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Adaptive Boundary Element Methods

A posteriori error estimators, adaptivity, convergence, and implementation

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Abstract This paper reviews the state of the art and discusses very recent mathematical developments in the field of adaptive boundary element methods. This includes an overview of available a posteriori error estimates as well as a state-of-the-art formulation of convergence and quasi-optimality of adaptive mesh-refining algorithms.

Keywords boundary element method · a posteriori error estimate · adaptive mesh refinement · convergence · optimal complexity

Mathematics Subject Classification (2000) 65N30 · 65N38 · 65N50 · 65R20 · 41A25

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1 Introduction

Many practically relevant PDEs[1] on bounded or unbounded domains $\Omega \subset \mathbb{R}^d$ can be equivalently formulated as integral equations on the $(d-1)$-dimensional boundary $\Gamma = \partial\Omega$. This reformulation is then discretized and solved numerically by BEM[2]. Striking advantages of BEM over FEM[3] rely on the dimension reduction, the natural treatment of unbounded domains, as well as a potentially high rate of convergence with respect to both, the natural energy norm as well as the pointwise error. On the other hand, high convergence rates are only achieved if the (given) data as well as the (unknown) exact solution are sufficiently smooth or if the possible singularities are appropriately resolved. In practice, one thus observes a huge gap between the theoretically possible optimal rate and the empirical convergence behavior, if the meshes are refined uniformly. The remedy is to use appropriately graded meshes which resolve the possible singularities of data and exact solution. To this end, a posteriori error estimation and related adaptive mesh-refinement have themselves proven to be important tools for scientific

[1] partial differential equation (PDE)
[2] boundary element method (BEM)
[3] finite element method (FEM)

computing, cf. [3,145]. First, they allow to monitor the actual error and to stop the computation if the computed solution is accurate enough. Second, they may also drive the problem-adapted discretization and thus the appropriate resolution of the possible singularities. While the convergence and quasi-optimality of AFEM[4] has been mathematically analyzed within the last decade [26,46,56,110,139], analogous results for ABEM[5] [7,65,66,70,76] have only been achieved very recently, see also [52,75,77,103] for adaptive wavelet-based BEM.

1.1 Galerkin BEM and Céa lemma

Throughout, our main focus is on Galerkin BEM. Here, the mathematical frame reads as follows: Let $\mathscr{X}$ be a real Hilbert space with norm $\|\cdot\|_{\mathscr{X}}$, which will be an appropriate Sobolev space in the applications in mind. Let $b : \mathscr{X} \times \mathscr{X} \to \mathbb{R}$ be a continuous and elliptic bilinear form, i.e., there are constants $C_{\rm cont}, C_{\rm ell} > 0$ such that

$$b(v,w) \le C_{\rm cont} \|v\|_{\mathscr{X}} \|w\|_{\mathscr{X}} \quad \text{for all } v,w \in \mathscr{X} \tag{1}$$

and

$$b(v,v) \ge C_{\rm ell} \|v\|_{\mathscr{X}}^2 \quad \text{for all } v \in \mathscr{X}. \tag{2}$$

Given a linear and continuous functional $F : \mathscr{X} \to \mathbb{R}$, the so-called weak formulation (or variational formulation) of the BIE[6] reads: Find the exact solution $u \in \mathscr{X}$ of

$$b(u,v) = F(v) \quad \text{for all } v \in \mathscr{X}. \tag{3}$$

Based on a triangulation $\mathscr{T}$ of the underlying spatial domain, let $\mathscr{X}_{\mathscr{T}} \subset \mathscr{X}$ be a finite-dimensional subspace. The Galerkin BEM discretization reads: Find $U \in \mathscr{X}_{\mathscr{T}}$ such that

$$b(U,V) = F(V) \quad \text{for all } V \in \mathscr{X}_{\mathscr{T}}. \tag{4}$$

For both, the continuous as well as the discrete formulations (3) and (4), the Lax-Milgram lemma applies and proves the existence and uniqueness of $u \in \mathscr{X}$ resp. $U \in \mathscr{X}_{\mathscr{T}}$. Moreover, a direct computation with the Galerkin orthogonality

$$b(u-U,V) = 0 \quad \text{for all } V \in \mathscr{X}_{\mathscr{T}}, \tag{5}$$

provides the Céa lemma

$$\frac{C_{\rm ell}}{C_{\rm cont}} \|u-U\|_{\mathscr{X}} \le \min_{V \in \mathscr{X}_{\mathscr{T}}} \|u-V\|_{\mathscr{X}} \le \|u-U\|_{\mathscr{X}} \tag{6}$$

i.e., the *computable* Galerkin solution $U \in \mathscr{X}_{\mathscr{T}}$ is a quasi-best approximation of u among all functions V in the discrete space $\mathscr{X}_{\mathscr{T}}$.

[4] adaptive finite element method (AFEM)
[5] adaptive boundary element method (ABEM)
[6] boundary integral equation (BIE)

1.2 Adaptive algorithm

As a consequence of the Céa lemma (6), a natural question is how to choose the discrete space $\mathscr{X}_{\mathscr{T}}$ (resp. the mesh $\mathscr{T}$). Ideally, one should choose the discrete space $\mathscr{X}_{\mathscr{T}}$ such that the best approximation error in (6) is minimal with respect to the number of degrees of freedom. Usually the necessary a priori knowledge is not available (even if the generic singularities appear to be known), such that it is infeasible to address this question. Another possibility is to choose a sequence of meshes such that the best approximation error shows an "optimal decay" with increasing dimension of $\mathscr{X}_{\mathscr{T}}$. Usually this question is empirically addressed by adaptive algorithms which start from an initial mesh $\mathscr{T}_0$ and generate a sequence of (locally) refined meshes $\mathscr{T}_\ell$ for $\ell \in \mathbb{N}_0$ by iterating the loop

$$\boxed{\texttt{solve}} \to \boxed{\texttt{estimate}} \to \boxed{\texttt{mark}} \to \boxed{\texttt{refine}}. \tag{7}$$

It provides a sequence of Galerkin solutions $U_\ell \in \mathscr{X}_\ell := \mathscr{X}_{\mathscr{T}_\ell}$ with nested discrete spaces $\mathscr{X}_\ell \subset \mathscr{X}_{\ell+1} \subset \mathscr{X}$ for all $\ell \ge 0$. Adaptive algorithms thus work with a sequence of meshes and need to solve in every step. Yet, it can be observed in model problems that they outperform algorithms which uniformly refine a coarse mesh up to a given number of degrees of freedom and finally solve only once. This superiority appears in terms of memory versus error as well as time consumption versus error, cf. [4,10].

The module $\boxed{\texttt{solve}}$ consists of the direct or iterative solution of the linear system corresponding to (4) to compute the (approximate) Galerkin solution $U_\ell \in \mathscr{X}_\ell$. Mathematical questions arise from the fact that, first, BEM matrices are densely populated (i.e., the number of non-zero entries is roughly equivalent to the overall number of entries) and hence have to be treated by matrix compression techniques like FMM[7] [79,117], $\mathscr{H}$-matrices[8] [83,84], panel clustering [85], or ACA[9] [19,21,20], see also the monograph [121] on this subject. In particular, this prevents the use of direct solvers for problems of practical interest. Second, the condition number of BEM matrices grows if the mesh is refined, i.e., one needs cheap and effective preconditioners which build on the hierarchical structure of the nested discrete spaces. Finally, the right-hand side F in (3) often involves evaluations of integral operators applied to the given data. Then, the computation of the right-hand side in (4) can hardly be done analytically. Instead, appropriate and reliable data approximation and/or quadrature has to be employed, and this additional consistency error has to be controlled.

[7] fast multipole method (FMM)
[8] hierarchical matrices ($\mathscr{H}$-matrices)
[9] adaptive cross approximation (ACA)

The module $\boxed{\texttt{estimate}}$ comprises the computation of a numerically computable a posteriori error estimator

$$\eta_\ell = \Big(\sum_{T\in\mathscr{T}_\ell} \eta_\ell(T)^2 \Big)^{1/2} \tag{8}$$

whose local contributions $\eta_\ell(T)$ measure —at least heuristically— the Galerkin error $u-U_\ell$ on an element T of the current triangulation $\mathscr{T}_\ell$. For this purpose, different types of error estimators have been proposed in the literature which range from simple two-grid error estimators over residual-based strategies to estimators which build on the BEM inherent Calderón system.

The module $\boxed{\texttt{mark}}$ uses the local refinement indicators $\eta_\ell(T)$ and selects certain elements for refinement, where refinement can either be a geometric bisection of the element (so-called h-refinement) or the increase of the local approximation order (so-called p-refinement).

Finally, the module $\boxed{\texttt{refine}}$ uses the prior information to generate a new mesh $\mathscr{T}_{\ell+1}$ as well as a related enriched space $\mathscr{X}_{\ell+1} \supset \mathscr{X}_\ell$. For now, we denote this by $\mathscr{T}_{\ell+1} \in \texttt{refine}(\mathscr{T}_\ell)$. In the later sections, this will be specified further. Usually, the numerical analysis requires certain care for the (otherwise simple) operation $\texttt{refine}(\cdot)$ to ensure that, e.g., hanging nodes are avoided, the quotient of the diameters of neighboring elements does not deteriorate and neither do the elements' angles. In particular, this leads to additional refinement of non-marked elements. As over-refinement might affect observed convergence rates with respect to the degrees of freedom, this requires mathematical care if it comes to the proof of optimal convergence rates.

The design of an adaptive algorithm usually consists in making appropriate choices for the different parts of the adaptive loop (7). For the h-version, which is the focus of this work (although we will also briefly discuss hp-versions, where a mixture of h- and p-refinement takes place), it is common to write the loop (7) in pseudo-code in the following form:

Algorithm 1 (Adaptive mesh refinement) INPUT*: initial mesh* $\mathscr{T}_0$ *and adaptivity parameter* $0<\theta\le 1$.
OUTPUT*: sequence of solutions* $(U_\ell)_{\ell\in\mathbb{N}_0}$*, sequence of estimators* $(\eta_\ell)_{\ell\in\mathbb{N}_0}$*, and sequence of meshes* $(\mathscr{T}_\ell)_{\ell\in\mathbb{N}_0}$.
ITERATION*: For all* $\ell=0,1,2,3,\dots$ *do* (i)–(iv).

(i) *Compute solution* U_ℓ *of* (4).
(ii) *Compute error indicators* $\eta_\ell(T)$ *for all elements* $T\in\mathscr{T}_\ell$.
(iii) *Find a set of (minimal) cardinality* $\mathscr{M}_\ell \subseteq \mathscr{T}_\ell$ *such that*

$$\theta\eta_\ell^2 \le \sum_{T\in\mathscr{M}_\ell} \eta_\ell(T)^2. \tag{9}$$

(iv) *Refine at least the marked elements* $T\in\mathscr{M}_\ell$ *to obtain the new mesh* $\mathscr{T}_{\ell+1} \in \texttt{refine}(\mathscr{T}_\ell)$.

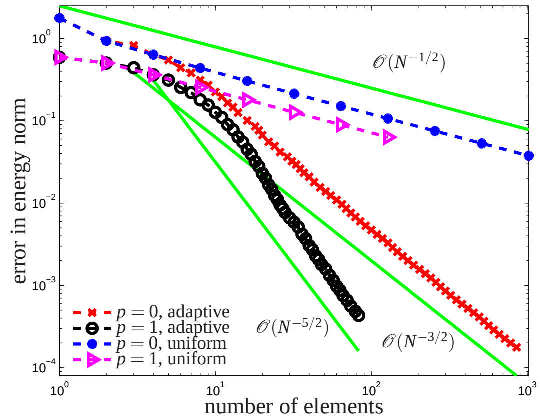

Fig. 1 BEM error for piecewise constants ($p=0$) and piecewise linears ($p=1$) on uniform and adaptive meshes for example (10).

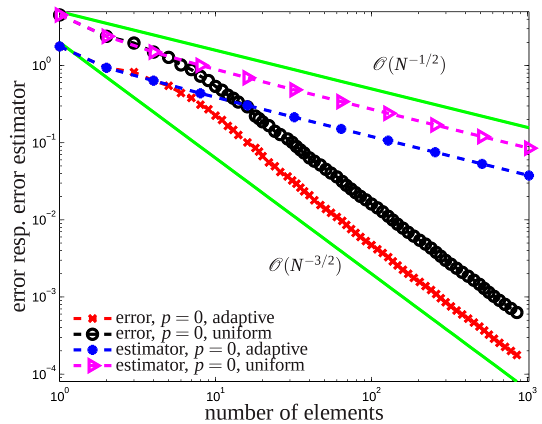

Fig. 2 BEM error and error estimator for piecewise constants ($p=0$) on uniform and adaptive meshes for example (10).

1.3 Mathematical questions

To illustrate some of the mathematical questions which have to be addressed, we consider simple toy problems for the 2D and 3D Laplacian: First, we consider the weakly singular integral equation

$$V\phi = f \quad \text{on the slit } \Gamma = (-1,1)\times\{0\}, \tag{10}$$

where V is the simple-layer integral operator of the 2D Laplacian (see Section 2.2 below). For given $f(s,0)=s$, the unique solution of (10) is known to be

$$u(s,0) = 2s/\sqrt{1-s^2}. \tag{11}$$

We consider BEM with piecewise constant ansatz and test functions ($p=0$) as well as with discontinuous piecewise linear ansatz and test functions ($p=1$). The adaptive mesh refinement is driven by some $(h-h/2)$-type error estimator (see Section 4.2.2 below). The initial mesh consists of one line segment of length 2.

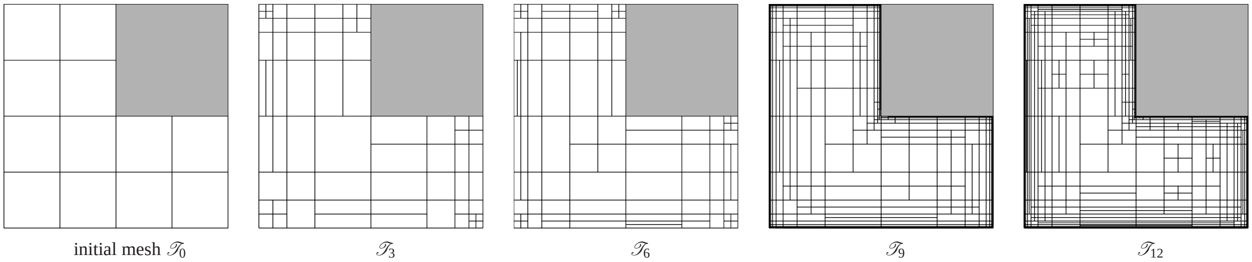

Fig. 3 Sequence of adaptively generated meshes for 3D BEM with anisotropic mesh refinement for example (15).

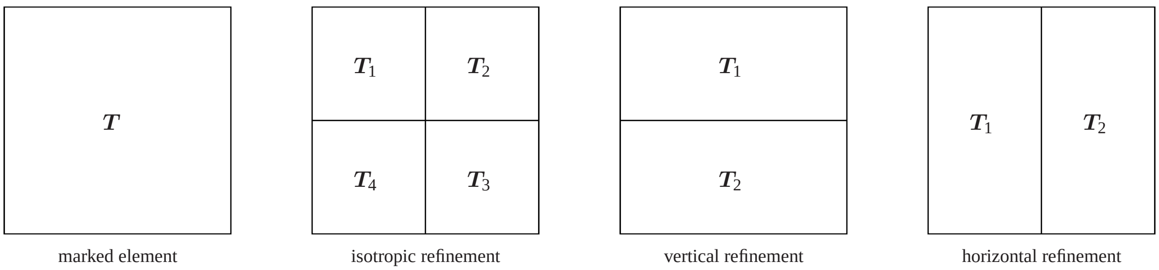

Fig. 4 For 3D BEM, each marked rectangle $T \in \mathcal{T}_\ell$ (left) is either refined isotropically into four elements or anisotropically into two elements.

Figs. 1 and 2 show the outcome of the numerical computations, where we compare uniform vs. adaptive mesh-refinement. We plot the error (measured in the natural $\widetilde{H}^{-1/2}$ norm) and the computed a posteriori error estimator versus the number N of elements. If u was smooth, the generically optimal order of convergence would be $\mathcal{O}(N^{-p-3/2})$, see [123]. However, in the present example, the exact solution has strong (generic) singularities at the tips of the slit and thus lacks the required regularity. For uniform mesh refinement, where all line segments are bisected to obtain $\mathcal{T}_{\ell+1}$ from $\mathcal{T}_\ell$, we observe a poor convergence rate of $\mathcal{O}(N^{-1/2})$ for both, piecewise constants and piecewise linears, see Fig. 1. Consequently, the use of higher-order polynomials does not pay on uniform meshes. However, if we use the adaptive algorithm which automatically enforces an appropriate grading of the mesh towards the singularities of u, we observe the optimal convergence behavior $\mathcal{O}(N^{-3/2})$ for piecewise constants $p = 0$ and $\mathcal{O}(N^{-5/2})$ for piecewise linears $p = 1$.

Mathematically, this observation gives rise to the following questions:

- Does the adaptive algorithm (Algorithm 1) guarantee convergence? More precisely, is it true that

$$\|u - U_\ell\|_{\mathcal{X}} \to 0 \quad \text{as } \ell \to \infty? \tag{12}$$

For uniform mesh refinement, the Céa lemma (6) and appropriate approximation results for smooth functions guarantee that Galerkin BEM always lead to convergence $U_\ell \to u$, independently of the overall regularity or possible singularities of the unknown solution u. As observed, this convergence can be slow. On the other hand, the adaptive algorithm does not guarantee that the mesh-size will tend to zero as $\ell \to \infty$. Consequently, the convergence analysis for uniform meshes does not carry over to adaptive meshes. However, a priori arguments guarantee that nestedness $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1}$ for all $\ell \geq 0$ implies convergence of U_ℓ towards some limit U_∞ (see Lemma 76), but raises the important question whether we can identify $u = U_\infty$. In particular, the numerical check for convergence will always be affirmative even if the adaptive algorithm does wrong, i.e., $u \neq U_\infty$.

- Empirically, the adaptive algorithm 1 does not only lead to convergence, but even ensures *linear* convergence, i.e., $\|u - U_{\ell+1}\|_{\mathcal{X}} \leq q\, \|u - U_\ell\|_{\mathcal{X}}$ for some uniform constant $0 < q < 1$.

For some symmetric and elliptic bilinear form $b(\cdot,\cdot)$ and the induced norm $\|v\|_{\mathcal{X}} = \sqrt{b(v,v)}$, the Céa lemma (6) holds with $C_{\mathrm{ell}} = 1 = C_{\mathrm{cont}}$. This implies at least $\|u - U_{\ell+1}\|_{\mathcal{X}} \leq \|u - U_\ell\|_{\mathcal{X}}$, and the numerical experiment also shows that pre-asymptotically even $q \approx 1$ can be observed, see Fig. 1.

- Does the adaptive algorithm 1 recover the optimal rate of convergence?

Clearly, the last two questions are strongly related to the a posteriori error estimator which drives the adaptive mesh refinement. As the adaptive algorithm does not see the actual error, but only the error estimator, an a posteriori error estimator is called *reliable*, if it provides an upper bound for the unknown error

$$\|u - U_\ell\|_{\mathcal{X}} \leq C_{\mathrm{rel}}\, \eta_\ell \tag{13}$$

up to some generic constant $C_{\mathrm{rel}} > 0$. If the adaptive algorithm thus drives the error estimator to zero, this implies

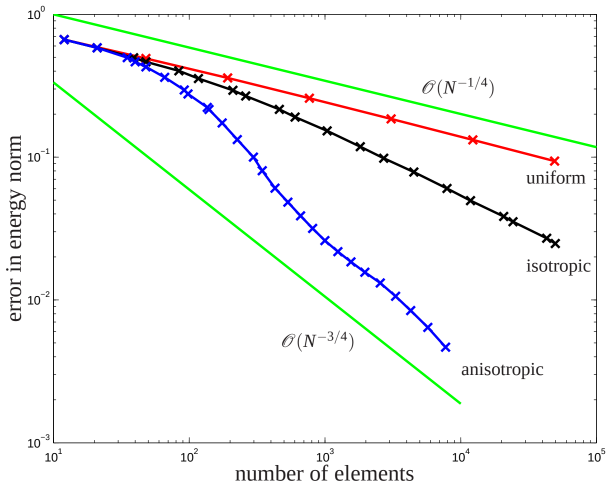

Fig. 5 BEM error for piecewise constants on uniform and adaptive meshes for example (15) with isotropic and anisotropic refinement.

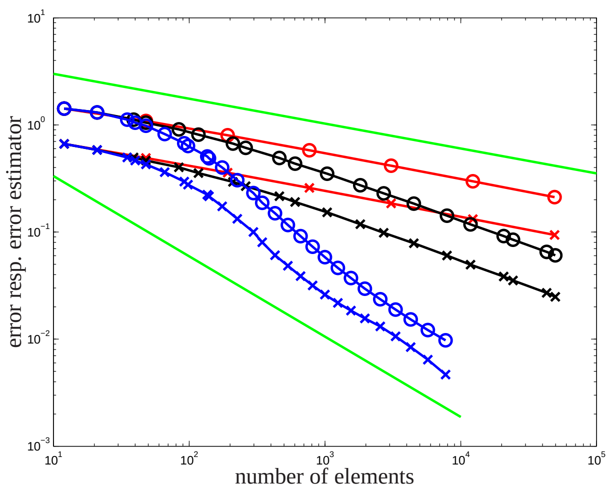

Fig. 6 BEM error and error estimator for piecewise constants on uniform and adaptive meshes for example (15) with isotropic and anisotropic refinement.

convergence of the overall scheme. Conversely, η_ℓ is called *efficient*, if it provides a lower bound for the unknown error

$$\eta_\ell \leq C_{\rm eff}\,\|u-U_\ell\|_{\mathscr{X}} \tag{14}$$

up to some generic constant $C_{\rm eff} > 0$. If η_ℓ is both, efficient and reliable, the adaptive algorithm monitors the convergence behavior.

Fig. 2 displays error as well as $(h-h/2)$-type error estimator for uniform and adaptive mesh refinement and lowest-order elements $p=0$. As the curves of error and error estimator are parallel, independently of the mesh refinement, it is observed that η_ℓ is reliable and efficient. The same observation is obtained for higher-order polynomials (not displayed).

- In which situations can it be mathematically guaranteed that the BEM error is estimated reliably and efficiently?

Mathematical details (and restrictions) for the $(h-h/2)$-error estimator are discussed in Section 4.2.2 below.

Moreover, optimal convergence behavior is also constrained by the mesh refinement used. To illustrate this, we consider the weakly singular integral equation

$$V\phi = f \text{ on the L-screen } \Gamma = \big((-1,1)^2\backslash(0,1)^2\big)\times\{0\}, \tag{15}$$

where V now is the simple-layer integral operator of the 3D Laplacian. We consider lowest-order BEM with piecewise constant ansatz and test functions. The adaptive mesh refinement is driven by some $(h-h/2)$-type error estimator (see Section 4.2.2 below). The initial mesh consists of 12 uniform squares with edge length $1/2$, see Fig. 3 (left).

It is well-known that the solutions of (15) suffer from edge singularities. For $f(x)=1$, we therefore compare uniform mesh refinement, where each square is divided into four similar squares of half edge length, with adaptive isotropic resp. anisotropic mesh refinement. In the isotropic case, marked squares are refined into four similar squares of half edge length. In the anisotropic case, we also allow that the (rectangular) elements are only refined along one edge into two rectangles, see Fig. 4. For the anisotropic refinement, some adaptively refined meshes are shown in Fig. 3.

The overall outcome of these computations is visualized in Figs. 5 and 6, where we compare uniform and adaptive isotropic and anisotropic mesh refinement. We plot the error (measured in the natural $\widetilde{H}^{-1/2}$-norm) and the computed a posteriori error estimator versus the number N of elements. If u was smooth, the generically optimal order of convergence would be $\mathcal{O}(h^{3/2})$ for the uniform mesh-size h. For 3D BEM, this corresponds to an optimal decay $\mathcal{O}(N^{-3/4})$ with respect to the number of elements. However, the exact solution exhibits generic singularities along the edges of Γ. For uniform mesh refinement, where all elements are refined isotropically, we observe a poor rate of convergence $\mathcal{O}(N^{-1/4})$ for the error. For the adaptive strategy with anisotropic elements, we observe the optimal rate of convergence $\mathcal{O}(N^{-3/4})$, while adaptive isotropic refinement leads to approximately $\mathcal{O}(N^{-1/2})$. We note that heuristic arguments show that $\mathcal{O}(N^{-1/2})$ is the optimal rate of convergence in the presence of generic edge singularities, if one restricts to isotropic elements [38]. This is also illustrated by the adaptive meshes shown in Fig. 7 as well as Fig. 8 which show adaptively generated anisotropic resp. isotropic meshes with (almost) the same number of elements.

Fig. 6 shows the BEM error as well as the $(h-h/2)$-type error estimators. As for the 2D example (10), we observe that the a posteriori error estimators used are reliable and efficient.

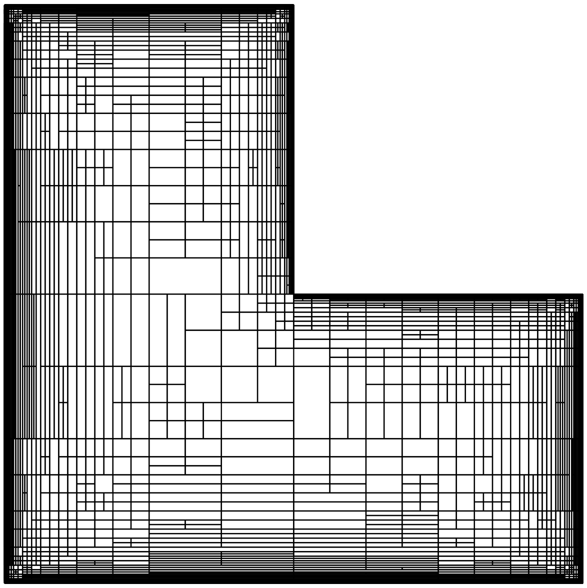

Fig. 7 Adaptively generated mesh with anisotropic elements.

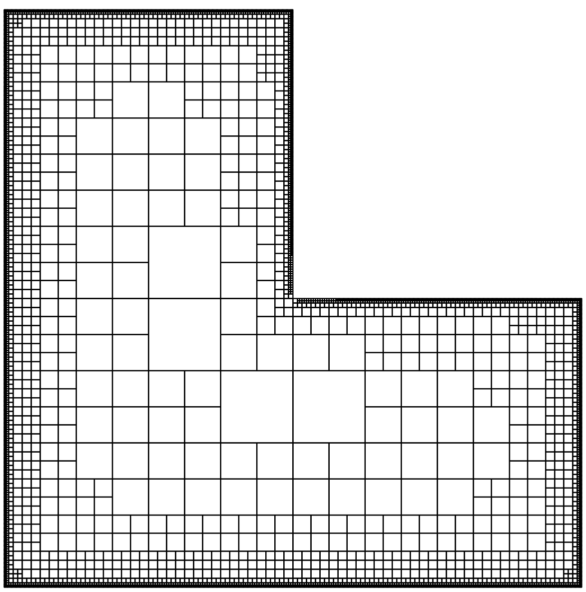

Fig. 8 Adaptively generated mesh with isotropic elements.

1.4 Outline

Essential ingredients of the mathematical theory of BEM will be collected in Section 2. The fundamental function spaces in BEM are Sobolev spaces, which will be introduced briefly in Section 2.1. There will be no further explanations on the connection of BIEs and PDEs, but in Section 2.2 we will define the boundary integral operators that constitute the equations that are to be solved (Sections 2.3 and 2.4). To emphasize the significance of local mesh refinement, Section 2.5 briefly summarizes the regularity theory in the context of this work. The terms associated with discrete spaces, such as meshes, piecewise polynomials, and so on, will be defined in Section 2.6, and the resulting discrete equations are given in Section 2.7.

As a basis for a posteriori error estimation, Section 3 focuses exclusively on the localization techniques for fractional order Sobolev norms. The results of this section will be used frequently in this work, and the two omnipresent approaches, localization by local fractional norms (Section 3.1) and localization by approximation (Section 3.2), will be treated in particular.

Section 4 gives an overview of the different a posteriori error estimators for BEM that have been proposed in the mathematical literature. The estimators are classified into five different groups:

- Residual error estimators (Section 4.1),
- estimators based on space enrichment (Section 4.2),
- averaging on large patches (Section 4.3),
- ZZ-type estimators (Section 4.4),
- and estimators based on the Calderón sytem (Section 4.5).

In addition, Section 4.6 deals with the question of how to estimate data approximation errors. The estimators presented up to this point might serve also in higher-order BEM, but are analyzed only with respect to mesh refinement. In contrast, a posteriori estimators and associated adaptive algorithms for p and hp-versions of the BEM are shown in Section 5.

The question of convergence of h-adaptive algorithms of the type (12) will be addressed in Section 6, which deals with the so-called *estimator reduction principle*. This is a rather general concept dealing with convergence of adaptive algorithms. This will be explained in detail in Section 6.4. The results of Section 6 are tailored to certain concrete model problems and estimators which are given in Sections 6.5–6.7 for a posteriori error estimation with $(h-h/2)$, ZZ, and weighted residual estimators, as well as in Sections 6.8 and 6.9 including data approximation. We also comment on convergence in the presence of anisotropic mesh refinement in Section 6.10.

The properties of module $\texttt{refine}(\cdot)$, responsible for mesh refinement, play an important role in the analysis of optimal rates of adaptive mesh-refining algorithms. Section 7 explains the requirements for $\texttt{refine}(\cdot)$ and summarizes available results from the literature to account for local refinement in 2D BEM as well as 3D BEM.

Regarding the question of optimal convergence of Algorithm 1, the following Section 8 introduces an abstract framework that was recently laid out even in a more general setting in [37]. At this point, we will have fixed all the parts of Algorithm 1 except $\boxed{\texttt{estimate}}$. Based on certain assumptions (called (A1)–(A4)) on the estimator η_ℓ that is employed in $\boxed{\texttt{estimate}}$, convergence of Algorithm 1 will be shown in Section 8.2, generalising the results of Section 6. Optimal convergence of Algorithm 1 within the abstract framework will be shown in Section 8.3. In

Sections 8.4–8.7, it is shown how to apply the abstract setting to concrete model problems, i.e., the assumptions (A1)–(A4) will be checked for different error estimators. More precisely, we obtain linear convergence for $(h-h/2)$-based estimators and optimal convergence for weighted residual estimators. From Section 8.8 on, we deal with convergence and optimality of ABEM including data approximation. To that end, an extended algorithm (Algorithm 123) will be formulated that differs from Algorithm 1 only in that Galerkin solutions are computed with respect to an approximate right-hand side and that error control for data approximation is included in the error estimation. For this extended error estimators, we again formulate assumptions (called $(\widetilde{\text{A1}})$–$(\widetilde{\text{A6}})$), and show not only convergence (Section 8.9) but also optimality (Section 8.10) of Algorithm 123. We show how to apply this abstract framework to concrete model problems in Sections 8.11 and 8.12.

The final Section 9 is devoted to details in implementation. We give detailed explanations how to implement the L_2-orthogonal projection (Section 9.1) as well as the Scott-Zhang projection (Section 9.2). Furthermore, we show how to implement the two-level error estimator, the $(h-h/2)$ based error estimator, as well as the weighted residual error estimator for $d=2$ and in the lowest order case. The ideas that we present for implementation transfer immediately to $d=3$ and higher-order polynomials.

2 Mathematical foundation of the BEM

This section briefly introduces the mathematical framework for boundary element methods. Definitive books in this respect are [98,108,114], which deal exclusively with boundary integral equations and their analytical underpinning, and [123,135], which focus to a great extent on boundary element discretizations. Let us also note that the analysis of finite elements for the discretization of boundary integral equations of the first kind goes back to Nédélec and Planchard [115], and Hsiao and Wendland [97].

2.1 Sobolev spaces

For a rigorous treatise of Sobolev spaces, we refer to the standard reference [1]. For $\Omega\subset\mathbb{R}^d$ an open and bounded set and $p\in[1,\infty]$, $L_p(\Omega)$ denotes the space of all measurable functions $u:\Omega\to\mathbb{R}$ whose p-th power is integrable, i.e., $\|u\|_{L_p(\Omega)}<\infty$, where

$$\|u\|_{L_p(\Omega)} := \begin{cases} \left(\int_\Omega |u|^p\right)^{1/p} & \text{for } p<\infty,\\ \inf_{\substack{M\subset\Omega\\ |M|=0}} \sup_{x\in\Omega\setminus M}|u(x)| & \text{for } p=\infty.\end{cases}$$

The space $L_2(\Omega)$ is a Hilbert space with inner product and norm

$$\langle u\,,w\rangle_\Omega := \int_\Omega uw\,dx, \qquad \|u\|_{L_2(\Omega)} := \langle u\,,u\rangle_\Omega^{1/2}.$$

The space $C_0^\infty(\Omega)$ is the space of smooth $\phi\in C^\infty(\Omega)$ with $\mathrm{supp}(\varphi)\subset\Omega$. If, for $u\in L_2(\Omega)$, a locally integrable function $w:\Omega\to\mathbb{R}^d$ exists such that, for all $\varphi\in C_0^\infty(\Omega)$

$$\int_\Omega u(x)\nabla\varphi(x)\,dx = -\int_\Omega w(x)\varphi(x)\,dx,$$

then w is called the *weak gradient* of u, abbreviated by $\nabla u:=w$. It follows from the fundamental lemma of calculus of variations that the weak gradient is uniquely defined almost everywhere, and integration by parts shows that it therefore coincides with the classical gradient of u if it exists. The space of all functions $u\in L_2(\Omega)$ with weak gradient $\nabla u\in L_2(\Omega)$ is the Sobolev space $H^1(\Omega)$. This is again a Hilbert space with inner product and norm

$$\langle u\,,w\rangle_{H^1(\Omega)} := \langle u\,,w\rangle_\Omega + \langle\nabla u\,,\nabla w\rangle_\Omega,$$
$$\|u\|_{H^1(\Omega)} := \langle u\,,u\rangle_{H^1(\Omega)}^{1/2}.$$

For $s\in(0,1)$, the fractional order Sobolev space $H^s(\Omega)$ consists of all $u\in L_2(\Omega)$ with $\|u\|_{H^s(\Omega)}<\infty$, where inner product and norm are

$$\langle u\,,w\rangle_{H^s(\Omega)} := \langle u\,,w\rangle_\Omega + \int_\Omega\int_\Omega \frac{(u(x)-u(y))(w(x)-w(y))}{|x-y|^{d+2s}}\,dx\,dy,$$
$$\|u\|_{H^s(\Omega)} := \langle u\,,u\rangle_{H^s(\Omega)}^{1/2}.$$

Generally, for a non-empty set $\omega\subset\Omega$ and $s\in(0,1)$, the associated seminorm is denoted by

$$|u|_{H^s(\omega)}^2 := \int_\omega\int_\omega \frac{(u(x)-u(y))^2}{|x-y|^{d+2s}}\,dx\,dy.$$

From now on we assume that Ω is simply connected and has a Lipschitz boundary $\partial\Omega$, i.e., local orthogonal coordinates may be introduced to represent $\partial\Omega$ locally as a Lipschitz function over a $(d-1)$-dimensional domain. Then, we can define Sobolev spaces $H^s(\Gamma)$ for $\Gamma\subseteq\partial\Omega$ and associated inner products and norms for $s\in[0,1)$ exactly as for $(d-1)$-dimensional domains but using surface integrals instead of integrals over domains. The definition of the surface integral does not depend on the parametrization used, so neither does the space $H^s(\Gamma)$ and its inner product or norm for $s\in[0,1)$. Independently of the chosen parametrization of Γ, a weak surface gradient ∇_Γ can be defined, cf. [144, Def. 1.9] or [47, Appendix A.3], and hence a space $H^1(\Gamma)$. The surface gradient is tangential to Γ, and for smooth functions u in $\mathbb{R}^d$ there holds $\nabla u = \nabla_\Gamma u + (\mathbf{n}\cdot\nabla u)\cdot\mathbf{n}$. For $d=2$, i.e., Γ a one-dimensional curve, the notation u' will be used to denote the gradient of u. For $s \in [0,1]$, define

$$\widetilde{H}^s(\Gamma) := \left\{u \in H^s(\partial\Omega) \mid \operatorname{supp}(u) \subset \overline{\Gamma}\right\}$$

with norm

$$\|u\|_{\widetilde{H}^s(\Gamma)} := \|\widetilde{u}\|_{H^s(\partial\Omega)}, \tag{16}$$

where $\widetilde{u}$ denotes the extension of u by zero on $\partial\Omega$. Clearly, if $\Gamma = \partial\Omega$ is the boundary of a bounded Lipschitz domain, it holds that $\widetilde{H}^s(\Gamma) = H^s(\Gamma)$ for all $s \in [0,1]$. The same space can be defined for a domain Ω instead of Γ by using $\mathbb{R}^d$ instead of $\partial\Omega$. A different characterization of the spaces $\widetilde{H}^s(\Omega)$ can be given; to that end, we introduce the *trace operator* γ_0, which is defined for smooth functions u as $\gamma_0 u := u|_{\partial\Omega}$. It can be shown that $\|\gamma_0 u\|_{H^{s-1/2}(\partial\Omega)} \le C_s \|u\|_{H^s(\Omega)}$ for $s \in (1/2,1]$, hence γ_0 can be extended to a linear and continuous operator from $H^s(\Omega)$ to $H^{s-1/2}(\Gamma)$ for $s \in (1/2,1]$. It is known that

$$\begin{aligned} \widetilde{H}^s(\Omega) &= \{u \in H^s(\Omega) \mid \gamma_0 u = 0\} && \text{for } s \in (1/2,1],\\ \widetilde{H}^s(\Omega) &= H^s(\Omega) && \text{for } s \in [0,1/2). \end{aligned}$$

Furthermore, in these cases, the norms on $\widetilde{H}^s(\Omega)$ from (16) and the norms on $H^s(\Omega)$ are equivalent, and the equivalence constants depend on s and Ω, cf. [80, Lem. 1.3.2.6 and Thm. 1.4.4.4]. If $s \in \{0,1\}$, the norms coincide. Note that the case $s = 1/2$ is excluded. Likewise, an operator γ_1 can be defined which extends the (co-)normal derivative.

For a linear operator $B : \mathscr{X} \to \mathscr{Y}$ between two normed linear spaces, denote its operator norm by

$$\|B\|_{\mathscr{X}\to\mathscr{Y}} := \sup_{0\neq x\in\mathscr{X}} \frac{\|B(x)\|_{\mathscr{Y}}}{\|x\|_{\mathscr{X}}}.$$

The operator B is called *bounded* if $\|B\|_{\mathscr{X}\to\mathscr{Y}} < \infty$. The dual space of a normed linear space $\mathscr{X}$, denoted by $\mathscr{X}'$, consists of all linear and bounded operators (so-called *functionals*) $f : \mathscr{X} \to \mathbb{R}$. A norm on $\mathscr{X}'$ is given by

$$\|f\|_{\mathscr{X}'} := \|f\|_{\mathscr{X}\to\mathbb{R}} = \sup_{0\neq x\in\mathscr{X}} \frac{|f(x)|}{\|x\|_{\mathscr{X}}}.$$

For the Sobolev space $H^s(\Gamma)$, the dual space can be characterized by the concept of the so-called *Gelfand triple*, cf. [123, Sec. 2.1.2.4], using the fact that for densely embedded Hilbert spaces $\mathscr{V} \subset \mathscr{U}$, their dual spaces are also densely embedded, i.e., $\mathscr{U}' \subset \mathscr{V}'$. Then, identifying $\mathscr{U}$ with its dual $\mathscr{U}'$, the scalar product $\langle\cdot\,,\cdot\rangle_{\mathscr{U}}$ can be extended to a duality pairing between $\mathscr{V}$ and its dual $\mathscr{V}'$. The space $\mathscr{U}$ is called *pivot space*. The described concept is used to define the duality pairing between $H^s(\Gamma)$ for $s > 0$ and its dual $\widetilde{H}^{-s}(\Gamma) := H^s(\Gamma)'$, where $L_2(\Gamma)$ is used as pivot space. The consequence is that, if $u \in H^s(\Gamma)$ and $v \in \widetilde{H}^{-s}(\Gamma)$ are both in $L_2(\Gamma)$, then $\langle u\,,v\rangle_{H^s(\Gamma)\times\widetilde{H}^{-s}(\Gamma)} = \langle u\,,v\rangle_\Gamma$ coincides with the $L_2(\Gamma)$ scalar product, and the last expression will be used from now on to denote the duality pairing. The dual space of $\widetilde{H}^s(\Gamma)$ for $s > 0$ will be denoted by $H^{-s}(\Gamma)$.

Certain equations that will be considered have a kernel, hence a quotient space will be needed to solve them. For $s \in [-1,1]$, define

$$H^s_0(\Gamma) := \{u \in H^s(\Gamma) \mid \langle u\,,1\rangle_\Gamma = 0\}.$$

Remark 1 In the literature, cf. [123, 135], Sobolev spaces are defined with a fixed parametrization and associated partition of unity on Γ. This gives an equivalent definition to ours, with constants that depend on the chosen parametrization. To see this, denote by a a specific parametrization and partition of unity on Γ, and associated norms $\|\cdot\|_{s,a}$. It follows immediately that $\|\cdot\|_{s,a} \le C_a \|\cdot\|_{H^s(\Gamma)}$, and the reverse inequality can be proven with the same arguments as in the proof of Theorem 17 below.

2.2 Boundary integral operators

From now on, $\Omega \subset \mathbb{R}^d$ will always denote a bounded, simply connected, d-dimensional domain with Lipschitz boundary $\partial\Omega$ and outer normal vector $\mathbf{n}(y)$ for $y \in \partial\Omega$, and Γ will denote a $(d-1)$-dimensional subset $\Gamma \subset \partial\Omega$. For simplicity, in the case $d = 2$, we assume that $\operatorname{cap}(\partial\Omega) < 1$, see [108], which can always be fulfilled by scaling Ω such that its diameter is smaller than 1. In order to transform a given PDE into an equivalent boundary integral equation, a fundamental solution of the PDE at hand needs to be available. For the Laplace operator $-\Delta$, the fundamental solution is given by

$$\mathrm{G}(z) := \begin{cases} -\frac{1}{2\pi}\log|z| & \text{for } d = 2,\\ \frac{1}{4\pi}\frac{1}{|z|} & \text{for } d = 3. \end{cases}$$

For densities $\phi, v : \Gamma \to \mathbb{R}$ and $x \in \mathbb{R}^d \setminus \Gamma$, define the following potentials:

- the **single layer potential** of ϕ as
$$\widetilde{V}\phi(x) := \int_\Gamma \mathrm{G}(x-y)\phi(y)\,d\Gamma(y),$$
- and the **double layer potential** of v as
$$\widetilde{K}v(x) := \int_\Gamma \partial_{\mathbf{n}(y)}\mathrm{G}(x-y)v(y)\,d\Gamma(y).$$

At least for $\phi, v \in L_1(\Gamma)$, these operators are smooth away from Γ, i.e., $\widetilde{V}\phi, \widetilde{K}v \in C^\infty(\mathbb{R}^d \setminus \Gamma)$, and also harmonic, i.e., $\Delta\widetilde{V}\phi = 0 = \Delta\widetilde{K}v$ on $\mathbb{R}^d \setminus \Gamma$. Starting from these definitions, boundary integral operators are defined as

$$\begin{aligned} V &:= \gamma_0\widetilde{V}, & K &:= 1/2 + \gamma_0\widetilde{K},\\ W &:= -\gamma_1\widetilde{K}, & K' &:= -1/2 + \gamma_1\widetilde{V}. \end{aligned}$$

The operator V is called the *single layer operator*, W the *hypersingular operator*, and K and K' the *double layer operator* and its *adjoint*, respectively. The two following results recall the stability and ellipticity properties of these boundary integral operators. For proofs and further references, we refer to [49, 108, 123, 144].

Theorem 2 *For $\Gamma = \partial\Omega$ a Lipschitz boundary and $s \in [-1/2, 1/2]$, the boundary integral operators are bounded as mappings*

$$\begin{aligned} V &: H^{-1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma)\\ K &: H^{1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma)\\ K' &: H^{-1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma)\\ W &: H^{1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma). \end{aligned}$$

If $\Gamma \subset \partial\Omega$, $\partial\Omega$ again a Lipschitz boundary, it holds

$$\begin{aligned} V &: \widetilde{H}^{-1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma)\\ W &: \widetilde{H}^{1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma). \end{aligned}$$

Theorem 3 *If $\Gamma = \partial\Omega$ is the boundary of a Lipschitz domain Ω, then there holds ellipticity*

$$\begin{aligned} \langle V\phi\,,\phi\rangle_\Gamma &\geq C_{\mathrm{ell}}\|\phi\|^2_{H^{-1/2}(\Gamma)} \quad \text{for all } \phi \in H^{-1/2}(\Gamma),\\ \langle Wu\,,u\rangle_\Gamma + \langle u\,,1\rangle^2_\Gamma &\geq C_{\mathrm{ell}}\|u\|^2_{H^{1/2}(\Gamma)} \quad \text{for all } u \in H^{1/2}(\Gamma). \end{aligned}$$

If $\Gamma \subsetneq \partial\Omega$ is only a subset, then there holds

$$\begin{aligned} \langle V\phi\,,\phi\rangle_\Gamma &\geq C_{\mathrm{ell}}\|\phi\|^2_{\widetilde{H}^{-1/2}(\Gamma)} \quad \text{for all } \phi \in \widetilde{H}^{-1/2}(\Gamma),\\ \langle Wu\,,u\rangle_\Gamma &\geq C_{\mathrm{ell}}\|u\|^2_{\widetilde{H}^{1/2}(\Gamma)} \quad \text{for all } u \in \widetilde{H}^{1/2}(\Gamma). \end{aligned}$$

The constant C_{ell} depends only on Γ.

2.3 Weakly singular integral equations

According to Theorems 2 and 3, $\langle V\cdot\,,\cdot\rangle_\Gamma$ is a scalar product on $\widetilde{H}^{-1/2}(\Gamma)$, such that the Riesz representation theorem immediately yields solutions to the following variational formulations.

Proposition 4 (Weakly singular integral equation) *Denote by $\Omega \subset \mathbb{R}^d$ a Lipschitz domain with $\mathrm{cap}(\partial\Omega) < 1$ for $d = 2$ and $\Gamma \subset \partial\Omega$. Given $f \in H^{1/2}(\Gamma)$, there is a unique solution $\phi \in \widetilde{H}^{-1/2}(\Gamma)$ of the variational problem*

$$\langle V\phi\,,\psi\rangle_\Gamma = \langle f\,,\psi\rangle_\Gamma \quad \text{for all } \psi \in \widetilde{H}^{-1/2}(\Gamma).$$

Proposition 5 (Dirichlet problem) *Denote by $\Omega \subset \mathbb{R}^d$ a Lipschitz domain with $\mathrm{cap}(\partial\Omega) < 1$ for $d = 2$ and $\Gamma = \partial\Omega$. Given $f \in H^{1/2}(\Gamma)$, there is a unique solution $\phi \in H^{-1/2}(\Gamma)$ of the variational problem*

$$\langle V\phi\,,\psi\rangle_\Gamma = \langle (1/2 + K)f\,,\psi\rangle_\Gamma \quad \text{for all } \psi \in H^{-1/2}(\Gamma).$$

2.4 Hypersingular integral equations

Likewise, Theorems 2 and 3 state that $\langle W\cdot\,,\cdot\rangle_\Gamma$ is a scalar product on $\widetilde{H}^{1/2}(\Gamma)$ if $\Gamma \subsetneq \partial\Omega$ is an open surface, while $\langle W\cdot\,,\cdot\rangle_\Gamma + \langle\cdot\,,1\rangle_\Gamma\langle\cdot\,,1\rangle_\Gamma$ is a scalar product on $H^{1/2}(\Gamma)$ in case of $\Gamma = \partial\Omega$.

Proposition 6 (Hypersingular integral equation) *Denote by $\Omega \subset \mathbb{R}^d$ a Lipschitz domain and $\Gamma \subsetneq \partial\Omega$ a simply connected, open surface. Given $\phi \in H^{-1/2}(\Gamma)$, there is a unique solution $u \in \widetilde{H}^{1/2}(\Gamma)$ of the variational problem*

$$\langle Wu\,,v\rangle_\Gamma = \langle\phi\,,v\rangle_\Gamma \quad \text{for all } v \in \widetilde{H}^{1/2}(\Gamma).$$

If $\Gamma = \partial\Omega$, then there is a unique solution $u \in H^{1/2}(\Gamma)$ of the variational problem

$$\langle Wu\,,v\rangle_\Gamma + \langle u\,,1\rangle_\Gamma\langle v\,,1\rangle_\Gamma = \langle\phi\,,v\rangle_\Gamma \quad \text{for all } v \in H^{1/2}(\Gamma).$$

Provided that $\phi \in H_0^{-1/2}(\Gamma)$, the solution satisfies $u \in H_0^{1/2}(\Gamma)$.

Proposition 7 (Neumann problem) *Denote by $\Omega \subset \mathbb{R}^d$ a Lipschitz domain and $\Gamma = \partial\Omega$ Given $\phi \in H_0^{-1/2}(\Gamma)$, there is a unique solution $u \in H_0^{1/2}(\Gamma)$ of the variational problem*

$$\langle Wu\,,v\rangle_\Gamma + \langle u\,,1\rangle_\Gamma\langle v\,,1\rangle_\Gamma = \langle(1/2 - K')\phi\,,v\rangle_\Gamma \quad \text{for all } v \in H^{1/2}(\Gamma).$$

2.5 Regularity of solutions

It is well known that solutions to BVPs[10] on non-smooth domains have in general limited regularity, even for smooth data. For polygonal/polyhedral domains and standard elliptic operators of second order there exists a precise regularity theory that proves that this regularity reduction is due to the presence of so-called corner singularities (on polygons and polyhedra) and corner-edge singularities (on polyhedra). In this paper we are studying the solution of integral equations of the first kind where unknowns are Cauchy data of BVP. Therefore, through trace operations (extended restriction and normal derivative), singular behavior of solutions to BVP imply in a natural way singular behavior of solutions to such integral equations.

For an overview of regularity theory for BVP on non-smooth domains we refer to the monograph by Dauge [53]. The singularity expressions by Dauge have been extensively studied by Stephan and von Petersdorff [146–148]. Their main contribution is tensor product expansions of singularities so that they are accessible to approximation analysis by piecewise polynomial functions. In this way, precise predictions can be made about convergence orders of FE and

[10] boundary value problem (BVP)

BE approximations. In two dimensions, the study of corner singularities goes back to the seminal paper by Kondratiev [104] and, of course, the structure of the appearing singularities is much simpler.

We note that for an optimal error analysis of hp-methods with geometric mesh refinement, a more specific regularity analysis based on countably normed weighted spaces is in order. We refer to [95,96] for a corresponding regularity theory of boundary integral equations on polygons. To our knowledge [106], Maischak and Stephan have a manuscript analyzing the case of the hypersingular integral equation (governing the Laplacian) on polyhedral surfaces.

Low-order methods severely suffer from the presence of singularities. They limit the order of convergence of the boundary element method when quasi-uniform meshes are used. Adaptive methods refine meshes locally by using information that stems from a posteriori error estimation. In this way, adaptivity aims at recovering the orders of convergence that one would obtain for smooth solutions and quasi-uniform meshes.

In the following, for some typical cases, we recall what are the principal singularities that one has to expect in solutions to the hypersingular and weakly singular boundary integral equations. These results stem from the previously mentioned publications [104,53,147,148].

Two space dimensions. Let Γ be the boundary of a simply connected polygon with edges Γ^j, vertices t_j and angles ω_j at the vertices ($j = 1,\dots,J$). We consider the weakly singular integral equation from Proposition 4 (with solution ϕ and right-hand side function f) and the hypersingular integral equation from Proposition 6 (with solution u and right-hand side function g). For piecewise analytic data f, g, the solutions ϕ and u behave singularly at the corners of the polygon and are smooth elsewhere.

To be precise we consider a partition of unity $(\chi_1,\dots,\chi_J)$ where χ_j is the restriction of a $C_0^\infty(\mathbb{R}^2)$ function to Γ such that $\chi_j = 1$ in a neighborhood of the vertex t_j and $\operatorname{supp}(\chi_j) \subset \Gamma^{j-1} \cup \{t_j\} \cup \Gamma^j$ ($\Gamma^0 = \Gamma^J$). In this way we may write any function φ on Γ like

$$\varphi = \sum_{j=1}^{J} (\varphi_-, \varphi_+)\chi_j$$

where a pair (φ_-, φ_+) corresponds to φ on $\Gamma^{j-1} \cup \{t_j\} \cup \Gamma^j$ with

$$\varphi_- = \varphi|_{\Gamma^{j-1}} \quad \text{and} \quad \varphi_+ = \varphi|_{\Gamma^j}.$$

From [50,94] we cite the following result.

Let $\alpha_{jk} := k\frac{\pi}{\omega_j}$ (integer $k \ge 1$, $j = 1,\dots,J$) and, for $t \ge 1/2$, let n be an integer with $n+1 > \frac{\omega_j}{\pi}(t-1/2) \ge n$.

(i) If f is a piecewise analytic function, then there exists a function ϕ_0 with $\phi_0|_{\Gamma^j} \in H^{t-1}(\Gamma^j)$ such that, for the solution ϕ of the weakly singular integral equation, there holds

$$\phi = \sum_{j=1}^{J} \sum_{k=1}^{n} \left((\phi_{jk})_-, (\phi_{jk})_+\right) \chi_j + \phi_0.$$

Here,

$$(\phi_{jk})_\pm(x) = c|x - t_j|^{\alpha_{jk}-1}$$

if α_{jk} is not an integer and

$$(\phi_{jk})_\pm(x) = c_1|x - t_j|^{\alpha_{jk}-1} + c_2|x - t_j|^{\alpha_{jk}-1} \log|x - t_j|$$

if α_{jk} is an integer.
(ii) If g is a piecewise analytic function, then there exists a function u_0 with $u_0|_{\Gamma^j} \in H^t(\Gamma^j)$ such that, for the solution u of the hypersingular integral equation, there holds

$$u = \sum_{j=1}^{J} \sum_{k=1}^{n} \left((u_{jk})_-, (u_{jk})_+\right) \chi_j + u_0.$$

Here,

$$(u_{jk})_\pm(x) = c|x - t_j|^{\alpha_{jk}}$$

if α_{jk} is not an integer and

$$(u_{jk})_\pm(x) = c_1|x - t_j|^{\alpha_{jk}} + c_2|x - t_j|^{\alpha_{jk}} \log|x - t_j|$$

if α_{jk} is an integer.

The constants c, c_1 and c_2 above (in (i) and (ii)) are generic.

The representation of singularities above is valid also in the case of open curves, by setting the angles $\omega_j = 2\pi$ at the endpoints. For example, Γ being an interval in $\mathbb{R}^2$, the solution ϕ of the weakly singular integral equation has (with t_0 being any endpoint of Γ) singularities of the form

$$\phi(x) \sim |t_0 - x|^{-1/2} \quad (x \text{ close to } t_0)$$

and the solution u of the hypersingular integral equation behaves like

$$u(x) \sim |t_0 - x|^{1/2} \quad (x \text{ close to } t_0).$$

An illustration of both cases is given in Figures 9, 10.

Concluding, the solutions of the integral equations are smooth away from the corners and have reduced regularity at the corners. In the case of the weakly singular equation, the solution can be unbounded at corners and in the case of the hypersingular equation, gradients (derivatives with respect to the arc-length) can be unbounded there. In the extreme case of an open polygon, the singularity $|\cdot - t_0|^{-1/2}$ prevents ϕ from being an $L_2(\Gamma)$-function and, similarly, u with its $|\cdot - t_0|^{1/2}$-singularity is not an element of $H^1(\Gamma)$.

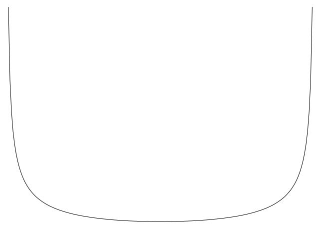

Fig. 9 Typical singular solution of the weakly singular integral equation on an interval

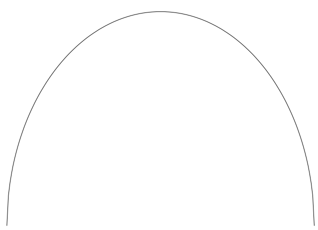

Fig. 10 Typical singular solution of the hypersingular integral equation on an interval

Three space dimensions. For simplicity we restrict our presentation of singularities in three dimensions to the case of Γ being a plane open surface with polygonal boundary. We use the results from [147,148] and follow the notation from [22,25], see also [127].

Again, we consider the weakly singular integral equation from Proposition 4 (with solution ϕ and right-hand side function f) and the hypersingular integral equation from Proposition 6 (with solution u and right-hand side function g). As before, we assume that f and g are sufficiently smooth. In the following we present the singularities of ϕ and u together.

Let V and E denote the sets of vertices and edges of Γ, respectively. For $v \in V$, let $E(v)$ denote the set of edges with v as an endpoint. Then, ϕ and u are of the form

$$\phi = \phi_{\mathrm{reg}} + \sum_{e\in E} \phi^e + \sum_{v\in V} \phi^v + \sum_{v\in V} \sum_{e\in E(v)} \phi^{ev},$$
$$u = u_{\mathrm{reg}} + \sum_{e\in E} u^e + \sum_{v\in V} u^v + \sum_{v\in V} \sum_{e\in E(v)} u^{ev},$$

where, using local polar and Cartesian coordinate systems (r_v, θ_v) and (x_{e1}, x_{e2}) with origin v, there hold the following representations:

(i) The regular parts satisfy $\phi_{\mathrm{reg}} \in H^k(\Gamma)$, $u_{\mathrm{reg}} \in H^{k+1}(\Gamma)$, with $k > 0$.

(ii) The edge singularities ϕ^e, u^e have the form

$$\phi^e = \sum_{j=1}^{m_e} \left(\sum_{s=0}^{s_j^e} b_{js}^e(x_{e1}) |\log x_{e2}|^s \right) x_{e2}^{\gamma_j^e - 1} \chi_1^e(x_{e1}) \chi_2^e(x_{e2}),$$
$$u^e = \sum_{j=1}^{m_e} \left(\sum_{s=0}^{s_j^e} b_{js}^e(x_{e1}) |\log x_{e2}|^s \right) x_{e2}^{\gamma_j^e} \chi_1^e(x_{e1}) \chi_2^e(x_{e2}),$$

where $\gamma_{j+1}^e \geq \gamma_j^e \geq \frac{1}{2}$, and m_e, s_j^e are integers. Here, χ_1^e, χ_2^e are C^∞ cut-off functions with $\chi_1^e = 1$ in a certain distance to the endpoints of e, and $\chi_1^e = 0$ in a neighbourhood of these vertices. Moreover, $\chi_2^e = 1$ for $0 \leq x_{e2} \leq \delta_e$ and $\chi_2^e = 0$ for $x_{e2} \geq 2\delta_e$ with some $\delta_e \in (0, \frac{1}{2})$. The functions $b_{js}^e \chi_1^e$ are in $H^m(e)$ for m as large as required.

(iii) The vertex singularities ϕ^v, u^v have the form

$$\phi^v = \chi^v(r_v) \sum_{i=1}^{n_v} \sum_{t=0}^{q_i^v} B_{it}^v |\log r_v|^t r_v^{\lambda_i^v - 1} w_{it}^v(\theta_v),$$
$$u^v = \chi^v(r_v) \sum_{i=1}^{n_v} \sum_{t=0}^{q_i^v} B_{it}^v |\log r_v|^t r_v^{\lambda_i^v} w_{it}^v(\theta_v),$$

where $\lambda_{i+1}^v \geq \lambda_i^v > 0$, n_v, $q_i^v \geq 0$ are integers, and B_{it}^v are real numbers. Here, χ^v is a C^∞ cut-off function with $\chi^v = 1$ for $0 \leq r_v \leq \tau_v$ and $\chi^v = 0$ for $r_v \geq 2\tau_v$ with some $\tau_v \in (0, \frac{1}{2})$. The functions w_{it}^v are in $H^q(0, \omega_v)$ for q as large as required. Here, ω_v denotes the interior angle (on Γ) between the edges meeting at v.

