# REGULARITY OF INVERSES OF SOBOLEV DEFORMATIONS WITH FINITE SURFACE ENERGY

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Abstract. Let $\mathbf{u}$ be a Sobolev $W^{1,p}$ map from a bounded open set $\Omega \subset \mathbb{R}^n$ to $\mathbb{R}^n$. We assume $\mathbf{u}$ to satisfy some invertibility properties that are natural in the context of nonlinear elasticity, namely, the topological condition INV and the orientation-preserving constraint $\det D\mathbf{u} > 0$. These deformations may present cavitation, which is the phenomenom of void formation. We also assume that the surface created by the cavitation process has finite area. If $p > n-1$, we show that a suitable defined inverse of $\mathbf{u}$ is a Sobolev map. A partial result is also given for the critical case $p = n-1$. The proof relies on the techniques used in the study of cavitation.

## 1. Introduction

A classic question in analysis and topology is to find out the regularity of the inverse function $\mathbf{u}^{-1}$ in terms of the regularity of the original function $\mathbf{u}$. In particular, the issue of ascertaining the optimal Sobolev or $BV$ regularity of $\mathbf{u}^{-1}$ given that of $\mathbf{u}$ has experienced a recent interest in the last decade. Most of the works in this question (see [17, 18, 23, 19, 8, 16, 26]) assume additionally that $\mathbf{u}$ is a homeomorphism. This implies, in particular, that $\mathbf{u}(\Omega)$ is open, so it makes sense to talk about a Sobolev or $BV$ space over $\mathbf{u}(\Omega)$.

In the context of nonlinear elasticity, one assumes that $\mathbf{u}$ is in the Sobolev space $W^{1,p}$ for some $p > 1$, but the assumption that $\mathbf{u}$ is a homeomorphism is not acceptable in general. Indeed, while Ball [2] proved that if $p > n$ and if other integrability conditions hold then deformations are homeomorphisms, in the case when $p < n$ there are interesting deformations in $W^{1,p}$ that present singularities, and, in particular, are not continuous. One such type of singularity is that of *cavitation*, which is the process of formation of voids in solids (see [3]). In fact, determining the conditions on the stored-energy function under which cavitation occurs was an important part of the motivation for the papers [24, 25, 22, 21] to study some regularity properties of a suitable defined inverse of $\mathbf{u}$; to be precise, the assumptions in [24, 25, 22] are incompatible with cavitation, while [21] does allow for cavitation. In those works, the deformation $\mathbf{u}$ was assumed to enjoy a certain property of invertibility much weaker than being a homeomorphism.

Following the steps of Müller & Spector [21], the authors [12, 13, 14, 15] carried out an existence theory for deformations allowing for fracture and cavitation. As happened with [21] (and earlier with Šverák [24]), that analysis lent itself to a study of the inverse of $\mathbf{u}$. In particular, in [13] we proved an $SBV$ regularity property of the inverse of an approximately differentiable map that was needed in order to carry out a geometric study of the surface created by the deformation. When the deformation $\mathbf{u}$ was assumed to be a Sobolev homeomorphism, it was shown in [14], as a by-product of the analysis of cavitation, that the inverse is actually Sobolev $W^{1,1}$. The same conclusion had been given by Csörnyei, Hencl & Malý [8], in fact, with weaker assumptions, using techniques of mappings of finite distortion.

In this paper we remove the assumption of being a homeomorphism; in particular, the deformations studied can present cavities. Specifically, we employ some techniques of [13, 14] to show that, under some assumptions on $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n)$ that are natural in the context of cavitation (namely, $\det D\mathbf{u} > 0$ a.e., the topological condition INV holds, $p \geq n-1$ and $\mathbf{u}$ has finite surface energy), an adequate definition $\tilde{\mathbf{u}}^{-1}$ of the inverse of $\mathbf{u}$ is a Sobolev map. A key ingredient is the use of the topological image $\operatorname{im}_{\mathrm{T}}(\mathbf{u}, \Omega)$ of $\mathbf{u}$ as the domain space for $\tilde{\mathbf{u}}^{-1}$. The topological image, which is defined as the set of points for which $\mathbf{u}$ has nonzero degree, coincides a.e. with the union of the image of $\mathbf{u}$ and the cavities created. The map $\tilde{\mathbf{u}}^{-1}$ is essentially the inverse of $\mathbf{u}$ outside the cavities, and it sends the whole cavity volume in the deformed configuration

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into the cavity point in the reference configuration. Thus, $\tilde{\mathbf{u}}^{-1}$ is not one-to-one a.e., but the amount of non-injectivity is well controlled.

If $p > n-1$, the set $\operatorname{im_T}(\mathbf{u}, \Omega)$ is open and, in this case, we prove that $\tilde{\mathbf{u}}^{-1} \in W^{1,1}(\operatorname{im_T}(\mathbf{u}, \Omega), \mathbb{R}^n)$. In the critical case $p = n-1$ the set $\operatorname{im_T}(\mathbf{u}, \Omega)$ is not open in general. Nevertheless, we prove that the extension of $\tilde{\mathbf{u}}^{-1}$ by zero to $\mathbb{R}^n$ is an $SBV$ function whose jump set does not intersect $\operatorname{im_T}(\mathbf{u}, \Omega)$; in particular, the restriction of the distributional derivative $D\tilde{\mathbf{u}}^{-1}$ to $\operatorname{im_T}(\mathbf{u}, \Omega)$ is an $L^1$ function.

As an example of the potential applications of the regularity properties proved in this paper, we mention that they can be used to improve the recent well-posedness results of Barchiesi & De Simone [4] in the theory of liquid crystal elastomers by making it possible to work with more realistic hypotheses on the stored-energy function and on the deformations. This will be shown in a future work.

## 2. Notation and preliminary results

In this section we set the notation and concepts of the paper, and state some preliminary results. Part of those results are standard in the theory of weakly differentiable functions, and part are collected from the works by [21, 7, 12, 13, 14, 15] on cavitation that are relevant for the regularity of inverses.

2.1. **General notation.** We will work in dimension $n \geq 2$, and $\Omega$ is a bounded open set of $\mathbb{R}^n$. Vector-valued and matrix-valued quantities will be written in boldface. Coordinates in the reference configuration will be denoted by $\mathbf{x}$, and in the deformed configuration by $\mathbf{y}$.

The closure of a set $A$ is denoted by $\bar{A}$, and its boundary by $\partial A$. Given two sets $U, V$ of $\mathbb{R}^n$, we will write $U \subset\subset V$ if $U$ is bounded and $\bar{U} \subset V$. The open ball of radius $r > 0$ centred at $\mathbf{x} \in \mathbb{R}^n$ is denoted by $B(\mathbf{x}, r)$. The function dist indicates the distance from a point to a set.

Given a square matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, its determinant is denoted by $\det \mathbf{A}$. The matrix $\operatorname{adj} \mathbf{A}$ is the matrix that satisfies $(\det \mathbf{A})\mathbf{1} = \mathbf{A} \operatorname{adj} \mathbf{A}$, where $\mathbf{1}$ denotes the identity matrix. The transpose of $\operatorname{adj} \mathbf{A}$ is denoted by $\operatorname{cof} \mathbf{A}$. If $\mathbf{A}$ is invertible, its inverse is denoted by $\mathbf{A}^{-1}$. The inner (dot) product of vectors and of matrices will be denoted by $\cdot$. The Euclidean norm of a vector $\mathbf{x}$ is denoted by $|\mathbf{x}|$, and the associated matrix norm is also denoted by $|\cdot|$. Given $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, the tensor product $\mathbf{a} \otimes \mathbf{b}$ is the $n \times n$ matrix whose component $(i, j)$ is $a_i\, b_j$. Note the elementary formula

$$\mathbf{a} \cdot (\mathbf{F}\mathbf{b}) = (\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{F}, \qquad \mathbf{a}, \mathbf{b} \in \mathbb{R}^n, \quad \mathbf{F} \in \mathbb{R}^{n\times n}. \tag{2.1}$$

The Lebesgue measure in $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$, the $(n-1)$-dimensional Hausdorff measure by $\mathcal{H}^{n-1}$, and the counting measure by $\mathcal{H}^0$. The Lebesgue $L^p$ and Sobolev $W^{1,p}$ spaces are defined in the usual way. So are the functions of class $C^k$, for $k \in \mathbb{N}$, and their versions $C^k_c$ of compact support. We will indicate the domain and target space, as in, for example, $L^p(\Omega, \mathbb{R}^n)$, except if the target space is $\mathbb{R}$, in which case we will simply write $L^p(\Omega)$. The identity function in $\mathbb{R}^n$ is denoted by $\mathbf{id}$.

If $\mu$ is a measure on a set $U$, and $V$ is a $\mu$-measurable subset of $U$, then the restriction of $\mu$ to $V$ is denoted by $\mu \llcorner V$. The measure $|\mu|$ denotes the total variation of $\mu$.

Given two sets $A, B$ of $\mathbb{R}^n$, we write $A \subset B$ a.e. if $\mathcal{L}^n(A \setminus B) = 0$, while $A = B$ a.e. means $A \subset B$ a.e. and $B \subset A$ a.e. Analogously, $A \,\tilde{\subset}\, B$ means $\mathcal{H}^{n-1}(A \setminus B) = 0$, while $A \cong B$ means $A \,\tilde{\subset}\, B$ and $B \,\tilde{\subset}\, A$.

2.2. **Density, boundary and perimeter.** Given a measurable set $A \subset \mathbb{R}^n$, its characteristic function will be denoted by $\chi_A$. Its *perimeter* is defined as

$$\operatorname{Per} A := \sup \left\{ \int_A \operatorname{div} \mathbf{g}(\mathbf{y})\, \mathrm{d}\mathbf{y} : \ \mathbf{g} \in C^1_c(\mathbb{R}^n, \mathbb{R}^n), \ \|\mathbf{g}\|_\infty \leq 1 \right\}.$$

The *density* of $A$ at an $\mathbf{x} \in \mathbb{R}^n$ is defined as

$$D(A, \mathbf{x}) := \lim_{r \searrow 0} \frac{\mathcal{L}^n(A \cap B(\mathbf{x}, r))}{\mathcal{L}^n(B(\mathbf{x}, r))}.$$

Half-spaces are denoted by

$$H^+(\mathbf{a}, \boldsymbol{\nu}) := \{\mathbf{x} \in \mathbb{R}^n : (\mathbf{x} - \mathbf{a}) \cdot \boldsymbol{\nu} \geq 0\}, \qquad H^-(\mathbf{a}, \boldsymbol{\nu}) := H^+(\mathbf{a}, -\boldsymbol{\nu}),$$

for a given $\mathbf{a} \in \mathbb{R}^n$ and a nonzero vector $\boldsymbol{\nu} \in \mathbb{R}^n$. The set of unit vectors in $\mathbb{R}^n$ is denoted by $\mathbb{S}^{n-1}$.

