

$$\lim_{t\to\infty} \frac{\bar{h}_i(t)}{t} = \infty, \qquad i \in \{1,2\}$$

and

$$\int_\Omega \bar{h}_1(|\operatorname{cof} \nabla(\chi_{A_\varepsilon} \mathbf{u}_\varepsilon)|) \, \mathrm{d}\mathbf{x} \leq \mathcal{L}^n(\Omega) \, h_1(1) + \int_{A_\varepsilon} W_\varepsilon \, \mathrm{d}\mathbf{x} \leq \mathcal{L}^n(\Omega) \, h_1(1) + \frac{1}{\delta^2} I_\varepsilon^E(\mathbf{u}_\varepsilon, v_\varepsilon) \lessapprox 1;$$

similarly,

$$\int_\Omega \bar{h}_2(\det \nabla(\chi_{A_\varepsilon} \mathbf{u}_\varepsilon)) \, \mathrm{d}\mathbf{x} \leq \mathcal{L}^n(\Omega) \, h_2(1) + \frac{1}{\delta^2} I_\varepsilon^E(\mathbf{u}_\varepsilon, v_\varepsilon) \lessapprox 1.$$

By De la Vallée–Poussin's criterion for equiintegrability, the sequences $\{\operatorname{cof} \nabla(\chi_{A_\varepsilon} \mathbf{u}_\varepsilon)\}_\varepsilon$ and $\{\det \nabla(\chi_{A_\varepsilon} \mathbf{u}_\varepsilon)\}_\varepsilon$ are equiintegrable. The rest of the components of $\{\boldsymbol{\mu}(\nabla(\chi_{A_\varepsilon} \mathbf{u}_\varepsilon))\}_\varepsilon$ are equiintegrable because $p \geq n-1$ and, due to Hölder's inequality, minors of order $k \in \mathbb{N}$ with $k < p$ are equiintegrable, as $\{\nabla(\chi_{A_\varepsilon} \mathbf{u}_\varepsilon)\}_\varepsilon$ is bounded in $L^p(\Omega, \mathbb{R}^{n\times n})$. □

Lemma 6.7. *For each ε, let $(\mathbf{u}_\varepsilon, v_\varepsilon) \in \mathcal{A}^E \times W^{1,q}(\Omega, [0,1])$ satisfy* (6.7)*. Let $\mathbf{u} \in SBV(\Omega, K)$ satisfy* (6.9)*. Let $\{A_\varepsilon\}_\varepsilon$ be a sequence of measurable subsets of Ω such that $\mathcal{L}^n(A_\varepsilon) \to \mathcal{L}^n(\Omega)$. Assume that*

$$\inf_\varepsilon \inf_{A_\varepsilon} v_\varepsilon > 0 \quad \text{and} \quad \sup_\varepsilon \operatorname{Per}(A_\varepsilon, \Omega) < \infty.$$

Then

$$\boldsymbol{\mu}_0(\nabla(\chi_{A_\varepsilon} \mathbf{u}_\varepsilon)) \rightharpoonup \boldsymbol{\mu}_0(\nabla\mathbf{u}) \text{ in } L^1(\Omega, \mathbb{R}^{\tau-1}).$$

Proof. We check that the sequence $\{\chi_{A_\varepsilon} \mathbf{u}_\varepsilon\}_\varepsilon$ satisfies the assumptions of Lemma 2.12.

Lemma 2.4 shows that $\chi_{A_\varepsilon} \mathbf{u}_\varepsilon \in SBV(\Omega, \mathbb{R}^n)$ and $\mathcal{H}^{n-1}(J_{\chi_{A_\varepsilon} \mathbf{u}_\varepsilon}) \leq \operatorname{Per}(A_\varepsilon, \Omega)$ for each ε. In addition, thanks to (6.9) and $\mathcal{L}^n(A_\varepsilon) \to \mathcal{L}^n(\Omega)$, we have that $\chi_{A_\varepsilon} \mathbf{u}_\varepsilon \to \mathbf{u}$ in $L^1(\Omega, \mathbb{R}^n)$. Therefore, using Lemma 6.6, we find that the sequence $\{\nabla(\chi_{A_\varepsilon} \mathbf{u}_\varepsilon)\}_\varepsilon$ is bounded in $L^p(\Omega, \mathbb{R}^{n\times n})$, and the sequence $\{\operatorname{cof} \nabla(\chi_{A_\varepsilon} \mathbf{u}_\varepsilon)\}_\varepsilon$ is equiintegrable. The conclusion is achieved thanks to Lemma 2.12. □

Proposition 6.8. *For each ε, let $(\mathbf{u}_\varepsilon, v_\varepsilon, w_\varepsilon) \in \mathcal{A}_\varepsilon$ satisfy* (6.4)*. Let $\mathbf{u} \in SBV(\Omega, K)$ satisfy* (6.9)*. Then $\mathbf{u}$ is one-to-one a.e.,* $\det D\mathbf{u} > 0$ *a.e.,*

$$\operatorname{Per} \operatorname{im_G}(\mathbf{u}, \Omega) + 2\,\mathcal{H}^{n-1}(J_{\mathbf{u}^{-1}}) \leq 6 \liminf_{\varepsilon\to 0} I_\varepsilon^W(w_\varepsilon), \tag{6.48}$$

$$\int_\Omega W(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x})) \, \mathrm{d}\mathbf{x} \leq \liminf_{\varepsilon\to 0} I_\varepsilon^E(\mathbf{u}_\varepsilon, v_\varepsilon) \tag{6.49}$$

and, for a subsequence,

$$w_\varepsilon \to \chi_{\operatorname{im_G}(\mathbf{u},\Omega)} \text{ in } L^1(Q). \tag{6.50}$$

Proof. Fix $0 < \delta_1 < \delta_2 < 1$. As in (6.17), using the coarea formula (2.4), we obtain that for each ε there exists $s_\varepsilon \in (\delta_1, \delta_2)$ such that the set $A_\varepsilon := \{\mathbf{x} \in \Omega : v_\varepsilon(\mathbf{x}) > s_\varepsilon\}$ satisfies $\sup_\varepsilon \operatorname{Per}(A_\varepsilon, \Omega) < \infty$ and, due to (6.8),

$$\mathcal{L}^n(A_\varepsilon) \to \mathcal{L}^n(\Omega). \tag{6.51}$$

Thanks to Lemma 6.7,

$$\boldsymbol{\mu}_0(\nabla(\chi_{A_\varepsilon}\mathbf{u}_\varepsilon)) \rightharpoonup \boldsymbol{\mu}_0(\nabla\mathbf{u}) \text{ in } L^1(\Omega,\mathbb{R}^{\tau-1}). \tag{6.52}$$

Again as in (6.17), for each ε there exists $t_\varepsilon \in (\delta_1,\delta_2)$ such that, defining

$$b_{\delta_1,\delta_2} := \int_{\delta_1}^{\delta_2} s(1-s)\,\mathrm{d}s, \qquad E_\varepsilon := \{\mathbf{y}\in Q : w_\varepsilon(\mathbf{y}) > t_\varepsilon\}, \qquad F_\varepsilon := \{\mathbf{x}\in\Omega : w_\varepsilon(\mathbf{u}_\varepsilon(\mathbf{x})) > t_\varepsilon\}$$

we have that

$$I_\varepsilon^W(w_\varepsilon) \geq \int_Q w_\varepsilon(1-w_\varepsilon)\,|Dw_\varepsilon|\,\mathrm{d}\mathbf{y} \geq b_{\delta_1,\delta_2}\operatorname{Per} E_\varepsilon. \tag{6.53}$$

We have also used the equality $\operatorname{Per} E_\varepsilon = \operatorname{Per}(E_\varepsilon, Q)$, which is true because conditions (4.10), (4.6) and the continuity of w_ε imply that $E_\varepsilon \subset\subset Q$. In particular, (6.53) shows that

$$\sup_\varepsilon \operatorname{Per} E_\varepsilon < \infty. \tag{6.54}$$

Thanks to (4.11), (4.12) and (6.2), we have that $w_\varepsilon \circ \mathbf{u}_\varepsilon - v_\varepsilon \to 0$ in $L^1(\Omega)$. With the convergence (6.8), we conclude that, for a subsequence, $w_\varepsilon\circ\mathbf{u}_\varepsilon \to 1$ in measure, hence

$$\mathcal{L}^n(F_\varepsilon) \to \mathcal{L}^n(\Omega). \tag{6.55}$$

Denoting by Δ the operator of symmetric difference of sets, we have, thanks to (4.11), that $v_\varepsilon|_{A_\varepsilon\Delta F_\varepsilon} \geq \delta_1$ for all ε, so Lemma 6.6 yields the equiintegrability of the sequence $\{\boldsymbol{\mu}_0(\chi_{A_\varepsilon\Delta F_\varepsilon} D\mathbf{u}_\varepsilon)\}_\varepsilon$. Therefore, using also (6.51) and (6.55),

$$\|\boldsymbol{\mu}_0(\nabla(\chi_{A_\varepsilon}\mathbf{u}_\varepsilon)) - \boldsymbol{\mu}_0(\nabla(\chi_{F_\varepsilon}\mathbf{u}_\varepsilon))\|_{L^1(\Omega,\mathbb{R}^{\tau-1})} = \int_{A_\varepsilon\Delta F_\varepsilon} |\boldsymbol{\mu}_0(D\mathbf{u}_\varepsilon)|\,\mathrm{d}\mathbf{x} \to 0,$$

which, together with (6.52), shows that

$$\boldsymbol{\mu}_0(\nabla(\chi_{F_\varepsilon}\mathbf{u}_\varepsilon)) \rightharpoonup \boldsymbol{\mu}_0(\nabla\mathbf{u}) \;\text{ in } L^1(\Omega,\mathbb{R}^{\tau-1}). \tag{6.56}$$

Now we verify the assumptions of Theorem 3.4 for the sequence $\{\mathbf{u}_\varepsilon\}_\varepsilon$ of maps and the sequence $\{F_\varepsilon\}_\varepsilon$ of sets. Using (4.10), it is easy to check that

$$\operatorname{im}_{\mathrm{G}}(\mathbf{u}_\varepsilon, F_\varepsilon) = E_\varepsilon \;\text{ a.e.}, \tag{6.57}$$

so

$$\operatorname{Per}\operatorname{im}_{\mathrm{G}}(\mathbf{u}_\varepsilon, F_\varepsilon) = \operatorname{Per} E_\varepsilon \tag{6.58}$$

and, recalling (6.54), we obtain that $\sup_\varepsilon \operatorname{Per}\operatorname{im}_{\mathrm{G}}(\mathbf{u}_\varepsilon, F_\varepsilon) < \infty$.

Now we show that $\mathbf{u}_{\varepsilon,F_\varepsilon}^{-1} \in SBV(\mathbb{R}^n,\mathbb{R}^n)$. Any $\mathbf{x}\in F_\varepsilon$ satisfies $v_\varepsilon(\mathbf{x}) > t_\varepsilon$, thanks to (4.11). As v_ε is continuous, any $\mathbf{x}\in\bar{F}_\varepsilon$ satisfies $v_\varepsilon(\mathbf{x}) \geq t_\varepsilon$, so $\mathbf{x}\notin\partial_N\Omega$, because of (4.9). Thus,

$$\bar{F}_\varepsilon \cap \partial_N\Omega = \varnothing. \tag{6.59}$$

Let now $\bar{\mathbf{u}}_\varepsilon \in W^{1,p}(\Omega_1,\mathbb{R}^n)$ be the extension of $\mathbf{u}_\varepsilon$ given by (4.4). Thanks to the relations $\Omega\cup\partial_D\Omega\subset\Omega_1$ and (6.59), as well as to the fact that $\partial_D\Omega$ and $\partial_N\Omega$ are closed disjoint sets, we can apply [38, Th. 2] to infer that, thanks to (4.5), there exists an open set $U_\varepsilon \subset\subset \Omega$ such that $F_\varepsilon \subset U_\varepsilon$ and $\bar{\mathbf{u}}_{\varepsilon,U_\varepsilon}^{-1} \in SBV(\mathbb{R}^n,\mathbb{R}^n)$. Using (6.57) and the inclusions

$$E_\varepsilon \subset \operatorname{im}_{\mathrm{G}}(\mathbf{u}_\varepsilon,\Omega) \subset \operatorname{im}_{\mathrm{G}}(\bar{\mathbf{u}}_\varepsilon, U_\varepsilon),$$

we obtain that $\operatorname{im}_{\mathrm{G}}(\mathbf{u}_\varepsilon, F_\varepsilon) = \operatorname{im}_{\mathrm{G}}(\bar{\mathbf{u}}_\varepsilon, U_\varepsilon)\cap E_\varepsilon$ a.e.; consequently, $\mathbf{u}_{\varepsilon,F_\varepsilon}^{-1} = \chi_{E_\varepsilon}\bar{\mathbf{u}}_{\varepsilon,U_\varepsilon}^{-1}$ a.e. Thus, by Lemma 2.4, we conclude that $\mathbf{u}_{\varepsilon,F_\varepsilon}^{-1} \in SBV(\mathbb{R}^n,\mathbb{R}^n)$.