(iv) The edge-vertex singularities ϕ^{ev}, u^{ev} have the form

$$\phi^{ev} = \phi_1^{ev} + \phi_2^{ev}, \quad u^{ev} = u_1^{ev} + u_2^{ev},$$

where

$$\phi_1^{ev} = \sum_{j=1}^{m_e} \sum_{i=1}^{n_v} \left(\sum_{s=0}^{s_j^e} \sum_{t=0}^{q_i^v} \sum_{l=0}^{s} B_{ijlts}^{ev} |\log x_{e1}|^{s+t-l} |\log x_{e2}|^l \right) x_{e1}^{\lambda_i^v - \gamma_j^e} x_{e2}^{\gamma_j^e - 1} \chi^v(r_v) \chi^{ev}(\theta_v),$$

$$u_1^{ev} = \sum_{j=1}^{m_e} \sum_{i=1}^{n_v} \left(\sum_{s=0}^{s_j^e} \sum_{t=0}^{q_i^v} \sum_{l=0}^{s} B_{ijlts}^{ev} |\log x_{e1}|^{s+t-l} |\log x_{e2}|^l \right) x_{e1}^{\lambda_i^v - \gamma_j^e} x_{e2}^{\gamma_j^e} \chi^v(r_v) \chi^{ev}(\theta_v)$$

and

$$\phi_2^{ev} = \sum_{j=1}^{m_e} \sum_{s=0}^{s_j^e} B_{js}^{ev}(r_v) |\log x_{e2}|^s x_{e2}^{\gamma_j^e - 1} \chi^v(r_v) \chi^{ev}(\theta_v),$$
$$u_2^{ev} = \sum_{j=1}^{m_e} \sum_{s=0}^{s_j^e} B_{js}^{ev}(r_v) |\log x_{e2}|^s x_{e2}^{\gamma_j^e} \chi^v(r_v) \chi^{ev}(\theta_v),$$

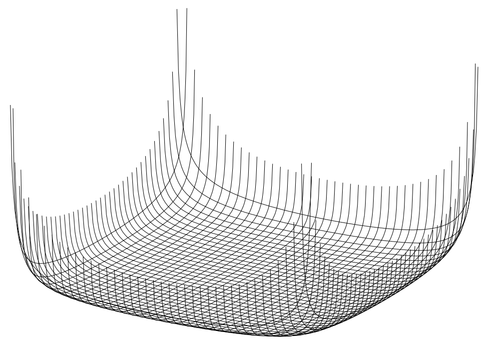

Fig. 11 Typical singular solution of the weakly singular integral equation on the open surface $(0,1)\times(0,1)\times\{0\}$

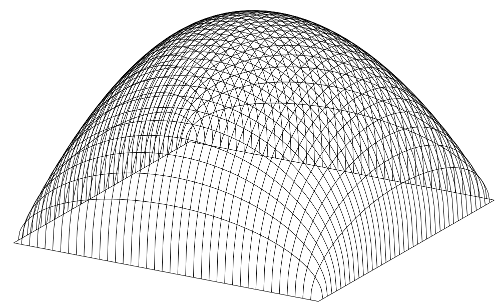

Fig. 12 Typical singular solution of the hypersingular integral equation on the open surface $(0,1)\times(0,1)\times\{0\}$

with

$$B^{ev}_{js}(r_v) = \sum_{l=0}^{s} B^{ev}_{jsl}(r_v)|\log r_v|^l.$$

Here, q^v_i, s^e_j, λ^v_i, γ^e_j, χ^v are as above, B^{ev}_{ijlts} are real numbers, and χ^{ev} is a C^∞ cut-off function with $\chi^{ev}=1$ for $0\le\theta_v\le\beta_v$ and $\chi^{ev}=0$ for $\frac{3}{2}\beta_v\le\theta_v\le\omega_v$ for some $\beta_v\in(0,\min\{\omega_v/2,\pi/8\}]$. The functions B^{ev}_{jsl} may be chosen such that

$$B^{ev}_{js}(r_v)\,\chi^v(r_v)\chi^{ev}(\theta_v) = \chi_{js}(x_{e1},x_{e2})\,\chi^e_2(x_{e2}),$$

where the extension of χ_{js} by zero onto

$$\mathbb{R}^{2+} := \{(x_{e1},x_{e2});\ x_{e2}>0\}$$

lies in $H^m(\mathbb{R}^{2+})$ for m as large as required. Here, χ^e_2 is a C^∞ cut-off function as in (ii).

Concluding, on open surfaces, both integral equations have solutions with singularities. The strongest ones are of the edge-type $\operatorname{dist}(\cdot,\partial\Gamma)^{-1/2}$ for the solution ϕ of the weakly singular equation. In this case, ϕ is not an element of $L_2(\Gamma)$, as in the case of two dimensions on open curves. Correspondingly, the strongest singularities of the solution u of the hypersingular equation are of the type $\operatorname{dist}(\cdot,\partial\Gamma)^{1/2}$ so that $u\notin H^1(\Gamma)$, again analogously to the case in two dimensions. In Figures 11, 12 we present typical solutions to both integral equations on the open surface $\Gamma=(0,1)\times(0,1)\times\{0\}$. It remains to mention that on polyhedral surfaces, singularities have the same structure but with larger exponents defining the edge and edge-vertex singularities, cf. [147,148] for more details.

2.6 Discrete spaces

The discrete spaces that will be used to approximate the solutions of the problems given by Propositions 4–7 are spaces of piecewise polynomials over a mesh of Γ. The related terms will be introduced in this section.

2.6.1 Meshes

Definition 8 *A mesh $\mathscr{T}$ on $\Gamma\subset\partial\Omega$ is a finite, mutually disjoint partition $\mathscr{T}=\{T_1,\dots,T_M\}$ with the following properties:*

- *every element $T\in\mathscr{T}$ is a d-simplex, i.e., the interior of the convex hull of d points $x_1,\dots,x_d$,*
- $\overline{\Gamma}=\bigcup_{i=1}^M\overline{T}_i$,
- *the intersection $\overline{T}\cap\overline{T}'$ is either empty, a common point, or a common edge of both T and T'.*

The collection of all points $\mathscr{N}:=\{x_1,\dots,x_N\}$ that constitute the elements is called the set of *nodes* of $\mathscr{T}$. Associated to a mesh $\mathscr{T}$ is the local mesh-width function $h_\mathscr{T}\in L_\infty(\Gamma)$, given $\mathscr{T}$-element-wise as $h_\mathscr{T}|_T := h_\mathscr{T}(T) := |T|^{1/(d-1)}$. On certain occasions the index $\mathscr{T}$ will be omitted if no confusion can arise, i.e., h_T will be used instead of $h_\mathscr{T}|_T$. The quantity

$$\sigma_\mathscr{T} := \begin{cases} \sup\limits_{T,T'\in\mathscr{T},\overline{T}\cap\overline{T}'\neq\emptyset} \dfrac{\operatorname{diam}(T)}{\operatorname{diam}(T')} & \text{for } d=2,\\ \max\limits_{T\in\mathscr{T}} \dfrac{\operatorname{diam}(T)^{d-1}}{|T|} & \text{for } d\ge 3,\end{cases}$$

is usually called the *shape-regularity constant* of $\mathscr{T}$, and it is a measure for the degeneracy of the elements T.

Remark 2 To say that *a constant C in a statement depends on shape-regularity* means that, given some constant $\sigma>0$, there is a constant $C(\sigma)$, depending only on σ, such that the statement holds true for all meshes $\mathscr{T}$ as long as $\sigma_\mathscr{T}\le\sigma$.

For a node $z \in \mathscr{N}$, the node-patch ω_z is the collection of all elements $T \in \mathscr{T}$ which share z, same idea for the element-patch ω_T, i.e.,

$$\omega_z := \left\{ T \in \mathscr{T} \mid z \in \overline{T} \right\}$$
$$\omega_T := \left\{ T' \in \mathscr{T} \mid \overline{T'} \cap \overline{T} \neq \emptyset \right\}.$$

An important concept is the so-called *reference element* T_{ref}, which is chosen to be fixed throughout, e.g., as the interior of the convex hull of $(0,0)$, $(1,0)$, and $(0,1)$. Every element $T \in \mathscr{T}$ with nodes $\{x_0, x_1, x_2\}$ is then the image $T = F_T(T_{\mathrm{ref}})$ of T_{ref} under the affine mapping

$$F_T : \begin{cases} \mathbb{R}^2 \to \mathbb{R}^3 \\ x \mapsto B_T x + x_0, \end{cases}$$

with matrix $B_T = (x_1 - x_0 \mid x_2 - x_0) \in \mathbb{R}^{3\times 2}$.

2.6.2 Polynomial spaces

Denoting for $p \geq 0$ the polynomial space on the reference element by

$$\mathscr{P}^p(T_{\mathrm{ref}}) := \operatorname{span}\left\{ (x,y) \in \mathbb{R}^2 \mapsto x^i y^k \mid 0 \leq i+k \leq p \right\},$$

polynomial spaces on a mesh $\mathscr{T}$ are defined by

$$\mathscr{P}^p(\mathscr{T}) := \{ u \in L_\infty(\Gamma) \mid u \circ F_T \in \mathscr{P}^p(T_{\mathrm{ref}}) \text{ for all } T \in \mathscr{T} \}$$
$$\mathscr{S}^p(\mathscr{T}) := \mathscr{P}^p(\mathscr{T}) \cap C^0(\Gamma).$$

For $\Gamma \subsetneq \partial\Omega$, define the space with vanishing boundary conditions

$$\widetilde{\mathscr{S}^p}(\mathscr{T}) := \widetilde{H}^{1/2}(\Gamma) \cap \mathscr{S}^p(\mathscr{T}).$$

Remark 3 If a mesh carries an index, e.g., $\mathscr{T}_\ell$, it's associated quantities are also equipped with this index, e.g., σ_ℓ denotes the shape-regularity constant, h_ℓ the mesh-width, $\mathscr{N}_\ell$ the set of nodes, and so on.

2.7 Galerkin formulation

Discrete approximations to the exact solutions ϕ and u of the Problems in Propositions 4–7 can be computed by changing the infinite dimensional spaces $\widetilde{H}^{\pm 1/2}(\Gamma)$ to the discrete spaces $\mathscr{P}^p(\mathscr{T})$ and $\mathscr{S}^p(\mathscr{T})$. The fact that we can also use functions of the discrete spaces in the variational formulations provides best-approximation estimates (Céa's Lemma).

Proposition 9 (Galerkin for weakly singular) *There is a unique solution $\Phi \in \mathscr{P}^p(\mathscr{T})$ of*

$$\langle V\Phi\,, \Psi \rangle_\Gamma = \langle f\,, \Psi \rangle_\Gamma \quad \text{for all } \Psi \in \mathscr{P}^p(\mathscr{T}).$$

Proposition 10 (Galerkin for Dirichlet) *There is a unique solution $\Phi \in \mathscr{P}^p(\mathscr{T})$ of*

$$\langle V\Phi\,, \Psi \rangle_\Gamma = \langle (1/2 + K) f\,, \Psi \rangle_\Gamma \quad \text{for all } \Psi \in \mathscr{P}^p(\mathscr{T}).$$

Lemma 11 *If $\phi \in \widetilde{H}^{-1/2}(\Gamma)$ is the solution of Proposition 4 or Proposition 5, and $\Phi \in \mathscr{P}^p(\mathscr{T})$ is the solution of Proposition 9 or 10, then*

$$\|\phi - \Phi\|_{\widetilde{H}^{-1/2}(\Gamma)} \leq \frac{C_{\mathrm{cont}}}{C_{\mathrm{ell}}} \min_{\Psi \in \mathscr{P}^p(\mathscr{T})} \|\phi - \Psi\|_{\widetilde{H}^{-1/2}(\Gamma)}.$$

Here, $C_{\mathrm{cont}} = \|V\|_{\widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)}$ is the stability constant of V, cf. Theorem 2, and C_{ell} is its ellipticity constant, cf. Theorem 3.

Proposition 12 (Galerkin for hypersingular) *In the case $\Gamma \subsetneq \partial\Omega$, there is a unique solution $U \in \widetilde{\mathscr{S}^p}(\mathscr{T})$ of*

$$\langle WU\,, V \rangle_\Gamma = \langle \phi\,, V \rangle_\Gamma \quad \text{for all } V \in \widetilde{\mathscr{S}^p}(\mathscr{T}).$$

In the case $\Gamma = \partial\Omega$, there is a unique solution $U \in \mathscr{S}^p(\mathscr{T})$ such that for all $V \in \mathscr{S}^p(\mathscr{T})$

$$\langle WU\,, V \rangle_\Gamma + \langle U\,, 1 \rangle_\Gamma \langle V\,, 1 \rangle_\Gamma = \langle \phi\,, V \rangle_\Gamma.$$

If $\phi \in H_0^{-1/2}(\Gamma)$, it holds that $\langle U\,, 1 \rangle_\Gamma = 0$.

Proposition 13 (Galerkin for Neumann) *There is a unique solution $U \in \mathscr{S}^p(\mathscr{T})$ such that for all $V \in \mathscr{S}^p(\mathscr{T})$*

$$\langle WU\,, V \rangle_\Gamma + \langle U\,, 1 \rangle_\Gamma \langle V\,, 1 \rangle_\Gamma = \langle (1/2 - K')\phi\,, V \rangle_\Gamma.$$

Lemma 14 *If $u \in \widetilde{H}^{1/2}(\Gamma)$ is the solution of Proposition 6 or Proposition 7, and $U \in \widetilde{\mathscr{S}^p}(\mathscr{T})$ resp. $U \in \mathscr{S}^p(\mathscr{T})$ is the solution of Proposition 12 or 13, then*

$$\|u - U\|_{\widetilde{H}^{1/2}(\Gamma)} \leq \frac{C_{\mathrm{cont}}}{C_{\mathrm{ell}}} \min_{V \in \mathscr{S}^p(\mathscr{T})} \|u - V\|_{\widetilde{H}^{1/2}(\Gamma)}.$$

Here, $C_{\mathrm{cont}} = \|W\|_{\widetilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)}$ is the stability constant of W, cf. Theorem 2, and C_{ell} is its ellipticity constant, cf. Theorem 3.

Note that the discrete formulations of Propositions 9– 13 are, indeed, linear systems of equations. A distinct feature of boundary element methods is that, due to the non-locality of the boundary integral operators, the system matrices are dense, and therefore sophisticated data compression techniques are used to reduce complexity for assembling and solving. In Propositions 10 and 13, also the right-hand sides contain boundary integral operators. There are fast methods to compute the right-hand sides, cf. [48,126], but if one wants to re-use the fast method that is employed for system matrices, the data f resp. ϕ needs to be approximated by discrete functions.

Proposition 15 (Galerkin for Dirichlet with data approximation) *Denote by $J_{\mathscr{T}}: H^{1/2}(\Gamma) \to \mathscr{S}^{p+1}(\mathscr{T})$ a $H^{1/2}(\Gamma)$ stable projection. Then, there is a unique solution $\Phi \in \mathscr{P}^p(\mathscr{T})$ of*

$$\langle V\Phi, \Psi \rangle_\Gamma = \langle (1/2+K)J_{\mathscr{T}} f, \Psi \rangle_\Gamma \quad \text{for all } \Psi \in \mathscr{P}^p(\mathscr{T}).$$

Proposition 16 (Galerkin for Neumann with data approximation) *Denote by $\pi_{\mathscr{T}}^{p-1}: L_2(\Gamma) \to \mathscr{P}^{p-1}(\mathscr{T})$ the $L_2(\Gamma)$-orthogonal projection. Then, there is a unique solution $U \in \mathscr{S}^p(\mathscr{T})$ such that for all $V \in \mathscr{S}^p(\mathscr{T})$*

$$\langle WU, V \rangle_\Gamma + \langle U, 1 \rangle_\Gamma \langle V, 1 \rangle_\Gamma = \langle (1/2-K')\pi_{\mathscr{T}}^{p-1}\phi, V \rangle_\Gamma.$$

3 Localization of fractional Sobolev norms

The numerical analysis of boundary element methods takes place in the Sobolev spaces $H^s(\Gamma)$ for $s \in [-1,1]$ that are defined in Section 2. Apart from the exceptional cases $H^0(\Gamma) = L_2(\Gamma)$ and $H^1(\Gamma)$, all other spaces are equipped with norms that are either non-local ($s \in (0,1)$) or additionally impossible to compute ($s \in [-1,0)$). A norm is understood to be non-local if it is not possible to split its square into contributions on the elements of a mesh, i.e., if one cannot write

$$\|u\|^2_{H^s(\Gamma)} \simeq \sum_{T \in \mathscr{T}} \|u\|^2_{H^s(T)}. \tag{17}$$

The spaces and dual spaces that are used in the variational formulations of integral equations are usually equipped with non-local norms. For example, the residual $V\Phi - f$ of a weakly singular integral equation is computable but has to be measured in the non-local norm of the space $H^{1/2}(\Gamma)$. However, only the knowledge of the residuals local contributions enables us to define local error indicators that can be used for local mesh refinement in adaptive algorithms. Different possibilities to localize a non-local norm are available and will be presented in this section.

3.1 Localization by local fractional norms

In general, (17) does not hold equivalently with constants that are independent of the mesh $\mathscr{T}$. Fortunately, this is no longer true if additional properties of the functions under consideration are assumed. The following result is shown in [63] for $d = 2$ and in [64] for $d = 3$.

Theorem 17 *If $\mathscr{T}$ is a mesh on Γ and $s \in (0,1)$, then it holds for all $v \in H^s(\Gamma)$ that*

$$|v|^2_{H^s(\Gamma)} \leq \sum_{z \in \mathscr{N}} |v|^2_{H^s(\omega_z)} + C_{\mathrm{loc}} \sum_{T \in \mathscr{T}} h_T^{-2s} \|v\|^2_{L_2(T)}, \tag{18}$$

where C_{loc} depends only on s and Γ.

Proof In the following, we use the abbreviation

$$\int_Y \int_X := \int_Y \int_X \frac{|v(x)-v(y)|^2}{|x-y|^{d-1+2s}} \, dx \, dy.$$

The idea of the proof is to write

$$|v|^2_{H^s(\Gamma)} = \sum_{T \in \mathscr{T}} \int_T \int_{\omega_T} + \sum_{T \in \mathscr{T}} \int_T \int_{\Gamma \setminus \omega_T} \tag{19}$$

and bound the second term via the triangle inequality

$$\int_T \int_{\Gamma \setminus \omega_T} \lesssim \int_T |v(y)|^2 \int_{\Gamma \setminus \omega_T} |x-y|^{-d+1-2s} \, dx \, dy + \int_{\Gamma \setminus \omega_T} |v(x)|^2 \int_T |x-y|^{-d+1-2s} \, dx \, dy.$$

One shows that the sum over all $T \in \mathscr{T}$ of the second part on the right-hand side is the same as the sum over the first part, hence

$$\sum_{T \in \mathscr{T}} \int_T \int_{\Gamma \setminus \omega_T} \lesssim \sum_{T \in \mathscr{T}} \int_T |v(y)|^2 \int_{\Gamma \setminus \omega_T} |x-y|^{-d+1-2s} \, dx \, dy.$$

Finally, direct calculation for $d = 2$ and the use of polar coordinates for $d = 3$ shows

$$\int_{\Gamma \setminus \omega_T} |x-y|^{-d+1-2s} \, dx \lesssim h_T^{-2s}.$$

The first term on the right-hand side of (19) can be estimated immediately via

$$\sum_{T \in \mathscr{T}} \int_T \int_{\omega_T} \leq \sum_{z \in \mathscr{N}} |v|^2_{H^s(\omega_z)},$$

which finishes the proof. □

The estimate (18) already provides a reliable localization of the non-local H^s-norm, independent of the shape-regularity of the mesh. However, choosing v constant on Γ shows that the reverse inequality to (18) cannot hold in general, i.e., the bound (18) is not efficient. However, efficiency can be shown to hold when certain orthogonality is available. More precisely, the following estimate from [64, Lemma 3.4] enables us to bound the L_2-terms on the right-hand side of (18) by local H^s-terms. The benefit will be twofold: First, it will enable us to show efficiency of the localization (18) on shape-regular meshes. Second, it provides us with another localization which is always efficient as well as reliable on shape-regular meshes (Theorem 19).

Lemma 18 *Let $\omega \subseteq \Gamma$ be a measurable set, $s \in (0,1)$, and $u \in H^s(\omega)$. Then,*

$$\|u\|^2_{L_2(\omega)} \leq \frac{\mathrm{diam}(\omega)^{d-1+2s}}{2\,|\omega|} |u|^2_{H^s(\omega)} + \frac{1}{|\omega|} \left(\int_\omega u(x) \, dx \right)^2.$$

In particular, if $\mathscr{T}$ is a mesh on Γ and $\langle u, \Psi_T\rangle_\Gamma = 0$ for $\Psi_T \in \mathscr{P}^0(\mathscr{T})$ the characteristic function of an element $T \in \mathscr{T}$,

$$\|u\|^2_{L_2(T)} \le \frac{\sigma_{\mathscr{T}} h_T^{2s}}{2} |u|^2_{H^s(T)}. \tag{20}$$

If $\langle u, \Psi_z\rangle_\Gamma = 0$ for $\Psi_z \in \mathscr{S}^1(\mathscr{T})$ the hat-function function of a node $z \in \mathscr{N}$,

$$\|u\|^2_{L_2(\omega_z)} \le C \frac{\operatorname{diam}(\omega_z)^{d-1+2s}}{|\omega_z|} |u|^2_{H^s(\omega_z)}, \tag{21}$$

where the constant $C > 0$ depends only on d.

Proof To see the first estimate, note that

$$\begin{aligned} 2\,|\omega|\,\|u\|^2_{L_2(\omega)} &= \int_\omega\int_\omega |u(x)|^2\,dx\,dy + \int_\omega\int_\omega |u(y)|^2\,dx\,dy \\ &= \int_\omega\int_\omega |u(x)-u(y)|^2\,dx\,dy \\ &\quad + 2\left(\int_\omega u(x)\,dx\right)^2. \end{aligned}$$

Since for all $x, y \in \omega$ with $x \ne y$ it holds

$$1 \le \operatorname{diam}(\omega)^{d-1+2s} \frac{1}{|x-y|^{d-1+2s}},$$

the first estimate follows. To show (20), choose $\omega = T$ and observe that

$$\frac{\operatorname{diam}(T)^{d-1+2s}}{2\,|T|} \le \frac{\sigma_{\mathscr{T}} h_T^{2s}}{2} \quad \text{and} \quad \langle u, \Psi_T\rangle_\Gamma = \int_T u(x)\,dx = 0.$$

To show (21) note first that

$$\int_{\omega_z} u\,\Psi_z\,dx = 0 \quad \text{for all } z \in \mathscr{N}_{\mathscr{T}},$$

where Ψ_z is the $\mathscr{T}$-piecewise linear and continuous basis function for z. The Cauchy-Schwarz inequality shows

$$\left|\int_{\omega_z} u\,dx\right| = \left|\int_{\omega_z} (1-\Psi_z)u\,dx\right| \le \|1-\Psi_z\|_{L_2(\omega_z)} \|u\|_{L_2(\omega_z)}.$$

A direct calculation shows that

$$\|1-\Psi_z\|^2_{L_2(\omega_z)} = \begin{cases} |\omega_z|/2 & \text{for } d = 3, \\ |\omega_z|/3 & \text{for } d = 2, \end{cases}$$

hence

$$\left|\int_{\omega_z} u\,dx\right| \le q\,|\omega_z|^{1/2}\,\|u\|_{L_2(\omega_z)},$$

for some $q < 1$ depending only on d. This concludes (21). □

The two preceding results show that the localization (18) is always reliable and, on shape-regular meshes, also efficient. We can combine the results also to obtain a localization which is always efficient, and, on shape-regular meshes also reliable.

Theorem 19 *Denote by $\mathscr{T}$ a mesh on Γ and let $s \in (0,1)$. Suppose that $\mathscr{X}_{\mathscr{T}}$ is a discrete space with $\mathscr{P}^p(\mathscr{T}) \subseteq \mathscr{X}_{\mathscr{T}}$ or $\mathscr{S}^p(\mathscr{T}) \subseteq \mathscr{X}_{\mathscr{T}}$. Then for all $v \in H^s(\Gamma)$ with $\int_\Gamma v\Psi\,dx = 0$ for all $\Psi \in \mathscr{X}_{\mathscr{T}}$, it holds that*

$$\|v\|^2_{H^s(\Gamma)} \simeq \sum_{z\in\mathscr{N}} |v|^2_{H^s(\omega_z)}. \tag{22}$$

The constant in the lower bound depends only on s and Γ, while the constant in the upper bound additionally depends on shape-regularity.

Proof The lower bound $\gtrsim$ follows immediately from the definition of the norms and the fact that every $T \in \mathscr{T}$ is part of at most d node-patches ω_z. Now, we show the upper bound if $\mathscr{P}^p(\mathscr{T}) \subseteq \mathscr{X}_{\mathscr{T}}$. Since $\mathscr{P}^0(\mathscr{T}) \subseteq \mathscr{P}^p(\mathscr{T})$, the estimate (20) can be used in (18) to show

$$|u|^2_{H^s(\Gamma)} \le \sum_{z\in\mathscr{N}} |u|^2_{H^s(\omega_z)} + C_{\mathrm{loc}} \frac{\sigma_{\mathscr{T}}}{2} \sum_{T\in\mathscr{T}} |u|^2_{H^s(T)},$$

and furthermore, due to $h_T^{2s} \le \sigma_{\mathscr{T}}^s\,|T|^s \le \sigma_{\mathscr{T}}^s\,|\Gamma|^s$,

$$\|u\|^2_{L_2(\Gamma)} \le \frac{\sigma_{\mathscr{T}}^{1+s}\,|\Gamma|^s}{2} \sum_{T\in\mathscr{T}} |u|^2_{H^s(T)}.$$

Using again the fact that every element $T \in \mathscr{T}$ it part of at most d patches ω_z finally shows the upper bound in (22).

Suppose now that $\mathscr{S}^p(\mathscr{T}) \subseteq \mathscr{X}_{\mathscr{T}}$. Since $\mathscr{S}^1(\mathscr{T}) \subseteq \mathscr{S}^p(\mathscr{T})$, the upper bound follows from the same arguments as in the case $\mathscr{P}^p(\mathscr{T}) \subseteq \mathscr{X}_{\mathscr{T}}$, using (21) instead of (20). □

The sets to which the norm on the left-hand side of (18) is localized are overlapping. This overlap can be omitted if further assumptions on u are imposed. The proof of the next Lemma first appeared in [146, Lemma 3.2] if norms are defined by a method called complex interpolation, and in [2, Thm. 4.1] if norms are defined by a method called real interpolation.

Lemma 20 *Suppose $u \in \widetilde{H}^s(\Gamma)$, $s \in [0,1]$ and $u|_T \in \widetilde{H}^s(T)$ for all elements $T \in \mathscr{T}$. Then,*

$$\|u\|^2_{\widetilde{H}^s(\Gamma)} \le C_{\mathrm{loc}} \sum_{T\in\mathscr{T}} \|u|_T\|^2_{\widetilde{H}^s(T)},$$

where C_{loc} is a constant independent of all the other involved quantities.

Proof The cases $s = 0,1$ are obvious, so choose $s \in (0,1)$. We will use the same abbreviation as in Theorem 17. As $\|\widetilde{u}\|^2_{H^s(\partial\Omega)} = \|\widetilde{u}\|^2_{L_2(\partial\Omega)} + |\widetilde{u}|^2_{H^s(\partial\Omega)}$, it suffices to consider the semi-norm on the right-hand side. As $\widetilde{u} = 0$ outside Γ, it follows $\widetilde{u}(x) - \widetilde{u}(y) = 0$ outside $\Gamma \times \partial\Omega \cup \partial\Omega \times \Gamma$, hence

$$|\widetilde{u}|^2_{H^s(\partial\Omega)} \le 2\int_\Gamma\int_{\partial\Omega} \frac{|\widetilde{u}(x)-\widetilde{u}(y)|^2}{|x-y|^{d-1+2s}}\,dx\,dy =: 2\int_\Gamma\int_{\partial\Omega}.$$

We split the double integral in

$$\int_\Gamma\int_{\partial\Omega} = \sum_{T\in\mathscr{T}}\int_T\int_T + \sum_{T\in\mathscr{T}}\int_T\int_{\Gamma\setminus T} + \sum_{T\in\mathscr{T}}\int_T\int_{\partial\Omega\setminus\Gamma}.$$

Now, for the first term,

$$\begin{aligned}\int_T\int_T &= \int_T\int_T \frac{|\widetilde{u}(x)-\widetilde{u}(y)|^2}{|x-y|^{d-1+2s}}\,dx\,dy\\ &= \int_T\int_T \frac{|u|_T(x)-u|_T(y)|^2}{|x-y|^{d-1+2s}}\,dx\,dy \le \|u|_T\|^2_{\widetilde{H}^s(T)}.\end{aligned}$$

For the second term, estimate as in the proof of Theorem 17

$$\sum_{T\in\mathscr{T}}\int_T\int_{\Gamma\setminus T} \lesssim \sum_{T\in\mathscr{T}}\int_T |\widetilde{u}(y)|^2\int_{\Gamma\setminus T}|x-y|^{-d+1-2s}\,dx\,dy$$

where we note that the integral on the right-hand side exists due to $\widetilde{u}|_T \in \widetilde{H}^s(T)$. We conclude

$$\begin{aligned}&\sum_{T\in\mathscr{T}}\int_T |\widetilde{u}(y)|^2\int_{\Gamma\setminus T}|x-y|^{-d+1-2s}\,dx\,dy \lesssim\\ &\quad\sum_{T\in\mathscr{T}}\int_{\partial\Omega}\int_{\partial\Omega} \frac{\left|\widetilde{u|_T}(x)-\widetilde{u|_T}(y)\right|^2}{|x-y|^{d-1+2s}}\,dx\,dy = \sum_{T\in\mathscr{T}}\|u|_T\|^2_{\widetilde{H}^s(T)}.\end{aligned}$$

For the last term, note that

$$\begin{aligned}\int_T\int_{\partial\Omega\setminus\Gamma} &= \int_T\int_{\partial\Omega\setminus\Gamma} \frac{|\widetilde{u}(x)-\widetilde{u}(y)|^2}{|x-y|^{d-1+2s}}\,dx\,dy\\ &= \int_{\partial\Omega}\int_{\partial\Omega} \frac{\left|\widetilde{u|_T}(x)-\widetilde{u|_T}(y)\right|^2}{|x-y|^{d-1+2s}}\,dx\,dy \lesssim \|u|_T\|^2_{\widetilde{H}^s(T)}.\end{aligned}$$

□

3.2 Localization by approximation

This localization technique is employed to derive a posteriori error estimators based on the so-called $(h-h/2)$ methodology, cf. Section 4.2.2. The idea of this approach is to compare the solution on the current mesh with a solution on a finer mesh. The energy norm of the difference of the two solutions is an efficient and (under a certain assumption) reliable error estimator. To localize the energy norm of the difference of two solutions, one uses approximation operators to bound the (non-local) energy norm from above by a stronger integer order (hence local) norm. This can be done by employing approximation estimates for the approximation operators used. The energy spaces of weakly singular and hypersingular equations require a different amount of smoothness, and therefore discontinuous as well as continuous approximation operators will be considered in this Section. To show efficiency of the $(h-h/2)$ type estimators, *inverse estimates* will be needed. These are the counterparts to approximation estimates and are important tools in finite and boundary elements.

Definition 21 *For a given mesh $\mathscr{T}$, denote by $\pi^p_{\mathscr{T}} : L_2(\Gamma) \to \mathscr{P}^p(\mathscr{T})$ and $\Pi^p_{\mathscr{T}} : L_2(\Gamma) \to \mathscr{S}^p(\mathscr{T})$ the $L_2(\Gamma)$-orthogonal projections, which are uniquely characterized by*

$$\begin{aligned}\langle \pi^p_{\mathscr{T}}\phi\,,\Phi\rangle_\Gamma &= \langle\phi\,,\Phi\rangle_\Gamma \quad \text{for all } \Phi\in\mathscr{P}^p(\mathscr{T}),\\ \langle \Pi^p_{\mathscr{T}}u\,,U\rangle_\Gamma &= \langle u\,,U\rangle_\Gamma \quad \text{for all } U\in\mathscr{S}^p(\mathscr{T}).\end{aligned}$$

We also write $\pi_{\mathscr{T}} := \pi^0_{\mathscr{T}}$ and $\Pi_{\mathscr{T}} := \Pi^1_{\mathscr{T}}$. The approximation properties of $\pi^p_{\mathscr{T}}$ can be stated as follows.

Lemma 22 *For $\phi \in H^s(\Gamma)$, $s\in[0,1]$, and $r\in(0,1]$ holds*

$$\begin{aligned}&\|\phi-\pi^p_{\mathscr{T}}\phi\|_{L_2(T)} \le C_{\rm apx}h_T^s|\phi|_{H^s(T)},\\ &\|\phi-\pi^p_{\mathscr{T}}\phi\|^2_{H^{-r}(\Gamma)} \le C_{\rm apx}\sum_{T\in\mathscr{T}}h_T^{2(s+r)}|\phi|^2_{H^s(T)}.\end{aligned}$$

The second estimate is also true if $\widetilde{H}^{-r}(\Gamma)$ is used instead of $H^{-r}(\Gamma)$. The constant $C_{\rm apx} > 0$ depends only on s, and in the cases $s\in(0,1)$ or $r\in(0,1)$ it additionally depends on the shape-regularity $\sigma_{\mathscr{T}}$.

Proof For $p = 1$, the first estimate is proven by a scaling argument, cf. [135, Thm. 10.2]. This special case extends immediately to general $p\in\mathbb{N}$ by the best approximation property of $\pi^p_{\mathscr{T}}$. If $s\in\{0,1\}$, the constant $C_{\rm apx}$ does only depend on s but not on shape-regularity, which is seen by a careful inspection of the scaling argument. The second estimate is a slight refinement of [135, Thm. 10.3]: For $v\in\widetilde{H}^r(\Gamma)$ holds

$$\begin{aligned}\langle\phi-\pi^p_{\mathscr{T}}\phi\,,v\rangle_\Gamma &= \langle h^r(\phi-\pi^p_{\mathscr{T}}\phi)\,,h^{-r}(v-\pi^p_{\mathscr{T}}v)\rangle_\Gamma\\ &\le \|h^r(\phi-\pi^p_{\mathscr{T}}\phi)\|_{L_2(\Gamma)}\|h^{-r}(v-\pi^p_{\mathscr{T}}v)\|_{L_2(\Gamma)}\\ &\lesssim \left(\sum_{T\in\mathscr{T}}h_T^{2(s+r)}|\phi|^2_{H^s(T)}\right)^{1/2}\|v\|_{\widetilde{H}^r(\Gamma)},\end{aligned}$$

The same estimate holds true if we choose $v\in H^r(\Gamma)$. The result follows by the dual definition of the $H^{-r}(\Gamma)$ and $\widetilde{H}^{-r}(\Gamma)$ norm. □

Inverse estimates in the context of the last lemma are proven in [78, Thm. 3.6]:

Lemma 23 *For $\mathscr{T}$ a mesh on Γ and $s\in[0,1]$ holds*

$$\|h^s_{\mathscr{T}}\Phi\|_{L_2(\Gamma)} \le C_{\rm inv}\|\Phi\|_{\widetilde{H}^{-s}(\Gamma)} \quad \text{for all } \Phi\in\mathscr{P}^p(\mathscr{T}).$$

The constant $C_{\rm inv} > 0$ depends only on the shape-regularity of $\mathscr{T}$, $p\in\mathbb{N}$, and s.

The approximation in fractional order spaces by continuous functions is a little bit more involved. For the proof of the following two lemmata, we refer to [100] and [12, Prop. 5 and Lem. 7].

Lemma 24 *For $s\in[0,1]$, each $\widetilde{H}^s(\Gamma)$-stable projection $J_{\mathscr{T}}:\widetilde{H}^s(\Gamma)\to\widetilde{\mathscr{S}}^p(\mathscr{T})$ satisfies*

$$\|v-J_{\mathscr{T}}v\|_{\widetilde{H}^s(\Gamma)}\le C_{\rm apx}\min_{V\in\mathscr{S}^p(\mathscr{T})}\|h_{\mathscr{T}}^{1-s}\nabla_\Gamma(v-V)\|_{L_2(\Gamma)}$$

for all $v\in\widetilde{H}^1(\Gamma)$. The constant $C_{\rm apx}$ depends only on Γ, $p\in\mathbb{N}$, s, shape-regularity of $\mathscr{T}$, and the stability constant of $J_{\mathscr{T}}$.

Lemma 25 *For $\mathscr{T}$ a mesh on Γ and $s\in[0,1]$ holds*

$$\|h_{\mathscr{T}}^{1-s}\nabla_\Gamma V\|_{L_2(\Gamma)}\le C_{\rm inv}\|V\|_{H^s(\Gamma)}\quad \textit{for all } V\in\mathscr{S}^p(\mathscr{T}).$$

The constant $C_{\rm inv}>0$ depends only on the shape-regularity of $\mathscr{T}$, $p\in\mathbb{N}$, and s.

Lemma 24 holds for any projection $J_{\mathscr{T}}:\widetilde{H}^s(\Gamma)\to\mathscr{S}^p(\mathscr{T})$ which is stable, i.e., for all $v\in\widetilde{H}^s(\Gamma)$ holds

$$\|J_{\mathscr{T}}v\|_{\widetilde{H}^s(\Gamma)}\le C_{\rm stab}\|v\|_{\widetilde{H}^s(\Gamma)},$$

and the constant $C_{\rm stab}$ does not depend on v. For an implementation of an associated $(h-h/2)$ error estimator, the operator $J_{\mathscr{T}}$ needs to be computed. Possible candidates are presented in the following.

3.2.1 The $L_2(\Gamma)$ projection onto $\mathscr{S}^p(\mathscr{T})$

The $L_2(\Gamma)$ orthogonal projection $\Pi_{\mathscr{T}}^p$ onto $\mathscr{S}^p(\mathscr{T})$ from Definition 21 is an easy-to-implement candidate for $J_{\mathscr{T}}$ in Lemma 24. The parameter s will subsequently be chosen to be greater than 0, such that the $\widetilde{H}^s(\Gamma)$-stability of $\Pi_{\mathscr{T}}^p$

$$\|\Pi_{\mathscr{T}}^p u\|_{\widetilde{H}^s(\Gamma)}\le C_{\rm stab}\|u\|_{\widetilde{H}^s(\Gamma)}, \tag{23}$$

needs to be available to use Lemma 24. While there holds (23) for $s=0$ and $C_{\rm stab}=1$ without any assumption on $\mathscr{T}$, this might not be the case for $s>0$. It can be shown that (23) holds for $s>0$ on a sequence of meshes where the quotient of the biggest and smallest element stays bounded, and $C_{\rm stab}$ depends on this bound, cf. [29]. However, as we will deal with adaptively refined meshes, this quotient will not stay bounded on the (infinite) sequence of meshes that we investigate. However, the fact that a sequence of adaptively refined meshes exhibits a strong structure can be used in order to show useful results. Existing works to this topic include [17,28,33,34,51,60,101,133]. We refer to Section 7.4.2 for a detailed discussion.

3.2.2 The Scott-Zhang projection

The Scott-Zhang projection, developed in [129], is widely used in numerical analysis. It is a linear and bounded projection onto $\mathscr{S}^p(\mathscr{T})$ which is defined on $H^{1/2+\varepsilon}(\Gamma)$ for $\varepsilon>0$. The energy spaces in BEM, usually variants of $H^{1/2}(\Gamma)$, lack the regularity necessary for the classical definition. However, if the energy space is $\widetilde{H}^{1/2}(\Gamma)$, an operator that maps into a space with zero boundary conditions is needed. A slightly modified derivation is therefore necessary and will be presented here. For ease of presentation, we present the details only for $\mathscr{S}^1(\mathscr{T})$. Suppose that $\{z_i\}_{i=1}^N$ is the collection of degrees of freedom for $\mathscr{S}^1(\mathscr{T})$, which are ordered in a way such that $\{z_i\}_{i=\widetilde{N}+1}^N$ are on the boundary $\partial\Gamma$ (if Γ is open, of course). For every z_i, we choose an element T_i with $z_i\in\overline{T}_i$. Denote by $\{\phi_{i,j}\}_{j=1}^d$ the nodal basis of $\mathscr{P}^1(T_i)$, and by $\{\psi_{i,k}\}_{k=1}^d$ an $L_2(T_i)$-dual basis defined by

$$\int_{T_i}\psi_{i,k}\phi_{i,j}\,dx=\delta_{k,j}. \tag{24}$$

Set ψ_i to be the dual basis function of $\phi_{i,j}$ with $\phi_{i,j}(z_i)=1$, and denote by $\{\eta_i\}_{i=1}^N$ the nodal basis of $\mathscr{S}^1(\mathscr{T})$ with $\eta_j(z_k)=\delta_{j,k}$. The Scott-Zhang operators are defined for $v\in L_1^{\rm loc}(\Gamma)$ via

$$J_{\mathscr{T}}v=\sum_{i=1}^N\eta_i\int_{T_i}\psi_i v\,dx\quad\text{and}\quad \widetilde{J}_{\mathscr{T}}v=\sum_{i=1}^{\widetilde{N}}\eta_i\int_{T_i}\psi_i v\,dx.$$

The following results can be shown using arguments from [129], cf. [12].

Lemma 26 *The operator $J_{\mathscr{T}}:L_1^{loc}(\Gamma)\to\mathscr{S}^p(\mathscr{T})$ is stable for all $s\in[0,1]$, i.e.,*

$$\|J_{\mathscr{T}}v\|_{H^s(\Gamma)}\le C_{\rm stab}\|v\|_{H^s(\Gamma)}\quad\textit{for all } v\in H^s(\Gamma).$$

The operator $\widetilde{J}_{\mathscr{T}}:L_1^{loc}(\Gamma)\to\widetilde{\mathscr{S}}^p(\mathscr{T})$ is stable for all $s\in[0,1]$, i.e.,

$$\|J_{\mathscr{T}}v\|_{\widetilde{H}^s(\Gamma)}\le C_{\rm stab}\|v\|_{\widetilde{H}^s(\Gamma)}\quad\textit{for all } v\in\widetilde{H}^s(\Gamma).$$

3.2.3 Nodal interpolation

The nodal interpolator $J_{\mathscr{T}}:C(\overline{\Gamma})\to\mathscr{S}^p(\mathscr{T})$ is without doubt the easiest approximation operator when it comes to implementation, as

$$J_{\mathscr{T}}v=\sum_{i=1}^N v(z_i)\eta_i,$$

where $\{z_i\}_{i=1}^N$ and $\{\eta_i\}_{i=1}^N$ are again the degrees of freedom and it's associated nodal basis, i.e., $\eta_j(z_k)=\delta_{j,k}$. However, $J_{\mathscr{T}}$ needs point evaluation, which is only a stable operation on curves (i.e., $d=2$, cf. Section 2.2) due to the Sobolev embedding theorem. The following result is essentially proved in [32, Thm. 1] and [42, Cor. 3.4] for $p=1$, but transfers verbatim to $p\ge1$.

Lemma 27 *For $d=2$, i.e., $\Gamma \subseteq \Omega$ a one-dimensional curve, it holds for all $v \in H^1(\Gamma)$ that*

$$\|v - J_{\mathscr{T}} v\|_{H^s(\Gamma)} \le C_{\rm apx} \|h_{\mathscr{T}}^{1-s} v'\|_{L_2(\Gamma)},$$

where $C_{\rm apx} > 0$ depends only Γ, $\sigma_{\mathscr{T}}$, and p.

Contrary, for surfaces, i.e., $d=3$, point evaluation is not a stable operation and a result analogous to the last one cannot hold. A remedy that can be used at least in $(h-h/2)$ error estimation is that nodal interpolation can be shown to be a stable operation when the function to be approximated is discrete on a finer scale, and the scales do not differ too much. The following result captures this idea in a mathematical sense, for a proof see [12].

Lemma 28 *Consider a mesh $\mathscr{T}$ together with its uniform refinement $\widehat{\mathscr{T}}$, cf. Section 7. For $q \ge p$, the nodal interpolation operator $J_{\mathscr{T}} : \widetilde{\mathscr{S}}^q(\widehat{\mathscr{T}}) \to \widetilde{\mathscr{S}}^p(\mathscr{T})$ satisfies for $s \in [0,1]$*

$$\|(1-J_{\mathscr{T}})\widehat{V}\|_{\widetilde{H}^s(\Gamma)} \le C_{\rm apx} \min_{V \in \mathscr{S}^p(\mathscr{T})} \|h_{\mathscr{T}}^{1-s} \nabla_\Gamma (\widehat{V} - V)\|_{L_2(\Gamma)}$$

as well as

$$\|h_{\mathscr{T}}^{1-s} \nabla_\Gamma (1-J_{\mathscr{T}})\widehat{V}\|_{L_2(\Gamma)} \le C_{\rm stab} \|h_{\mathscr{T}}^{1-s} (1-\Pi_{\mathscr{T}}^p) \nabla_\Gamma \widehat{V}\|_{L_2(\Gamma)}$$

for all $\widehat{V} \in \widetilde{\mathscr{S}}^q(\widehat{\mathscr{T}})$. The constant $C_{\rm apx} > 0$ depends only on Γ, p, q, s, and the shape-regularity $\sigma_{\mathscr{T}}$, whereas the constant $C_{\rm stab}$ depends only on the shape-regularity $\sigma_{\mathscr{T}}$.

The implementation of the nodal interpolation operator is straight forward and will not be discussed further.

3.3 Localization by Multilevel norms

The localization techniques discussed in Sections 3.1–3.3 depend on either orthogonality and/or approximation. If neither of those properties is available, one can still use so-called *multilevel norms* to localize fractional-order Sobolev norms of discrete functions. The following theorem and its proof are found in [118]. We will not give the proof here, as it involves deeper mathematical results such as Besov spaces.

Theorem 29 *Let $(\mathscr{T}_\ell)_{\ell \in \mathbb{N}_0}$ be a uniform sequence of meshes on Γ with corresponding mesh-width h_ℓ. Denote by $\pi_\ell : L_2(\Gamma) \to \mathscr{P}^0(\mathscr{T}_\ell)$ the L_2 orthogonal projections and $\pi_{-1} = 0$. Then, there are constants $C_1, C_2 > 0$ such that for all $L \ge 0$ and all $\Phi_L \in \mathscr{P}^0(\mathscr{T}_L)$ it holds that*

$$\begin{aligned} C_1 \|\Phi_L\|^2_{\widetilde{H}^{-1/2}(\Gamma)} &\le \sum_{\ell=0}^{L} h_\ell \|(\pi_\ell - \pi_{\ell-1})\Phi_L\|^2_{L_2(\Gamma)} \\ &\le C_2 (L+1)^2 \|\Phi_L\|^2_{\widetilde{H}^{-1/2}(\Gamma)}. \end{aligned}$$

The last theorem gives a reliable, computable bound for the $\widetilde{H}^{-1/2}$ norm of a discrete function. The upper bound in contrast depends on the number of levels L that are involved. This upper bound cannot be improved in general, cf. [118].

4 A posteriori error estimators for the h-version

The continuous solution u of any of our model problems is in general unknown, and so is the error $u - U$, where U is a discrete approximation to u from a discrete space $\mathscr{X}_{\mathscr{T}}$ based on a mesh $\mathscr{T}$. The idea of a posteriori error estimation is to estimate the error $u - U$ in order to

- use an estimate for the *global* error as a stopping criterion, or
- to use local contributions of the error to decide where to refine the mesh locally.

Adaptive algorithms clearly need estimators that provide local contributions, which will be written as

$$\eta_{\mathscr{T}} = \Big(\sum_{T \in \mathscr{T}} \eta_T^2\Big)^{1/2} \quad \text{or } \eta_{\mathscr{T}} = \Big(\sum_{j \in J} \eta_j^2\Big)^{1/2},$$

where J is a certain index set (e.g., the set of nodes). If the error estimator does not underestimate the error, i.e.,

$$\|u - U\| \le C_{\rm rel} \eta_{\mathscr{T}}$$

holds true, then $\eta_{\mathscr{T}}$ is said to be *reliable*. Likewise, if the error estimator does not overestimate the error, i.e.,

$$\eta_{\mathscr{T}} \le C_{\rm eff} \|u - U\|$$

holds true, it is called *efficient*. Here, $\|\cdot\|$ is a norm of interest, typically the energy norm of the problem. Usually, $C_{\rm rel}, C_{\rm eff} > 0$ are unknown (except for $(h-h/2)$ estimators, where $C_{\rm eff} = 1$), but do not depend on the current mesh $\mathscr{T}$.

In this Section, we present different approaches for a posteriori error estimation in boundary element methods that are available in the mathematical literature. The focus is to show reliability and efficiency and give an overview on the available approaches. Frequently, we will identify a bilinear form $b : \mathscr{X} \times \mathscr{X} \to \mathbb{R}$ with an operator $B : \mathscr{X} \to \mathscr{X}'$ via

$$b(v,w) = \langle Bv\,, w \rangle_{\mathscr{X}' \times \mathscr{X}},$$

where $\langle \cdot\,, \cdot \rangle_{\mathscr{X}' \times \mathscr{X}}$ is the chosen duality pairing.

4.1 Residual type estimators

While the exact solution u of an equation is unknown, the residual $R := F - BU$ is a computable quantity. Here, B is the involved operator (i.e., the simple layer V or the hypersingular operator W), and F is the corresponding right-hand side. In boundary element methods, the residual is usually measured in a non-local fractional Sobolev norm, and the different approaches for residual error estimation differ in their approach for localization of this norm.

4.1.1 Babuška-Rheinboldt-Estimators

In [62], Faermann extended the estimators that were developed for finite element methods by Babuška and Rheinboldt [14] to fractional order Sobolev norms. We will sketch the ideas for the case of the hypersingular integral equation and lowest-order discretization, and comment on the other cases afterwards. To that end, denote by $u \in H^{1/2}(\Gamma)$ the exact solution to the Neumann problem, see Proposition 7, and by $U \in \mathscr{S}^1(\mathscr{T})$ the Galerkin solution, see Proposition 13

Definition 30 *Denote by $\{\Psi_j\}_{j=1}^N$ the nodal basis of $\mathscr{S}^1(\mathscr{T})$ and by $R := (1/2 - K')\phi - WU$ the residual. The BR-type error estimator is defined by*

$$\eta_{\mathscr{T}}^2 = \sum_{j=1}^N \eta_j^2, \quad \textit{where} \quad \eta_j := \sup_{\substack{v \in H^{1/2}(\Gamma) \\ \Psi_j v \neq 0}} \frac{\langle R\,, \Psi_j v\rangle_\Gamma}{\|\Psi_j v\|_{H^{1/2}(\Gamma)}}$$

For the hypersingular integral operator, the BR-type estimators are reliable and efficient.

Theorem 31 *There are constants $C_{\mathrm{rel}}, C_{\mathrm{eff}} > 0$ such that*

$$C_{\mathrm{eff}}^{-2} \eta_{\mathscr{T}}^2 \le \|u - U\|_{\widetilde{H}^{1/2}(\Gamma)}^2 \le C_{\mathrm{rel}}^2 \eta_{\mathscr{T}}^2. \tag{25}$$

The constant C_{rel} depends on the shape-regularity of $\mathscr{T}$, whereas the constant C_{eff} depends also on Γ.

Proof We sketch the ideas of [36, Sect. 6]. The exact solution satisfies $u \in H_0^{1/2}(\Gamma)$, hence $Wu = (1/2 - K')\phi$ in $H^{1/2}(\Gamma)$. As W^{-1} is linear and bounded, it follows that $\|u - U\|_{\widetilde{H}^{1/2}(\Gamma)} \lesssim \|R\|_{H^{-1/2}(\Gamma)}$, and

$$\begin{aligned}\|R\|_{H^{-1/2}(\Gamma)} &= \sup_{\|v\|_{H^{1/2}(\Gamma)}=1} \langle R\,, v\rangle_\Gamma \\ &\le \eta_{\mathscr{T}} \sup_{\|v\|_{H^{1/2}(\Gamma)}=1} \left(\sum_{j=1}^N \|(v - z_j)\Psi_j\|_{H^{1/2}(\Gamma)}^2 \right)^{1/2},\end{aligned}$$

with arbitrary $z_j \in \mathbb{R}$ for $j = 1, \dots, N$. Now, with $\omega_j = \mathrm{supp}(\Psi_j)$, it holds

$$\begin{aligned}|(v - z_j)\Psi_j|_{H^{1/2}(\Gamma)} &\lesssim |v|_{H^{1/2}(\omega_j)} \\ &\quad + \mathrm{diam}(\omega_j)^{-1/2} \|v - z_j\|_{L_2(\omega_j)}.\end{aligned}$$

Choosing z_j according to the variant of the Poincaré inequality of [57, Thm. 7.1] shows the upper bound in (25). To show the lower bound, note that the index set $J = \{1, \dots, N\}$ can be decomposed into at most M pairwise disjoint subsets J_k, $k = 1, \dots, M$, and M depends only on the shape-regularity of $\mathscr{T}$, such that the supports of the basis functions $\{\Psi_j\}_{j \in J_k}$ are pairwise disjoint. Due to the latter property, it is possible to choose for an arbitrary collection of functions $v_j \in H^{1/2}(\Gamma)$, $j \in J_k$, a function $w_j \in H^{1/2}(\Gamma)$ such that on the support of Ψ_j it holds

$$w_j = \frac{\langle R\,, \Psi_j v_j\rangle_\Gamma}{\|\Psi_j v_j\|_{H^{1/2}(\Gamma)}^2} v_j.$$

It follows that

$$\sum_{j \in J_k} \|\Psi_j w_j\|_{H^{1/2}(\Gamma)}^2 = \sum_{j \in J_k} \frac{\langle R\,, \Psi_j v_j\rangle_\Gamma^2}{\|\Psi_j v_j\|_{H^{1/2}(\Gamma)}^2} = \langle R\,, \sum_{j \in J_k} \Psi_j w_j\rangle_\Gamma,$$

and hence

$$\begin{aligned}\sum_{j \in J_k} \frac{\langle R\,, \Psi_j v_j\rangle_\Gamma^2}{\|\Psi_j v_j\|_{H^{1/2}(\Gamma)}^2} &= \frac{\left(\langle R\,, \sum_{j \in J_k} \Psi_j w_j\rangle_\Gamma\right)^2}{\sum_{j \in J_k} \|\Psi_j w_j\|_{H^{1/2}(\Gamma)}^2} \\ &\le C_{\mathrm{loc}} \|R\|_{H^{-1/2}(\Gamma)}^2,\end{aligned}$$

where C_{loc} is the constant from Lemma 20. Since the v_j can be chosen arbitrarily, we conclude that

$$\sum_{j \in J_K} \eta_j^2 \le C_{\mathrm{loc}} \|R_N\|_{H^{-1/2}(\Gamma)}^2 \le C_{\mathrm{loc}} \|W\|^2 \|u - U_N\|_{\widetilde{H}^{1/2}(\Gamma)}^2,$$

where $\|W\| = \|W\|_{\widetilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)}$. Hence, the lower bound in (25) follows with $C_{\mathrm{eff}}^2 = M C_{\mathrm{loc}} \|W\|^2$. $\square$

In [62], a result like Theorem 31 is proven for bijective, continuous operators $B : H^\alpha(\Gamma) \to H^{-\alpha}(\Gamma)$, $\alpha \in \mathbb{R}$, that satisfy the Gårding inequality

$$|b(v,v)| \ge C_{\mathrm{ell}} \|v\|_{H^\alpha(\Gamma)}^2 - C_{\mathrm{g}} \|v\|_{H^{\alpha-\delta}(\Gamma)} \tag{26}$$

for all $v \in H^\alpha(\Gamma)$ and some $\delta > 0$, where $C_{\mathrm{ell}} > 0$ and $C_{\mathrm{g}} \ge 0$. As in the case of the hypersingular operator, estimators of BR-type are associated to a basis $\{\Psi_j\}_{j=1}^N$ of the discrete trial space $\mathscr{X}_{\mathscr{T}}$, which is supposed to fulfill the following assumptions.

Assumption 32 *There are constants $M \in \mathbb{N}$ and $C_\alpha > 0$ such that*

(i) The basis $\{\Psi_j\}_{j=1}^N$ can be partitioned into M sets of basis functions with mutually disjoint support, i.e., there are at most M disjoint subsets I_k, $k = 1, \dots, M$ with $I_k \subseteq \{1, \dots, N\}$ such that

$$\mathrm{supp}\,(\Psi_m) \cap \mathrm{supp}\,(\Psi_n) = \emptyset \quad \textit{for } m, n \in I_k, m \neq n.$$

(ii) The basis $\{\Psi_j\}_{j=1}^N$ is a partition of unity, i.e.,

$$\sum_{j=1}^N \Psi_j(x) = 1 \quad \textit{for almost all } x \in \Gamma.$$

(iii) For every function $v \in H^{\max\{\alpha,0\}}(\Gamma)$ there is a function $V \in \mathscr{X}_{\mathscr{T}}$ such that

$$\sum_{j=1}^N \|\Psi_j(v - V)\|_{H^\alpha(\Gamma)}^2 \le C_\alpha \|v\|_{H^\alpha(\Gamma)}^2.$$

(iv) *For all $v \in H^{\alpha+\delta}(\Gamma)$ exists a function $V_N \in \mathscr{X}_{\mathscr{T}}$ such that*

$$\|v - V_N\|_{H^\alpha(\Gamma)} \le C_{\mathrm{apx}} \max_{T\in\mathscr{T}} h^\delta \|v\|_{H^{\alpha+\delta}(\Gamma)},$$

and C_{apx} depends only on the shape-regularity $\sigma_{\mathscr{T}}$ and Γ.

For $d = 2$, the standard bases of $\mathscr{P}^p(\mathscr{T})$ or $\mathscr{S}^p(\mathscr{T})$ always fulfill *(i)*. By standard bases, we mean functions having support only on one element for $\mathscr{P}^p(\mathscr{T})$, and the classical finite-element *hat-functions* for $\mathscr{S}^p(\mathscr{T})$. For $d = 3$, the standard basis of $\mathscr{P}^p(\mathscr{T})$ always fulfills *(i)*, while for the standard basis of $\mathscr{S}^p(\mathscr{T})$ the constant M depends on shape-regularity. Independent of d, the assumption *(ii)* can always be fulfilled as long as there are no boundary conditions imposed, i.e., in the case of Γ being not a closed boundary, the space $\widetilde{\mathscr{S}}^p(\mathscr{T})$ cannot fulfill *(ii)*. Assumption *(iii)* holds for $d = 2$ for $\mathscr{P}^0(\mathscr{T})$ and $\alpha = -1/2$. In the case of $\mathscr{S}^1(\mathscr{T})$ and $\alpha = 1/2$, the constant C_α depends on the shape-regularity of $\mathscr{T}$. In $d = 3$, it holds for $\mathscr{P}^0(\mathscr{T})$ and $\alpha = -1/2$ or $\mathscr{S}^1(\mathscr{T})$ and $\alpha = 1/2$, and the constant C_α depends on the shape-regularity of $\mathscr{T}$ in both cases.

Remark 4 Note that the basis Ψ_j is only needed for the computation of the BR-indicators.

In the general case, the following result together with a proof can be found in [62, Thm. 5.2].

Theorem 33 *Denote by $B : H^\alpha(\Gamma) \to H^{-\alpha}(\Gamma)$, $\alpha \in \mathbb{R}$, a linear and bounded operator which satisfies the Gårding inequality* (26)*. If $C_{\mathrm{g}} > 0$, assume that $B : H^{\alpha-\delta}(\Gamma) \to H^{-\alpha-\delta}(\Gamma)$ is bijective and continuous, and denote by $\mathscr{X}_{\mathscr{T}} \subset H^\alpha(\Gamma)$ a discrete space which fulfills Assumption 32. If $u \in H^{\max\{\alpha,0\}}(\Gamma)$ and $U \in \mathscr{X}_{\mathscr{T}}$ are the exact and the Galerkin solution of*

$$\begin{aligned} \langle Bu\,, v\rangle_\Gamma &= \langle F\,, v\rangle_\Gamma \quad \text{for all } v \in \widetilde{H}^{-\alpha}(\Gamma),\\ \langle BU\,, V\rangle_\Gamma &= \langle F\,, V\rangle_\Gamma \quad \text{for all } V \in \mathscr{X}_{\mathscr{T}}, \end{aligned}$$

denote the residual by $R := F - BU$ and define a BR-type estimator by

$$\eta_{\mathscr{T}}^2 = \sum_{j=1}^N \eta_j^2, \quad \text{where} \quad \eta_j := \sup_{\substack{v\in H^\alpha(\Gamma)\\ \Psi_j v\neq 0}} \frac{\langle R\,, \Psi_j v\rangle_\Gamma}{\|\Psi_j v\|_{H^\alpha(\Gamma)}}.$$

Then, the following holds:

– *If B is elliptic, i.e., $C_{\mathrm{g}} = 0$, $\eta_{\mathscr{T}}$ is reliable, i.e.,*

$$\|u - U\|_{H^\alpha(\Gamma)} \le C_{\mathrm{rel}} \eta_{\mathscr{T}}. \tag{27}$$

– *If B is not elliptic but satisfies a Gårding inequality, then* (27) *holds if $h_{\mathscr{T}}$ is sufficiently small.*

– *For $\alpha \ge 0$ holds efficiency*

$$\eta_{\mathscr{T}} \le C_{\mathrm{eff}} \|u - U\|_{H^\alpha(\Gamma)}, \tag{28}$$

where C_{eff} depends on B, Γ, and M.

– *For $\alpha \in \mathbb{R}$ holds efficiency* (28)*, where C_{eff} depends on* $\dim(X)$.

– *For $d = 2$ and $\alpha \in (-1/2, 0)$ holds* (28) *if the mesh is sufficiently small, where C_{eff} depends on A, Γ, α, and the shape-regularity of $\mathscr{T}$.*

As the BR-type estimators are defined as a supremum, they are not computable. By definition, any $v \in H^\alpha(\Gamma)$ with $\Psi_j v \neq 0$ fulfills

$$\frac{\langle R\,, \Psi_j v\rangle_\Gamma}{\|\Psi_j v\|_{H^\alpha(\Gamma)}} \le \eta_j,$$

providing a lower, computable bound by choosing, e.g., $v = \Psi_j$. Computable upper bounds are more involved. In [62], it is shown that for $d = 2$, it holds

$$\eta_j \lesssim \begin{cases} \mathrm{diam}(\omega_j)^\alpha \|R_N\|_{L_2(\omega_j)} & \text{for } \alpha \ge 0,\\ |R_N|^2_{H^{-\alpha}(\omega_j)} + \\ \quad \sum_{j=0}^{\lceil -\alpha\rceil} \mathrm{diam}(\omega_j)^{2(j+\alpha)} |R_N|^2_{H^j(\omega_j)} & \text{for } \alpha < 0. \end{cases}$$

For the case of the hypersingular integral operator ($\alpha = 1/2$), this upper bound corresponds to the weighted residual error estimator which will be considered in Section 4.1.3.