The *reduced boundary* $\partial^* A$ of $A$ is the set of $\mathbf{y} \in \mathbb{R}^n$ for which there exists $\boldsymbol{\nu}_A(\mathbf{y}) \in \mathbb{S}^{n-1}$ (necessarily unique) such that

$$D(A \cap H^-(\mathbf{y}, \boldsymbol{\nu}_A(\mathbf{y})), \mathbf{y}) = \frac{1}{2} \quad \text{and} \quad D(A \cap H^+(\mathbf{y}, \boldsymbol{\nu}_A(\mathbf{y})), \mathbf{y}) = 0.$$

This definition may differ from other notions of *reduced* or *essential* or *measure-theoretic* boundary used in the literature, but, thanks to Federer's [10] theorem (see also [1, Th. 3.61] or [27, Sect. 5.6]), they all coincide $\mathcal{H}^{n-1}$-a.e. for sets of finite perimeter. In particular, if $\operatorname{Per} A < \infty$ then $\operatorname{Per} A = \mathcal{H}^{n-1}(\partial^* A)$, and if $A$ is an open set with a $C^1$ boundary then $\partial A = \partial^* A$.

2.3. **Approximate differentiability and functions of bounded variation.** We assume that the reader has some familiarity with the set $BV$ of functions of bounded variation, and of special bounded variation $SBV$; see [10, 27, 1], if necessary, for the definitions. This subsection is meant primarily to set some notation.

**Definition 2.1.** *Let $A$ be a measurable set in $\mathbb{R}^n$, and $\mathbf{u} : A \to \mathbb{R}^n$ a measurable function. Let $\mathbf{x}_0 \in \mathbb{R}^n$ satisfy $D(A, \mathbf{x}_0) = 1$.*

*a) We say that $\mathbf{x}_0$ is an approximate continuity point of $\mathbf{u}$ if there exists $\mathbf{y}_0 \in \mathbb{R}^n$ such that*

$$D\left(\{\mathbf{x} \in A : |\mathbf{u}(\mathbf{x}) - \mathbf{y}_0| \geq \delta\}, \mathbf{x}_0\right) = 0$$

*for all $\delta > 0$. In this case, $\mathbf{y}_0$ is uniquely determined and called the approximate limit of $\mathbf{u}$ at $\mathbf{x}_0$. The complement in $A$ of the sets of approximate continuity points of $\mathbf{u}$ is denoted by $S_{\mathbf{u}}$.*

*b) We say that $\mathbf{x}_0$ is an approximate jump point of $\mathbf{u}$ if there exist $\mathbf{u}^+(\mathbf{x}_0), \mathbf{u}^-(\mathbf{x}_0) \in \mathbb{R}^n$ and $\boldsymbol{\nu}_{\mathbf{u}}(\mathbf{x}_0) \in \mathbb{S}^{n-1}$ such that $\mathbf{u}^+(\mathbf{x}_0) \neq \mathbf{u}^-(\mathbf{x}_0)$ and*

$$D\left(\left\{\mathbf{x} \in A \cap H^\pm(\mathbf{x}_0, \boldsymbol{\nu}_{\mathbf{u}}(\mathbf{x}_0)) : \left|\mathbf{u}(\mathbf{x}) - \mathbf{u}^\pm(\mathbf{x}_0)\right| \geq \delta\right\}, \mathbf{x}_0\right) = 0$$

*for all $\delta > 0$. The set of approximate jump points of $\mathbf{u}$ is denoted by $J_{\mathbf{u}}$.*

*c) We say that $\mathbf{u}$ is approximately differentiable at $\mathbf{x}_0 \in A$ if there exist $\mathbf{y}_0 \in \mathbb{R}^n$ and $\mathbf{L} \in \mathbb{R}^{n\times n}$ such that*

$$D\left(\left\{\mathbf{x} \in A \setminus \{\mathbf{x}_0\} : \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{y}_0 - \mathbf{L}(\mathbf{x} - \mathbf{x}_0)|}{|\mathbf{x} - \mathbf{x}_0|} \geq \delta\right\}, \mathbf{x}_0\right) = 0$$

*for all $\delta > 0$. In this case, $\mathbf{y}_0$ is the approximate limit of $\mathbf{u}$ at $\mathbf{x}_0$, and $\mathbf{L}$, which is also uniquely determined, is called the approximate differential of $\mathbf{u}$ at $\mathbf{x}_0$, and is denoted by $\nabla\mathbf{u}(\mathbf{x}_0)$.*

If $\mathbf{u} \in BV(\Omega, \mathbb{R}^n)$, we denote by $D\mathbf{u}$ the distributional derivative of $\mathbf{u}$, which is a Radon measure in $\Omega$. Standard results in the theory of $BV$ functions show that $\mathbf{u}$ is approximately differentiable a.e. and there exist Borel maps $\mathbf{u}^\pm : J_{\mathbf{u}} \to \mathbb{R}^n$ and $\boldsymbol{\nu}_{\mathbf{u}} : J_{\mathbf{u}} \to \mathbb{S}^{n-1}$ satisfying the conditions of Definition 2.1 b). Note that $\boldsymbol{\nu}_{\mathbf{u}}(\mathbf{x})$ is uniquely determined up to a sign, for each $\mathbf{x} \in J_{\mathbf{u}}$; we will always assume that a Borel choice of $\boldsymbol{\nu}_{\mathbf{u}}$ has been done, in which case $\mathbf{u}^\pm(\mathbf{x})$ are uniquely determined. Moreover, if $\mathbf{u} \in SBV(\Omega, \mathbb{R}^n)$, we have that $J_{\mathbf{u}} \cong S_{\mathbf{u}}$ and the following decomposition holds (see, e.g., [1, Sect. 4.1]):

$$D\mathbf{u} = \nabla\mathbf{u}\, \mathcal{L}^n \llcorner \Omega + (\mathbf{u}^+ - \mathbf{u}^-) \otimes \boldsymbol{\nu}_{\mathbf{u}}\, \mathcal{H}^{n-1} \llcorner J_{\mathbf{u}}. \tag{2.2}$$

2.4. **Area formulas and geometric image.** We will use the following version of Federer's [10] area formula.

**Proposition 2.2.** *Let $\mathbf{u} : \Omega \to \mathbb{R}^n$ be measurable, approximately differentiable a.e., and denote the set of approximate differentiability points of $\mathbf{u}$ by $\Omega_d$. Then, for any measurable set $A \subset \Omega$ and any measurable function $\varphi : \mathbb{R}^n \to \mathbb{R}$,*

$$\int_A \varphi(\mathbf{u}(\mathbf{x}))\, |\det D\mathbf{u}(\mathbf{x})|\, \mathrm{d}\mathbf{x} = \int_{\operatorname{im}_{\mathrm{G}}(\mathbf{u}, A)} \varphi(\mathbf{y})\, \mathcal{H}^0\left(\{\mathbf{x} \in \Omega_d \cap A : \mathbf{u}(\mathbf{x}) = \mathbf{y}\}\right) \mathrm{d}\mathbf{y},$$

*whenever either integral exists. Moreover, if $\psi : A \to \mathbb{R}$ is measurable $\bar{\psi} : \mathbf{u}(\Omega_d \cap A) \to \mathbb{R}$ is given by*

$$\bar{\psi}(\mathbf{y}) := \sum_{\substack{\mathbf{x} \in \Omega_d \cap A \\ \mathbf{u}(\mathbf{x}) = \mathbf{y}}} \psi(\mathbf{x}),$$

*then $\bar{\psi}$ is measurable and*

$$\int_A \psi(\mathbf{x})\, \varphi(\mathbf{u}(\mathbf{x}))\, |\det D\mathbf{u}(\mathbf{x})|\, \mathrm{d}\mathbf{x} = \int_{\mathbf{u}(\Omega_d \cap A)} \bar{\psi}(\mathbf{y})\, \varphi(\mathbf{y})\, \mathrm{d}\mathbf{y},$$

*whenever the integral of the left-hand side exists.*

We present the notion of the *geometric image* of a set (see [21, 7, 14]).

**Definition 2.3.** *Let $\mathbf{u} \in W^{1,1}(\Omega, \mathbb{R}^n)$ and suppose that $\det D\mathbf{u} > 0$ a.e. Define $\Omega_0$ as the set of $\mathbf{x} \in \Omega$ for which the following are satisfied:*

*i) the approximate differential of $\mathbf{u}$ at $\mathbf{x}$ exists and equals $D\mathbf{u}(\mathbf{x})$,*
*ii) there exist $\mathbf{w} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and a compact set $K \subset \Omega$ of density 1 at $\mathbf{x}$ such that $\mathbf{u}|_K = \mathbf{w}|_K$ and $D\mathbf{u}|_K = D\mathbf{w}|_K$, and*
*iii) $\det D\mathbf{u}(\mathbf{x}) > 0$.*

*For any measurable set $A$ of $\Omega$, we define the geometric image of $A$ under $\mathbf{u}$ as $\mathbf{u}(A \cap \Omega_0)$, and denote it by $\operatorname{im}_{\mathrm{G}}(\mathbf{u}, A)$.*

Standard arguments, essentially due to Federer [10, Thms. 3.1.8 and 3.1.16] (see also [21, Prop. 2.4] and [7, Rk. 2.5]), show that the set $\Omega_0$ in the Definition 2.3 is of full measure in $\Omega$.

Before stating the change of variables formula in $(n-1)$-dimensional surfaces for approximately differentiable maps, we present the notion of tangential approximate differentiability (cf. [10, Def. 3.2.16]).

**Definition 2.4.** *Let $S \subset \mathbb{R}^n$ be a $C^1$ differentiable manifold of dimension $n-1$, and let $\mathbf{x}_0 \in S$. Let $T_{\mathbf{x}_0} S$ be the linear tangent space of $S$ at $\mathbf{x}_0$. A map $\mathbf{u} : S \to \mathbb{R}^n$ is said to be $\mathcal{H}^{n-1} \llcorner S$-approximately differentiable at $\mathbf{x}_0$ if there exist $\mathbf{y}_0 \in \mathbb{R}^n$ and $\mathbf{L} \in \mathbb{R}^{n\times n}$ such that for all $\delta > 0$,*

$$\lim_{r \searrow 0} \frac{1}{r^{n-1}} \mathcal{H}^{n-1} \left( \left\{ \mathbf{x} \in S \cap B(\mathbf{x}_0, r) : \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{y}_0 - \mathbf{L}(\mathbf{x} - \mathbf{x}_0)|}{|\mathbf{x} - \mathbf{x}_0|} \geq \delta \right\} \right) = 0.$$

*In this case, $\mathbf{y}_0$ is the approximate limit of $\mathbf{u}$ at $\mathbf{x}_0$, and the linear map $\mathbf{L}|_{T_{\mathbf{x}_0} S} : T_{\mathbf{x}_0} S \to \mathbb{R}^n$ is uniquely determined and called the tangential approximate derivative of $\mathbf{u}$ at $\mathbf{x}_0$. We denote it by $\nabla \mathbf{u}(\mathbf{x}_0)$.*

Starting from Federer's [10, Cor. 3.2.20] change of variables formula in surfaces and applying the standard technique of approximating nonnegative functions by a simple functions, we obtain the following result. Its formulation is taken from [14, Prop. 2.9].