As $\mathcal{E}(\bar{\mathbf{u}}_\varepsilon) = 0$, we can apply now [38, Th. 3] to obtain that $\mathcal{H}^{n-1}(\Gamma_I(\bar{\mathbf{u}}_\varepsilon)) = 0$. Here Γ_I denotes the invisible surface, as defined in [38, Def. 9]. For the purposes of the proof, here it suffices to know that $\Gamma_I(\bar{\mathbf{u}}_\varepsilon)$ is the set of $\mathbf{y} \in J_{\bar{\mathbf{u}}_\varepsilon^{-1}}$ such that both lateral traces $(\bar{\mathbf{u}}_\varepsilon)^\pm(\mathbf{y})$ belong to Ω_1. Now, any $\mathbf{y} \in J_{(\mathbf{u}_\varepsilon|_{F_\varepsilon})^{-1}}$ satisfies that the lateral traces $((\mathbf{u}_\varepsilon|_{F_\varepsilon})^{-1})^\pm(\mathbf{y})$ exist, are distinct and belong to $\bar{F}_\varepsilon$, and, hence, to Ω_1, due to (6.59). Thus, $\mathbf{y} \in \Gamma_I(\bar{\mathbf{u}}_\varepsilon)$. Therefore, $J_{(\mathbf{u}_\varepsilon|_{F_\varepsilon})^{-1}} \subset \Gamma_I(\bar{\mathbf{u}}_\varepsilon)$ and, consequently,

$$\mathcal{H}^{n-1}(J_{(\mathbf{u}_\varepsilon|_{F_\varepsilon})^{-1}}) = 0. \tag{6.60}$$

Due to (4.11) and Lemma 6.6, there exists $\theta \in L^1(\Omega)$ such that, for a subsequence, $\chi_{F_\varepsilon} \det D\mathbf{u}_\varepsilon \rightharpoonup \theta$ in $L^1(\Omega)$. Moreover, $\theta \geq 0$ a.e. If θ were zero in a set $A \subset \Omega$ of positive measure, using (6.51) and (6.55), we would have (for a subsequence) $\det D\mathbf{u}_\varepsilon \to 0$ a.e. in A and $\chi_{A_\varepsilon} \to 1$ a.e. in Ω; hence by assumption (W2), we would obtain $\chi_{A_\varepsilon} h_2(\det D\mathbf{u}_\varepsilon) \to \infty$ a.e. in A, and, by Fatou's lemma,

$$\lim_{\varepsilon \to 0} \int_{A_\varepsilon \cap A} h_2(\det D\mathbf{u}_\varepsilon) \, \mathrm{d}\mathbf{x} = \infty,$$

but for each ε, recalling the notation (6.6),

$$I_\varepsilon^E(\mathbf{u}_\varepsilon, v_\varepsilon) \geq \int_{A_\varepsilon} v_\varepsilon^2 \, W_\varepsilon \, \mathrm{d}\mathbf{x} \geq \delta_1^2 \int_{A_\varepsilon} W_\varepsilon \, \mathrm{d}\mathbf{x} \geq \delta_1^2 \int_{A_\varepsilon} h_2(\det D\mathbf{u}_\varepsilon) \, \mathrm{d}\mathbf{x} \geq \delta_1^2 \int_{A_\varepsilon \cap A} h_2(\det D\mathbf{u}_\varepsilon) \, \mathrm{d}\mathbf{x},$$

which is a contradiction with (6.4). Thus, $\theta > 0$ a.e. We can therefore apply Theorem 3.4 and (6.60) in order to conclude that $\theta = \det \nabla \mathbf{u}$ a.e., $\mathbf{u}$ is one-to-one a.e.,

$$\chi_{\mathrm{im_G}(\mathbf{u}_\varepsilon, F_\varepsilon)} \to \chi_{\mathrm{im_G}(\mathbf{u}, \Omega)} \quad \text{a.e. and in } L^1(\mathbb{R}^n), \tag{6.61}$$

up to a subsequence, and

$$\operatorname{Per} \mathrm{im_G}(\mathbf{u}, \Omega) + 2\, \mathcal{H}^{n-1}(J_{\mathbf{u}^{-1}}) \leq \liminf_{\varepsilon \to 0} \operatorname{Per} \mathrm{im_G}(\mathbf{u}_\varepsilon, F_\varepsilon). \tag{6.62}$$

In particular,

$$\det(\chi_{F_\varepsilon} D\mathbf{u}_\varepsilon) \rightharpoonup \det \nabla \mathbf{u} \ \text{ in } L^1(\Omega). \tag{6.63}$$

Having in mind (6.53) and (6.58), we obtain

$$\operatorname{Per} \mathrm{im_G}(\mathbf{u}_\varepsilon, F_\varepsilon) \leq \frac{1}{b_{\delta_1, \delta_2}} I_\varepsilon^W(w_\varepsilon). \tag{6.64}$$

Putting together (6.62) and (6.64), and letting $\delta_1 \to 0$ and $\delta_2 \to 1$, we obtain inequality (6.48).

We prove now (6.49). Convergences (6.55), (6.56) and (6.63) show that

$$\boldsymbol{\mu}(\chi_{F_\varepsilon} D\mathbf{u}_\varepsilon) \rightharpoonup \boldsymbol{\mu}(\nabla \mathbf{u}) \ \text{ in } L^1(\Omega, \mathbb{R}^\tau) \quad \text{and} \quad \chi_{F_\varepsilon} \mathbf{u}_\varepsilon \to \mathbf{u} \ \text{ a.e.} \tag{6.65}$$

Let $\{\tilde{F}_\varepsilon\}_\varepsilon$ be the increasing sequence of sets obtained from $\{F_\varepsilon\}_\varepsilon$, i.e., $\tilde{F}_\varepsilon := \bigcup_{\varepsilon' \geq \varepsilon} F_{\varepsilon'}$. Naturally, (6.55) and (6.65) yield

$$\mathcal{L}^n(\tilde{F}_\varepsilon) \to \mathcal{L}^n(\Omega), \qquad \boldsymbol{\mu}(\chi_{\tilde{F}_\varepsilon} D\mathbf{u}_\varepsilon) \rightharpoonup \boldsymbol{\mu}(\nabla \mathbf{u}) \ \text{ in } L^1(\Omega, \mathbb{R}^\tau) \quad \text{and} \quad \chi_{\tilde{F}_\varepsilon} \mathbf{u}_\varepsilon \to \mathbf{u} \ \text{ a.e.} \tag{6.66}$$

Now fix an element ε_1 of the sequence $\{\varepsilon\}_\varepsilon$. Convergences (6.66) and assumption (W1) allow us to use the lower semicontinuity theorem of [12, Th. 5.4] applied to the function $\tilde{W}_{\varepsilon_1} : \Omega \times K \times \mathbb{R}_+^\tau \to \mathbb{R}$ defined as $\tilde{W}_{\varepsilon_1}(\mathbf{x}, \mathbf{y}, \boldsymbol{\mu}) := \chi_{\tilde{F}_{\varepsilon_1}}(\mathbf{x}) \tilde{W}(\mathbf{x}, \mathbf{y}, \boldsymbol{\mu})$, so as to obtain that

$$\int_{\tilde{F}_{\varepsilon_1}} W(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla \mathbf{u}(\mathbf{x})) \, \mathrm{d}\mathbf{x} \leq \liminf_{\varepsilon \to 0} \int_{\tilde{F}_{\varepsilon_1}} W(\mathbf{x}, (\chi_{\tilde{F}_\varepsilon} \mathbf{u}_\varepsilon)(\mathbf{x}), (\chi_{\tilde{F}_\varepsilon} \nabla \mathbf{u}_\varepsilon)(\mathbf{x})) \, \mathrm{d}\mathbf{x}. \tag{6.67}$$

Moreover, for each $\varepsilon \leq \varepsilon_1$ we have $\tilde{F}_{\varepsilon_1} \subset \tilde{F}_\varepsilon$, so using assumption (4.11), we find that

$$\int_{\tilde{F}_{\varepsilon_1}} W(\mathbf{x}, (\chi_{\tilde{F}_\varepsilon} \mathbf{u}_\varepsilon)(\mathbf{x}), (\chi_{\tilde{F}_\varepsilon} \nabla \mathbf{u}_\varepsilon)(\mathbf{x}))\,\mathrm{d}\mathbf{x} = \int_{\tilde{F}_{\varepsilon_1}} W_\varepsilon \,\mathrm{d}\mathbf{x} \leq \int_{\tilde{F}_\varepsilon} W_\varepsilon \,\mathrm{d}\mathbf{x} \leq \frac{1}{\delta_1^2} \int_{\tilde{F}_\varepsilon} v_\varepsilon^2 W_\varepsilon \,\mathrm{d}\mathbf{x} \leq \frac{1}{\delta_1^2} I_\varepsilon^E(\mathbf{u}_\varepsilon, v_\varepsilon). \tag{6.68}$$

On the other hand, by (6.66) and the monotone convergence theorem,

$$\lim_{\varepsilon_1 \to 0} \int_{\tilde{F}_{\varepsilon_1}} W(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla \mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x} = \int_\Omega W(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla \mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}. \tag{6.69}$$

Equations (6.67), (6.68) and (6.69) show that

$$\int_\Omega W(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla \mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x} \leq \frac{1}{\delta_1^2} \liminf_{\varepsilon \to 0} I_\varepsilon^E(\mathbf{u}_\varepsilon, v_\varepsilon).$$

Letting $\delta_1 \to 1$ and $\delta_2 \to 1$ we conclude (6.49).

We pass to prove (6.50). As $\sup_\varepsilon I_\varepsilon^W(w_\varepsilon) < \infty$, a well-known argument going back to Modica [44, Th. I and Prop. 3] (see also [1, Sect. 4.5]) shows that there exists a measurable set $V \subset Q$ such that, for a subsequence,

$$w_\varepsilon \to \chi_V \quad \text{a.e. and in } L^1(Q). \tag{6.70}$$

Take a $\mathbf{y} \in Q$ for which convergences (6.61) and (6.70) hold at $\mathbf{y}$. If $\mathbf{y} \in \mathrm{im}_\mathrm{G}(\mathbf{u}, \Omega)$, applying (6.61), for all sufficiently small ε we have that $\mathbf{y} \in \mathrm{im}_\mathrm{G}(\mathbf{u}_\varepsilon, F_\varepsilon)$. The definition of F_ε shows that $w_\varepsilon(\mathbf{y}) \geq \delta_1$, and, due to (6.70) we must have $w_\varepsilon(\mathbf{y}) \to 1$ and $\mathbf{y} \in V$. Let now $\mathbf{y} \notin \mathrm{im}_\mathrm{G}(\mathbf{u}, \Omega)$. Applying (6.61), for all sufficiently small ε we have that $\mathbf{y} \notin \mathrm{im}_\mathrm{G}(\mathbf{u}_\varepsilon, F_\varepsilon)$. If $\mathbf{y} \notin \mathrm{im}_\mathrm{G}(\mathbf{u}_\varepsilon, \Omega)$ then $w_\varepsilon(\mathbf{y}) = 0$ because of (4.10), whereas if $\mathbf{y} \in \mathrm{im}_\mathrm{G}(\mathbf{u}_\varepsilon, \Omega \setminus F_\varepsilon)$ then $w_\varepsilon(\mathbf{y}) \leq \delta_2$. In either case, due to (6.70), necessarily $w_\varepsilon(\mathbf{y}) \to 0$ and $\mathbf{y} \notin V$. This shows that $\chi_{\mathrm{im}_\mathrm{G}(\mathbf{u}, \Omega)} = \chi_V$ a.e. in Q and concludes the proof. □

It is clear that Propositions 6.2, 6.5 and 6.8 complete the proof of Theorem 6.1.

7 Upper bound

In this section we prove the upper bound inequality for some particular but illustrating cases. For simplicity, and to underline the main ideas of the constructions, we assume the space dimension n to be 2. This is mainly a simplification for the notation, since the deformations considered enjoy many symmetries that lend themselves to natural n-dimensional versions. Moreover, we assume that the stored-energy function $W : \mathbb{R}^{2\times 2}_+ \to [0, \infty]$ depends only on the deformation gradient, and there exist $c_1 > 0$, $p_1, p_2 \geq 1$, and a continuous function $h : (0, \infty) \to [0, \infty)$ satisfying

($\bar{\mathrm{W}}$1) $W(\mathbf{F}) \leq c_1 |\mathbf{F}|^{p_1} + h(\det \mathbf{F})$ for all $\mathbf{F} \in \mathbb{R}^{2\times 2}_+$,

($\bar{\mathrm{W}}$2) $\displaystyle\limsup_{t \to \infty} \frac{h(t)}{t^{p_2}} < \infty$, and

($\bar{\mathrm{W}}$3) for every $\alpha_0 > 1$ there exists $C(\alpha_0) > 0$ such that $h(\alpha t) \leq C(\alpha_0)(h(t) + 1)$ for all $\alpha \in (\alpha_0^{-1}, \alpha_0)$ and all $t \in (0, \infty)$.

Assumptions ($\bar{\mathrm{W}}$1)–($\bar{\mathrm{W}}$2) are somehow the upper bound counterpart of assumption (W2) of Section 4. Assumption ($\bar{\mathrm{W}}$3) does not have an analogue in the lower bound inequality, and it is used here to conclude that if the determinant of the gradient of two deformations are similar, then their energies are also similar. It allows, for example, a polynomial or a logarithmic growth of W in $\det \mathbf{F}$.

Since our main motivation is the study of cavitation and fracture, the deformations $\mathbf{u}$ chosen for the analysis present cavitation and fracture of various types. For those deformations, we prove that for each ε there exists $(\mathbf{u}_\varepsilon, v_\varepsilon, w_\varepsilon) \in \mathcal{A}_\varepsilon$ such that (6.5) holds and

$$\int_\Omega W(\nabla \mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x} + \lambda_1 \left[\mathcal{H}^1(J_\mathbf{u}) + \mathcal{H}^1\left(\{\mathbf{x} \in \partial_D\Omega : \mathbf{u}(\mathbf{x}) \neq \mathbf{u}_0(\mathbf{x})\}\right) + \frac{1}{2}\mathcal{H}^1(\partial_N\Omega)\right] \\ + \lambda_2 \left[\operatorname{Per}\operatorname{im_G}(\mathbf{u}, \Omega) + 2\,\mathcal{H}^1(J_{\mathbf{u}^{-1}})\right] = \lim_{\varepsilon \to 0} I_\varepsilon(\mathbf{u}_\varepsilon, v_\varepsilon, w_\varepsilon). \tag{7.1}$$

The calculations leading to (7.1) are lengthy, and will only be sketched. It is also cumbersome to check that each element $(\mathbf{u}_\varepsilon, v_\varepsilon, w_\varepsilon)$ of the recovery sequence actually belongs to $\mathcal{A}_\varepsilon$, so the proof of this is left to the reader. Moreover, in the constructions of this section, the container sets K and Q (see Section 4) do not play an essential role, so we will not specify them.

For convenience, the notation of (6.3) will be further simplified. Since the functionals I_ε^E, I_ε^V and I_ε^W will always be evaluated at $(\mathbf{u}_\varepsilon, v_\varepsilon)$, v_ε and w_ε, respectively, for any measurable sets $A \subset \Omega$ and $B \subset Q$, the quantities $I_\varepsilon^E(\mathbf{u}_\varepsilon, v_\varepsilon; A)$, $I_\varepsilon^V(v_\varepsilon; A)$ and $I_\varepsilon^W(w_\varepsilon; B)$ will be simply denoted by $I_\varepsilon^E(A)$, $I_\varepsilon^V(A)$ and $I_\varepsilon^W(B)$, respectively.