4.1.2 RYW-Estimators

These types of estimators were the first ones available for boundary element methods. Developed and analyzed by Rank [120] and Wendland-Yu [149], they were labeled *RYW*-Estimators in [62]. These estimators are connected to the Babuška-Rheinboldt estimators from Section 4.1.1. Again we sketch the ideas for the case of the hypersingular integral equation and lowest-order discretization first. To that end, denote by $u \in H^{1/2}(\Gamma)$ the exact solution to the Neumann problem of Proposition 7, and by $U \in \mathscr{S}^1(\mathscr{T})$ the Galerkin solution of the discrete version of Proposition 13.

Definition 34 *Denote by $\{\Psi_j\}_{j=1}^N$ the nodal basis of $\mathscr{S}^1(\mathscr{T})$. For every $j = 1, \ldots, N$, consider the space*

$$H_j := \widetilde{H}^{1/2}(\mathrm{supp}(\Psi_j)^\circ),$$

which is a closed subspace of $H^{1/2}(\Gamma)$. Define $\zeta_j \in H_j$ as the unique solution of

$$\langle W\zeta_j\,, v_j\rangle_\Gamma = \langle W(u - U_N)\,, v_j\rangle_\Gamma \quad \text{for all } v_j \in H_j,$$

and set

$$\eta_{\mathscr{T}}^2 = \sum_{j=1}^N \eta_j^2 \quad \text{where } \eta_j := \|\zeta_j\|_{H^{1/2}(\Gamma)}.$$

Theorem 35 *There are constants $C_{\mathrm{rel}}, C_{\mathrm{eff}} > 0$ such that*

$$C_{\mathrm{eff}}^{-2}\eta_{\mathscr{T}}^2 \le \|u-U\|_{H^{1/2}(\Gamma)}^2 \le C_{\mathrm{rel}}^2\eta_{\mathscr{T}}^2. \tag{29}$$

The constant $C_{\mathrm{rel}} > 0$ depends only on the shape-regularity of $\mathscr{T}$.

Proof Denote by η_{BR} the Babuška-Rheinboldt estimator from Definition 30 with its local contributions $\eta_{\mathrm{BR},j}$ and by $R := (1/2-K')\phi - WU_N$ the residual. Now, if $v \in H^{1/2}(\Gamma)$ with $\Psi_j v \neq 0$, it follows $\Psi_j v \in H_j$ and hence

$$\langle R\,,\Psi_j v\rangle_\Gamma = \langle W\zeta_j\,,\Psi_j v\rangle_\Gamma \lesssim \|\zeta_j\|_{H^{1/2}(\Gamma)}\|\Psi_j v\|_{H^{1/2}(\Gamma)}.$$

Taking the supremum over all those v yields

$$\eta_{\mathrm{BR},j} \le \eta_j,$$

such that reliability, i.e., the upper bound in (29), follows from Theorem 31. To show efficiency, choose for $\delta > 0$ a function $v_j^{(\delta)}$ such that

$$\|\Psi_j v_j^{(\delta)} - \zeta_j\|_{H^{1/2}(\Gamma)} \le \delta.$$

Then,

$$\eta_j^2 \simeq \langle W\zeta_j\,,\zeta_j\rangle_\Gamma = \langle W\zeta_j\,,\Psi_j v_j^{(\delta)}\rangle_\Gamma + \langle W\zeta_j\,,\zeta_j - \Psi_j v_j^{(\delta)}\rangle_\Gamma,$$

and

$$\begin{aligned}\langle W\zeta_j\,,\Psi_j v_j^{(\delta)}\rangle_\Gamma &\le \eta_{\mathrm{BR},j}\|\Psi_j v_j^{(\delta)}\|_{H^{1/2}(\Gamma)},\\ \langle W\zeta_j\,,\zeta_j - \Psi_j v_j^{(\delta)}\rangle_\Gamma &\lesssim \delta\|\zeta_j\|_{H^{1/2}(\Gamma)}.\end{aligned}$$

Due to $\|\Psi_j v_j^{(\delta)}\|_{H^{1/2}(\Gamma)} \le \eta_j + \delta$ it follows that

$$\eta_j^2 \lesssim \eta_{\mathrm{BR},j}(\eta_j+\delta) + \delta\eta_j,$$

such that the limit $\delta \to 0$ finishes the proof of efficiency. □

In [149], estimators of this type are analyzed for bijective, continuous operators $B : H^\alpha(\Gamma) \to H^{-\alpha}(\Gamma)$, $\alpha \in \mathbb{R}$, that satisfy the Gårding inequality (26) with discretizations that satisfy (ii) and (iii) of Assumption 32. The following Theorem summarizes the available results on the RYW estimators.

Theorem 36 ([62,149]) *Denote by $B : H^\alpha(\Gamma) \to H^{-\alpha}(\Gamma)$, $\alpha \in \mathbb{R}$, a linear and bounded operator which satisfies the Gårding inequality* (26)*. If $C_{\mathrm{g}} > 0$, assume that $B : H^{\alpha-\delta}(\Gamma) \to H^{-\alpha-\delta}(\Gamma)$ is bijective and continuous, and denote by $\mathscr{X}_{\mathscr{T}} \subset H^\alpha(\Gamma)$ a discrete space which fulfills (ii) and (iii) of Assumption 32. Suppose that there is a basis $\{\Psi_j\}_{j=1}^N$ of $\mathscr{X}_{\mathscr{T}}$ such that there are at most M disjoint subsets I_k, $k = 1,\dots,M$, $I_k \subseteq \{1,\dots,N\}$, which satisfy the strengthened Cauchy-Schwarz inequality*

$$\langle Av_m\,,v_n\rangle_\Gamma \le (\#I_k)^{-1}\langle Av_m\,,v_m\rangle_\Gamma^{1/2}\langle Av_n\,,v_n\rangle_\Gamma^{1/2}$$

for all $m \neq n \in I_k$ and $v_j \in H_j$, with

$$H_j := \widetilde{H}^\alpha(\mathrm{supp}(\Psi_j)^\circ).$$

If $u \in H^\alpha(\Gamma)$ and $U \in \mathscr{X}_{\mathscr{T}}$ are the exact and the Galerkin solution of

$$\begin{aligned}\langle Au\,,v\rangle_\Gamma &= \langle F\,,v\rangle_\Gamma \quad \text{for all } v \in \widetilde{H}^{-\alpha}(\Gamma),\\ \langle AU\,,V\rangle_\Gamma &= \langle F\,,V\rangle_\Gamma \quad \text{for all } V \in \mathscr{X}_{\mathscr{T}},\end{aligned}$$

denote the residual by $R := F - BU$ and define a RYW-type estimator by

$$\eta_{\mathscr{T}}^2 = \sum_{j=1}^N \eta_j^2, \quad \text{where} \quad \eta_j := \|\zeta_j\|_{H^\alpha(\Gamma)},$$

where $\zeta_j \in H_j$ is defined by

$$\langle A\zeta_j\,,v_j\rangle_\Gamma = \langle R\,,v_j\rangle_\Gamma \quad \text{for all } v_j \in H_j.$$

Then, the following holds:

- *If B is elliptic, i.e., $C_{\mathrm{g}} = 0$, $\eta_{\mathscr{T}}$ is reliable, i.e.,*

$$\|u-U\|_{\widetilde{H}^\alpha(\Gamma)} \le C_{\mathrm{rel}}\eta_{\mathscr{T}}. \tag{30}$$

- *If B is not elliptic but satisfies a Gårding inequality, i.e., $C_{\mathrm{g}} > 0$, then* (30) *holds if $h_{\mathscr{T}}$ is sufficiently small.*
- *For $\alpha \ge 0$ holds efficiency*

$$\eta_{\mathscr{T}} \le C_{\mathrm{eff}}\|u-U\|_{\widetilde{H}^\alpha(\Gamma)}, \tag{31}$$

  where C_{eff} depends on B, Γ, and M.
- *For $\alpha \in \mathbb{R}$ holds efficiency* (28)*, where C_{eff} depends on M.*
- *For $d = 2$ and $\alpha \in (-1/2, 0)$ efficiency* (31) *holds if the mesh is sufficiently small. The efficiency constant $C_{\mathrm{eff}} > 0$ depends on B, Γ, α and on the shape-regularity of $\mathscr{T}$.*

Proof The proof was first shown in [149], with efficiency always dependent on M. Later, Faermann [62] showed equivalence of the RYW and the BR estimators, thereby obtaining efficiency without dependence on M for $\alpha \ge 0$ and $d = 2$ and $\alpha \in (-1/2, 0)$. □

The constant M in the last theorem can always be chosen as $M = N$ by decomposing the set of degrees of freedom into it's single elements. In [150] it is postulated that M can be chosen even much smaller.

4.1.3 Weighted-residual estimators

Estimators of this kind usually employ orthogonality properties to localize the residuals' fractional norm by a weighted norm of integer order. The very first paper in this sense is [44], where the following idea was carried out in a more general Banach space setting: Suppose that $\mathscr{X}_{\mathscr{T}} \subset \mathscr{X}$ is a discrete space and denote by U the Galerkin approximation to $u \in \mathscr{X}$, cf. (4), and by $R := Bu - BU$ the residual. Due to the open mapping theorem, B^{-1} is bounded and it holds

$$\|u-U\|_{\mathscr{X}} \le \|B^{-1}\|_{\mathscr{X}' \to \mathscr{X}} \|R\|_{\mathscr{X}'}.$$

We assume that there are spaces $\mathscr{X}_0, \mathscr{X}_1$ with $\mathscr{X}_0' \supseteq \mathscr{X}' \supseteq \mathscr{X}_1'$, such that $\mathscr{X}_{\mathscr{T}} \subset \mathscr{X}_0$, $R \in \mathscr{X}_1'$, and

$$\begin{aligned} \|u\|_{\mathscr{X}'} &\le C\|u\|^{1-s}_{\mathscr{X}_0'} \|u\|^s_{\mathscr{X}_1'} \quad \text{for all } u \in \mathscr{X}_1', \\ 0 &= \langle V, R\rangle_{\mathscr{X}_0 \times \mathscr{X}_0'} \quad \text{for all } V \in \mathscr{X}_{\mathscr{T}}. \end{aligned} \tag{32}$$

Then, according to the theorem of Hahn-Banach, there is $\rho \in \mathscr{X}_0$ with

$$\|\rho\|^2_{\mathscr{X}_0} = \|R\|^2_{\mathscr{X}_0'} = \langle \rho, R\rangle_{\mathscr{X}_0 \times \mathscr{X}_0'} = \langle \rho - V, R\rangle_{\mathscr{X}_0 \times \mathscr{X}_0'}$$

for all $V \in \mathscr{X}_{\mathscr{T}}$, and from (32) we infer, using the Cauchy-Schwarz inequality, that

$$\|u-U\|_{\mathscr{X}} \lesssim \|R\|^s_{\mathscr{X}_1'} \inf_{V \in \mathscr{X}_{\mathscr{T}}} \|\rho - V\|^{1-s}_{\mathscr{X}_0}.$$

For example, in the context of weakly singular integral equations in $d \ge 2$ one chooses $B = V$, $\mathscr{X} = \widetilde{H}^{-1/2}(\Gamma)$, $\mathscr{X}_0' = L_2(\Gamma)$, $\mathscr{X}_1' = H^1(\Gamma)$, $s = 1/2$, and $\mathscr{X}_{\mathscr{T}} := \mathscr{P}^p(\mathscr{T})$. Then, $\rho = R$ and due to Lemma 22,

$$\inf_{V \in \mathscr{P}^p(\mathscr{T})} \|\rho - V\|_{L_2(\Gamma)} \lesssim \|h_{\mathscr{T}} \nabla_\Gamma R\|_{L_2(\Gamma)}.$$

This shows the following, cf. [44, Thm. 2].

Theorem 37 *If $\phi \in \widetilde{H}^{-1/2}(\Gamma)$ is the exact solution of Proposition 4 or 5 with $f \in H^1(\Gamma)$ and $\Phi \in \mathscr{P}^p(\mathscr{T})$ is the respective Galerkin approximation from Proposition 9 or 10, then*

$$\|\phi - \Phi\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim \|R\|^{1/2}_{H^1(\Gamma)} \|h_{\mathscr{T}} \nabla_\Gamma R\|^{1/2}_{L_2(\Gamma)}$$

This method can be applied to problems involving hypersingular integrals [44, Thms. 3, 4] as well as transmission problems [44, Sec. 5]. However, the a posteriori error estimates based on this method are of the form

$$\eta^2_{\mathscr{T}} := \Big(\sum_{T \in \mathscr{T}} \eta_T^2\Big)^{1/2} \cdot \Big(\sum_{T \in \mathscr{T}} h_T^2 \eta_T^2\Big)^{1/2},$$

which reflects the fact that this method does not fully localize a fractional norm, see the discussions in [44, Sec. 6] and [45, Sec. 1]. This issue can be overcome when considering a uniform sequence of meshes, where the result of Theorem 37 clearly reduces to

$$\|\phi - \Phi\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim \eta_{\mathscr{T}} := \Big(\sum_{T \in \mathscr{T}} \eta_T^2\Big)^{1/2} \quad \text{with } \eta_T := \|h^{1/2}_{\mathscr{T}} \nabla_\Gamma R\|_{L_2(T)}. \tag{33}$$

Further works on weighted residual error estimation in BEM focus on establishing the reliability estimate (33) also for locally refined meshes. The first one to mention is [45]. For weakly singular equations, it is shown that for $d = 2$ it holds

$$\|\phi - \Phi\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim \sigma^{1/2}_{\mathscr{T}} \sum_{T \in \mathscr{T}} h_T^{1/2} (h_T^2 + 1)^{1/4} \|R'\|_{L_2(T)},$$

where $\sigma_{\mathscr{T}}$ is the shape-regularity constant of $\mathscr{T}$ and $(\cdot)'$ abbreviates the arclength derivative $\nabla_\Gamma(\cdot)$ for $d = 2$. An analogous result holds for equations involving the hypersingular operator. The most advanced results regarding reliable a posteriori estimation by weighted residuals are due to [32] for $d = 2$ and [38,39] for $d = 3$. The first theorem that will be presented is concerned with the a posteriori error estimation for weakly singular integral equations, cf. [32, Ex. 1] for $d = 2$ and [39, Cor. 4.2] for $d = 3$. The idea of the proof will be presented briefly.

Theorem 38 *If $\phi \in \widetilde{H}^{-1/2}(\Gamma)$ is the exact solution of Proposition 4 or 5 with $f \in H^1(\Gamma)$ and $\Phi \in \mathscr{P}^p(\mathscr{T})$ is the respective Galerkin approximation from Proposition 9 or 10, then*

$$\|\phi - \Phi\|_{\widetilde{H}^{-1/2}(\Gamma)} \le C_{\mathrm{rel}} \|h^{1/2}_{\mathscr{T}} \nabla_\Gamma R\|_{L_2(\Gamma)} =: \eta_{\mathscr{T}},$$

where R denotes the residual, i.e., $R = f - V\Phi$ in the case of Proposition 4 and $R = (1/2 + K)f - V\Phi$ in the case of Proposition 5. The constant $C_{\mathrm{rel}} > 0$ depends only on Γ, the shape-regularity $\sigma_{\mathscr{T}}$, and on the polynomial degree p.

Proof We will show the result for $\Gamma = \partial\Omega$ a closed boundary. The case of an open boundary then follows easily. Stability of V^{-1} shows

$$\|\phi - \Phi\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim \|R\|_{H^{1/2}(\Gamma)}.$$

The set of nodes $\mathscr{N}$ of the mesh $\mathscr{T}$ can be split into $m > 0$ subsets, m depending only on $\sigma_{\mathscr{T}}$, into sets $\mathscr{N}^i$, $i = 1, \ldots, m$, such that

$$\mathscr{N} = \bigcup_{i=1}^m \mathscr{N}^i,$$

and $\operatorname{supp}(\varphi_{z_1}) \cap \operatorname{supp}(\varphi_{z_2}) = \emptyset$ for $z_1, z_2 \in \mathscr{N}^i$, where ϕ_z denotes the hat function associated to a vertex $z \in \mathscr{N}$. As

$\left(\sum_{j=1}^m a_j\right)^2 \le m\left(\sum_{j=1}^m a_j^2\right)$, it follows from the triangle inequality and Lemma 20 that

$$\begin{aligned}\|R\|^2_{H^{1/2}(\Gamma)} &\le m\sum_{i=1}^m \|\sum_{z\in\mathscr{N}^i}\varphi_z R\|^2_{\widetilde{H}^{1/2}(\Gamma)}\\ &\le C_{\mathrm{loc}} m \sum_{z\in\mathscr{N}}\|\varphi_z R\|^2_{\widetilde{H}^{1/2}(\omega_z)},\end{aligned}$$

where $\omega_z := \operatorname{supp}(\varphi_z)$. Friedrich's inequality shows

$$\|\varphi_z R\|^2_{\widetilde{H}^{1/2}(\omega_z)} \lesssim h_z(1+h_z^2)^{1/2}\|\nabla_\Gamma(\varphi_z R)\|^2_{L_2(\omega_z)},$$

where $h_z := \operatorname{diam}(\omega_z)$. Now, as R is orthogonal to piecewise constants, a Poincaré inequality shows

$$\|R\|_{L_2(\omega_z)} \lesssim \operatorname{diam}(\omega_z)\|\nabla_\Gamma R\|_{L_2(\omega_z)},$$

and taking into account $\|\varphi_z\|_{L_\infty(\Gamma)} \simeq 1$ and $\|\nabla_\Gamma\varphi_z\|_{L_\infty(\Gamma)} \simeq h_z^{-1}$ shows the result. $\square$

An analogous estimate holds for hypersingular integral equations, cf. [32, Ex. 5] for $d=2$ and [38, Thm. 4.2] for $d=3$.

Theorem 39 *If $u\in\widetilde{H}^{1/2}(\Gamma)$ is the exact solution of Proposition 6 or 7 with $\phi\in L_2(\Gamma)$ and $U\in\mathscr{S}^p(\mathscr{T})$ is the respective Galerkin approximation from Proposition 12 or 13, then*

$$\|u-U\|_{\widetilde{H}^{1/2}(\Gamma)} \le C_{\mathrm{rel}}\|h_{\mathscr{T}}^{1/2}R\|_{L_2(\Gamma)},$$

where R denotes the residual, i.e., $R=\phi-WU$ in the case of Prop. 6 and $R=(1/2-K')\phi-WU$ in the case of Prop. 7. The constant $C_{\mathrm{rel}}>0$ depends only on Γ, the shape-regularity $\sigma_{\mathscr{T}}$, and on the polynomial degree p.

The preceding two theorems provide reliable and fully localized error estimators for Galerkin methods for weakly singular and hypersingular integral equations. Up to now these estimators are the only ones which can be mathematically shown to drive adaptive BEM algorithms with optimal rates, cf. Section 8.5. The efficiency of this type of estimator is more involved and requires a careful analysis of the (possible) singular behavior of the solutions of the problem at hand. In the current optimality theory for adaptive algorithms, efficiency can be used to characterize approximation classes and therefore provides a means to work only with the error estimator, cf. [7].

Efficiency results for the estimators of Theorem 38 and 39 have first been proved for $d=2$ and globally quasi-uniform meshes in [31]. We state the idea in the context and with notation of Theorem 38. As we consider globally quasi-uniform meshes, we treat the mesh-with $h_{\mathscr{T}}$ of mesh $\mathscr{T}$ as a constant rather than a function. For a given mesh $\mathscr{T}$ with mesh size $h_{\mathscr{T}}$, suppose that $\mathscr{T}_\star \ge \mathscr{T}$ is a finer mesh with mesh size $h_{\mathscr{T}_\star} \le h_{\mathscr{T}}$. Recall that $\pi_{\mathscr{T}}$ and $\pi_{\mathscr{T}_\star}$ denote the L_2 projections onto $\mathscr{P}^0(\mathscr{T})$ and $\mathscr{P}^0(\mathscr{T}_\star)$, respectively, and that R denotes the residual on $\mathscr{T}$, using the Galerkin solution $\Phi\in\mathscr{P}^p(\mathscr{T})$. As $(1-\pi_{\mathscr{T}_\star}) = (1-\pi_{\mathscr{T}_\star})(1-\pi_{\mathscr{T}_\star})$, it follows from the approximation properties of $\pi_{\mathscr{T}_\star}$ that

$$\|(1-\pi_{\mathscr{T}_\star})\phi\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim h_{\mathscr{T}_\star}^{1/2}\|(1-\pi_{\mathscr{T}_\star})\phi\|_{L_2(\Gamma)},$$

and an inverse estimate then shows

$$\begin{aligned}&h_{\mathscr{T}}^{1/2}\|\pi_{\mathscr{T}_\star}\phi-\Phi\|_{L_2(\Gamma)} \lesssim\\ &\quad h_{\mathscr{T}}^{1/2}\|\pi_{\mathscr{T}_\star}\phi-\phi\|_{\widetilde{H}^{-1/2}(\Gamma)} + \left(\frac{h_{\mathscr{T}}}{h_{\mathscr{T}_\star}}\right)^{1/2}\|\phi-\Phi\|_{\widetilde{H}^{-1/2}(\Gamma)}.\end{aligned}$$

Finally, this gives

$$\begin{aligned}h_{\mathscr{T}}^{1/2}\|R\|_{H^1(\Gamma)} &\lesssim h_{\mathscr{T}}^{1/2}\|\phi-\Phi\|_{L_2(\Gamma)}\\ &\lesssim h_{\mathscr{T}}^{1/2}\|\phi-\pi_{\mathscr{T}_\star}\phi\|_{L_2(\Gamma)} + \left(\frac{h_{\mathscr{T}}}{h_{\mathscr{T}_\star}}\right)^{1/2}\|\phi-\Phi\|_{\widetilde{H}^{-1/2}(\Gamma)}.\end{aligned} \tag{34}$$

Given $\mathscr{T}$ and an arbitrary $q<1$, the fine mesh $\mathscr{T}_\star$ can always be chosen such that

$$\|\phi-\pi_{\mathscr{T}_\star}\phi\|_{L_2(\Gamma)} \le q\|\phi-\pi_{\mathscr{T}}\phi\|_{L_2(\Gamma)} \le q\|\phi-\Phi\|_{L_2(\Gamma)} \tag{35}$$

holds. It follows that (34) and (35) yield

$$h_{\mathscr{T}}^{1/2}\|R\|_{H^1(\Gamma)} \lesssim \left(\frac{h_{\mathscr{T}}}{h_{\mathscr{T}_\star}}\right)^{1/2}\|\phi-\Phi\|_{\widetilde{H}^{-1/2}(\Gamma)}. \tag{36}$$

The mesh $\mathscr{T}_\star$ depends on $\mathscr{T}$ and on q (it is a refinement of $\mathscr{T}$ that fulfills (35)). However, from (36) we see that efficiency can only hold if it is guaranteed that $h_{\mathscr{T}} \le C h_{\mathscr{T}_\star}$, where $0<C<1$ only depends on q. This can be done by exploiting explicit knowledge of the qualitative behavior of ϕ, cf. [31, Prop. 1] for globally quasi-uniform meshes and weakly singular and hypersingular equations. The presented approach is analyzed in a local fashion in [7] to obtain efficiency results on locally refined meshes for the weakly singular case. The corresponding result is the following.

Theorem 40 *Suppose $d=2$ and that the data fulfill $f\in H^1(\Gamma)$ and $f\in H^s$ for some $s>2$ on the different sides of the polygonal boundary Γ. Then,*

$$\begin{aligned}C_{\mathrm{eff}}^{-1}\|h_{\mathscr{T}}^{1/2}\nabla_\Gamma R\|_{L_2(\Gamma)} \le &\|\phi-\Phi\|_{\widetilde{H}^{-1/2}}+\\ &C(s,\varepsilon)\left(\sum_{T\in\mathscr{T}} h_{\mathscr{T}}(T)^{\min\{2s,5\}-1-\varepsilon}\right)\end{aligned}$$

for all $\varepsilon>0$. The constant $C_{\mathrm{eff}}>0$ depends only Γ and $\sigma_{\mathscr{T}}$, whereas $C(s,\varepsilon)$ additionally depends on s and ε.

4.1.4 Faermann's local double norm estimators

It was suggested in [72] to split the outer integral of the H^s norm of the residual in contributions on different faces of the mesh. The authors used this approach to present an adaptive boundary element algorithm for the solution of the Helmholtz equation, but nevertheless this procedure does not give fully localized indicators. Their approach was refined in [63,64], where localization techniques for Sobolev-Slobodeckij norms (cf. Section 3) were deduced and put into action to derive fully localized error indicators. Up to now, this is the only way to obtain localized estimators that are both reliable and efficient (on shape-regular meshes) without further conditions or additional analysis. Their operational area is restricted to continuous and bijective operators

$$B : \widetilde{H}^{s+2\alpha}(\Gamma) \to H^s(\Gamma), \quad s \in (0,1), \alpha \in \mathbb{R}.$$

The upper bound on s stems from the fact that we deal with Lipschitz domains, but this bound can be enlarged on smoother domains. For arbitrary d, denote by $\phi \in \widetilde{H}^{s+2\alpha}(\Gamma)$ the exact solution of the equation

$$\langle B\phi\,,\psi\rangle_\Gamma = \langle F\,,\psi\rangle_\Gamma \quad \text{for all } \psi \in \widetilde{H}^{-s}(\Gamma).$$

For a discrete space $\mathscr{X}_\mathscr{T} \subseteq \widetilde{H}^{s+2\alpha}(\Gamma)$, denote by $\Phi \in \mathscr{X}_\mathscr{T}$ the Galerkin solution

$$\langle B\Phi\,,V\rangle_\Gamma = \langle F\,,V\rangle \quad \text{for all } V \in \mathscr{X}_\mathscr{T}.$$

Denote by $R := F - B\Phi \in H^s(\Gamma)$ the residual. The localization result of Theorem 17 immediately provides a localized a posteriori estimator.

Theorem 41 *Suppose that the assumptions and notations from the beginning of this section hold. For a mesh $\mathscr{T}$, define the a posteriori error estimator*

$$\eta_\mathscr{T}^2 := \sum_{z\in\mathscr{N}} \eta_z^2 + \sum_{T\in\mathscr{T}} \eta_T^2 \quad \textit{with}$$
$$\eta_z^2 := |R|^2_{H^s(\omega_z)}, \quad \eta_T^2 := h_T^{-2s} \|R\|^2_{L_2(T)}.$$

Then, $\eta_\mathscr{T}$ is always reliable, i.e.,

$$\|\phi - \Phi\|_{\widetilde{H}^{s+2\alpha}(\Gamma)} \le C_{\rm rel}\,\eta_\mathscr{T},$$

and $C_{\rm rel}$ depends only on s and Γ. If $\mathscr{P}^p(\mathscr{T}) \subseteq \mathscr{X}_\mathscr{T}$ or $\mathscr{S}^p(\mathscr{T}) \subseteq \mathscr{X}_\mathscr{T}$, then we have efficiency

$$\eta_\mathscr{T} \le C_{\rm eff} \|\phi - \Phi_N\|_{\widetilde{H}^{s+2\alpha}(\Gamma)},$$

and $C_{\rm eff}$ depends only on the shape-regularity $\sigma_\mathscr{T}$.

Proof To show reliability, note first that B^{-1} is bounded due to the bounded inverse theorem. This gives

$$\|\phi - \Phi_N\|^2_{\widetilde{H}^{s+2\alpha}(\Gamma)} \lesssim \|R_N\|^2_{H^s(\Gamma)} = \|R_N\|^2_{L_2(\Gamma)} + |R_N|^2_{H^s(\Gamma)}.$$

The H^s-part can be bounded immediately with Theorem 17. There is a constant $C(\Gamma) > 0$ such that for all $T \in \mathscr{T}$ it holds that $h_T \le C(\Gamma)$, and the L_2-part can be hence bounded by

$$\|R\|^2_{L_2(\Gamma)} \le C(\Gamma)^{2s} \sum_{T\in\mathscr{T}} h_T^{-2s} \|R\|^2_{L_2(T)}.$$

This yields reliability. To show efficiency, we note that due to the assumptions $\mathscr{P}^p(\mathscr{T}) \subseteq \mathscr{X}_\mathscr{T}$ or $\mathscr{S}^p(\mathscr{T}) \subseteq \mathscr{X}_\mathscr{T}$ it follows that $\langle R\,,\Psi\rangle_\Gamma = 0$ for all discrete functions $\Psi \in \mathscr{X}_\mathscr{T}$. Lemma 18 shows that

$$\sum_{T\in\mathscr{T}} h_T^{-2s} \|R_N\|^2_{L_2(T)} \le C(\sigma_\mathscr{T}) \sum_{z\in\mathscr{N}} |R_N|^2_{H^s(\omega_z)}.$$

Hence,

$$\eta_\mathscr{T}^2 \lesssim \sum_{z\in\mathscr{N}} |R_N|^2_{H^s(\omega_z)} \lesssim \|R_N\|^2_{H^s(\Gamma)},$$

and continuity of B shows the efficiency. □

The estimator of the last theorem is always reliable, and on shape-regular meshes it is also efficient. With Theorem 19, the reverse situation can be generated.

Theorem 42 *Suppose that the assumptions and notations from the beginning of this section hold. For a mesh $\mathscr{T}$, define the a posteriori error estimator*

$$\eta_\mathscr{T}^2 := \sum_{z\in\mathscr{N}} \eta_z^2 \quad \textit{with} \quad \eta_z^2 := |R|^2_{H^s(\omega_z)}.$$

Then, $\eta_\mathscr{T}$ is always efficient, i.e.,

$$\eta_\mathscr{T} \le C_{\rm eff} \|\phi - \Phi\|_{\widetilde{H}^{s+2\alpha}(\Gamma)},$$

and $C_{\rm eff}$ depends only on s and Γ. If $\mathscr{P}^p(\mathscr{T}) \subseteq \mathscr{X}_\mathscr{T}$ or $\mathscr{S}^p(\mathscr{T}) \subseteq \mathscr{X}_\mathscr{T}$, then it is also reliable,

$$\|\phi - \Phi\|_{\widetilde{H}^{s+2\alpha}(\Gamma)} \le C_{\rm rel}\,\eta_\mathscr{T},$$

and $C_{\rm rel}$ depends only on the shape-regularity $\sigma_\mathscr{T}$.

4.2 Estimators based on space enrichment

The principal idea for the construction of error estimators based on space enrichment is that, for a given approximation, the Galerkin error can be approximated by replacing the exact solution with an improved approximation from an enriched discrete space. In the following, we introduce the basic setting for this methodology. Afterwards, in Subsections 4.2.1 and 4.2.2, we discuss specific variants within this framework.

To fix notation, let us consider the variational problem specified in Section 1.1, i.e., find u in a Hilbert space $\mathscr{X}$ such that u is a solution of equation (3), where b is a continuous and elliptic bilinear form, i.e. (1) and (2) hold true with constants $C_{\rm cont}, C_{\rm ell} > 0$. Recall that for a discrete approximation $U \in \mathscr{X}_\mathscr{T} \subset \mathscr{X}$, there holds Céa's Theorem (6). Now, for the error estimation, one considers an enriched approximation space $\mathscr{X}_\mathscr{T} \subset \widehat{\mathscr{X}_\mathscr{T}} \subset \mathscr{X}$ with corresponding Galerkin approximation $\widehat{U} \in \widehat{\mathscr{X}_\mathscr{T}}$. Under appropriate conditions,

$$\overline{\eta}_\mathscr{T} := \|\widehat{U} - U\|_\mathscr{X} \tag{37}$$

is a good approximation of the error $\|u - U\|_\mathscr{X}$.

The estimator $\overline{\eta}_\mathscr{T}$ is usually not practical for two reasons. First, it requires the calculation of the improved approximation $\widehat{U}$, which is expensive. Second, considering boundary element methods for integral equations of the first kind, the $\mathscr{X}$-norm is non-local so that $\overline{\eta}_\mathscr{T}$ does not immediately provide local informations that could be used for adaptivity. Therefore, further techniques are needed to avoid these problems. The resulting methods are called *error estimators based on space enrichment*. First and common step for their analysis is to study reliability and efficiency of $\overline{\eta}_\mathscr{T}$ for the estimation of $\|u - U\|_\mathscr{X}$. Reliability of the estimator is based on the "richness" of $\widehat{\mathscr{X}_\mathscr{T}}$ which is usually formulated as the following *saturation assumption*.

Assumption 43 (saturation) *Let $(\mathscr{X}_\ell)_{\ell=1}^\infty$ be a sequence of approximation spaces $\mathscr{X}_\ell \subset \mathscr{X}$ with corresponding sequence of enriched spaces $(\widehat{\mathscr{X}_\ell})_\ell^\infty$ and Galerkin projections $U_\ell \in \mathscr{X}_\ell$, $\widehat{U}_\ell \in \widehat{\mathscr{X}_\ell}$. There exists a constant $C_{\rm sat} \in [0,1)$ such that*

$$\|u - \widehat{U}_\ell\|_\mathscr{X} \le C_{\rm sat}\|u - U_\ell\|_\mathscr{X} \quad \text{for all } \ell \in \mathbb{N}.$$

In the following, when referring to this assumption and to simplify notation, we will simply write

$$\|u - \widehat{U}\|_\mathscr{X} \le C_{\rm sat}\|u - U\|_\mathscr{X}$$

in the sense that $U \in \mathscr{X}_\mathscr{T}$ is an element of a family of Galerkin approximations and that $\widehat{U}$ is an improved approximation from an enriched space $\widehat{\mathscr{X}_\mathscr{T}}$.

Reliability and efficiency of $\overline{\eta}_\mathscr{T}$ (or variants) in this or similar situations are based on the saturation assumption and have been studied many times in the literature, to our knowledge first in [16]. An immediate consequence of the saturation assumption, combined with the triangle inequality, is the following two-sided estimate, showing reliability and efficiency of the global estimator (37).

Proposition 44 *The estimator $\overline{\eta}_\mathscr{T}$ is efficient, i.e.,*

$$\overline{\eta}_\mathscr{T} \le \frac{C_{\rm cont}}{C_{\rm ell}}\|u - U\|_\mathscr{X}.$$

In the situation of Assumption 43, $\overline{\eta}_\mathscr{T}$ is also reliable, i.e.,

$$\|u - U\|_\mathscr{X} \le (1 - C_{\rm sat})^{-1}\overline{\eta}_\mathscr{T}$$

Hence, $\overline{\eta}_\mathscr{T}$ is an efficient measure for the error, whereas it's reliability hinges on the saturation assumption. In the case that the bilinear form $b(\cdot,\cdot)$ is symmetric, both estimates can be improved when they are formulated in terms of the so-called *energy norm* $\|\cdot\|_b := \sqrt{b(\cdot,\cdot)}$. In many publications, the saturation assumption is formulated with respect to this norm anyhow, whereas Assumption 43 uses the $\mathscr{X}$-norm. Of course, both norms are equivalent, i.e.,

$$C_{\rm ell}\|v\|_\mathscr{X}^2 \le \|v\|_b^2 \le C_{\rm cont}\|v\|_\mathscr{X}^2 \text{ for all } v \in \mathscr{X}. \tag{38}$$

Using this norm equivalence, Assumption 43 yields

$$\|u - \widehat{U}\|_b \le \sqrt{C_{\rm cont}/C_{\rm ell}}\, C_{\rm sat}\|u - U\|_b.$$

However, from (1) and (2) it follows that

$$1 \le C_{\rm cont}/C_{\rm ell},$$

which does not provide a saturation assumption in the energy norm. Proposition 46 below shows how to overcome this problem For convenience, we separately formulate the saturation assumption for the energy norm first.

Assumption 45 (saturation in energy norm) *Let us consider the situation of Assumption 43 and let the bilinear form $b(\cdot,\cdot)$ be symmetric. We assume that there exists a constant $C_{\rm sata} \in [0,1)$ such that*

$$\|u - \widehat{U}_\ell\|_b \le C_{\rm sata}\|u - U_\ell\|_b \quad \text{for all } \ell \in \mathbb{N}.$$

As previously in Assumption 43, we will use this estimate for a single discrete space $\mathscr{X}_\mathscr{T}$ understanding that it is an element of a family of spaces (with corresponding Galerkin approximations and enrichments).

We are ready to present the results corresponding to Proposition 44 in the case of a symmetric bilinear form and in terms of the energy norm.

Proposition 46 *Let the bilinear form $b(\cdot,\cdot)$ be symmetric and $\widehat{\mathscr{X}_\mathscr{T}}$ be an enriched space of $\mathscr{X}_\mathscr{T} \subset \mathscr{X}$. Define an estimator by*

$$\eta_\mathscr{T} := \|\widehat{U} - U\|_b.$$

Then, the estimator $\eta_\mathscr{T}$ is efficient, i.e.,

$$\eta_\mathscr{T} \le \|u - U\|_b.$$

If additionally Assumption 45 holds, then $\eta_\mathscr{T}$ is also reliable, i.e.,

$$\|u - U\|_b \le (1 - C_{\rm sata}^2)^{-1/2}\eta_\mathscr{T}.$$

Furthermore, Assumption 43 implies Assumption 45.

Proof Symmetry of $b(\cdot,\cdot)$, Galerkin orthogonality and the saturation assumption 45 immediately yield

$$\|\widehat{U}-U\|_b^2 \le \|u-U\|_b^2 \le C_{\rm sata}^2\|u-U\|_b^2+\|\widehat{U}-U\|_b^2.$$

This proves both reliability and efficiency. To show that saturation in the norm $\|\cdot\|_{\mathscr{X}}$ implies saturation in the norm $\|\cdot\|_b$, use the reliability of Proposition 44 and the norm equivalence (38) to see

$$\|u-U\|_b \le \frac{\sqrt{C_{\rm cont}/C_{\rm ell}}}{1-C_{\rm sat}}\|\widehat{U}-U\|_b.$$

Galerkin orthogonality then yields

$$\begin{aligned}\|u-\widehat{U}\|_b^2 &= \|u-U\|_b^2-\|\widehat{U}-U\|_b^2\\ &\le \|u-U\|_b^2-\left(\frac{\sqrt{C_{\rm cont}/C_{\rm ell}}}{1-C_{\rm sat}}\right)^{-2}\|u-U\|_b^2\end{aligned}$$

and saturation in the energy norm $\|\cdot\|_b$ follows with

$$C_{\rm sata} = \left(1-\left(\frac{\sqrt{C_{\rm cont}/C_{\rm ell}}}{1-C_{\rm sat}}\right)^{-2}\right)^{1/2}.$$

As $1\le C_{\rm cont}/C_{\rm ell}$ and $C_{\rm sat}\in[0,1)$ it follows that $C_{\rm sata}\in[0,1)$. □

4.2.1 Two-level estimators

The term two-level estimator refers to the fact that, using the notation $\widehat{\mathscr{X}_{\mathscr{T}}}$ and $\mathscr{X}_{\mathscr{T}}\subset\mathscr{X}$ from Section 4.2, the space $\widehat{\mathscr{X}_{\mathscr{T}}}$ is generated like

$$\widehat{\mathscr{X}_{\mathscr{T}}} = \mathscr{X}_{\mathscr{T}}\oplus\mathscr{Z}_{\mathscr{T}}. \tag{39}$$

That means $\widehat{\mathscr{X}_{\mathscr{T}}}$ is generated by adding to the approximation space $\mathscr{X}_{\mathscr{T}}$ a second level as enrichment. In other words, $\widehat{\mathscr{X}_{\mathscr{T}}}$ has a hierarchical two-level decomposition like (39). In order to produce local contributions to the final error estimator, the second level is usually further decomposed so that

$$\widehat{\mathscr{X}_{\mathscr{T}}} = \mathscr{Z}_{\mathscr{T},0}\oplus\mathscr{Z}_{\mathscr{T},1}\oplus\mathscr{Z}_{\mathscr{T},2}\oplus\cdots\oplus\mathscr{Z}_{\mathscr{T},L} \tag{40}$$

with $\mathscr{Z}_{\mathscr{T},0}:=\mathscr{X}_{\mathscr{T}}$ if we want to be consistent with (39). Here, the number L of subspaces $\mathscr{Z}_{\mathscr{T},j}\subset\mathscr{Z}_{\mathscr{T}}$ can be fixed or can vary with the dimension of $\mathscr{X}_{\mathscr{T}}$.

In the following, let us consider the simplest case of a symmetric (and elliptic, continuous) bilinear form $b(\cdot,\cdot)$. Based on the decomposition (40) one defines error indicators

$$\eta_j := \|P_j(\widehat{U}-U)\|_b,\quad j=0,\dots,L, \tag{41}$$

with

$$P_j:\ \widehat{\mathscr{X}_{\mathscr{T}}}\to\mathscr{Z}_{\mathscr{T},j}:\quad b(P_jv,w)=b(v,w)\quad\text{for all } w\in\mathscr{Z}_{\mathscr{T},j}.$$

The projectors P_j are called *additive Schwarz projectors* and $P:=\sum_{j=0}^L P_j$ is the *additive Schwarz operator* corresponding to the decomposition (40), cf. [119,131,143]. The operator P corresponds to a preconditioned stiffness matrix and is related to techniques from domain decomposition when (40) is constructed via such a decomposition. However, in principle, (40) can be generated by any means, in particular to allow for indicators aimed at anisotropic mesh refinement, cf., e.g., [61]. Finally, having at hand the indicators η_j, an error estimator is defined by

$$\eta_{\mathscr{T}} := \Big(\sum_{j=0}^L \eta_j^2\Big)^{1/2}. \tag{42}$$

The following simple result shows that the calculation of $\eta_{\mathscr{T}}$ is not expensive if the dimensions of $\mathscr{Z}_{\mathscr{T},j}$ $(j>0)$ are small and $\mathscr{Z}_{\mathscr{T},0}=\mathscr{X}_{\mathscr{T}}$. In particular, there is no need to calculate the improved Galerkin approximation $\widehat{U}\in\widehat{\mathscr{X}_{\mathscr{T}}}$.

Lemma 47 *The additive Schwarz projections $P_j(\widehat{U}-U)$ can be calculated by solving problems in the subspaces $\mathscr{Z}_{\mathscr{T},j}$ without knowing $\widehat{U}$,*

$$b(P_j(\widehat{U}-U),V)=L(V)-b(U,V)\quad\textit{for all } V\in\mathscr{Z}_{\mathscr{T},j}.$$

Moreover, if $\mathscr{Z}_{\mathscr{T},0}\subset\mathscr{X}_{\mathscr{T}}$ then $\eta_0=0$.

Proof These properties follow immediately by the definition of the projectors and the Galerkin orthogonality. □

To show reliability and efficiency of η one usually shows stability of the decomposition (40). This can be formulated in different equivalent ways as follows (see, e.g., [119,131, 143]).

Proposition 48 *Let P be the additive Schwarz operator related to the symmetric bilinear form $b(\cdot,\cdot)$ and discrete space $\widehat{\mathscr{X}_{\mathscr{T}}}$ with decomposition* (40)*. Then, for two positive numbers λ_0, λ_1 the following statements are equivalent.*
(i) There hold the bounds $\lambda_{\min}(P)\ge\lambda_0$ and $\lambda_{\max}(P)\le\lambda_1$ for the minimum and maximum eigenvalues of P, respectively.

$$(ii)\quad \lambda_0\sum_{j=0}^L b(v_j,v_j)\le b(v,v)\le\lambda_1\sum_{j=0}^L b(v_j,v_j)$$

for all $v=\sum_{j=0}^L v_j\in\widehat{\mathscr{X}_{\mathscr{T}}}$ with $v_j\in\mathscr{Z}_{\mathscr{T},j}$ $(j=0,\dots,L)$.

$$(iii)\quad \lambda_0\|v\|_b^2\le b(Pv,v)\le\lambda_1\|v\|_b^2\quad\textit{for all } v\in\widehat{\mathscr{X}_{\mathscr{T}}}.$$

For a decomposition of $\widehat{\mathscr{X}_{\mathscr{T}}}$ that, unlike (40), is not direct, the spectral properties of P are characterized slightly differently. In the following we will consider only direct decompositions (40) of $\widehat{\mathscr{X}_{\mathscr{T}}}$.

In most applications, the stability of (40) is ensured by two independent steps, first the enrichment of $\mathscr{X}_{\mathscr{T}}$ by a second level $\mathscr{Z}_{\mathscr{T}}$ so that the decomposition (39) is stable and, second, a stable decomposition of the second level,

$$\mathscr{Z}_{\mathscr{T}} = \mathscr{Z}_{\mathscr{T},1} \oplus \cdots \oplus \mathscr{Z}_{\mathscr{T},L}. \tag{43}$$

Of course, the stability of (39) is optimal when $\mathscr{X}_{\mathscr{T}}$ and $\mathscr{Z}_{\mathscr{T}}$ are orthogonal,

$$b(v,v) = b(x,x) + b(z,z)$$

for all $v = x + z \in \widehat{\mathscr{X}_{\mathscr{T}}}$ with $x \in \mathscr{X}_{\mathscr{T}}$ and $z \in \mathscr{Z}_{\mathscr{T}}$, cf. Proposition 48, (iii). A generalization of this case is the so-called *strengthened Cauchy-Schwarz inequality*.

Definition 49 *The decomposition* (39) *satisfies a* strengthened Cauchy-Schwarz inequality *if there exists a number* $\gamma \in [0,1)$ *such that*

$$b(x,z) \le \gamma \|x\|_b \|z\|_b \quad \forall x \in \mathscr{X}_{\mathscr{T}},\, z \in \mathscr{Z}_{\mathscr{T}}.$$

Immediate implication of the strengthened Cauchy-Schwarz inequality is the stability of the two-level decomposition.

Lemma 50 *Let the decomposition* (39) *satisfy a strengthened Cauchy-Schwarz inequality (with constant γ) and let* (43) *be a stable decomposition with constants $\lambda_0^{\mathscr{Z}}$ and $\lambda_1^{\mathscr{Z}}$,*

$$\lambda_0^{\mathscr{Z}} \sum_{j=1}^{L} b(v_j, v_j) \le b(v,v) \le \lambda_1^{\mathscr{Z}} \sum_{j=1}^{L} b(v_j, v_j) \tag{44}$$

for all $v = \sum_{j=1}^{L} v_j \in \mathscr{Z}_{\mathscr{T}}$ with $v_j \in \mathscr{Z}_{\mathscr{T},j}$ $(j = 1,\ldots,L)$. Then, (40) *is stable in the sense of Proposition 48 with*

$$\lambda_0 \ge (1-\gamma)\min\{1,\lambda_0^{\mathscr{Z}}\} \quad \text{and} \quad \lambda_1 \le (1+\gamma)\max\{1,\lambda_1^{\mathscr{Z}}\}.$$

Proof The strengthened Cauchy-Schwarz inequality implies that

$$\begin{aligned}(1-\gamma)\big(\|v_0\|_b^2 + \|v_{\mathscr{Z}}\|_b^2\big) &\le \|v_0 + v_{\mathscr{Z}}\|_b^2 \\ &\le (1+\gamma)\big(\|v_0\|_b^2 + \|v_{\mathscr{Z}}\|_b^2\big)\end{aligned}$$

for all $v_0 \in \mathscr{X}_{\mathscr{T}}$ and $v_{\mathscr{Z}} \in \mathscr{Z}_{\mathscr{T}}$. The assertion then follows immediately by application of (44). □

A combination of Propositions 46 and 48 leads to the following general result on the efficiency and reliability of a two-level error estimator.

Theorem 51 *Let the bilinear form $b(\cdot,\cdot)$ be symmetric and $\widehat{\mathscr{X}_{\mathscr{T}}}$ be an enriched space of $\mathscr{X}_{\mathscr{T}} \subset \mathscr{X}$. Assume that the decomposition* (40) *is stable in the sense that there exist positive numbers λ_0, λ_1 that satisfy the relations of Proposition 48. Then the estimator $\eta_{\mathscr{T}}$ from* (42) *defined by the local projections* (41) *is efficient,*

$$\lambda_1^{-1/2} \eta_{\mathscr{T}} \le \|u - U\|_b.$$

If, additionally, Assumption 45 holds then $\eta_{\mathscr{T}}$ is also reliable,

$$\|u - U\|_b \le (1 - C_{\mathrm{sata}}^2)^{-1/2} \lambda_0^{-1/2}\, \eta_{\mathscr{T}}.$$

Here, $U \in \mathscr{X}_{\mathscr{T}}$ is the Galerkin projection of the exact solution $u \in \mathscr{X}$ of the abstract problem (3)*.*

Proof By definition of η_ℓ and the projectors P_j, and using the characterization by Proposition 48, (iii), there holds

$$\begin{aligned}\eta_{\mathscr{T}}^2 &= \sum_{j=0}^{L} \eta_j^2 = \sum_{j=0}^{L} b\big(P_j(\widehat{U} - U), P_j(\widehat{U} - U)\big) \\ &= \sum_{j=0}^{L} b\big(\widehat{U} - U, P_j(\widehat{U} - U)\big) = b\big(\widehat{U} - U, P(\widehat{U} - U)\big) \\ &\begin{cases} \le \lambda_1 \|\widehat{U} - U\|_b^2 = \lambda_1 \eta_{\mathscr{T}}^2, \\ \ge \lambda_0 \|\widehat{U} - U\|_b^2 = \lambda_0 \eta_{\mathscr{T}}^2, \end{cases}\end{aligned}$$

where η_ℓ is the estimator defined in Proposition 46. The assertions follow from the properties of η_ℓ. □

Having set the abstract (additive Schwarz) framework for two-level error estimators we proceed considering the specific cases of low order approximations to solutions of weakly singular and hypersingular integral equations.

Weakly singular operator: Let us consider the weakly singular integral equation (see Proposition 4) on an open or closed polyhedral surface Γ, with solution $\phi \in \widetilde{H}^{-1/2}(\Gamma)$. For simplicity we write $\widetilde{H}^{-1/2}(\Gamma) = H^{-1/2}(\Gamma)$ also on a closed surface. For a mesh $\mathscr{T}$ of shape-regular triangles and quadrilaterals, and discrete space $\mathscr{P}^0(\mathscr{T})$ of piecewise constant functions, $\Phi \in \mathscr{P}^0(\mathscr{T})$ denotes the Galerkin approximation of ϕ, cf. Proposition 9. We stress the fact that the mesh needs not be quasi-uniform and the **quadrilaterals can be anisotropic** but must be convex and satisfy a minimum angle condition In the notation introduced previously,

$$b(u,v) = \langle Vu\,, v\rangle_\Gamma, \quad \mathscr{X} = \widetilde{H}^{-1/2}(\Gamma), \quad \mathscr{X}_{\mathscr{T}} = \mathscr{P}^0(\mathscr{T}).$$

Now, in order to define a two-level estimator for the error $\|\phi - \Phi\|_{\widetilde{H}^{-1/2}(\Gamma)}$, we define the second level space $\mathscr{Z}_{\mathscr{T}}$ as piecewise constant functions on a refined mesh $\widehat{\mathscr{T}}$ with the restriction that the functions have integral-mean zero on any element of $\mathscr{T}$:

$$\mathscr{Z}_{\mathscr{T}} = \{V \in \mathscr{P}^0(\widehat{\mathscr{T}}) \mid \langle V\,, 1\rangle_T = 0 \ \text{ for all } T \in \mathscr{T}\}.$$

The enriched space is

$$\widehat{\mathscr{X}_{\mathscr{T}}} = \mathscr{P}^0(\mathscr{T}) \oplus \mathscr{Z}_{\mathscr{T}} = \mathscr{P}^0(\widehat{\mathscr{T}}).$$

Here, we generate $\widehat{\mathscr{T}}$ by refining every element of $\mathscr{T}$ in such a way that elements of $\widehat{\mathscr{T}}$ are shape-regular, see Figure 13. In this enrichment step the objective is two-fold. Essential is to make the saturation assumption hold. Second,

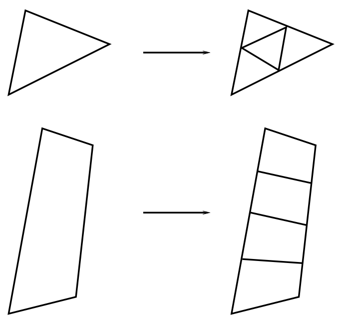

Fig. 13 Some elements of $\mathscr{T}$ (on the left) and their refinements to shape-regular elements of $\widetilde{\mathscr{T}}$ (on the right)

if one wants to perform anisotropic mesh refinement then one needs sufficiently many unknowns on every old element that allow for direction indicators. Some more details will be given below.

In this relatively general setting one can show reliability (based on saturation) and efficiency of the element-based error estimator

$$\eta_{\mathscr{T}} := \Big(\sum_{T\in\mathscr{T}} \eta_T^2\Big)^{1/2}, \quad \eta_T := \|P_T(\widehat{\Phi}-\Phi)\|_b. \tag{45}$$

Here, $\widehat{\Phi} \in \widehat{\mathscr{X}_{\mathscr{T}}}$ is the improved Galerkin approximation and, for any $T \in \mathscr{T}$ and $\mathscr{Z}_T := \{v \in \mathscr{Z}_{\mathscr{T}} \mid \mathrm{supp}(v) \subset \bar{T}\}$,

$$P_T : \widehat{\mathscr{X}_{\mathscr{T}}} \to \mathscr{Z}_T : \quad \langle VP_T v, w\rangle_T = \langle Vv, w\rangle_T \quad \forall w \in \mathscr{Z}_T$$

and

$$\|v\|_b^2 = \langle Vv, v\rangle_T \quad \text{for all } v \in \mathscr{Z}_T.$$

Theorem 52 *The error estimator $\eta_{\mathscr{T}}$ defined by* (45) *is efficient: there exists a constant $C_{\mathrm{eff}} > 0$ such that, for any mesh $\mathscr{T}$ with shape-regular refinement $\widehat{\mathscr{T}}$, there holds*

$$\eta_{\mathscr{T}} \le C_{\mathrm{eff}} \|\phi - \Phi\|_b.$$

Furthermore, if Assumption 45 holds, then $\eta_{\mathscr{T}}$ is also reliable: there exists a constant $c > 0$ such that, with $C_{\mathrm{rel}} = (1 - C_{\mathrm{sata}}^2)^{-1/2} c$, there holds for any mesh $\mathscr{T}$ with shape-regular refinement $\widetilde{\mathscr{T}}$ the estimate

$$\|\phi - \Phi\|_b \le C_{\mathrm{rel}}\, \eta_{\mathscr{T}}.$$

For a detailed proof of Theorem 52 we refer to [61], where the vector case of the weakly singular operator for the Stokes problem is analyzed. As indicated by Theorem 51, a proof boils down to a stability analysis of the underlying decomposition

$$\widehat{\mathscr{X}_{\mathscr{T}}} = \mathscr{P}^0(\mathscr{T}) \oplus \bigoplus_{T\in\mathscr{T}} \mathscr{Z}_T. \tag{46}$$

This analysis uses estimates for norms from fractional order Sobolev spaces. Therefore, a major ingredient is to find a Sobolev norm that is equivalent to the energy norm $\|\cdot\|_b$. For a fixed surface Γ, this is the $\widetilde{H}^{-1/2}(\Gamma)$-norm according to Theorems 2 and 3. However, for an element $v \in \mathscr{Z}_T$, there holds the equivalence

$$\|v\|_b^2 = \langle Vv, v\rangle_T \simeq \|v\|^2_{\widetilde{H}^{-1/2}(T)},$$

and it is not immediately clear how the corresponding equivalence numbers depend on T. One has to find a Sobolev norm that is uniformly equivalent to the energy norm for shape-regular elements $T \in \widehat{\mathscr{T}}$. By an affine mapping of T to a reference element T_{ref} one finds that

$$\|v\|_b^2 \simeq h_T^{2d-3} \|\hat{v}\|_b^2.$$

Here, $\hat{v}$ is the affinely transformed function defined on T_{ref}. This equivalence is immediate by the two Jacobians of the double integral in $\langle V\cdot, \cdot\rangle_T$ and by the scaling property of the weakly singular kernel,

$$\frac{1}{|x-y|} = \frac{1}{|F_T(\hat{x}) - F_T(\hat{y})|} \simeq h_T^{-1} \frac{1}{|\hat{x}-\hat{y}|},$$

where $x = F_T(\hat{x}), y = F_T(\hat{y})$. On the other hand,

$$\|v\|_{\widetilde{H}^{-1/2}(T)} = \sup_{\varphi \in H^{1/2}(T)\setminus\{0\}} \frac{\langle v, \varphi\rangle_T}{\|\varphi\|_{H^{1/2}(T)}}$$

is certainly not uniformly equivalent to the energy norm since the duality in the numerator scales under affine transformations but the denominator does not (the semi-norm $|\cdot|_{H^{1/2}(T)}$ behaves differently from the $L_2(T)$-norm under affine mappings). To fix this mismatch, one uses an $H^{1/2}(T)$-norm with weighted $L_2(T)$-term,

$$\|v\|^2_{H_h^{1/2}(T)} := h_T^{-1} \|v\|^2_{L_2(T)} + |v|^2_{H^{1/2}(T)},$$

and defines a scalable $\widetilde{H}^{-1/2}(T)$-norm by duality:

$$\|v\|_{\widetilde{H}_h^{-1/2}(T)} := \sup_{\varphi \in H^{1/2}(T)\setminus\{0\}} \frac{\langle v, \varphi\rangle_T}{\|\varphi\|_{H_h^{1/2}(T)}}.$$

This norm is uniformly equivalent to the energy norm under affine mappings that maintain shape regularity, as long as the functions under consideration have integral-mean zero. This integral-mean zero condition is essential and the reason for the particular construction of our second level space $\mathscr{Z}$.

A proof of stability of the decomposition (46) then reduces to the following three steps.

1. Replace the energy norm by the uniformly equivalent scalable Sobolev norm $\|\cdot\|_{\widetilde{H}_h^{-1/2}(T)}$ in the spaces $\mathscr{Z}_T$.
2. One shows (see [61, Lemma 3.2]) that

   $$\|v\|^2_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim \sum_{T\in\mathscr{T}} \|v|_T\|^2_{\widetilde{H}_h^{-1/2}(T)}$$

   for all $v \in \widetilde{H}^{-1/2}(\Gamma)$ with $v|_T \in \widetilde{H}^{-1/2}(T)$ and $\langle v, 1\rangle_T = 0$ for all $T \in \mathscr{T}$.

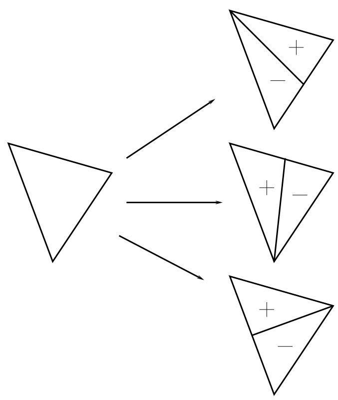

Fig. 14 A triangle of $\mathscr{T}$ (on the left) and its three partitions (on the right) for the construction of error indicators with direction control.

3. By scalability and equivalence of norms in finite-dimensional spaces one proves (see [61, (3.16)]) that

$$\sum_{T\in\mathscr{T}} \|V|_T\|^2_{\widetilde{H}_h^{-1/2}(T)} \lesssim \|V\|^2_{\widetilde{H}^{-1/2}(\Gamma)} \quad \forall V \in \mathscr{Z}_{\mathscr{T}}.$$

Finally, having shown the stability of (46) and making use of the saturation assumption, Theorem 52 is proved by application of Theorem 51.

Remark 5 The indicators η_T defined so far give information only with respect to the location of elements. By simple changes, it is easy to define indicators with respect to directions, so that anisotropic refinements can be considered. One only has to use slightly different local spaces $\mathscr{Z}_T$, further split so that corresponding projections give the direction indicators. In Figure 14 we have illustrated this for a single triangle (on the left) which is decomposed into two triangles in three different ways (on the right). The plus and minus signs indicate that one has to use piecewise constant functions (positive on the triangle with the plus sign and negative on the other) so that the function has integral-mean value zero. In this way, on each triangle $T \in \mathscr{T}$, one has three spaces and together they generate the second level $\mathscr{Z}$ on T. A refinement algorithm with direction control would consider, e.g., the triangle refinement that corresponds to the space among the three whose error indicator is largest. Similar constructions work on quadrilaterals. Throughout, in the refinement procedure with direction steering, one has to consider a minimum angle condition.

The inclusion of this direction control in the stability analysis of the decomposition of $\mathscr{Z}_{\mathscr{T}}$ is straightforward by selecting the previously scalable Sobolev norm. One only has to use an argument from equivalence of norms in finite-dimensional spaces. For more details we refer to [61].

Let us comment on other publications on two-level error estimators for weakly singular integral equations. In [113], Mund, Stephan and Weiße analyze the situation we have considered above for the particular case of uniform meshes of squares. In this case, several of the required norm estimates can be calculated exactly so that general arguments from fractional order Sobolev spaces (that we have discussed above) can be avoided. Also, [113] reports on numerical experiments on curved surfaces.

In [112], Mund and Stephan study two-level error estimators for the coupling of finite elements and boundary elements. The model problem is a transmission problem in two dimensions with nonlinear behavior in a bounded domain, coupled with the Laplacian in the exterior. The variational formulation and its discretization involves the weakly singular operator (on a curve). The proposed error estimator is of the two-level kind with additive Schwarz theory. Here, the authors prove stability of the boundary element contribution up to a perturbation of the type $h^{-\varepsilon}$ with $\varepsilon > 0$ and h being the mesh size.