**Proposition 2.5.** *Let $S \subset \Omega$ be an orientable $C^1$ differentiable manifold of dimension $n-1$ oriented by the unit vector field $\boldsymbol{\nu}$, and let $\mathbf{u} \in W^{1,1}(\Omega, \mathbb{R}^n)$ satisfy $\det D\mathbf{u} > 0$ a.e. Let $\Omega_0$ be the set of Definition 2.3. Suppose that a set $S_d \subset \Omega_0 \cap S$ exists such that $\mathcal{H}^{n-1}(S \setminus S_d) = 0$, and such that for every $\mathbf{x} \in S_d$ the restriction $\mathbf{u}|_S$ is $\mathcal{H}^{n-1} \llcorner S$-approximately differentiable at $\mathbf{x}$, and $\nabla(\mathbf{u}|_S)(\mathbf{x}) = D\mathbf{u}(\mathbf{x})|_{T_{\mathbf{x}} S}$. Assume that $\operatorname{cof} D\mathbf{u} \in L^1(S, \mathbb{R}^{n\times n})$. Then, for every bounded and $\mathcal{H}^{n-1}$-measurable $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$, and any $\mathcal{H}^{n-1}$-measurable subset $A \subset S$,*

$$\int_A \mathbf{g}(\mathbf{u}(\mathbf{x})) \cdot (\operatorname{cof} D\mathbf{u}(\mathbf{x})\, \boldsymbol{\nu}(\mathbf{x}))\, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}) = \int_{\mathbf{u}(S_d \cap A)} \mathbf{g}(\mathbf{y}) \cdot \tilde{\boldsymbol{\nu}}(\mathbf{y})\, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}), \tag{2.3}$$

*provided that the integral on the left-hand side of (2.3) exists, and where*

$$\tilde{\boldsymbol{\nu}}(\mathbf{y}) := \sum_{\substack{\mathbf{x} \in S_d \cap A \\ \mathbf{u}(\mathbf{x}) = \mathbf{y}}} \frac{(\operatorname{cof} D\mathbf{u}(\mathbf{x}))\boldsymbol{\nu}(\mathbf{x})}{|(\operatorname{cof} D\mathbf{u}(\mathbf{x}))\boldsymbol{\nu}(\mathbf{x})|}, \qquad \mathbf{y} \in \mathbf{u}(S_d \cap A).$$

We will see in Subsection 2.6 that the equality $\nabla(\mathbf{u}|_S)(\mathbf{x}) = D\mathbf{u}(\mathbf{x})|_{T_{\mathbf{x}} S}$ holds for most points $\mathbf{x}$ if $\mathbf{u}$ is a Sobolev map.

2.5. **Topological image and condition INV.** Even though in this paper we do not make an explicit use of degree theory, we ought to say that behind this theory there is the underlying concept of degree for $W^{1,p}$ maps with $p > n-1$, or for $W^{1,n-1} \cap L^\infty$ maps, which in fact is a particular case of the Brezis–Nirenberg [6] degree. We refer to [21, Prop. 2.1], [7, Def. 3.1] or [14, Prop. 2.10], but, just for completeness, we state an axiomatic definition.

**Proposition 2.6.** *Let $U \subset\subset \mathbb{R}^n$ be a nonempty open set with a $C^1$ boundary. Let $p > n-1$ and suppose that $\mathbf{u} \in W^{1,p}(\partial U, \mathbb{R}^n)$ or $\mathbf{u} \in W^{1,n-1}(\partial U, \mathbb{R}^n) \cap L^\infty(\partial U, \mathbb{R}^n)$. Then there exists a unique integer-valued $BV(\mathbb{R}^n)$ function, denoted by $\deg(\mathbf{u}, \partial U, \cdot)$, such that*

$$\int_{\partial U} \mathbf{g}(\mathbf{u}(\mathbf{x})) \cdot (\Lambda_{n-1}(D\mathbf{u}(\mathbf{x}))\,\boldsymbol{\nu}(\mathbf{x}))\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}) = \int_{\mathbb{R}^n} \deg(\mathbf{u}, \partial U, \mathbf{y})\,\mathrm{div}\,\mathbf{g}(\mathbf{y})\,\mathrm{d}\mathbf{y}$$

*for all $\mathbf{g} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, where $\boldsymbol{\nu}$ denotes the unit exterior normal to $U$.*

In Proposition 2.6, $D\mathbf{u}(\mathbf{x})$ denotes the distributional derivative of $\mathbf{u}$ at $\mathbf{x}$, which is a linear map from the tangent space $T_\mathbf{x}\partial U$ to $\mathbb{R}^n$. The linear map $\Lambda_{n-1}(D\mathbf{u}(\mathbf{x})) : \Lambda_{n-1}(T_\mathbf{x}\partial U) \to \mathbb{R}^n$ is defined by

$$\Lambda_{n-1}(D\mathbf{u}(\mathbf{x}))\,(\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_{n-1}) = D\mathbf{u}(\mathbf{x})\mathbf{a}_1 \wedge \cdots \wedge D\mathbf{u}(\mathbf{x})\mathbf{a}_{n-1}, \qquad \mathbf{a}_1, \dots, \mathbf{a}_{n-1} \in T_\mathbf{x}\partial U.$$

Here $\wedge$ denotes the exterior product between vectors in $\mathbb{R}^n$, and $\Lambda_{n-1}(T_\mathbf{x}\partial U)$ is the space of all alternating $(n-1)$ tensors in $T_\mathbf{x}\partial U$. In practice, one identifies the one-dimensional subspace $\Lambda_{n-1}(T_\mathbf{x}\partial U)$ with $\{\lambda\boldsymbol{\nu}(\mathbf{x}) : \lambda \in \mathbb{R}\}$ and finds that if $\tilde{\mathbf{L}} : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map extending $D\mathbf{u}(\mathbf{x})$, then

$$\Lambda_{n-1}(D\mathbf{u}(\mathbf{x}))\,\boldsymbol{\nu}(\mathbf{x}) = (\mathrm{cof}\,\tilde{\mathbf{L}})\,\boldsymbol{\nu}(\mathbf{x}).$$

The concept of topological image was introduced by Šverák [24] (see also [21] and [7]).

**Definition 2.7.** *Let $U \subset\subset \mathbb{R}^n$ be a nonempty open set with a $C^1$ boundary.*

*a) If $\mathbf{u} \in W^{1,p}(\partial U, \mathbb{R}^n)$ for some $p > n-1$, we define $\mathrm{im_T}(\mathbf{u}, U)$, the topological image of $U$ under $\mathbf{u}$, as the set of $\mathbf{y} \in \mathbb{R}^n$ such that $\deg(\mathbf{u}, \partial U, \mathbf{y}) \neq 0$.*

*b) If $\mathbf{u} \in W^{1,n-1}(\partial U, \mathbb{R}^n) \cap L^\infty(\partial U, \mathbb{R}^n)$, we define $\mathrm{im_T}(\mathbf{u}, U)$, the topological image of $U$ under $\mathbf{u}$, as the set of $\mathbf{y} \in \mathbb{R}^n$ such that $D(A_{\mathbf{u},U}, \mathbf{y}) = 1$, where $A_{\mathbf{u},U} := \{\mathbf{y} \in \mathbb{R}^n : \deg(\mathbf{u}, \partial U, \mathbf{y}) \neq 0\}$.*

In case *a)*, due to the continuity of the degree for $W^{1,p}$ maps when $p > n-1$, the set $\mathrm{im_T}(\mathbf{u}, U)$ is open, while in case *b)*, thanks to the following lemma, a point $\mathbf{y} \in \mathbb{R}^n$ belongs to $\mathrm{im_T}(\mathbf{u}, U)$ if and only if $D(\mathrm{im_T}(\mathbf{u}, U), \mathbf{y}) = 1$.

**Lemma 2.8.** *Let $A$ be a measurable set of $\mathbb{R}^n$, and let $B$ the set of $\mathbf{x} \in \mathbb{R}^n$ such that $D(A, \mathbf{x}) = 1$. Then $B$ coincides with the set of $\mathbf{x} \in \mathbb{R}^n$ such that $D(B, \mathbf{x}) = 1$.*

*Proof.* By Lebesgue's density theorem, $A = B$ a.e. Therefore, for each $\mathbf{x} \in \mathbb{R}^n$ and $r > 0$,

$$\frac{\mathcal{L}^n(B \cap B(\mathbf{x}, r))}{\mathcal{L}^n(B(\mathbf{x}, r))} = \frac{\mathcal{L}^n(A \cap B(\mathbf{x}, r))}{\mathcal{L}^n(B(\mathbf{x}, r))}.$$

Taking limits when $r \searrow 0$ in the above expression, we find that $D(A, \mathbf{x}) = 1$ if and only if $D(B, \mathbf{x}) = 1$, and, hence, the conclusion of the statement follows. ☐

Condition INV (see [21, 7]) is defined as follows.

**Definition 2.9.** *Let $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n)$ with $p > n-1$ or $\mathbf{u} \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$. We say that $\mathbf{u}$ satisfies condition INV provided that for every $\mathbf{x}_0 \in \Omega$ and a.e. $r \in (0, \mathrm{dist}(\mathbf{x}_0, \partial\Omega))$, the following conditions hold:*

*i) $\mathbf{u}(\mathbf{x}) \in \mathrm{im_T}(\mathbf{u}, B(\mathbf{x}_0, r))$ for a.e. $\mathbf{x} \in B(\mathbf{x}_0, r)$.*

*ii) $\mathbf{u}(\mathbf{x}) \notin \mathrm{im_T}(\mathbf{u}, B(\mathbf{x}_0, r))$ for a.e. $\mathbf{x} \in \Omega \setminus B(\mathbf{x}_0, r)$.*

2.6. **A class of good open sets.** In the following definition, given a nonempty open set $U \subset\subset \Omega$ with a $C^2$ boundary, we call $d : \Omega \to \mathbb{R}$ the function given by

$$d(\mathbf{x}) := \begin{cases} \mathrm{dist}(\mathbf{x}, \partial U) & \text{if } \mathbf{x} \in U \\ 0 & \text{if } \mathbf{x} \in \partial U \\ -\mathrm{dist}(\mathbf{x}, \partial U) & \text{if } \mathbf{x} \in \Omega \setminus \bar{U} \end{cases}$$

and $U_t := \{\mathbf{x} \in \Omega : d(\mathbf{x}) > t\}$, for each $t \in \mathbb{R}$. We note that there exists $\delta > 0$ such that for all $t \in (-\delta, \delta)$, the set $U_t$ is open, compactly contained in $\Omega$ and has a $C^2$ boundary.