This section has the following parts. In Subsection 7.1 we construct the optimal profile for the phase-field functions v_ε and w_ε to vary from 0 to 1. Subsection 7.2 reviews some well-known concepts and formulas related to curves in the plane. In Subsections 7.3–7.6 we construct the recovery sequence for four particular deformations, each of them with a specific kind of singularity: a cavity, a crack on the boundary, an interior crack and a crack joining two cavities. All constructions follow the same general lines, which are explained in Subsection 7.3 and then adapted in Subsections 7.4–7.6.

7.1 Optimal profile of the transition layer

We introduce the functions that will give the optimal profile for v_ε and w_ε to go from 0 to 1. The construction is purely one-dimensional, so that v_ε and w_ε will only depend on the distance to the singular set through a function called, respectively, $\sigma_{\varepsilon,V}$ and $\sigma_{\varepsilon,W}$. These functions solve an ordinary differential equation, which is presented in this subsection, and determine the optimal transition, in terms of energy, of going from 0 to 1. The construction is standard and goes back to Modica & Mortola [45] for the approximation of the perimeter; it was then used by Ambrosio & Tortorelli [8] for the approximation of the fracture term.

We start using the fundamental theorem of Calculus: as $1 < q' < 2$ the function

$$s \mapsto \int_0^s \frac{1}{(1-\xi)^{q'-1}}\,\mathrm{d}\xi$$

is a homeomorphism from $[0,1]$ onto $[0, \int_0^1 \frac{\mathrm{d}\xi}{(1-\xi)^{q'-1}}]$. Its inverse σ_V is of class C^1 and, by definition,

$$\sigma_V^{-1}(s) = \int_0^s \frac{1}{(1-\xi)^{q'-1}}\,\mathrm{d}\xi, \qquad s \in [0,1].$$

Analogously, there exists a homeomorphism σ_W from $[0, \int_0^1 \frac{\mathrm{d}\xi}{\xi^{q'-1}(1-\xi)^{q'-1}}]$ onto $[0,1]$ of class C^1 such that

$$\sigma_W^{-1}(s) = \int_0^s \frac{1}{\xi^{q'-1}(1-\xi)^{q'-1}}\,\mathrm{d}\xi, \qquad s \in [0,1].$$

We note that σ_V and σ_V^{-1} can be given a closed-form expression, but not σ_W or σ_W^{-1}. Notice that

$$\sigma_V(0) = 0, \quad \sigma_V' = (1-\sigma_V)^{q'-1}, \qquad \sigma_W(0) = 0, \quad \sigma_W' = \sigma_W^{q'-1}(1-\sigma_W)^{q'-1}. \tag{7.2}$$

As an aside, we mention that the initial value problem satisfied by σ_W (the last two equations of (7.2)) does not enjoy uniqueness, since the nonlinearity is not Lipschitz. In fact, the function σ_W thus constructed is the maximal solution of those satisfying the initial value problem.

For each ε, define $\sigma_{\varepsilon,V} : [0, \varepsilon\sigma_V^{-1}(1)] \to [0,1]$ and $\sigma_{\varepsilon,W} : [0, \varepsilon\sigma_W^{-1}(1)] \to [0,1]$ as

$$\sigma_{\varepsilon,V}(t) := \sigma_V\left(\frac{t}{\varepsilon}\right), \qquad \sigma_{\varepsilon,W}(t) := \sigma_W\left(\frac{t}{\varepsilon}\right).$$

Both $\sigma_{\varepsilon,V}$ and $\sigma_{\varepsilon,W}$ are homeomorphisms of class C^1 such that

$$\sigma_{\varepsilon,V}^{-1}(s) = \varepsilon\sigma_V^{-1}(s), \quad \sigma_{\varepsilon,W}^{-1}(s) = \varepsilon\sigma_W^{-1}(s), \qquad 0 \le s \le 1.$$

In particular,

$$\sigma_{\varepsilon,V}^{-1}(1) \approx \sigma_{\varepsilon,W}^{-1}(1) \approx \varepsilon. \tag{7.3}$$

Moreover, by (7.2),

$$\sigma_{\varepsilon,V}(0) = 0, \quad \sigma'_{\varepsilon,V} = \frac{(1-\sigma_{\varepsilon,V})^{q'-1}}{\varepsilon}, \qquad \sigma_{\varepsilon,W}(0) = 0, \quad \sigma'_{\varepsilon,W} = \frac{\sigma_{\varepsilon,W}^{q'-1}(1-\sigma_{\varepsilon,W})^{q'-1}}{\varepsilon}. \tag{7.4}$$

7.2 Some notation about curves

We recall some definitions and facts about plane curves. Given $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$, we define $\mathbf{a} \wedge \mathbf{b}$ as the determinant of the matrix $(\mathbf{a}, \mathbf{b})$ whose columns are $\mathbf{a}$ and $\mathbf{b}$. The matrix $\binom{\mathbf{a}}{\mathbf{b}}$ has rows $\mathbf{a}$ and $\mathbf{b}$. We define $\mathbf{a}^\perp := (-a_2, a_1)$ whenever $\mathbf{a} = (a_1, a_2)$. Note that

$$\mathbf{a} \wedge \mathbf{b} = \mathbf{a}^\perp \cdot \mathbf{b} = -\mathbf{a} \cdot \mathbf{b}^\perp = \mathbf{a}^\perp \wedge \mathbf{b}^\perp \qquad \text{and} \qquad (\mathbf{a}, \mathbf{b})^{-1} = \frac{1}{\mathbf{a} \wedge \mathbf{b}} \begin{pmatrix} -\mathbf{b}^\perp \\ \mathbf{a}^\perp \end{pmatrix}.$$

Let Θ be a C^2 differentiable manifold of dimension 1, and let $\bar{\mathbf{u}} \in C^{1,1}(\Theta, \mathbb{R}^2)$ satisfy $\bar{\mathbf{u}}'(\theta) \neq \mathbf{0}$ for all $\theta \in \Theta$. The *normal* $\boldsymbol{\nu} \in C^{0,1}(\Theta, \mathbb{S}^1)$ to $\bar{\mathbf{u}}$ and the *signed curvature* $\kappa : \Theta \to \mathbb{R}$ of $\bar{\mathbf{u}}$ are defined as

$$\boldsymbol{\nu} := -\frac{(\bar{\mathbf{u}}')^\perp}{|\bar{\mathbf{u}}'|}, \qquad \kappa := \frac{\bar{\mathbf{u}}' \wedge \bar{\mathbf{u}}''}{|\bar{\mathbf{u}}'|^3}. \tag{7.5}$$

The following identities hold a.e.:

$$\boldsymbol{\nu} \cdot \boldsymbol{\nu}' = 0, \quad \boldsymbol{\nu} \wedge \bar{\mathbf{u}}' = |\bar{\mathbf{u}}'|, \quad \boldsymbol{\nu}' = -\frac{1}{|\bar{\mathbf{u}}'|}(\bar{\mathbf{u}}'')^\perp - \frac{\bar{\mathbf{u}}' \cdot \bar{\mathbf{u}}''}{|\bar{\mathbf{u}}'|^2}\boldsymbol{\nu}, \quad \frac{\bar{\mathbf{u}}' \cdot \boldsymbol{\nu}'}{|\bar{\mathbf{u}}'|^2} = \frac{\boldsymbol{\nu} \wedge \boldsymbol{\nu}'}{|\bar{\mathbf{u}}'|} = \kappa, \quad |\boldsymbol{\nu}'| = |\bar{\mathbf{u}}'|\,|\kappa|. \tag{7.6}$$

Given an interval I and a differentiable function $g : I \to \mathbb{R}$, we consider the function

$$\mathbf{Y} : I \times \Theta \to \mathbb{R}^2, \qquad \mathbf{Y}(t, \theta) := \bar{\mathbf{u}}(\theta) + g(t)\,\boldsymbol{\nu}(\theta),$$

and find the gradient of its inverse $\mathbf{y} \mapsto (t, \theta)$ by writing Dt and $D\theta$ as a linear combination of $\frac{\bar{\mathbf{u}}'}{|\bar{\mathbf{u}}'|}$ and $\boldsymbol{\nu}$ and solving the linear system

$$\begin{cases} Dt \cdot \frac{\partial \mathbf{Y}}{\partial t} = 1, & Dt \cdot \frac{\partial \mathbf{Y}}{\partial \theta} = 0, \\ D\theta \cdot \frac{\partial \mathbf{Y}}{\partial t} = 0, & D\theta \cdot \frac{\partial \mathbf{Y}}{\partial \theta} = 1, \end{cases}$$

which yields

$$Dt = \frac{1}{g'(t)}\boldsymbol{\nu}, \qquad D\theta = \frac{1}{|\bar{\mathbf{u}}'|\,(1 + g(t)\kappa)} \frac{\bar{\mathbf{u}}'}{|\bar{\mathbf{u}}'|}. \tag{7.7}$$

We also have that

$$\frac{\partial \mathbf{Y}}{\partial t} = g'(t)\,\boldsymbol{\nu}(\theta), \qquad \frac{\partial \mathbf{Y}}{\partial \theta} = \bar{\mathbf{u}}'(\theta) + g(t)\,\boldsymbol{\nu}'(\theta), \qquad \frac{\partial \mathbf{Y}}{\partial t} \wedge \frac{\partial \mathbf{Y}}{\partial \theta} = g'(t)\,|\bar{\mathbf{u}}'(\theta)|\,(1 + g(t)\kappa(\theta)). \tag{7.8}$$

7.3 Cavitation

We consider a typical deformation creating a cavity. Let Θ be the differentiable manifold defined as the topological quotient space obtained from $[-\pi, \pi]$ with the identification $-\pi \sim \pi$, and note that Θ is diffeomorphic to $\mathbb{S}^1$. Functions defined on Θ will be identified with 2π-periodic functions defined on $\mathbb{R}$, in the obvious way. We assume the existence of a homeomorphism $\mathbf{u}_0$ as in Section 4. Moreover, Ω is a Lipschitz domain containing $\gamma := \{\mathbf{0}\}$, we take $\partial_D\Omega = \partial\Omega$ and $p_1 < 2$. Suppose, further, that:

(D1) $\mathbf{u} \in C^{1,1}(\bar{\Omega} \setminus \gamma, \mathbb{R}^2)$ is one-to-one in $\bar{\Omega} \setminus \gamma$, satisfies $\det \nabla\mathbf{u} > 0$ a.e. in Ω, and

$$\int_\Omega [|D\mathbf{u}|^{p_1} + h(\det D\mathbf{u})]\,\mathrm{d}\mathbf{x} < \infty. \tag{7.9}$$

(D2) There exist $\rho \in C^{1,1}(\Theta, (0,\infty))$ and $\varphi \in C^{1,1}(\mathbb{R})$ with $\varphi' > 0$ and $\varphi(\cdot + 2\pi) - \varphi(\cdot) = 2\pi$ such that, when we define $\bar{\mathbf{u}} : \Theta \to \mathbb{R}^2$ as $\bar{\mathbf{u}}(\theta) := \rho(\theta)e^{i\varphi(\theta)}$, we have that

$$\lim_{t\to 0^+} \sup_{\theta\in\Theta} \left|\mathbf{u}(te^{i\theta}) - \bar{\mathbf{u}}(\theta)\right| = 0.$$

(D3) $\bar{\mathbf{u}}$ is a Jordan curve, and $\mathbf{u}(\bar{\Omega} \setminus \gamma)$ lies on the unbounded component of $\mathbb{R}^2 \setminus \bar{\mathbf{u}}(\Theta)$.

(D4) $\limsup_{t\to 0^+} \sup_{\theta\in\Theta} \left(\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}(te^{i\theta})\right| + \left|\frac{\mathrm{d}}{\mathrm{d}\theta}\mathbf{u}(te^{i\theta})\right| \right) < \infty$.

(D5) The inverse of $\mathbf{u}$ has a continuous extension $\mathbf{v} : \overline{\mathbf{u}(\Omega \setminus \gamma)} \to \bar{\Omega}$.

The reader can check that a typical deformation creating a cavity at γ satisfies indeed assumptions (D1)–(D5), the only artificial assumption may be (D2), which implies that the cavity is star-shaped. Note, in particular, that the assumptions imply that $\mathbf{u} \in W^{1,p_1}(\Omega, \mathbb{R}^2)$, $\mathcal{H}^1(J_{\mathbf{u}^{-1}}) = 0$ and $\operatorname{im}_{\mathrm{G}}(\mathbf{u}, \Omega) = \mathbf{u}(\Omega \setminus \gamma)$ a.e.

For the approximated functional I_ε and the admissible set $\mathcal{A}_\varepsilon$, the sequences $\{\eta_\varepsilon\}_\varepsilon$ and $\{b_\varepsilon\}_\varepsilon$ of (6.1)–(6.2) are chosen to satisfy

$$\eta_\varepsilon \ll \varepsilon^{p_2-1} \quad \text{and} \quad \varepsilon \ll b_\varepsilon. \tag{7.10}$$

Under these assumptions, the following result holds. We remark that the notation of the proof is chosen so that some of its parts can be used for the constructions of Subsections 7.4–7.6.

Proposition 7.1. *For each ε there exists $(\mathbf{u}_\varepsilon, v_\varepsilon, w_\varepsilon) \in \mathcal{A}_\varepsilon$ satisfying* (6.5) *and* (7.1).

Sketch of proof. The construction requires five steps, which will correspond to five independent zones Z_1^ε–Z_5^ε in the domain Ω. These zones follow one another in order of increasing distance $t = |\mathbf{x}|$ to the singular set γ.