So far we have only discussed the case of symmetric and elliptic bilinear forms. This theory can be extended to indefinite problems. In particular, in [16] Bank and Smith analyze in an abstract setting the general case of a variational form with bounded bilinear form that only satisfies the continuous and discrete inf-sup conditions (to guarantee existence and uniqueness of a continuous and discrete solution). Apart from the saturation assumption in the form of Assumption 43 (with respect to a Sobolev norm rather than energy norm), the analysis is based on a strengthened Cauchy-Schwarz inequality in the corresponding Sobolev norm (cf. Definition 49 in the energy norm). More specifically, for the boundary element method, Maischak, Mund and Stephan analyze in [107] two-level error estimators for weakly singular integral equations governing the Helmholtz problem with small wave number. Their theory follows the setting from [16], by showing that it is enough to have a stable decomposition corresponding to the elliptic part of the operator, and that the compact perturbation due to non-zero wave number does not change the behavior of the two-level error estimator. However, proofs are given for the two-dimensional case. In three dimensions, numerical results verify the expected behavior of the error estimator.

Finally, we note that in [99] the authors have studied two-level error estimators for boundary element discretizations of weakly singular and hypersingular operators in two dimensions. However, there are several unresolved theoretical hickups involving subtle issues with fractional order Sobolev spaces that the authors replaced with several assumptions. We prefer not to discuss the outcomes in detail.

Hypersingular operator: In two dimensions, that means for boundary integral equations on curves, additive Schwarz theory for weakly singular operators is equivalent to the one for hypersingular operators. This is due to the fact that Sobolev spaces of orders plus and minus one half are being mapped among them by differentiation and integration with respect to the arc-length. Correspondingly, basis functions are being transformed. The only, purely technical, difficulty is an integral-mean zero condition for functions in $H^{-1/2}$ along the elements or the curve. For an early observation and application of this fact, see [137].

In three dimensions, however, the situation is different. In this case, rather than simple differentiation and integration, pseudo-differential operators act as appropriate mappings. Possible operators are the square root of the negative Laplacian (more precisely, of the negative Laplace-Beltrami operator) and its inverse operator. These operators do not map piecewise polynomials onto piecewise polynomials. Therefore, in three dimensions on surfaces, the stability analyses of two-level decompositions of discrete spaces in $H^{1/2}$ and $H^{-1/2}$ are substantially different.

We do not know of any mathematical publication on two-level error estimators for hypersingular integral equations on surfaces that do not also consider the p-version (where approximations are improved by increasing polynomial degrees). Therefore, we postpone the discussion of this case to Section 5 which deals with the hp-version.

4.2.2 $(h-h/2)$ estimators

The starting point for this type of error estimators is Proposition 46, which states

$$\|\widehat{U}-U\|_b \le \|u-U\|_b \le (1-C_{\mathrm{sata}}^2)^{-1/2}\|\widehat{U}-U\|_b$$

for a symmetric bilinear form $b(\cdot,\cdot)$, where the upper bound holds under the saturation assumption 45. Here, $U \in \mathscr{X}_{\mathscr{T}}$ denotes the Galerkin solution with respect to a mesh $\mathscr{T}$ and $\widehat{U} \in \widehat{\mathscr{X}_{\mathscr{T}}} := \mathscr{X}_{\widehat{\mathscr{T}}}$ denotes the Galerkin solution with respect to a uniformly refined mesh $\widehat{\mathscr{T}}$. The term

$$\eta := \|\widehat{U}-U\|_b$$

is computable in the sense that it does not contain any unknowns and that it can be evaluated easily as a matrix-vector product

$$\|\widehat{U}-U\|_b^2 = (\widehat{\mathbf{U}}-\mathbf{U})\cdot\widehat{\mathbf{B}}\cdot(\widehat{\mathbf{U}}-\mathbf{U}),$$

where $\widehat{\mathbf{B}}$ is the Galerkin matrix on the space $\widehat{\mathscr{X}_{\mathscr{T}}}$ and $\widehat{\mathbf{U}}$ and $\mathbf{U}$ are the coefficient vectors of the Galerkin solutions with respect to the chosen basis for $\widehat{\mathscr{X}_{\mathscr{T}}}$. Now, the idea of $(h-h/2)$ estimators is to overcome the following two problems:

- The computation of both, U and $\widehat{U}$ is necessary.
- The inherent non-locality of the norm prevents us to use $\|\widehat{U}-U\|_b$ as refinement indicator in an adaptive algorithm.

To motivate a remedy for the first problem, we note that, due to best approximation properties of Galerkin solutions, $\widehat{U}$ will always be a better solution than U which therefore becomes only a temporary result. Furthermore, as soon as $\widehat{U}$ is computed, the computational cost of computing U is quite high in contrast to a simple postprocessing of $\widehat{U}$. Hence, to avoid the (expensive) computation of U, we use $\Pi\widehat{U}$ instead, where Π is a (preferably cheap) projection onto the space $\mathscr{X}_{\mathscr{T}}$, which is supposed to fulfill the following properties for all $\widehat{U} \in \widehat{\mathscr{X}_{\mathscr{T}}}$:

$$\|(1-\Pi)\widehat{U}\|_b \le C_{\mathrm{apx}} \min_{V\in\mathscr{X}_{\mathscr{T}}} \|\widehat{U}-V\|_{h^s} \tag{47}$$

$$\|\widehat{U}\|_{h^s} \le C_{\mathrm{inv}}\|\widehat{U}\|_b, \tag{48}$$

where $\|\cdot\|_{h^s}$ denotes an adequate h^s-weighted, integer order seminorm. In BEM, the norm $\|\cdot\|_b$ is equivalent to a fractional order Sobolev norm. Hence, the first estimate in (47) is an approximation property for the operator Π, whereas the estimate (48) corresponds to an inverse estimate. Note, however, that $\widehat{U} \in \widehat{\mathscr{X}_{\mathscr{T}}}$ is based on a fine mesh $\widehat{\mathscr{T}}$, whereas h in (48) corresponds to the mesh $\mathscr{T}$. In other words, (48) requires that the mesh-size $\widehat{h}$ of $\widehat{\mathscr{T}}$ must not be too small in comparison with the mesh-size h of $\mathscr{T}$. Now, the best approximation properties of Galerkin methods show immediately that

$$\eta \le \widetilde{\eta} := \|(1-\Pi)\widehat{U}\|_b.$$

From the estimate (47) follows immediately that

$$\widetilde{\eta} = \|(1-\Pi)\widehat{U}\|_b \lesssim \|(1-\Pi)\widehat{U}\|_{h^s} =: \widetilde{\mu}.$$

The estimator $\widetilde{\mu}$ has all the desired properties as it is local and avoids the computation of U. Next, the estimates (48) and (47) show

$$\widetilde{\mu} = \|(1-\Pi)\widehat{U}\|_{h^s} \lesssim \|(1-\Pi)\widehat{U}\|_b \lesssim \|\widehat{U}-U\|_{h^s} =: \mu.$$

Finally it follows from estimate (48) that

$$\mu = \|\widehat{U}-U\|_{h^s} \lesssim \|\widehat{U}-U\|_b = \eta$$

The strength of the resulting estimators is that they are conceptually simple and require nearly no overhead in implementation. In the following, we will specify the involved quantities to obtain estimates in the weakly singular and hypersingular case.

Weakly singular operator: For weakly singular integral equations, i.e., integral equations involving the single layer potential V, the presented approach was analyzed in detail for $d=2$ in [58] and for $d=3$ in [74]. The cited works discuss only the lowest-order case $p=0$, therefore we sketch the proof for general $p\ge 0$. The energy norm is given in this case by

$$\|u\|_b^2 := \|u\|_V^2 := \langle Vu\,,u\rangle_\Gamma.$$

The first result regarding reliability and efficiency of the estimator η follows directly from Proposition 46, cf. [74, Prop. 1.1] and [58, Prop. 3.1].

Theorem 53 *Suppose that $\phi \in \widetilde{H}^{-1/2}(\Gamma)$ is the exact solution of Proposition 4 or 5. Given a mesh $\mathscr{T}$ and its uniform refinement $\widehat{\mathscr{T}}$, denote by $\Phi \in \mathscr{P}^p(\mathscr{T})$ and $\widehat{\Phi} \in \mathscr{P}^p(\widehat{\mathscr{T}})$ the respective Galerkin approximations from Proposition 9 or 10. Then,*

$$\eta_{\mathscr{T}} := \|\Phi - \widehat{\Phi}\|_V \leq \|\phi - \Phi\|_V,$$

i.e., the estimator $\eta_{\mathscr{T}}$ is efficient with $C_{\mathrm{eff}} = 1$. Under the saturation assumption 45, $\eta_{\mathscr{T}}$ is reliable, i.e.,

$$\|\phi - \Phi\|_V \leq (1 - C_{\mathrm{sata}}^2)^{-1/2}\eta_{\mathscr{T}}.$$

The localization of $\eta_{\mathscr{T}}$ and the avoidance of the computation of Φ is done by using the L_2-orthogonal projection $\pi_{\mathscr{T}}^p$ from Definition 21, and the seminorm $\|\cdot\|_{h^s}$ will be the $h_{\mathscr{T}}^{1/2}$-weighted L_2 norm in this case. Note that by Theorem 2 and 3, $\|\cdot\|_b$ is an equivalent norm on $\widetilde{H}^{-1/2}(\Gamma)$. Lemma 22 with $r = 1/2$ and $s = 0$ provides the approximation properties for the derivation of (47), and Lemma 23 provides the inverse estimate that is needed in (48). Note that the projection property of $\pi_{\mathscr{T}}^p$ is used to arrive at the minima. The resulting estimators and equivalences are stated in the following theorem, cf. [74, Thms. 3.2, 3.4].

Theorem 54 *Define the following a posteriori error estimators:*

$$\begin{aligned} \eta_{\mathscr{T}} &:= \|\widehat{\Phi} - \Phi\|_V, & \mu_{\mathscr{T}} &:= \|h_{\mathscr{T}}^{1/2}(\widehat{\Phi} - \Phi)\|_{L_2(\Gamma)}, \\ \widetilde{\eta}_{\mathscr{T}} &:= \|(1 - \pi_{\mathscr{T}}^p)\widehat{\Phi}\|_V, & \widetilde{\mu}_{\mathscr{T}} &:= \|h_{\mathscr{T}}^{1/2}(1 - \pi_{\mathscr{T}}^p)\widehat{\Phi}\|_{L_2(\Gamma)}. \end{aligned}$$

Then, it holds

$$\begin{aligned} \eta_{\mathscr{T}} &\leq \widetilde{\eta}_{\mathscr{T}} \leq C_V^{1/2} C_{\mathrm{apx}} \widetilde{\mu}_{\mathscr{T}}, \\ \widetilde{\mu}_{\mathscr{T}} &\leq \mu_{\mathscr{T}} \leq C_{\mathrm{inv}} C_{\mathrm{ell}}^{-1/2} \eta_{\mathscr{T}}, \end{aligned}$$

where $C_V = \|V\|_{\widetilde{H}^{-1/2}(\Gamma)\to H^{1/2}(\Gamma)}$ and C_{ell} are the stability and ellipticity constants of the single layer operator V, C_{apx} is the constant of Lemma 22, and C_{inv} is the constant of the inverse estimate of Lemma 23.

The last theorem shows that all estimators are equivalent up to constants that depend only on Γ, p, and the shape-regularity constant $\sigma_{\mathscr{T}}$. In particular, Theorem 53 shows that all estimators are efficient and (under the saturation assumption 45) reliable.

Remark 6 The work [74] uses the quantity $\rho_{\mathscr{T}}$ instead of $h_{\mathscr{T}}$ to define the estimators $\mu_{\mathscr{T}}$ and $\widetilde{\mu}_{\mathscr{T}}$, where $\rho_{\mathscr{T}}$ is defined $\mathscr{T}$-elementwise as the diameter of the largest sphere centered at a point in $T \in \mathscr{T}$ whose intersection with Γ lies entirely in T. The reason for this is that [74] also uses the estimator $\widetilde{\mu}_{\mathscr{T}}$ to steer an adaptive anisotropic mesh refinement on quadrilaterals, for which $\rho_{\mathscr{T}}$ is more appropriate than $h_{\mathscr{T}}$. After an element has been selected for refinement, the choice on the refinement directions is based on the expansion of $\widehat{\Phi}$ in a series of functions on $\widehat{\mathscr{T}}$ indicating the possible refinement directions. The resulting adaptive algorithms behave reasonable and the authors observe the optimal convergence rate $\mathscr{O}(N^{-3/2})$, where N is the number of degrees of freedom. We refer to [11,74] as well as Section 6.10 for further details.

Hypersingular operator: For hypersingular integral equations, the analysis of $(h-h/2)$-type estimators is given in [42, 59] for $d = 2$ and the lowest-order case $p = 1$, and in [12] for $d = 3$ and general $p \geq 1$. The energy norm is given by

$$\|u\|_b^2 := \|u\|_W^2 := \begin{cases} \langle Wu\,, u\rangle_\Gamma & \text{for } \Gamma \subsetneq \partial\Omega \\ \langle Wu\,, u\rangle_\Gamma + \langle u\,, 1\rangle_\Gamma^2 & \text{for } \Gamma = \partial\Omega, \end{cases}$$

cf. Section 2. Again, Proposition 46 shows reliability and efficiency of the estimator.

Theorem 55 *Suppose that $u \in \widetilde{H}^{1/2}(\Gamma)$ is the exact solution of Proposition 6 or 7. Given a mesh $\mathscr{T}$ and its uniform refinement $\widehat{\mathscr{T}}$, denote by $U \in \widetilde{\mathscr{S}}^p(\mathscr{T})$ and $\widehat{U} \in \widetilde{\mathscr{S}}^p(\widehat{\mathscr{T}})$ the respective Galerkin approximations from Proposition 12 or 13. Then,*

$$\eta_{\mathscr{T}} := \|U - \widehat{U}\|_W \leq \|u - U\|_W,$$

i.e., the estimator $\eta_{\mathscr{T}}$ is efficient with $C_{\mathrm{eff}} = 1$. Under the saturation assumption 45, $\eta_{\mathscr{T}}$ is reliable, i.e.,

$$\|u - U\|_W \leq (1 - C_{\mathrm{sata}}^2)^{-1/2}\eta_{\mathscr{T}}.$$

The estimator $\eta_{\mathscr{T}}$ will be localized by the seminorm

$$\|\cdot\|_{h^s} = \|h_{\mathscr{T}}^{1/2}\nabla_\Gamma(\cdot)\|_{L_2(\Gamma)}.$$

Note first that $\|\cdot\|_b$ is an equivalent norm on $\widetilde{H}^{1/2}(\Gamma)$ by Theorem 2 and 3. Estimate (48) is valid due to the inverse estimate of Lemma 25 with $s = 1/2$, as long as Π is a projection. To show (47), Lemma 24 with $s = 1/2$ can be employed as long as Π is an $\widetilde{H}^{1/2}$ stable projection. Sections 3.2.1–3.2.3 present different projection operators that can be used in this context.

- The Scott-Zhang operator $\widetilde{J}_{\mathscr{T}}$ (resp. $J_{\mathscr{T}}$), which is an $\widetilde{H}^{1/2}(\Gamma)$ stable projection due to Lemma 26.
- For $d = 2$, the nodal interpolation operator $J_{\mathscr{T}}$, which is $\widetilde{H}^{1/2}(\Gamma)$ stable due to Lemma 27.
- On a sequence of meshes that is generated by certain mesh refinement rules, the the L_2 projection $\Pi_{\mathscr{T}}^p$ onto $\widetilde{\mathscr{S}}^p(\mathscr{T})$ can be shown to be stable in $\widetilde{H}^{1/2}(\Gamma)$. Present proofs for this property require certain restrictions on the mesh refinement and the polynomial degree, cf. Section 7 for details.

– For $d=3$, the nodal interpolation operator $J_{\mathscr{T}}$ is not $\widetilde{H}^{1/2}(\Gamma)$ stable. However, Lemma 28 with $s=1/2$ and $q=p$ can be used. Indeed, the estimate (47) is given explicitly in Lemma 28.

The resulting estimators and equivalences are summarized in the following Theorem, cf. [42,59] for $d=2, p=1$, and [12] for $d=3, p\geq 1$.

Theorem 56 *Denote by $P_{\mathscr{T}}$ either*

(a) the Scott-Zhang operator or
(b) the L_2 orthogonal projection onto $\mathscr{S}^p(\mathscr{T})$ (given that it is $\widetilde{H}^{1/2}(\Gamma)$ stable, cf. Section 7)
(c) the nodal interpolation operator.

Define the following a posteriori error estimators:

$$\eta_{\mathscr{T}} := \|\widehat{U}-U\|_W, \qquad \mu_{\mathscr{T}} := \|h_{\mathscr{T}}^{1/2}\nabla_\Gamma(\widehat{U}-U)\|_{L_2(\Gamma)},$$
$$\widetilde{\eta}_{\mathscr{T}} := \|(1-P_{\mathscr{T}})\widehat{U}\|_W, \quad \widetilde{\mu}_{\mathscr{T}} := \|h_{\mathscr{T}}^{1/2}\nabla_\Gamma(1-P_{\mathscr{T}})\widehat{U}\|_{L_2(\Gamma)}.$$

Then, it holds that

$$\eta_{\mathscr{T}} \leq \widetilde{\eta}_{\mathscr{T}} \leq C_W^{1/2} C_{\rm apx}\widetilde{\mu}_{\mathscr{T}},$$
$$\widetilde{\mu}_{\mathscr{T}} \leq C_{\rm stab}\mu_{\mathscr{T}} \leq C_{\rm inv}C_{\rm ell}^{-1/2}\eta_{\mathscr{T}},$$

where $C_W = \|W\|_{\widetilde{H}^{1/2}(\Gamma)\to H^{-1/2}(\Gamma)}$ and $C_{\rm ell}$ are the stability and ellipticity constants of the hypersingular operator W, $C_{\rm inv}$ is the constant of the inverse estimate of Lemma 25, and, depending on the choice of $P_{\mathscr{T}}$,

(a) $C_{\rm apx}$ is the constant of Lemma 24 and $C_{\rm stab}$ depends solely on the operator norm $\|J_{\mathscr{T}}\|_{H^1(\Gamma)}$, or
(b) $C_{\rm apx}$ is the constant of Lemma 24 and $C_{\rm stab} = C_{\rm inv}C_{\rm apx}$.
(c) For $d=2$, $C_{\rm apx}$ is the constant from Lemma 27 and $C_{\rm stab} = 1$, and for $d=3$, $C_{\rm apx}$ and $C_{\rm stab}$ are the constants from Lemma 28.

The estimators of the last theorem always apply an operator $P_{\mathscr{T}}$ to the solution $\widehat{U}$. However, as Lemma 28 shows, also the gradient $\nabla_\Gamma\widehat{U}$ could be projected locally on the coarse mesh, which is much cheaper.

Theorem 57 *Define the a posteriori error estimator*

$$\overline{\mu}_{\mathscr{T}} := \|h_{\mathscr{T}}^{1/2}(1-\pi_{\mathscr{T}}^{p-1})\nabla_\Gamma\widehat{U}\|_{L_2(\Gamma)}.$$

Then, it holds that

$$\eta_{\mathscr{T}} \leq C_W C_{\rm apx} C_{\rm stab}\overline{\mu}_{\mathscr{T}} \leq C_{\rm inv}C_{\rm ell}^{-1/2}\eta_{\mathscr{T}},$$

where $C_{\rm apx}, C_{\rm stab} > 0$ are the constants from Lemma 28, $C_{\rm inv} > 0$ is the constant of the inverse estimate of Lemma 25, and $C_W = \|W\|_{\widetilde{H}^{1/2}(\Gamma)\to H^{-1/2}(\Gamma)}$ and $C_{\rm ell}$ are the stability and ellipticity constants of the hypersingular operator W,

Remark 7 The concept of $((h-h/2))$ type error estimators has recently been extended to nonconforming boundary element methods for hypersingular integral equations, see [55, 90].

4.3 Averaging estimators

The advantage of space-enrichment based error estimators (Section 4.2) is that their implementation essentially only requires a simple postprocessing of the Galerkin data and the computed Galerkin solution. For the $(h-h/2)$-type error estimators from Section 4.2.2, one theoretical drawback is that the Galerkin solution has to computed on the fine-mesh $\widehat{\mathscr{T}}$, while the error estimators only estimates the coarse-mesh error, cf. Theorem 53 for the weakly singular integral equation and Theorem 55 for the hypersingular integral equation. Although the two-level error estimators from Section 4.2.1 avoid the computation of the fine-mesh solution, their computation requires the assembly of the fine-mesh Galerkin data. Since the latter is the most time consuming part of BEM computations, neither of these error estimators seems to be attractive at the first glance.

This section discusses error estimation by averaging on large patches. On an abstract level, the approach can be outlined as follows: Let $u\in\mathscr{X}$ denote the unknown exact solution of (3). Suppose that $\mathscr{T}$ is a given mesh with uniform refinement $\widehat{\mathscr{T}}$ and that we are given a space $\mathscr{X}(\widehat{\mathscr{T}})$ with low-order polynomials on the fine mesh and a space $\widehat{\mathscr{X}}(\mathscr{T})$ with higher-order polynomials on the coarse mesh. The goal is to derive a computable error estimator $\eta_{\mathscr{T}}$ which estimates the fine-mesh error $\|u-\widehat{U}\|_{\mathscr{X}}$ of the Galerkin solution $\widehat{U}\in\mathscr{X}(\widehat{\mathscr{T}})$ of (4) with $\mathscr{X}=\mathscr{X}(\widehat{\mathscr{T}})$. To that end, let $G:\mathscr{X}\to\widehat{\mathscr{X}}(\mathscr{T})$ denote the Galerkin projection, i.e., for all $w\in\mathscr{X}$, $Gw\in\widehat{\mathscr{X}}(\mathscr{T})$ is the unique solution of the linear system

$$b(Gw,v) = b(w,v) \quad \text{for all } v\in\widehat{\mathscr{X}}(\mathscr{T}). \tag{49}$$

With this notation, we define the computable error estimator

$$\eta_{\mathscr{T}} := \|(1-G)\widehat{U}\|_{\mathscr{X}}. \tag{50}$$

The following abstract theorem is found, e.g., in [41, Thm. 2.1].

Theorem 58 *Define the quantities*

$$q := \frac{\|(1-G)u\|_{\mathscr{X}}}{\|u-\widehat{U}\|_{\mathscr{X}}}, \tag{51}$$

$$\lambda := \max_{V\in\widehat{\mathscr{X}}(\mathscr{T})}\min_{\widehat{V}\in\mathscr{X}(\widehat{\mathscr{T}})}\frac{\|V-\widehat{V}\|_{\mathscr{X}}}{\|V\|_{\mathscr{X}}}. \tag{52}$$

Then, the error estimator $\eta_{\mathscr{T}}$ is efficient

$$\eta_{\mathscr{T}} \leq (C_{\rm cont}/C_{\rm ell}+q)\,\|u-\widehat{U}\|_{\mathscr{X}}. \tag{53}$$

Provided that the ellipticity and continuity constant of $b(\cdot,\cdot)$ satisfy $q+\lambda < C_{\rm ell}/C_{\rm cont}$, there also holds reliability

$$\|u-\widehat{U}\|_{\mathscr{X}} \leq \frac{C_{\rm cont}}{C_{\rm ell}-C_{\rm cont}(q+\lambda)}\,\eta_{\mathscr{T}}. \tag{54}$$

Proof Let $\langle\cdot\,,\cdot\rangle_{\mathscr{X}}$ denote the scalar product on the Hilbert space $\mathscr{X}$ which gives rise to the norm $\|\cdot\|_{\mathscr{X}}$. Recall that the Céa lemma (6) also applies for $\widehat{\mathscr{X}}(\mathscr{T})$ and hence G. The efficiency estimate (53) therefore follows from the triangle inequality

$$\begin{aligned}\eta_{\mathscr{T}} &\le \|(1-G)(u-\widehat{U})\|_{\mathscr{X}} + \|(1-G)u\|_{\mathscr{X}}\\ &\le (C_{\rm cont}/C_{\rm ell}+q)\,\|u-\widehat{U}\|_{\mathscr{X}}.\end{aligned}$$

For the proof of the reliability estimate (54), we define $e := u-\widehat{U}$. Let $E\in\widehat{\mathscr{X}}(\mathscr{T})$ denote the best approximation in $\widehat{\mathscr{X}}(\mathscr{T})$, i.e.,

$$\|e-E\|_{\mathscr{X}} = \min_{V\in\widehat{\mathscr{X}}(\mathscr{T})} \|e-V\|_{\mathscr{X}}. \tag{55}$$

Recall that E is then characterized by the orthogonality

$$\langle e-E\,,V\rangle_{\mathscr{X}} = 0 \quad \text{for all } V\in\widehat{\mathscr{X}}(\mathscr{T})$$

which implies the Pythagoras theorem

$$\|e-E\|_{\mathscr{X}}^2 + \|E\|_{\mathscr{X}}^2 = \|e\|_{\mathscr{X}}^2.$$

First, note that that the best approximation property (55) and the triangle inequality for $V = Gu + G\widehat{U}$ prove

$$\begin{aligned}C_{\rm cont}^{-1}\, b(e,e-E) &\le \|e\|_{\mathscr{X}}\,\|e-E\|_{\mathscr{X}}\\ &\le \|e\|_{\mathscr{X}}\,(\|(1-G)u\|_{\mathscr{X}} + \eta_{\mathscr{T}})\\ &\le \|e\|_{\mathscr{X}}\,(q\,\|e\|_{\mathscr{X}} + \eta_{\mathscr{T}}).\end{aligned}$$

Second, observe that by definition of λ the Galerkin orthogonality for $e = u-\widehat{U}$ as well as the estimate $\|E\|_{\mathscr{X}}\le\|e\|_{\mathscr{X}}$ prove

$$\begin{aligned}C_{\rm cont}^{-1}\, b(e,E) = C_{\rm cont}^{-1} \min_{\widehat{V}\in\mathscr{X}(\widehat{\mathscr{T}})} b(e,E-\widehat{V}) &\le \lambda\,\|e\|_{\mathscr{X}}\|E\|_{\mathscr{X}}\\ &\le \lambda\,\|e\|_{\mathscr{X}}^2.\end{aligned}$$

Altogether, we see

$$\begin{aligned}C_{\rm ell}\|e\|_{\mathscr{X}}^2 \le b(e,e) &= b(e,e-E) + b(e,E)\\ &\le C_{\rm cont}(q+\lambda)\|e\|_{\mathscr{X}}^2 + C_{\rm cont}\,\eta_{\mathscr{T}}\,\|e\|_{\mathscr{X}}.\end{aligned}$$

Rearranging this estimate, we conclude the proof. □

In practice, higher-order polynomials lead to higher-order convergence rates if the unknown solution u is smooth or if the mesh $\mathscr{T}$ is appropriately graded. Therefore, one may expect that the constant q from (51) satisfies $q\to 0$ if the mesh is adaptively refined. The constant λ from (52) satisfies $0\le\lambda\le 1$ by definition. Geometrically, $\lambda<1$ corresponds to a strengthened Cauchy inequality, cf. [41, Sect 4]. In practice, $\lambda\ll 1$ follows if the mesh $\widehat{\mathscr{T}}$ is sufficiently fine with respect to $\mathscr{T}$. We refer to the discussion below. In conclusion, the assumption $q+\lambda < C_{\rm ell}/C_{\rm cont}$ required for the reliability estimate (54) can be satisfied in practice.

As for the $(h-h/2)$-error estimator from Section 4.2.2, a practical BEM application has, first, to replace the non-local norm $\|\cdot\|_{\mathscr{X}}$ by some easily computable local norm, e.g., some locally weighted L_2-norm resp. H^1-seminorm. Moreover, the computationally expensive Galerkin projection G has to be replaced by some numerically cheaper operator $\Pi:\mathscr{X}(\widehat{\mathscr{T}})\to\widehat{\mathscr{X}}(\mathscr{T})$. Both aspects are discussed for the weakly singular and hypersingular model problem in the following subsections.

We finally note that for our applications, i.e., weakly singular and hypersingular integral equation, averaging on large patches turns out to be equivalent to $(h-h/2)$-type error estimation.

Weakly singular operator: Averaging on large patches for weakly singular integral equations in 2D and 3D BEM has first been proposed and analyzed in [40]. We also refer to [11] for the discussion on anisotropic mesh refinement. In [40], it holds $\mathscr{X}(\widehat{\mathscr{T}}) = \mathscr{P}^p(\widehat{\mathscr{T}})$ and $\widehat{\mathscr{X}}(\mathscr{T}) = \mathscr{P}^{p+1}(\mathscr{T})$. We suppose that $\widehat{\mathscr{T}}$ is obtained from k uniform refinements of $\mathscr{T}$, i.e., the corresponding mesh-sizes satisfy

$$\widehat{h} = 2^{-k}h. \tag{56}$$

Let $\pi^p_{\widehat{\mathscr{T}}}$ be the L_2-projection onto $\mathscr{P}^p(\widehat{\mathscr{T}})$. Fix $V\in\mathscr{P}^{p+1}(\mathscr{T})$. The approximation estimate from Lemma 22 yields

$$\begin{aligned}\min_{\widehat{V}\in\mathscr{P}^p(\widehat{\mathscr{T}})} \|V-\widehat{V}\|_{\widetilde{H}^{-1/2}(\Gamma)} &\le \|(1-\pi^p_{\widehat{\mathscr{T}}})V\|_{\widetilde{H}^{-1/2}(\Gamma)}\\ &\lesssim \|\widehat{h}^{1/2}V\|_{L_2(\Gamma)} \le \|(\widehat{h}/h)^{1/2}\|_{L^\infty(\Gamma)}\,\|h^{1/2}V\|_{L_2(\Gamma)}.\end{aligned}$$

By choice of $\widehat{\mathscr{T}}$, it holds $\|(\widehat{h}/h)^{1/2}\|_{L^\infty(\Gamma)}\le 2^{-k/2}$. The inverse estimate of Lemma 23 proves

$$\|h^{1/2}V\|_{L_2(\Gamma)} \lesssim \|V\|_{\widetilde{H}^{-1/2}(\Gamma)}.$$

Combining these observations, we see that the constant λ from (52) satisfies, for k sufficiently large,

$$\lambda := \max_{V\in\mathscr{P}^{p+1}(\mathscr{T})}\ \min_{\widehat{V}\in\mathscr{P}^p(\widehat{\mathscr{T}})} \frac{\|V-\widehat{V}\|_{\widetilde{H}^{-1/2}(\Gamma)}}{\|V\|_{\widetilde{H}^{-1/2}(\Gamma)}} \lesssim 2^{-k/2} \ll 1,$$

where the hidden constant depends only on Γ, shape regularity of $\mathscr{T}$, and the polynomial degree p. Moreover, standard approximation results prove (see e.g. [123]) that, at least for smooth solutions u, the constant q from (51) satisfies $q = \mathscr{O}(h^{p+5/2}/\widehat{h}^{p+3/2}) = \mathscr{O}(h)$.

The following theorem is first found in [40, Sect 5] and formulated in the energy norm $\|\cdot\|_V \simeq \|\cdot\|_{\widetilde{H}^{-1/2}(\Gamma)}$. Note that $\eta_{\mathscr{T}}$ corresponds to the abstract error estimator $\eta_{\mathscr{T}}$ from the abstract Theorem 58. Since the proof is similar to that of Theorem 54, we omit the details.

Theorem 59 *Let $\pi_{\mathscr{T}}^{p+1}$ denote the L_2-orthogonal projection onto $\mathscr{P}^{p+1}(\mathscr{T})$. Let $G_{\mathscr{T}}^{p+1}$ denote the Galerkin projection* (49) *onto $\mathscr{P}^{p+1}(\mathscr{T})$. Then, the estimators*

$$\eta_{\mathscr{T}} := \|(1-G_{\mathscr{T}}^{p+1})\widehat{\Phi}\|_V, \quad \mu_{\mathscr{T}} := \|h_{\mathscr{T}}^{1/2}(1-G_{\mathscr{T}}^{p+1})\widehat{\Phi}\|_{L_2(\Gamma)},$$
$$\widetilde{\eta}_{\mathscr{T}} := \|(1-\pi_{\mathscr{T}}^{p+1})\widehat{\Phi}\|_V, \quad \widetilde{\mu}_{\mathscr{T}} := \|h_{\mathscr{T}}^{1/2}(1-\pi_{\mathscr{T}}^{p+1})\widehat{\Phi}\|_{L_2(\Gamma)},$$

satisfy the equivalence estimates

$$\eta_{\mathscr{T}} \le \widetilde{\eta}_{\mathscr{T}} \le C_V^{1/2} C_{\mathrm{apx}} \widetilde{\mu}_{\mathscr{T}},$$
$$\widetilde{\mu}_{\mathscr{T}} \le \mu_{\mathscr{T}} \le 2^{k/2} C_{\mathrm{inv}} C_{\mathrm{ell}}^{-1/2} \eta_{\mathscr{T}},$$

where $C_V = \|V\|_{\widetilde{H}^{-1/2}(\Gamma)\to H^{1/2}(\Gamma)}$ and C_{ell} are the stability and ellipticity constants of the single layer operator V, C_{apx} is the constant of Lemma 22, and C_{inv} is the constant of the inverse estimate of Lemma 23.

The numerical experiments in [40,41,58,11] give empirical evidence that $k = 2$ seems to be sufficient in practice. As first observed in [58, Thm. 5.3] for lowest-order 2D BEM $p = 0$, one can prove that averaging on large patches is equivalent to $(h-h/2)$-error estimation. The argument also transfers to 3D and arbitrary polynomial degree $p \ge 0$.

Corollary 60 *For all $T \in \mathscr{T}$, it holds*

$$\begin{aligned}\|(1-\pi_{\mathscr{T}}^{p+1})\widehat{\Phi}\|_{L_2(T)} &\le \|(1-\pi_{\mathscr{T}}^{p})\widehat{\Phi}\|_{L_2(T)} \\ &\le C_{\mathrm{equiv}} \|(1-\pi_{\mathscr{T}}^{p+1})\widehat{\Phi}\|_{L_2(T)},\end{aligned} \tag{57}$$

where the constant C_{equiv} depends only on the polynomial degree p. Comparing the error estimators $\widetilde{\mu}_{\mathscr{T}}$ of Theorem 54 and Theorem 59, this proves that all eight error estimators are equivalent. In particular, the estimate (57) *shows that the equivalence of the respective $\widetilde{\mu}_{\mathscr{T}}$ estimators holds even elementwise.*

Proof The lower bound in (57) follows from the local best approximation property

$$\|(1-\pi_{\mathscr{T}}^{p+1})\widehat{\Phi}\|_{L_2(T)} = \min_{\Psi\in\mathscr{P}^{p+1}(T)} \|\widehat{\Phi}-\Psi\|_{L_2(T)}$$

of the L_2-projection $\pi_{\mathscr{T}}^{p+1}$ and nestedness $\mathscr{P}^p(T) \subseteq \mathscr{P}^{p+1}(T)$. To prove the upper bound in (57), observe that

$$\|(1-\pi_{\mathscr{T}}^{p+1})\widehat{\Phi}\|_{L_2(T)} = 0 \quad \Longleftrightarrow \quad \|(1-\pi_{\mathscr{T}}^{p})\widehat{\Phi}\|_{L_2(T)} = 0.$$

Therefore, the equivalence follows from scaling arguments and equivalence of seminorms on finite dimensional spaces. □

Hypersingular operator: For hypersingular integral equations, averaging on large patches has been proposed and analyzed for lowest-order 2D BEM in [42]. The equivalence of $(h-h/2)$-type error estimators (cf. Theorem 56) and averaging on large patches has been proved in [59]. These results have been generalized to 3D BEM and arbitrary polynomial order $p \ge 1$ in [12]. Altogether, the results from Theorem 59 and Corollary 60 hold accordingly. For these reasons, we leave the details to the reader and refer to the given references.

4.4 ZZ-type error estimator

The idea of the ZZ-type error estimator (in the context of FEM also *gradient recovery* estimator) is to *recover* a smoother approximation of the computed solution and to compare it with the discrete solution. Since the seminal work [151], the ZZ-type error estimators for FEM became very popular within the engineering community due to their implementational ease. Although ZZ-type error estimators are mathematically well-developed for FEM, see e.g. [18,35,30,122], there was no theory for BEM until [67] which treats the 2D case and lowest-order elements. In our presentation, we extend the approach to $d = 2,3$ but stick with lowest-order elements $p = 0$ for weakly singular integral equations resp. $p = 1$ for hypersingular integral equations.

Weakly singular operator: The ZZ-type error estimator from [67] reads

$$\eta_\ell^2 := \sum_{T\in\mathscr{T}_\ell} \eta_\ell(T)^2 := \sum_{T\in\mathscr{T}_\ell} h_T \|(1-A_\ell)\Phi_\ell\|_{L_2(T)}^2,$$

where the smoothing operator $A_\ell : L_2(\Gamma) \to \mathscr{P}^1(\mathscr{T}_\ell)$ is defined as follows: Let z denote a node of $\mathscr{T}_\ell$ and let $\omega_z := T_1 \cup \ldots \cup T_{\#\omega_z}$ be the node patch.

- If the normal vector of Γ does not jump at z, define

$$(A_\ell \psi)(z) := |\omega_z|^{-1} \int_{\omega_z} \psi \, dz. \tag{58}$$

- If the normal vector of Γ jumps at z, find sets $C_1, \ldots, C_{m_z}$, with $m_z \le \#\omega_z$ and $\bigcup_{i=1}^{m_z} C_i = \omega_z$ such that the normal vector does not jump on the C_i, $i = 1, \ldots, m_z$. Then, define for all $i = 1, \ldots, m_z$

$$(A_\ell \psi)|_{C_i}(z) := |C_i|^{-1} \int_{C_i} \psi \, dz. \tag{59}$$

This definition is useful since Φ_ℓ approximates a normal derivative and is supposed to jump at corners and edges of Γ.

Remark 8 For $d = 2$, the definition of A_ℓ simplifies as one only has to check if the normal vector jumps at a given node. Then, one integrates separately over the two adjacent elements. The search for continuity components C_i is no longer required. Also for $d = 3$, one may save some implementational efforts by just setting $C_i = \overline{T_i}$ for all $T_i \subseteq \omega_z$. This might not be the optimal solution, but still works in practice.

Theorem 61 *Let $\mathscr{T}_\ell$ be the uniform refinement of some mesh $\mathscr{T}_\ell'$. Then, there holds*

$$\|\Phi_\ell - \Phi_\ell'\|_{\widetilde{H}^{-1/2}(\Gamma)} \le C_{\mathrm{ZZ}} \eta_\ell$$

for the corresponding Galerkin solutions Φ_ℓ and Φ_ℓ'. Under the saturation assumption (Assumption 45), this implies

$$\|\phi - \Phi_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)} \le \widetilde{C}_{\mathrm{ZZ}} \eta_\ell.$$

The constant $C_{\mathrm{ZZ}} > 0$ depends only on Γ and all possible shapes of element patches in $\mathscr{T}_\ell$, while $\widetilde{C}_{\mathrm{ZZ}} > 0$ depends additionally on C_{sata} from Assumption 45.

Proof The complete proof for the 2D situation can be found in [67, Thm. 5]. Here, we only provide a brief sketch. First, we use Theorem 54 to see

$$\|\boldsymbol{\Phi}_\ell - \boldsymbol{\Phi}'_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)} \simeq \|h_\ell^{1/2}(1-\pi_\ell^{0\prime})\boldsymbol{\Phi}_\ell\|_{L_2(\Gamma)},$$

where $\pi_\ell^{0\prime} : L_2(\Gamma) \to \mathscr{P}^0(\mathscr{T}'_\ell)$. With the element patch $\omega_T := \bigcup\{T' \in \mathscr{T}_\ell : \overline{T} \cap \overline{T}' \neq \emptyset\}$, the elementwise estimate

$$\|h_\ell^{1/2}(1-\pi_\ell^{0\prime})\boldsymbol{\Phi}_\ell\|_{L_2(T)}^2 \lesssim \sum_{T' \subseteq \omega_T} \eta_\ell(T')^2$$

then follows by scaling arguments, and the hidden constant depends on the number of different patch shapes of $\mathscr{T}_\ell$. This proves

$$\|\boldsymbol{\Phi}_\ell - \boldsymbol{\Phi}'_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim \eta_\ell.$$

Under the saturation assumption, we derive

$$\|\phi - \boldsymbol{\Phi}_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim \|\phi - \boldsymbol{\Phi}'_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim \|\boldsymbol{\Phi}_\ell - \boldsymbol{\Phi}'_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)}.$$

This concludes the proof. □

Theorem 62 *There holds*

$$C_{\mathrm{ZZ}}^{-1}\eta_\ell \le \|\phi - \boldsymbol{\Phi}_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)} + \min_{\boldsymbol{\Psi} \in \mathscr{S}^1(\mathscr{T}_\ell)} \|\phi - \boldsymbol{\Psi}\|_{\widetilde{H}^{-1/2}(\Gamma)}.$$

The constant $C_{\mathrm{ZZ}} > 0$ depends only on Γ and all possible patch shapes of $\mathscr{T}_\ell$.

Proof The complete proof can be found in [67, Thm. 7]. Here, we only provide a brief sketch. Elementwise arguments show

$$\eta_\ell \lesssim \|\boldsymbol{\Phi}_\ell - \boldsymbol{\Psi}\|_{\widetilde{H}^{-1/2}(\Gamma)}$$

for all $\boldsymbol{\Psi} \in \mathscr{S}^1(\mathscr{T}_\ell)$. With this, we obtain

$$\begin{aligned}\eta_\ell &\lesssim \min_{\boldsymbol{\Psi} \in \mathscr{S}^1(\mathscr{T}_\ell)} \|\boldsymbol{\Phi}_\ell - \boldsymbol{\Psi}\|_{\widetilde{H}^{-1/2}(\Gamma)} \\ &\le \min_{\boldsymbol{\Psi} \in \mathscr{S}^1(\mathscr{T}_\ell)} \|\phi - \boldsymbol{\Psi}\|_{\widetilde{H}^{-1/2}(\Gamma)} + \|\phi - \boldsymbol{\Phi}_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)}.\end{aligned}$$

This concludes the proof. □

Hypersingular operator: The ZZ-type error estimator from [67] reads

$$\eta_\ell^2 := \sum_{T \in \mathscr{T}_\ell} \eta_\ell(T)^2 := \sum_{T \in \mathscr{T}_\ell} h_T \|(1-A_\ell)\nabla U_\ell\|_{L_2(T)}^2,$$

where the smoothing operator $A_\ell : \big(L_2(\Gamma)\big)^d \to \big(\mathscr{S}^1(\mathscr{T}_\ell)\big)^d$ is defined nodewise by

$$(A_\ell\psi)(z) := |\omega_z|^{-1}\int_{\omega_z} \psi\, dz$$

for all nodes z of $\mathscr{T}_\ell$. The difference to the weakly singular case is the fact that $A_\ell\psi \in \mathscr{S}^1(\mathscr{T}_\ell)$ is continuous on Γ, independently of jumps of the normal vector.

Theorem 63 *Let $\mathscr{T}_\ell$ be the uniform refinement of some mesh $\mathscr{T}'_\ell$. Then, there holds*

$$\|U_\ell - U'_\ell\|_{\widetilde{H}^{1/2}(\Gamma)} \le C_{\mathrm{ZZ}}\eta_\ell.$$

Under the saturation assumption (Assumption 45), this implies

$$\|u - U_\ell\|_{\widetilde{H}^{1/2}(\Gamma)} \le \widetilde{C}_{\mathrm{ZZ}}\eta_\ell.$$

The constant $C_{\mathrm{ZZ}} > 0$ depends only on Γ and all possible shapes of element patches in $\mathscr{T}_\ell$, while $\widetilde{C}_{\mathrm{ZZ}} > 0$ depends additionally on C_{sata} from Assumption 45.

Proof The proof is similar to the weakly singular case in Theorem 62 and can be found in [67, Thm. 1]. □

Theorem 64 *There holds*

$$C_{\mathrm{ZZ}}^{-1}\eta_\ell \le \|u - U_\ell\|_{H^{1/2}(\Gamma)} + \min_{V \in \widetilde{\mathscr{S}}^{2,1}(\mathscr{T}_\ell)} \|u - V\|_{\widetilde{H}^{-1/2}(\Gamma)}.$$

The space $\widetilde{\mathscr{S}}_0^{2,1}(\mathscr{T}_\ell) := \mathscr{S}^2(\mathscr{T}_\ell) \cap C^1(\Gamma) \cap \widetilde{H}^{1/2}(\Gamma)$ denotes the space of all piecewise quadratics which are globally differentiable with zero trace at $\partial\Gamma$. If Γ is closed, i.e. $\partial\Gamma = \emptyset$, we have $\widetilde{\mathscr{S}}^{2,1}(\mathscr{T}_\ell) = \mathscr{S}^{2,1}(\mathscr{T}_\ell)$. The constant $C_{\mathrm{ZZ}} > 0$ depends only on Γ and all possible patch shapes of $\mathscr{T}_\ell$.

Proof The proof is similar to the weakly singular case in Theorem 62 and can be found in [67, Thm. 3] for $d = 2$. □

4.5 Two-equation estimators

In this section, we consider only the Dirichlet- or Neumann problem, i.e., Γ is always the boundary of a bounded domain. In these cases, there is a method that differs completely from residual-based methods or approaches based on space enrichment.

In [124, 125, 132] it is shown that the error of, e.g., the Dirichlet problem, i.e., $\phi - \boldsymbol{\Phi}$, fulfills a second-kind integral equation. Indeed, the representation formula allows to define a potential based on the approximate Neumann data and the exact Dirichlet data via $\widetilde{u} = \widetilde{V}\Phi - \widetilde{K}g$. The trace and normal derivative of this potential fulfill

$$\gamma_0\widetilde{u} = V\Phi + (1/2-K)g$$
$$V\gamma_1\widetilde{u} = (1/2+K)\gamma_0\widetilde{u},$$

where γ_0 denotes the trace operator and γ_1 denotes the (co-) normal derivative. The combination of this equations and the identities

$$(1/2+K)(1/2-K) = VW, \quad KV = VK'$$

show

$$V\gamma_1\widetilde{u} = V(1/2+K')\Phi + VWg,$$

and as $Wg = (1/2-K')\phi$, this yields the second-kind integral equation for the error

$$\gamma_1\widetilde{u} - \Phi = (1/2-K')(\phi - \Phi),$$

cf. [124, Lemma 2.1]. This equation has to be solved approximately in $H^{-1/2}(\Gamma)$ and the corresponding norm of the solution to be localized. The approximate solution is based on the following observation, which is proved in [136, Thm. 3.1].

Theorem 65 *There is a constant $C_{\rm cnt} < 1$, such that for all $\phi \in H^{-1/2}(\Gamma)$ holds*

$$\|(1/2+K')\phi\|_V \le C_{\rm cnt}\|\phi\|_V.$$

Remark 9 The notation used in this section is bounded to Galerkin methods. Error estimators of the type presented here do not use orthogonality and can therefore be defined also for collocation or qualocation methods, where, instead of the factor $1/2$, a function has to be used which represents the curvature of the boundary.

According to the last theorem, the Neumann series

$$(1/2-K')^{-1} = \sum_{j=0}^{\infty}(1/2+K')^j$$

converges in the norm $\|\cdot\|_V$, so that one may define for $J \in \mathbb{N}_0$ the global error estimator

$$\eta^{(J)} := \|\sum_{j=0}^{J}(1/2+K')^j(\gamma_1\widetilde{u} - \Phi)\|_V.$$

Due to representation via a Neumann series, the estimator is efficient and reliable, as is shown in [124].

Theorem 66 *The estimator $\eta^{(J)}$ is efficient and reliable,*

$$\frac{1}{1+C_{\rm cnt}^{J+1}}\eta^{(J)} \le \|\phi - \Phi\|_V \le \frac{1}{1-C_{\rm cnt}^{J+1}}\eta^{(J)}$$

The same arguments also apply for the Neumann problem, where the error estimator for an approximation U is defined by

$$\eta^{(J)} := \|\sum_{j=0}^{J}\big(P(1/2-K)^jP(\gamma_0\widetilde{u} - U)\big)\|_W,$$

cf. [125, 132], where P is an operator that ensures vanishing integral mean. The analogue to Theorem 66 is of course valid.

On the implementational side, one has to introduce an approximation of the application of the Neumann series. If, e.g., $\Phi \in \mathscr{P}^p(\mathscr{T})$ is an approximation of the solution of a Dirichlet problem, the $L_2(\Gamma)$-projection π on a space finer than $\mathscr{P}^p(\mathscr{T})$ may be used to compute

$$\widetilde{\eta}^{(J)} := \|\sum_{j=0}^{J}\big(\pi(1/2+K')\big)^j\pi(\gamma_1\widetilde{u} - \Phi)\|_V,$$

in which case the following result is valid, cf. [124, Thm. 3.3].

Theorem 67 *Let $(\mathscr{T}_\ell)_{\ell\in\mathbb{N}_0}$ be a uniform sequence of meshes with mesh-width h_ℓ and $(\widehat{\mathscr{T}_\ell})_{\ell\in\mathbb{N}_0}$ be a uniform sequence of meshes with mesh-width $\widehat{h}_\ell$ such that $\mathscr{T}_\ell \subseteq \widehat{\mathscr{T}_\ell}$ for $\ell \in \mathbb{N}_0$. If $\Phi_\ell \in \mathscr{P}^p(\mathscr{T}_\ell)$ is an approximation to the exact solution ϕ of a Dirichlet problem with data g, $\widetilde{u}_\ell := \widetilde{V}\Phi_\ell - \widetilde{K}g$, and $\widehat{\pi}_\ell : L_2 \to \mathscr{P}^p(\widehat{\mathscr{T}_\ell})$ is the L_2 orthogonal projection, the estimator*

$$\widetilde{\eta}_\ell^{(J)} := \|\sum_{j=0}^{J}\big(\widehat{\pi}_\ell(1/2+K')\big)^j\widehat{\pi}_\ell(\gamma_1\widetilde{u}_\ell - \Phi_\ell)\|_V,$$

is efficient and reliable in the sense that there exists a constant $C_3 > 0$ such that

$$\frac{1}{1+C_{\rm cnt}^{J+1}}\left\{\widetilde{\eta}_\ell^{(J)} - C_3J\widehat{h}_\ell\eta_\ell^{(0)}\right\} \le \|\phi - \Phi_\ell\|_V$$
$$\le \frac{1}{1-C_{\rm cnt}^{J+1}}\left\{\widetilde{\eta}_\ell^{(J)} + C_3J\widehat{h}_\ell\eta_\ell^{(0)}\right\}$$

The localization of $\widetilde{\eta}_\ell^{(J)}$ in this case is a more subtle matter, as no orthogonality or approximation property can be used. Indeed, the derivation of this type of estimator assumed no whatsoever special approximation property. In [124] the authors use the localization

$$\left(\widetilde{\eta}_\ell^{(J)}\right)^2 = \sum_{T\in\mathscr{T}_\ell}\widetilde{\eta}_T^{(J)}$$

where

$$\widetilde{\eta}_T^{(J)} = \langle Ve_\ell^{(J)}, e_\ell^{(J)}\rangle_T,$$
$$e_\ell^{(J)} = \sum_{j=0}^{J}\big(\pi(1/2+K')\big)^j\pi(\gamma_1\widetilde{u} - \Phi_\ell)$$

This is indeed not a fully localized estimator since it involves the single layer operator V. In [132], it is suggested to use the multilevel localization from Section 3.3. With the notation from Theorem 67, define the operator

$$A_\ell^s \phi := \sum_{k=0}^{\ell} h_k^{-2s}(\pi_k - \pi_{k-1})\phi + \widehat{h}_\ell^{-2s}(\widehat{\pi}_\ell - \pi_\ell)\phi$$

and note that for $\widehat{\Phi}_\ell \in \mathscr{P}^0(\widehat{\mathscr{T}_\ell})$ holds

$$\begin{aligned}\sum_{k=0}^{\ell} h_k^{-2s}\|(\pi_k - \pi_{k-1})\Phi_\ell\|_{\ell_2(\Gamma)}^2 + \widehat{h}_\ell^{-2s}\|(\widehat{\pi}_\ell - \pi_\ell)\Phi_\ell\|_{\ell_2(\Gamma)}^2 \\ = \langle A_\ell^s \widehat{\Phi}_\ell, \widehat{\Phi}_\ell\rangle_{\ell_2(\Gamma)} = \langle A_\ell^{s/2}\widehat{\Phi}_\ell, A_\ell^{s/2}\widehat{\Phi}_\ell\rangle_{\ell_2(\Gamma)},\end{aligned}$$

where the last identity follows from the properties of the π_k, cf. [132, Prop. 2.1]. Finally, Theorem 29 states that the $H^{-1/2}$ norm of $e_\ell^{(J)}$ can be bounded by

$$\begin{aligned}&\langle A_\ell^{-1/4} e_\ell^{(J)}, A_\ell^{-1/4} e_\ell^{(J)}\rangle_{\ell_2(\Gamma)} \\ &\quad = \sum_{T\in\mathscr{T}_\ell} \langle A_\ell^{-1/4} e_\ell^{(J)}, A_\ell^{-1/4} e_\ell^{(J)}\rangle_{\ell_2(T)},\end{aligned}$$

via

$$\sum_{T\in\mathscr{T}_\ell} \eta_T^2 \lesssim \left(\widetilde{\eta}_\ell^{(J)}\right)^2 \lesssim (\ell+2)^2 \sum_{T\in\mathscr{T}_\ell} \eta_T^2.$$

4.6 A posteriori error control of data approximation

In practice, the right-hand side F of (4) cannot be computed analytically. For the weakly singular integral equation from Proposition 5, it holds for instance $F = (K+1/2)f$ for some given $f \in H^{1/2}(\Gamma)$. For the hypersingular integral equation from Proposition 7, it holds for instance $F = (K'-1/2)f$ for some given $f \in H_0^{-1/2}(\Gamma)$. In either case, the action of the integral operator to the continuous data f is well-defined, but hardly computable. In practice, the given data f is therefore replaced by some piecewise polynomial data f_ℓ. This leads to a computable right-hand side for the Galerkin discretization (4), where F is replaced by some approximation F_ℓ. The following short sections give insight in how to control this additional approximation error.

4.6.1 Inverse estimates for integral operators

The following inverse-type estimates have independently first been shown in [70,76] for piecewise polynomials. While [70] considered lowest-order polynomials on piecewise polygonal geometries, [76] covers arbitrary-order piecewise polynomials but is restricted to smooth boundaries Γ. In [5], the results of [70,76] are generalized to general densities instead of piecewise polynomials.

Lemma 68 *There exists a constant $C_4 > 0$ such that for all $v \in \widetilde{H}^1(\Gamma)$ and all $\psi \in L_2(\Gamma)$*

$$\begin{aligned}\|h_\ell^{1/2}\nabla_\Gamma V\psi\|_{L_2(\Gamma)} + \|h_\ell^{1/2}(1/2-K')\psi\|_{L_2(\Gamma)} \\ \le C_4\big(\|\psi\|_{\widetilde{H}^{-1/2}(\Gamma)} + \|h_\ell^{1/2}\psi\|_{L_2(\Gamma)}\big), \\ \|h_\ell^{1/2}Wv\|_{L_2(\Gamma)} + \|h_\ell^{1/2}\nabla_\Gamma(1/2+K)v\|_{L_2(\Gamma)} \\ \le C_4\big(\|v\|_{\widetilde{H}^{1/2}(\Gamma)} + \|h_\ell^{1/2}\nabla_\Gamma v\|_{L_2(\Gamma)}\big).\end{aligned} \tag{60}$$

The constant C_4 depends only on the shape regularity of $\mathscr{T}_\ell$ and on Γ. In the special case $v = V_\ell \in \widetilde{\mathscr{S}^p}(\mathscr{T}_\ell)$ and $\psi = \Psi_\ell \in \mathscr{P}^p(\mathscr{T}_\ell)$, there even holds

$$\begin{aligned}\|h_\ell^{1/2}\nabla_\Gamma V\Psi_\ell\|_{L_2(\Gamma)} + \|h_\ell^{1/2}(1/2-K')\Psi_\ell\|_{L_2(\Gamma)} \\ \le C_5\|\Psi_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)}, \\ \|h_\ell^{1/2}WV_\ell\|_{L_2(\Gamma)} + \|h_\ell^{1/2}\nabla_\Gamma(1/2+K)V_\ell\|_{L_2(\Gamma)} \\ \le C_5\|V_\ell\|_{\widetilde{H}^{1/2}(\Gamma)}.\end{aligned} \tag{61}$$

The constant $C_5 >$ depends only on Γ, the shape regularity of $\mathscr{T}_\ell$, and on the polynomial degree p.

4.6.2 Weakly singular integral equation

We show two eligible ways of data approximation for the weakly singular integral equation from Proposition 10 with $\Gamma = \partial\Omega$. First, the approximation for the right-hand side F can be done via the Scott-Zhang projection $J_\ell : L_2(\Gamma) \to \mathscr{S}^{p+1}(\mathscr{T}_\ell)$ from Section 3.2.2, i.e.

$$F_\ell := (1/2+K)J_\ell f, \tag{62}$$

or via the L_2-orthogonal projection $\Pi_\ell^{p+1} : L_2(\Gamma) \to \mathscr{S}^{p+1}(\mathscr{T}_\ell)$ from Section 3.2.1, i.e.,

$$F_\ell := (1/2+K)\Pi_\ell^{p+1} f, \tag{63}$$

where we additionally assume that Π_ℓ^{p+1} is H^1-stable (cf., Section 3.2.1). In the following, we denote with P_ℓ either the Scott-Zhang projection $P_\ell = J_\ell$ or the L_2-orthogonal projection $P_\ell = \Pi_\ell^{p+1}$. Let $\widetilde{\Phi}_\ell \in \mathscr{P}^p(\mathscr{T}_\ell)$ denote the solution of (4) with right-hand side (62) or (63). The introduced approximation error can be controlled with the following result

Lemma 69 *There exists a constant $C_6 > 0$ such that*

$$C_6^{-1}\|\Phi_\ell - \widetilde{\Phi}_\ell\|_{H^{-1/2}(\Gamma)} \le \|h_\ell^{1/2}\nabla_\Gamma(1-P_\ell)f\|_{L_2(\Gamma)}. \tag{64a}$$

Moreover, there exists a constant $C_7 > 0$ such that

$$C_7^{-1}\|\Phi_\ell - \widetilde{\Phi}_\ell\|_{H^{-1/2}(\Gamma)} \le \|h_\ell^{1/2}(1-\pi_\ell^p)\nabla_\Gamma f\|_{L_2(\Gamma)}. \tag{64b}$$

The constant C_6 depends only on Γ, the shape regularity of $\mathscr{T}_\ell$, and on the polynomial degree p. The constant C_7 depends additionally on all possible shapes of element patches in $\mathscr{T}_\ell$.

Proof The Galerkin formulation (4) shows

$$b(\Phi_\ell - \widetilde{\Phi}_\ell\,,\Phi_\ell - \widetilde{\Phi}_\ell) = \langle (1/2+K)(f-P_\ell f)\,,\Phi_\ell - \widetilde{\Phi}_\ell\rangle_\Gamma \lesssim \|f-P_\ell f\|_{H^{1/2}(\Gamma)}\|\Phi_\ell - \widetilde{\Phi}_\ell\|_{H^{-1/2}(\Gamma)},$$

where we used the stability of K from Theorem 2. With ellipticity from Theorem 3, this yields

$$\|\Phi_\ell - \widetilde{\Phi}_\ell\|_{H^{-1/2}(\Gamma)} \lesssim \|f-P_\ell f\|_{H^{1/2}(\Gamma)}.$$

We use the assumption on H^1-stability on Π_ℓ^{p+1} or, in case of $P_\ell = J_\ell$, the H^1-stability of J_ℓ from Lemma 26. With Lemma 24, it holds

$$\|\Phi_\ell - \widetilde{\Phi}_\ell\|_{H^{-1/2}(\Gamma)} \lesssim \min_{V_\ell\in\mathscr{S}^{p+1}(\mathscr{T}_\ell)} \|h_\ell^{1/2}\nabla_\Gamma(f-V_\ell)\|_{L_2(\Gamma)}.$$

This shows (64a). For (64b), the result [8, Proposition 8] finally shows

$$\min_{V_\ell\in\mathscr{S}^{p+1}(\mathscr{T}_\ell)} \|h_\ell^{1/2}\nabla_\Gamma(f-V_\ell)\|_{L_2(\Gamma)} \lesssim \|h_\ell^{1/2}(1-\pi_\ell^p)\nabla_\Gamma f\|_{L_2(\Gamma)},$$

where the hidden constant depends on the shapes of the element patches in $\mathscr{T}_\ell$. This concludes the proof. □

Also the error estimators satisfy certain stability properties.

Lemma 70 *Let η_ℓ denote one of the $(h-h/2)$-type error estimators defined in Theorem 53 or the weighted residual error estimator from Theorem 38. Moreover, let $\widetilde{\eta}_\ell$ denote the perturbed version of the respective error estimator computed with the perturbed Galerkin approximation $\widetilde{\Phi}_\ell$ and the perturbed data F_ℓ. Then, there exists a constant $C_8 > 0$ such that*

$$|\widetilde{\eta}_\ell - \eta_\ell| \le C_8\|h_\ell^{1/2}\nabla_\Gamma(1-P_\ell)f\|_{L_2(\Gamma)}. \tag{65a}$$

Moreover, there exists a constant $C_9 > 0$ such that

$$|\widetilde{\eta}_\ell - \eta_\ell| \le C_9\|h_\ell^{1/2}(1-\pi_\ell^p)\nabla_\Gamma f\|_{L_2(\Gamma)}. \tag{65b}$$

The constant C_8 depends only on Γ, the shape regularity of $\mathscr{T}_\ell$, and on p. The constant C_9 depends additionally on all possible shapes of element patches in $\mathscr{T}_\ell$.