**Definition 2.10.** *Let $p > n-1$. Let $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n)$ or $\mathbf{u} \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ be such that* $\det D\mathbf{u} > 0$ *a.e. We define $\mathcal{U}$ as the family of nonempty open sets $U \subset\subset \Omega$ with a $C^2$ boundary that satisfy the following conditions:*

*(1)* $\mathbf{u}|_{\partial U} \in W^{1,p}(\partial U, \mathbb{R}^n)$ *or* $\mathbf{u}|_{\partial U} \in W^{1,n-1}(\partial U, \mathbb{R}^n) \cap L^\infty(\partial U, \mathbb{R}^n)$ *(according to whether* $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n)$ *or* $\mathbf{u} \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$*), and* $(\operatorname{cof} D\mathbf{u})|_{\partial U} \in L^1(\partial U, \mathbb{R}^{n\times n})$*.*

*(2)* $\partial U \tilde{\subset} \Omega_0$*, where* $\Omega_0$ *is the set of Definition 2.3, and* $D(\mathbf{u}|_{\partial U})(\mathbf{x})$ *coincides with the orthogonal projection of* $D\mathbf{u}(\mathbf{x})$ *onto* $T_{\mathbf{x}}\partial U$ *for* $\mathcal{H}^{n-1}$*-a.e.* $\mathbf{x} \in \partial U$*.*

*(3)* $\displaystyle\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^\varepsilon \left| \int_{\partial U_t} |\operatorname{cof} D\mathbf{u}| \, \mathrm{d}\mathcal{H}^{n-1} - \int_{\partial U} |\operatorname{cof} D\mathbf{u}| \mathrm{d}\, \mathcal{H}^{n-1} \right| \mathrm{d}t = 0.$

*(4)* *For every* $\mathbf{g} \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$*,*

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^\varepsilon \left| \int_{\partial U_t} \mathbf{g}(\mathbf{u}(\mathbf{x})) \cdot (\operatorname{cof} D\mathbf{u}(\mathbf{x})\, \boldsymbol{\nu}_t(\mathbf{x})) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}) - \int_{\partial U} \mathbf{g}(\mathbf{u}(\mathbf{x})) \cdot (\operatorname{cof} D\mathbf{u}(\mathbf{x})\, \boldsymbol{\nu}(\mathbf{x})) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}) \right| \mathrm{d}t = 0,$$

*where* $\boldsymbol{\nu}_t$ *denotes the unit outward normal to* $U_t$ *for each* $t \in (0, \varepsilon)$*, and* $\boldsymbol{\nu}$ *the unit outward normal to* $U$*.*

The family $\mathcal{U}$ depends on $\mathbf{u}$, but since $\mathbf{u}$ will be fixed throughout the paper, we do not emphasize this dependence. The following result guarantees that there are enough sets in $\mathcal{U}$ (see [13, Lemma 2 and Def. 11] or [14, Lemma 2.16]).

**Lemma 2.11.** *Let* $p > n - 1$*. Let* $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n)$ *or* $\mathbf{u} \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ *be such that* $\det D\mathbf{u} > 0$ *a.e. Let* $U \subset\subset \Omega$ *be a nonempty open set with a* $C^2$ *boundary. Then,* $U_t \in \mathcal{U}$ *for a.e.* $t \in (-\delta, \delta)$*. Moreover, for each compact* $K \subset \Omega$ *there exists* $U \in \mathcal{U}$ *such that* $K \subset U$*.*

A consequence of Lemma 2.11 is that there exists an increasing family $\{U_k\}_{k\in\mathbb{N}}$ in $\mathcal{U}$ such that $\Omega = \bigcup_{k\in\mathbb{N}} U_k$. For the rest of the paper, we fix that family and call it $\mathcal{U}_0$. For future reference, we note that

$$\Omega = \bigcup_{U \in \mathcal{U}_0} U, \qquad \operatorname{im_G}(\mathbf{u}, \Omega) = \bigcup_{U \in \mathcal{U}_0} \operatorname{im_G}(\mathbf{u}, U). \tag{2.4}$$

2.7. **Surface energy.** The following concepts were defined in [12, 15].

**Definition 2.12.** *Let* $\mathbf{u} : \Omega \to \mathbb{R}^n$ *be measurable and approximately differentiable a.e. Suppose that* $\det \nabla\mathbf{u} \in L^1(\Omega)$ *and* $\operatorname{cof} \nabla\mathbf{u} \in L^1(\Omega, \mathbb{R}^{n\times n})$*. For every* $\mathbf{f} \in C^1_c(\bar{\Omega} \times \mathbb{R}^n, \mathbb{R}^n)$*, define*

$$\mathcal{E}(\mathbf{u}, \mathbf{f}) := \int_\Omega [\operatorname{cof} \nabla\mathbf{u}(\mathbf{x}) \cdot D\mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x})) + \det \nabla\mathbf{u}(\mathbf{x}) \operatorname{div} \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x}))] \, \mathrm{d}\mathbf{x} \tag{2.5}$$

*and*

$$\mathcal{E}(\mathbf{u}) := \sup \left\{ \mathcal{E}(\mathbf{u}, \mathbf{f}) : \ \mathbf{f} \in C^1_c(\Omega \times \mathbb{R}^n, \mathbb{R}^n), \ \|\mathbf{f}\|_\infty \leq 1 \right\},$$
$$\bar{\mathcal{E}}(\mathbf{u}) := \sup \left\{ \mathcal{E}(\mathbf{u}, \mathbf{f}) : \ \mathbf{f} \in C^1_c(\bar{\Omega} \times \mathbb{R}^n, \mathbb{R}^n), \ \|\mathbf{f}\|_\infty \leq 1 \right\}.$$

In equation (2.5) and elsewhere in the paper, $D\mathbf{f}(\mathbf{x}, \mathbf{y})$ denotes the derivative of $\mathbf{f}(\cdot, \mathbf{y})$ evaluated at $\mathbf{x}$, while div always denotes the divergence operator in the deformed configuration, so $\operatorname{div} \mathbf{f}(\mathbf{x}, \mathbf{y})$ is the divergence of $\mathbf{f}(\mathbf{x}, \cdot)$ evaluated at $\mathbf{y}$.

The functional $\mathcal{E}$ was introduced in [12] to measure the creation of new surface of a deformation. The functional $\bar{\mathcal{E}}$ was introduced in [15], and its difference with respect to $\mathcal{E}$ is that $\bar{\mathcal{E}}$ also takes into account the stretching of $\partial\Omega$ by $\mathbf{u}$.

2.8. **Properties of the topological image.** We recall the notion of topological image of a point (see [24, p. 115], [21, p. 33] or [7, Def. 3.13]).

**Definition 2.13.** *Let* $p > n - 1$*. Let* $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n)$ *or* $\mathbf{u} \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$*. Suppose that* $\det D\mathbf{u} > 0$ *a.e., and let* $\mathbf{x} \in \Omega$*. The topological image of* $\mathbf{x}$ *under* $\mathbf{u}$ *is defined as*

$$\operatorname{im_T}(\mathbf{u}, \mathbf{x}) := \bigcap_{\substack{U \in \mathcal{U} \\ \mathbf{x} \in U}} \operatorname{im_T}(\mathbf{u}, U).$$

*We define* $C(\mathbf{u})$ *as the set of* $\mathbf{x} \in \Omega$ *such that* $\mathcal{L}^n(\operatorname{im_T}(\mathbf{u}, \mathbf{x})) > 0$*.*

It can be shown that the set $C(\mathbf{u})$ can be characterized as the atoms of the distributional determinant $\operatorname{Det} D\mathbf{u}$, but in this work we make no explicit use of $\operatorname{Det} D\mathbf{u}$.

The following was proved in [14, Prop. 2.17, Th. 3.2, Th. 4.2, Lemma 4.10].

**Proposition 2.14.** *Let $p > n-1$. Let $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n)$ or $\mathbf{u} \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$. Assume $\mathbf{u}$ satisfies condition INV, $\det D\mathbf{u} > 0$ a.e. and $\mathcal{E}(\mathbf{u}) < \infty$. Then*

*i) $C(\mathbf{u})$ is countable, $\operatorname{im_T}(\mathbf{u}, \mathbf{a})$ is of finite perimeter for each $\mathbf{a} \in C(\mathbf{u})$ and*

$$\mathcal{E}(\mathbf{u}) = \sum_{\mathbf{a} \in C(\mathbf{u})} \operatorname{Per} \operatorname{im_T}(\mathbf{u}, \mathbf{a}).$$

*ii) $\operatorname{im_T}(\mathbf{u}, \mathbf{a}) \cap \operatorname{im_T}(\mathbf{u}, \mathbf{b}) = \varnothing$ and $\partial^* \operatorname{im_T}(\mathbf{u}, \mathbf{a}) \cap \partial^* \operatorname{im_T}(\mathbf{u}, \mathbf{b}) \cong \varnothing$ for any $\mathbf{a}, \mathbf{b} \in C(\mathbf{u})$ with $\mathbf{a} \neq \mathbf{b}$.*

*iii) $\operatorname{im_T}(\mathbf{u}, U) \subset \operatorname{im_T}(\mathbf{u}, V)$ if $U, V \in \mathcal{U}$ satisfy $U \subset V$.*

*iv) For each $U \in \mathcal{U}$, the set $\operatorname{im_T}(\mathbf{u}, U)$ is of finite perimeter, $\partial^* \operatorname{im_T}(\mathbf{u}, U) \cong \operatorname{im_G}(\mathbf{u}, \partial U)$ and*

$$\operatorname{im_T}(\mathbf{u}, U) = \operatorname{im_G}(\mathbf{u}, U) \cup \bigcup_{\mathbf{a} \in C(\mathbf{u}) \cap U} \operatorname{im_T}(\mathbf{u}, \mathbf{a}) \quad \text{a.e.}$$

*v) $\operatorname{im_G}(\mathbf{u}, \Omega) \cap \operatorname{im_T}(\mathbf{u}, \mathbf{a}) = \operatorname{im_G}(\mathbf{u}, \{\mathbf{a}\})$ for each $\mathbf{a} \in C(\mathbf{u})$.*

Define

$$\operatorname{im_T}(\mathbf{u}, \Omega) := \bigcup_{U \in \mathcal{U}} \operatorname{im_T}(\mathbf{u}, U), \tag{2.6}$$

which, thanks to the definition of $\mathcal{U}_0$, also satisfies

$$\operatorname{im_T}(\mathbf{u}, \Omega) = \bigcup_{U \in \mathcal{U}_0} \operatorname{im_T}(\mathbf{u}, U). \tag{2.7}$$

Equalities (2.4), (2.7) and Proposition 2.14 imply that

$$\operatorname{im_T}(\mathbf{u}, \Omega) = \operatorname{im_G}(\mathbf{u}, \Omega) \cup \bigcup_{\mathbf{a} \in C(\mathbf{u})} \operatorname{im_T}(\mathbf{u}, \mathbf{a}) \quad \text{a.e.} \tag{2.8}$$

with disjoint union up to a countable set.

2.9. **Inverses of one-to-one a.e. maps.** The following result comprises results of [21, Lemma 3.4], [7, Lemma 3.9], [13, Lemma 3] and [14, Th. 2].

**Lemma 2.15.** *Let $p > n-1$. Let $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n)$ or $\mathbf{u} \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$. Let $\mathbf{u}$ satisfy condition INV and $\det D\mathbf{u} > 0$ a.e. Let $\Omega_0$ be as in Definition 2.3. Then $\mathbf{u}|_{\Omega_0}$ is one-to-one. Moreover, for each $\mathbf{y} \in \operatorname{im_G}(\mathbf{u}, \Omega)$,*

$$\nabla \mathbf{u}^{-1}(\mathbf{y}) = \nabla \mathbf{u}(\mathbf{u}^{-1}(\mathbf{y}))^{-1}.$$

Under the assumptions of Lemma 2.15, the inverse $\mathbf{u}^{-1}$ is defined on $\operatorname{im_G}(\mathbf{u}, \Omega)$. Moreover, for any $U \in \mathcal{U}$ or $U = \Omega$ define $\tilde{\mathbf{u}}_U^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\tilde{\mathbf{u}}_U^{-1}(\mathbf{y}) := \begin{cases} \mathbf{u}^{-1}(\mathbf{y}) & \text{if } \mathbf{y} \in \operatorname{im_G}(\mathbf{u}, U), \\ \mathbf{a} & \text{if } \mathbf{y} \in \operatorname{im_T}(\mathbf{u}, \mathbf{a}) \text{ for some } \mathbf{a} \in C(\mathbf{u}) \cap U, \\ \mathbf{0} & \text{otherwise.} \end{cases} \tag{2.9}$$

Thanks to Proposition 2.14 and (2.8), the function $\tilde{\mathbf{u}}_U^{-1}$ is well defined a.e.