Let $\{a_\varepsilon\}_\varepsilon$ be any sequence such that

$$\eta_\varepsilon \ll a_\varepsilon^{2p_2-2}, \qquad a_\varepsilon \ll \varepsilon^{\frac{1}{2}}, \tag{7.11}$$

which is possible thanks to (7.10). Introduce the auxiliary function

$$f_\varepsilon : [a_\varepsilon, \infty) \to [0,\infty), \qquad f_\varepsilon(t) := t^2 - a_\varepsilon^2. \tag{7.12}$$

The values of t at which one zone ends and the other begins are

$$a_\varepsilon, \qquad a_{\varepsilon,V} := a_\varepsilon + \sigma_{\varepsilon,V}^{-1}(1), \qquad a_{\varepsilon,W} := f_\varepsilon^{-1}\Big(f_\varepsilon(a_{\varepsilon,V}) + \sigma_{\varepsilon,W}^{-1}(1)\Big), \qquad 2a_{\varepsilon,W}. \tag{7.13}$$

More precisely,

$$\begin{aligned} &Z_1^\varepsilon := \{\mathbf{x} \in \Omega : \operatorname{dist}(\mathbf{x},\gamma) < a_\varepsilon\}, \quad Z_2^\varepsilon := \{\mathbf{x} : a_\varepsilon \le \operatorname{dist}(\mathbf{x},\gamma) < a_{\varepsilon,V}\}, \\ &Z_3^\varepsilon := \{\mathbf{x} : a_{\varepsilon,V} \le \operatorname{dist}(\mathbf{x},\gamma) < a_{\varepsilon,W}\}, \quad Z_4^\varepsilon := \{\mathbf{x} : a_{\varepsilon,W} \le \operatorname{dist}(\mathbf{x},\gamma) < 2a_{\varepsilon,W}\}, \quad Z_5^\varepsilon := \Omega \setminus \bigcup_{i=1}^4 Z_i^\varepsilon. \end{aligned} \tag{7.14}$$

Thanks to (7.3) and (7.11), we have that $a_{\varepsilon,V} \approx \max\{a_\varepsilon, \varepsilon\}$ and $a_{\varepsilon,W} \approx \varepsilon^{\frac{1}{2}}$.

Step 1: regularization of $\mathbf{u}$. It is in Z_1^ε where the singularity of $\mathbf{u}$ at γ is smoothed out, so that $\mathbf{u}_\varepsilon$ fills the hole created by $\mathbf{u}$. More precisely, we set

$$\mathbf{X}(t,\theta) := t\,e^{i\theta}, \quad \mathbf{u}_\varepsilon(\mathbf{X}(t,\theta)) := \frac{t}{a_\varepsilon}\,\bar{\mathbf{u}}(\theta), \quad v_\varepsilon(\mathbf{X}(t,\theta)) := 0, \quad w_\varepsilon(\mathbf{u}_\varepsilon(\mathbf{X}(t,\theta))) := 0, \qquad (t,\theta) \in [0, a_\varepsilon) \times \Theta. \tag{7.15}$$

The reason why $v_\varepsilon = 0$ in Z_1^ε is that $\det D\mathbf{u}_\varepsilon$ is roughly the area of the cavity (of order 1) divided by the area of Z_1^ε (of order a_ε^{-2}), so $\det D\mathbf{u}_\varepsilon \approx a_\varepsilon^{-2}$, and $W(\mathbf{F})$ normally grows superlinearly in $\det \mathbf{F}$; it is thus necessary that $v_\varepsilon = 0$ so as to make $I_\varepsilon^E(Z_1^\varepsilon)$ small. The precise calculations are

$$D\mathbf{u}_\varepsilon(\mathbf{X}(t,\theta)) = \frac{\mathrm{d}\mathbf{u}_\varepsilon}{\mathrm{d}t} \otimes Dt + \frac{\mathrm{d}\mathbf{u}_\varepsilon}{\mathrm{d}\theta} \otimes D\theta, \qquad \begin{pmatrix} Dt \\ D\theta \end{pmatrix} = \left(\frac{\partial \mathbf{X}}{\partial t}, \frac{\partial \mathbf{X}}{\partial \theta} \right)^{-1} = \frac{1}{\frac{\partial \mathbf{X}}{\partial t} \wedge \frac{\partial \mathbf{X}}{\partial \theta}} \begin{pmatrix} -\frac{\partial \mathbf{X}}{\partial \theta}^\perp \\ \frac{\partial \mathbf{X}}{\partial t}^\perp \end{pmatrix}. \tag{7.16}$$

From (7.15), we find that

$$\frac{\partial \mathbf{X}}{\partial t} = e^{i\theta}, \quad \frac{\partial \mathbf{X}}{\partial \theta} = t\,ie^{i\theta}, \quad \frac{\partial \mathbf{X}}{\partial t} \wedge \frac{\partial \mathbf{X}}{\partial \theta} = t, \quad \frac{\mathrm{d}\mathbf{u}_\varepsilon}{\mathrm{d}t} = \frac{1}{a_\varepsilon}\,\bar{\mathbf{u}}, \quad \frac{\mathrm{d}\mathbf{u}_\varepsilon}{\mathrm{d}\theta} = \frac{t}{a_\varepsilon}\,\bar{\mathbf{u}}', \quad \frac{\mathrm{d}\mathbf{u}_\varepsilon}{\mathrm{d}t} \wedge \frac{\mathrm{d}\mathbf{u}_\varepsilon}{\mathrm{d}\theta} = \frac{t}{a_\varepsilon^2}\,\bar{\mathbf{u}} \wedge \bar{\mathbf{u}}', \tag{7.17}$$

so $Dt = e^{i\theta}$ and $D\theta = t^{-1} ie^{i\theta}$. Consequently, using (7.16)–(7.17) as well,

$$\left| D\mathbf{u}_\varepsilon(te^{i\theta}) \right| \lessapprox a_\varepsilon^{-1} + t a_\varepsilon^{-1} t^{-1} \approx a_\varepsilon^{-1}. \tag{7.18}$$

On the other hand, considering that

$$\frac{\mathrm{d}\mathbf{u}_\varepsilon}{\mathrm{d}t} \wedge \frac{\mathrm{d}\mathbf{u}_\varepsilon}{\mathrm{d}\theta} = \left((D\mathbf{u}_\varepsilon) \frac{\partial \mathbf{X}}{\partial t} \right) \wedge \left((D\mathbf{u}_\varepsilon) \frac{\partial \mathbf{X}}{\partial \theta} \right) = \det D\mathbf{u}_\varepsilon \left(\frac{\partial \mathbf{X}}{\partial t} \wedge \frac{\partial \mathbf{X}}{\partial \theta} \right), \tag{7.19}$$

we find from (7.17) and (D2) that $\det D\mathbf{u}_\varepsilon = a_\varepsilon^{-2} \bar{\mathbf{u}} \wedge \bar{\mathbf{u}}' = a_\varepsilon^{-2} \rho^2 \varphi'$, so

$$\det D\mathbf{u}_\varepsilon \approx a_\varepsilon^{-2}. \tag{7.20}$$

Using ($\bar{\mathrm{W}}$1)–($\bar{\mathrm{W}}$2), (7.18) and (7.20) we find that

$$W(D\mathbf{u}_\varepsilon) \lessapprox |D\mathbf{u}_\varepsilon|^{p_1} + (\det D\mathbf{u}_\varepsilon)^{p_2} \lessapprox a_\varepsilon^{-p_1} + a_\varepsilon^{-2p_2} \lessapprox a_\varepsilon^{-2p_2}.$$

Therefore, thanks to (7.11) we conclude that

$$I_\varepsilon^E(Z_1^\varepsilon) \lessapprox \eta_\varepsilon\, a_\varepsilon^{-2p_2}\, \mathcal{L}^2(Z_1^\varepsilon) \approx \eta_\varepsilon\, a_\varepsilon^{2-2p_2} \ll 1, \qquad I_\varepsilon^V(Z_1^\varepsilon) \approx \varepsilon^{-1}\, \mathcal{L}^2(Z_1^\varepsilon) \approx \varepsilon^{-1}\, a_\varepsilon^2 \ll 1, \qquad I_\varepsilon^W(\mathbf{u}_\varepsilon(Z_1^\varepsilon)) = 0.$$

Step 2: transition of v_ε from 0 *to* 1. It is very expensive for v to be equal to zero, hence we set

$$v_\varepsilon(\mathbf{x}) := \begin{cases} \sigma_{\varepsilon,V}(t(\mathbf{x}) - a_\varepsilon), & \text{if } a_\varepsilon \le t(\mathbf{x}) < a_{\varepsilon,V}, \\ 1, & \text{if } t(\mathbf{x}) \ge a_{\varepsilon,V}, \end{cases} \tag{7.21}$$

which satisfies

$$|Dv_\varepsilon(\mathbf{x})| = \sigma_{\varepsilon,V}'(t(\mathbf{x}) - a_\varepsilon), \quad \text{if } a_\varepsilon \le t(\mathbf{x}) < a_{\varepsilon,V}.$$

Since

$$ab = \frac{a^q}{q} + \frac{b^{q'}}{q'} \quad \text{whenever } a, b \ge 0 \text{ with } a^q = b^{q'} \tag{7.22}$$

and (7.4) holds, we have that

$$\frac{\left(\varepsilon^{1-\frac{1}{q}}\, |Dv_\varepsilon| \right)^q}{q} + \frac{\left(\varepsilon^{-\frac{1}{q'}}\, (1 - v_\varepsilon) \right)^{q'}}{q'} = |Dv_\varepsilon|\,(1 - v_\varepsilon)\,. \tag{7.23}$$

Consequently, thanks to the coarea formula (2.2),

$$I_\varepsilon^V(\Omega\setminus Z_1^\varepsilon)=\int_0^1(1-s)\,\mathcal{H}^1(\{\mathbf{x}\in Z_2^\varepsilon : v_\varepsilon(\mathbf{x})=s\})\,\mathrm{d}s=\int_0^1(1-s)\,2\pi\,(a_\varepsilon+\sigma_{\varepsilon,V}^{-1}(s))\,\mathrm{d}s\ll 1. \tag{7.24}$$

Step 3: transition of w_ε *from* 0 *to* 1. In $Z_2^\varepsilon\cup Z_3^\varepsilon$ we are not able to construct $\mathbf{u}_\varepsilon$ as a close approximation of $\mathbf{u}$. Instead, we define

$$\mathbf{u}_\varepsilon(\mathbf{X}(t,\theta)):=\mathbf{Y}(f_\varepsilon(t),\theta),\quad (t,\theta)\in[a_\varepsilon,a_{\varepsilon,W})\times\Theta;\qquad \mathbf{Y}(\tau,\theta):=\bar{\mathbf{u}}(\theta)+\tau\boldsymbol{\nu}(\theta),\quad \tau\ge 0, \tag{7.25}$$

with f_ε and $\boldsymbol{\nu}$ as in (7.12) and (7.5). This definition is partly motivated be the explicit construction of incompressible angle-preserving maps in [41, Sect. 4]. In this way, the deformation $\mathbf{u}_\varepsilon$ follows the geometry of the cavity, while $\det D\mathbf{u}_\varepsilon$ remains controlled. Note that there exists $\delta_{\bar{\mathbf{u}}}>0$ such that $\mathbf{Y}$ is a homeomorphism from $[0,\delta_{\bar{\mathbf{u}}}]\times\Theta$ onto its image.

As for w_ε, we recall that $v_\varepsilon(\mathbf{x})$ was constructed as a function of the distance $t=|\mathbf{x}|$ from $\mathbf{x}$ to γ, and notice that I_ε^W is minimized when $w_\varepsilon(\mathbf{y})$ is a function of the distance from $\mathbf{y}$ to the cavity surface $\bar{\mathbf{u}}(\Theta)$. Since we want $w_\varepsilon\circ\mathbf{u}_\varepsilon$ to coincide with v_ε in a subset of Ω with almost full measure, it is convenient that the level sets of the function $\mathbf{x}\mapsto\operatorname{dist}(\mathbf{x},\gamma)$ are mapped by $\mathbf{u}_\varepsilon$ to level sets of $\mathbf{y}\mapsto\operatorname{dist}(\mathbf{y},\bar{\mathbf{u}}(\Theta))$. This is precisely the main virtue of the definition (7.25) of $\mathbf{u}_\varepsilon$.

The radial function f_ε was defined as (7.12) so as to maintain $\det D\mathbf{u}_\varepsilon$ bounded and far away from zero. Indeed, by (7.8), (7.17), (7.19) and (7.25) it can be seen that

$$\det D\mathbf{u}_\varepsilon=\frac{f_\varepsilon'(t)}{t}|\bar{\mathbf{u}}'|(1+f_\varepsilon(t)\kappa(\theta))\approx 1.$$

At the same time, (7.7), (7.8), (7.16), (7.17) and (7.25) yield $|D\mathbf{u}_\varepsilon(te^{i\theta})|\lessapprox t^{-1}$. Therefore, recalling ($\bar{\mathrm{W}}$1)–($\bar{\mathrm{W}}$2) and (7.17), and changing variables, we find that

$$I_\varepsilon^E(Z_2^\varepsilon\cup Z_3^\varepsilon)\lessapprox\int_{a_\varepsilon}^{a_{\varepsilon,W}}t^{1-p_1}\,\mathrm{d}t\approx a_{\varepsilon,W}^{2-p_1}\approx\varepsilon^{1-\frac{p_1}{2}}.$$

Due to the choice of f_ε in (7.12), the image of Z_2^ε by $\mathbf{u}_\varepsilon$ is an annular region of width $a_{\varepsilon,V}^2-a_\varepsilon^2\approx\max\{a_\varepsilon^2,\varepsilon^2\}$, where w_ε does not have enough room to do an optimal transition. This is why we let the transition of v_ε and w_ε occur independently: first v_ε in Z_2^ε, and then w_ε in $\mathbf{u}_\varepsilon(Z_3^\varepsilon)$. So we set $w_\varepsilon=0$ in $\mathbf{u}_\varepsilon(Z_2^\varepsilon)$ and

$$w_\varepsilon(\bar{\mathbf{u}}(\theta)+\tau\boldsymbol{\nu}(\theta)):=\sigma_{\varepsilon,W}(\tau-f_\varepsilon(a_{\varepsilon,V})),\qquad f_\varepsilon(a_{\varepsilon,V})\le\tau<f_\varepsilon(a_{\varepsilon,W}). \tag{7.26}$$

In order to calculate I_ε^W, first we fix $s\in(0,1)$ and observe that the level set $\{\mathbf{y}\in\mathbf{u}_\varepsilon(Z_3^\varepsilon):w_\varepsilon(\mathbf{y})=s\}$ can be parametrized by $\mathbf{y}=\bar{\mathbf{u}}(\theta)+\tau_\varepsilon(s)\boldsymbol{\nu}(\theta)$, for $\theta\in\Theta$ and $\tau_\varepsilon(s):=f_\varepsilon(a_{\varepsilon,V})+\sigma_{\varepsilon,W}^{-1}(s)\lessapprox\varepsilon$. Thus,

$$\lim_{\varepsilon\to0}\mathcal{H}^1(\{\mathbf{y}\in\mathbf{u}_\varepsilon(Z_3^\varepsilon):w_\varepsilon(\mathbf{y})=s\})=\lim_{\varepsilon\to0}\int_\Theta|\bar{\mathbf{u}}'(\theta)+\tau_\varepsilon(s)\boldsymbol{\nu}'(\theta)|\,\mathrm{d}\theta=\int_\Theta|\bar{\mathbf{u}}'(\theta)|\,\mathrm{d}\theta=\mathcal{H}^1(\bar{\mathbf{u}}(\Theta)).$$