Proof As example for $(h-h/2)$-type estimators, we choose e.g., $\eta_\ell = \|h_\ell^{1/2}(1-\pi_\ell^p)\widehat{\Phi}_\ell\|_{L_2(\Gamma)}$. Let $\widetilde{\widehat{\Phi}}_\ell$ denote the solution of (4) on the uniformly refined space $\widehat{\mathscr{X}_\ell} := \mathscr{P}^p(\widehat{\mathscr{T}_\ell})$ with right-hand side F_ℓ. The inverse triangle inequality combined with the inverse estimate from Lemma 23 shows

$$\begin{aligned}\big|\|h_\ell^{1/2}(1-\pi_\ell^p)\widetilde{\widehat{\Phi}}_\ell\|_{L_2(\Gamma)} - \|h_\ell^{1/2}(1-\pi_\ell^p)\widetilde{\Phi_\ell}\|_{L_2(\Gamma)}\big| &\le \|h_\ell^{1/2}(\widetilde{\widehat{\Phi}}_\ell - \widehat{\Phi}_\ell)\|_{L_2(\Gamma)}\\ &\lesssim \|\widetilde{\widehat{\Phi}}_\ell - \widehat{\Phi}_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)}.\end{aligned}$$

The remaining statement follows from (64a)–(64b).

The estimate for the weighted residual error estimator is similar. There holds

$$\begin{aligned}\big|\|h_\ell^{1/2}\nabla_\Gamma(V\widetilde{\Phi}_\ell - F_\ell)\|_{L_2(\Gamma)} - \|h_\ell^{1/2}\nabla_\Gamma(V\Phi_\ell - F)\|_{L_2(\Gamma)}\big| &\lesssim \|h_\ell^{1/2}\nabla_\Gamma V(\widetilde{\Phi}_\ell - \Phi_\ell)\|_{L_2(\Gamma)}\\ &\quad + \|h_\ell^{1/2}\nabla_\Gamma(1/2+K)(f-P_\ell f)\|_{L_2(\Gamma)}.\end{aligned}$$

We apply the inverse estimate for V from (61) and for $(1/2+K)$ from (60) to obtain

$$\begin{aligned}\big|\|h_\ell^{1/2}\nabla_\Gamma(V\widetilde{\Phi}_\ell - F_\ell)\|_{L_2(\Gamma)} - \|h_\ell^{1/2}\nabla_\Gamma(V\Phi_\ell - F)\|_{L_2(\Gamma)}\big| &\lesssim \|\widetilde{\Phi}_\ell - \Phi_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)}\\ &\quad + \|f-P_\ell f\|_{H^{1/2}(\Gamma)} + \|h_\ell^{1/2}\nabla_\Gamma(f-P_\ell f)\|_{L_2(\Gamma)}.\end{aligned}$$

The remainder follows as in the proof of Lemma 69. □

4.6.3 Hypersingular integral equation

For the hypersingular integral equation from Proposition 13 with $\Gamma = \partial\Omega$, the most useful method for data approximation employs the L_2-orthogonal projection $\pi_\ell^{p-1} : L_2(\Gamma) \to \mathscr{P}^{p-1}(\mathscr{T}_\ell)$, i.e.,

$$F_\ell := (1/2-K')\pi_\ell^{p-1}f. \tag{66}$$

Note that if $f\in H_0^{-1/2}(\Gamma)$, then also $\pi_\ell^{p-1}f \in H_0^{-1/2}(\Gamma)$. Let $U_\ell \in \mathscr{S}^p(\mathscr{T}_\ell)$ denote the solution of (4) with right-hand side (66). The introduced approximation error can be controlled with the following result.

Lemma 71 *There exists a constant $C_{10} > 0$ such that*

$$C_{10}^{-1}\|U_\ell - \widetilde{U}_\ell\|_{H^{1/2}(\Gamma)} \le \|h_\ell^{1/2}(1-\pi_\ell^{p-1})f\|_{L_2(\Gamma)}. \tag{67}$$

The constant C_{10} depends only on Γ, the shape regularity of $\mathscr{T}_\ell$, and on the polynomial degree p.

Proof The proof follows as for the weakly singular case in Lemma 69. □

Again, also the error estimators satisfy certain stability properties.

Lemma 72 *Let η_ℓ denote one of the $(h-h/2)$-type error estimators defined in Theorem 55 or the weighted residual error estimator from Theorem 39. Moreover, let $\widetilde{\eta}_\ell$ denote the perturbed version of the respective error estimator computed with the perturbed Galerkin approximation $\widetilde{U}_\ell$ and the perturbed data F_ℓ. Then, there exists a constant $C_{11} > 0$ such that*

$$|\widetilde{\eta}_\ell - \eta_\ell| \le C_{11}\|h_\ell^{1/2}(1-\pi_\ell^{p-1})f\|_{L_2(\Gamma)}. \tag{68}$$

The constant C_{11} depends only on Γ, the shape regularity of $\mathscr{T}_\ell$, and on p.

Proof The proof follows as for the weakly singular case in Lemma 70. □

5 A posteriori error estimators for the p and hp-versions

The p-version of the boundary element method is the extreme case of improving approximation properties only by increasing polynomial degrees, of piecewise polynomials on a fixed mesh. A combination of mesh refinement with increasing polynomial degrees is called hp-version. Higher order polynomial degrees are particularly suited for the approximation of singular functions, the ones that appear due to corners and edges of domains, recall Section 2.5 for details.

As is well known from finite element error analysis, the p-version converges twice as fast as the h-version for problems with singularities when the meshes match the singularity locations. This is also true of boundary elements. For a first analysis in two dimensions (on curves) considering hypersingular and weakly singular operators see [137]. An optimal analysis of this case has been provided in [81]. In three dimensions (on surfaces) the first p-version analysis for the hypersingular operator appeared in [128]. Here, only closed surfaces are considered, implying $H^1(\Gamma)$ regularity of the solution. Later, this gap has been closed in [22] for hypersingular operators and [23] presents an analysis for weakly singular operators. Of course, the pure p-version is mainly of theoretical interest since in practice, mesh refinement is easier to implement than polynomials of high degree in a stable way. Combining the h- and the p-version one can choose quasi-uniform or non-uniformly refined meshes. The hp-version with quasi-uniform meshes combines the convergence orders of both variants (twice the rate with respect to degrees in comparison to mesh refinement). The corresponding analyses have been given in [138] and [82] (preliminary and optimal estimates, respectively) in two dimensions for both operators. In three dimensions, and for open surfaces with the strongest singularities, the publications are [24] (hypersingular operator) and [25].

The hp-version gives full flexibility in choosing any mesh and degree combination, with analysis for quasi-uniform meshes provided by the publications mentioned before. In the so-called hp-version with geometric meshes one selects a specific combination of geometrically graded meshes with polynomial degrees that are larger on larger elements. In three dimensions (on surfaces) this implies the use of anisotropic elements and polynomial degrees that are different in different directions on the same element. In this way, an exponential rate of convergence (faster than any algebraic order in terms of numbers of unknowns) can be achieved, cf. [91].

Finite and boundary element analysis for meshes including anisotropic elements is challenging. This is due to the fact that no simple scaling arguments related to affine mappings onto reference elements apply. In many cases, different scaling properties in different directions get mixed up and make the analysis on distorted elements cumbersome.

In the case of the p-version there is another difficulty. The analysis of low order methods employs scaling arguments in order to use arguments from the equivalence of norms in finite-dimensional spaces, defined on reference elements. When considering the p-version, by definition dimensions of approximation spaces on elements are not bounded. This means that simple arguments from the equivalence of different norms do not apply. Analytical tools for the analysis of p- and hp-versions are usually different, and scaling arguments form only a part of the story.

Considering both remarks, on anisotropic elements and on the difficulties with the p-version, is becomes clear that error estimation for the hp-version with geometric meshes is a non-trivial issue. In fact, we are not aware of any publication analyzing this situation in a satisfying way, neither using residual-based estimators nor enrichment-based methods. In particular, nothing is known for the a posteriori p-error estimation of weakly singular operators in three dimensions. Additive Schwarz theory is the most advanced area dealing with p-approximations of boundary integral operators in three dimensions. This, in particular, is the case with hypersingular operators. However, let us recall that there is a satisfying analysis of two-level error estimation on anisotropic meshes for weakly singular integral equations [61], as discussed in Section 4.2.1, cf. Figure 13 and Theorem 52.

In the following we discuss the existing theory for two-level error estimation of the p- and hp-version with quasi-uniform meshes for the solution of hypersingular integral equations on surfaces [89,93]. For the p- and hp-version of the BEM, solving integral equations on curves, we refer to [92], see also [86].

Our model problem is the hypersingular integral equation considered in Proposition 6, and for simplicity we consider an open flat surface Γ with polygonal boundary. The meshes $\mathscr{T}$ are assumed to be quasi-uniform with shape-regular elements. Triangles and convex quadrilaterals are allowed. We will use the notation and framework introduced in Section 1.1. That means we are considering

$$b(u,v) = \langle Wu\,,v\rangle_\Gamma, \quad \mathscr{X} = \widetilde{H}^{1/2}(\Gamma), \quad \mathscr{X}^p_{\mathscr{T}} = \widetilde{\mathscr{S}}^p(\mathscr{T}).$$

Here, the index p in the discrete space refers to polynomial degrees $p \geq 1$ that can be different on different elements, and even different in different directions. The exact solution of the problem is $u \in \widetilde{H}^{1/2}(\Gamma)$ and the discrete solution is denoted by $U \in \widetilde{\mathscr{S}}^p(\mathscr{T})$.

Now, in order to define a two-level estimator for the error $\|u-U\|_{\widetilde{H}^{1/2}(\Gamma)}$, we consider as in Section 4.2.1 an enriched discrete space $\widehat{\mathscr{X}}^p_{\mathscr{T}}$ with $\mathscr{X}^p_{\mathscr{T}} \subset \widehat{\mathscr{X}}^p_{\mathscr{T}} \subset \widetilde{H}^{1/2}(\Gamma)$ and decomposition

$$\widehat{\mathscr{X}}^p_{\mathscr{T}} = \mathscr{Z}_{\mathscr{T},0} \oplus \mathscr{Z}_{\mathscr{T},1} \oplus \mathscr{Z}_{\mathscr{T},2} \oplus \cdots \oplus \mathscr{Z}_{\mathscr{T},L}. \tag{69}$$

The resulting error estimator is

$$\eta_{\mathscr{T}} := \Big(\sum_{j=0}^{L} \eta_j^2\Big)^{1/2}, \quad \eta_j := \|P_j(\widehat{U}-U)\|_b. \tag{70}$$

Here,

$$P_j : \widehat{\mathscr{X}_{\mathscr{T}}^p} \to \mathscr{Z}_{\mathscr{T},j} : \quad \langle W P_j v, w\rangle_\Gamma = \langle W v, w\rangle_\Gamma \quad \forall w \in \mathscr{Z}_{\mathscr{T},j}$$

and

$$\|v\|_b^2 = \langle W v, v\rangle_\Gamma \quad \forall v \in \mathscr{Z}_{\mathscr{T},j} \quad (j = 0,\dots,L).$$

Moreover, $\eta_0 = 0$ corresponds to $\mathscr{Z}_0 \subset \mathscr{X}_{\mathscr{T}}^p$, cf. Lemma 47.

There are different issues to be considered when selecting the enriched space and a decomposition.

– Basis functions for $\widehat{\mathscr{X}_{\mathscr{T}}^p} \subset \widetilde{H}^{1/2}(\Gamma)$ must be continuous. This fact restricts the possibility of having stable decompositions of $\widehat{\mathscr{X}_{\mathscr{T}}^p}$ with subspaces that are localized on elements.
– Decompositions (69) for the p-version based on the separation of basis functions are inherently unstable for a standard basis (cf. [13] for the finite element method), or require specific basis functions that are partially orthogonal (cf. [87, 88]).
– Partially orthogonal basis functions (as mentioned before) are not hierarchical. They can be constructed a priorily for rectangles (through tensor product representations) or a posteriorily through a Schur complement step. This Schur complement is not a local construction for boundary integral operators and, thus, expensive.
– Increasing polynomial degrees by a finite number, e.g., from p to $p+1$, for the generation of the enriched space $\widehat{\mathscr{X}_{\mathscr{T}}^p}$ does in general not satisfy the saturation assumption. On the other hand, increasing polynomial degrees by a fixed factor is not practical since polynomials of higher degrees are inherently difficult to implement in a stable and efficient way.

For the reasons above, we suggest to consider two different enrichments with corresponding decompositions. One for error estimation with focus on satisfying the saturation assumption (let's call this estimator $\eta_{\rm est}$) and another one to generate indicators steering the mesh refinement (and/or increase of polynomial degrees) with focus on providing local information (let's call this estimator $\eta_{\rm ref}$). The estimator $\eta_{\rm est}$ can be relatively expensive since it will be used only for a stopping criterion, it is not necessary to calculate it after each refinement step. On the other hand, $\eta_{\rm ref}$ is needed for every refinement step and should be cheap. In the following we discuss both cases.

Error estimator $\eta_{\rm est}$. We consider the enriched space $\widehat{\mathscr{X}_{\mathscr{T}}^p} := \widehat{\mathscr{S}^{p}}(\widehat{\mathscr{T}})$ that one obtains by refining the mesh $\mathscr{T}$ uniformly, i.e., subdividing every triangle and quadrilateral in an isotropic way. Here, $\widehat{\mathscr{T}}$ denotes the refined mesh. Polynomial degrees $\widehat{p}$ can be inherited from father elements or one can select the maximum polynomial degree from the actual space $\mathscr{X}_{\mathscr{T}}^p$ for the enriched space, $\widehat{p} = \max\{p\}$. In this way, numerical experiments indicate good compliance with the saturation property, cf. [89]. Uniformly stable decompositions can be obtained through overlapping domain decomposition. To this end, let ω_j ($z_j \in \widehat{\mathscr{N}}$) denote the patches of elements that are adjacent to interior nodes of the refined mesh $\widehat{\mathscr{T}}$. Here, for simplicity, we assume that Γ is convex to avoid the appearance of special situations at incoming corners. This is only for technical reasons and not essential. Then, we consider the decomposition $\widehat{\mathscr{X}_{\mathscr{T}}^p} = \mathscr{Z}_{\mathscr{T},0} \cup \mathscr{Z}_{\mathscr{T},1} \cup \dots \cup \mathscr{Z}_{\mathscr{T},L}$ with

$$\mathscr{Z}_{\mathscr{T},0} = \widetilde{\mathscr{S}^1}(\widehat{\mathscr{T}}) \quad \text{and} \quad \mathscr{Z}_{\mathscr{T},j} = \widetilde{\mathscr{S}^{\widehat{p}}}(\widehat{\mathscr{T}}|_{\omega_j}), \quad z_j \in \widehat{\mathscr{N}}. \tag{71}$$

Two comments are in order. First, this decomposition is not direct. This results in a slightly more complicated additive Schwarz theory than presented in Section 4.2.1. Second, we do not have the inclusion $\mathscr{Z}_{\mathscr{T},0} \subset \mathscr{X}_{\mathscr{T}}^p$ so that the error indicator

$$\eta_{\rm est,0} = \|P_0(\widehat{U}-U)\|_b$$

corresponding to this subspace does not vanish in general. $\mathscr{Z}_{\mathscr{T},0}$ is the so-called coarse grid space of the decomposition and, since it is defined with lowest order polynomial degree, its calculation is not too expensive. Additionally, this step can be accelerated by using efficient low order implementations (though this has not been studied in this particular situation).

Theorem 73 *Let $\widehat{\mathscr{X}_{\mathscr{T}}^p} = \widetilde{\mathscr{S}^{\widehat{p}}}(\widehat{\mathscr{T}})$ be defined with uniform degree $\widehat{p} = \max\{p\}$. The error estimator $\eta_{\rm est}$ defined through the decomposition of $\widehat{\mathscr{X}_{\mathscr{T}}^p}$ with subspaces* (71) *is efficient: there exists a constant $C_{\rm eff} > 0$ such that, for any quasi-uniform mesh $\mathscr{T}$ of shape-regular elements with shape-regular refinement $\widehat{\mathscr{T}}$, there holds for any polynomial degree $\widehat{p}$*

$$\eta_{\rm est} \le C_{\rm eff} \|u-U\|_b.$$

Furthermore, let $\widehat{\mathscr{T}}$ be sufficiently refined so that Assumption 45 holds. Then $\eta_{\rm est}$ is also reliable: there exists a constant $c > 0$ such that, with $C_{\rm rel} = (1-C_{\rm sata}^2)^{-1/2} c$, there holds for any mesh $\mathscr{T}$ with shape-regular refinement $\widehat{\mathscr{T}}$ and for any polynomial degree $\widehat{p}$ the estimate

$$\|u-U\|_b \le C_{\rm rel}\ \eta_{\rm est}.$$

For a proof of Theorem 73 we refer to [89].

Error indicator η_{ind}. In order to generate error indicators that are local and useful for adaptive steering we increase locally polynomial degrees. In particular, we aim at error indicators that indicate also in which direction to refine (literally an element or in the sense of increasing polynomial degrees in a certain direction on elements). Here, we do not focus on satisfying the saturation assumption. In the following, to keep things simpler, we consider only rectangular elements. For meshes consisting of triangles, or rectangles and triangles, we refer to [89].

Our enriched space $\widehat{\mathscr{X}_{\mathscr{T}}^{p}}$ and decomposition will be like

$$\widehat{\mathscr{X}_{\mathscr{T}}^{p}} = \mathscr{Z}_{\mathscr{T},0} \oplus \bigoplus_{T\in\mathscr{T};\, i=1,2} \mathscr{Z}_{Ti}. \tag{72}$$

Here, $\mathscr{Z}_{T1}$ and $\mathscr{Z}_{T2}$ consist of functions with support on (the closure of) an element $T \in \mathscr{T}$ and the two spaces will generate direction indicators. The space $\mathscr{Z}_{\mathscr{T},0}$ consists of functions with support on (the closure of) Γ. In order to have conformity $\mathscr{Z}_{Ti} \subset \widetilde{H}^{1/2}(\Gamma)$ and locality of functions, the elements of $\mathscr{Z}_{Ti}$ must vanish on the boundary of T (so that they can be continuously extended by zero onto $\Gamma \setminus T$). Of course we are considering polynomials on T, and in that case these functions (with vanishing trace on the boundary of T) are called bubble functions. The generation of bubble functions requires a minimum polynomial degree. On a triangle the lowest order bubble function has degree three, and on rectangles one uses tensor products of polynomials of at least degree two in both directions. In both cases, the minimum degree allows for only one linearly independent bubble function. Therefore, in order to have enough unknowns for indicators in different directions, we need slightly higher polynomial degrees.

For ease of presentation let us now assume that T is a rectangle that is oriented in the x_1-x_2 plane. Furthermore, $\mathscr{P}^{p_1,p_2}(T)$ indicates the space of polynomials on T with degrees up to p_i in x_i-direction, $i = 1,2$. We then define for any $T \in \mathscr{T}$ the spaces $\mathscr{Z}_{T1}$, $\mathscr{Z}_{T2}$ as follows.

$$\mathscr{Z}_{T1} := \begin{cases} \operatorname{span}\left\{\mathscr{P}^{p_1+1,p_2}(T)\setminus\mathscr{P}^{p_1,p_2}(T)\right\}\cap H_0^1(T) \\ \qquad \text{if } p_1>1, p_2>1 \quad \text{(a)} \\ \operatorname{span}\left\{\mathscr{P}^{p_1+1,2}(T)\setminus\mathscr{P}^{p_1,2}(T)\right\}\cap H_0^1(T) \\ \qquad \text{if } p_1>1, p_2=1 \quad \text{(b)} \\ \mathscr{P}^{2,p_2}(T)\cap H_0^1(T) \\ \qquad \text{if } p_1=1, p_2>1 \quad \text{(c)} \\ \mathscr{P}^{3,2}(T)\cap H_0^1(T) \\ \qquad \text{if } p_1=1, p_2=1 \quad \text{(d)} \end{cases}$$

is the space to generate an indicator on T in x_1-direction and

$$\mathscr{Z}_{T2} := \begin{cases} \operatorname{span}\left\{\mathscr{P}^{p_1,p_2+1}(T)\setminus\mathscr{P}^{p_1,p_2}(T)\right\}\cap H_0^1(T) \\ \qquad \text{if } p_1>1, p_2>1 \quad \text{(a)} \\ \mathscr{P}^{p_1,2}(T)\cap H_0^1(T) \\ \qquad \text{if } p_1>1, p_2=1 \quad \text{(b)} \\ \operatorname{span}\left\{\mathscr{P}^{2,p_2+1}(T)\setminus\mathscr{P}^{2,p_2}(T)\right\}\cap H_0^1(T) \\ \qquad \text{if } p_1=1, p_2>1 \quad \text{(c)} \\ \mathscr{P}^{2,3}(T)\cap H_0^1(T) \\ \qquad \text{if } p_1=1, p_2=1 \quad \text{(d)} \end{cases}$$

will generate an indicator in x_2-direction. Here, (p_1, p_2) are the polynomial degrees in $\widetilde{\mathscr{S}^{p}}(\mathscr{T})$ on T. They can be different on every element. As basis functions for the subspaces $\mathscr{Z}_{T1}$, $\mathscr{Z}_{T2}$ we take affine images of the tensor products

$$\psi_{p_1}(x_1)\psi_{p_2}(x_2) \quad \text{with} \quad \psi_q(s) := \int_{-1}^{s} L_{q-1}(t)\,dt$$

for $p_1, p_2 \geq 2$ defined on $(-1,1)^2$ with L_{q-1} being the Legendre polynomial of degree $q-1$.

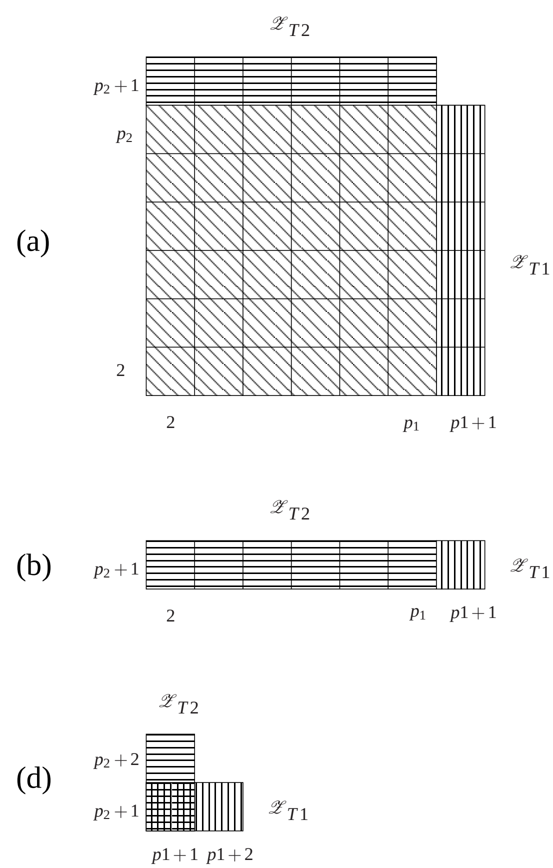

Fig. 15 Illustration of p-enrichment and decomposition on an element T for direction control.

In Figure 15 we illustrate the increase of polynomial degrees for the generation of $\mathscr{Z}_{T1}$ and $\mathscr{Z}_{T2}$ in different situations. The marked regions represent pairs of polynomial degrees for the two spaces. We represent only degrees larger than one, that means we illustrate only bubble functions. Case (a) is the general case when the polynomial degrees p_1, p_2 on an element T are larger than one. We increase the polynomial degrees by one in both directions and define the local spaces $\mathscr{Z}_{T1}$ and $\mathscr{Z}_{T2}$ by the indicated degrees. The remaining situations, where p_1 or p_2 is one, are illustrated by (b), (c) and (d). These cases are not covered by (a) and we need a special p-enrichment to produce subspaces that indicate different directions for refinement. Case (c) is analogous to the case (b) when exchanging p_1 and p_2, and is omitted. Only in the case (a) (with $p_1 \geq 2$ and $p_2 \geq 2$) bubble functions of the previous space $\mathscr{X}^p_{\mathscr{T}}$ are present. This is indicated by the diagonal shading. Case (d) is the only situation where the decomposition of $\mathscr{Z}_T = \mathscr{Z}_{T1} \cup \mathscr{Z}_{T2}$ is not direct.

Now, for elements not being aligned with the x_1-x_2 directions, the construction of the two spaces $\mathscr{Z}_{T1}$, $\mathscr{Z}_{T2}$ works analogously. Concluding, we have defined the decomposition (72) with the exception of $\mathscr{Z}_{\mathscr{T},0}$. In [89] and [93], different strategies have been considered to generate $\mathscr{Z}_{\mathscr{T},0}$ so that $\mathscr{X}^p_{\mathscr{T}} \subset \widehat{\mathscr{X}}^p_{\mathscr{T}}$ and (72) is (almost) stable. These strategies are partial or full orthogonalizations and Schur complement steps. Here we are not interested in reliable error estimation (which is being provided by the error estimator based on mesh refinement) and therefore, finish this section with recalling stability of the decomposition of the enrichment level

$$\mathscr{Z}_{\mathscr{T}} = \bigoplus_{T\in\mathscr{T}} \Big(\mathscr{Z}_{T1} \cup \mathscr{Z}_{T2}\Big) \tag{73}$$

and assuming a stable construction of $\mathscr{Z}_{\mathscr{T},0}$ without giving more details. This then implies efficient and reliable estimation of $\|\widehat{U}-U\|_b$ by $\eta_{\rm ind}$. Note that in some cases, as discussed above, the decomposition $\mathscr{Z}_{T1} \cup \mathscr{Z}_{T2}$ can be non-direct.

As said before, we consider meshes consisting only of rectangles. As in [89,93] this can be generalized to meshes including quadrilaterals and triangles.

Theorem 74 *Let $\mathscr{Z}_{\mathscr{T}}$ be defined through decomposition* (73) *with local spaces $\mathscr{Z}_{Ti}$ ($i = 1,2$) as defined previously, and assume that the construction of $\mathscr{Z}_{\mathscr{T},0}$ in* (72) *is stable. Then the corresponding error indicator*

$$\eta_{\rm ind} := \Big(\|P_0(\widehat{U}-U)\|_b^2 + \sum_{T\in\mathscr{T},\, i=1,2} \|P_{Ti}(\widehat{U}-U)\|_b^2\Big)^{1/2}$$

is reliable and efficient for the estimation of $\|\widehat{U}-U\|_b$ in the following sense. Assume that the mesh $\mathscr{T}$ is locally quasi-uniform, i.e. the ratio of largest side length and smallest side length on each element is bounded from above by a global positive constant. Then there exist constants $c_1, c_2 > 0$ which are independent of the mesh and polynomial degrees p such that

$$c_1\eta_{\rm ind} \leq \|\widehat{U}-U\|_b \leq c_2(1+\log p_{\max})\eta_{\rm ind}.$$

Here, $p_{\max}$ is the maximum of all polynomial degrees in $\mathscr{X}^p_{\mathscr{T}}$.

For a proof we refer to [89]. We finish this section with presenting a three-level refinement algorithm that decides where to add unknowns and whether to refine the mesh or increase polynomial degrees at those places. This algorithm has proved to work appropriately in standard situations. The definition and analysis of optimal algorithms for direction control and decision for h or p refinement in boundary element methods is an open problem.

Three-fold algorithm [89]: Define an initial ansatz space $\widetilde{\mathscr{S}^p}(\mathscr{T})$ with initial mesh $\mathscr{T}$ and low polynomial degrees. Choose an error tolerance $\varepsilon > 0$ and steering parameters δ_1, δ_2, δ_3 with $0 < \delta_2 < \delta_1 < 1$ and $0 < \delta_3 < 1$. Then perform the following steps.

1. Galerkin solution. Compute the Galerkin solution $U \in \widetilde{\mathscr{S}^p}(\mathscr{T})$.
2. Error estimator. Calculate the terms $\eta_j = \|P_j(\widehat{U}-U)\|_b$ and the estimator $\eta_{\rm est}$, based on the decomposition (71).
Stop if $\eta_{\rm est} \leq \varepsilon$.
3. Adaption steps.

(i) Indicators. Compute the error indicators $\eta_{Ti} = \|P_{Ti}(\widehat{U}-U)\|_b$, $\eta_T := (\eta_{T1}^2 + \eta_{T2}^2)^{1/2}$ ($T \in \mathscr{T}$, $i = 1,2$) based on the decomposition (73), and set $\eta_{\max} := \max_{T\in\mathscr{T}} \eta_T$.

(ii) Classification of elements. Classify quadrilaterals T as follows (in pseudo Fortran90 language, directions are understood with respect to local coordinates):

```
if (ηT > δ1 ηmax) then   ! h-refinement
   if (ηT1 < δ3 ηT2) then
      classify T for horizontal intersection
   elseif (ηT2 < δ3 ηT1) then
      classify T for vertical intersection
   else
      classify T for intersections in both directions
   endif
elseif (ηT > δ2 ηmax) then   ! p-increase
   if (ηT1 < δ3 ηT2) then
      classify T for increase of polynomial degree
in vertical direction
   elseif (ηT2 < δ3 ηT1) then
      classify T for increase of polynomial degree
in horizontal direction
   else
```

classify T for increase of polynomial degrees in both directions

```
    endif
  endif
```

Triangles are classified without direction control, i.e., they are refined by halving all their edges if $\eta_T > \delta_1 \eta_{\max}$ and the polynomial degree is increased if $\delta_2 \eta_{\max} < \eta_T \le \delta_1 \eta_{\max}$.

(iii) Adaption.

(a) Go through all the elements and refine as classified.

(b) Go through all the elements and refine as necessary to remove hanging nodes.

(c) Go through all the elements and increase polynomial degrees as classified if the corresponding element has not been refined in (b).

goto 1.

Remark 10 When an element is refined then polynomial degrees for the new elements need to be given. To avoid high polynomial degrees on small elements one can inherit the degrees reduced by one for the refinement direction whenever possible (i.e., when the degree is larger than one). A more sophisticated algorithm may include the refinement of quadrilaterals into triangles and vice versa. This has been studied on [89, 93].

Remark 11 An adaptive h-version can be realized by choosing $\delta_2 \ge \delta_1$. Pure p-adaptivity occurs when choosing $\delta_1 > 1$. Isotropic adaption (no direction control) can be performed by taking $\delta_3 = 0$.

6 Estimator reduction

This section explains the concept of estimator reduction and its use to prove plain convergence of ABEM, i.e., the validity of (12). The general idea that will be presented here applies to error estimators whose local contributions are weighted by the local mesh-size h_ℓ. The approach is illustrated for some $(h-h/2)$-type error estimators from Section 4.2.2, the ZZ-type error estimators from Section 4.4, and the weighted residual error estimators from Section 4.1.3. To that end, we consider a sequence of meshes $\mathscr{T}_\ell$ which, e.g., is generated by the adaptive Algorithm 1. We only need some minor assumptions on the mesh refinement operation $\mathtt{refine}(\cdot)$.

6.1 Assumptions on mesh refinement

We say that a mesh $\mathscr{T}_\star$ is a refinement of another mesh $\mathscr{T}_\ell$, written $\mathscr{T}_\star \in \mathtt{refine}(\mathscr{T}_\ell)$, if the following applies:

– For all $T \in \mathscr{T}_\ell$, there holds
$$\overline{T} = \bigcup\{\overline{T}' : T' \in \mathscr{T}_\star \text{ with } T' \subseteq T\}, \tag{74}$$
i.e., each coarse-mesh element $T \in \mathscr{T}_\ell$ is basically the union of fine-mesh elements $T' \in \mathscr{T}_\star$.

– For all $T \in \mathscr{T}_\ell$ and $T' \in \mathscr{T}_\star$, there holds
$$T' \subsetneqq T \quad \Longrightarrow \quad |T'| \le |T|/2, \tag{75}$$
i.e., the area of refined elements is at least halved.

A sequence of meshes $(\mathscr{T}_\ell)_{\ell \in \mathbb{N}_0}$ is called nested, if for all $\ell \in \mathbb{N}_0$ it holds $\mathscr{T}_{\ell+1} \in \mathtt{refine}(\mathscr{T}_\ell)$ and if the shape regularity constant σ_ℓ from Section 2.6 is uniformly bounded $\sup_{\ell \in \mathbb{N}_0} \sigma_\ell < \infty$.

Recall that with each mesh $\mathscr{T}_\star$, we associate the local mesh-size function $h_\star \in \mathscr{P}^0(\mathscr{T}_\star)$ defined by $h_\star|_T := h_T = |T|^{1/(d-1)}$.

6.2 Abstract error estimator

Given the mesh $\mathscr{T}_\ell$, suppose that there exists a computable error estimator
$$\eta_\ell := \Big(\sum_{T \in \mathscr{T}_\ell} \eta_\ell(T)^2\Big)^{1/2} \text{ with } \eta_\ell(T) \in [0, \infty) \text{ for all } T \in \mathscr{T}_\ell.$$
The estimator usually depends on the computed solution U_ℓ of (4) as well as on the right-hand side F.

6.3 Abstract adaptive algorithm

Convergence of the adaptive algorithm 1 has first been addressed in the frame of AFEM in the pioneering work [56] which also introduced the bulk chasing (9). While [56] only proved convergence up to the resolution of the given data on the initial mesh $\mathscr{T}_0$, the work [110] included the adaptive resolution of the data and contained the first plain convergence result. For ABEM, convergence of this algorithm has mathematically been addressed first in [73] and [11] for $(h-h/2)$-type and averaging error estimators, while the analysis of [43] relied on an additional (and practically artificial and unnecessary) feedback control.

Remark 12 In practice, step (iv) of the Algorithm 1 provides the *coarsest* refinement $\mathscr{T}_{\ell+1}$ of $\mathscr{T}_\ell$ such that all marked elements have been refined by the mesh refinement strategy used, written $\mathscr{T}_{\ell+1} := \mathtt{refine}(\mathscr{T}_\ell, \mathscr{M}_\ell)$. We refer to Section 7 for possible concrete strategies for local mesh refinement of 2D and 3D BEM meshes.

Remark 13 To find a set $\mathscr{M}_\ell \subseteq \mathscr{T}_\ell$ which satisfies the bulk chasing (9), one arbitrarily adds elements $T \in \mathscr{T}_\ell$ to $\mathscr{M}_\ell$ until (9) is satisfied (at least $\mathscr{M}_\ell = \mathscr{T}_\ell$ will do the job). If one seeks a set of minimal cardinality $\mathscr{M}_\ell$, it is necessary to sort the elementwise error indicators, i.e., $\eta_\ell(T_1) \ge \eta_\ell(T_2) \ge \ldots \ge \eta_\ell(T_{\#\mathscr{T}_\ell})$. Then, determine the minimal $1 \le n \le \#\mathscr{T}_\ell$ such that $\theta \eta_\ell^2 \le \sum_{j=1}^n \eta_\ell(T_j)^2$. By construction,

$\mathscr{M}_\ell := \{T_1, \dots, T_n\}$ satisfies (9) with minimal cardinality. Obviously, the set $\mathscr{M}_\ell$ is not unique in general. This may lead to non-symmetric meshes even for completely symmetric problems.

6.4 Estimator reduction principle

The estimator reduction principle [11] is an elementary yet very useful starting point for the a posteriori analysis of any error estimator. The following lemma states the main idea of the principle.

Lemma 75 *Given a sequence of error estimators $(\eta_\ell)_{\ell\in\mathbb{N}_0}$, suppose a contraction constant $0 < q_{\rm est} < 1$ as well as a perturbation sequence $(\alpha_\ell)_{\ell\in\mathbb{N}_0} \subset \mathbb{R}$ such that the error estimator satisfies the perturbed contraction*

$$\eta_{\ell+1}^2 \le q_{\rm est}\eta_\ell^2 + \alpha_\ell^2 \quad \textit{for all } \ell \in \mathbb{N}_0. \tag{76}$$

Then, $\lim_{\ell\to\infty} \alpha_\ell^2 = 0$ *implies estimator convergence*

$$\lim_{\ell\to\infty} \eta_\ell = 0. \tag{77}$$

Proof Apply the limes superior $\overline{\lim}$ on both sides of the estimator reduction (76) to obtain

$$\overline{\lim_{\ell\to\infty}}\,\eta_{\ell+1}^2 \le q_{\rm est}\overline{\lim_{\ell\to\infty}}\,\eta_\ell^2 + \overline{\lim_{\ell\to\infty}}\,\alpha_\ell^2.$$

Since α_ℓ^2 converges towards zero, there holds $\overline{\lim}_{\ell\to\infty}\alpha_\ell^2 = 0$. This implies

$$\overline{\lim_{\ell\to\infty}}\,\eta_{\ell+1}^2 \le q_{\rm est}\overline{\lim_{\ell\to\infty}}\,\eta_\ell^2 = q_{\rm est}\overline{\lim_{\ell\to\infty}}\,\eta_{\ell+1}^2.$$

Since $0 < q_{\rm est} < 1$, this leaves the possibilities $\overline{\lim}_{\ell\to\infty}\eta_{\ell+1}^2 = 0$ or $\overline{\lim}_{\ell\to\infty}\eta_{\ell+1}^2 = \infty$. To rule out the second option, apply the estimator reduction (76) inductively to see

$$\begin{aligned}
\eta_\ell^2 &\le q_{\rm est}\eta_{\ell-1}^2 + \alpha_{\ell-1}^2\\
&\le q_{\rm est}^2\eta_{\ell-2}^2 + q_{\rm est}\alpha_{\ell-2}^2 + \alpha_{\ell-1}^2\\
&\le q_{\rm est}^\ell\eta_0^2 + \sum_{k=0}^{\ell-1} q_{\rm est}^k\alpha_{\ell-k-1}^2.
\end{aligned}$$

Convergence of α_ℓ^2 implies the boundedness $\sup_{\ell\in\mathbb{N}_0}\alpha_\ell^2 < \infty$ and the convergence of the geometric series concludes

$$\eta_\ell^2 \le q_{\rm est}^\ell\eta_0^2 + \frac{1}{1-q_{\rm est}}\sup_{\ell\in\mathbb{N}_0}\alpha_\ell^2 < \infty.$$

This proves $\overline{\lim}_{\ell\to\infty}\eta_{\ell+1}^2 = 0$ and elementary calculus yields

$$0 \le \lim_{\ell\to\infty}\eta_\ell^2 \le \overline{\lim_{\ell\to\infty}}\,\eta_{\ell+1}^2 = 0.$$

This concludes the proof. □

Before we apply Lemma 75 to concrete error estimators η_ℓ, we collect a number of auxiliary results on the convergence of projections and quasi-interpolants. Later, these will prove that the perturbation terms α_ℓ vanish as $\ell\to\infty$. The first lemma has already been proved in the pioneering work [15] for symmetric problems and reinvented in [111, 43].

Lemma 76 *Suppose a sequence of nested spaces $(\mathscr{X}_\ell)_{\ell\in\mathbb{N}_0} \subset \mathscr{X}$, i.e., $\mathscr{X}_\ell \subseteq \mathscr{X}_{\ell+1}$ for all $\ell\in\mathbb{N}_0$. Then, the Galerkin approximations U_ℓ of* (4) *satisfy*

$$\lim_{\ell\to\infty} \|U_\infty - U_\ell\|_{\mathscr{X}} = 0 \tag{78}$$

for an a priori limit $U_\infty \in \mathscr{X}$ which is unknown in general and depends on the concrete sequence of spaces.

Proof Define the closed space $\mathscr{X}_\infty := \overline{\bigcup_{\ell\in\mathbb{N}_0}\mathscr{X}_\ell} \subseteq \mathscr{X}$, where the closure is understood in the space $\mathscr{X}$. The Lax-Milgram lemma guarantees a unique solution $U_\infty \in \mathscr{X}_\infty$ of (4), where $\mathscr{X}_\ell$ is replaced with $\mathscr{X}_\infty$. By use of the Galerkin orthogonality, we prove the Céa lemma (6) also for U_∞, i.e., any $V_\ell \in \mathscr{X}_\ell$ satisfies

$$\begin{aligned}
C_{\rm ell}\|U_\infty - U_\ell\|_{\mathscr{X}}^2 &\le b(U_\infty - U_\ell\,, U_\infty - U_\ell)\\
&= b(U_\infty - U_\ell\,, U_\infty - V_\ell)\\
&\le C_{\rm cont}\|U_\infty - U_\ell\|_{\mathscr{X}}\|U_\infty - V_\ell\|_{\mathscr{X}}.
\end{aligned}$$

Hence, we are led to

$$\|U_\infty - U_\ell\|_{\mathscr{X}} \lesssim \min_{V_\ell\in\mathscr{X}_\ell}\|U_\infty - V_\ell\|_{\mathscr{X}}.$$

Let $\varepsilon > 0$. The definition of $\mathscr{X}_\infty$ implies the existence of $k \in \mathbb{N}$ and $W_k \in \mathscr{X}_k$ such that $\|U_\infty - W_k\|_{\mathscr{X}} \le \varepsilon$. Combining this with the nestedness $\mathscr{X}_k \subseteq \mathscr{X}_\ell$ for $\ell \ge k$ and the Céa lemma, we obtain $\|U_\infty - U_\ell\|_{\mathscr{X}} \lesssim \varepsilon$. This concludes the proof. □

The following lemma provides a similar result for quasi-interpolation operators and is proved in [67, Proposition 11].

Lemma 77 *Given a sequence of nested meshes $(\mathscr{T}_\ell)_{\ell\in\mathbb{N}_0}$ as well as corresponding linear operators $(J_\ell : L_2(\Gamma) \to L_2(\Gamma))_{\ell\in\mathbb{N}}$ which satisfy for all $T\in\mathscr{T}_\ell$ the following properties (i)–(iii):*

(i) local L_2-stability

$$\|J_\ell v\|_{L_2(T)} \le C_{\rm J}\|v\|_{L_2(\omega_T)} \quad \textit{for all } v \in L_2(\Gamma);$$

(ii) local first-order approximation property

$$\|(1-J_\ell)v\|_{L_2(T)} \le C_{\rm J}\|h_\ell\nabla_\Gamma v\|_{L_2(\omega_T)} \quad \textit{for all } v \in H^1(\Gamma);$$

(iii) local definition, i.e., $(J_\ell v)|_T$ depends only on the values of $v|_{\omega_T}$.

Then, there exists a linear and continuous limit operator $J_\infty : L_2(\Gamma) \to L_2(\Gamma)$ *with*

$$\lim_{\ell\to\infty} \|J_\infty v - J_\ell v\|_{L_2(\Gamma)} = 0 \quad \text{for all } v \in L_2(\Gamma). \tag{79}$$

Suppose in addition that $J_\ell(L_2(\Gamma)) \subseteq H^1(\Gamma)$ *and that the following property holds:*

(iv) local H^1*-stability*

$$\|\nabla_\Gamma (J_\ell v)\|_{L_2(T)} \le C_{\mathrm{J}} \|v\|_{H^1(\omega_T)} \quad \text{for all } v \in H^1(\Gamma).$$

Then, the limit operator J_∞ *has the following additional properties:*

- $J_\infty : H^s(\Gamma) \to H^s(\Gamma)$ *is well-defined and continuous for all* $0 \le s \le 1$*;*
- *for all* $0 \le s < 1$*,* J_∞ *is the pointwise limit of* J_ℓ*, i.e.,*

$$\lim_{\ell\to\infty} \|J_\infty v - J_\ell v\|_{H^s(\Gamma)} = 0 \quad \text{for all } v \in H^s(\Gamma); \tag{80}$$

- *for* $s = 1$ *and all* $v \in H^1(\Gamma)$*,* $J_\ell v$ *converges weakly to* $J_\infty v$ *as* $\ell \to \infty$*.*

The Scott-Zhang projection J_ℓ from Lemma 26 satisfies even stronger convergence results. We stress that J_ℓ satisfies the assumptions (i)–(iv) from Lemma 77. The following lemma is proved in [71, Lemma 18].

Lemma 78 *Suppose a sequence of nested meshes* $(\mathscr{T}_\ell)_{\ell\in\mathbb{N}_0}$ *as well as the corresponding Scott-Zhang operators* $(J_\ell : L_2(\Gamma) \to L_2(\Gamma))_{\ell\in\mathbb{N}}$*. Then, the limit operator* $J_\infty : L_2(\Gamma) \to L_2(\Gamma)$*, which exists due to Lemma 77, is a projection in the sense of* $J_\infty v = v$ *for all* $v \in \mathscr{S}^p_\infty := \overline{\bigcup_{\ell\in\mathbb{N}_0} \mathscr{S}^p(\mathscr{T}_\ell)} \subseteq L_2(\Gamma)$ *where the closure is understood with respect to* $L_2(\Gamma)$*, and satisfies pointwise convergence*

$$\lim_{\ell\to\infty} \|J_\infty v - J_\ell v\|_{H^s(\Gamma)} = 0 \quad \text{for all } v \in H^s(\Gamma) \tag{81}$$

and all $0 \le s \le 1$*.*

Finally, also the non-local L_2-orthogonal projection $\Pi^p_\ell : L_2(\Gamma) \to \mathscr{S}^p(\mathscr{T}_\ell)$ from Definition 21 satisfies the convergence properties of Lemma 77–78. The following lemma improves an observation of [100] to general $0 \le s \le 1$. We note that Lemma 79 does not follow from Lemma 77, since Π^p_ℓ is a *non-local* operator and fails to satisfy the *local* properties (i)–(iv) from Lemma 77.

Lemma 79 *Given a sequence of nested meshes* $(\mathscr{T}_\ell)_{\ell\in\mathbb{N}_0}$*, suppose uniform* H^1*-stability of the* L_2*-orthogonal projection* $\Pi^p_\ell : L_2(\Gamma) \to \mathscr{S}^p(\mathscr{T}_\ell)$ *for all* $\ell \in \mathbb{N}_0$*, i.e.,*

$$\|\nabla_\Gamma \Pi^p_\ell v\|_{L_2(\Gamma)} \le C_{\mathrm{stab}} \|v\|_{H^1(\Gamma)} \quad \text{for all } v \in H^1(\Gamma). \tag{82}$$

Then, there exists a linear and continuous limit operator $\Pi^p_\infty : L_2(\Gamma) \to L_2(\Gamma)$ *which is a projection in the sense of* $\Pi^p_\infty v = v$ *for all* $v \in \mathscr{S}^p_\infty := \overline{\bigcup_{\ell\in\mathbb{N}_0} \mathscr{S}^p(\mathscr{T}_\ell)} \subseteq L_2(\Gamma)$ *where the closure is understood with respect to* $L_2(\Gamma)$*, and satisfies*

- $\Pi^p_\infty : H^s(\Gamma) \to H^s(\Gamma)$ *is well-defined and continuous for all* $0 \le s \le 1$
- *For all* $0 \le s \le 1$*,* Π^p_∞ *is the pointwise limit of* Π^p_ℓ*, i.e.*

$$\lim_{\ell\to\infty} \|\Pi^p_\infty v - \Pi^p_\ell v\|_{H^s(\Gamma)} = 0 \quad \text{for all } v \in H^s(\Gamma) \tag{83}$$

Proof Since Π^p_ℓ is an orthogonal projection for the L_2-scalar product, one proves analogously to Lemma 76 that there exists an operator $\Pi^p_\infty : L_2(\Gamma) \to L_2(\Gamma)$ such that

$$\lim_{\ell\to\infty} \|\Pi^p_\ell v - \Pi^p_\infty v\|_{L_2(\Gamma)} = 0 \quad \text{for all } v \in L_2(\Gamma). \tag{84}$$

Clearly, this and the projection property of Π^p_ℓ imply in particular that $v = \Pi^p_\infty v$ for all $v \in \mathscr{S}^p_\infty$. As Π^p_ℓ is stable in $L_2(\Gamma)$ by definition and stable in $H^1(\Gamma)$ by assumption (82), it follows from deeper mathematical techniques (see, e.g., [142]) that it is also stable in $H^s(\Gamma)$ for $s \in (0,1)$. For general $v \in H^s(\Gamma)$, the boundedness $\sup_{\ell\in\mathbb{N}_0} \|\Pi^p_\ell v\|_{H^s(\Gamma)} < \infty$ implies for a subsequence $\Pi^p_{\ell_k} v \rightharpoonup w$ weakly in $H^s(\Gamma)$ and hence $\Pi^p_\infty v = w \in H^s(\Gamma)$. To see H^s-convergence for all $0 \le s \le 1$, we apply the projection property of Π^p_ℓ to see

$$\begin{aligned}\|\Pi^p_\infty v - \Pi^p_\ell v\|_{H^s(\Gamma)} &= \|(1-\Pi^p_\ell)\Pi^p_\infty v\|_{H^s(\Gamma)} \\ &= \|(1-\Pi^p_\ell)(1-J_\ell)\Pi^p_\infty v\|_{H^s(\Gamma)},\end{aligned}$$

where $J_\ell : L_2(\Gamma) \to \mathscr{S}^p(\mathscr{T}_\ell)$ denotes the Scott-Zhang projection from Lemma 26. The H^s-stability of Π^p_ℓ then shows

$$\begin{aligned}\|\Pi^p_\infty v - \Pi^p_\ell v\|_{H^s(\Gamma)} &\lesssim \|(1-J_\ell)\Pi^p_\infty v\|_{H^s(\Gamma)} \\ &= \|(J_\infty - J_\ell)\Pi^p_\infty v\|_{H^s(\Gamma)} \to 0,\end{aligned}$$

as $\Pi^p_\infty v \in \mathscr{S}^p_\infty$ and hence $\Pi^p_\infty v = J_\infty \Pi^p_\infty v$ by Lemma 78. □

Remark 14 For 2D BEM, the H^1-stability (82) is well-analyzed and found in [51]. For 3D BEM, available results include [17,28,29,33,34,101], and we refer to Section 7.4.2 below.

Remark 15 Suppose that $J_\ell : L_2(\Gamma) \to L_2(\Gamma)$ satisfies $J_\ell(H^1_0(\Gamma)) \subseteq H^1_0(\Gamma)$, i.e., J_ℓ incorporates homogeneous Dirichlet conditions. Suppose that J_ℓ satisfies the properties (i)–(iv) of Lemma 77 with $H^1(\Gamma)$ replaced by $H^1_0(\Gamma) = \widetilde{H}^1(\Gamma)$. Then, the according a priori convergence holds in $\widetilde{H}^s(\Gamma)$ for $0 \le s \le 1$ instead of $H^s(\Gamma)$. This observation applies, in particular, for the Scott-Zhang projection from Lemma 78, and we refer to [12] for stable Scott-Zhang projectors in $\widetilde{H}^s(\Gamma)$. Finally, also Lemma 79 transfers to this case, if Π^p_ℓ denotes the L_2-orthogonal projection onto $\widetilde{\mathscr{S}^p}(\mathscr{T}_\ell)$. We refer to [102] for H^1-stability of this L_2-projection for the lowest-order case $p = 1$, see also Section 7.4.2 below.

Remark 16 For 2D BEM and lowest-order elements, nodal interpolation $J_\ell : C(\overline{\Gamma}) \to \mathscr{S}^1(\mathscr{T}_\ell)$ from Section 3.2.3 satisfies the identity $(J_\ell v)' = \pi^0_\ell(v')$ for all $v \in H^1(\Gamma)$, where $\pi^0_\ell : L_2(\Gamma) \to \mathscr{P}^0(\mathscr{T}_\ell)$ denotes the L_2-orthogonal projection onto $\mathscr{P}^0(\mathscr{T}_\ell)$. Given a sequence of nested meshes $(\mathscr{T}_\ell)_{\ell\in\mathbb{N}_0}$, it is part of [10, Proof of Prop. 5.2] that therefore the limit of $J_\ell v$ exists in $H^1(\Gamma)$, i.e., $\|v_\infty - J_\ell v\|_{H^1(\Gamma)} \to 0$ as $\ell\to\infty$ for some appropriate $v_\infty \in H^1(\Gamma)$.

6.5 $(h-h/2)$-type error estimators

This section follows [10,11,100] and discusses the estimator reduction (76) for the easy-to-implement $(h-h/2)$ error estimator from [12,59,74]. Given any $\mathscr{T}_\ell$, we assume that $\widehat{\mathscr{T}_\ell} \in \texttt{refine}(\mathscr{T}_\ell)$ is the uniform refinement of $\mathscr{T}_\ell$, i.e., it holds nestedness

$$\mathscr{P}^0(\mathscr{T}_\ell) \subseteq \mathscr{P}^0(\mathscr{T}_{\ell+1}) \subseteq \mathscr{P}^0(\widehat{\mathscr{T}_\ell}) \subseteq \mathscr{P}^0(\widehat{\mathscr{T}_{\ell+1}}), \tag{85}$$

and for all $T\in\mathscr{T}_\ell$ and $T'\in\widehat{\mathscr{T}_\ell}$ holds

$$T' \subseteq T \quad \Longrightarrow \quad q\,|T| \le |T'| \le |T|/2, \tag{86}$$

for some fixed and ℓ-independent $0<q\le 1/2$, i.e., the local mesh-sizes of $\mathscr{T}_\ell$ and $\widehat{\mathscr{T}_\ell}$ are comparable.

6.5.1 *Weakly-singular integral equation*

As model problem serves the weakly singular integral equation from Proposition 9. The corresponding $(h-h/2)$-type error estimator from Theorem 54 employs the $L_2(\Gamma)$-orthogonal projection $\pi_\ell^p := \pi_{\mathscr{T}_\ell}^p : L_2(\Gamma) \to \mathscr{P}^p(\mathscr{T}_\ell)$ from Lemma 21 as well as the solution $\widehat{\Phi}_\ell$ of (4), where $\mathscr{X}_\ell = \mathscr{P}^p(\mathscr{T}_\ell)$ is replaced with the uniform refinement $\widehat{\mathscr{X}_\ell} = \mathscr{P}^p(\widehat{\mathscr{T}_\ell})$ corresponding to $\widehat{\mathscr{T}_\ell}$, i.e.,

$$\eta_\ell^2 := \sum_{T\in\mathscr{T}_\ell} \eta_\ell(T)^2 := \sum_{T\in\mathscr{T}_\ell} h_T \|(1-\pi_\ell^p)\widehat{\Phi}_\ell\|_{L_2(T)}^2, \tag{87}$$

where $h_T := |T|^{1/(d-1)} \simeq \mathrm{diam}(T)$. The following lemma proves the estimator reduction estimate (76). The proof reveals that the contraction constant $0<q_{\mathrm{set}}<1$ essentially follows from the contraction (75) of the local mesh-size on refined elements.

Lemma 80 *Given a sequence of nested meshes* $(\mathscr{T}_\ell)_{\ell\in\mathbb{N}}$, *which additionally satisfy the bulk chasing* (9) *for all* $\ell\in\mathbb{N}$ *and some* $0<\theta\le 1$, *the* $(h-h/2)$ *error estimator* η_ℓ *from* (87) *satisfies the estimator reduction* (76) *with* $\alpha_\ell := C_{\mathrm{est}}\|\widehat{\Phi}_{\ell+1}-\widehat{\Phi}_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)}$. *While* q_{est} *depends only on the marking parameter* θ, *the constant* C_{est} *depends additionally on* Γ, *the polynomial degree* p, *and the uniform shape regularity of* $\mathscr{T}_\ell$.

Proof The triangle inequality yields

$$\begin{aligned}\eta_{\ell+1} \le &\|h_{\ell+1}^{1/2}(1-\pi_{\ell+1}^p)\widehat{\Phi}_\ell\|_{L_2(\Gamma)}\\ &+\|h_{\ell+1}^{1/2}(1-\pi_{\ell+1}^p)(\widehat{\Phi}_{\ell+1}-\widehat{\Phi}_\ell)\|_{L_2(\Gamma)}.\end{aligned} \tag{88}$$

Note that the projection $\pi_{\ell+1}^p$ is even the $\mathscr{T}_{\ell+1}$-piecewise best approximation, i.e.,

$$\|(1-\pi_{\ell+1}^p)\psi\|_{L_2(T')} = \min_{\Psi_{\ell+1}\in\mathscr{P}^p(T')} \|\psi - \Psi_{\ell+1}\|_{L_2(T')}.$$

This and the inverse estimate from Lemma 23 prove

$$\begin{aligned}&\|h_{\ell+1}^{1/2}(1-\pi_{\ell+1}^p)(\widehat{\Phi}_{\ell+1}-\widehat{\Phi}_\ell)\|_{L_2(\Gamma)}\\ &\qquad \le \|h_{\ell+1}^{1/2}(\widehat{\Phi}_{\ell+1}-\widehat{\Phi}_\ell)\|_{L_2(\Gamma)} \lesssim \|\widehat{\Phi}_{\ell+1}-\widehat{\Phi}_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)}.\end{aligned}$$

The first summand in (88) is split into the contributions on refined and non-refined elements

$$\begin{aligned}\|h_{\ell+1}^{1/2}(1-\pi_{\ell+1}^p)\widehat{\Phi}_\ell\|_{L_2(\Gamma)}^2 = &\\ \sum_{T\in\mathscr{T}_\ell\setminus\mathscr{T}_{\ell+1}} &\|h_{\ell+1}^{1/2}(1-\pi_{\ell+1}^p)\widehat{\Phi}_\ell\|_{L_2(T)}^2\\ &+\sum_{T\in\mathscr{T}_\ell\cap\mathscr{T}_{\ell+1}} \|h_{\ell+1}^{1/2}(1-\pi_{\ell+1}^p)\widehat{\Phi}_\ell\|_{L_2(T)}^2.\end{aligned}$$

For non-refined elements $T\in\mathscr{T}_\ell\cap\mathscr{T}_{\ell+1}$ holds

$$\begin{aligned}\|h_{\ell+1}^{1/2}(1-\pi_{\ell+1}^p)\widehat{\Phi}_\ell\|_{L_2(T)}^2 &= \|h_\ell^{1/2}(1-\pi_\ell^p)\widehat{\Phi}_\ell\|_{L_2(T)}^2\\ &= \eta_\ell(T)^2.\end{aligned}$$

For refined elements $T\in\mathscr{T}_\ell\setminus\mathscr{T}_{\ell+1}$ holds

$$\begin{aligned}\|h_{\ell+1}^{1/2}(1-\pi_{\ell+1}^p)\widehat{\Phi}_\ell\|_{L_2(T)}^2 &\le 2^{-1/(d-1)}\,\|h_\ell^{1/2}(1-\pi_\ell^p)\widehat{\Phi}_\ell\|_{L_2(T)}^2\\ &= 2^{-1/(d-1)}\,\eta_\ell(T)^2.\end{aligned} \tag{89}$$

Combining this with the bulk chasing (9) and $\mathscr{M}_\ell \subseteq \mathscr{T}_\ell\setminus\mathscr{T}_{\ell+1}$, we obtain

$$\begin{aligned}&\|h_{\ell+1}^{1/2}(1-\pi_{\ell+1}^p)\widehat{\Phi}_\ell\|_{L_2(\Gamma)}^2\\ &\qquad\le \eta_\ell^2 - (1-2^{-1/(d-1)})\sum_{T\in\mathscr{T}_\ell\setminus\mathscr{T}_{\ell+1}} \eta_\ell(T)^2\\ &\qquad\le \big(1-\theta(1-2^{-1/(d-1)})\big)\,\eta_\ell^2.\end{aligned}$$

This concludes the proof with $q_{\mathrm{est}} = \sqrt{1-\theta(1-2^{-1/(d-1)})}$. □

Proposition 81 *Algorithm 1 guarantees convergence* $\lim_{\ell\to\infty}\eta_\ell = 0$ *of the* $(h-h/2)$*-type estimator* η_ℓ *from* (87).

Proof According to Lemma 75 and Lemma 80, it remains to prove $\alpha_\ell\to 0$ as $\ell\to\infty$. By nestedness (85), Lemma 76 provides some limit $\widehat{\Phi}_\infty \in \widetilde{H}^{-1/2}(\Gamma)$ such that $\lim_{\ell\to\infty}\|\widehat{\Phi}_\infty - \widehat{\Phi}_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)} = 0$. Hence,

$$\|\widehat{\Phi}_{\ell+1}-\widehat{\Phi}_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)} \to 0$$

as $\ell\to\infty$. This concludes the proof. □

Remark 17 Usual implementations of uniform mesh-refinement ensure $\mathscr{T}_{\ell+1} \backslash \mathscr{T}_\ell \subseteq \widehat{\mathscr{T}_\ell}$. This implies

$$\|h_{\ell+1}^{1/2}(1-\pi_{\ell+1}^p)\widehat{\Phi}_\ell\|_{L_2(T)}^2 = 0$$

in (89) for all refined elements $T \in \mathscr{T}_\ell \backslash \mathscr{T}_{\ell+1}$ and thus leads to $q_{\rm est} = \sqrt{1-\theta}$ in Lemma 80.

Remark 18 Analogous results hold for other $(h-h/2)$-type estimators like $\eta_\ell = \|h_\ell^{1/2}(\widehat{\Phi}_\ell - \Phi_\ell)\|_{L_2(\Gamma)}$, where $\alpha_\ell \simeq \|\widehat{\Phi}_{\ell+1} - \widehat{\Phi}_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)} + \|\Phi_{\ell+1} - \Phi_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)} \to 0$. We note that, in practice, the variant from (87) is preferred as it avoids the computation of the coarse-mesh Galerkin solution Φ_ℓ.