In [15, Prop. 3.2], the following regularity result is proved.

**Proposition 2.16.** *Let $\mathbf{u} \in L^\infty(\Omega, \mathbb{R}^n)$ be measurable, approximately differentiable a.e., one-to-one a.e., and such that $\det \nabla \mathbf{u} > 0$ a.e., $\operatorname{cof} \nabla \mathbf{u} \in L^1(\Omega, \mathbb{R}^{n \times n})$ and $\bar{\mathcal{E}}(\mathbf{u}) < \infty$. Then the function $\mathbf{u}_\Omega^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ defined as*

$$\mathbf{u}_\Omega^{-1}(\mathbf{y}) := \begin{cases} \mathbf{u}^{-1}(\mathbf{y}) & \text{if } \mathbf{y} \in \operatorname{im_G}(\mathbf{u}, \Omega), \\ \mathbf{0} & \text{if } \mathbf{y} \in \mathbb{R}^n \setminus \operatorname{im_G}(\mathbf{u}, \Omega) \end{cases}$$

*is in $SBV(\mathbb{R}^n, \mathbb{R}^n)$.*

## 3. Regularity of inverses

In this section we prove the main results of the paper. We start with an identity that was somewhat implicit in the proof of [14, Prop. 5.1]. Below and in the rest of the section, the divergence of an $\mathbb{R}^{n\times n}$-valued function is defined as the $\mathbb{R}^n$-valued function whose components are the divergences of the rows.

**Lemma 3.1.** *Let $p > n-1$. Let $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n)$ or $\mathbf{u} \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$. Assume $\mathbf{u}$ satisfies condition INV, $\det D\mathbf{u} > 0$ a.e. and $\mathcal{E}(\mathbf{u}) < \infty$. Let $U \in \mathcal{U}$ and $\mathbf{G} \in C^1_c(\mathbb{R}^n, \mathbb{R}^{n\times n})$. Then*

$$\begin{aligned} &-\int_{\partial U} \mathbf{x} \cdot (\mathbf{G}(\mathbf{u}(\mathbf{x})) \operatorname{cof} D\mathbf{u}(\mathbf{x})\, \boldsymbol{\nu}_U(\mathbf{x}))\, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}) + \int_U \operatorname{adj} D\mathbf{u}(\mathbf{x}) \cdot \mathbf{G}(\mathbf{u}(\mathbf{x}))\, \mathrm{d}\mathbf{x} \\ &+ \int_U \mathbf{x} \cdot \operatorname{div} \mathbf{G}(\mathbf{u}(\mathbf{x})) \det D\mathbf{u}(\mathbf{x})\, \mathrm{d}\mathbf{x} = -\sum_{\mathbf{a}\in C(\mathbf{u})\cap U} \mathbf{a} \cdot \int_{\partial^* \operatorname{im_T}(\mathbf{u},\mathbf{a})} \mathbf{G}(\mathbf{y})\, \boldsymbol{\nu}_{\operatorname{im_T}(\mathbf{u},\mathbf{a})}(\mathbf{y})\, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}). \end{aligned} \tag{3.1}$$

*Proof.* Let $\varphi \in C^1(\mathbb{R})$ satisfy $\varphi(t) = 0$ for $t \leq 0$, $\varphi(t) = 1$ for $t \geq 1$, and $\varphi' \geq 0$. For each $j \in \mathbb{N}$, define $\eta_j : \Omega \to \mathbb{R}$ as $\eta_j(\mathbf{x}) := \varphi(j \operatorname{dist}(\mathbf{x}, \partial U))$ and $\boldsymbol{\phi}_j : \Omega \to \mathbb{R}^n$ as $\boldsymbol{\phi}_j(\mathbf{x}) := \eta_j(\mathbf{x})\, \mathbf{x}$. It is easy to show that there exists $j_0 \in \mathbb{N}$ such that the functions $\eta_j$ and $\boldsymbol{\phi}_j$ are of class $C^1_c$ for all $j \geq j_0$. For each $1 \leq \alpha \leq n$, call $\mathbf{g}_\alpha$ the $\alpha$-th row of $\mathbf{G}$, and $\phi_j^\alpha$ the $\alpha$-th component of $\boldsymbol{\phi}_j$. A direct computation from (2.5) using (2.1) yields

$$\begin{aligned} \mathcal{E}(\mathbf{u}, \phi_j^\alpha\, \mathbf{g}_\alpha) = &\int_\Omega x^\alpha\, \mathbf{g}_\alpha(\mathbf{u}(\mathbf{x})) \cdot (\operatorname{cof} D\mathbf{u}(\mathbf{x})\, D\eta_j(\mathbf{x}))\, \mathrm{d}\mathbf{x} \\ &+ \int_\Omega [\eta_j(\mathbf{x})\, (\operatorname{cof} D\mathbf{u}(\mathbf{x})\, \mathbf{e}_\alpha) \cdot \mathbf{g}_\alpha(\mathbf{u}(\mathbf{x})) + x^\alpha\, \eta_j(\mathbf{x}) \operatorname{div} \mathbf{g}_\alpha(\mathbf{u}(\mathbf{x})) \det D\mathbf{u}(\mathbf{x})]\, \mathrm{d}\mathbf{x}, \end{aligned}$$

for each $j \geq j_0$ and $1 \leq \alpha \leq n$. Here $\mathbf{e}_\alpha \in \mathbb{R}^n$ is the $\alpha$-th vector of the canonical basis, and $x^\alpha := \mathbf{x} \cdot \mathbf{e}_\alpha$. It was shown in the proof of [13, Th. 2] that

$$\begin{aligned} \lim_{j\to\infty} \mathcal{E}(\mathbf{u}, \phi_j^\alpha\, \mathbf{g}_\alpha) = &-\int_{\partial U} x^\alpha\, \mathbf{g}_\alpha(\mathbf{u}(\mathbf{x})) \cdot (\operatorname{cof} D\mathbf{u}(\mathbf{x})\, \boldsymbol{\nu}_U(\mathbf{x}))\, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}) \\ &+ \int_U [(\operatorname{cof} D\mathbf{u}(\mathbf{x})\, \mathbf{e}_\alpha) \cdot \mathbf{g}_\alpha(\mathbf{u}(\mathbf{x})) + x^\alpha \operatorname{div} \mathbf{g}_\alpha(\mathbf{u}(\mathbf{x})) \det D\mathbf{u}(\mathbf{x})]\, \mathrm{d}\mathbf{x}. \end{aligned} \tag{3.2}$$

On the other hand, it was shown in the proof of [14, Prop. 5.1] that

$$\lim_{j\to\infty} \mathcal{E}(\mathbf{u}, \phi_j^\alpha\, \mathbf{g}_\alpha) = -\sum_{\mathbf{a}\in C(\mathbf{u})\cap U} a^\alpha \int_{\partial^* \operatorname{im_T}(\mathbf{u},\mathbf{a})} \mathbf{g}_\alpha(\mathbf{y}) \cdot \boldsymbol{\nu}_{\operatorname{im_T}(\mathbf{u},\mathbf{a})}(\mathbf{y})\, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}). \tag{3.3}$$

Comparing (3.2) and (3.3), and taking sums in $\alpha$, we obtain equality (3.1). $\square$

We now calculate the distributional derivative of the function $\tilde{\mathbf{u}}_U^{-1}$ of (2.9).

**Proposition 3.2.** *Let $p > n-1$. Let $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n)$ or $\mathbf{u} \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$. Assume $\mathbf{u}$ satisfies condition INV, $\det D\mathbf{u} > 0$ a.e. and $\mathcal{E}(\mathbf{u}) < \infty$. Let $U \in \mathcal{U}$. Then $\tilde{\mathbf{u}}_U^{-1} \in SBV(\mathbb{R}^n, \mathbb{R}^n)$ with $J_{\tilde{\mathbf{u}}_U^{-1}} \cong \operatorname{im_G}(\mathbf{u}, \partial U)$ and*

$$\int_{\mathbb{R}^n} \tilde{\mathbf{u}}_U^{-1} \cdot \operatorname{div} \mathbf{G}\, \mathrm{d}\mathbf{y} = \int_{\operatorname{im_G}(\mathbf{u},\partial U)} \mathbf{u}^{-1} \cdot (\mathbf{G}\, \tilde{\boldsymbol{\nu}})\, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}) - \int_{\operatorname{im_G}(\mathbf{u},U)} \left(D\mathbf{u} \circ \mathbf{u}^{-1}\right)^{-1} \cdot \mathbf{G}\, \mathrm{d}\mathbf{y} \tag{3.4}$$

*for all $\mathbf{G} \in C^1_c(\mathbb{R}^n, \mathbb{R}^{n\times n})$, where*

$$\tilde{\boldsymbol{\nu}}(\mathbf{y}) := \frac{\operatorname{cof} D\mathbf{u}(\mathbf{x})\, \boldsymbol{\nu}_U(\mathbf{x})}{|\operatorname{cof} D\mathbf{u}(\mathbf{x})\, \boldsymbol{\nu}_U(\mathbf{x})|}, \qquad \mathbf{y} = \mathbf{u}(\mathbf{x}), \quad \mathbf{x} \in \partial U \cap \Omega_0. \tag{3.5}$$

*Proof.* Let $\mathbf{G} \in C_c^1(\mathbb{R}^n, \mathbb{R}^{n\times n})$. By Propositions 2.14, 2.2 and 2.5, the Gauss–Green theorem (e.g., [9, Th. 5.8.1]) and Lemma 3.1, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \tilde{\mathbf{u}}_U^{-1}(\mathbf{y}) \cdot \operatorname{div} \mathbf{G}(\mathbf{y}) \, \mathrm{d}\mathbf{y} \\
&= \int_{\mathrm{im_G}(\mathbf{u},U)} \mathbf{u}^{-1}(\mathbf{y}) \cdot \operatorname{div} \mathbf{G}(\mathbf{y}) \, \mathrm{d}\mathbf{y} + \sum_{\mathbf{a}\in C(\mathbf{u})\cap U} \mathbf{a} \cdot \int_{\mathrm{im_T}(\mathbf{u},\mathbf{a})} \operatorname{div} \mathbf{G}(\mathbf{y}) \, \mathrm{d}\mathbf{y} \\
&= \int_U \mathbf{x} \cdot \operatorname{div} \mathbf{G}(\mathbf{u}(\mathbf{x})) \det D\mathbf{u}(\mathbf{x}) \, \mathrm{d}\mathbf{x} + \sum_{\mathbf{a}\in C(\mathbf{u})\cap U} \mathbf{a} \cdot \int_{\partial^* \mathrm{im_T}(\mathbf{u},\mathbf{a})} \mathbf{G}(\mathbf{y}) \, \boldsymbol{\nu}_{\mathrm{im_T}(\mathbf{u},\mathbf{a})}(\mathbf{y}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}) \\
&= \int_{\partial U} \mathbf{x} \cdot (\mathbf{G}(\mathbf{u}(\mathbf{x})) \operatorname{cof} D\mathbf{u}(\mathbf{x}) \, \boldsymbol{\nu}_U(\mathbf{x})) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}) - \int_U \operatorname{adj} D\mathbf{u}(\mathbf{x}) \cdot \mathbf{G}(\mathbf{u}(\mathbf{x})) \, \mathrm{d}\mathbf{x} \\
&= \int_{\mathrm{im_G}(\mathbf{u},\partial U)} \mathbf{u}^{-1}(\mathbf{y}) \cdot (\mathbf{G}(\mathbf{y}) \, \tilde{\boldsymbol{\nu}}(\mathbf{y})) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}) - \int_{\mathrm{im_G}(\mathbf{u},U)} \big(D\mathbf{u}(\mathbf{u}^{-1}(\mathbf{y}))\big)^{-1} \cdot \mathbf{G}(\mathbf{y}) \, \mathrm{d}\mathbf{y},
\end{aligned}$$

where we used the notation (3.5). This shows (3.4), which can be rewritten as

$$D\tilde{\mathbf{u}}_U^{-1} = \big(D\mathbf{u} \circ \mathbf{u}^{-1}\big)^{-1} \mathcal{L}^n \llcorner \mathrm{im_G}(\mathbf{u}, U) - \mathbf{u}^{-1} \otimes \tilde{\boldsymbol{\nu}} \, \mathcal{H}^{n-1} \llcorner \mathrm{im_G}(\mathbf{u}, \partial U). \tag{3.6}$$