Inverting the map $(\tau,\theta)\mapsto\mathbf{y}=\bar{\mathbf{u}}(\theta)+\tau\boldsymbol{\nu}(\theta)$ we obtain that $\tau(\mathbf{y})$ is the distance from $\mathbf{y}$ to the cavity surface $\bar{\mathbf{u}}(\Theta)$ and that $D\tau(\mathbf{y})=\boldsymbol{\nu}(\theta(\mathbf{y}))$ (see also (7.7)), hence $|Dw_\varepsilon|=\sigma_{\varepsilon,W}'(\tau)$. Using (7.22) and the differential equation (7.4) for $\sigma_{\varepsilon,W}$, we find, in an analogous calculation to that of (7.23)–(7.24), that

$$\lim_{\varepsilon\to0}I_\varepsilon^W(\mathbf{u}(Z_\varepsilon^3))=\left(\int_0^1 s(1-s)\,\mathrm{d}s\right)\mathcal{H}^1(\bar{\mathbf{u}}(\Theta))=\frac{1}{6}\mathcal{H}^1(\bar{\mathbf{u}}(\Theta)). \tag{7.27}$$

Step 4: back to the original deformation. In the fourth zone, $\mathbf{u}_\varepsilon$ must find a way to attain all the material points in $\mathbf{u}(Z_1^\varepsilon\cup Z_2^\varepsilon\cup Z_3^\varepsilon\cup Z_4^\varepsilon)$ using only those points in Z_4^ε. The resulting map $\mathbf{u}_\varepsilon$ needs to be continuous at the interface between Z_3^ε and Z_4^ε, and the regions $\mathbf{u}_\varepsilon(Z_2^\varepsilon \cup Z_3^\varepsilon)$ and $\mathbf{u}_\varepsilon(Z_4^\varepsilon)$ must not overlap. To this end, we introduce the auxiliary functions

$$\mathbf{G}_\varepsilon(\bar{\mathbf{u}}(\theta)+\tau\boldsymbol{\nu}(\theta)) := \begin{cases} \bar{\mathbf{u}}(\theta)+(f_\varepsilon(a_{\varepsilon,W})+\tau/2)\boldsymbol{\nu}(\theta), & 0 \le \tau \le 2f_\varepsilon(a_{\varepsilon,W}), \\ \bar{\mathbf{u}}(\theta)+\tau\boldsymbol{\nu}(\theta), & \tau \ge 2f_\varepsilon(a_{\varepsilon,W}), \end{cases} \tag{7.28}$$

and

$$\mathbf{F}_\varepsilon(\mathbf{X}(t,\theta)) := \mathbf{X}(r(t),\theta), \qquad r(t) := \begin{cases} \frac{2}{\sqrt{3}}\sqrt{t^2-a_{\varepsilon,W}^2}, & a_{\varepsilon,W} < t < 2a_{\varepsilon,W}, \\ t, & t \ge 2a_{\varepsilon,W}. \end{cases} \tag{7.29}$$

For any $a > 2f_\varepsilon(a_{\varepsilon,W})$, the function $\mathbf{G}_\varepsilon$ retracts $\mathbf{Y}([0,a]\times\Theta)$ onto $\mathbf{Y}([f_\varepsilon(a_{\varepsilon,W}),a]\times\Theta)$, while $\mathbf{F}_\varepsilon$ expands $\{\mathbf{x} : \operatorname{dist}(\mathbf{x},\gamma) > a_{\varepsilon,W}\}$ onto $\{\mathbf{x} : \operatorname{dist}(\mathbf{x},\gamma) > 0\}$. Moreover, $\mathbf{G}_\varepsilon = \mathbf{id}$ in $\mathbf{Y}([2f_\varepsilon(a_{\varepsilon,W}),\infty)\times\Theta)$ and $\mathbf{F}_\varepsilon = \mathbf{id}$ in Z_5^ε. Define $\mathbf{u}_\varepsilon := \mathbf{G}_\varepsilon \circ \mathbf{u} \circ \mathbf{F}_\varepsilon$ in $Z_4^\varepsilon \cup Z_5^\varepsilon$. Note that $\mathbf{u}_\varepsilon = \mathbf{u}$ in Z_5^ε, and that, thanks to (D2), $\mathbf{u}_\varepsilon$ is continuous on $\bar{Z}_3^\varepsilon \cap \bar{Z}_4^\varepsilon$.

As in (7.16), writing $\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}r} := \left(D\mathbf{u}\left(r(t)e^{i\theta}\right)\right)e^{i\theta}$, in region Z_4^ε we have that

$$D\mathbf{u}(r(t)e^{i\theta}) = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}r}\otimes e^{i\theta} + r^{-1}\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\theta}\otimes ie^{i\theta}, \qquad D\mathbf{F}_\varepsilon(te^{i\theta}) = r' e^{i\theta}\otimes e^{i\theta} + \frac{r}{t}ie^{i\theta}\otimes ie^{i\theta}.$$

Hence $\det D\mathbf{F}_\varepsilon = r'\frac{r}{t} = \frac{4}{3}$ and, thanks to (D4), we conclude that

$$\left|D(\mathbf{u}\circ\mathbf{F}_\varepsilon)(te^{i\theta})\right| \le r'\left|\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}r}\right| + \frac{1}{t}\left|\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\theta}\right| \lessapprox \max\{r',\frac{1}{t}\} = r' \lessapprox a_{\varepsilon,W}^{\frac{1}{2}}(t-a_{\varepsilon,W})^{-\frac{1}{2}}.$$

Analogously, the gradient of $\mathbf{G}_\varepsilon$ can be calculated as in (7.7) (with $g(\tau)=\tau$, which corresponds to the definition of $\mathbf{Y}(\tau,\theta)$ of (7.25)) and (7.16):

$$D\mathbf{G}_\varepsilon(\mathbf{Y}(\tau,\theta)) = \frac{\mathrm{d}\mathbf{G}_\varepsilon}{\mathrm{d}\tau}\otimes\boldsymbol{\nu} + \frac{1}{|\bar{\mathbf{u}}'|(1+\tau\kappa)}\frac{\mathrm{d}\mathbf{G}_\varepsilon}{\mathrm{d}\theta}\otimes\frac{\bar{\mathbf{u}}'}{|\bar{\mathbf{u}}'|},$$

hence

$$|D\mathbf{G}_\varepsilon(\mathbf{Y}(\tau,\theta))| \le \left|\frac{\mathrm{d}\mathbf{G}_\varepsilon}{\mathrm{d}\tau}\right| + \frac{1}{|\bar{\mathbf{u}}'|(1+\tau\kappa)}\left|\frac{\mathrm{d}\mathbf{G}_\varepsilon}{\mathrm{d}\theta}\right| \lessapprox 1. \tag{7.30}$$

Moreover, the analogue of (7.19) and (7.8) (applied to $g(\tau)=\tau$ in the denominator and $g(\tau)=f_\varepsilon(a_{\varepsilon,W})+\tau/2$ in the numerator) yields

$$\det D\mathbf{G}_\varepsilon = \frac{\frac{\mathrm{d}\mathbf{G}_\varepsilon}{\mathrm{d}\tau}\wedge\frac{\mathrm{d}\mathbf{G}_\varepsilon}{\mathrm{d}\theta}}{|\bar{\mathbf{u}}'|(1+\tau\kappa)} \simeq \bar{\mathbf{u}}\wedge\frac{\bar{\mathbf{u}}'}{|\bar{\mathbf{u}}'|} + \frac{1}{2} \approx 1. \tag{7.31}$$

The above calculations imply that

$$|D\mathbf{u}_\varepsilon| \lessapprox a_{\varepsilon,W}^{\frac{1}{2}}(t-a_{\varepsilon,W})^{-\frac{1}{2}}, \qquad \det D\mathbf{u}_\varepsilon(\mathbf{X}(t,\theta)) = (\det D\mathbf{G}_\varepsilon)(\det D\mathbf{u})(\det D\mathbf{F}_\varepsilon) \approx \det\nabla\mathbf{u}(\mathbf{X}(r(t),\theta)).$$

Hence, thanks to ($\bar{\mathrm{W}}$1)–($\bar{\mathrm{W}}$3),

$$W(D\mathbf{u}_\varepsilon(\mathbf{X}(t,\theta)) \lessapprox a_{\varepsilon,W}^{\frac{p_1}{2}}(t-a_{\varepsilon,W})^{-\frac{p_1}{2}} + h\left(\det D\mathbf{u}(\mathbf{X}(r(t),\theta))\right).$$

Therefore, by the last assumption in (D1), considering that $\mathcal{L}^2(\bigcup_{i=1}^4 Z_i^\varepsilon) \approx a_{\varepsilon,W}^2 \approx \varepsilon$,

$$I_\varepsilon^E(Z_4^\varepsilon) \lessapprox \int_{a_{\varepsilon,W}}^{2a_{\varepsilon,W}} a_{\varepsilon,W}^{\frac{p_1}{2}}(t-a_{\varepsilon,W})^{-\frac{p_1}{2}}\,t\,\mathrm{d}t + \frac{3}{4}\int_{\bigcup_{i=1}^4 Z_i^\varepsilon} h(\det\nabla\mathbf{u}(\mathbf{z}))\,\mathrm{d}\mathbf{z} \ll a_{\varepsilon,W}^2 + 1 \approx 1.$$

Step 5: transition of w_ε from 1 to 0 close to the outer boundary. A further transition is needed in order for w_ε to satisfy the boundary condition (4.10). Let $\boldsymbol{\nu}_Q(\mathbf{y})$ denote the unit normal to $\mathbf{y}\in\mathbf{u}_0(\partial\Omega)$ pointing towards $\mathbb{R}^2\setminus\mathbf{u}(\Omega\setminus\gamma)$. Call also

$$Y_\varepsilon := \{\mathbf{y}-\tau\boldsymbol{\nu}_Q(\mathbf{y}) : \mathbf{y}\in\mathbf{u}_0(\partial\Omega),\, 0\le\tau\le\sigma_{\varepsilon,W}^{-1}(1)\} \tag{7.32}$$

Set $w_\varepsilon = 1$ in $\mathbf{u}_\varepsilon(Z_4^\varepsilon \cup Z_5^\varepsilon) \setminus Y_\varepsilon$ and

$$w_\varepsilon(\mathbf{y} - \tau \boldsymbol{\nu}_Q(\mathbf{y})) := \sigma_{\varepsilon,W}(\tau), \quad 0 \le \tau \le \sigma_{\varepsilon,W}^{-1}(1). \tag{7.33}$$

Proceeding as in the argument leading to (7.27), one can show that

$$\lim_{\varepsilon \to 0} I_\varepsilon^W(Y_\varepsilon) = \frac{1}{6}\mathcal{H}^1(\mathbf{u}(\partial\Omega)). \tag{7.34}$$

Concluding remarks. Based on the results obtained, it can be seen that $(\mathbf{u}_\varepsilon, v_\varepsilon, w_\varepsilon)$ fulfils the conclusion of the proposition. Here we will show only that $\partial \operatorname{im}_{\mathrm{G}}(\mathbf{u}, \Omega) = \bar{\mathbf{u}}(\Theta) \cup \mathbf{u}_0(\partial\Omega)$. First note that for all $\theta \in \Theta$,

$$\mathbf{v}(\bar{\mathbf{u}}(\theta)) = \mathbf{v}\left(\lim_{r \to 0} \mathbf{u}(re^{i\theta})\right) = \lim_{r \to 0} \mathbf{v}(\mathbf{u}(re^{i\theta})) = \lim_{r \to 0} re^{i\theta} = \mathbf{0}.$$

It follows from (D2) that $\bar{\mathbf{u}}(\Theta) \subset \overline{\mathbf{u}(\Omega \setminus \gamma)}$. Moreover, $\bar{\mathbf{u}}(\Theta) \cap \mathbf{u}(\Omega \setminus \gamma) = \varnothing$, since otherwise there would exist $\mathbf{y} \in \bar{\mathbf{u}}(\Theta)$ and $\mathbf{x} \in \Omega \setminus \{\mathbf{0}\}$ such that $\mathbf{y} = \mathbf{u}(\mathbf{x})$; as seen before, $\mathbf{v}(\mathbf{y}) = \mathbf{0}$, but on the other hand, $\mathbf{v}(\mathbf{y}) = \mathbf{v}(\mathbf{u}(\mathbf{x})) = \mathbf{x}$, which is a contradiction. Therefore,

$$\bar{\mathbf{u}}(\Theta) \subset \overline{\mathbf{u}(\Omega \setminus \gamma)} \setminus \mathbf{u}(\Omega \setminus \gamma) = \partial \mathbf{u}(\Omega \setminus \gamma),$$

the latter equality being due to the invariance of domain theorem. It is easy to see that $\mathbf{u}_0(\partial\Omega)$ is also contained in $\partial\mathbf{u}(\Omega \setminus \gamma)$, since every $\mathbf{x} \in \partial\Omega$ is the limit of a sequence $\{\mathbf{x}_j\}_{j\in\mathbb{N}} \subset \Omega$, $\mathbf{u}_0(\mathbf{x}) = \mathbf{u}(\mathbf{x})$, and $\mathbf{u} : \bar{\Omega} \setminus \gamma \to \mathbb{R}^2$ is continuous and injective.