6.5.2 Hyper-singular integral equation

As model problem serves the hypersingular integral equation from Proposition 12. One possible $(h-h/2)$-type error estimator from Theorem 57 employs the $L_2(\Gamma)$-orthogonal projection $\pi_\ell^{p-1} := \pi_{\mathscr{T}_\ell}^{p-1} : L_2(\Gamma) \to \mathscr{P}^{p-1}(\mathscr{T}_\ell)$ as well as the solution $\widehat{U}_\ell$ of (4), where $\mathscr{X}_\ell = \widetilde{\mathscr{S}}^p(\mathscr{T}_\ell)$ is replaced with the uniform refinement $\widehat{\mathscr{X}_\ell} = \widetilde{\mathscr{S}}^p(\widehat{\mathscr{T}_\ell})$ and reads

$$\eta_\ell^2 := \sum_{T \in \mathscr{T}_\ell} \eta_\ell(T)^2 := \sum_{T \in \mathscr{T}_\ell} h_T \|(1-\pi_\ell^{p-1})\nabla_\Gamma \widehat{U}_\ell\|_{L_2(T)}^2, \quad (90)$$

where $h_T := |T|^{1/(d-1)} \simeq \mathrm{diam}(T)$.

Lemma 82 *Given a sequence of nested meshes $(\mathscr{T}_\ell)_{\ell\in\mathbb{N}}$, which additionally satisfy the bulk chasing* (9) *for all $\ell \in \mathbb{N}$ and some $0 < \theta \le 1$, the $(h-h/2)$ error estimator η_ℓ from* (90) *satisfies the estimator reduction* (76) *with $\alpha_\ell := C_{\rm est}\|\widehat{U}_{\ell+1} - \widehat{U}_\ell\|_{\widetilde{H}^{1/2}(\Gamma)}$. The constant $q_{\rm est}$ depends only on the marking parameter θ, while $C_{\rm est}$ additionally depends on Γ, the polynomial degree p, and uniform shape regularity of $\mathscr{T}_\ell$.*

Proof The proof is very similar to that for the weakly singular integral equation from Lemma 80 and therefore omitted. The only difference is that at the present case, we need the inverse estimate from Lemma 25. □

Proposition 83 *Algorithm 1 guarantees convergence $\lim_{\ell\to\infty} \eta_\ell = 0$ of the $(h-h/2)$-type estimator η_ℓ from* (90).

Proof As the proof of Proposition 81, the statement follows with Lemma 76 and Lemma 82. □

Remark 19 The proofs and assertions of Lemma 82 and Proposition 83 also transfer to other $(h-h/2)$-type error estimators from Theorem 56, e.g.,

$$\eta_\ell = \|h_\ell^{1/2}\nabla_\Gamma(1-J_\ell)\widehat{U}_\ell\|_{L_2(\Gamma)},$$

where

$$\begin{aligned}\eta_{\ell+1} \le &\|h_{\ell+1}^{1/2}\nabla_\Gamma(1-J_\ell)\widehat{U}_\ell\|_{L_2(\Gamma)} \\ &+ \|h_{\ell+1}^{1/2}\nabla_\Gamma\big((1-J_{\ell+1})\widehat{U}_{\ell+1} - (1-J_\ell)\widehat{U}_\ell\big)\|_{L_2(\Gamma)}.\end{aligned}$$

Arguing with the mesh-size reduction as in the proof of Lemma 80, one sees $\|h_{\ell+1}^{1/2}\nabla_\Gamma(1-J_\ell)\widehat{U}_\ell\|_{L_2(\Gamma)} \le q_{\rm est}\,\eta_\ell$. Suppose that J_ℓ satisfies the properties (i)–(iv) of Lemma 77, e.g., J_ℓ is the Scott-Zhang projection from Section 3.2.2. Then, the second term in the above estimate is bounded by

$$\begin{aligned}&\|h_{\ell+1}^{1/2}\nabla_\Gamma\big((1-J_{\ell+1})\widehat{U}_{\ell+1} - (1-J_\ell)\widehat{U}_\ell\big)\|_{L_2(\Gamma)} \\ &\lesssim \|h_{\ell+1}^{1/2}\nabla_\Gamma(J_{\ell+1}-J_\ell)\widehat{U}_{\ell+1}\|_{L_2(\Gamma)} + \|h_{\ell+1}^{1/2}(\widehat{U}_{\ell+1} - \widehat{U}_\ell)\|_{L_2(\Gamma)} \\ &\lesssim \|(J_{\ell+1}-J_\ell)\widehat{U}_{\ell+1}\|_{\widetilde{H}^{1/2}(\Gamma)} + \|\widehat{U}_{\ell+1} - \widehat{U}_\ell\|_{\widetilde{H}^{1/2}(\Gamma)} =: \alpha_\ell\end{aligned}$$

where we have used the inverse estimate of Lemma 25. With the a priori convergence results of Lemma 76 and Lemma 77, one sees that $\alpha_\ell \to 0$ as $\ell \to \infty$. This concludes the proof of the estimator reduction also for other variants of the $(h-h/2)$ error estimator.

6.6 ZZ-type error estimators

For this section, we consider only the lowest-order case, which is $p = 0$ for the weakly singular integral equation and $p = 1$ for the hypersingular integral equation.

6.6.1 Weakly singular integral equation

We consider the problem from Proposition 9. The ZZ-type error estimator from Section 4.4 reads

$$\eta_\ell^2 := \sum_{T \in \mathscr{T}_\ell} \eta_\ell(T)^2 := \sum_{T \in \mathscr{T}_\ell} h_T \|(1-A_\ell)\Phi_\ell\|_{L_2(T)}^2, \quad (91)$$

where the smoothing operator $A_\ell : L_2(\Gamma) \to \mathscr{P}^1(\mathscr{T}_\ell)$ is defined in (58)–(59).

Lemma 84 *Given a sequence of nested meshes $(\mathscr{T}_\ell)_{\ell\in\mathbb{N}}$, which additionally satisfy the bulk chasing* (9) *for all $\ell \in \mathbb{N}$ and some $0 < \theta \le 1$, the ZZ-type error estimator η_ℓ from* (91) *satisfies the estimator reduction* (76) *with $\alpha_\ell := (\|h_{\ell+1}^{1/2}(1-A_\ell)(\Phi_{\ell+1} - \Phi_\ell)\|_{L_2(\Gamma)} + \|h_{\ell+1}^{1/2}(A_{\ell+1} - A_\ell)\Phi_{\ell+1}\|_{L_2(\Gamma)})$. The constant $q_{\rm est}$ depends only on θ.*

Proof The same arguments as used in the proof of Lemma 80 apply. The triangle inequality and reduction of the mesh-size (75) on marked elements result in

$$\begin{aligned}\eta_{\ell+1} &\le q_{\rm est}\,\eta_\ell + \|h_{\ell+1}^{1/2}\big((1-A_{\ell+1})\Phi_{\ell+1} - (1-A_\ell)\Phi_\ell\|_{L_2(\Gamma)} \\ &\le q_{\rm est}\,\eta_\ell + \alpha_\ell.\end{aligned}$$

This concludes the proof. □

Proposition 85 *Algorithm 1 guarantees convergence* $\lim_{\ell\to\infty}\eta_\ell = 0$ *of the ZZ-type estimator* η_ℓ *from* (91).

Proof Lemma 75 and Lemma 84 prove that it suffices to show $\alpha_\ell \to 0$ as $\ell\to\infty$. The operator A_ℓ satisfies the assumptions (i)–(iv) of Lemma 77 (see [67]). For the first contribution of α_ℓ, we use the L_2-stability (i) from Lemma 77 and the inverse estimate from Lemma 23 to see

$$\begin{aligned}\|h_\ell^{1/2}(1-A_\ell)(\Phi_{\ell+1}-\Phi_\ell)\|_{L_2(\Gamma)} &\lesssim \|h_\ell^{1/2}(\Phi_{\ell+1}-\Phi_\ell)\|_{L_2(\Gamma)}\\ &\lesssim \|\Phi_{\ell+1}-\Phi_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)}.\end{aligned}$$

Moreover, Lemma 77 provides a limit operator $A_\infty : L_2(\Gamma)\to L_2(\Gamma)$. For any $k\le\ell$, there holds

$$\begin{aligned}&\|h_\ell^{1/2}(A_{\ell+1}-A_\ell)\Phi_\ell\|_{L_2(\Gamma)}\\ &\quad\le \|h_\ell^{1/2}(A_{\ell+1}-A_\ell)\Phi_k\|_{L_2(\Gamma)}\\ &\qquad + \|h_\ell^{1/2}(A_{\ell+1}-A_\ell)(\Phi_k-\Phi_\ell)\|_{L_2(\Gamma)}\\ &\quad\lesssim \|h_\ell^{1/2}(A_{\ell+1}-A_\ell)\Phi_k\|_{L_2(\Gamma)} + \|h_\ell^{1/2}(\Phi_k-\Phi_\ell)\|_{L_2(\Gamma)}.\end{aligned}$$

where we used the local stability (i) from Lemma 77. The inverse estimate from Lemma 23 shows

$$\begin{aligned}&\|h_\ell^{1/2}(A_{\ell+1}-A_\ell)\Phi_\ell\|_{L_2(\Gamma)}\\ &\qquad\lesssim \|h_\ell^{1/2}(A_{\ell+1}-A_\ell)\Phi_k\|_{L_2(\Gamma)} + \|\Phi_k-\Phi_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)}.\end{aligned}$$

Given any $\varepsilon>0$, Lemma 76 allows to choose $k\in\mathbb{N}$ sufficiently large such that $\|\Phi_\ell-\Phi_k\|_{\widetilde{H}^{-1/2}(\Gamma)}<\varepsilon$ for all $\ell\ge k$. For sufficiently large ℓ, there also holds due to Lemma 77

$$\|h_\ell^{1/2}(A_{\ell+1}-A_\ell)\Phi_k\|_{L_2(\Gamma)} \lesssim \|(A_{\ell+1}-A_\ell)\Phi_k\|_{L_2(\Gamma)} \le \varepsilon.$$

Altogether, we get $\alpha_\ell \lesssim \varepsilon$ for all $\ell\ge k$ and therefore conclude $\alpha\to 0$ as $\ell\to\infty$. □

6.6.2 Hyper singular integral equation

We consider the problem from Proposition 12. The ZZ-type error estimator from Section 4.4 reads

$$\eta_\ell^2 := \sum_{T\in\mathscr{T}_\ell}\eta_\ell(T)^2 := \sum_{T\in\mathscr{T}_\ell} h_T\|(1-A_\ell)\nabla_\Gamma U_\ell\|_{L_2(T)}^2, \tag{92}$$

where the smoothing operator $A_\ell : L_2(\Gamma)\to\mathscr{S}^1(\mathscr{T}_\ell)$ is defined by

$$(A_\ell\psi)(z) := |\omega_z|^{-1}\int_{\omega_z}\psi\,dz \quad \text{for all nodes } z \text{ of } \mathscr{T}_\ell$$

with the node patch $\omega_z := \bigcup\{T\in\mathscr{T}_\ell : z\in\overline{T}\}$. The difference to the weakly singular case is, that $A_\ell\psi$ is continuous everywhere on Γ.

Lemma 86 *Given a sequence of nested meshes* $(\mathscr{T}_\ell)_{\ell\in\mathbb{N}}$, *which additionally satisfy the bulk chasing* (9) *for all* $\ell\in\mathbb{N}$ *and some* $0<\theta\le 1$, *the ZZ-type error estimator* η_ℓ *from* (92) *satisfies the estimator reduction* (76) *with* $\alpha_\ell := (\|h_{\ell+1}^{1/2}(1-A_\ell)\nabla_\Gamma(U_{\ell+1}-U_\ell)\|_{L_2(\Gamma)} + \|h_{\ell+1}^{1/2}(A_{\ell+1}-A_\ell)\nabla_\Gamma U_{\ell+1}\|_{L_2(\Gamma)})$. *The constant* $q_{\rm est}$ *depends only on* θ.

Proof The same arguments as in the proof of Lemma 80, prove the estimator reduction (76) for the ZZ-type error estimator. □

Proposition 87 *Algorithm 1 guarantees convergence* $\lim_{\ell\to\infty}\eta_\ell = 0$ *of the ZZ-type estimator* η_ℓ *from* (92).

Proof The proof follows analogously to the proof of Proposition 85. □

6.7 Weighted-residual error estimators

The weighted residual error estimator for BEM is more complex than $(h-h/2)$-type- or ZZ-type error estimators in the sense that it requires the evaluation of a non-local integral operator. Therefore, the techniques are more involved in this section.

6.7.1 Weakly singular integral equation

We consider the problem from Proposition 9. The standard weighted residual error estimator from Section 4.1.3 for this problem reads

$$\eta_\ell^2 := \sum_{T\in\mathscr{T}_\ell}\eta_\ell(T)^2 := \sum_{T\in\mathscr{T}_\ell} h_T\|\nabla_\Gamma(V\Phi_\ell - f)\|_{L_2(T)}^2, \tag{93}$$

where $\nabla_\Gamma(\cdot)$ denotes the surface gradient on Γ. Note that, while (9) is well-stated for $f\in H^{1/2}(\Gamma)$, the definition of η_ℓ requires additional regularity $f\in H^1(\Gamma)$ of the data.

Lemma 88 *Given a sequence of nested meshes* $(\mathscr{T}_\ell)_{\ell\in\mathbb{N}}$, *which additionally satisfy the bulk chasing* (9) *for all* $\ell\in\mathbb{N}$ *and some* $0<\theta\le 1$, *the weighted residual error estimator* η_ℓ *from* (93) *satisfies the estimator reduction* (76) *with* $\alpha_\ell := C_{\rm est}\|\Phi_{\ell+1}-\Phi_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)}$. *The constant* $q_{\rm est}$ *depends only on* θ, *while* $C_{\rm est}$ *depends additionally on* Γ, *the polynomial degree* p, *and uniform shape regularity of* $\mathscr{T}_\ell$.

Proof The proof follows the lines of the proof of Lemma 80. The triangle inequality and contraction (75) of the mesh-size on marked elements result in

$$\eta_{\ell+1} \le q_{\rm est}\,\eta_\ell + \|h_{\ell+1}^{1/2}\nabla V(\Phi_{\ell+1}-\Phi_\ell)\|_{L_2(\Gamma)}.$$

Instead of the standard inverse estimates, one needs to employ (61) to obtain

$$\|h_{\ell+1}^{1/2}\nabla V(\Phi_{\ell+1}-\Phi_\ell)\|_{L_2(\Gamma)} \lesssim \|\Phi_{\ell+1}-\Phi_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)}.$$

This concludes the proof. □

Proposition 89 *Algorithm 1 guarantees convergence* $\lim_{\ell\to\infty}\eta_\ell = 0$ *of the weighted residual error estimator* η_ℓ *from* (93)*.*

6.7.2 Hyper-singular integral equation

We consider the problem from Proposition 12. The standard weighted residual error estimator from Section 4.1.3 for this problem reads

$$\eta_\ell^2 := \sum_{T\in\mathscr{T}_\ell} \eta_\ell(T)^2 := \sum_{T\in\mathscr{T}_\ell} h_T \|WU_\ell - f\|_{L_2(T)}^2. \tag{94}$$

Note that, while (12) is well-stated for $f\in\widetilde{H}^{-1/2}(\Gamma)$, the definition of η_ℓ requires additional regularity $f\in L_2(\Gamma)$ of the data.

Lemma 90 *Given a sequence of nested meshes* $(\mathscr{T}_\ell)_{\ell\in\mathbb{N}}$*, which additionally satisfy the bulk chasing* (9) *for all* $\ell\in\mathbb{N}$ *and some* $0<\theta\le 1$*, the weighted residual error estimator* η_ℓ *from* (94) *satisfies the estimator reduction* (76) *with* $\alpha_\ell := C_{\rm est}\|U_{\ell+1}-U_\ell\|_{\widetilde{H}^{1/2}(\Gamma)}$*. The constant* $q_{\rm est}$ *depends only on* θ*, while* $C_{\rm est}$ *depends additionally on* Γ*, the polynomial degree* p*, and uniform shape regularity of* $\mathscr{T}_\ell$*.*

Proof The proof works analogously to the proof of Lemma 88, only this time employ the inverse-type estimate

$$\|h_{\ell+1}^{1/2} W(U_{\ell+1}-U_\ell)\|_{L_2(\Gamma)} \lesssim \|U_{\ell+1}-U_\ell\|_{\widetilde{H}^{1/2}(\Gamma)}.$$

from (61). □

Arguing as before, Lemma 90 allows to derive convergence of the related ABEM with Lemma 75.

Proposition 91 *Algorithm 1 guarantees convergence* $\lim_{\ell\to\infty}\eta_\ell = 0$ *of the weighted residual error estimator* η_ℓ *from* (94)*.*

6.8 Approximation of right-hand side data with $((h-h/2))$-type estimators

In many cases, the right-hand side F in (3) involves the application of integral operators to the given data which can hardly be computed analytically in practice. To circumvent this bottleneck, the aim of data approximation is to replace the right-hand side F in (3) with some computable approximation F_ℓ on any mesh $\mathscr{T}_\ell$ and to solve

$$b(U_\ell, V) = F_\ell(V) \quad \text{for all } V\in\mathscr{X}_\ell$$

instead of (4).

6.8.1 Weakly singular integral equation

We consider the problem from Proposition 10, where $\Gamma = \partial\Omega$ and the right-hand side in (3) reads $F(\psi) := \langle (1/2+K)f\,,\psi\rangle_\Gamma$ for all $\psi\in\widetilde{H}^{-1/2}(\Gamma)$. We approximate the right-hand side by approximating $f\in H^{1/2}(\Gamma)$ via the Scott-Zhang projection J_ℓ^{p+1}, i.e., $f_\ell := J_\ell^{p+1} f \in \mathscr{S}^{p+1}(\mathscr{T}_\ell)$, and hence $F_\ell(\psi) := \langle (1/2+K)f_\ell\,,\psi\rangle_\Gamma$. We thus end up with the formulation given in Proposition 15. The additional error is controlled via extending the $((h-h/2))$-type error estimator from (87) by a data oscillation term

$$\begin{aligned}\eta_\ell^2 &:= \sum_{T\in\mathscr{T}_\ell} \eta_\ell(T)^2 \\ &:= \sum_{T\in\mathscr{T}_\ell} h_T\big(\|(1-\pi_\ell^p)\widehat{\Phi}_\ell\|_{L_2(T)}^2 + \|\nabla_\Gamma(1-J_\ell^{p+1})f\|_{L_2(T)}^2\big),\end{aligned} \tag{95}$$

cf. Section 4.6.2.

Lemma 92 *Given a sequence of nested meshes* $(\mathscr{T}_\ell)_{\ell\in\mathbb{N}}$*, which additionally satisfy the bulk chasing* (9) *for all* $\ell\in\mathbb{N}$ *and some* $0<\theta\le 1$*, the* $((h-h/2))$*-type error estimator* η_ℓ *with data oscillation term from* (95) *satisfies the estimator reduction* (76) *with* $\alpha_\ell := C_{\rm est}\big(\|\widehat{\Phi}_{\ell+1}-\widehat{\Phi}_\ell\|_{H^{-1/2}(\Gamma)} + \|(J_{\ell+1}^{p+1}-J_\ell^{p+1})f\|_{H^{1/2}(\Gamma)}\big)$*. The constant* $q_{\rm est}$ *depends only on* θ*, while* $C_{\rm est}$ *depends additionally on* Γ*, the polynomial degree* p*, and uniform shape regularity of* $\mathscr{T}_\ell$*.*

Proof The same arguments as used in the proof of Lemma 80 apply. The triangle inequality and reduction of the mesh-size (75) on marked elements result in

$$\begin{aligned}\eta_{\ell+1} \le q_{\rm est}\,\eta_\ell &+ \|h_{\ell+1}^{1/2}(1-\pi_{\ell+1}^p)(\widehat{\Phi}_{\ell+1}-\widehat{\Phi}_\ell)\|_{L_2(\Gamma)} \\ &+ \|h_{\ell+1}^{1/2}\nabla_\Gamma(J_{\ell+1}^{p+1}-J_\ell^{p+1})f\|_{L_2(\Gamma)}\end{aligned}$$

As before, the inverse estimate from Lemma 23 proves

$$\|h_{\ell+1}^{1/2}(1-\pi_{\ell+1}^p)(\widehat{\Phi}_{\ell+1}-\widehat{\Phi}_\ell)\|_{L_2(\Gamma)} \lesssim \|\widehat{\Phi}_{\ell+1}-\widehat{\Phi}_\ell\|_{H^{-1/2}(\Gamma)}.$$

Moreover, the inverse estimate from Lemma 25 gives

$$\|h_{\ell+1}^{1/2}\nabla_\Gamma(J_{\ell+1}^{p+1}-J_\ell^{p+1})f\|_{L_2(\Gamma)} \lesssim \|(J_{\ell+1}^{p+1}-J_\ell^{p+1})f\|_{H^{1/2}(\Gamma)}$$

and concludes the proof. □

Proposition 93 *Algorithm 1 guarantees convergence* $\lim_{\ell\to\infty}\eta_\ell = 0$ *of the* $((h-h/2))$*-type estimator* η_ℓ *with data approximation from* (95)*.*

Proof Lemma 78 proves a priori convergence

$$\|(J_\infty - J_\ell)f\|_{H^{1/2}(\Gamma)} \xrightarrow{\ell\to\infty} 0.$$

Consequently, it holds

$$\|(J_{\ell+1} - J_\ell)f\|_{H^{1/2}(\Gamma)} \xrightarrow{\ell\to\infty} 0.$$

It remains to prove that

$$\|\widehat{\Phi}_{\ell+1} - \widehat{\Phi}_\ell\|_{H^{-1/2}(\Gamma)}.$$

Note that one cannot directly employ Lemma 76, since $\widehat{\Phi}_{\ell+1}$ and $\widehat{\Phi}_\ell$ are computed with respect to *different* right-hand sides. To tackle this issue, let $\widehat{\Phi}_{\ell,\infty} \in \mathscr{P}^p(\widehat{\mathscr{T}_\ell})$ be the unique solution of

$$b(\widehat{\Phi}_{\ell,\infty}, V) = \langle (1/2+K)J_\infty f, V\rangle_{L_2(\Gamma)} \quad \text{for all } V \in \widehat{\mathscr{X}_\ell}.$$

For this, Lemma 76 applies and proves convergence $\|\widehat{\Phi}_\infty - \widehat{\Phi}_{\ell,\infty}\|_{H^{-1/2}(\Gamma)} \to 0$ as $\ell \to \infty$. The triangle inequality proves

$$\begin{aligned}
&\|\widehat{\Phi}_{\ell+1} - \widehat{\Phi}_\ell\|_{H^{-1/2}(\Gamma)} \\
&\quad \le \|\widehat{\Phi}_{\ell,\infty} - \widehat{\Phi}_\ell\|_{H^{-1/2}(\Gamma)} + \|\widehat{\Phi}_{\ell+1} - \widehat{\Phi}_{\ell+1,\infty}\|_{H^{-1/2}(\Gamma)} \\
&\qquad + \|\widehat{\Phi}_{\ell,\infty} - \widehat{\Phi}_{\ell+1,\infty}\|_{H^{-1/2}(\Gamma)}.
\end{aligned}$$

The third term on the right-hand side already vanishes as $\ell \to \infty$. For the remaining to terms, the stability of the problem and Lemma 79 show

$$\begin{aligned}
&\|\widehat{\Phi}_{\ell,\infty} - \widehat{\Phi}_\ell\|_{H^{-1/2}(\Gamma)} + \|\widehat{\Phi}_{\ell+1} - \widehat{\Phi}_{\ell+1,\infty}\|_{H^{-1/2}(\Gamma)} \\
&\quad \lesssim \|f_\ell - J_\infty f\|_{H^{1/2}(\Gamma)} + \|f_{\ell+1} - J_\infty f\|_{H^{1/2}(\Gamma)} \to 0
\end{aligned}$$

as $\ell \to \infty$. Altogether, we obtain $\lim_{\ell\to\infty} \alpha_\ell = 0$ and conclude the proof. □

Remark 20 As a consequence of Proposition 93, one obtains that $\|f - J_\ell f\|_{H^{1/2}(\Gamma)} \lesssim \eta_\ell \to 0$ as $\ell \to \infty$, i.e., $J_\infty f = f$.

Remark 21 If the L_2-orthogonal projection $\Pi_\ell^{p+1} : L_2(\Gamma) \to \mathscr{S}^{p+1}(\mathscr{T}_\ell)$ is H^1-stable (82), Lemma 92 and Proposition 93 transfer to data approximation with $f_\ell = \Pi_\ell^{p+1} f$. In practice, this approach is preferred, since it might lead to superconvergence for pointwise errors inside of Ω.

Remark 22 The proofs of Lemma 92 and Proposition 93 transfer to situations, where the approximation error of $f_\ell = J_\ell f \approx f$ is controlled by $\|h_\ell^{1/2}(1-\pi_\ell^p)\nabla_\Gamma f\|_{L_2(\Gamma)}$ in (95) instead of $\|h_\ell^{1/2}\nabla_\Gamma(1-J_\ell)f\|_{L_2(\Gamma)}$. This requires additional care with the mesh-refinement to ensure equivalence of these two norms, cf. Section 4.6.2. Possible mesh refinement strategies are discussed in Section 7 below. In this case, one may even use more general $H^{1/2}$-stable projections $J_\ell : H^{1/2}(\Gamma) \to \mathscr{S}^{p+1}(\mathscr{T}_\ell)$ instead of the Scott-Zhang projection to discretize the data, see [8, 65]. This approach will be presented in Section 6.9.1.

6.8.2 Hyper singular integral equation

We consider the problem from Proposition 13, where $\Gamma = \partial\Omega$ and the right-hand side in (3) reads $F(\psi) := \langle (1/2 - K')f, v\rangle_\Gamma$ for all $v \in H^{1/2}(\Gamma)$. We approximate the right-hand side by approximating $f \in H^{-1/2}(\Gamma)$ by $f_\ell := \pi_\ell^{p-1} f \in \mathscr{P}^{p-1}(\mathscr{T}_\ell)$ and let $F_\ell(v) := \langle (1/2-K')f_\ell, v\rangle_\Gamma$. Recall that π_ℓ^{p-1} is the $L_2(\Gamma)$-orthogonal projection onto $\mathscr{P}^{p-1}(\mathscr{T}_\ell)$. We thus end up with the formulation given in Proposition 16. The additional error is controlled via extending the error estimator by a data oscillation term

$$\begin{aligned}
\eta_\ell^2 &:= \sum_{T\in\mathscr{T}_\ell} \eta_\ell(T)^2 \\
&:= \sum_{T\in\mathscr{T}_\ell} h_T\big(\|(1-\pi_\ell^{p-1})\nabla_\Gamma \widehat{U}_\ell\|_{L_2(T)}^2 \\
&\qquad\qquad + \|(1-\pi_\ell^{p-1})f\|_{L_2(T)}^2\big),
\end{aligned} \tag{96}$$

cf. Section 4.6.3

Lemma 94 *Given a sequence of nested meshes $(\mathscr{T}_\ell)_{\ell\in\mathbb{N}}$, which additionally satisfy the bulk chasing* (9) *for all $\ell \in \mathbb{N}$ and some $0 < \theta \le 1$, the $(h-h/2)$-type error estimator with data oscillation term from* (96) *satisfies the estimator reduction* (76) *with $\alpha_\ell := C_{\rm est}\|\widehat{U}_{\ell+1} - \widehat{U}_\ell\|_{H^{1/2}(\Gamma)}$. While $q_{\rm est}$ depends only on θ, the constant $C_{\rm est}$ depends additionally on Γ, the polynomial degree p, the marking parameter θ, and uniform shape regularity of $\mathscr{T}_\ell$.*

Proof The proof works analogously to that of the weakly singular case from Lemma 92, but may additionally use that $\|(1-\pi_{\ell+1}^{p-1})f\|_{L_2(T)} \le \|(1-\pi_\ell^{p-1})f\|_{L_2(T)}$ for all $T \in \mathscr{T}_\ell$. This leads to an improved perturbation term α_ℓ. □

As for the weakly singular case in Proposition 93, we obtain convergence of data perturbed ABEM for the hypersingular integral equation.

Proposition 95 *Algorithm 1 guarantees convergence* $\lim_{\ell\to\infty}\eta_\ell = 0$ *of the ($(h-h/2)$)-type estimator η_ℓ with data approximation from* (96).

6.9 Approximation of right-hand side data with weighted residual estimators

6.9.1 Weakly singular integral equation

We consider the problem from Proposition 10, where $\Gamma = \partial\Omega$ and the right-hand side is given by $F(\psi) := \langle (1/2 + K)f, \psi\rangle_\Gamma$ for all $\psi \in \widetilde{H}^{-1/2}(\Gamma)$. Let $J_\ell^{p+1} : H^{1/2}(\Gamma) \to \mathscr{S}^{p+1}(\mathscr{T}_\ell)$ be an arbitrary $H^{1/2}(\Gamma)$ stable projection such that the pointwise limit

$$J_\infty^{p+1} v = \lim_{\ell\to\infty} J_\ell^{p+1} v$$

exists for any $v \in H^{1/2}(\Gamma)$. We approximate the right-hand side by approximating $f \in H^{1/2}(\Gamma)$ by $f_\ell := J_\ell^{p+1} f \in \mathscr{S}^{p+1}(\mathscr{T}_\ell)$ and let $F_\ell(\psi) := \langle (1/2+K) f_\ell\,, \psi\rangle_\Gamma$, i.e., we arrive at the discrete formulation of Proposition 15. This additional error is controlled via extending the error estimator by a data oscillation term

$$\eta_\ell^2 := \sum_{T\in\mathscr{T}_\ell} \eta_\ell(T)^2 \qquad (97)$$
$$:= \sum_{T\in\mathscr{T}_\ell} h_T \big(\|\nabla_\Gamma (V\Phi_\ell - (1/2+K) f_\ell)\|_{L_2(T)}^2 + \|(1-\pi_\ell^p)\nabla_\Gamma f\|_{L_2(T)}^2\big).$$

Possible example for J_ℓ^{p+1} include the Scott-Zhang projection onto $\mathscr{S}^{p+1}(\mathscr{T}_\ell)$, cf. Lemma 78, as well as the L_2 - orthogonal projection onto $\mathscr{S}^{p+1}(\mathscr{T}_\ell)$, provided that the latter is $H^1(\Gamma)$ stable, cf Lemma 79.

Lemma 96 *Given a sequence of nested meshes* $(\mathscr{T}_\ell)_{\ell\in\mathbb{N}}$, *which additionally satisfy the bulk chasing* (9) *for all* $\ell \in \mathbb{N}$ *and some* $0 < \theta \le 1$, *the weighted residual error estimator* η_ℓ *with data oscillation term from* (97) *satisfies the estimator reduction* (76) *with* $\alpha_\ell := C_{\rm est}(\|\Phi_{\ell+1} - \Phi_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)} + \|f_{\ell+1} - f_\ell\|_{H^{1/2}(\Gamma)})$. *While* $q_{\rm est}$ *depends only on* θ, *the constant* $C_{\rm est}$ *depends additionally on* Γ, *the polynomial degree* p, *the marking parameter* θ, *and uniform shape regularity of* $\mathscr{T}_\ell$.

Proof The data oscillation term is treated as in the proof of Lemma 92. For the estimator, one needs additionally the inverse estimate (61) to estimate

$$\|h_{\ell+1}^{1/2}\nabla_\Gamma (1/2+K)(f_{\ell+1} - f_\ell)\|_{L_2(\Gamma)} \lesssim \|f_{\ell+1} - f_\ell\|_{H^{1/2}(\Gamma)}.$$

The remainder however, follows exactly the lines of the proof of Lemma 88. □

Proposition 97 *Algorithm 1 guarantees convergence* $\lim_{\ell\to\infty}\eta_\ell = 0$ *of the weighted residual estimator with data approximation from* (97).

Proof The proof follows the lines of Proposition 93. Additionally, we need to employ the convergence $\lim_{\ell\to\infty}\|f_{\ell+1} - f_\ell\|_{H^{1/2}(\Gamma)}^2 = 0$ from Lemma 79. □

6.9.2 Hyper-singular integral equation

We consider the problem from Proposition 13, where the right-hand side in (3) reads $F(v) := \langle (1/2-K') f\,, v\rangle_\Gamma$ for all $v \in H^{1/2}(\Gamma)$. We approximate the right-hand side by approximating $f \in \widetilde{H}^{-1/2}(\Gamma)$ by $f_\ell := \pi_\ell^{p-1} f \in \mathscr{P}^{p-1}(\mathscr{T}_\ell)$ and let $F_\ell(v) := \langle (1/2-K') f_\ell\,, v\rangle_\Gamma$, i.e., we arrive at the discrete formulation of Proposition 16. The additional error is controlled via extending the error estimator by a data oscillation term

$$\eta_\ell^2 := \sum_{T\in\mathscr{T}_\ell} \eta_\ell(T)^2 \qquad (98)$$
$$:= \sum_{T\in\mathscr{T}_\ell} h_T \big(\|\nabla_\Gamma (WU_\ell - (1/2-K') f_\ell)\|_{L_2(T)}^2 + \|(1-\pi_\ell^{p-1}) f\|_{L_2(T)}^2\big).$$

Lemma 98 *Given a sequence of nested meshes* $(\mathscr{T}_\ell)_{\ell\in\mathbb{N}}$, *which additionally satisfy the bulk chasing* (9) *for all* $\ell \in \mathbb{N}$ *and some* $0 < \theta \le 1$, *the weighted residual error estimator with data oscillation term* η_ℓ *from* (98) *satisfies the estimator reduction* (76) *with* $\alpha_\ell := C_{\rm est}(\|U_{\ell+1} - U_\ell\|_{H^{1/2}(\Gamma)} + \|f_{\ell+1} - f_\ell\|_{H^{-1/2}(\Gamma)})$. *The constants* $q_{\rm est}, C_{\rm est}$ *depend only on* Γ, *the marking parameter* θ, *the polynomial degree* p, *and the uniform* σ_ℓ*-shape regularity.*

Proof The data oscillation term is treated as in the proof of Lemma 92. For the estimator, one needs additionally the inverse estimate (61) to estimate

$$\|h_{\ell+1}^{1/2}(1/2-K')(f_{\ell+1} - f_\ell)\|_{L_2(\Gamma)} \lesssim \|f_{\ell+1} - f_\ell\|_{H^{-1/2}(\Gamma)}.$$

The remainder however, follows exactly the lines of the proof of Lemma 90. □

Proposition 99 *Algorithm 1 guarantees convergence* $\lim_{\ell\to\infty}\eta_\ell = 0$ *of the weighted residual estimator with data approximation from* (98).

Proof The proof follows along the lines of Proposition 93. Additionally, we need to employ the convergence $\lim_{\ell\to\infty}\|f_{\ell+1} - f_\ell\|_{H^{-1/2}(\Gamma)}^2 \le \lim_{\ell\to\infty}\|f_{\ell+1} - f_\ell\|_{L_2(\Gamma)}^2 = 0$ from Lemma 79. □

6.10 Anisotropic mesh refinement

The presence of edge singularities in solutions of simple problems like $V\phi = 1$ on some boundary $\Gamma := \partial\Omega$ with $\Omega \subseteq \mathbb{R}^3$ makes it necessary to allow for anisotropic mesh refinement if one aims to achieve optimal convergence rates. This implies that the shape-regularity constant σ_ℓ from Section 2.6 cannot remain bounded for a given sequence of meshes $(\mathscr{T}_\ell)_{\ell\in\mathbb{N}_0}$, but satisfies $\sup_{\ell\in\mathbb{N}_0}\sigma_\ell = \infty$. The following variant of the $((h-h/2))$-type error estimator accounts for this:

$$\eta_\ell^2 := \sum_{T\in\mathscr{T}_\ell} \eta_\ell(T)^2 := \sum_{T\in\mathscr{T}_\ell} \rho_T \|(1-\pi_\ell^0)\widehat{\Phi}_\ell\|_{L_2(T)}^2, \qquad (99)$$

where $\rho_T > 0$ denotes the radius of the largest inscribed circle of the element $T \in \mathscr{T}_\ell$. Obviously, there holds $\rho_T \le h_T$, and $\sup_{T\in\mathscr{T}_\ell} h_T/\rho_T$ depends only on σ_ℓ. We briefly discuss the lowest-order case $p = 0$ for rectangular elements $T \in$

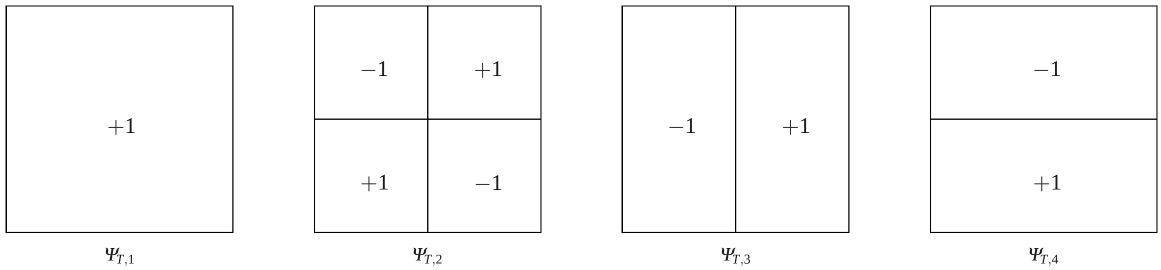

Fig. 16 Functions $\Psi_{T,i}$ with their values on the element $T \in \mathscr{T}_\ell$ used for the computation of the error estimator from Section 6.10.

$\mathscr{T}_\ell$, which provides an easy-to-implement criterion to decide how to refine the elements, while the general case $p \geq 0$ is discussed in [11]. As depicted in Figure 16, we define four element functions $\Psi_{T,i} \in \mathscr{P}^0(\widehat{\mathscr{T}_\ell})$ with $\mathrm{supp}(\Psi_{T,i}) \subseteq T$. Note that $\{\Psi_{T,i} : T \in \mathscr{T}_\ell, i = 1, \ldots, 4\}$ defines a basis of $\mathscr{P}^0(\widehat{\mathscr{T}_\ell})$. Hence, for each $T \in \mathscr{T}_\ell$, there exist (computable) coefficients

$$c_{T,i} := \frac{\int_T \Psi_{T,i}\, \widehat{\Phi}_\ell \, dx}{\|\Psi_{T,i}\|_{L_2(T)}} \quad \text{for } i = 1,2,3,4,$$

such that

$$\widehat{\Phi}_\ell = \sum_{i=1}^{4} c_{T,i}\Psi_{T,i} \quad \text{on } T.$$

If one intends to refine T (i.e., T is marked for refinement by the bulk chasing (9)), the following set of rules decides the direction of refinement: Choose an additional parameter $0 < \tau < 1$ which steers the sensitivity to directional refinement (cf. Figure 4 from the introduction).

(R1) If $c_{T,2}^2 + c_{T,3}^2 \leq \tau/(1-\tau)c_{T,4}^2$, split T along the vertical direction to generate two sons $T_1, T_2 \in \mathscr{T}_{\ell+1}$.
(R2) If $c_{T,2}^2 + c_{T,4}^2 \leq \tau/(1-\tau)c_{T,3}^2$, split T along the horizontal direction to generate two sons $T_1, T_2 \in \mathscr{T}_{\ell+1}$.
(R3) If none of the above applies split T along both directions to generate four sons $T_1, T_2, T_3, T_4 \in \mathscr{T}_{\ell+1}$.

We note that (R1) and (R2) are exclusive, i.e., if the criterion from (R1) is satisfied, the criterion from (R2) fails to hold (cf. [11]). Moreover, we need to ensure the following two refinement rules to guarantee the validity of the inverse estimate [78, Thm. 3.6] of Lemma 23 on anisotropic meshes.

(R4) Hanging nodes are at most of order one, i.e., each side e of an elements $T \in \mathscr{T}_\ell$ contains at most one node z which is not an endpoint of e.
(R5) K-mesh property: there holds for some $\kappa_\ell > 0$

$$\frac{\rho_T}{\rho_{T'}} + \frac{h_T}{h_{T'}} \leq \kappa_\ell < \infty,$$

for all $T, T' \in \mathscr{T}_\ell$ with $T \cap T' \neq \emptyset$.

The following lemma is proved in [74, 11].

Lemma 100 *Given a sequence of meshes $(\mathscr{T}_\ell)_{\ell\in\mathbb{N}}$ with $\mathscr{T}_{\ell+1} \in \mathtt{refine}(\mathscr{T}_\ell)$ for all $\ell \in \mathbb{N}_0$ and $K := \sup_{\ell\in\mathbb{N}_0} \kappa_\ell < \infty$, which additionally satisfy the bulk chasing* (9) *for all $\ell \in \mathbb{N}$ and some $0 < \theta \leq 1$. Suppose that all marked elements $\mathscr{M}_\ell \subseteq \mathscr{T}_\ell \setminus \mathscr{T}_{\ell+1}$ are refined according to the rules (R1)–(R5). Then, the modified $(h - h/2)$ error estimator η_ℓ from* (99) *satisfies the estimator reduction* (76) *with $\alpha_\ell := C_{\mathrm{est}}\|\widehat{\Phi}_{\ell+1} - \widehat{\Phi}_\ell\|^2_{\widetilde{H}^{-1/2}(\Gamma)}$. The constants q_{est} depends only on θ and τ, while C_{est} depends additionally on Γ, the polynomial degree p, and the K-mesh constant K.*

As before, Lemma 75 implies convergence of ABEM.

Proposition 101 *Algorithm 1 guarantees convergence $\lim_{\ell\to\infty} \eta_\ell = 0$ of the modified ($(h - h/2)$)-type estimator η_ℓ from* (99) *for anisotropic mesh refinement.*

7 Mesh refinement

When it comes to the mathematical proof of optimal convergence rates of ABEM (Section 8), it is clear that this requires certain properties of the mesh refinement which go beyond the elementary properties from Section 6.1. While those are sufficient to prove plain convergence of ABEM by means of the estimator reduction principle from Section 6, they formally do not prevent that marking of *one single* element $\mathscr{M}_\ell = \{T\} \subset \mathscr{T}_\ell$ results in a refinement $\mathscr{T}_{\ell+1}$, where *all* elements have been refined, i.e., $\mathscr{T}_\ell \backslash \mathscr{T}_{\ell+1} = \mathscr{T}_\ell$. Moreover, the contemporary mathematical proofs of optimal convergence rates require certain additional properties.

7.1 General notation

Suppose a fixed mesh refinement strategy $\mathtt{refine}(\cdot)$ and an admissible mesh $\mathscr{T}$, i.e., the mesh refinement $\mathtt{refine}(\cdot)$ can be used to refine $\mathscr{T}$ and provides a refined admissible mesh. For $\mathscr{M} \subseteq \mathscr{T}$ being a set of marked elements, we write $\mathscr{T}' = \mathtt{refine}(\mathscr{T}, \mathscr{M})$ if $\mathscr{T}'$ is the coarsest admissible mesh which is obtained from $\mathscr{T}$ by refinement of at least the marked elements $\mathscr{M}$, i.e., $\mathscr{T} \backslash \mathscr{T}' \supseteq \mathscr{M}$. For some admissible mesh $\mathscr{T}$, we write $\mathscr{T}' \in \mathtt{refine}(\mathscr{T})$, if there exists some $k \in \mathbb{N}_0$ and sets of marked elements $\widetilde{\mathscr{M}}_0, \dots, \widetilde{\mathscr{M}}_{k-1}$ as well as meshes $\widetilde{\mathscr{T}}_0, \widetilde{\mathscr{T}}_1, \dots, \widetilde{\mathscr{T}}_k$ such that $\widetilde{\mathscr{M}}_j \subseteq \widetilde{\mathscr{T}}_j$ and $\widetilde{\mathscr{T}}_{j+1} = \mathtt{refine}(\widetilde{\mathscr{T}}_j, \widetilde{\mathscr{M}}_j)$ for all $j = 0, \dots, k-1$, with $\mathscr{T} = \widetilde{\mathscr{T}}_0$ and $\mathscr{T}' = \widetilde{\mathscr{T}}_k$.

7.2 Optimality conditions on mesh refinement

Besides the naive properties from Section 6.1, the contemporary mathematical proofs of optimal convergence rates require certain additional properties of the mesh refinement. Suppose that $\mathscr{T}_0$ is a given admissible initial mesh for the adaptive algorithm and that $\mathscr{T}_\ell$ for $\ell \geq 1$ is obtained inductively by $\mathscr{T}_\ell = \mathtt{refine}(\mathscr{T}_{\ell-1}, \mathscr{M}_{\ell-1})$. Let η_ℓ be the a posteriori error estimator used to mark elements $\mathscr{M}_\ell \subseteq \mathscr{T}_\ell$ for refinement. Then, the analysis of Section 8 relies on the following three properties of $\mathtt{refine}(\cdot)$, where $\#(\cdot)$ denotes the number of elements of a finite set:

- **Bounded shape-regularity:** The mesh refinement strategy has to ensure that all estimator related constants in, e.g., reliability or efficiency estimates (13)–(14), remain uniformly bounded as $\ell \to \infty$.
- **Mesh-closure estimate:** The number of refined elements can (at least in average and up to some multiplicative constant) be controlled by the number of marked elements in the sense that
$$\#\mathscr{T}_{\ell+1} - \#\mathscr{T}_0 \leq C_{\mathrm{nvb}} \sum_{k=0}^{\ell} \#\mathscr{M}_k \tag{100}$$
for some constant $C_{\mathrm{nvb}} > 0$.
- **Overlay estimate:** To compare the adaptively generated meshes $\mathscr{T}_\ell$ with some (purely theoretical) optimal mesh, one requires that for all $\mathscr{T}_\star \in \mathtt{refine}(\mathscr{T}_0)$ exists a coarsest common refinement $\mathscr{T}_\star \oplus \mathscr{T}_\ell$ of both $\mathscr{T}_\star$ and $\mathscr{T}_\ell$ such that
$$\#(\mathscr{T}_\star \oplus \mathscr{T}_\ell) \leq \#\mathscr{T}_\star + \#\mathscr{T}_\ell - \#\mathscr{T}_0. \tag{101}$$

Different methods for local mesh refinement are available in the literature. To the best of our knowledge, only three strategies are available which ensure these properties: for 2D BEM, the extended 1D bisection algorithm from [7]; for 3D BEM, the 2D NVB algorithm[11], see e.g. [140, 101], as well as red-refinement with hanging nodes of maximum order 1, see [27]. In the following, we shall discuss the extended 1D bisection algorithm from [7] as well as the results of [101] on 2D NVB.

We close this section with some historical remarks. The mesh-closure estimate (100) has first been proved in [26] for 2D NVB and later for NVB and general dimension $d \geq 2$ in [140]. Either work requires an additional assumption on the initial mesh $\mathscr{T}_0$. For 2D, this assumption has recently been removed in [101]. The overlay estimate (101) first appeared in [139] for 2D NVB. In [46], the proof is generalized to NVB in arbitrary dimension $d \geq 2$. Bisection in 1D has only been considered and analyzed in [7]. Even though the above mesh refinement strategies seem fairly arbitrary, to the best of the authors' knowledge, NVB is the only refinement strategy for $d \geq 2$ known to satisfy (100)–(101). Even the simple red-green-blue refinement, see e.g. [34], fails to satisfy (101), while the mesh-closure estimate (100) can still be proved, see [101] and the references therein.

7.3 Extended 1D bisection for 2D BEM

In 2D BEM, the constants in the a priori or a posteriori error analysis usually depend on a uniform upper bound $\sigma > 0$ of the shape-regularity constant (or: bounded local mesh-ratio)
$$\frac{\mathrm{diam}(T)}{\mathrm{diam}(T')} \leq \sigma \text{ for all neighbors } T, T' \in \mathscr{T}_\ell \text{ and } \ell \geq 0. \tag{102}$$

Since this property is not guaranteed by simple 1D bisection algorithms, it has to be ensured explicitly. With the shape-regularity constant of the initial mesh
$$\sigma_{\mathscr{T}_0} := \max\{\frac{\mathrm{diam}(T)}{\mathrm{diam}(T')} : T, T' \in \mathscr{T}_0 \text{ with } \overline{T} \cap \overline{T}' \neq \emptyset\},$$
we use the following algorithm from [7].

Algorithm 102 (Extended 1D biesction) INPUT*: local mesh-ratio* $\sigma_{\mathscr{T}_0}$*, current mesh* $\mathscr{T}_\ell$*, and set of marked elements* $\mathscr{M}_\ell \subseteq \mathscr{T}_\ell$*.*
OUTPUT*: refined mesh* $\mathscr{T}_{\ell+1}$*.*

(o) *Set counter* $k := 0$ *and define* $\mathscr{M}_\ell^{(0)} := \mathscr{M}_\ell$ *.*
(i) *Define* $\mathscr{U}^{(k)} := \bigcup_{T \in \mathscr{M}_\ell^{(k)}} \{T' \in \mathscr{T}_\ell \backslash \mathscr{M}_\ell^{(k)}$ *neighbor of* T : $\mathrm{diam}(T') > \sigma_{\mathscr{T}_0} \mathrm{diam}(T)\}$ *and* $\mathscr{M}_\ell^{(k+1)} := \mathscr{M}_\ell^{(k)} \cup \mathscr{U}^{(k)}$.
(ii) *If* $\mathscr{M}_\ell^{(k)} \subsetneqq \mathscr{M}_\ell^{(k+1)}$*, increase counter* $k \mapsto k+1$ *and goto (i).*
(iii) *Otherwise bisect all elements* $T \in \mathscr{M}^{(k)}$ *to obtain the new mesh* $\mathscr{T}_{\ell+1}$*.*

The following result is proved in [7, Thm. 2.3].

Theorem 103 *Suppose that* $\mathscr{T}_0$ *is a partition of* Γ*, and* $(\mathscr{T}_\ell)_{\ell \in \mathbb{N}_0}$ *is generated by Algorithm 102, i.e., for all* $\ell \in \mathbb{N}_0$ *holds*
$$\mathscr{T}_{\ell+1} = \mathtt{refine}(\mathscr{T}_\ell, \mathscr{M}_\ell).$$
Then, bounded shape regularity (102) *with* $\sigma = 2\sigma_{\mathscr{T}_0}$ *is guaranteed. As a consequence of bisection and* (102)*, only finitely many shapes of node and element patches can occur. Moreover, the mesh closure estimate* (100) *as well as the overlay*

[11] newest vertex bisection (NVB).

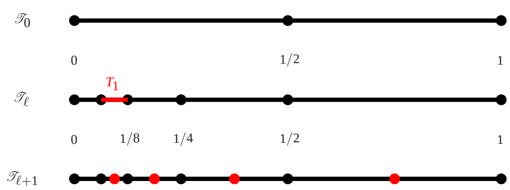

Fig. 17 For 1D bisection, refined elements $\mathscr{T}_\ell \setminus \mathscr{T}_{\ell+1}$ are bisected into two sons, whence $\#\mathscr{M}_\ell \le \#(\mathscr{T}_\ell \setminus \mathscr{T}_{\ell+1}) = \#\mathscr{T}_{\ell+1} - \#\mathscr{T}_\ell$. With $\sigma_{\mathscr{T}_\ell} \le 2\,\sigma_{\mathscr{T}_0}$, the converse inequality $\#\mathscr{T}_{\ell+1} - \#\mathscr{T}_\ell \lesssim \#\mathscr{M}_\ell$ cannot hold in general as the following elementary example proves: Let $\mathscr{T}_0$ denote the partition of $[0,1]$ in two elements of length $1/2$, i.e., $\sigma_{\mathscr{T}_0} = 1$. Repeated marking of the leftmost elements of $\mathscr{T}_0, \mathscr{T}_1, \dots, \mathscr{T}_{\ell-1}$ generates the mesh $\mathscr{T}_\ell$ with $\sigma_{\mathscr{T}_\ell} = 2$ and $\#\mathscr{T}_\ell = \ell$. Marking the highlighted element $T_1 \in \mathscr{T}_\ell$ results in the mesh $\mathscr{T}_{\ell+1} := \mathtt{refine}(\mathscr{T}_\ell, \{T_1\})$, where $\ell - 1$ elements are refined to ensure $\sigma_{\mathscr{T}_{\ell+1}} = 2$. Consequently, the number of additional refinements can be arbitrarily large, and (104) cannot hold in general.

estimate (101) *are valid, where the constant $C_{\rm nvb} > 0$ depends only on $\mathscr{T}_0$. Finally, the coarsest common refinement $\mathscr{T}_\star \oplus \mathscr{T}_\ell$ of $\mathscr{T}_\ell, \mathscr{T}_\star \in \mathtt{refine}(\mathscr{T}_0)$ is the overlay*

$$\mathscr{T}_\star \oplus \mathscr{T}_\ell = \big\{T \in \mathscr{T}_\star \cup \mathscr{T}_\ell : \forall T' \in \mathscr{T}_\star \cup \mathscr{T}_\ell \quad (T' \subseteq T \Rightarrow T' = T)\big\}, \tag{103}$$

i.e., the union of the locally finest elements. □

We note that, while the mesh-closure estimate (100) is true, a stepwise variant

$$\#\mathscr{T}_{\ell+1} - \#\mathscr{T}_\ell \le C_{\rm nvb}\,\#\mathscr{M}_\ell \quad \text{for all } \ell \in \mathbb{N}_0 \tag{104}$$

cannot hold with an ℓ-independent constant $C_{\rm nvb} > 0$. We refer to a simple counter example from [7] which is also illustrated in Fig. 17.

7.4 2D newest vertex bisection for 3D BEM

Denote by $\mathscr{T}_\ell$ a given mesh and by $\mathscr{E}_\ell$ its edges. Suppose that for each triangle $T \in \mathscr{T}_\ell$, there is a so-called *reference edge* $e_T \in \mathscr{E}_\ell$ with $e_T \subset \partial\overline{T}$. To refine a specific element $T \in \mathscr{T}_\ell$, the midpoint m_e of its reference edge e_T becomes a new node, and T is bisected along m_e and the node opposite to e_T into its two sons, see Fig. 18 (left). The edges opposite to m_e become the reference edges of the two sons of T.

If the marked elements $\mathscr{M}_\ell$ of a mesh are refined according to this rule, the new mesh automatically inherits a distribution of reference edges. Hence, only the initial mesh $\mathscr{T}_0$, with which an adaptive algorithm would be initialized, needs to be equipped with a distribution of reference edges.

In order to keep the mesh conforming, i.e., to avoid hanging nodes, different approaches are available: Sewell [130] proposes to bisect T either if its reference edge is on the boundary, or if it is *compatibly divisible*, i.e., the neighbor T' on the other side of the reference edge e_T of T also uses the common edge as reference edge. This approach was refined by Mitchell in [109], who proposes to recursively call the bisection algorithm on the neighbor T' of T until a compatibly divisible element is found. This approach is reasonable under certain conditions, however, situations exists where the recursion per se cannot terminate, cf. [105]. However, if all elements in $\mathscr{T}_0$ are compatibly divisible, the recursion terminates on every following mesh $\mathscr{T}_\ell$. Distributing the reference edges on $\mathscr{T}_0$ this way, i.e., all elements end up being compatibly divisible, is always possible as proven in [26]. However, no scalable algorithm is known which performs this task. To circumvent this problem, as shown in [101], it is possible to cast the 2D NVB into an iterative algorithm, which does not need a special distribution of the reference edges to terminate:

Algorithm 104 (Iterative formulation of 2D NVB) INPUT*: mesh $\mathscr{T}_\ell$ and set of marked elements $\mathscr{M}_\ell \subseteq \mathscr{T}_\ell$.*
OUTPUT*: refined mesh $\mathscr{T}_{\ell+1}$.*

(o) *Set counter $k := 0$ and define set of marked reference edges $\mathscr{M}_\ell^{(0)} := \{e_T \mid T \in \mathscr{M}_\ell\}$.*
(i) *Define $\mathscr{M}_\ell^{(k+1)} := \Big\{e_T \mid \exists e \in \mathscr{M}_\ell^{(k)} \text{ with } e \subset \partial T\Big\}$.*
(ii) *If $\mathscr{M}_\ell^{(k)} \subsetneqq \mathscr{M}_\ell^{(k+1)}$, increase counter $k \mapsto k+1$ and goto (i).*
(iii) *Otherwise and with $\mathscr{M}_\ell^{(k)}$ being the set of marked edges, use newest vertex bisection to refine all elements $T \in \mathscr{T}_\ell$ with $e_T \in \mathscr{M}_\ell^{(k)}$ according to Fig. 18 to obtain the new mesh $\mathscr{T}_{\ell+1}$.*

The following proposition collects the elementary properties of NVB, and we also refer to Fig. 19.

Proposition 105 *Algorithm 104 terminates regardless of the distribution of reference edges. The output $\mathscr{T}_{\ell+1} = \mathtt{refine}(\mathscr{T}_\ell, \mathscr{M}_\ell)$ is the coarsest conforming mesh such that all elements in $\mathscr{M}_\ell$ are refined. Moreover, NVB ensures that only finitely many shapes of elements (and hence also patches) may occur. In particular, NVB generated meshes are uniformly σ-shape regular, cf. Section 2.6, where $\sigma > 0$ depends only on the initial mesh $\mathscr{T}_0$.*

7.4.1 Mesh closure and overlay estimate

Another look back to Algorithm 104 reveals that, while only advised to refine elements of $\mathscr{M}_\ell$, it refines all the elements $T \in \mathscr{T}_\ell$ with $e_T \in \mathscr{M}_\ell^{(k)}$. It does this to circumvent the generation of hanging nodes. We refer to a counter example in [116, p. 462] that, as in 1D, an elementary estimate of the type

$$\#\mathscr{T}_{\ell+1} - \#\mathscr{T}_\ell \le C_{\rm nvb}\,\#\mathscr{M}_\ell \quad \text{for all } \ell \in \mathbb{N}_0$$

cannot hold with an ℓ-independent constant $C_{\rm nvb} > 0$. The following theorem is proved in [101] for 2D NVB.

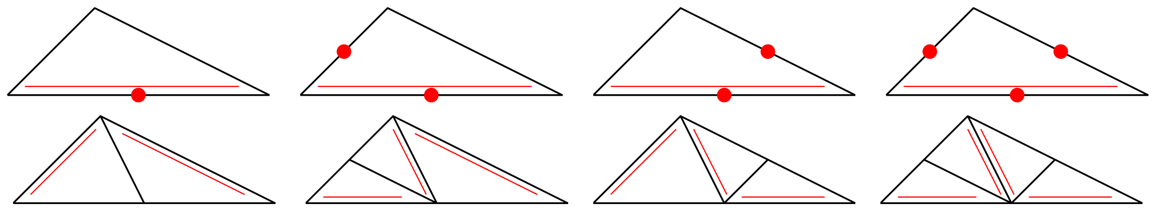

Fig. 18 For each triangle $T \in \mathscr{T}_\ell$, there is one fixed *reference edge*, indicated by the double line (left, top). Refinement of T is done by bisecting the reference edge, where its midpoint becomes a new node. The reference edges of the son triangles $T' \in \mathscr{T}_{\ell+1}$ are opposite to this newest vertex (left, bottom). To avoid hanging nodes, one proceeds as follows: We assume that certain edges of T, but at least the reference edge, are marked for refinement (top). Using iterated newest vertex bisection, the element is then split into 2, 3, or 4 son triangles (bottom). If all elements are refined by three bisections (right, bottom), we obtain the so-called uniform bisec(3)-refinement which is denoted by $\widehat{\mathscr{T}_\ell}$.

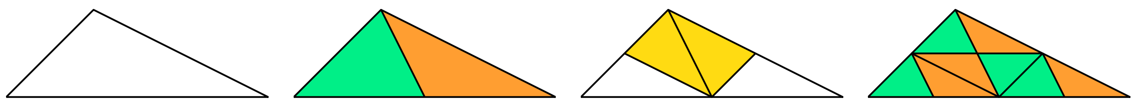

Fig. 19 NVB refinement only leads to finitely many shapes of triangles for the family of all possible triangulations obtained by arbitrary newest vertex bisections. To see this, we start from a macro element (left), where the bottom edge is the reference edge. Using iterated newest vertex bisection, one observes that only four similarity classes of triangles occur, which are indicated by the coloring. After three steps of bisections (right), no additional similarity class appears.

Theorem 106 *Suppose that $\mathscr{T}_0$ is a mesh on Γ with an arbitrary distribution of reference edges, and $(\mathscr{T}_\ell)_{\ell\in\mathbb{N}_0}$ is generated by Algorithm 104, i.e., for all $\ell \in \mathbb{N}_0$ holds*

$$\mathscr{T}_{\ell+1} = \texttt{refine}(\mathscr{T}_\ell, \mathscr{M}_\ell).$$

Then, the mesh-closure estimate (100) *is valid, and the constant $C_{\rm nvb} > 0$ depends only on $\mathscr{T}_0$.*

Theorem 106 was first proved in [26] for $d = 2$, under the additional assumption that the distribution of reference edges in $\mathscr{T}_0$ is such that all elements are compatibly divisible. In [140], the theorem was extended to $d \geq 2$, and in [54] it was shown to hold also if additional refinements are made to keep the mesh mildly graded. The work [101] finally removed the assumption on the special distribution of the reference edges in $d = 2$.

The following theorem is proved in [26] for $d = 2$ and [46] for $d \geq 3$. To guarantee termination of their recursive formulations of the NVB algorithm, these works require that the distribution of reference edges in $\mathscr{T}_0$ is such that all elements are compatibly divisible. However, their proofs of the overlay estimate (101) do not use this assumption and also apply to the inductive formulation of 2D NVB from [101].