Now, by Proposition 2.2,

$$\int_{\mathrm{im_G}(\mathbf{u},U)} \Big| \big(D\mathbf{u}(\mathbf{u}^{-1}(\mathbf{y}))\big)^{-1} \Big| \, \mathrm{d}\mathbf{y} = \int_U \big| D\mathbf{u}(\mathbf{x})^{-1} \det D\mathbf{u}(\mathbf{x}) \big| \, \mathrm{d}\mathbf{x} = \int_U |\operatorname{adj} D\mathbf{u}(\mathbf{x})| \, \mathrm{d}\mathbf{x} < \infty, \tag{3.7}$$

while, thanks to Propositions 2.5 and 2.14,

$$\int_{\mathrm{im_G}(\mathbf{u},\partial U)} \big| \mathbf{u}^{-1} \otimes \tilde{\boldsymbol{\nu}} \big| \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}) \le \|\mathbf{id}\|_{L^\infty(\Omega,\mathbb{R}^n)} \, \mathcal{H}^{n-1}(\mathrm{im_G}(\mathbf{u}, \partial U)) = \|\mathbf{id}\|_{L^\infty(\Omega,\mathbb{R}^n)} \operatorname{Per} \mathrm{im_T}(\mathbf{u}, U) < \infty. \tag{3.8}$$

From (3.6), (3.7) and (3.8), we conclude that $\tilde{\mathbf{u}}_U^{-1} \in SBV(\mathbb{R}^n, \mathbb{R}^n)$. Therefore, a comparison between expressions (3.6) and (2.2) reveals that

$$\big((\tilde{\mathbf{u}}_U^{-1})^+ - (\tilde{\mathbf{u}}_U^{-1})^-\big) \otimes \boldsymbol{\nu}_{\tilde{\mathbf{u}}_U^{-1}} \, \mathcal{H}^{n-1} \llcorner J_{\tilde{\mathbf{u}}_U^{-1}} = -\mathbf{u}^{-1} \otimes \tilde{\boldsymbol{\nu}} \, \mathcal{H}^{n-1} \llcorner \mathrm{im_G}(\mathbf{u}, \partial U). \tag{3.9}$$

As $\mathbf{u}^{-1} : \mathrm{im_G}(\mathbf{u}, \Omega) \to \mathbb{R}^n$ is one-to-one, it takes the value $\mathbf{0}$ in at most one point. In particular, $\mathbf{u}^{-1}(\mathbf{y}) \neq \mathbf{0}$ for $\mathcal{H}^{n-1}$-a.e. $\mathbf{y} \in \mathrm{im_G}(\mathbf{u}, \partial U)$. We conclude from (3.9) that $J_{\tilde{\mathbf{u}}_U^{-1}} \cong \mathrm{im_G}(\mathbf{u}, \partial U)$. □

In Proposition 3.2, the introduction of the open set $U$ is somewhat artificial. The difficulty is that $\Omega \notin \mathcal{U}$ and, in particular, there is no guarantee that any of conditions of Definition 2.10 is satisfied for $U = \Omega$. The way to overcome this obstacle is different for the cases $p > n-1$ and $p = n-1$. The following theorem presents the regularity result when $p > n-1$. In this case, thanks to the definition (2.6) and the continuity of the degree (see Subsection 2.5), the set $\mathrm{im_T}(\mathbf{u}, \Omega)$ is open and the support of any function in $C_c^1(\mathrm{im_T}(\mathbf{u}, \Omega))$ is contained in $\mathrm{im_T}(\mathbf{u}, U)$ for a certain $U \in \mathcal{U}$.

**Theorem 3.3.** *Let $p > n-1$. Let $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R})$ satisfy condition INV and be such that $\det D\mathbf{u} > 0$ a.e. and $\mathcal{E}(\mathbf{u}) < \infty$. Define $\tilde{\mathbf{u}}^{-1} : \mathrm{im_T}(\mathbf{u}, \Omega) \to \mathbb{R}^n$ as*

$$\tilde{\mathbf{u}}^{-1}(\mathbf{y}) := \begin{cases} \mathbf{u}^{-1}(\mathbf{y}) & \text{if } \mathbf{y} \in \mathrm{im_G}(\mathbf{u}, \Omega), \\ \mathbf{a} & \text{if } \mathbf{y} \in \mathrm{im_T}(\mathbf{u}, \mathbf{a}) \text{ for some } \mathbf{a} \in C(\mathbf{u}). \end{cases} \tag{3.10}$$

*Then $\tilde{\mathbf{u}}^{-1} \in W^{1,1}(\mathrm{im_T}(\mathbf{u}, \Omega), \mathbb{R}^n)$ and*

$$D\tilde{\mathbf{u}}^{-1}(\mathbf{y}) = \begin{cases} D\mathbf{u}(\mathbf{u}^{-1}(\mathbf{y}))^{-1} & \text{if } \mathbf{y} = \mathrm{im_G}(\mathbf{u}, \Omega), \\ \mathbf{0} & \text{otherwise.} \end{cases} \tag{3.11}$$

*Proof.* Note that, in the notation of (2.9), we have $\tilde{\mathbf{u}}^{-1} = \tilde{\mathbf{u}}_\Omega^{-1}|_{\mathrm{im_T}(\mathbf{u},\Omega)}$, and, in particular, $\tilde{\mathbf{u}}^{-1}$ is well defined a.e. thanks to equality (2.8).

Let $\mathbf{G} \in C_c^1(\mathrm{im_T}(\mathbf{u},\Omega), \mathbb{R}^{n\times n})$, and let $\bar{\mathbf{G}} \in C_c^1(\mathbb{R}^n, \mathbb{R}^{n\times n})$ be the extension of $\mathbf{G}$ by zero. Thanks to Lemma 2.11, there exists $U \in \mathcal{U}$ such that the support of $\mathbf{G}$ is contained in $\mathrm{im_T}(\mathbf{u}, U)$. By Proposition 3.2, and, in particular, equality (3.4),

$$\int_{\mathbb{R}^n} \tilde{\mathbf{u}}_U^{-1} \cdot \operatorname{div} \bar{\mathbf{G}} \, \mathrm{d}\mathbf{y} = \int_{\mathrm{im_G}(\mathbf{u},\partial U)} \mathbf{u}^{-1} \cdot \big(\bar{\mathbf{G}}\, \tilde{\boldsymbol{\nu}}\big) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}) - \int_{\mathrm{im_G}(\mathbf{u},U)} \big(D\mathbf{u} \circ \mathbf{u}^{-1}\big)^{-1} \cdot \bar{\mathbf{G}} \, \mathrm{d}\mathbf{y}, \tag{3.12}$$

where $\tilde{\boldsymbol{\nu}}$ is as in (3.5). Now, the definitions of $\bar{\mathbf{G}}$ and $\tilde{\mathbf{u}}^{-1}$ yield

$$\int_{\mathbb{R}^n} \tilde{\mathbf{u}}_U^{-1} \cdot \operatorname{div} \bar{\mathbf{G}} \, \mathrm{d}\mathbf{y} = \int_{\mathrm{im_T}(\mathbf{u},U)} \tilde{\mathbf{u}}_U^{-1} \cdot \operatorname{div} \mathbf{G} \, \mathrm{d}\mathbf{y} = \int_{\mathrm{im_T}(\mathbf{u},U)} \tilde{\mathbf{u}}^{-1} \cdot \operatorname{div} \mathbf{G} \, \mathrm{d}\mathbf{y} = \int_{\mathrm{im_T}(\mathbf{u},\Omega)} \tilde{\mathbf{u}}^{-1} \cdot \operatorname{div} \mathbf{G} \, \mathrm{d}\mathbf{y} \tag{3.13}$$

and

$$\int_{\mathrm{im_G}(\mathbf{u},U)} \big(D\mathbf{u} \circ \mathbf{u}^{-1}\big)^{-1} \cdot \bar{\mathbf{G}} \, \mathrm{d}\mathbf{y} = \int_{\mathrm{im_G}(\mathbf{u},U)} \big(D\mathbf{u} \circ \mathbf{u}^{-1}\big)^{-1} \cdot \mathbf{G} \, \mathrm{d}\mathbf{y} = \int_{\mathrm{im_G}(\mathbf{u},\Omega)} \big(D\mathbf{u} \circ \mathbf{u}^{-1}\big)^{-1} \cdot \mathbf{G} \, \mathrm{d}\mathbf{y}. \tag{3.14}$$

Moreover, since $\bar{\mathbf{G}}$ vanishes in $\mathbb{R}^n \setminus \mathrm{im_T}(\mathbf{u}, U)$, and, hence, in $\partial^* \mathrm{im_T}(\mathbf{u}, U)$, we apply Proposition 2.14 to obtain that

$$\int_{\mathrm{im_G}(\mathbf{u},\partial U)} \mathbf{u}^{-1} \cdot \big(\bar{\mathbf{G}}\, \tilde{\boldsymbol{\nu}}\big) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}) = \int_{\partial^* \mathrm{im_T}(\mathbf{u},U)} \mathbf{u}^{-1} \cdot \big(\bar{\mathbf{G}}\, \tilde{\boldsymbol{\nu}}\big) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}) = 0. \tag{3.15}$$

Equalities (3.12), (3.13), (3.14) and (3.15) yield

$$\int_{\mathrm{im_T}(\mathbf{u},\Omega)} \tilde{\mathbf{u}}^{-1} \cdot \operatorname{div} \mathbf{G} \, \mathrm{d}\mathbf{y} = - \int_{\mathrm{im_G}(\mathbf{u},\Omega)} \big(D\mathbf{u} \circ \mathbf{u}^{-1}\big)^{-1} \cdot \mathbf{G} \, \mathrm{d}\mathbf{y}. \tag{3.16}$$

This shows that $D\tilde{\mathbf{u}}^{-1} = \big(D\mathbf{u} \circ \mathbf{u}^{-1}\big)^{-1} \mathcal{L}^n \llcorner \mathrm{im_G}(\mathbf{u},\Omega)$. As in (3.7), we obtain that

$$\int_{\mathrm{im_G}(\mathbf{u},\Omega)} \Big| \big(D\mathbf{u}(\mathbf{u}^{-1}(\mathbf{y}))\big)^{-1} \Big| \, \mathrm{d}\mathbf{y} < \infty.$$

Consequently, $\tilde{\mathbf{u}}^{-1} \in W^{1,1}(\mathrm{im_T}(\mathbf{u},\Omega), \mathbb{R}^n)$ and (3.11) is true. $\square$

When $p = n-1$, the sets $\mathrm{im_T}(\mathbf{u},\Omega)$ and $\mathrm{im_T}(\mathbf{u},U)$ for $U \in \mathcal{U}$ need not be open. Instead of showing Sobolev regularity for $\tilde{\mathbf{u}}^{-1}$, we show an analogue of Proposition 3.2 for $U = \Omega$. Even though $\Omega \notin \mathcal{U}$, we have some control of $\mathbf{u}$ on $\partial\Omega$ thanks to the stronger assumption $\bar{\mathcal{E}}(\mathbf{u}) < \infty$ (see Subsection 2.7). Since the choice of $\mathbf{0}$ as the value of $\tilde{\mathbf{u}}_\Omega^{-1}$ outside $\mathrm{im_T}(\mathbf{u},\Omega)$ is arbitrary (see (2.9)), we have added the assumption $\mathbf{0} \notin \bar{\Omega}$, so that $\mathbf{0}$ does not interfere with the actual values of $\tilde{\mathbf{u}}^{-1}$ (see (3.10)).