Conversely, let $\mathbf{y} \in \partial\mathbf{u}(\Omega \setminus \gamma)$. Then there exist a sequence $\{\mathbf{x}_j\}_{j\in\mathbb{N}}$ in $\Omega \setminus \gamma$ converging to some $\mathbf{x} \in \overline{\Omega}$ such that $\mathbf{y} = \lim_{j\to\infty} \mathbf{u}(\mathbf{x}_j)$. Since $\partial\mathbf{u}(\Omega \setminus \gamma) \cap \mathbf{u}(\Omega \setminus \gamma) = \varnothing$, necessarily $\mathbf{x} \in \{\mathbf{0}\} \cup \partial\Omega$. If $\mathbf{x} \in \partial\Omega$, then $\mathbf{y} \in \mathbf{u}_0(\partial\Omega)$ since $\mathbf{u} : \bar{\Omega} \setminus \gamma \to \mathbb{R}^2$ is continuous. If $\mathbf{x} = \mathbf{0}$ then $r_j := |\mathbf{x}_j| \to 0$ as $j \to \infty$. For each $j \in \mathbb{N}$ let $\theta_j \in \Theta$ be such that $\mathbf{x}_j = r_j e^{i\theta_j}$. Using (D2) and the inequality

$$|\mathbf{y} - \bar{\mathbf{u}}(\theta_j)| \le |\mathbf{y} - \mathbf{u}(\mathbf{x}_j)| + |\mathbf{u}(r_j e^{i\theta_j}) - \bar{\mathbf{u}}(\theta_j)|$$

we find that $\mathbf{y} = \lim_{j\to\infty} \bar{\mathbf{u}}(\theta_j)$, so $\mathbf{y} \in \overline{\bar{\mathbf{u}}(\Theta)} = \bar{\mathbf{u}}(\Theta)$. This completes our sketch of proof. □

7.4 Fracture at the boundary

We illustrate the role of the term $\mathcal{H}^{n-1}(\{\mathbf{x} \in \partial_D\Omega : \mathbf{u} \neq \mathbf{u}_0\})$ in (7.1) by means of a simple example in which the Dirichlet condition is not satisfied. Let $\Omega = B(\mathbf{0}, 1)$, $\partial_D\Omega = \partial\Omega$, $\rho > 0$, and consider the functions

$$\bar{r}(t) := \sqrt{t^2 + \rho^2}, \qquad \mathbf{u}(te^{i\theta}) := \bar{r}(t)e^{i\theta}, \qquad \mathbf{u}_0(\mathbf{x}) := \lambda_0 \mathbf{x},$$

and a number $\lambda_0 > \bar{r}(1)$. Call $\bar{\mathbf{u}}(\theta) := \rho e^{i\theta}$ for $\theta \in \Theta$, and Θ as in Subsection 7.3. This choice of $\mathbf{u}$ satisfies hypotheses (D1)–(D5) of Subsection 7.3. Call $p := \max\{p_1, p_2\}$ and assume that

$$\eta_\varepsilon \ll \varepsilon^{p-1}, \qquad \varepsilon \ll b_\varepsilon. \tag{7.35}$$

Take sequences $\{a_\varepsilon\}_\varepsilon$ and $\{c_\varepsilon\}_\varepsilon$ of positive numbers satisfying $a_\varepsilon \ll \varepsilon^{\frac{1}{2}}$, $c_\varepsilon \ll \varepsilon$ and $\eta_\varepsilon \ll c_\varepsilon^{p-1}$. The numbers $a_{\varepsilon,V}$ and $a_{\varepsilon,W}$, and the transition levels are defined as in (7.13), the zones Z_1^ε–Z_5^ε as in (7.14), the functions f_ε as in (7.12), $\mathbf{X}$ as in (7.15) and $\mathbf{G}_\varepsilon$, $\mathbf{F}_\varepsilon$, r as in (7.28)–(7.29). Finally, set

$$d_\varepsilon^+ := 1 - \sigma_{\varepsilon,V}^{-1}(1), \qquad d_\varepsilon^- := d_\varepsilon^+ - c_\varepsilon.$$

In zones Z_1^ε–Z_4^ε, define $\mathbf{u}_\varepsilon$, v_ε, and w_ε as in Subsection 7.3. The definition of $(\mathbf{u}_\varepsilon, v_\varepsilon, w_\varepsilon)$ in Z_5^ε needs to be modified, due to the following considerations. On the one hand, $\mathbf{u}_\varepsilon$ has to satisfy the Dirichlet condition violated by $\mathbf{u}$: $\mathbf{u}_\varepsilon(\mathbf{x}) = \lambda_0\mathbf{x}$ if $|\mathbf{x}| = 1$; on the other hand, most of the time $\mathbf{u}_\varepsilon$ should coincide with $\mathbf{u}$. Since $\mathbf{u}_\varepsilon$ must be continuous, we will define it in such a way that it stretches the material contained in $\{d_\varepsilon^- \le |\mathbf{x}| \le d_\varepsilon^+\}$ in order to fill the gap between $\mathbf{u}(\Omega) = B(\mathbf{0}, \bar r(1))$ and $\mathbf{u}_0(\partial\Omega) = \partial B(\mathbf{0}, \lambda_0)$. This stretching of material comes with large gradients that are prohibitively expensive in terms of elastic energy, unless $v_\varepsilon = 0$ in that annular region. Because of restriction (4.11), we need to produce first a transition for w_ε from 1 to 0 before the transition of v_ε from 1 to 0. After the stretching takes place, v_ε must go back from 0 to 1 due to condition (4.8).

In the region $\{2a_{\varepsilon,W} \le |\mathbf{x}| \le d_\varepsilon^-\}$ we set $\mathbf{u}_\varepsilon := \mathbf{G}_\varepsilon \circ \mathbf{u} \circ \mathbf{F}_\varepsilon$, as in Step 4 of the proof of Proposition 7.1. It is easy to see that $\mathbf{u}_\varepsilon(te^{i\theta}) = \mathbf{u}(te^{i\theta})$ if $\bar r(t) - \rho \ge 2f_\varepsilon(a_{\varepsilon,W})$. Since $\bar r(d_\varepsilon^-) \to \bar r(1)$ and $f_\varepsilon(a_{\varepsilon,W}) \ll 1$, it is clear that $\mathbf{u}_\varepsilon(te^{i\theta}) = \mathbf{u}(te^{i\theta})$ long before t reaches the value d_ε^-. In $\{d_\varepsilon^- \le |\mathbf{x}| \le d_\varepsilon^+\}$, define $\mathbf{u}_\varepsilon(te^{i\theta})$ as $r_\varepsilon(t)e^{i\theta}$, where r_ε is the linear interpolation such that $\bar r_\varepsilon(d_\varepsilon^-) = \bar r(d_\varepsilon^-)$ and $\bar r_\varepsilon(d_\varepsilon^+) = \bar r(d_\varepsilon^+) + \lambda_0 - \bar r(1)$. In the remaining annulus $\{d_\varepsilon^+ \le |\mathbf{x}| \le 1\}$, set $r_\varepsilon(t) = \bar r(t) + \lambda_0 - \bar r(1)$. To sum up, $\mathbf{u}_\varepsilon(te^{i\theta}) = r_\varepsilon(t)e^{i\theta}$ in Z_5^ε, with

$$r_\varepsilon(t) := \begin{cases} \frac{\bar r(t)+\rho}{2} + f_\varepsilon(a_{\varepsilon,W}), & \text{if } \bar r(t) - \rho \le 2f_\varepsilon(a_{\varepsilon,W}), \\ \bar r(t), & \text{if } \bar r(t) - \rho \ge 2f_\varepsilon(a_{\varepsilon,W}) \text{ and } t \le d_\varepsilon^-, \\ \frac{d_\varepsilon^+ - t}{d_\varepsilon^+ - d_\varepsilon^-}\bar r(d_\varepsilon^-) + \frac{t - d_\varepsilon^-}{d_\varepsilon^+ - d_\varepsilon^-}(\bar r(d_\varepsilon^+) + \lambda_0 - \bar r(1)), & d_\varepsilon^- \le t \le d_\varepsilon^+, \\ \bar r(t) + \lambda_0 - \bar r(1), & d_\varepsilon^+ \le t \le 1. \end{cases}$$

The definition for v_ε is as in (7.15) and (7.21) in zones $Z_1^\varepsilon \cup Z_2^\varepsilon$ and

$$v_\varepsilon(te^{i\theta}) := \begin{cases} 1, & a_{\varepsilon,V} \le t \le d_\varepsilon^- - \sigma_{\varepsilon,V}^{-1}(1), \\ \sigma_{\varepsilon,V}(d_\varepsilon^- - t), & d_\varepsilon^- - \sigma_{\varepsilon,V}^{-1}(1) \le t \le d_\varepsilon^-, \\ 0, & d_\varepsilon^- \le t \le d_\varepsilon^+, \\ \sigma_{\varepsilon,V}(t - d_\varepsilon^+), & d_\varepsilon^+ \le t \le 1. \end{cases}$$

The assumption on $\{c_\varepsilon\}_\varepsilon$ is such that

$$I_\varepsilon^E(\{d_\varepsilon^- \le |\mathbf{x}| \le d_\varepsilon^+\}) + I_\varepsilon^V(\{d_\varepsilon^- \le |\mathbf{x}| \le d_\varepsilon^+\}) \lessapprox \eta_\varepsilon c_\varepsilon \left(c_\varepsilon^{-p_1} + c_\varepsilon^{-p_2}\right) + c_\varepsilon \varepsilon^{-1} \ll 1.$$

The definition of w_ε is 0 in $\mathbf{u}_\varepsilon(Z_\varepsilon^1 \cup Z_\varepsilon^2)$, as in (7.26) in $\mathbf{u}_\varepsilon(Z_\varepsilon^3)$, 1 in $\mathbf{u}_\varepsilon(Z_\varepsilon^4)$, and in $\mathbf{u}_\varepsilon(Z_\varepsilon^5)$ it is

$$w_\varepsilon(\tau e^{i\theta}) := \begin{cases} 1, & \bar r(2a_{\varepsilon,W}) \le \tau \le \bar r(d_\varepsilon^- - \sigma_{\varepsilon,V}^{-1}(1)) - \sigma_{\varepsilon,W}^{-1}(1), \\ \sigma_{\varepsilon,W}(\bar r(d_\varepsilon^- - \sigma_{\varepsilon,V}^{-1}(1)) - \tau), & \bar r(d_\varepsilon^- - \sigma_{\varepsilon,V}^{-1}(1)) - \sigma_{\varepsilon,W}^{-1}(1) \le \tau \le \bar r(d_\varepsilon^- - \sigma_{\varepsilon,V}^{-1}(1)), \\ 0, & \bar r(d_\varepsilon^- - \sigma_{\varepsilon,V}^{-1}(1)) \le \tau \le \bar r(1). \end{cases}$$

With respect to the analysis of Subsection 7.3, the only extra term appearing in the energy estimates is

$$I_\varepsilon^V\left(\{d_\varepsilon^- - \sigma_{\varepsilon,V}^{-1}(1) \le |\mathbf{x}| \le d_\varepsilon^-\} \cup \{d_\varepsilon^+ \le |\mathbf{x}| \le 1\}\right) = 2\pi\left(d_\varepsilon^- + d_\varepsilon^+\right)\int_0^1 (1-s)\,\mathrm{d}s \to \mathcal{H}^1(\partial\Omega).$$

This completes the sketch of proof of (7.1) in this example of fracture at the boundary.

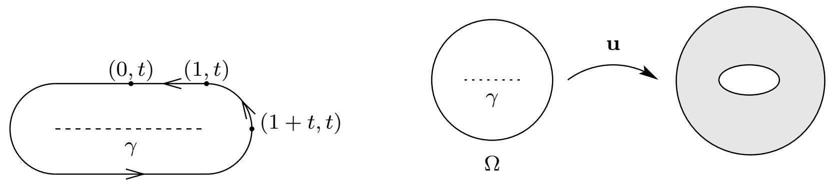
7.5 Fracture in the interior

In this subsection we consider a deformation creating a crack in the interior of the body. To be precise, the reference configuration is $\Omega = B(\mathbf{0}, 2)$ with $\partial_D\Omega = \partial\Omega$. We fix $\lambda > 1$ and declare $\mathbf{u}_0 = \lambda\mathbf{id}$. We set $\gamma = [-1, 1] \times \{0\}$. Let Θ be the topological quotient space obtained from $[-2, 2]$ with the identification $-2 \sim 2$. Define $\mathbf{X} : [0, \infty) \times \Theta \to \mathbb{R}^2$, first for $\theta \in [0, 1]$ by

$$\mathbf{X}(t,\theta) := \begin{cases} (1,0) + te^{i\beta(t,\theta)}, & \theta \in \Theta_0(t) := [0, \frac{\pi t}{2+\pi t}], \\ \left((1-\theta)(1 + \frac{\pi}{2}t), t\right), & \theta \in \Theta_1(t) := [\frac{\pi t}{2+\pi t}, 1], \end{cases} \qquad \beta(t,\theta) := (t^{-1} + \frac{\pi}{2})\theta, \tag{7.36}$$

and then extended to all $[0, \infty) \times \Theta$ by symmetry:

$$\mathbf{X}(t,\theta) := \begin{cases} \left(-x_1(t, 2-\theta), x_2(t, 2-\theta)\right), & \theta \in [1, 2], \\ \left(x_1(t, -\theta), -x_2(t, -\theta)\right), & \theta \in [-2, 0], \end{cases} \tag{7.37}$$



(a) Representation of $\mathbf{X}(t,\cdot)$. (b) Representation of $\mathbf{u}$.

Figure 2: Representation of $\mathbf{X}$ and $\mathbf{u}$ corresponding to Section 7.5.

where we have called x_1, x_2 the components of $\mathbf{X}$. A representation of $\mathbf{X}$ is shown in Figure 2(a). Note that $\mathbf{X}(t,\cdot)$ is a parametrization of the level curve $\{\mathbf{x} \in \Omega : \operatorname{dist}(\mathbf{x},\gamma) = t\}$, which is close to being of arc-length. The assumptions for the deformation are the following:

(F1) $\mathbf{u} \in C^{1,1}(\bar{\Omega} \setminus \gamma, \mathbb{R}^2)$ is one-to-one in $\bar{\Omega} \setminus \gamma$, satisfies $\det \nabla \mathbf{u} > 0$ a.e. in Ω, and (7.9) holds.

(F2) There exist $t_0 \in (0, \operatorname{dist}(\gamma, \partial\Omega))$, $\rho \in C^2([0,t_0] \times \Theta, (0,\infty))$ and $\varphi \in C^2([0,t_0] \times \mathbb{R})$ such that

$$\frac{\partial \varphi}{\partial \theta}(t,\theta) > 0, \quad \varphi(t,\theta+4) = \varphi(t,\theta) + 2\pi, \qquad (t,\theta) \in [0,t_0] \times \mathbb{R}$$

and

$$\mathbf{u}(\mathbf{X}(t,\theta)) = \rho(t,\theta)\, e^{i\varphi(t,\theta)}, \qquad (t,\theta) \in (0,t_0] \times \Theta.$$

(F3) For all $t \in (0,t_0)$, the curvature κ_t of $\mathbf{u}(\mathbf{X}(t,\cdot))$ (as defined in (7.5)) satisfies $\kappa_t > 0$ a.e.