Theorem 107 *Suppose that $\mathscr{T}_0$ is a mesh on Γ with an arbitrary distribution of reference edges and that $(\mathscr{T}_\ell)_{\ell\in\mathbb{N}_0}$ is generated by Algorithm 104, i.e., for all $\ell \in \mathbb{N}_0$ holds*

$$\mathscr{T}_{\ell+1} = \texttt{refine}(\mathscr{T}_\ell, \mathscr{M}_\ell).$$

Then, the overlay estimate (101) *is valid. Moreover, the coarsest common refinement $\mathscr{T}_\star \oplus \mathscr{T}_\ell$ of $\mathscr{T}_\ell, \mathscr{T}_\star \in \texttt{refine}(\mathscr{T}_0)$ is the overlay* (103)*.*

7.4.2 H^s stability of the L_2 projection

If the $L_2(\Gamma)$ projection onto $\mathscr{S}^p(\mathscr{T})$ (or $\widetilde{\mathscr{S}^p}(\mathscr{T})$) is used for localization of a fractional order Sobolev norm, it needs to fulfill the approximation estimate from Lemma 24. According to this Lemma, stability in $H^s(\Gamma)$ (or $\widetilde{H}^s(\Gamma)$) is a sufficient condition, and it is seen easily that it is also necessary. Indeed, choosing $s = 1$ in Lemma 24, it holds

$$\begin{aligned}\|\Pi^p_{\mathscr{T}} v\|_{H^1(\Gamma)} &\leq \|v\|_{H^1(\Gamma)} + \|v - \Pi^p_{\mathscr{T}} v\|_{H^1(\Gamma)} \\ &\lesssim (1 + \mathrm{diam}(\Gamma))\|v\|_{H^1(\Gamma)}.\end{aligned}$$

By deeper mathematical results, it follows from this estimate that $\Pi^p_{\mathscr{T}}$ is $H^s(\Gamma)$ stable. Hence there is no other way than analyzing $\Pi^p_{\mathscr{T}}$'s stability in H^1. For quasi-uniform meshes, it follows with arguments from [29] that $\Pi^p_{\mathscr{T}}$ is $H^1(\Gamma)$-stable. In fact, with an arbitrary $H^1(\Gamma)$-stable Clément-type operator $J_{\mathscr{T}}$ (e.g., the Scott-Zhang projection from 3.2.2), it follows with the inverse estimate from Lemma 25 that

$$\begin{aligned}\|\nabla \Pi^p_{\mathscr{T}} v\|_{L_2(\Gamma)} &\leq \|\nabla(\Pi^p_{\mathscr{T}} - J_{\mathscr{T}})v\|_{L_2(\Gamma)} + \|\nabla J_{\mathscr{T}} v\|_{L_2(\Gamma)} \\ &\lesssim \|h^{-1}_{\mathscr{T}}\|_{L_\infty(\Gamma)}\|(\Pi^p_{\mathscr{T}} - J_{\mathscr{T}})v\|_{L_2(\Gamma)} + \|\nabla J_{\mathscr{T}} v\|_{L_2(\Gamma)},\end{aligned}$$

and due to the projection property of $\Pi^p_{\mathscr{T}}$ and its $L_2(\Gamma)$-stability,

$$\begin{aligned}\|(\Pi^p_{\mathscr{T}} - J_{\mathscr{T}})v\|_{L_2(\Gamma)} &\leq \|(1 - J_{\mathscr{T}})v\|_{L_2(\Gamma)} \\ &\lesssim \|h_{\mathscr{T}}\|_{L_\infty(\Gamma)}\|v\|_{H^1(\Gamma)},\end{aligned}$$

where we have finally used the first-order approximation property of $J_{\mathscr{T}}$. Combining these two estimates and regarding the fact that for quasi-uniform meshes

$$\|h^{-1}_{\mathscr{T}}\|_{L_\infty(\Gamma)}\|h_{\mathscr{T}}\|_{L_\infty(\Gamma)} \lesssim 1,$$

the $H^1(\Gamma)$-stability of $\Pi^p_{\mathscr{T}}$ follows.

Unfortunately, this argument cannot be used in this straight forward manner on adaptively refined meshes. The $H^s(\Gamma)$-stability can be shown, though, under certain conditions on the mesh. There are basically two approaches:

– Imposing global or local growth-conditions on the mesh. This approach is used in the works [28,33,51,60,133,134].
– Using only a sequence of adaptively generated meshes such that an equivalent mesh-size function can be used, which takes care of the fact that the mesh is not quasi-uniform. This approach is used in the works [17,34,101,102].

The strength of the first approach is that it can be used for an *arbitrary* sequence of meshes which does not have to be the output of an adaptive mesh refinement strategy. However, certain growth-conditions may be too restrictive if already the coarsest mesh violates them. Therefore, the second approach will yield more general results when it comes to adaptive mesh refinement.

Theorem 108 *Suppose that $\mathscr{T}_0$ is a mesh on Γ with an arbitrary distribution of reference edges. Then, if $(\mathscr{T}_\ell)_{\ell\in\mathbb{N}_0}$ is generated by Algorithm 104, the sequence of $L_2(\Gamma)$-orthogonal projections Π_ℓ onto $\mathscr{S}^1(\mathscr{T}_\ell)$ is uniformly $H^1(\Gamma)$-stable, i.e., for all $\ell\in\mathbb{N}_0$ holds*

$$\|\Pi_\ell v\|_{H^1(\Gamma)} \le C_{\mathrm{stab}}\|v\|_{H^1(\Gamma)}, \quad \text{for all } v\in H^1(\Gamma),$$

and the constant C_{stab} depends only on $\mathscr{T}_0$. The same result holds for the $L_2(\Gamma)$-projection onto $\widetilde{\mathscr{S}^p}(\mathscr{T})$.

The first proof of this type of result is due to Carstensen [34]. In the latter work, the distribution of reference edges is supposed to fulfill an additional assumption, and instead of 2D NVB, a modified 2D red-green-blue mesh refinement strategy is considered. In the work [101], the assumptions on the initial distribution of reference edges has been removed, and 2D NVB was considered as underlying refinement strategy. In [102] the analysis of [101] has been generalized to NVB in arbitrary dimension $d\ge 3$.

The last theorem can be employed in lowest-order Galerkin boundary element methods, but higher-order methods require the $H^1(\Gamma)$-stability of the $L_2(\Gamma)$-projection onto $\mathscr{S}^p(\mathscr{T})$. Results of this kind have been shown by Bank and Yserentant in [17]. To state their result, the concept of the so-called *level-function* $\mathrm{gen}_\ell : \mathscr{T}_\ell \to \mathbb{N}_0$ has to be introduced, which measures the number of bisections needed to create a specific element. For all $T\in\mathscr{T}_0$, define $\mathrm{gen}_0(T) := 0$. Then, the two sons T_1 and T_2 of an element T that arise due to a bisection, see Fig. 18 (left), have level $\mathrm{gen}_{\ell+1}(T_1) = \mathrm{gen}_{\ell+1}(T_2) := \mathrm{gen}_\ell(T)+1$. The following theorem is the main result of [17].

Theorem 109 *Assume that for the sequence of meshes $(\mathscr{T}_\ell)_{\ell\in\mathbb{N}_0}$ holds*

$$\big|\mathrm{gen}_\ell(T)-\mathrm{gen}_\ell(T')\big| \le 1 \quad \text{if } \overline{T}\cap\overline{T'}\neq 0.$$

Then, the sequence $\big(\Pi_\ell^p\big)_{\ell\in\mathbb{N}_0}$ of $L_2(\Gamma)$-orthogonal projections onto $\mathscr{S}^p(\mathscr{T}_\ell)$ is uniformly $H^1(\Gamma)$ stable for $p\le 12$ in $d=2$ and for $p\le 7$ in $d=3$.

The assumption on the level function in Theorem 109 suggests that also elements sharing a vertex need to have a difference in their level-functions of at most 1. This assumption needs to be enforced via additional refinements, as suggested in [17].

8 Optimal convergence of adaptive BEM

Whereas plain convergence of error estimators was the concern of Section 6, this section deals with convergence of Algorithm 1 even with optimal rates. The first result on convergence rates [26] considered AFEM for the 2D Poisson model problem and required an additional coarsening step which has later been proved to be unnecessary [139]. The latter work introduced the assumption that the set of marked elements in Step (iii) of Algorithm 1 has minimal cardinality and proved that the marking criterion (9) is (in some sense) even necessary (see Lemma 114 below).

For ABEM, optimal convergence rates for weighted-residual error estimators have independently first been proved in [70,76]. While [70] considers ABEM for the 3D Laplacian on polyhedral domains, [76] considers ABEM for general operators, but the analysis requires smooth boundaries.

The goal of this section is to explain the concept of convergence with optimal rates and to provide abstract results which cover the BEM model problems from Section 8.4–8.7. Since most of the analysis can be done in an abstract mathematical setting, we stick with the frame and the notation of the Lax-Milgram lemma from Section 1.

8.1 Necessary approximation property

We suppose that the discrete spaces $\mathscr{X}_\ell$ are nested in the sense that $\mathscr{X}_\ell\subseteq\mathscr{X}_\star\subseteq\mathscr{X}$ if $\mathscr{T}_\star\in\mathtt{refine}(\mathscr{T}_\ell)$ and that they satisfy the following approximation property: For all $\mathscr{T}_\ell\in\mathtt{refine}(\mathscr{T}_0)$ and all $\varepsilon>0$, there exists $\mathscr{T}_\star\in\mathtt{refine}(\mathscr{T}_\ell)$ such that

$$\|u-U_\star\|_{\mathscr{X}} \le \varepsilon, \tag{105}$$

where $U_\star\in\mathscr{X}_\star$ is the solution of (4) on the mesh $\mathscr{T}_\star$.

Remark 23 Although one could theoretically construct spaces $\mathscr{X}$, where assumption (105) is violated, the authors are unaware of any practical example, where this is the case. In practice, (105) is satisfied for $\mathscr{T}_\star$ being a sufficiently fine uniform refinement of $\mathscr{T}_\ell$.

8.1.1 Assumptions on the adaptive algorithm

Algorithm 1 needs to be modified to allow for convergence with optimal rates. In contrast to Section 6, we now have to ensure that we choose a set of minimal cardinality $\mathscr{M}_\ell$ in Step (iii) of Algorithm 1 (see also Remark 13 for details on the realization). In step (iv) of Algorithm 1, we suppose that $\mathscr{T}_{\ell+1}$ is the coarsest refinement of $\mathscr{T}_\ell$ such that all marked elements $T \in \mathscr{M}_\ell$ have been refined, written $\mathscr{T}_{\ell+1} = \texttt{refine}(\mathscr{T}_\ell, \mathscr{M}_\ell)$. Finally, we suppose that the mesh refinement strategy used satisfies the properties of Section 7.2 plus the fact that it produces only finitely many shapes of element patches. These assumptions are, for example, satisfied for the bisection strategies discussed in Section 7.

8.1.2 Assumptions on the error estimator

The following four assumptions on the error estimator η_ℓ are first found in [37] and distilled from the literature on AFEM [8,46,69,71,139] and ABEM [7,65,66,70,76]. We will use these assumptions to prove the main results on convergence (Theorem 111) and optimal rates (Theorem 113). In Section 8.4–8.7 below, these assumptions will be verified for concrete model problems.

(A1) Stability on non-refined elements: There exists a constant $C_{12} > 0$ such that any refinement $\mathscr{T}_\star \in \texttt{refine}(\mathscr{T}_\ell)$ of $\mathscr{T}_\ell \in \texttt{refine}(\mathscr{T}_0)$ satisfies

$$\Big|\Big(\sum_{T\in\mathscr{T}_\ell\cap\mathscr{T}_\star}\eta_\ell(T)^2\Big)^{1/2} - \Big(\sum_{T\in\mathscr{T}_\ell\cap\mathscr{T}_\star}\eta_\star(T)^2\Big)^{1/2}\Big| \le C_{12}\|U_\star - U_\ell\|_{\mathscr{X}}.$$

(A2) Reduction on refined elements: There exist constants $C_{13} > 0$ and $0 < q_{\rm red} < 1$ such that any refinement $\mathscr{T}_\star \in \texttt{refine}(\mathscr{T}_\ell)$ of $\mathscr{T}_\ell \in \texttt{refine}(\mathscr{T}_0)$ satisfies

$$\sum_{T\in\mathscr{T}_\star\setminus\mathscr{T}_\ell}\eta_\star(T)^2 \le q_{\rm red}\sum_{T\in\mathscr{T}_\ell\setminus\mathscr{T}_\star}\eta_\ell(T)^2 + C_{13}\|U_\star - U_\ell\|_{\mathscr{X}}^2.$$

(A3) Reliability: There exists a constant $C_{14} > 0$ such that any mesh $\mathscr{T}_\ell \in \texttt{refine}(\mathscr{T}_0)$ satisfies

$$\|u - U_\ell\|_{\mathscr{X}} \le C_{14}\eta_\ell.$$

(A4) Discrete reliability: There exist constants $C_{15} > 0$ and $C_{16} > 0$ such that any refinement $\mathscr{T}_\star \in \texttt{refine}(\mathscr{T}_\ell)$ of $\mathscr{T}_\ell \in \texttt{refine}(\mathscr{T}_0)$ satisfies

$$\|U_\star - U_\ell\|_{\mathscr{X}} \le C_{15}\Big(\sum_{T\in\mathscr{R}(\ell,\star)}\eta_\ell(T)^2\Big)^{1/2},$$

where the set $\mathscr{R}(\ell,\star) \supseteq \mathscr{T}_\ell \setminus \mathscr{T}_\star$ satisfies $\#\mathscr{R}(\ell,\star) \le C_{16}\#(\mathscr{T}_\ell \setminus \mathscr{T}_\star)$.

Remark 24 The assumptions (A1)–(A2) are fulfilled by many error estimators with *h*-weighting factor, e.g., weighted residual error-estimators, $(h-h/2)$-type error estimator, and ZZ-type error estimators as shown implicitly in Section 6. They form the main ingredients for the abstract proof of estimator reduction (76). Discrete reliability (A4) is stronger than reliability (A3) as is shown in the following. So far, it is only proved for the weighted residual error estimator with $\mathscr{R}(\ell,\star)$ being the refined elements plus one additional element layer around them.

Lemma 110 *The discrete reliability* (A4) *implies the reliability* (A3) *with* $C_{14} = C_{15}$.

Proof Suppose (A4) and use the approximation property (105) for $\varepsilon > 0$ to see

$$\begin{aligned}\|u - U_\ell\|_{\mathscr{X}} &\le \|u - U_\star\|_{\mathscr{X}} + \|U_\star - U_\ell\|_{\mathscr{X}}\\ &\le \varepsilon + C_{15}\Big(\sum_{T\in\mathscr{R}(\ell,\star)}\eta_\ell(T)^2\Big)^{1/2} \le \varepsilon + C_{15}\eta_\ell.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this implies (A3) with $C_{14} = C_{15}$. □

8.2 Convergence of error estimator and error

Theorem 111 *Suppose that the error estimator satisfies stability* (A1) *and reduction* (A2)*. Then, Algorithm 1 drives the estimator to zero, i.e.*

$$\lim_{\ell\to\infty}\eta_\ell = 0. \tag{106}$$

Suppose that the error estimator additionally satisfies reliability (A3)*. Then, Algorithm 1 converges even R-linearly in the sense that there exist constants $C_{17} > 0$ and $0 < q_{\rm R} < 1$ such that*

$$C_{14}^{-2}\|u - U_{\ell+n}\|_{\mathscr{X}}^2 \le \eta_{\ell+n}^2 \le C_{17}q_{\rm R}^n\eta_\ell^2 \quad \text{for all } \ell, n \in \mathbb{N}_0, \tag{107}$$

which particularly implies

$$\|u - U_\ell\|_{\mathscr{X}}^2 \le C_{14}^2 C_{17}\eta_0^2 q_{\rm R}^\ell \quad \text{for all } \ell \in \mathbb{N}_0. \tag{108}$$

The constants C_{17} and $q_{\rm R}$ depend only on θ as well as on the constants in (A1)–(A3)*.*

The proof of the above theorem is split into several steps. The first lemma states that the assumptions (A1) and (A2) imply the estimator reduction (76) and thus render an abstract version of the results in Lemmas 80–90. Such an estimate is first but implicitly found in [46]. Together with the a priori convergence of Lemma 76, it proves that the adaptive algorithm drives the estimator to zero (106). This so-called *estimator reduction principle* is discussed in Section 6 and has been proposed in [11] for $(h-h/2)$-type estimators and was generalized afterwards to data-perturbed

BEM [10,100], residual-based error estimators [70], as well as the FEM-BEM coupling [9,6]. Instead of the respective concrete settings, the following proof relies only on the abstract assumptions (A1)–(A2) as well as the marking criterion (9).

Lemma 112 *Suppose* (A1)–(A2)*. Then, there exist constants* $C_{18} > 0$ *and* $0 < q_{\rm est} < 1$ *such that Algorithm 1 satisfies*

$$\eta_{\ell+1}^2 \le q_{\rm est}\eta_\ell^2 + C_{18}\|U_{\ell+1} - U_\ell\|_{\mathscr{X}}^2 \tag{109}$$

for all $\ell \in \mathbb{N}$*. The constant* $q_{\rm est}$ *depends only on* θ *and* $q_{\rm red}$ *from* (A2)*. The constant* C_{18} *depends additionally on* C_{13} *as well as* C_{12}*.*

Proof We split the error estimator into refined and non-refined elements and apply (A1)–(A2). With Young's inequality $(a+b)^2 \le (1+\delta)a^2 + (1+\delta^{-1})b^2$ for all $a, b \in \mathbb{R}$ and all $\delta > 0$, this shows

$$\begin{aligned}\eta_{\ell+1}^2 &= \sum_{T\in\mathscr{T}_{\ell+1}\setminus\mathscr{T}_\ell} \eta_{\ell+1}(T)^2 + \sum_{T\in\mathscr{T}_{\ell+1}\cap\mathscr{T}_\ell} \eta_{\ell+1}(T)^2\\ &\le q_{\rm red}\sum_{T\in\mathscr{T}_\ell\setminus\mathscr{T}_{\ell+1}} \eta_\ell(T)^2 + (1+\delta)\sum_{T\in\mathscr{T}_\ell\cap\mathscr{T}_{\ell+1}} \eta_\ell(T)^2\\ &\qquad + (C_{13} + (1+\delta^{-1})C_{12}^2)\|U_{\ell+1} - U_\ell\|_{\mathscr{X}}^2.\end{aligned}$$

The bulk chasing (9) together with $\mathscr{M}_\ell \subseteq \mathscr{T}_{\ell+1}\setminus\mathscr{T}_\ell$ implies

$$\begin{aligned}\eta_{\ell+1}^2 &\le (q_{\rm red} - (1+\delta))\sum_{T\in\mathscr{T}_\ell\setminus\mathscr{T}_{\ell+1}} \eta_\ell(T)^2 + (1+\delta)\eta_\ell^2\\ &\qquad + (C_{13} + (1+\delta^{-1})C_{12}^2)\|U_{\ell+1} - U_\ell\|_{\mathscr{X}}^2\\ &\le ((1+\delta) - \theta((1+\delta) - q_{\rm red}))\eta_\ell^2 + C_{18}\|U_{\ell+1} - U_\ell\|_{\mathscr{X}}^2,\end{aligned}$$

where $C_{18} := (C_{13} + (1+\delta^{-1})C_{12}^2)$ and $q_{\rm est} := (1+\delta) - \theta((1+\delta) - q_{\rm red}) \in (0,1)$ for $\delta > 0$ sufficiently small. □

With the previous results, we are able to prove the first part of Theorem 111.

Proof (of estimator convergence (106)*)* By assumption (Section 8.1), the discrete spaces are nested. Therefore, Lemma 76 proves a priori convergence of U_ℓ towards some limit U_∞, and hence the perturbation term in (109) vanishes. Consequently, Lemma 75 applies and concludes the proof. □

The next goal is the R-linear convergence (107). To this end, the literature usually employs the Pythagoras identity

$$\|u - U_{\ell+1}\|_{\mathscr{X}}^2 + \|U_{\ell+1} - U_\ell\|_{\mathscr{X}}^2 = \|u - U_\ell\|_{\mathscr{X}}^2, \tag{110}$$

see, e.g., [46,70,76,139]. Under reliability (A3), estimator convergence (106) implies

$$\lim_{\ell\to\infty} \|u - U_\ell\|_{\mathscr{X}} = 0. \tag{111}$$

Consequently, (110) results in

$$\sum_{k=\ell}^\infty \|U_{k+1} - U_k\|_{\mathscr{X}}^2 = \|u - U_\ell\|_{\mathscr{X}}^2. \tag{112}$$

However, (110) and thus (112) rely heavily on the symmetry of the bilinear form $b(\cdot\,,\cdot)$. This may suffice for many problems, however, when it comes to mixed boundary value problems or the FEM-BEM coupling, (110) is wrong in general. The recent work [68] proves a weaker version of (112) which holds for general continuous and elliptic bilinear forms $b(\cdot\,,\cdot)$ and is sufficient for the upcoming analysis. With (1)–(2) and as the sequence $(U_\ell)_{\ell\in\mathbb{N}_0}$ is convergent (111), there exists a constant $C_{19} > 0$ such that all $\ell \in \mathbb{N}_0$ satisfy

$$\sum_{k=\ell}^\infty \|U_{k+1} - U_k\|_{\mathscr{X}}^2 \le C_{19}\|u - U_\ell\|_{\mathscr{X}}^2. \tag{113}$$

We note that (113) is weaker than (112), but avoids the use of the symmetry of $b(\cdot\,,\cdot)$. This will be employed in the following proof.

Proof (of R-linear estimator convergence (107)*)* Let $N, \ell \in \mathbb{N}$. Use the estimator reduction (109) to see

$$\sum_{k=\ell+1}^{\ell+N} \eta_k^2 \le \sum_{k=\ell+1}^{\ell+N} \Big(q_{\rm est}\eta_{k-1}^2 + C_{18}\|U_k - U_{k-1}\|_{\mathscr{X}}^2\Big).$$

This implies

$$(1 - q_{\rm est})\sum_{k=\ell+1}^{\ell+N} \eta_k^2 \le \eta_\ell^2 + C_{18}\sum_{k=\ell+1}^{\ell+N} \|U_k - U_{k-1}\|_{\mathscr{X}}^2.$$

The estimate (113) together with reliability (A3) then shows

$$\sum_{k=\ell}^{\ell+N} \eta_k^2 \le \frac{2 + C_{18}C_{19}C_{14}^2}{1 - q_{\rm est}}\eta_\ell^2.$$

Since the right-hand side does not depend on N, there also holds with $C_{17} := (2 + C_{18}C_{19}C_{14}^2)(1 - q_{\rm est})^{-1} \ge 1$

$$\sum_{k=\ell}^\infty \eta_k^2 \le C_{17}\eta_\ell^2.$$

This result is employed several times to conclude the proof. Mathematical induction on $n \in \mathbb{N}$ shows

$$\begin{aligned}\eta_{\ell+n}^2 &\le \sum_{k=\ell+n-1}^\infty \eta_k^2 - \eta_{\ell+n-1}^2\\ &\le (1 - C_{17}^{-1})\sum_{k=\ell+n-1}^\infty \eta_k^2\\ &\;\;\vdots\\ &\le (1 - C_{17}^{-1})^n\sum_{k=\ell}^\infty \eta_k^2 \le C_{17}(1 - C_{17}^{-1})^n\eta_\ell^2.\end{aligned}$$

This proves (107) with $q_{\rm R} = (1 - C_{17}^{-1})$. □

8.3 Convergence with optimal rates

Having fixed the error estimator η for Step (ii) and the mesh refinement strategy for Step (iv) of Algorithm 1, the overall goal in this section is to prove algebraic convergence rates. For $s>0$, we define the approximability norm

$$\|\eta\|_{\mathbb{A}_s} := \sup_{N\in\mathbb{N}_0} (N+1)^s \Big(\inf_{\substack{\mathscr{T}_\star\in\mathtt{refine}(\mathscr{T}_0)\\ \#\mathscr{T}_\star-\#\mathscr{T}_0\le N}} \eta_\star \Big) \in [0,\infty]. \tag{114}$$

By definition, $\|\eta\|_{\mathbb{A}_s}<\infty$ means that one can find a sequence $(\overline{\mathscr{T}}_\ell)_{\ell\in\mathbb{N}_0}$ of meshes such that the corresponding sequence of estimators $(\overline{\eta_\ell})_{\ell\in\mathbb{N}_0}$ satisfies

$$\overline{\eta_\ell} \lesssim (\#\overline{\mathscr{T}}_\ell - \#\mathscr{T}_0)^{-s} \quad \text{for all } \ell\in\mathbb{N}_0, \tag{115}$$

i.e., the estimators decay with algebraic rate $-s$. We note that the meshes $\overline{\mathscr{T}}_\ell$ are not necessarily nested.

The following theorem proves that for each $s>0$, $\|\eta\|_{\mathbb{A}_s}<\infty$ is equivalent to (115) for $\overline{\mathscr{T}}_\ell := \mathscr{T}_\ell$ being the adaptively generated mesh, i.e., each possible algebraic convergence rate (constrained by estimator and mesh refinement) will in fact by achieved by Algorithm 1 (which produces only nested meshes). In particular, adaptive mesh refinement is superior to uniform mesh refinement.

Theorem 113 *Suppose* (A1)–(A2) *as well as discrete reliability* (A4)*. Let the adaptivity parameter satisfy $\theta<\theta_0 := (1+C_{12}^2C_{15}^2)^{-1}$. Then, Algorithm 1 converges with the best possible rate in the sense that for all $s>0$, it holds $\|\eta\|_{\mathbb{A}_s}<\infty$ if and only if*

$$\eta_\ell \le C_{20}(\#\mathscr{T}_\ell - \#\mathscr{T}_0)^{-s} \quad \textit{for all } \ell\in\mathbb{N}, \tag{116}$$

where $C_{20}>0$ depends only on the constants in (A1)–(A4) *as well as on $\|\eta\|_{\mathbb{A}_s}$ and θ.*

An essential part of the proof of the above theorem is that the bulk chasing criterion (9) does not mark too many elements. This is the concern of the first lemma, which states that if one observes linear convergence (107), the refined elements satisfy the bulk chasing (9). In this respect the marking strategy (9) appears to be sufficient as well as necessary for linear convergence (107). We note that at this stage the discrete reliability (A4) enters. This observation has first been proved for AFEM in [139]. Unlike the AFEM literature [139, 46], our statement and proof relies only on the error estimator and avoids the use of any efficiency estimate (or lower error bound) for the estimator.

Lemma 114 *Let the error estimator satisfy stability* (A1) *and discrete reliability* (A4)*. Then, there exists $0<\kappa_0<1$ such that any refinement $\mathscr{T}_\star\in\mathtt{refine}(\mathscr{T}_\ell)$ which satisfies*

$$\eta_\star^2 \le \kappa_0\eta_\ell^2 \tag{117}$$

fulfils the bulk chasing (9) *in the sense*

$$\theta\eta_\ell^2 \le \sum_{T\in\mathscr{R}(\ell,\star)} \eta_\ell(T)^2 \tag{118}$$

for all $0\le\theta<\theta_0$. The constant θ_0 is defined in Theorem 113 whereas $\mathscr{R}(\ell,\star)\subseteq\mathscr{T}_\ell$ is guaranteed by (A4)*.*

Proof Similar to the proof of the estimator reduction in Lemma 112, we split the error estimator and apply (A1) together with Young's inequality $(a+b)^2\le(1+\delta)a^2+(1+\delta^{-1})b^2$ for all $a,b\in\mathbb{R}$ and $\delta>0$. This yields

$$\begin{aligned}\eta_\ell^2 &= \sum_{T\in\mathscr{T}_\ell\setminus\mathscr{T}_\star}\eta_\ell(T)^2 + \sum_{T\in\mathscr{T}_\ell\cap\mathscr{T}_\star}\eta_\ell(T)^2\\ &\le \sum_{T\in\mathscr{T}_\ell\setminus\mathscr{T}_\star}\eta_\ell(T)^2 + (1+\delta)\eta_\star^2\\ &\qquad + (1+\delta^{-1})C_{12}^2\|U_\star-U_\ell\|_{\mathscr{X}}^2.\end{aligned}$$

The assumption (117) as well as (A4) apply and show

$$\begin{aligned}\eta_\ell^2 &\le \sum_{T\in\mathscr{T}_\ell\setminus\mathscr{T}_\star}\eta_\ell(T)^2 + (1+\delta)\kappa_0\eta_\ell^2\\ &\qquad + (1+\delta^{-1})C_{12}^2\|U_\star-U_\ell\|_{\mathscr{X}}^2\\ &\le (1+(1+\delta^{-1})C_{12}^2C_{15}^2)\sum_{T\in\mathscr{R}(\ell,\star)}\eta_\ell(T)^2 + (1+\delta)\kappa_0\eta_\ell^2.\end{aligned}$$

This implies

$$\frac{1-(1+\delta)\kappa_0}{1+(1+\delta^{-1})C_{12}^2C_{15}^2}\eta_\ell^2 \le \sum_{T\in\mathscr{R}(\ell,\star)}\eta_\ell(T)^2.$$

If $\theta<\theta_0$, there exist $\delta>0$ and $\kappa_0>0$ such that

$$\theta \le \frac{1-(1+\delta)\kappa_0}{1+(1+\delta^{-1})C_{12}^2C_{15}^2} \le \frac{1}{1+C_{12}^2C_{15}^2} =: \theta_0.$$

The combination of the last two estimates concludes the proof. □

Proof (of Theorem 113) The very technical proof of Theorem 113 is found in great detail in [37, Section 4]. Therefore, we only provide a brief sketch here. Given $\lambda>0$ and by use of $\|\eta\|_{\mathbb{A}_s}<\infty$, one can find a mesh $\mathscr{T}_\varepsilon\in\mathtt{refine}(\mathscr{T}_0)$ with $\eta_\varepsilon^2\le\varepsilon^2:=\lambda\eta_\ell^2$ and $\#\mathscr{T}_\varepsilon-\#\mathscr{T}_0\lesssim\varepsilon^{-1/s}$. The overlay property (101) proves for $\mathscr{T}_\star := \mathscr{T}_\varepsilon\oplus\mathscr{T}_\ell\in\mathtt{refine}(\mathscr{T}_\ell)$ that

$$\#(\mathscr{T}_\ell\setminus\mathscr{T}_\star) \le \#\mathscr{T}_\varepsilon-\#\mathscr{T}_0 \lesssim \varepsilon^{-1/s}. \tag{119}$$

The arguments of the proof of Lemma 112 apply also for $\mathscr{T}_\star\in\mathtt{refine}(\mathscr{T}_\varepsilon)$ and show

$$\eta_\star^2 \lesssim \eta_\varepsilon^2 + \|U_\star-U_\varepsilon\|_{\mathscr{X}}^2.$$

Then, the discrete reliability (A4) implies

$$\eta_\star^2 \lesssim \eta_\varepsilon^2 \le \varepsilon^2 = \lambda\eta_\ell^2.$$

Finally, we choose $\lambda > 0$ sufficiently small such that there holds

$$\eta_\star^2 \leq \kappa_0 \eta_\ell^2. \tag{120}$$

Note that the choice of λ depends only on the constants in (A1)–(A2) and (A4), as well as on $\|\eta\|_{\mathbb{A}_s}$. With (120), Lemma 114 applies and proves that $\mathscr{R}(\ell,\star)$ satisfies the bulk chasing (118). Since $\mathscr{M}_\ell$ is chosen in Step (iii) of Algorithm 1 as a set with minimal cardinality which satisfies the bulk chasing criterion, there holds with (119)

$$\#\mathscr{M}_\ell \leq \#\mathscr{R}(\ell,\star) \leq C_{16}\#(\mathscr{T}_\ell \setminus \mathscr{T}_\star) \lesssim \varepsilon^{-1/s} = \lambda^{-1/2s}\eta_\ell^{-1/s}.$$

Next, the mesh closure estimate (100) provides

$$\#\mathscr{T}_\ell - \#\mathscr{T}_0 \lesssim \sum_{k=0}^{\ell-1} \#\mathscr{M}_k \lesssim \sum_{k=0}^{\ell-1} \#\eta_k^{-1/s} \quad \text{for all } \ell \in \mathbb{N}_0$$

The R-linear convergence (107) shows $\eta_\ell^2 \lesssim q_{\rm R}^{\ell-k}\eta_k^2$, which implies $\eta_k^{-1/s} \lesssim q_{\rm R}^{(\ell-k)/(2s)}\eta_\ell^{-1/s}$. By convergence of the geometric series, this concludes

$$\#\mathscr{T}_\ell - \#\mathscr{T}_0 \lesssim \eta_\ell^{-1/s}\sum_{k=0}^{\ell-1} q_{\rm R}^{(\ell-k)/(2s)} \leq \eta_\ell^{-1/s}\frac{1}{1-q_{\rm R}^{1/(2s)}}.$$

Taking the estimate to the power of $-s$ shows

$$\eta_\ell \lesssim (\#\mathscr{T}_\ell - \#\mathscr{T}_0)^{-s} \quad \text{for all } \ell \in \mathbb{N}_0.$$

This concludes the proof. □

8.4 Linear convergence of a $(h-h/2)$-type error estimator for the weakly singular integral equation

This section discusses the $(h-h/2)$ error estimator from [74], cf. Section 4.2.2 and extends the estimator reduction result from Section 6.5.1 in the context of the abstract framework. In the terms of the previous section, it holds according to Proposition 9

$$\begin{aligned} b(\phi\,,\psi) &:= \langle V\phi\,,\psi\rangle_\Gamma \quad \text{for all } \phi,\psi \in \mathscr{X} := \widetilde{H}^{-1/2}(\Gamma),\\ F(\psi) &:= \langle f\,,\psi\rangle_\Gamma \quad \text{for all } \psi \in \mathscr{X}, \end{aligned} \tag{121}$$

where $f \in H^{1/2}(\Gamma)$. The exact solution $\phi \in \mathscr{X}$ satisfies

$$b(\phi\,,\psi) = F(\psi) \quad \text{for all } \psi \in \mathscr{X}. \tag{122}$$

Since the $(h-h/2)$ error-estimator defined in (87) uses the uniformly refined mesh $\widehat{\mathscr{T}_\ell} := \texttt{refine}(\mathscr{T}_\ell,\mathscr{T}_\ell)$ instead of the original mesh $\mathscr{T}_\ell$, it is natural to consider the discrete spaces $\mathscr{X}_\ell := \widehat{\mathscr{X}_\ell} := \mathscr{P}^p(\widehat{\mathscr{T}_\ell})$ for the discrete Galerkin formulation (4) which reads in this setting: find $\widehat{\Phi}_\ell \in \widehat{\mathscr{X}_\ell}$ such that

$$b(\widehat{\Phi}_\ell\,,\Psi) = F(\Psi) \quad \text{for all } \Psi \in \widehat{\mathscr{X}_\ell}. \tag{123}$$

Recall the $(h-h/2)$-type error estimator from Theorem 54, i.e.,

$$\eta_\ell^2 := \sum_{T\in\mathscr{T}_\ell}\eta_\ell(T)^2 := \sum_{T\in\mathscr{T}_\ell} h_T\|(1-\pi_\ell^p)\widehat{\Phi}_\ell\|_{L_2(T)}^2, \tag{124}$$

where $h_T := |T|^{1/(d-1)} \simeq \operatorname{diam}(T)$. The next step is to prove the assumptions (A1)–(A2) for η_ℓ.

Lemma 115 *The $(h-h/2)$ error estimator η_ℓ from* (124) *satisfies stability* (A1) *and reduction* (A2)*. The constants $C_{12}, q_{\rm red}, C_{13}$ depend only on Γ, the polynomial degree p, and on the shape regularity of $\mathscr{T}_\ell$.*

Proof Stability (A1) and reduction (A2) are implicitly shown in the proof of Lemma 80. □

Theorem 116 *For all $0 < \theta \leq 1$, Algorithm 1 with the $(h-h/2)$ error estimator η_ℓ converges in the sense*

$$\lim_{\ell\to\infty}\eta_\ell = 0. \tag{125}$$

If the saturation assumption (Assumption 45) is satisfied, then η_ℓ from (124) *satisfies reliability* (A3)*, and there holds R-linear convergence*

$$C_{14}^{-2}\|\phi - \Phi_{\ell+n}\|_{\widetilde{H}^{-1/2}(\Gamma)}^2 \leq \eta_{\ell+n}^2 \leq C_{17}q_{\rm R}^n\eta_\ell^2 \tag{126}$$

for all $\ell, n \in \mathbb{N}_0$. The constant C_{14} depends only on the saturation constant $C_{\rm sata}$, Γ, the shape regularity of $\mathscr{T}_\ell$, and the polynomial degree p. The constants C_{17} and $q_{\rm R}$ depend additionally on θ.

Proof The convergence (125) follows from Theorem 111, since η_ℓ satisfies the assumptions (A1)–(A2). The reliability of η_ℓ under the saturation assumption is proved in Theorem 53 together with the equivalence of $\|\cdot\|_V \simeq \|\cdot\|_{\widetilde{H}^{-1/2}(\Gamma)}$. The remaining statement follows from Theorem 111. □

8.5 Optimal convergence of weighted residual error estimator for the weakly singular integral equation

We consider the model problem (122), but in contrast to the previous section, the discrete problem employs the original mesh $\mathscr{T}_\ell$ instead of its uniform refinement $\widehat{\mathscr{T}_\ell}$, i.e., find $\Phi_\ell \in \mathscr{X}_\ell := \mathscr{P}^p(\mathscr{T}_\ell)$ such that

$$b(\Phi_\ell\,,\Psi) = F(\Psi) \quad \text{for all } \Psi \in \mathscr{X}_\ell. \tag{127}$$

As in Section 6.7.1, the standard weighted residual error estimator from Section 4.1.3 reads

$$\eta_\ell^2 := \sum_{T\in\mathscr{T}_\ell}\eta_\ell(T)^2 := \sum_{T\in\mathscr{T}_\ell} h_T\|\nabla_\Gamma(V\Phi_\ell - f)\|_{L_2(T)}^2, \tag{128}$$

where $\nabla_\Gamma(\cdot)$ denotes the surface gradient on Γ. Note that, while (3) and (127) are well-stated for $f \in H^{1/2}(\Gamma)$, the definition of η_ℓ requires additional regularity $f \in H^1(\Gamma)$ of the data.

Lemma 117 *The weighted residual error estimator η_ℓ from* (128) *satisfies the assumptions* (A1)–(A4). *The set $\mathscr{R}(\ell,\star)$ from* (A4) *satisfies $\mathscr{R}(\ell,\star) := \{T \in \mathscr{T}_\ell : \exists T' \in \mathscr{T}_\ell \setminus \mathscr{T}_\star, \overline{T}' \cap \overline{T} \neq \emptyset\}$. The constant C_{16} depends only on the shape regularity of the mesh $\mathscr{T}_\ell$, whereas the constants $C_{12}, C_{13}, q_{\mathrm{red}}, C_{14}$, C_{15} depend additionally on Γ and the polynomial degree p.*

Proof Reliability (A3) is well-known for η_ℓ since its invention in [45] for $d = 2$ and [39] for $d = 3$. The assumptions (A1)–(A2) are shown in the proof of Lemma 88. The proof of discrete reliability (A4) analyzes the original reliability proof from [39]. The technical proof is found in [70, 76] for $p = 0$ and in [65] for general $p \geq 0$. □

Theorem 118 *For all $0 < \theta \leq 1$, Algorithm 1 with the residual error estimator η_ℓ from* (128) *converges in the sense*

$$C_{14}^{-2} \|\phi - \Phi_{\ell+n}\|^2_{\widetilde{H}^{-1/2}(\Gamma)} \leq \eta_{\ell+n}^2 \leq C_{17} q_{\mathrm{R}}^n \eta_\ell^2 \tag{129}$$

for all $\ell, n \in \mathbb{N}_0$. For $0 < \theta < \theta_0$, Algorithm 1 converges with the best possible rate $s > 0$ in the sense that $\|\eta\|_{\mathbb{A}_s} < \infty$ if and only if

$$\eta_\ell \leq C_{20}(\#\mathscr{T}_\ell - \#\mathscr{T}_0)^{-s} \quad \textit{for all } \ell \in \mathbb{N}, \tag{130}$$

where the constants C_{17}, q_{R} depend only on Γ, the shape regularity of the meshes $\mathscr{T}_\ell$, the polynomial degree p, and θ. The constant $C_{20} > 0$ depends additionally on $\|\eta\|_{\mathbb{A}_s}$.

Proof Lemma 117 shows that the assumption (A1)–(A4) are satisfied. Theorem 111 and Theorem 113 prove the statements. □

8.6 Linear convergence of a $(h - h/2)$-type error estimator for the hyper singular integral equation

This section extends the estimator reduction result from Section 6.5.2 in the context of the abstract framework. In the terms of previous section, the variational formulation reads according to Proposition 12

$$b(u\,,v) := \begin{cases} \langle Wu\,,v\rangle_\Gamma & \text{for } \Gamma \subsetneq \partial\Omega, \\ \langle Wu\,,v\rangle_\Gamma + \langle u\,,1\rangle_\Gamma \langle v\,,1\rangle_\Gamma & \text{for } \Gamma = \partial\Omega, \end{cases}$$
$$F(v) := \langle \phi\,,v\rangle_\Gamma \quad \text{for all } u,v \in \mathscr{X} := \widetilde{H}^{1/2}(\Gamma), \tag{131}$$

where $\phi \in H^{-1/2}(\Gamma)$, respectively $\phi \in H_0^{-1/2}(\Gamma) := \{\psi \in H^{-1/2}(\Gamma) : \langle \psi\,,1\rangle_\Gamma = 0\}$. The exact solution $u \in \mathscr{X}$ satisfies

$$b(u\,,v) = F(v) \quad \text{for all } v \in \mathscr{X}. \tag{132}$$

Since the $(h - h/2)$ error-estimator from Theorem 56 uses the uniformly refined mesh $\widehat{\mathscr{T}_\ell} := \texttt{refine}(\mathscr{T}_\ell, \mathscr{T}_\ell)$ instead of the original mesh $\mathscr{T}_\ell$, it is natural to consider the discrete spaces $\mathscr{X}_\ell := \widehat{\mathscr{X}_\ell} := \widetilde{\mathscr{S}}^p(\widehat{\mathscr{T}_\ell})$ for the discrete Galerkin formulation (4) which reads in this setting: find $\widehat{U}_\ell \in \widehat{\mathscr{X}_\ell}$ such that

$$b(\widehat{U}_\ell\,, V) = F(V) \quad \text{for all } V \in \widehat{\mathscr{X}_\ell}. \tag{133}$$

Recall the $(h - h/2)$-type error estimator from Theorem 56

$$\eta_\ell^2 := \sum_{T \in \mathscr{T}_\ell} \eta_\ell(T)^2 := \sum_{T \in \mathscr{T}_\ell} h_T \|(1 - \pi_\ell^p)\nabla_\Gamma \widehat{U}_\ell\|^2_{L_2(T)}, \tag{134}$$

where ∇_Γ denotes the surface gradient on Γ. The next step is to prove the assumptions (A1)–(A2) for η_ℓ.

Lemma 119 *The $(h - h/2)$ error estimator η_ℓ from 134 satisfies stability* (A1) *and reduction* (A2). *The constants C_{12}, q_{red}, C_{13} depend only on Γ, the polynomial degree p, and on the shape regularity of $\mathscr{T}_\ell$.*

Proof The statement is essentially proved in Lemma 82. □

Theorem 120 *For all $0 < \theta \leq 1$, Algorithm 1 with the $(h - h/2)$ error estimator η_ℓ from* (134) *converges in the sense*

$$\lim_{\ell \to \infty} \eta_\ell = 0. \tag{135}$$

If the saturation assumption (Assumption 45) is satisfied, then η_ℓ satisfies reliability (A3), *and there holds R-linear convergence*

$$C_{14}^{-2} \|u - U_{\ell+n}\|^2_{\widetilde{H}^{1/2}(\Gamma)} \leq \eta_{\ell+n}^2 \leq C_{17} \eta_\ell^2\, q_{\mathrm{R}}^n \tag{136}$$

for all $\ell, n \in \mathbb{N}_0$. The constant C_{14} depends only on the saturation constant C_{sata}, Γ, the shape regularity of $\mathscr{T}_\ell$, and the polynomial degree p. The constants C_{17} and q_{R} depend additionally on θ.

Proof The convergence (135) follows from Theorem 111, since η_ℓ satisfies the assumptions (A1)–(A2). The reliability of η_ℓ under the saturation assumption is proved in Theorem 55 together with the equivalence of $\|\cdot\|_W \simeq \|\cdot\|_{\widetilde{H}^{1/2}(\Gamma)}$, and the remaining statement follows from Theorem 111. □

8.7 Optimal convergence of weighted residual error estimator for the hyper singular integral equation

We consider the model problem (132), but in contrast to the previous section, the discrete problem employs the original mesh $\mathscr{T}_\ell$ instead of its uniform refinement $\widehat{\mathscr{T}_\ell}$, i.e., find $U_\ell \in \mathscr{X}_\ell := \widetilde{\mathscr{S}}^p(\mathscr{T}_\ell)$ such that

$$b(U_\ell\,, V) = F(V) \quad \text{for all } V \in \mathscr{X}_\ell. \tag{137}$$

As in Section 6.7.2, the weighted residual error estimator from Section 4.1.3 requires more regularity, i.e., $\phi \in L_2(\Gamma)$ needs to be assumed. The error estimator then reads

$$\eta_\ell^2 := \sum_{T \in \mathscr{T}_\ell} \eta_\ell(T)^2 := \sum_{T \in \mathscr{T}_\ell} h_T \|WU_\ell - \phi\|^2_{L_2(T)}. \tag{138}$$

Lemma 121 *The weighted residual error estimator η_ℓ from* (138) *satisfies the assumptions* (A1)–(A4)*. Moreover,* (A4) *holds with $\mathscr{R}(\ell,\star) := \mathscr{T}_\ell \setminus \mathscr{T}_\star$ and $C_{16} = 1$. The constants C_{12}, C_{13}, $q_{\rm red}$, C_{14}, C_{15} depend only on the shape regularity of the mesh $\mathscr{T}_\ell$, Γ and the polynomial degree p.*

Proof Reliability (A3) is well-known for η_ℓ since its invention in [45] for $d = 2$ and [38] for $d = 3$. The assumptions (A1)–(A2) are shown in the proof of Lemma 90. The proof of discrete reliability (A4) employs the Scott-Zhang projection from Lemma 26 to obtain the local statement. The technical proof refines the arguments from [38] and is found in [66, Proposition 4]. Alternatively, the proof of [76] built on the localization techniques from [63,64]. This, however, restricts the analysis to lowest-order elements $p = 1$, where $\mathscr{R}(\ell,\star)$ consists of $\mathscr{T}_\ell\setminus\mathscr{T}_\star$ plus one layer of non-refined elements. □

Theorem 122 *For all $0 < \theta \le 1$, Algorithm 1 with the residual error estimator η_ℓ from* (138) *converges in the sense*

$$C_{14}^{-2}\|u-U_{\ell+n}\|^2_{\widetilde{H}^{1/2}(\Gamma)} \le \eta^2_{\ell+n} \le C_{17} q_{\rm R}^n \eta_\ell^2 \tag{139}$$

for all $\ell, n \in \mathbb{N}_0$. For $0 < \theta < \theta_0$, Algorithm 1 converges with the best possible rate $s > 0$ in the sense that $\|\eta\|_{\mathbb{A}_s} < \infty$ if and only if

$$\eta_\ell \le C_{20}(\#\mathscr{T}_\ell - \#\mathscr{T}_0)^{-s} \quad \textit{for all } \ell \in \mathbb{N}, \tag{140}$$

where the constants $C_{17}, q_{\rm R}$ depend only on Γ, the shape regularity of the meshes $\mathscr{T}_\ell$, the polynomial degree p, and θ. The constant $C_{20} > 0$ depends additionally on $\|\eta\|_{\mathbb{A}_s}$.

Proof Lemma 121 shows that the assumption (A1)–(A4) are satisfied. Theorem 111 and Theorem 113 prove the statements. □

8.8 Inclusion of data approximation

Also the data approximation, which is already discussed in Section 6.8, can be analyzed towards optimal convergence rates. As in Section 6.8, we replace the right-hand side F in (3) with some computable approximation F_ℓ on any mesh $\mathscr{T}_\ell$ and solve

$$b(\widetilde{U}_\ell, V) = F_\ell(V) \quad \text{for all } V \in \mathscr{X}_\ell \tag{141}$$

instead of (4). To control the additional error $\|\widetilde{U}_\ell - U_\ell\|_{\mathscr{X}}$ introduced by this approximation, the error estimator η_ℓ is extended by some data approximation term

$$\mathrm{data}_\ell^2 := \sum_{T\in\mathscr{T}_\ell} \mathrm{data}_\ell(T)^2 \ge C_{\rm data}^{-1}\|\widetilde{U}_\ell - U_\ell\|^2_{\mathscr{X}}. \tag{142}$$

This is an abstract approach to the concrete results of Section 4.6, where several examples for data_ℓ are given. The extended error estimator reads elementwise for all $T \in \mathscr{T}_\ell$

$$\widetilde{\eta}_\ell(T)^2 := \eta_\ell(T)^2 + \mathrm{data}_\ell(T)^2,$$

where $\eta_\ell(T)$ uses the computable approximate solution $\widetilde{U}_\ell$ and the approximate data F_ℓ instead of the non-computable solution U_ℓ. Note the difference with the notation of Section 4.6. With this, the extension of Algorithm 1 reads:

Algorithm 123 (adaptive mesh refinement) INPUT*: initial mesh $\mathscr{T}_0$ and adaptivity parameter* $0 < \theta \le 1$.
OUTPUT*: sequence of solutions $(\widetilde{U}_\ell)_{\ell\in\mathbb{N}_0}$, sequence of estimators $(\widetilde{\eta}_\ell)_{\ell\in\mathbb{N}_0}$, and sequence of meshes $(\mathscr{T}_\ell)_{\ell\in\mathbb{N}_0}$.*
ITERATION*: For all $\ell = 0,1,2,3,\dots$ do* (i)–(iv)

(i) *Compute solution $\widetilde{U}_\ell$ of* (141)*.*
(ii) *Compute error indicators $\widetilde{\eta}_\ell(T)$ for all elements $T \in \mathscr{T}_\ell$.*
(iii) *Find a set of minimal cardinality $\mathscr{M}_\ell \subseteq \mathscr{T}_\ell$ such that*

$$\theta\widetilde{\eta}_\ell^2 \le \sum_{T\in\mathscr{M}_\ell} \widetilde{\eta}_\ell(T)^2. \tag{143}$$

(iv) *Refine (cf. Section 8.1.1) at least the marked elements to obtain the new mesh $\mathscr{T}_{\ell+1} := \mathtt{refine}(\mathscr{T}_\ell, \mathscr{M}_\ell)$.*

To account for the new estimator term, we have to adopt the assumptions from Section 8.1.2 slightly. To that end, we introduce a theoretical data approximation term $\widetilde{\mathrm{data}}_\ell^2$ which satisfies $C_{\rm data}^{-1}\mathrm{data}_\ell^2 \le \widetilde{\mathrm{data}}_\ell^2 \le C_{\rm data}\mathrm{data}_\ell^2$ for some constant $C_{\rm data} > 0$. The only reason for this is that we want to allow ourselves to use a slightly different oscillation term for implementation than we use for the analysis. This simplifies the realization of Algorithm 123.

($\widetilde{\rm A1}$) Stability on non-refined elements: There exists a constant $C_{12} > 0$ such that any refinement $\mathscr{T}_\star \in \mathtt{refine}(\mathscr{T}_\ell)$ of $\mathscr{T}_\ell \in \mathtt{refine}(\mathscr{T}_0)$ satisfies

$$\Big|\Big(\sum_{T\in\mathscr{T}_\ell\cap\mathscr{T}_\star}\widetilde{\eta}_\ell(T)^2\Big)^{1/2} - \Big(\sum_{T\in\mathscr{T}_\ell\cap\mathscr{T}_\star}\widetilde{\eta}_\star(T)^2\Big)^{1/2}\Big|^2 \le C_{12}^2\Big(\|\widetilde{U}_\star - \widetilde{U}_\ell\|^2_{\mathscr{X}} + \widetilde{\mathrm{data}}_\ell^2 - \widetilde{\mathrm{data}}_\star^2\Big).$$

($\widetilde{\rm A2}$) Reduction on refined elements: There exist constants $C_{13} > 0$ and $0 < q_{\rm red} < 1$ such that any refinement $\mathscr{T}_\star \in \mathtt{refine}(\mathscr{T}_\ell)$ of $\mathscr{T}_\ell \in \mathtt{refine}(\mathscr{T}_0)$ satisfies

$$\sum_{T\in\mathscr{T}_\star\setminus\mathscr{T}_\ell}\widetilde{\eta}_\star(T)^2 \le q_{\rm red}\sum_{T\in\mathscr{T}_\ell\setminus\mathscr{T}_\star}\widetilde{\eta}_\ell(T)^2 + C_{13}\Big(\|\widetilde{U}_\star - \widetilde{U}_\ell\|^2_{\mathscr{X}} + \widetilde{\mathrm{data}}_\ell^2 - \widetilde{\mathrm{data}}_\star^2\Big).$$

($\widetilde{\rm A3}$) Reliability: There exists a constant $C_{14} > 0$ such that any mesh $\mathscr{T}_\ell \in \mathtt{refine}(\mathscr{T}_0)$ satisfies

$$\|u - \widetilde{U}_\ell\|_{\mathscr{X}} \le C_{14}\widetilde{\eta}_\ell.$$

($\widetilde{\text{A4}}$) Discrete reliability: There exist constants $C_{15} > 0$ and $C_{16} > 0$ such that any refinement $\mathscr{T}_\star \in \texttt{refine}(\mathscr{T}_\ell)$ of $\mathscr{T}_\ell \in \texttt{refine}(\mathscr{T}_0)$ satisfies

$$\|\widetilde{U}_\star - \widetilde{U}_\ell\|^2_{\mathscr{X}} + \widetilde{\text{data}}^2_\ell - \widetilde{\text{data}}^2_\star \le C^2_{15} \sum_{T\in\mathscr{R}(\ell,\star)} \widetilde{\eta}_\ell(T)^2,$$

where the set $\mathscr{R}(\ell,\star) \supseteq \mathscr{T}_\ell \setminus \mathscr{T}_\star$ satisfies $\#\mathscr{R}(\ell,\star) \le C_{16}\#(\mathscr{T}_\ell \setminus \mathscr{T}_\star)$.

Moreover, and in contrast to the unperturbed case in Section 8.1.2, the a priori convergence (78) as well as the generalized Pythagoras estimate (113) are not available in this general setting. Hence, we also have to verify

($\widetilde{\text{A5}}$) A priori convergence of data: There exists a continuous linear functional $F_\infty : \mathscr{X}_\infty \to \mathbb{R}$ such that

$$\lim_{\ell\to\infty} \|F_\infty - F_\ell\|_{\mathscr{X}'_\ell} := \lim_{\ell\to\infty} \sup_{\substack{V\in\mathscr{X}_\ell\\ \|V\|_{\mathscr{X}}=1}} |F_\infty(V) - F_\ell(V)| = 0,$$

where $\mathscr{X}_\infty := \overline{\bigcup_{\ell\in\mathbb{N}_0} \mathscr{X}_\ell} \subseteq \mathscr{X}$ (the closure is understood with respect to $\mathscr{X}$). Moreover, there exists $\widetilde{\text{data}}_\infty \ge 0$ such that

$$\lim_{\ell\to\infty} \widetilde{\text{data}}_\ell = \widetilde{\text{data}}_\infty.$$

($\widetilde{\text{A6}}$) Pythagoras estimate: For all $\varepsilon > 0$, there exists a constant $C_{19}(\varepsilon) > 0$ such that for all $k \in \mathbb{N}$ holds

$$\sum_{k=\ell}^{\infty} \|\widetilde{U}_{k+1} - \widetilde{U}_k\|^2_{\mathscr{X}} - \varepsilon\widetilde{\eta}^2_k \le C_{19}(\varepsilon)\widetilde{\eta}^2_\ell.$$

8.9 Data approximation and convergence of ABEM

The proofs in this section differ only slightly from the unperturbed case in Section 8.2. Therefore, we only highlight the differences.

Theorem 124 *Suppose that there hold* ($\widetilde{\text{A1}}$)–($\widetilde{\text{A2}}$) *as well as* ($\widetilde{\text{A5}}$)*. Then, Algorithm 123 drives the estimator to zero, i.e.,*

$$\lim_{\ell\to\infty} \widetilde{\eta}_\ell = 0. \tag{144}$$

Suppose that the error estimator additionally satisfies ($\widetilde{\text{A3}}$) *and* ($\widetilde{\text{A6}}$)*. Then, Algorithm 123 converges even linearly in the sense that there exist constants $C_{17} > 0$ and $0 < q_{\rm R} < 1$ such that*

$$C^{-2}_{14}\|u - \widetilde{U}_{\ell+n}\|^2_{\mathscr{X}} \le \widetilde{\eta}^2_{\ell+n} \le C_{17} q^n_{\rm R} \widetilde{\eta}^2_\ell \text{ for all } \ell, n \in \mathbb{N}_0, \tag{145}$$

which particularly implies

$$\|u - \widetilde{U}_\ell\|^2_{\mathscr{X}} \le C^2_{14} C_{17} \widetilde{\eta}^2_0\, q^\ell_{\rm R} \quad \text{for all } \ell \in \mathbb{N}_0. \tag{146}$$

The constants C_{17} and $q_{\rm R}$ depend only on θ as well as on the constants in ($\widetilde{\text{A1}}$)–($\widetilde{\text{A3}}$) *and* ($\widetilde{\text{A6}}$)*.*

Again, an important ingredient is the a priori convergence of $\widetilde{U}_\ell$.

Lemma 125 (a priori convergence) *Suppose* ($\widetilde{\text{A5}}$)*. Then, there exists $\widetilde{U}_\infty \in \mathscr{X}$ such that Algorithm 123 satisfies*

$$\lim_{\ell\to\infty} \|\widetilde{U}_\infty - \widetilde{U}_\ell\|_{\mathscr{X}} = 0. \tag{147}$$

Proof Replace F in (4) by F_∞ from ($\widetilde{\text{A5}}$) and consider the corresponding solution $(U_{\infty,\ell})_{\ell\in\mathbb{N}}$. Lemma 76 shows the existence of $\widetilde{U}_\infty \in \mathscr{X}$ such that

$$\lim_{\ell\to\infty} \|\widetilde{U}_\infty - U_{\ell,\infty}\|_{\mathscr{X}} = 0. \tag{148}$$

From stability

$$\|U_{\ell,\infty} - \widetilde{U}_\ell\|_{\mathscr{X}} \le \|F_\infty - F_\ell\|_{\mathscr{X}'_\ell},$$

it follows

$$\begin{aligned}\|\widetilde{U}_\infty - \widetilde{U}_\ell\|_{\mathscr{X}} &\le \|\widetilde{U}_\infty - U_{\ell,\infty}\|_{\mathscr{X}} + \|U_{\ell,\infty} - \widetilde{U}_\ell\|_{\mathscr{X}}\\ &\lesssim \|\widetilde{U}_\infty - U_{\ell,\infty}\|_{\mathscr{X}} + \|F_\infty - F_\ell\|_{\mathscr{X}'_\ell} \to 0\end{aligned}$$

as $\ell \to \infty$ by assumption ($\widetilde{\text{A5}}$) and (148). □

Also the estimator reduction follows accordingly.

Lemma 126 *Suppose* ($\widetilde{\text{A1}}$)–($\widetilde{\text{A2}}$)*. Then, there exist constants $C_{18} > 0$ and $0 < q_{\rm est} < 1$ such that Algorithm 123 satisfies*

$$\widetilde{\eta}^2_{\ell+1} \le q_{\rm est}\widetilde{\eta}^2_\ell + C_{18}\Big(\|\widetilde{U}_{\ell+1} - \widetilde{U}_\ell\|^2_{\mathscr{X}} + \widetilde{\text{data}}^2_\ell - \widetilde{\text{data}}^2_{\ell+1}\Big) \tag{149}$$

for all $\ell \in \mathbb{N}$. The constant $q_{\rm est}$ depends only on θ and $q_{\rm red}$ from ($\widetilde{\text{A2}}$)*. The constant C_{18} depends additionally on C_{13} as well as C_{12}.*

Proof The proof is identical to that of Lemma 112. □

This implies the first part of Theorem 124.

Proof (of estimator convergence (144)*)* Due to ($\widetilde{\text{A5}}$), there holds $\widetilde{\text{data}}^2_\ell - \widetilde{\text{data}}^2_{\ell+1} \to 0$ as $\ell \to \infty$. With this and the arguments of the proof of (106), we see with a priori convergence (147) for the limes superior $\overline{\lim}_{\ell\to\infty}\widetilde{\eta}^2_{\ell+1} = 0$ or $\overline{\lim}_{\ell\to\infty}\widetilde{\eta}^2_{\ell+1} = \infty$. To rule out the second option, apply the estimator reduction iteratively to see

$$\begin{aligned}\widetilde{\eta}^2_\ell &\le q^\ell_{\rm est}\widetilde{\eta}^2_0 + C_{18}\sum_{k=0}^{\ell-1} q^k_{\rm est}\Big(\|\widetilde{U}_{\ell-k} - \widetilde{U}_{\ell-k-1}\|^2_{\mathscr{X}}\\ &\qquad\qquad + \widetilde{\text{data}}^2_{\ell-k-1} - \widetilde{\text{data}}^2_{\ell-k}\Big)\\ &\le q^\ell_{\rm est}\widetilde{\eta}^2_0 + C_{18}\Big(\widetilde{\text{data}}^2_0 + \sum_{k=0}^{\ell-1} q^k_{\rm est}\|\widetilde{U}_{\ell-k} - \widetilde{U}_{\ell-k-1}\|^2_{\mathscr{X}}\Big),\end{aligned}$$

by exploiting the telescoping series. The a priori convergence of Lemma 125 implies $\sup_{\ell\in\mathbb{N}} \|\widetilde{U}_\ell - \widetilde{U}_{\ell-1}\|_{\mathscr{X}}^2 \le C_{\max} < \infty$. This and the convergence of the geometric series show

$$\widetilde{\eta}_\ell^2 \le \widetilde{\eta}_0^2 + 2C_{18}(\mathrm{data}_0^2 + C_{\max}) < \infty \quad \text{for all } \ell \in \mathbb{N}$$

and consequently $\overline{\lim}_{\ell\to\infty}\eta_\ell^2 = 0$. This concludes the proof of (144). □

The convergence (144) leads us to the *R*-linear convergence.