**Theorem 3.4.** *Assume $\mathbf{0} \notin \bar{\Omega}$. Let $\mathbf{u} \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ satisfy condition INV and be such that* $\det D\mathbf{u} > 0$ *a.e. and* $\bar{\mathcal{E}}(\mathbf{u}) < \infty$*. Then* $\mathrm{im_G}(\mathbf{u},\Omega)$ *and* $\mathrm{im_T}(\mathbf{u},\Omega)$ *have finite perimeter,* $\tilde{\mathbf{u}}_\Omega^{-1} \in SBV(\mathbb{R}^n, \mathbb{R}^n)$ *with*

$$\partial^* \mathrm{im_T}(\mathbf{u},\Omega) \stackrel{\sim}{\subset} J_{\tilde{\mathbf{u}}_\Omega^{-1}} \stackrel{\sim}{\subset} \{\mathbf{y} \in \mathbb{R}^n \setminus \mathrm{im_T}(\mathbf{u},\Omega) : D(\mathrm{im_T}(\mathbf{u},\Omega), \mathbf{y}) = 1\} \cup \partial^* \mathrm{im_T}(\mathbf{u},\Omega) \tag{3.17}$$

*and*

$$D\tilde{\mathbf{u}}_\Omega^{-1} \llcorner \mathrm{im_T}(\mathbf{u},\Omega) = \big(D\mathbf{u} \circ \mathbf{u}^{-1}\big)^{-1} \mathcal{L}^n \llcorner \mathrm{im_G}(\mathbf{u},\Omega). \tag{3.18}$$

*Proof.* By Proposition 2.16, the function $\mathbf{u}_\Omega^{-1}$ defined therein belongs to $SBV(\mathbb{R}^n, \mathbb{R}^n)$. By the chain rule (see, e.g., [1, Sect. 3.10]), $|\mathbf{u}_\Omega^{-1}| \in BV(\mathbb{R}^n)$, so, as a consequence of the coarea formula (see, e.g., [1, Th. 3.40]), almost all superlevel sets of $|\mathbf{u}_\Omega^{-1}|$ have finite perimeter. As $\mathbf{0} \notin \bar{\Omega}$, for all $t > 0$ sufficiently small we have that

$$\big\{\mathbf{y} \in \mathbb{R}^n : \big|\mathbf{u}_\Omega^{-1}(\mathbf{y})\big| > t\big\} = \mathrm{im_G}(\mathbf{u},\Omega).$$

Consequently, $\mathrm{im_G}(\mathbf{u},\Omega)$ has finite perimeter.

Using (2.8) and Proposition 2.14, we find that

$$\operatorname{Per} \mathrm{im_T}(\mathbf{u},\Omega) \le \operatorname{Per} \mathrm{im_G}(\mathbf{u},\Omega) + \sum_{\mathbf{a} \in C(\mathbf{u})} \operatorname{Per} \mathrm{im_T}(\mathbf{u},\mathbf{a}) = \operatorname{Per} \mathrm{im_G}(\mathbf{u},\Omega) + \mathcal{E}(\mathbf{u}) < \infty,$$

so $\mathrm{im_T}(\mathbf{u},\Omega)$ has finite perimeter as well.

Now, the function $\mathbf{v}:\mathbb{R}^n\to\mathbb{R}^n$ defined as $\mathbf{v}:=\sum_{\mathbf{a}\in C(\mathbf{u})}\mathbf{a}\,\chi_{\operatorname{im_T}(\mathbf{u},\mathbf{a})}$ is in $SBV(\mathbb{R}^n,\mathbb{R}^n)$. Indeed, by Proposition 2.14, for each finite set $F\subset C(\mathbf{u})$, the function $\mathbf{v}_F:=\sum_{\mathbf{a}\in F}\mathbf{a}\,\chi_{\operatorname{im_T}(\mathbf{u},\mathbf{a})}$ is in $SBV(\mathbb{R}^n,\mathbb{R}^n)$ with

$$D\mathbf{v}_F=-\sum_{\mathbf{a}\in F}\mathbf{a}\otimes\boldsymbol{\nu}_{\operatorname{im_T}(\mathbf{u},\mathbf{a})}\,\mathcal{H}^{n-1}\llcorner\,\partial^*\operatorname{im_T}(\mathbf{u},\mathbf{a})$$

(see, e.g., [1, Sect. 3.5]) and, hence,

$$|D\mathbf{v}_F|\,(\mathbb{R}^n)\leq\sum_{\mathbf{a}\in F}|\mathbf{a}|\,\mathcal{H}^{n-1}(\partial^*\operatorname{im_T}(\mathbf{u},\mathbf{a}))\leq\|\mathbf{id}\|_{L^\infty(\Omega,\mathbb{R}^n)}\,\mathcal{E}(\mathbf{u}).$$

As $C(\mathbf{u})$ is countable, by the closure theorem in $SBV$ (see, e.g., [1, Th. 4.7]), we obtain that $\mathbf{v}\in SBV(\mathbb{R}^n,\mathbb{R}^n)$. As both $\mathbf{u}_\Omega^{-1}$ and $\mathbf{v}$ are in $SBV(\mathbb{R}^n,\mathbb{R}^n)$, by [1, Th. 3.84], $\tilde{\mathbf{u}}_\Omega^{-1}$ is in $SBV(\mathbb{R}^n,\mathbb{R}^n)$, too. Using the representation (2.2), we find that

$$D\tilde{\mathbf{u}}_\Omega^{-1}=\nabla\tilde{\mathbf{u}}_\Omega^{-1}\,\mathcal{L}^n+\left((\tilde{\mathbf{u}}_\Omega^{-1})^+-(\tilde{\mathbf{u}}_\Omega^{-1})^-\right)\otimes\boldsymbol{\nu}_{\tilde{\mathbf{u}}_\Omega^{-1}}\,\mathcal{H}^{n-1}\llcorner\,J_{\tilde{\mathbf{u}}_\Omega^{-1}}. \tag{3.19}$$

We pass to calculate $\nabla\tilde{\mathbf{u}}_\Omega^{-1}$. By Lebesgue's density theorem, a.e. $\mathbf{y}_0\in\mathbb{R}^n\setminus\operatorname{im_T}(\mathbf{u},\Omega)$ satisfies $D(\operatorname{im_T}(\mathbf{u},\Omega),\mathbf{y}_0)=0$. For such a $\mathbf{y}_0$ we have that

$$D\left(\left\{\mathbf{y}\in\mathbb{R}^n:\tilde{\mathbf{u}}_\Omega^{-1}(\mathbf{y})=\mathbf{0}\right\},\mathbf{y}_0\right)=1. \tag{3.20}$$

Consequently, $\nabla\tilde{\mathbf{u}}_\Omega^{-1}(\mathbf{y}_0)=\mathbf{0}$. Similarly, consider $U\in\mathcal{U}_0$ and note that, thanks to Proposition 3.2, a.e. $\mathbf{y}_0\in\operatorname{im_T}(\mathbf{u},U)$ is a point of approximate differentiability of $\tilde{\mathbf{u}}_U^{-1}$. Take such a $\mathbf{y}_0$. By Lemma 2.8, $D(\operatorname{im_T}(\mathbf{u},U),\mathbf{y}_0)=1$ and, hence,

$$D\left(\left\{\mathbf{y}\in\mathbb{R}^n:\tilde{\mathbf{u}}_\Omega^{-1}(\mathbf{y})=\tilde{\mathbf{u}}_U^{-1}(\mathbf{y})\right\},\mathbf{y}_0\right)=1.$$

Consequently, $\nabla\tilde{\mathbf{u}}_\Omega^{-1}(\mathbf{y}_0)=\nabla\tilde{\mathbf{u}}_U^{-1}(\mathbf{y}_0)$. Using (2.4) and Proposition 3.2 (in particular, (3.6)), we conclude that $\nabla\tilde{\mathbf{u}}_\Omega^{-1}=(D\mathbf{u}\circ\mathbf{u}^{-1})^{-1}\,\chi_{\operatorname{im_G}(\mathbf{u},\Omega)}$ a.e. in $\operatorname{im_T}(\mathbf{u},\Omega)$. In total,

$$\nabla\tilde{\mathbf{u}}_\Omega^{-1}(\mathbf{y})=\begin{cases}D\mathbf{u}(\mathbf{u}^{-1}(\mathbf{y}))^{-1}, & \text{a.e. } \mathbf{y}\in\operatorname{im_G}(\mathbf{u},\Omega),\\ \mathbf{0}, & \text{a.e. } \mathbf{y}\in\mathbb{R}^n\setminus\operatorname{im_G}(\mathbf{u},\Omega).\end{cases} \tag{3.21}$$

Now we show that

$$\bigcup_{U\in\mathcal{U}_0}\left(\operatorname{im_T}(\mathbf{u},U)\setminus J_{\tilde{\mathbf{u}}_U^{-1}}\right)\,\tilde{\subset}\,\operatorname{im_T}(\mathbf{u},\Omega)\setminus J_{\tilde{\mathbf{u}}_\Omega^{-1}}. \tag{3.22}$$

Indeed, let $U\in\mathcal{U}_0$ and $\mathbf{y}_0\in\operatorname{im_T}(\mathbf{u},U)\setminus S_{\tilde{\mathbf{u}}_U^{-1}}$ (recall Definition 2.1). Then $\mathbf{y}_0\in\operatorname{im_T}(\mathbf{u},\Omega)$ and there exists $\mathbf{x}_0\in\mathbb{R}^n$ such that for all $\delta>0$,

$$D\left(\left\{\mathbf{y}\in\mathbb{R}^n:\left|\tilde{\mathbf{u}}_U^{-1}(\mathbf{y})-\mathbf{x}_0\right|\geq\delta\right\},\mathbf{y}_0\right)=0.$$

Since $\tilde{\mathbf{u}}_U^{-1}$ and $\tilde{\mathbf{u}}_\Omega^{-1}$ coincide in $\operatorname{im_T}(\mathbf{u},U)$, and $D\left(\operatorname{im_T}(\mathbf{u},U),\mathbf{y}_0\right)=1$ (thanks to Lemma 2.8), we conclude that

$$D\left(\left\{\mathbf{y}\in\mathbb{R}^n:\left|\tilde{\mathbf{u}}_\Omega^{-1}(\mathbf{y})-\mathbf{x}_0\right|\geq\delta\right\},\mathbf{y}_0\right)=0.$$

Therefore, $\mathbf{y}_0\notin S_{\tilde{\mathbf{u}}_\Omega^{-1}}$. This shows that

$$\operatorname{im_T}(\mathbf{u},U)\setminus S_{\tilde{\mathbf{u}}_U^{-1}}\subset\operatorname{im_T}(\mathbf{u},\Omega)\setminus S_{\tilde{\mathbf{u}}_\Omega^{-1}}$$

and, consequently,

$$\operatorname{im_T}(\mathbf{u},U)\setminus J_{\tilde{\mathbf{u}}_U^{-1}}\,\tilde{\subset}\,\operatorname{im_T}(\mathbf{u},\Omega)\setminus J_{\tilde{\mathbf{u}}_\Omega^{-1}},$$

which implies (3.22).