(F4) The inverse of $\mathbf{u}$ has a continuous extension $\mathbf{v} : \overline{\mathbf{u}(\Omega \setminus \gamma)} \to \overline{\Omega}$.

(F5) For each $a \in [-1,1]$, the limits

$$\mathbf{u}^+(a,0) := \lim_{\substack{(x_1,x_2)\to(a,0)\\ x_2>0}} \mathbf{u}(x_1,x_2), \qquad \mathbf{u}^-(a,0) := \lim_{\substack{(x_1,x_2)\to(a,0)\\ x_2<0}} \mathbf{u}(x_1,x_2)$$

exist.

A representation of $\mathbf{u}$ is shown in Figure 2(b). Thanks to (F1) and (F5) one can easily show that $\mathbf{u} \in SBV(\Omega,\mathbb{R}^2)$ and $J_{\mathbf{u}} = \gamma$ $\mathcal{H}^1$-a.e. Furthermore, using also (F4) and reasoning as in the last part of the proof Proposition 7.1, we can check the equalities

$$\operatorname{Per} \operatorname{im}_{\mathrm{G}}(\mathbf{u},\Omega) = \operatorname{Per} \mathbf{u}(\Omega \setminus \gamma) = \mathcal{H}^1(\mathbf{u}^-(\gamma)) + \mathcal{H}^1(\mathbf{u}^+(\gamma)) + \mathcal{H}^1(\mathbf{u}_0(\partial\Omega)) \quad \text{and} \quad \mathcal{H}^1(J_{\mathbf{u}^{-1}}) = 0. \tag{7.38}$$

Call $p := \max\{p_1,p_2\}$ and assume that (7.35).

Proposition 7.2. *For each ε there exists $(\mathbf{u}_\varepsilon, v_\varepsilon, w_\varepsilon) \in \mathcal{A}_\varepsilon$ satisfying* (6.5) *and* (7.1).

Sketch of proof. The construction of $(\mathbf{u}_\varepsilon, v_\varepsilon, w_\varepsilon)$ follows the same scheme of Proposition 7.1. Let $\{a_\varepsilon\}_\varepsilon$ be any sequence such that

$$\eta_\varepsilon^{\frac{1}{p-1}} \ll a_\varepsilon \ll \varepsilon, \tag{7.39}$$

Instead of (7.12), define $f_\varepsilon(t) := t - a_\varepsilon$. Define $a_{\varepsilon,V}$ and $a_{\varepsilon,W}$ as in (7.13), and Z_1^ε–Z_5^ε as in (7.14). Note that $a_{\varepsilon,V} \approx a_{\varepsilon,W} \approx \varepsilon$.

Step 1. Define $\mathbf{u}_\varepsilon$ in Z_1^ε by

$$\mathbf{u}_\varepsilon(\ell \mathbf{X}(a_\varepsilon,\theta)) := \ell \bar{\mathbf{u}}(\theta), \qquad \bar{\mathbf{u}}(\theta) := \mathbf{u}(\mathbf{X}(a_\varepsilon,\theta)), \qquad (\ell,\theta) \in [0,1] \times \Theta.$$

Let $v_\varepsilon = 0$ in Z_1^ε and $w_\varepsilon = 0$ in $\mathbf{u}_\varepsilon(Z_1^\varepsilon)$. As in (7.16), we have that $D\mathbf{u}_\varepsilon = \bar{\mathbf{u}} \otimes D\ell + \ell \bar{\mathbf{u}}' \otimes D\theta$, with

$$\begin{pmatrix} D\ell \\ D\theta \end{pmatrix} = \frac{1}{\mathbf{X}(a_\varepsilon,\theta) \wedge \ell \frac{\partial \mathbf{X}}{\partial \theta}} \begin{pmatrix} -(\ell \frac{\partial \mathbf{X}}{\partial \theta})^\perp \\ \mathbf{X}(a_\varepsilon,\theta)^\perp \end{pmatrix} = \begin{cases} \frac{1}{a_\varepsilon + \cos\beta} \begin{pmatrix} \cos\beta & \sin\beta \\ \frac{-a_\varepsilon \sin\beta}{\ell(1+\frac{\pi}{2}a_\varepsilon)} & \frac{1+a_\varepsilon\cos\beta}{\ell(1+\frac{\pi}{2}a_\varepsilon)} \end{pmatrix}, & \theta \in \Theta_0(a_\varepsilon), \\ \frac{1}{a_\varepsilon} \begin{pmatrix} 0 & 1 \\ \frac{-a_\varepsilon}{\ell(1+\frac{\pi}{2}a_\varepsilon)} & \frac{1-\theta}{\ell} \end{pmatrix}, & \theta \in \Theta_1(a_\varepsilon), \end{cases}$$

the result in the rest of Θ being analogous. Taking (F2) into account we obtain that $|D\mathbf{u}_\varepsilon| \lessapprox a_\varepsilon^{-1}$. From the analogue of (7.19) it follows that

$$\det D\mathbf{u}_\varepsilon = \frac{\bar{\mathbf{u}} \wedge \ell \bar{\mathbf{u}}'}{\mathbf{X}(a_\varepsilon,\theta) \wedge \ell \frac{\partial \mathbf{X}}{\partial \theta}} = \begin{cases} \frac{\rho^2 \frac{\partial \varphi}{\partial \theta}(a_\varepsilon,\theta)}{(1+\frac{\pi}{2}a_\varepsilon)} \frac{1}{a_\varepsilon + \cos\beta}, & \theta \in \Theta_0(a_\varepsilon), \\ \frac{\rho^2 \frac{\partial \varphi}{\partial \theta}(a_\varepsilon,\theta)}{a_\varepsilon(1+\frac{\pi}{2}a_\varepsilon)}, & \theta \in \Theta_1(a_\varepsilon). \end{cases}$$

Hence, by (F2),

$$\frac{1}{2} (\inf \rho)^2 \inf \frac{\partial \varphi}{\partial \theta} \le \det D\mathbf{u}_\varepsilon \lessapprox a_\varepsilon^{-1}.$$

In addition, the geometry of γ shows that $\mathcal{L}^2(Z_1^\varepsilon) \approx a_\varepsilon$. Therefore, thanks to (7.39),

$$I_\varepsilon^E(Z_1^\varepsilon) + I_\varepsilon^V(Z_1^\varepsilon) + I_\varepsilon^W(\mathbf{u}_\varepsilon(Z_1^\varepsilon)) \lessapprox \eta_\varepsilon \left(a_\varepsilon^{-p_1} + a_\varepsilon^{-p_2} \right) a_\varepsilon + \varepsilon^{-1} a_\varepsilon \ll 1.$$

Step 2. Define v_ε in Z_2^ε as in (7.21). The analysis is the same as in Proposition 7.1, save that now we have that for all $t \in (a_\varepsilon, a_{\varepsilon,V})$,

$$\mathcal{H}^1(\{\mathbf{x} \in \Omega : \operatorname{dist}(\mathbf{x},\gamma) = t\}) = 2\left(\mathcal{H}^1(\gamma) + \pi t\right),$$

hence

$$\lim_{\varepsilon\to 0} I_\varepsilon^V(Z_2^\varepsilon) = \lim_{\varepsilon \to 0} \int_0^1 (1-s)\, \mathcal{H}^1(\{\mathbf{x} \in \Omega : \operatorname{dist}(\mathbf{x},\gamma) = a_\varepsilon + \sigma_{\varepsilon,V}^{-1}(s)\})\, \mathrm{d}s = \mathcal{H}^1(\gamma).$$

Step 3. Define $\mathbf{u}_\varepsilon$ in $Z_\varepsilon^2 \cup Z_\varepsilon^3$ and $\mathbf{Y}(\tau,\theta)$ as in (7.25), recalling that now $f_\varepsilon(t) = t - a_\varepsilon$, and $\mathbf{X}$ is given by (7.36)–(7.37). The function v_ε is defined as 1 in $Z_3^\varepsilon \cup Z_4^\varepsilon \cup Z_5^\varepsilon$, and w_ε as in (7.26) in $\mathbf{u}_\varepsilon(Z_3^\varepsilon)$. By (7.6) and (F3) we have that $|\boldsymbol{\nu}'| = \kappa_{a_\varepsilon} |\bar{\mathbf{u}}'|$. Observe from (F2) that $|\bar{\mathbf{u}}'|$ is bounded from below by $\inf(\rho \frac{\partial\varphi}{\partial\theta}) > 0$. Therefore,

$$\sup_\varepsilon \sup \kappa_{a_\varepsilon} \le \sup_{t \in (0,t_0]} \sup \kappa_t < \infty.$$

On the other hand, $\left|\frac{\partial \mathbf{X}}{\partial t}\right| \le 1 + \theta/t \le 1 + \frac{\pi}{2}$ in $\Theta_0(t)$. Therefore,

$$\frac{\partial \mathbf{X}}{\partial t} \wedge \frac{\partial \mathbf{X}}{\partial \theta} = 1 + \frac{\pi}{2} t, \qquad \left| \frac{\partial \mathbf{X}}{\partial t} \right| \le 1 + \frac{\pi}{2} \quad \text{and} \quad \left| \frac{\partial \mathbf{X}}{\partial \theta} \right| = 1 + \frac{\pi}{2} t \qquad \text{in } [0,\infty) \times \Theta. \tag{7.40}$$

Using now (7.16) and (F2) we find that

$$|D\mathbf{u}_\varepsilon(\mathbf{X}(t,\theta))| \le \frac{1}{\frac{\partial \mathbf{X}}{\partial t} \wedge \frac{\partial \mathbf{X}}{\partial \theta}} \left(\left| \frac{\partial \mathbf{X}}{\partial \theta} \right| + |\bar{\mathbf{u}}'| \left(1 + (t - a_\varepsilon)\kappa_{a_\varepsilon}\right) \left| \frac{\partial \mathbf{X}}{\partial t} \right| \right) \lessapprox 1 + \sup \left(\left| \frac{\partial \rho}{\partial \theta} \right| + \rho \frac{\partial \varphi}{\partial \theta} \right) \lessapprox 1.$$

On the other hand, (7.19), (7.8), (F2), and (F3) imply that

$$\det D\mathbf{u}_\varepsilon = \frac{|\bar{\mathbf{u}}'|(1 + (t-a_\varepsilon)\kappa_{a_\varepsilon})}{1 + \frac{\pi}{2} t} \approx 1.$$

Hence

$$I_\varepsilon^E(Z_2^\varepsilon \cup Z_3^\varepsilon) \lessapprox \mathcal{L}^2(Z_2^\varepsilon \cup Z_3^\varepsilon) \lessapprox \varepsilon.$$

The analysis for I_ε^W is the same as in (7.26)–(7.27), except that we need (F2) in order to conclude that

$$\lim_{\varepsilon\to 0}\mathcal{H}^1(\{\mathbf{y}\in\mathbf{u}_\varepsilon(Z_3^\varepsilon):w_\varepsilon(\mathbf{y})=s\})=\lim_{\varepsilon\to 0}\int_\Theta\left|\frac{\partial(\mathbf{u}\circ\mathbf{X})}{\partial\theta}(a_\varepsilon,\theta)\right|\mathrm{d}\theta=\lim_{\varepsilon\to 0}\mathcal{H}^1\left((\mathbf{u}\circ\mathbf{X})(a_\varepsilon,\cdot)(\Theta)\right)$$
$$=\mathcal{H}^1(\mathbf{u}^-(\gamma))+\mathcal{H}^1(\mathbf{u}^+(\gamma)).$$

Step 4. Define $\mathbf{u}_\varepsilon:=\mathbf{G}_\varepsilon\circ\mathbf{u}\circ\mathbf{F}_\varepsilon$ in $Z_4^\varepsilon\cup Z_5^\varepsilon$, with $\mathbf{F}_\varepsilon$ and $\mathbf{G}_\varepsilon$ as in (7.28)–(7.29), but changing $r(t)$ to

$$r(t):=\begin{cases}2(t-a_{\varepsilon,W})+a_\varepsilon(2-\frac{t}{a_{\varepsilon,W}}), & a_{\varepsilon,W}<t<2a_{\varepsilon,W},\\ t, & t\geq 2a_{\varepsilon,W}.\end{cases}\tag{7.41}$$

By (7.16) (applied to $\mathbf{F}_\varepsilon$), (7.41), and (7.40),

$$|D\mathbf{F}_\varepsilon(\mathbf{X}(t,\theta))|\leq\frac{1}{\frac{\partial\mathbf{X}}{\partial t}\wedge\frac{\partial\mathbf{X}}{\partial\theta}}\left(\left|\frac{\partial\mathbf{X}}{\partial t}(r(t),\theta)\right||r'(t)|\left|\frac{\partial\mathbf{X}}{\partial\theta}\right|+\left|\frac{\partial\mathbf{X}}{\partial\theta}(r(t),\theta)\right|\left|\frac{\partial\mathbf{X}}{\partial t}\right|\right)\lessapprox 1.$$

Using now (7.19) we find that

$$\det D\mathbf{F}_\varepsilon=\frac{(1+\frac{\pi}{2}r(t))(2-\frac{a_\varepsilon}{a_{\varepsilon,W}})}{1+\frac{\pi}{2}t}\approx 1.$$

Having also in mind the estimates (7.30) and (7.31), we find that

$$|D\mathbf{u}_\varepsilon|\lessapprox|D\mathbf{u}|\quad\text{and}\quad\det D\mathbf{u}_\varepsilon\approx\det D\mathbf{u}.$$

On the other hand, the definition of $\mathbf{G}_\varepsilon$ and $\mathbf{F}_\varepsilon$ are so that $\mathbf{u}_\varepsilon(\mathbf{x})=\mathbf{u}(\mathbf{x})$ whenever $\mathbf{x}=\mathbf{X}(t,\theta)$ with $t\geq 2a_{\varepsilon,W}$ and $\mathbf{u}(\mathbf{x})=\bar{\mathbf{u}}(\theta)+\tau\boldsymbol{\nu}(\theta)$ with $\tau\geq 2(a_{\varepsilon,W}-a_\varepsilon)$. Therefore, the set N^ε of $\mathbf{x}\in Z_4^\varepsilon\cup Z_5^\varepsilon$ such that $\mathbf{u}_\varepsilon(\mathbf{x})\neq\mathbf{u}(\mathbf{x})$ satisfies $\mathcal{L}^2(N^\varepsilon)\ll 1$. Using ($\bar{\text{W}}$1) and (F1), we conclude that

$$I_\varepsilon^E(N^\varepsilon)\lessapprox\int_{N^\varepsilon\setminus\gamma}\left[|D\mathbf{u}|^{p_1}+h\left(\det D\mathbf{u}\right)\right]\mathrm{d}\mathbf{x}\ll 1.$$

Step 5. This is exactly the same as in the proof of Proposition 7.1. The function w_ε is defined as 1 in $\mathbf{u}_\varepsilon(Z_4^\varepsilon\cup Z_5^\varepsilon)\setminus Y_\varepsilon$, and as (7.33) in Y_ε, where the region Y_ε is defined as (7.32). We thus arrive at (7.34). This concludes our sketch of proof. □

7.6 Coalescence

Coalescence is the process by which two or more cavities are joined to form a bigger cavity or else a crack. In this subsection we present a simple example of a deformation that forms a crack joining two preexisting cavities.