Now we show that

$$\operatorname{im_T}(\mathbf{u},\Omega)\,\tilde{\subset}\bigcup_{U\in\mathcal{U}_0}\left(\operatorname{im_T}(\mathbf{u},U)\setminus J_{\tilde{\mathbf{u}}_U^{-1}}\right). \tag{3.23}$$

Indeed, let $U_0\in\mathcal{U}_0$ and choose $U\in\mathcal{U}_0$ such that $U_0\subset\subset U$. From Proposition 3.2 we find that $J_{\tilde{\mathbf{u}}_U^{-1}}\cong\operatorname{im_G}(\mathbf{u},\partial U)$, from Proposition 2.14 we obtain that $\operatorname{im_T}(\mathbf{u},U_0)\subset\operatorname{im_T}(\mathbf{u},U)$ and $\operatorname{im_G}(\mathbf{u},\partial U)\cong\partial^*\operatorname{im_T}(\mathbf{u},U)$, whereas Lemma 2.8 implies that $\operatorname{im_T}(\mathbf{u},U)\cap\partial^*\operatorname{im_T}(\mathbf{u},U)=\varnothing$. In total,

$$\operatorname{im_T}(\mathbf{u},U_0)\cap J_{\tilde{\mathbf{u}}_U^{-1}}\cong\operatorname{im_T}(\mathbf{u},U_0)\cap\partial^*\operatorname{im_T}(\mathbf{u},U)\subset\operatorname{im_T}(\mathbf{u},U)\cap\partial^*\operatorname{im_T}(\mathbf{u},U)=\varnothing,$$

and, hence, $\operatorname{im}_T(\mathbf{u}, U_0) \tilde{\subset} \operatorname{im}_T(\mathbf{u}, U) \setminus J_{\tilde{\mathbf{u}}_U^{-1}}$. Therefore,

$$\operatorname{im}_T(\mathbf{u}, U_0) \tilde{\subset} \bigcup_{U \in \mathcal{U}_0} \left( \operatorname{im}_T(\mathbf{u}, U) \setminus J_{\tilde{\mathbf{u}}_U^{-1}} \right),$$

which, thanks to (2.7), implies (3.23).

A combination of (3.22) and (3.23) yields $\operatorname{im}_T(\mathbf{u}, \Omega) \tilde{\subset} \operatorname{im}_T(\mathbf{u}, \Omega) \setminus J_{\tilde{\mathbf{u}}_\Omega^{-1}}$, so

$$\operatorname{im}_T(\mathbf{u}, \Omega) \cap J_{\tilde{\mathbf{u}}_\Omega^{-1}} \cong \varnothing. \tag{3.24}$$

Now, if $D(\operatorname{im}_T(\mathbf{u}, \Omega), \mathbf{y}_0) = 0$, then, (3.20) also holds, so for all $\delta > 0$,

$$D\left( \left\{ \mathbf{y} \in \mathbb{R}^n : \left| \tilde{\mathbf{u}}_\Omega^{-1}(\mathbf{y}) \right| \geq \delta \right\}, \mathbf{y}_0 \right) = 0.$$

We have therefore proved that

$$\text{if } D(\operatorname{im}_T(\mathbf{u}, \Omega), \mathbf{y}_0) = 0 \text{ then } \mathbf{y}_0 \notin J_{\tilde{\mathbf{u}}_\Omega^{-1}}. \tag{3.25}$$

As $\operatorname{im}_T(\mathbf{u}, \Omega)$ has finite perimeter, $\mathcal{H}^{n-1}$-a.e. point of $\mathbb{R}^n$ has density 1 in $\operatorname{im}_T(\mathbf{u}, \Omega)$ or density 0 in $\operatorname{im}_T(\mathbf{u}, \Omega)$ or belongs to $\partial^* \operatorname{im}_T(\mathbf{u}, \Omega)$. Likewise, as $\tilde{\mathbf{u}}_\Omega^{-1}$ is of special bounded variation, $\mathcal{H}^{n-1}$-a.e. point of $\mathbb{R}^n$ is an approximate continuity point of $\tilde{\mathbf{u}}_\Omega^{-1}$ or a jump point of $\tilde{\mathbf{u}}_\Omega^{-1}$. In either case, both $\left(\tilde{\mathbf{u}}_\Omega^{-1}\right)^+$ and $\left(\tilde{\mathbf{u}}_\Omega^{-1}\right)^-$ exist at those points: they coincide for approximate continuity points, and differ for jump points. Take such a point $\mathbf{y}_0$. If $\mathbf{y}_0 \in \partial^* \operatorname{im}_T(\mathbf{u}, \Omega)$ then it is clear that $\left(\tilde{\mathbf{u}}_\Omega^{-1}\right)^+(\mathbf{y}_0) = \mathbf{0}$, while, $\left(\tilde{\mathbf{u}}_\Omega^{-1}\right)^-(\mathbf{y}_0)$, which we are assuming to exist, must belong to $\bar{\Omega}$. Since $\mathbf{0} \notin \bar{\Omega}$, we have that $\mathbf{y}_0 \in J_{\tilde{\mathbf{u}}_\Omega^{-1}}$. Thus,

$$\partial^* \operatorname{im}_T(\mathbf{u}, \Omega) \tilde{\subset} J_{\tilde{\mathbf{u}}_\Omega^{-1}}. \tag{3.26}$$

The discussion above, and, in particular, equations (3.24), (3.25) and (3.26) show the validity of inclusions (3.17). When we restrict equality (3.19) to $\operatorname{im}_T(\mathbf{u}, \Omega)$ and use (3.21) and (3.24), we conclude that equality (3.18) is satisfied. $\square$

It is tempting to think that, in the setting of Theorem 3.4, one can conclude that $J_{\tilde{\mathbf{u}}_\Omega^{-1}} \cong \partial^* \operatorname{im}_T(\mathbf{u}, \Omega)$. However, this is not the case as the following simple example shows. In $\mathbb{R}^2$, let $\Omega := (1, 2) \times (0, 2\pi)$, and $\mathbf{u} : \Omega \to \mathbb{R}^2$ be the diffeomorphism given by $\mathbf{u}(x_1, x_2) := (x_1 \cos x_2, x_1 \sin x_2)$. It is easy to check that

$$\mathbf{u}(\Omega) = \operatorname{im}_G(\mathbf{u}, \Omega) = \operatorname{im}_T(\mathbf{u}, \Omega) = B(\mathbf{0}, 2) \setminus \left( \bar{B}(\mathbf{0}, 1) \cup ((1, 2) \times \{0\}) \right),$$

and that the set of points of density 1 in $\operatorname{im}_T(\mathbf{u}, \Omega)$ is $B(\mathbf{0}, 2) \setminus \bar{B}(\mathbf{0}, 1)$. As a consequence,

$$\partial^* \operatorname{im}_T(\mathbf{u}, \Omega) = \partial B(\mathbf{0}, 2) \cup \partial B(\mathbf{0}, 1),$$

but a direct calculation shows that the jump set of $\tilde{\mathbf{u}}_\Omega^{-1}$ is

$$\partial B(\mathbf{0}, 2) \cup \partial B(\mathbf{0}, 1) \cup ((1, 2) \times \{0\}).$$

This example was used in [13] to show that $D(\operatorname{im}_G(\mathbf{u}, \Omega), \mathbf{y}) = 1$ for all $\mathbf{y} \in (1, 2) \times \{0\}$, but $D(\operatorname{im}_G(\mathbf{u}, U), \mathbf{y}) = 0$ for every $U \in \mathcal{U}$.

Without the assumption $\mathbf{0} \notin \bar{\Omega}$, the inclusion $\partial^* \operatorname{im}_T(\mathbf{u}, \Omega) \tilde{\subset} J_{\tilde{\mathbf{u}}_\Omega^{-1}}$ in (3.17) does not hold in general. For example, consider an open halfspace $H$ such that $\mathbf{0} \in \partial H$, take $\Omega = H \cap B(\mathbf{0}, 1)$ and let $\mathbf{u} : \Omega \to \mathbb{R}^n$ be defined as $\mathbf{u}(\mathbf{x}) := \mathbf{x} + \frac{\mathbf{x}}{|\mathbf{x}|}$. A simple calculation shows that

$$\operatorname{im}_T(\mathbf{u}, \Omega) = H \cap B(\mathbf{0}, 2) \setminus \bar{B}(\mathbf{0}, 1) \quad \text{but} \quad \partial^* \operatorname{im}_T(\mathbf{u}, \Omega) = J_{\tilde{\mathbf{u}}_\Omega^{-1}} \cup (H \cap \partial B(\mathbf{0}, 1)),$$

with disjoint union. Nevertheless, the rest of the conclusions of Theorem 3.4 remain true. Indeed, the only other step of the proof where the assumption $\mathbf{0} \notin \bar{\Omega}$ is used was to show that $\operatorname{im}_G(\mathbf{u}, \Omega)$ has finite perimeter, and this can be achieved by choosing any $\mathbf{a} \in \mathbb{R}^n$ such that $\mathbf{0} \notin \bar{\Omega} + \mathbf{a}$ and arguing with the translated function $\mathbf{w} : \Omega + \mathbf{a} \to \mathbb{R}^n$ defined as $\mathbf{w}(\mathbf{x}) := \mathbf{u}(\mathbf{x} - \mathbf{a})$.

Theorem 3.4 is close to saying that $\tilde{\mathbf{u}}_\Omega^{-1}$ is Sobolev in $\operatorname{im}_T(\mathbf{u}, \Omega)$, since the distributional derivative of $\tilde{\mathbf{u}}_\Omega^{-1}$ restricted to $\operatorname{im}_T(\mathbf{u}, \Omega)$ is an $L^1$ function. The problem is that $\operatorname{im}_T(\mathbf{u}, \Omega)$ is not, in general, an open set. Although there are several definitions of Sobolev spaces over non-open sets (see, in particular, the monographs [20, 11, 5] and the references therein), we have decided to leave the conclusion of Theorem 3.4 without a mention to Sobolev spaces, since we believe that the current statement is more transparent.

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