Let $\underline{r}>0$, $\mu>0$ and $h>0$. Let Ω be a Lipschitz domain such that

$$(-1,1)\times\{0\}\subset\Omega,\qquad\Omega\cap\left(\bar{B}((-1-\underline{r},0),\underline{r})\cup\bar{B}((1+\underline{r},0),\underline{r})\right)=\varnothing$$

and

$$\partial B((-1-\underline{r},0),\underline{r})\cup\partial B((1+\underline{r},0),\underline{r})\subset\bar{\Omega}.$$

Set

$$\partial_N\Omega=\partial B((-1-\underline{r},0),\underline{r})\cup\partial B((1+\underline{r},0),\underline{r}),\qquad\partial\Omega_D=\partial\Omega\setminus\partial_N\Omega,\qquad\gamma:=[-1,1]\times\{0\}.$$

We assume

(L1) $\mathbf{u}\in C^{1,1}(\bar{\Omega}\setminus\gamma,\mathbb{R}^2)$ is one-to-one in $\bar{\Omega}\setminus\gamma$, satisfies $\det\nabla\mathbf{u}>0$ a.e. in Ω, and (7.9) holds.

(L2) The inverse of $\mathbf{u}$ has a continuous extension $\mathbf{v}:\overline{\mathbf{u}(\Omega\setminus\gamma)}\to\bar{\Omega}$.

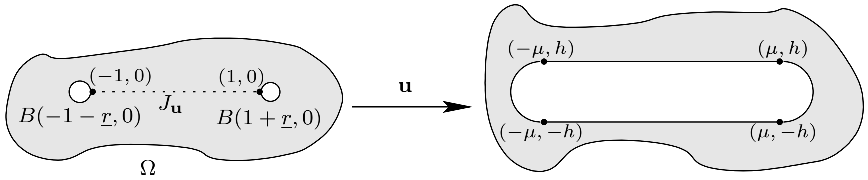
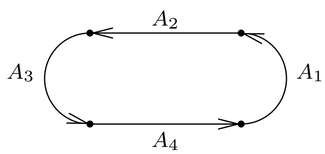
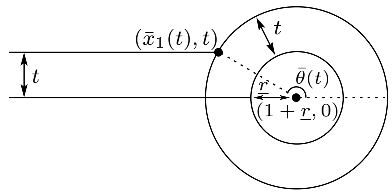


Figure 3: Representation of $\mathbf{u}$ in the construction of Subsection 7.6.



(a) Representation of Θ.



(b) Construction of $\bar{x}_1$ and $\bar{\theta}$.

Figure 4: Representations of Θ, $\bar{x}_1$ and $\bar{\theta}$, corresponding to Subsection 7.6

(L3) When we define $\mathbf{u}^{\pm}:\gamma\to\mathbb{R}^2$ as

$$\mathbf{u}^{\pm}(x_1,0)=(\mu x_1,\pm h),\qquad x_1\in(-1,1),$$

we have that for all $x_1\in(-1,1)$,

$$\lim_{\substack{\mathbf{x}\to(x_1,0)\\ \pm x_2\geq 0}}\mathbf{u}(\mathbf{x})=\mathbf{u}^{\pm}(x_1,0).$$

(L4) The deformation $\mathbf{u}$ can be continuously extended to $\partial_N\Omega\setminus\{(-1,0),(1,0)\}$ by

$$\begin{cases}\mathbf{u}\left((-1-\underline{r},0)+\underline{r}e^{(2\theta-\pi)i}\right):=(-\mu,0)+he^{i\theta}, & \theta\in(\frac{\pi}{2},\frac{3\pi}{2}),\\ \mathbf{u}\Big((1+\underline{r},0)+\underline{r}e^{2\theta i}\Big):=(\mu,0)+he^{i\theta}, & \theta\in(-\frac{\pi}{2},\frac{\pi}{2}).\end{cases}$$

A representation of $\mathbf{u}$ is shown in Figure 3. As in Subsection 7.5, it is easy to check that $\mathbf{u}\in SBV(\Omega,\mathbb{R}^2)$, $J_{\mathbf{u}}=\gamma$ $\mathcal{H}^1$-a.e. and (7.38) holds.

Assume (7.35). The following result holds.

Proposition 7.3. *For each ε there exists $(\mathbf{u}_\varepsilon,v_\varepsilon,w_\varepsilon)\in\mathcal{A}_\varepsilon$ satisfying* (6.5) *and* (7.1).

Sketch of proof. We define first a parametrization $\mathbf{X}(t,\theta)$ of the domain in which the parameter t represents the distance from $\mathbf{X}(t,\theta)$ to $\gamma\cup\partial_N\Omega$. To this aim, define Θ as the quotient space obtained by taking the union $A_1\cup A_2\cup A_3\cup A_4$, where

$$A_1:=[-\frac{\pi}{2},\frac{\pi}{2}]\times\{1\},\qquad A_2:=[-1,1]\times\{2\},\qquad A_3:=[\frac{\pi}{2},\frac{3\pi}{2}]\times\{3\},\qquad A_4:=[-1,1]\times\{4\},$$

and identifying the points

$$(\frac{\pi}{2},1)\sim(-1,2),\qquad(1,2)\sim(\frac{\pi}{2},3),\qquad(\frac{3\pi}{2},3)\sim(-1,4),\qquad(1,4)\sim(-\frac{\pi}{2},1).$$

A representation of Θ is shown in Figure 4(a). Note that Θ is diffeomorphic to $\mathbb{S}^1$.

Define $\bar{x}_1 : [0,\infty) \to [0,\infty)$ and $\bar{\theta} : [0,\infty) \to \mathbb{S}^1$ as

$$\bar{x}_1(t) := 1 + \underline{r} - \sqrt{\underline{r}^2 + 2\underline{r}t}, \qquad \bar{\theta}(t) := \pi - \arctan \frac{t}{\sqrt{\underline{r}^2 + 2\underline{r}t}}. \tag{7.42}$$

The point $(\bar{x}_1(t), t)$ lies on the circle of centre $(1+\underline{r}, 0)$ and radius $\underline{r}+t$, whereas $\bar{\theta}(t)$ is the angle of $(\bar{x}_1(t), t)$ with respect to $(1+\underline{r}, 0)$; see Figure 4(b). The parabola $(\bar{x}_1(t), t)$ represents, therefore, the interface between the set of points that are closer to γ and those that are closer to $\partial B((1+\underline{r}, 0), \underline{r})$.

Define $\mathbf{X} : [0,\infty) \times \Theta \to \mathbb{R}^2$ and $\mathbf{Y} : [-h,\infty) \times \Theta \to \mathbb{R}^2$ as

$$\mathbf{X}(t,\theta) := \begin{cases} (1+\underline{r}, 0) + (\underline{r}+t)e^{i\frac{2\bar{\theta}(t)}{\pi}\theta} & \text{if } \theta \in A_1, \\ (-\bar{x}_1(t)\theta, t) & \text{if } \theta \in A_2, \\ \text{by symmetry} & \text{if } \theta \in A_3 \cup A_4, \end{cases} \qquad \mathbf{Y}(\tau,\theta) := \begin{cases} (\mu, 0) + (h+\tau)e^{i\theta} & \text{if } \theta \in A_1, \\ (-\mu\theta, h+\tau) & \text{if } \theta \in A_2, \\ \text{by symmetry} & \text{if } \theta \in A_3 \cup A_4. \end{cases}$$

In both definitions, we have identified A_1 with $[-\frac{\pi}{2}, \frac{\pi}{2}]$, A_2 with $[-1,1]$ and so on. Let $\{a_\varepsilon\}_\varepsilon$ be any sequence such that (7.39). As in Subsection 7.5, write $a_{\varepsilon,V} := a_\varepsilon + \sigma_{\varepsilon,V}^{-1}(1)$ and $a_{\varepsilon,W} := a_{\varepsilon,V} + \sigma_{\varepsilon,W}^{-1}(1)$. Let

$$\bar{\mathbf{u}}(\theta) := \mathbf{Y}(0,\theta) = \begin{cases} \mathbf{u}(\mathbf{X}(0,\theta)), & \theta \in \operatorname{Int} A_1 \cup \operatorname{Int} A_3, \\ \mathbf{u}^+(\mathbf{X}(0,\theta)), & \theta \in A_2, \\ \mathbf{u}^-(\mathbf{X}(0,\theta)), & \theta \in A_4, \end{cases} \qquad \boldsymbol{\nu}(\theta) := \begin{cases} e^{i\theta}, & \theta \in A_1 \cup A_3, \\ (0,1), & \theta \in A_2, \\ (0,-1), & \theta \in A_4, \end{cases}$$

where $\operatorname{Int} A_1$ stands for $(-\frac{\pi}{2}, \frac{\pi}{2}) \times \{1\}$, which is further identified with $(-\frac{\pi}{2}, \frac{\pi}{2})$, and analogously for $\operatorname{Int} A_3$. Let $\mathbf{G}_\varepsilon$ be as in (7.28), where f_ε is given by $f_\varepsilon(t) := t - a_\varepsilon$. The recovery sequence is defined as

$$\mathbf{u}_\varepsilon(\mathbf{X}(t,\theta)) := \begin{cases} \mathbf{Y}(h(\frac{t}{a_\varepsilon} - 1), \theta), & (t,\theta) \in (0, a_\varepsilon] \times \Theta, \\ \mathbf{Y}(t - a_\varepsilon, \theta), & (t,\theta) \in (a_\varepsilon, a_{\varepsilon,W}] \times \Theta, \\ \mathbf{G}_\varepsilon \circ \mathbf{u}\left(\mathbf{X}\left(2(t - a_{\varepsilon,W}), \theta\right)\right), & (t,\theta) \in (a_{\varepsilon,W}, 2a_{\varepsilon,W}] \times \Theta, \\ \mathbf{G}_\varepsilon \circ \mathbf{u}\left(\mathbf{X}(t,\theta)\right), & (t,\theta) \in (2a_{\varepsilon,W}, \infty) \times \Theta \cap \mathbf{X}^{-1}(\Omega), \end{cases}$$

$$v_\varepsilon(\mathbf{x}) := \begin{cases} 0, & \text{if } \operatorname{dist}(\mathbf{x}, \gamma \cup \partial_N \Omega) < a_\varepsilon, \\ \sigma_{\varepsilon,V}(\operatorname{dist}(\mathbf{x}, \gamma \cup \partial_N \Omega) - a_\varepsilon), & \text{if } a_\varepsilon \le \operatorname{dist}(\mathbf{x}, \gamma \cup \partial_N \Omega) \le a_{\varepsilon,V}, \\ 1, & \text{if } \operatorname{dist}(\mathbf{x}, \gamma \cup \partial_N \Omega) > a_{\varepsilon,V}, \end{cases}$$

and

$$w_\varepsilon(\mathbf{y}) := \begin{cases} 0, & \text{in } \mathbf{Y}([0, a_{\varepsilon,V} - a_\varepsilon] \times \Theta), \\ \sigma_{\varepsilon,W}\left(\operatorname{dist}\left(\mathbf{y}, \bar{\mathbf{u}}(\Theta)\right) - (a_{\varepsilon,V} - a_\varepsilon)\right), & \text{in } \mathbf{Y}([a_{\varepsilon,V} - a_\varepsilon, a_{\varepsilon,W} - a_\varepsilon] \times \Theta), \\ \sigma_{\varepsilon,W}\left(\operatorname{dist}\left(\mathbf{y}, \mathbf{u}(\partial_D \Omega)\right)\right), & \text{if } \mathbf{y} \in \mathbf{u}(\Omega \setminus \gamma) \text{ and } \operatorname{dist}\left(\mathbf{y}, \mathbf{u}(\partial_D \Omega)\right) \le \sigma_{\varepsilon,W}^{-1}(1), \\ 1, & \text{in any other case in } \mathbf{u}(\Omega \setminus \gamma). \end{cases}$$

From (7.42) we obtain

$$\bar{x}_1'(t) = -\frac{\underline{r}}{\sqrt{\underline{r}^2 + 2\underline{r}t}}, \qquad \bar{\theta}'(t) = -\frac{\underline{r}}{(\underline{r}+t)\sqrt{\underline{r}^2 + 2\underline{r}t}}.$$

Standard calculations show that

$$\left|\frac{\partial \mathbf{X}}{\partial t}\right| \lessapprox 1, \qquad \left|\frac{\partial \mathbf{X}}{\partial \theta}\right| \lessapprox 1, \qquad \frac{\partial \mathbf{X}}{\partial t} \wedge \frac{\partial \mathbf{X}}{\partial \theta} \approx 1$$

in compact subsets of $(t,\theta) \in [0,\infty) \times \Theta$, and

$$\left|\frac{\partial \mathbf{Y}}{\partial \tau}\right| \lessapprox 1, \qquad \left|\frac{\partial \mathbf{Y}}{\partial \theta}\right| \lessapprox 1, \qquad \frac{\partial \mathbf{Y}}{\partial \tau} \wedge \frac{\partial \mathbf{Y}}{\partial \theta} \approx 1.$$

in compact subsets of $(\tau,\theta) \in [-h,\infty) \times \Theta$. Using this, the result can be established exactly as in Subsection 7.5. □

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