Proof (of R-linear estimator convergence (145)*)* Let $N,\ell \in \mathbb{N}$. Use the estimator reduction (149) to see

$$\sum_{k=\ell+1}^{\ell+N} \widetilde{\eta}_k^2 \le \sum_{k=\ell+1}^{\ell+N} \Big(q_{\mathrm{est}}\widetilde{\eta}_{k-1}^2 + C_{18}\big(\|\widetilde{U}_k - \widetilde{U}_{k-1}\|_{\mathscr{X}}^2 + \widetilde{\mathrm{data}}_{k-1}^2 - \widetilde{\mathrm{data}}_k^2\big)\Big).$$

By use of the telescoping series, this implies

$$(1-q_{\mathrm{est}}-C_{18}\varepsilon)\sum_{k=\ell+1}^{\ell+N}\widetilde{\eta}_k^2 \le \widetilde{\eta}_\ell^2 + C_{18}\Big(\widetilde{\mathrm{data}}_\ell^2 + \sum_{k=\ell+1}^{\ell+N}\big(\|\widetilde{U}_k - \widetilde{U}_{k-1}\|_{\mathscr{X}}^2 - \varepsilon\widetilde{\eta}_k^2\big)\Big).$$

The assumption ($\widetilde{\mathrm{A6}}$) together with $C_{\mathrm{data}}^{-1}\widetilde{\mathrm{data}}_\ell^2 \le \mathrm{data}_\ell^2 \le \widetilde{\eta}_\ell^2$ then shows

$$\sum_{k=\ell}^{\ell+N}\widetilde{\eta}_k^2 \le \frac{2+C_{18}(C_{19}(\varepsilon)+C_{\mathrm{data}})}{1-q_{\mathrm{est}}-C_{18}\varepsilon}\widetilde{\eta}_\ell^2 := C_{17}\widetilde{\eta}_\ell^2.$$

Clearly, $C_{17} \ge 1$ for sufficiently small $\varepsilon > 0$. Moreover, the right-hand side is independent of N and hence

$$\sum_{k=\ell}^{\infty}\widetilde{\eta}_k^2 \le C_{17}\widetilde{\eta}_\ell^2.$$

The remainder of the proof follows as in the proof of (107). □

8.10 Data approximation and optimal rates

The approximability norm now also contains the data approximation term data_ℓ, i.e.,

$$\|\widetilde{\eta}\|_{\mathbb{A}_s} := \sup_{N\in\mathbb{N}_0}(N+1)^s\big(\inf_{\substack{\mathscr{T}_\star\in\mathtt{refine}(\mathscr{T}_0)\\ \#\mathscr{T}_\star-\#\mathscr{T}_0\le N}}\widetilde{\eta}_\star\big) \in [0,\infty]. \tag{150}$$

This allows us to formulate the following theorem.

Theorem 127 *Suppose* ($\widetilde{\mathrm{A1}}$)–($\widetilde{\mathrm{A2}}$) *as well as discrete reliability* ($\widetilde{\mathrm{A4}}$) *and* ($\widetilde{\mathrm{A5}}$)–($\widetilde{\mathrm{A6}}$). *Let the adaptivity parameter satisfy $\theta < \theta_0 := (1+C_{12}^2C_{15}^2)^{-1}$. Then, Algorithm 1 converges with the best possible rate in the sense that for all $s > 0$, it holds $\|\widetilde{\eta}\|_{\mathbb{A}_s} < \infty$ if and only if*

$$\eta_\ell \le C_{20}(\#\mathscr{T}_\ell - \#\mathscr{T}_0)^{-s} \quad \text{for all } \ell\in\mathbb{N}, \tag{151}$$

where $C_{20} > 0$ depends only on the constants in ($\widetilde{\mathrm{A1}}$)–($\widetilde{\mathrm{A6}}$) *as well as on $\|\widetilde{\eta}\|_{\mathbb{A}_s}$ and θ.*

The optimality of the marking criterion still holds with data approximation.

Lemma 128 *Let the error estimator satisfy stability* ($\widetilde{\mathrm{A1}}$) *and discrete reliability* ($\widetilde{\mathrm{A4}}$). *Then, there exists $0 < \kappa_0 < 1$ such that any refinement $\mathscr{T}_\star \in \mathtt{refine}(\mathscr{T}_\ell)$ which satisfies*

$$\widetilde{\eta}_\star^2 \le \kappa_0\widetilde{\eta}_\ell^2, \tag{152}$$

fulfils the bulk chasing (143) *in the sense*

$$\theta\widetilde{\eta}_\ell^2 \le \sum_{T\in\mathscr{R}(\ell,\star)}\widetilde{\eta}_\ell(T)^2 \tag{153}$$

for all $0 \le \theta < \theta_0$. The constant θ_0 is defined in Theorem 127 whereas $\mathscr{R}(\ell,\star) \subseteq \mathscr{T}_\ell$ is defined in ($\widetilde{\mathrm{A4}}$).

Proof The proof is identical to that of Lemma 114. □

Proof (of Theorem 127) The proof combines only the previous results and is therefore identical to the proof of Theorem 127. □

The final lemma proves that the overall best rate is now determined by the respective best rates for data approximation terms and for the non-perturbed problem.

Lemma 129 *Suppose that for $s_1, s_2 > 0$, there holds*

$$\sup_{N\in\mathbb{N}_0}(N+1)^{s_1}\big(\inf_{\substack{\mathscr{T}_\star\in\mathtt{refine}(\mathscr{T}_0)\\ \#\mathscr{T}_\star-\#\mathscr{T}_0\le N}}\eta_\star\big) < \infty \tag{154}$$

as well as

$$\sup_{N\in\mathbb{N}_0}(N+1)^{s_2}\big(\inf_{\substack{\mathscr{T}_\star\in\mathtt{refine}(\mathscr{T}_0)\\ \#\mathscr{T}_\star-\#\mathscr{T}_0\le N}}\mathrm{data}_\star\big) < \infty. \tag{155}$$

Then, this implies $\|\widetilde{\eta}\|_{\mathbb{A}_s} < \infty$ for $s := \min\{s_1,s_2\}$. Conversely, $\|\widetilde{\eta}\|_{\mathbb{A}_s} < \infty$ implies (154)–(155) *with $s_1 = s = s_2$.*

Proof The proof is technical and can be found in [65], but essentially only relies on the overlay estimate (101). □

8.11 Optimal convergence of weighted residual error estimator for the weakly singular integral equation with data approximation

As in Section 6.9.1, we consider the model problem from Proposition 10, i.e.

$$b(\phi\,,\psi) := \langle V\phi\,,\psi\rangle \quad \text{for all } \phi,\psi\in\mathscr{X} := \widetilde{H}^{-1/2}(\Gamma),$$
$$F(\psi) := \langle (1/2+K)f\,,\psi\rangle \quad \text{for all } \phi\in\mathscr{X},$$

where $f\in H^1(\Gamma)$. In contrast to Section 6.9.1, the data approximation is done via the Scott-Zhang operator $J_\ell^{p+1} := J_{\mathscr{T}_\ell}^{p+1} : L_2(\Gamma)\to\mathscr{S}^{p+1}(\mathscr{T}_\ell)$ from Lemma 26. We define

$$F_\ell(\psi) := \langle (1/2+K)J_\ell^{p+1}f\,,\psi\rangle \quad \text{for all } \ell\in\mathbb{N}_0.$$

With $\mathscr{X}_\ell := \mathscr{P}^p(\mathscr{T}_\ell)$, the discrete version (141) reads: Find $\widetilde{\Phi}_\ell\in\mathscr{X}_\ell$ such that

$$b(\widetilde{\Phi}_\ell\,,\Psi) = F_\ell(\Psi) \quad \text{for all } \Psi\in\mathscr{X}_\ell.$$

There holds $J_\ell f\in H^1(\Gamma)$ and hence the standard weighted residual error estimator reads

$$\eta_\ell^2 := \sum_{T\in\mathscr{T}_\ell}\eta_\ell(T)^2 := \sum_{T\in\mathscr{T}_\ell} h_T\|\nabla_\Gamma(V\widetilde{\Phi}_\ell-(1/2+K)J_\ell^{p+1}f)\|_{L_2(T)}^2,$$

where $\nabla_\Gamma(\cdot)$ denotes the surface gradient on Γ. The data approximation term is defined as

$$\mathrm{data}_\ell^2 := \sum_{T\in\mathscr{T}_\ell}\mathrm{data}_\ell(T)^2 := \sum_{T\in\mathscr{T}_\ell} h_T\|(1-\pi_\ell^p)\nabla_\Gamma f\|_{L_2(T)}^2.$$

It is proved in Lemma 69, that $C_{\mathrm{data}}^{-1}\|\Phi_\ell-\widetilde{\Phi}_\ell\|_{\mathscr{X}}^2 \le \mathrm{data}_\ell^2$, where the constant $C_{\mathrm{data}} = C_7 > 0$ depends only on the polynomial degree p, on $\mathscr{T}_0$ (since Proposition 105 states that Algorithm 104 produces only finitely many different shapes of element patches), and on the shape regularity of $\mathscr{T}_\ell$. Altogether, the extended error estimator reads

$$\widetilde{\eta}_\ell^2 = \|h_\ell^{1/2}\nabla_\Gamma(V\widetilde{\Phi}_\ell-(1/2+K)J_\ell^{p+1}f)\|_{L_2(\Gamma)}^2 + \|h_\ell^{1/2}(1-\pi_\ell^p)\nabla_\Gamma f\|_{L_2(T)}^2. \tag{156}$$

For the abstract analysis of Section 8.8, we define an elementwise equivalent data approximation term, which is only of theoretical purpose and does not have to be computed at all. This term is defined as

$$\widetilde{\mathrm{data}}_\ell^2 = \|\widetilde{h}_\ell^{1/2}(1-\pi_\ell^p)\nabla_\Gamma f\|_{L_2(T)}^2,$$

where we exchanged the mesh-size function h_ℓ with the modified mesh-size function $\widetilde{h}_\ell$ from [37, Section 8] in the data approximation term. This modified mesh-width function satisfies the following.

Lemma 130 *The modified mesh-size function $\widetilde{h}_\ell\in\mathscr{P}^0(\mathscr{T}_\ell)$ from [37, Section 8] satisfies for $\mathscr{T}_\star\in\mathtt{refine}(\mathscr{T}_\ell)$ the following properties* (i)–(iii)*:*

(i) Elementwise equivalence: $C_{\mathrm{h}}^{-1}h_\ell|_T\le\widetilde{h}_\ell|_T\le h_\ell|_T$ for all $T\in\mathscr{T}_\ell$;
(ii) Monotonicity: $\widetilde{h}_\star|_T\le\widetilde{h}_\ell|_T$ for all $T\in\mathscr{T}_\ell$;
(iii) Reduction: $\widetilde{h}_\star\le q_{\mathrm{h}}\widetilde{h}_\ell$ on $\mathscr{R}(\ell,\star)$,

where $\mathscr{R}(\ell,\star)\supseteq\mathscr{T}_\ell\setminus\mathscr{T}_\star$ is defined as

$$\mathscr{R}(\ell,\star) := \{T\in\mathscr{T}_\ell : \textit{exists } T_1=T,T_2,\dots,T_6 \textit{ with } T_6\in\mathscr{T}_\ell\setminus\mathscr{T}_\star \textit{ and } \overline{T}_j\cap\overline{T}_{j+1}\ne\emptyset \textit{ for all } j=0,\dots,4\} \tag{157}$$

which roughly means $\mathscr{T}_\ell\setminus\mathscr{T}_\star$ plus five additional layers of elements around $\mathscr{T}_\ell\setminus\mathscr{T}_\star$. The constants $C_{\mathrm{h}}>0$ and $0<q_{\mathrm{h}}<1$ depend only on the σ_0-shape regularity of $\mathscr{T}_0$ and the space dimension d.

Due to (i) in the above Lemma 130, there holds obviously $\widetilde{\mathrm{data}}_\ell\simeq\mathrm{data}_\ell$.

Lemma 131 *The weighted residual error estimator $\widetilde{\eta}_\ell$ from* (156) *satisfies the assumptions* $(\widetilde{\mathrm{A1}})$–$(\widetilde{\mathrm{A6}})$*. The set $\mathscr{R}(\ell,\star)$ from* (A4) *is defined in* (157)*. The constant C_{16} depends only on the shape regularity of the mesh $\mathscr{T}_\ell$, whereas the constants $C_{12}, C_{13}, q_{\mathrm{red}}, C_{14}, C_{15}, C_{19}$ depend additionally on Γ and the polynomial degree p.*

Proof (of $(\widetilde{\mathrm{A1}})$–$(\widetilde{\mathrm{A2}})$*)* The assumptions $(\widetilde{\mathrm{A1}})$–$(\widetilde{\mathrm{A2}})$ are proved very similar to the assumptions (A1)–(A2) for the unperturbed case in Lemma 131. The proof can be found in great detail in [65] and is only sketched in the following.

The main difference to the proofs of (A1)–(A1) in the unperturbed case of Lemma 117 is that one obtains analogously to the proof of Lemma 70 the additional term

$$\|J_\star f - J_\ell f\|_{H^{1/2}(\Gamma)}^2$$

on the right-hand side. The elementwise equivalence of h_ℓ and $\widetilde{h}_\ell$ from Lemma 130 together with an approximation property of the Scott-Zhang projection proved in [65, Lemma 2] imply

$$\|J_\star f - J_\ell f\|_{H^{1/2}(\Gamma)}^2 \lesssim \|h_\ell^{1/2}(1-\pi_\ell)\nabla_\Gamma f\|_{L_2(\bigcup\mathscr{R}(\ell,\star))}^2 \simeq \|\widetilde{h}_\ell^{1/2}(1-\pi_\ell)\nabla_\Gamma f\|_{L_2(\bigcup\mathscr{R}(\ell,\star))}^2$$

The fact that $\widetilde{h}_\star\le q_{\mathrm{h}}\widetilde{h}_\ell$ on $\mathscr{R}(\ell,\star)$ together with the monotonicity of $\widetilde{h}_\ell$ show

$$(1-q_{\mathrm{h}})\widetilde{h}_\ell|_{\mathscr{R}(\ell,\star)}\le\widetilde{h}_\ell-\widetilde{h}_\star$$

pointwise almost everywhere on $\mathscr{R}(\ell,\star)$. Hence, we get

$$
\begin{aligned}
\|\widetilde{h}_\ell^{1/2}&(1-\pi_\ell)\nabla_\Gamma f\|^2_{L_2(\bigcup\mathscr{R}(\ell,\star))}\\
&= \int_{\bigcup\mathscr{R}(\ell,\star)} \widetilde{h}_\ell\big((1-\pi_\ell)\nabla_\Gamma f\big)^2\,dx\\
&\le (1-q_{\rm h})^{-1}\int_\Gamma(\widetilde{h}_\ell-\widetilde{h}_\star)\big((1-\pi_\ell)\nabla_\Gamma f\big)^2\,dx\\
&\lesssim \|\widetilde{h}_\ell^{1/2}(1-\pi_\ell)\nabla_\Gamma f\|^2_{L_2(\Gamma)} - \|\widetilde{h}_\star^{1/2}(1-\pi_\ell)\nabla_\Gamma f\|^2_{L_2(\Gamma)}\\
&\le \widetilde{\mathrm{data}}_\ell^2 - \widetilde{\mathrm{data}}_\star^2.
\end{aligned}
\tag{158}
$$

This result is the main ingredient for the proof. □

Proof (of $(\widetilde{\mathrm{A3}})$*)* Define $\phi_\ell\in H^{-1/2}(\Gamma)$ as solution of (122) when replacing the right-hand side with $F_\ell := (1/2+K)J_\ell f$. By definition of data_ℓ and the reliability of the unperturbed problem (A3) from Lemma 117, there holds

$$
\begin{aligned}
\|\phi-\widetilde{\Phi}_\ell\|_{\mathscr{X}} &\le \|\phi-\widetilde{\phi}_\ell\|_{\mathscr{X}} + \|\widetilde{\phi}_\ell-\widetilde{\Phi}_\ell\|_{\mathscr{X}}\\
&\lesssim \widetilde{\eta}_\ell + \mathrm{data}_\ell,
\end{aligned}
$$

where we used Lemma 69 for the last estimate. This proves $(\widetilde{\mathrm{A3}})$. □

Proof (of $(\widetilde{\mathrm{A4}})$*)* The proof is very similar to the unperturbed case but additionally utilizes Lemma 69. It can be found in [65]. □

Proof (of $(\widetilde{\mathrm{A5}})$*)* The proof of a priori convergence $(\widetilde{\mathrm{A5}})$ for $F_\ell := (1/2+K)J_\ell f$ uses the a priori convergence of the Scott-Zhang projection proved in Lemma 78 in the sense

$$
\lim_{\ell\to\infty} J_\ell f = J_\infty f \in H^{1/2}(\Gamma).
$$

This and the stability of $K : H^{1/2}(\Gamma)\to H^{1/2}(\Gamma)$ imply for $F_\infty := (1/2+K)J_\infty f$

$$
\|F_\infty - F_\ell\|_{H^{1/2}(\Gamma)} \lesssim \|J_\infty f - J_\ell f\|_{H^{1/2}(\Gamma)} \to 0
$$

as $\ell\to\infty$.

We define $\widetilde{\mathrm{data}}_\infty := \lim_{\ell\to\infty}\|\widetilde{h}_\ell^{1/2}(1-\pi_\infty^p)\nabla_\Gamma f\|^2_{L_2(\Gamma)}$, where $\pi_\infty^p : L_2(\Gamma)\to\mathscr{X}_\infty$ is the L_2-orthogonal projection. By definition of $\mathscr{X}_\infty$, there holds $\lim_{\ell\to\infty}\widetilde{\mathrm{data}}_\ell = \widetilde{\mathrm{data}}_\infty$. This concludes the proof. □

Proof (of $(\widetilde{\mathrm{A6}})$*)* Define $\widetilde{\phi}_k\in H^{-1/2}(\Gamma)$ as solution of (122) when replacing the right-hand side with $F_k := (1/2+K)J_k f$. Then, there holds for all $k\in\mathbb{N}_0$ that $b(\phi_{k+1}-\widetilde{\Phi}_{k+1}, \widetilde{\Phi}_{k+1}-\widetilde{\Phi}_k)=0$ and hence

$$
\begin{aligned}
\|\widetilde{\Phi}_{k+1}-\widetilde{\Phi}_k\|^2_{\widetilde{V}} = \|\widetilde{\phi}_{k+1}-\widetilde{\Phi}_k\|^2_{\widetilde{V}}\\
-\|\widetilde{\phi}_{k+1}-\widetilde{\Phi}_{k+1}\|^2_{\widetilde{V}}.
\end{aligned}
$$

Young's inequality $(a+b)^2\le(1+\delta)a^2+(1-\delta^{-1})b^2$ for all $a,b\in\mathbb{R}$ and $\delta>0$ shows

$$
\begin{aligned}
\|\widetilde{\Phi}_{k+1}&-\widetilde{\Phi}_k\|^2_{\widetilde{V}}\\
&\le (1+\delta)\|\widetilde{\phi}_k-\widetilde{\Phi}_k\|^2_{\widetilde{V}} - \|\widetilde{\phi}_{k+1}-\widetilde{\Phi}_{k+1}\|^2_{\widetilde{V}}\\
&\quad + (1+\delta^{-1})\|\widetilde{\phi}_{k+1}-\widetilde{\phi}_k\|^2_{\widetilde{V}}.
\end{aligned}
$$

The stability of the problem (3) implies $\|\widetilde{\phi}_{k+1}-\widetilde{\phi}_k\|^2_{\widetilde{V}}\simeq \|\widetilde{\phi}_{k+1}-\widetilde{\phi}_k\|^2_{\widetilde{V}}\lesssim \|J_{k+1}f-J_kf\|_{H^{1/2}(\Gamma)}$ and as in the proof of $(\widetilde{\mathrm{A1}})$–$(\widetilde{\mathrm{A2}})$ above, we see

$$
\|\widetilde{\phi}_{k+1}-\widetilde{\phi}_k\|^2_{\widetilde{V}} \lesssim \widetilde{\mathrm{data}}_k^2 - \widetilde{\mathrm{data}}_{k+1}^2.
$$

As in the proof of $(\widetilde{\mathrm{A3}})$, one shows that $\|\widetilde{\phi}_k-\widetilde{\Phi}_k\|_{\widetilde{V}}\le C_{14}\widetilde{\eta}_k$. Altogether, this shows for $\varepsilon C_{14}^{-1}=\delta$

$$
\begin{aligned}
\sum_{k=\ell}^\infty&(\|\widetilde{\Phi}_{k+1}-\widetilde{\Phi}_k\|^2_{\widetilde{V}}-\varepsilon\widetilde{\eta}_k^2)\\
&\le \sum_{k=\ell}^\infty\big(\|\widetilde{\Phi}_{k+1}-\widetilde{\Phi}_k\|^2_{\widetilde{V}}-\delta\|\widetilde{\phi}_k-\widetilde{\Phi}_k\|^2_{\widetilde{V}}\big)\\
&\lesssim \sum_{k=\ell}^\infty\big(\|\widetilde{\phi}_k-\widetilde{\Phi}_k\|^2_{\widetilde{V}}-\|\widetilde{\phi}_{k+1}-\widetilde{\Phi}_{k+1}\|^2_{\widetilde{V}}\\
&\qquad + \widetilde{\mathrm{data}}_k^2-\widetilde{\mathrm{data}}_{k+1}^2\big).
\end{aligned}
$$

The telescoping series reveals

$$
\begin{aligned}
\sum_{k=\ell}^\infty&(\|\widetilde{\Phi}_{k+1}-\widetilde{\Phi}_k\|^2_{\widetilde{V}}-\varepsilon\widetilde{\eta}_k^2)\\
&\qquad\lesssim \|\widetilde{\phi}_\ell-\widetilde{\Phi}_\ell\|^2_{\widetilde{V}} + \widetilde{\mathrm{data}}_\ell^2 \lesssim \widetilde{\eta}_\ell^2.
\end{aligned}
$$

This concludes the proof. □

Theorem 132 *For $0<\theta\le 1$, Algorithm 1 with the residual error estimator $\widetilde{\eta}_\ell$ from* (156) *converges in the sense of*

$$
C_{14}^{-2}\|\phi-\widetilde{\Phi}_{\ell+n}\|^2_{H^{-1/2}(\Gamma)} \le \widetilde{\eta}_{\ell+n}^2 \le C_{17}q_{\rm R}^n\widetilde{\eta}_\ell^2 \tag{159}
$$

for all $\ell,n\in\mathbb{N}_0$. For $0<\theta<\theta_0$, Algorithm 1 converges with the best possible rate $s>0$ in the sense that $\|\widetilde{\eta}\|_{\mathbb{A}_s}<\infty$ if and only if

$$
\widetilde{\eta}_\ell \le C_{20}(\#\mathscr{T}_\ell-\#\mathscr{T}_0)^{-s} \quad \textit{for all } \ell\in\mathbb{N}, \tag{160}
$$

where the constants $C_{17}, q_{\rm R}$ depend only on Γ, the shape regularity of the meshes $\mathscr{T}_\ell$, the polynomial degree p, and θ. The constant $C_{20}>0$ depends additionally on $\|\widetilde{\eta}\|_{\mathbb{A}_s}$.

Proof Lemma 131 shows that the assumption $(\widetilde{\mathrm{A1}})$–$(\widetilde{\mathrm{A6}})$ are satisfied. Theorem 124 and Theorem 127 prove the statements. □

8.12 Optimal convergence of weighted residual error estimator for the hyper singular integral equation with data approximation

As in Section 6.9.2, we consider the model problem from Proposition 13, i.e.

$$b(u,v) := \langle Wu, v\rangle_\Gamma + \langle u,1\rangle_\Gamma \langle v,1\rangle_\Gamma \quad \text{for all } u,v \in \mathscr{X},$$
$$F(v) := \langle (1/2-K')\phi, v\rangle \quad \text{for all } v \in \mathscr{X} := H^{1/2}(\Gamma),$$

where $\phi \in L_2(\Gamma)$. The data approximation is done via the L_2-orthogonal projection $\pi_\ell := \pi^{p-1}_{\mathscr{T}_\ell} : L_2(\Gamma) \to \mathscr{P}^{p-1}(\mathscr{T}_\ell)$. We define

$$F_\ell(\psi) := \langle (1/2-K')\pi_\ell\phi, \psi\rangle \quad \text{for all } \ell \in \mathbb{N}_0.$$

With $\mathscr{X}_\ell := \mathscr{S}^p(\mathscr{T}_\ell)$ for $p \geq 1$, the discrete version (141) of (132) reads: Find $\widetilde{U}_\ell \in \mathscr{X}_\ell$ such that

$$b(\widetilde{U}_\ell, V) = F_\ell(V) \quad \text{for all } V \in \mathscr{X}_\ell.$$

The standard weighted residual error estimator reads

$$\eta_\ell^2 := \sum_{T\in\mathscr{T}_\ell} \eta_\ell(T)^2 := \sum_{T\in\mathscr{T}_\ell} h_T \|W\widetilde{U}_\ell - (1/2-K')\pi_\ell\phi)\|^2_{L_2(T)}.$$

The data approximation term is defined as

$$\mathrm{data}_\ell^2 := \sum_{T\in\mathscr{T}_\ell} \mathrm{data}_\ell(T)^2 := \sum_{T\in\mathscr{T}_\ell} h_T \|(1-\pi_\ell)\phi\|^2_{L_2(T)},$$

with $\widetilde{\mathrm{data}}_\ell = \mathrm{data}_\ell$. Lemma 71 shows that $C^{-1}_{\mathrm{data}}\|U_\ell - \widetilde{U}_\ell\|^2_{\mathscr{X}} \leq \mathrm{data}_\ell^2$, where the constant $C_{\mathrm{data}} = C_{10} > 0$ depends only on the polynomial degree p and on the shape regularity of $\mathscr{T}_\ell$. Altogether, the extended error estimator reads

$$\widetilde{\eta}_\ell^2 = \|h_\ell^{1/2}(W\widetilde{U}_\ell - (1/2-K')\pi_\ell\phi)\|^2_{L_2(\Gamma)} + \|h_\ell^{1/2}(1-\pi_\ell)\phi\|^2_{L_2(T)}. \tag{161}$$

Lemma 133 *The weighted residual error estimator $\widetilde{\eta}_\ell$ from (161) satisfies the assumptions $(\widetilde{\mathrm{A1}})$–$(\widetilde{\mathrm{A6}})$. The set $\mathscr{R}(\ell,\star)$ from $(\widetilde{\mathrm{A4}})$ satisfies $\mathscr{R}(\ell,\star) := \mathscr{T}_\ell \setminus \mathscr{T}_\star$ and $C_{16} = 1$. The constants C_{12}, C_{13}, q_{red}, C_{14}, C_{15}, C_{19} depend on Γ, the shape regularity of $\mathscr{T}_\ell$, and the polynomial degree p.*

Proof The assumptions $(\widetilde{\mathrm{A1}})$–$(\widetilde{\mathrm{A3}})$ are straightforward to prove. A detailed proof is found in [66]. The proof of the discrete reliability is very similar to the unperturbed case and is also found in [66]. To see $(\widetilde{\mathrm{A5}})$, define $\mathscr{P}^p(\mathscr{T}_\infty) := \overline{\bigcup_{\ell\in\mathbb{N}_0} \mathscr{P}^p(\mathscr{T}_\ell)} \subseteq L_2(\Gamma)$ and $F_\infty(v) := \langle (1/2-K')\pi_\infty\phi, v\rangle_\Gamma$, where $\pi_\infty : L_2(\Gamma) \to \mathscr{P}^p(\mathscr{T}_\infty)$ denotes the L_2-orthogonal projection. By definition and with the stability of $K' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, there holds

$$\|F_\infty - F_\ell\|_{\mathscr{X}'_\ell} \leq \|(1/2-K')(\pi_\infty - \pi_\ell)\phi\|_{H^{-1/2}(\Gamma)} \lesssim \|(\pi_\infty - \pi_\ell)\phi\|^2_{L_2(\Gamma)}.$$

The term on the right-hand side tends to zero as $\ell \to \infty$. The convergence of $\widetilde{\mathrm{data}}_\ell$ follows as in the weakly singular case in the proof of Lemma 131. This shows $(\widetilde{\mathrm{A5}})$. Finally, $(\widetilde{\mathrm{A6}})$ follows analogously to the proof for the weakly singular case in Lemma 131. With $u_k \in H^{1/2}(\Gamma)$ the solution of (132) with right-hand side $F_k := (1/2-K')\pi_k\phi$, the only difference is the proof of $\|u_{k+1} - u_k\|_{H^{1/2}(\Gamma)} \lesssim \mathrm{data}_k^2 - \mathrm{data}_{k+1}^2$, which is much easier now. By ellipticity of $b(\cdot,\cdot)$, there holds

$$\|u_{k+1} - u_k\|_{H^{1/2}(\Gamma)} \lesssim \|(\pi_{k+1} - \pi_k)\phi\|_{H^{-1/2}(\Gamma)} = \|(1-\pi_k)\pi_{k+1}\phi\|_{H^{-1/2}(\Gamma)}.$$

The approximation property of the L_2-orthogonal projection π_k (see Lemma 22) implies

$$\|(1-\pi_k)\pi_{k+1}\phi\|_{H^{-1/2}(\Gamma)} \lesssim \|h_\ell^{1/2}(\pi_{k+1} - \pi_k)\phi\|_{L_2(\Gamma)} = \|h_\ell^{1/2}(\pi_{k+1} - \pi_k)\phi\|_{L_2(\bigcup(\mathscr{T}_k\setminus\mathscr{T}_{k+1})},$$

where we used the elementwise definition of π_k and $\pi_{k+1} = \pi_k$ on $\mathscr{T}_k \cap \mathscr{T}_{k+1}$. There exists a constant $0 < q < 1$ which depends on the space dimension d, such that $h_{k+1}|_T \leq q h_k|_T$ for all $T \in \mathscr{R}(k,k+1) = \mathscr{T}_k \setminus \mathscr{T}_{k+1}$. Hence there holds

$$(1-q)h_k|_{\mathscr{R}(k,k+1)} \leq h_k - h_{k+1}$$

and therefore

$$\|h_\ell^{1/2}(\pi_{k+1} - \pi_k)\phi\|^2_{L_2(\bigcup(\mathscr{R}(k,k+1)))} \lesssim \mathrm{data}_k^2 - \mathrm{data}_{k+1}^2.$$

The remainder follows analogously to the proof for the weakly singular case in Lemma 131. □

Theorem 134 *For all $0 < \theta \leq 1$, Algorithm 123 with the residual error estimator $\widetilde{\eta}_\ell$ from (161) converges in the sense*

$$C_{14}^{-2}\|u - \widetilde{U}_{\ell+n}\|^2_{H^{1/2}(\Gamma)} \leq \widetilde{\eta}^2_{\ell+n} \leq C_{17} q_{\mathrm{R}}^n \widetilde{\eta}_\ell^2 \tag{162}$$

for all $\ell, n \in \mathbb{N}_0$. For $0 < \theta < \theta_0$, Algorithm 123 converges with the best possible rate $s > 0$ in the sense that $\|\widetilde{\eta}\|_{\mathbb{A}_s} < \infty$ if and only if

$$\widetilde{\eta}_\ell \leq C_{20}(\#\mathscr{T}_\ell - \#\mathscr{T}_0)^{-s} \quad \text{for all } \ell \in \mathbb{N}, \tag{163}$$

where the constants C_{17}, q_{R} depend only on Γ, the shape regularity of the meshes $\mathscr{T}_\ell$, the polynomial degree p, and θ. The constant $C_{20} > 0$ depends additionally on $\|\widetilde{\eta}\|_{\mathbb{A}_s}$.

Proof Lemma 133 shows that the assumption $(\widetilde{\mathrm{A1}})$–$(\widetilde{\mathrm{A6}})$ are satisfied. Theorem 124 and Theorem 127 prove the statements. □

9 Implementational details

This section deals with implementational issues for the L_2-orthogonal projection π_ℓ^p and the Scott-Zhang operator J_ℓ as well as for some of the error estimators discussed above. For the ease of presentation, we consider only $d=2$ and give precise examples for the lowest-order cases $p \in \{0,1\}$. However, the following considerations are elementary and most of the ideas directly transfer to higher-order discretizations as well as $d \geq 3$.

9.1 Implementation of L_2-orthogonal projection π_ℓ^p

Let $\{\Psi_j^{\mathrm{ref}}\}_{j=1}^m \in \mathscr{P}^p(T_{\mathrm{ref}})$ denote a basis of the space of piecewise polynomials on the reference element T_{ref} with

$$m := \dim(\mathscr{P}^p(T_{\mathrm{ref}})) = \begin{cases} p+1 & d=2, \\ \frac{1}{2}(p+1)(p+2) & d=3. \end{cases} \tag{164}$$

Define the mass matrix $\mathbf{M}_{T_{\mathrm{ref}}} \in \mathbb{R}^{m\times m}$ associated to the reference element T_{ref} by

$$(\mathbf{M}_{T_{\mathrm{ref}}})_{jk} = \langle \Psi_k^{\mathrm{ref}}, \Psi_j^{\mathrm{ref}} \rangle_{T_{\mathrm{ref}}}. \tag{165}$$

Recall the affine mapping $F_T : T_{\mathrm{ref}} \to T$ from Section 2.6, which maps the reference element T_{ref} to $T \in \mathscr{T}_\ell$. Define the basis $\{\Psi_j^T\}_{j=1}^m$ of $\mathscr{P}^p(T)$ by $\Psi_j^T \circ F_T = \Psi_j^{\mathrm{ref}}$. Let $\mathbf{M}_T \in \mathbb{R}^{m\times m}$ denote the local mass matrix with entries

$$(\mathbf{M}_T)_{jk} := \langle \Psi_k^T, \Psi_j^T \rangle_T. \tag{166}$$

By using the transformation F_T, the computation of the entries $(\mathbf{M}_T)_{jk}$ can be reduced to the computation of $(\mathbf{M}_{T_{\mathrm{ref}}})_{jk}$, i.e.,

$$\begin{aligned} \langle \Psi_k^T, \Psi_j^T \rangle_T &= \int_T \Psi_k^T \Psi_j^T \, dx = \sqrt{\det(B_T^T B_T)} \int_{T_{\mathrm{ref}}} \Psi_k^{\mathrm{ref}} \Psi_j^{\mathrm{ref}} \, dy \\ &= \sqrt{\det(B_T^T B_T)} \langle \Psi_k^{\mathrm{ref}}, \Psi_j^{\mathrm{ref}} \rangle_{T_{\mathrm{ref}}}. \end{aligned} \tag{167}$$

Hence,

$$\mathbf{M}_T = \sqrt{\det(B_T^T B_T)} \mathbf{M}_{T_{\mathrm{ref}}} = \frac{|T|}{|T_{\mathrm{ref}}|} \mathbf{M}_{T_{\mathrm{ref}}}. \tag{168}$$

Note that $\mathbf{M}_{T_{\mathrm{ref}}}$ can be computed analytically. This is useful in practice, as we have to compute $\mathbf{M}_{T_{\mathrm{ref}}}$ only once and then multiply it by $|T|/|T_{\mathrm{ref}}|$ to get $\mathbf{M}_T$ for each element $T \in \mathscr{T}_\ell$.

Let $g \in L_2(\Gamma)$ and let $\pi_\ell^p : L_2(\Gamma) \to \mathscr{P}^p(\mathscr{T}_\ell)$ denote the L_2-orthogonal projection onto $\mathscr{P}^p(\mathscr{T}_\ell)$. Then, $\pi_\ell^p g$ satisfies

$$\langle \pi_\ell^p g, \Psi \rangle_\Gamma = \langle g, \Psi \rangle_\Gamma \quad \text{for all } \Psi \in \mathscr{P}^p(\mathscr{T}_\ell). \tag{169}$$

A basis $\{\Psi_j\}_{j=1}^M$ with $M := m\#\mathscr{T}_\ell$ is given by the combination of all basis functions Ψ_k^T for each element $T \in \mathscr{T}_\ell$. Define the mass matrix $\mathbf{M} \in \mathbb{R}^{M\times M}$ by

$$\mathbf{M}_{jk} = \langle \Psi_k, \Psi_j \rangle_\Gamma. \tag{170}$$

Then, (169) is equivalent to

$$\mathbf{M}\mathbf{x} = \mathbf{g} \quad \text{with } \mathbf{g}_j := \langle g, \Psi_j \rangle_\Gamma, \tag{171}$$

where $\pi_\ell^p g = \sum_{j=1}^M \mathbf{x}_j \Psi_j$. However, a simple calculation shows that π_ℓ^p is local in the sense that

$$(\pi_\ell^p) g|_T = \pi_{\ell,T}^p g = \pi_{\ell,T}^p (g|_T), \tag{172}$$

where $\pi_{\ell,T}^p$ denotes the L_2-orthogonal projection onto $\mathscr{P}^p(T)$. This allows us to reduce the computation of the orthogonal projection $\pi_\ell^p g$ to the solution of the local problems

$$\mathbf{M}_T \mathbf{x}_T = \mathbf{g}_T \quad \text{for all } T \in \mathscr{T}_\ell, \tag{173}$$

where $(\mathbf{g}_T)_j = \langle g, \Psi_j^T \rangle_T$ and $\pi_{\ell,T}^p g = \sum_{j=1}^m (\mathbf{x}_T)_j \Psi_j^T$. The coefficients $\langle g, \Psi_j \rangle_T$ can be computed by use of, e.g., Gaussian quadrature rules, cf. [141], as follows: First, we transform the integral over T to the reference element T_{ref}. Then, we apply an appropriate quadrature rule of order $q \in \mathbb{N}$ with weights $\{w_i\}_{i=1}^q$ and evaluation points $\{y_i\}_{i=1}^q \subseteq T_{\mathrm{ref}}$, i.e.

$$\langle g, \Psi_j^T \rangle_T = \int_{T_{\mathrm{ref}}} g \circ F_T \Psi_j^{\mathrm{ref}} \, dy \approx \sum_{j=1}^q w_i (g \circ F_T)(y_i) \Psi_j^{\mathrm{ref}}(y_i). \tag{174}$$

We give some examples for the matrix $\mathbf{M}_T$ for $d = 2,3$ and $p = 0,1$. Let $d = 2$ with the reference element $T_{\mathrm{ref}} = (-1,1)$. For $p = 0$ and the basis function $\Psi_1^{\mathrm{ref}}(x) = 1$ for $x \in T_{\mathrm{ref}}$, we get

$$\mathbf{M}_T = |T|. \tag{175}$$

For $d = 2$ and $p = 1$ with the basis

$$\Psi_1^{\mathrm{ref}}(x) = 1 \quad \text{and} \quad \Psi_2^{\mathrm{ref}}(x) = x \quad \text{for } x \in T_{\mathrm{ref}},$$

the local mass matrix reads

$$\mathbf{M}_T = |T| \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}. \tag{176}$$

Let $d = 3$ with reference element given by

$$T_{\mathrm{ref}} = \mathrm{conv}\{(0,0),(1,0),(0,1)\}.$$

For $p = 0$ with basis $\Psi_1^{\mathrm{ref}}(x,y) = 1$ for $(x,y) \in T_{\mathrm{ref}}$, we have

$$\mathbf{M}_T = |T|. \tag{177}$$

For $p = 1$ with basis

$$\Psi_1^{\mathrm{ref}}(x,y) = 1, \quad \Psi_2^{\mathrm{ref}}(x,y) = x, \quad \Psi_3^{\mathrm{ref}}(x,y) = y$$

for $(x,y) \in T_{\mathrm{ref}}$, we get

$$\langle \Psi_k^T, \Psi_j^T \rangle_T = 2|T| \int_0^1 \int_0^{1-x} \Psi_k^{\mathrm{ref}} \Psi_j^{\mathrm{ref}} \, dy \, dx. \tag{178}$$

Thus, the local mass matrix reads

$$\mathbf{M}_T = \frac{|T|}{12} \begin{pmatrix} 12 & 4 & 4 \\ 4 & 2 & 1 \\ 4 & 1 & 2 \end{pmatrix}. \tag{179}$$

9.2 Implementation of the Scott-Zhang projection $J_{\mathscr{T}}$

The implementation of the Scott-Zhang projection defined in Section 3.2.2 requires the computation of an L_2-dual basis and a numerical integration. We stick with the setting and notations of Section 3.2.2. Suppose that $T_i \in \mathscr{T}$ is the element chosen for the computation of $J_{\mathscr{T}} v$ at the node z_i and $\{\phi_{i,j}\}_{j=1}^d$ is the nodal basis of $\mathscr{P}^1(T_i)$. Recall the definition

$$\int_{T_i} \psi_{i,k}\phi_{i,j}\,dx = \delta_{k,j}$$

of the dual basis $\{\psi_{i,k}\}_{k=1}^d$ from (24). Hilbert space theory predicts the representation

$$\psi_{i,k} = \sum_{m=1}^{d} a_{k,m}^{(i)}\phi_{i,m} \quad \text{with} \quad a_{k,m}^{(i)} \in \mathbb{R}.$$

Using the duality (24), the coefficients $a_{k,m}^{(i)}$ can be computed by solving the d systems of $d \times d$ linear equations

$$\sum_{m=1}^{d} a_{k,m}^{(i)} \int_{T_i} \phi_{i,m}(x)\phi_{i,j} = \delta_{k,j}.$$

Denoting by $\mathbf{A}^{(i)} \in \mathbb{R}^{d\times d}$ the matrix with $\mathbf{A}_{k,m}^{(i)} = a_{k,m}^{(i)}$, this yields

$$\mathbf{A}^{(i)} = |T_i|^{-1}\begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix} \text{ for } d = 2,$$

$$\mathbf{A}^{(i)} = |T_i|^{-1}\begin{pmatrix} 18 & -6 & -6 \\ -6 & 18 & -6 \\ -6 & -6 & 18 \end{pmatrix} \text{ for } d = 3.$$

Let ψ_i denote the dual basis function $\psi_{i,k}$ with $k \in \{1,\dots,d\}$ such that $\phi_{i,k}(z_i) = 1$. Hence,

$$\psi_i = \sum_{m=1}^{d} \mathbf{A}_{k,m}^{(i)}\phi_{i,m}.$$

To compute $\int_{T_i} \psi_i v\,dx$, standard quadrature rules on intervals respectively triangles can be used, cf. [141].

9.3 Assumptions on uniform refinement for $d = 2$

We suppose that the uniform refinement $\widehat{\mathscr{T}_\ell}$ of $\mathscr{T}_\ell$ is obtained by splitting each element $T \in \mathscr{T}_\ell$ into k sons $T' \in \widehat{\mathscr{T}_\ell}$ for some fixed $k \geq 2$. A natural approach for $d = 2$ employs $k = 2$ and ensures $h_{\widehat{\mathscr{T}_\ell}} = h_{\mathscr{T}_\ell}/2$ for the respective mesh-size functions $h_\ell, \widehat{h}_\ell : \Gamma \to \mathbb{R}$ with, e.g., $h_\ell|_T = h_T = \operatorname{diam}(T)$ for all $T \in \mathscr{T}_\ell$. For the remainder of this section, let $T^+, T^- \in \widehat{\mathscr{T}_\ell}$ denote the unique elements with $\overline{T}^+ \cup \overline{T}^- = \overline{T}$ for all $T \in \mathscr{T}_\ell$.

9.4 Two-level estimator

Recall the hierarchical two-level decomposition

$$\widehat{\mathscr{X}_{\mathscr{T}}} = \mathscr{X}_{\mathscr{T}} \oplus \mathscr{Z}_{\mathscr{T}} \tag{180}$$

from (39), where Z is further decomposed into

$$\mathscr{Z}_{\mathscr{T}} = \mathscr{Z}_{\mathscr{T},1} \oplus \cdots \oplus \mathscr{Z}_{\mathscr{T},L} \quad \text{with } \dim(\mathscr{Z}_{\mathscr{T},i}) = 1. \tag{181}$$

Let $\Psi_j \neq 0$ denote an appropriate element of $\mathscr{Z}_{\mathscr{T},j}$. The basis functions Ψ_j, hence $\mathscr{Z}_{\mathscr{T},j}$, will be specified accordingly for the weakly singular integral equation as well as the hypersingular integral equation later on. Note that $\widehat{\mathscr{X}_{\mathscr{T}}} = \mathscr{X}_{\widehat{\mathscr{T}}}$ denotes the space associated to the uniformly refined mesh $\widehat{\mathscr{T}}$.

The computation of the local indicators

$$\eta_j = \|P_j(\widehat{U} - U)\|_b, \tag{182}$$

where $b(\cdot,\cdot)$ denotes the bilinear form corresponding to the weakly singular or hypersingular integral equation, requires the representation of the projection operators P_j. Since $\mathscr{Z}_{\mathscr{T},j}$ is one-dimensional, the subsequent identity follows immediately from the definition of P_j,

$$P_j\widehat{V} = \frac{b(\widehat{V},\Psi_j)}{\|\Psi_j\|_b^2}\Psi_j \quad \text{for all } \widehat{V} \in \widehat{\mathscr{X}_{\mathscr{T}}}. \tag{183}$$

With $\widehat{V} = \widehat{U} - U$, Lemma 47 shows

$$\eta_j = \frac{|f(\Psi_j) - b(U,\Psi_j)|}{\|\Psi_j\|_b^2}. \tag{184}$$

The computation of η_j involves the assembling of the Galerkin matrix corresponding to $b(\cdot,\cdot)$ with ansatz space $\mathscr{X}_{\mathscr{T}}$ and test function Ψ_j as well as the computation of $b(\Psi_j,\Psi_j)$. Therefore, we have to compute the entries

$$b(V_k,\Psi_j) \quad \text{as well as} \quad b(\Psi_j,\Psi_j),$$

where $\{V_k\}_{k=1}^{\dim(\mathscr{X}_{\mathscr{T}})}$ denotes a basis of $\mathscr{X}_{\mathscr{T}}$. We note that this quantities can be obtained from the Galerkin matrix with respect to the fine space $\widehat{\mathscr{X}_{\mathscr{T}}}$.

9.4.1 2D weakly singular integral equation

We consider $\mathscr{X}_{\mathscr{T}} = \mathscr{P}^0(\mathscr{T})$ and $\widehat{\mathscr{X}_{\mathscr{T}}} = \mathscr{X}_{\widehat{\mathscr{T}}} = \mathscr{P}^0(\widehat{\mathscr{T}})$. For each $T_j \in \mathscr{T}$, let $T_j^\pm \in \widehat{\mathscr{T}}$ denote the two elements with $\overline{T}_j^+ \cup \overline{T}_j^- = \overline{T}_j$ and $|T_j^\pm| = |T_j|/2$. Define the basis function $\Psi_j \in \widehat{\mathscr{X}_{\mathscr{T}}}$ by

$$\Psi_j|_{T_j^+} = +1 \quad \Psi_j|_{T_j^-} = -1 \quad \Psi_j|_{\Gamma\setminus\overline{T}_j} = 0. \tag{185}$$

It holds

$$\langle \Psi_j\,,1\rangle_T = 0. \tag{186}$$

The stability of the corresponding decomposition (180) is proved in [92] resp. [58] for $d = 2$ and [61] for $d = 3$.

9.4.2 2D hypersingular integral equation

We consider $\mathscr{X}_{\mathscr{T}} = \mathscr{S}^1(\mathscr{T})$ and $\widehat{\mathscr{X}_{\mathscr{T}}} = \mathscr{S}^1(\widehat{\mathscr{T}})$. For each $T_j \in \mathscr{T}$, let $T_j^{\pm} \in \widehat{\mathscr{T}}$ denote the two elements with $\overline{T}_j^+ \cup \overline{T}_j^- = \overline{T}$ and $|T_j^{\pm}| = |T_j|/2$. Define the midpoint $m_j = \overline{T}_j^+ \cap \overline{T}_j^-$ and the basis function $\Psi_j \in \widehat{\mathscr{X}_{\mathscr{T}}}$ by

$$\Psi_j(m_j) = 1 \quad \text{and} \quad \Psi_j|_{\Gamma \setminus \overline{T}_j} = 0. \tag{187}$$

The stability of the corresponding decomposition (180) is proved in [92] resp. [58] for $d = 2$ and [89] resp. [12] for $d = 3$.

9.5 $(h-h/2)$ error estimators in 2D

For the $(h-h/2)$-based error estimators μ_ℓ, $\widetilde{\mu}_\ell$ from Section 4.2.2, we have to compute local refinement indicators of the form

$$h_T \|\widehat{\Psi}\|^2_{L_2(T)} \quad \text{for all } T \in \mathscr{T}_\ell$$

with $\widehat{\Psi} \in \mathscr{P}^p(\widehat{\mathscr{T}_\ell})$ and $h_T = \mathrm{diam}(T)$.

Let $T^{\pm} \in \widehat{\mathscr{T}_\ell}$ denote the son elements of the father $T \in \mathscr{T}_\ell$, i.e., $\overline{T} = \overline{T}^+ \cup \overline{T}^-$. We define the local mesh $\widehat{\mathscr{T}_T}$ as the restriction of $\widehat{\mathscr{T}_\ell}$ to T, i.e., $\widehat{\mathscr{T}_T} := \{T^+, T^-\}$. Let $\{\widehat{\Psi}_j\}_{j=1}^{2(p+1)}$ denote a basis of the local subspace $\mathscr{P}^p(\widehat{\mathscr{T}_T})$. With the representation of $\widehat{\Psi} \in \mathscr{P}^p(\widehat{\mathscr{T}_T})$ on the father element $T \in \mathscr{T}_\ell$

$$\widehat{\Psi}|_T = \sum_{j=1}^{2(p+1)} \alpha_j \widehat{\Psi}_j, \tag{188}$$

and the local mass matrix $\widehat{\mathbf{M}}_T \in \mathbb{R}^{2(p+1)\times 2(p+1)}_{\mathrm{sym}}$ with entries

$$(\widehat{\mathbf{M}}_T)_{jk} = \langle \widehat{\Psi}_j \,, \widehat{\Psi}_k \rangle_T, \tag{189}$$

the computation of the local indicators read

$$h_T \|\widehat{\Psi}\|^2_{L_2(T)} = h_T \alpha^T \widehat{\mathbf{M}}_T \alpha. \tag{190}$$

In the following, this observation is employed for the weakly singular and hypersingular integral equation. Note that the estimators for the hypersingular integral equation require the computation of the arclength derivative $\nabla_\Gamma(\cdot)$ of a discrete function.

9.5.1 2D weakly singular integral equation

We consider the local refinement indicators of the estimator

$$\mu_\ell^2 = \sum_{T \in \mathscr{T}_\ell} h_T \|\widehat{\Phi}_\ell - \Phi_\ell\|^2_{L_2(T)} =: \sum_{T \in \mathscr{T}_\ell} h_T \|\widehat{\Psi}\|^2_{L_2(T)}, \tag{191}$$

where $\Phi_\ell \in \mathscr{P}^p(\mathscr{T}_\ell)$ and $\widehat{\Phi}_\ell \in \mathscr{P}^p(\widehat{\mathscr{T}_\ell})$ are the respective Galerkin solutions.

For the lowest-order case $p = 0$, we choose the characteristic functions as basis, i.e., $\widehat{\Psi}_i$ is the characteristic function on the element $T_i \in \widehat{\mathscr{T}_T}$. The corresponding local mass matrix reads $\widehat{\mathbf{M}}_T = h_T/2\,\mathbf{I}$, where $\mathbf{I}$ denotes the 2×2 identity matrix. Hence, the computation of $\mu_\ell^2(T) = h_T \|\widehat{\Phi}_\ell - \Phi_\ell\|^2_{L_2(T)}$ reads

$$\begin{aligned} \mu_\ell^2(T) &= h_T \alpha^T \cdot \widehat{\mathbf{M}}_T \alpha = h_T^2/2\, \alpha^T \cdot \mathbf{I}\alpha \\ &= \frac{h_T^2}{2}\Big((\widehat{\Phi}_\ell|_{T^+} - \Phi_\ell|_T)^2 + (\widehat{\Phi}_\ell|_{T^-} - \Phi_\ell|_T)^2 \Big). \end{aligned} \tag{192}$$

The computation of the local refinement indicators

$$\widetilde{\mu}_\ell^2(T) = h_T \|\widehat{\Phi}_\ell - \pi_\ell \widehat{\Phi}_\ell\|^2_{L_2(T)} \tag{193}$$

of the estimator $\widetilde{\mu}_\ell$ is done in the same manner. For the computation of $\pi_\ell \widehat{\Phi}_\ell$ we proceed as in Section 9.1. Let Ψ_T denote the characteristic function on $T \in \mathscr{T}_\ell$. It holds $\langle \widehat{\Phi}_\ell \,, \Psi_T \rangle_T = h_T/2(\widehat{\Phi}_\ell|_{T^+} + \widehat{\Phi}_\ell|_{T^-})$. Hence,

$$\pi_\ell \widehat{\Phi}_\ell|_T = \frac{1}{2}(\widehat{\Phi}_\ell|_{T^+} + \widehat{\Phi}_\ell|_{T^-}). \tag{194}$$

For the local refinement indicators in (193), we get

$$\begin{aligned} \widetilde{\mu}_\ell^2(T) &= \frac{h_T^2}{2}(\widehat{\Phi}_\ell|_{T^+} - \frac{1}{2}(\widehat{\Phi}_\ell|_{T^+} + \widehat{\Phi}_\ell|_{T^-}))^2 \\ &\qquad + \frac{h_T^2}{2}(\widehat{\Phi}_\ell|_{T^-} - \frac{1}{2}(\widehat{\Phi}_\ell|_{T^+} + \widehat{\Phi}_\ell|_{T^-}))^2 \\ &= \frac{h_T^2}{4}(\widehat{\Phi}_\ell|_{T^+} - \widehat{\Phi}_\ell|_{T^-})^2. \end{aligned} \tag{195}$$

From a practical point of view, the estimator $\widetilde{\mu}_\ell$ is more attractive than μ_ℓ, since the computation involves only the Galerkin solution on the fine mesh $\widehat{\mathscr{T}_\ell}$.

For $p = 1$, we use the basis $\{\widehat{\Psi}_j\}_{j=1}^4 \subseteq \mathscr{P}^1(\widehat{\mathscr{T}_T})$ with

$$\begin{aligned} \widehat{\Psi}_1|_{T^+} &= 1, & \widehat{\Psi}_1|_{T^-} &= 0, \\ \widehat{\Psi}_2|_{T^+} &= 0, & \widehat{\Psi}_1|_{T^-} &= 1, \\ \widehat{\Psi}_3|_{T^+} \circ F_{T^+}(x) &= x, & \widehat{\Psi}_3|_{T^-} &= 0, \\ \widehat{\Psi}_4|_{T^+} &= 0, & \widehat{\Psi}_4|_{T^-} \circ F_{T^-}(x) &= x, \end{aligned} \tag{196}$$

for all $x \in T_{\mathrm{ref}} = (-1,1)$. The local mass matrix $\widehat{\mathbf{M}}_T$ then reads

$$\widehat{\mathbf{M}}_T = \frac{h_T}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}, \tag{197}$$

and with $\widehat{\Psi} = \widehat{\Phi}_\ell|_T - \Phi_\ell|_T = \sum_{j=1}^4 \alpha_j \widehat{\Psi}_j$, we get

$$\mu_\ell^2(T) = \frac{h_T}{2} \|\widehat{\Psi}\|^2_{L_2(T)} = \frac{h_T^2}{2}(\alpha_1^2 + \alpha_2^2 + \frac{1}{3}(\alpha_3^2 + \alpha_4^2)). \tag{198}$$

For the computation of the local error indicator $\widetilde{\mu}_\ell^2(T)$, we proceed as before and compute π_ℓ^1. Let $\beta \in \mathbb{R}^4$ satisfy $\widehat{\Phi}_\ell|_T = \sum_{j=1}^4 \beta_j \widehat{\Psi}_j$. We end up with

$$\widetilde{\mu}_\ell^2(T) = \frac{h_T^2}{4}((\beta_1-\beta_2)^2 + \frac{1}{3}(\beta_3-\beta_4)^2). \tag{199}$$

9.5.2 2D hypersingular integral equation

We consider the local refinement indicators of the estimator

$$\mu_\ell^2 = \sum_{T\in\mathscr{T}_\ell} h_T \|\nabla_\Gamma(\widehat{U}_\ell - U_\ell)\|_{L_2(T)}^2 =: \sum_{T\in\mathscr{T}_\ell} h_T \|\widehat{\Psi}\|_{L_2(T)}^2, \tag{200}$$

where $U_\ell \in \mathscr{S}^{p+1}(\mathscr{T}_\ell)$ resp. $\widehat{U}_\ell \in \mathscr{S}^{p+1}(\widehat{\mathscr{T}_\ell})$ are the respective Galerkin solutions. Hence, $\widehat{\Psi} := \nabla_\Gamma(\widehat{U}_\ell - U_\ell) \in \mathscr{P}^p(\widehat{\mathscr{T}_\ell})$. To apply (190), it remains to provide a formula to compute the arclength derivative $\nabla_\Gamma(\widehat{U}_\ell - U_\ell)|_T \in \mathscr{P}^p(\widehat{\mathscr{T}_T})$.

Let $p=0$ and let $\{\eta_1,\eta_2,\eta_3\}$ denote a basis of the local subspace $\mathscr{S}^1(\widehat{\mathscr{T}_T})$. We define the matrix $\widehat{\mathbf{G}}_T \in \mathbb{R}^{2\times 3}$, which represents the gradients of the basis $\{\eta_j\}$ with respect to the basis $\{\widehat{\Psi}_1,\widehat{\Psi}_2\}$ of $\mathscr{P}^0(\widehat{\mathscr{T}_T})$, i.e.

$$\nabla_\Gamma \eta_j = \sum_{k=1}^2 (\widehat{\mathbf{G}}_T)_{kj}\widehat{\Psi}_k. \tag{201}$$

Then, $\nabla_\Gamma(\widehat{U}_\ell - U_\ell) = \sum_j \beta_j \nabla_\Gamma \eta_j$, where $\alpha = \widehat{\mathbf{G}}_T\beta$. Testing equation (201) with $\widehat{\Psi}_j$ in $L_2(T)$, we obtain the equivalent matrix equation

$$\widehat{\mathbf{D}}_T = \widehat{\mathbf{M}}_T\widehat{\mathbf{G}}_T \tag{202}$$

with $(\widehat{\mathbf{D}}_T)_{jk} = \langle \nabla_\Gamma \eta_k, \widehat{\Psi}_j\rangle_T$. Hence, $\widehat{\mathbf{G}}_T = (\widehat{\mathbf{M}}_T)^{-1}\widehat{\mathbf{D}}_T$. Together with (190), the computation of the local indicators in (200) with $\widehat{\Psi} = \nabla_\Gamma(\widehat{U} - U)$ reads

$$\begin{aligned} h_T\|\widehat{\Psi}\|_{L_2(T)}^2 &= h_T\alpha^T\widehat{\mathbf{M}}_T\alpha = h_T\beta^T\widehat{\mathbf{G}}_T^T\widehat{\mathbf{M}}_T\widehat{\mathbf{G}}_T\beta \\ &= h_T\beta^T(\widehat{\mathbf{M}}_T^{-1}\widehat{\mathbf{D}}_T)^T\widehat{\mathbf{M}}_T\widehat{\mathbf{M}}_T^{-1}\widehat{\mathbf{D}}_T\beta \\ &= h_T\beta^T\widehat{\mathbf{D}}_T^T\widehat{\mathbf{M}}_T^{-1}\widehat{\mathbf{D}}_T\beta. \end{aligned} \tag{203}$$

Let x_1,x_2 denote the endpoints of the element $T\in\mathscr{T}_\ell$ and let $x_3 = (x_1+x_2)/2$ denote the midpoint of T. We choose the nodal basis $\{\eta_j\}$ with respect to the nodes $\{x_j\}$. Let $\widehat{\Psi}_1,\widehat{\Psi}_2$ denote the characteristic functions on $T^+,T^- \in \widehat{\mathscr{T}_\ell}$ with $\overline{T}^+\cup\overline{T}^- = \overline{T}$ and $x_1\in\overline{T}^+$, $x_2\in\overline{T}^-$. Then,

$$\widehat{\mathbf{D}}_T = \begin{pmatrix} -1 & 0 & +1 \\ 0 & +1 & -1 \end{pmatrix}, \qquad \widehat{\mathbf{M}}_T = \frac{h_T}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and (203) becomes with $\beta_j = (\widehat{U}_\ell - U_\ell)(x_j)$

$$\begin{aligned} h_T\beta^T\widehat{\mathbf{D}}_T^T\widehat{\mathbf{M}}_T^{-1}\widehat{\mathbf{D}}_T\beta &= 2\beta^T\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}\beta \\ &= 2(\beta_1-\beta_3)^2 + 2(\beta_2-\beta_3)^2. \end{aligned} \tag{204}$$

Using $U_\ell(x_3) = (U_\ell(x_1) + U(x_2))/2$, the indicator $\mu_\ell(T)^2$ is computed by

$$\begin{aligned} \mu_\ell(T)^2 = {} & 2(\widehat{U}_\ell(x_1) - \widehat{U}_\ell(x_3) + (U_\ell(x_2) - U_\ell(x_1))/2)^2 \\ & + 2(\widehat{U}_\ell(x_2) - \widehat{U}_\ell(x_3) + (U_\ell(x_1) - U_\ell(x_2))/2)^2. \end{aligned} \tag{205}$$

For the computation of the local error indicators

$$\widetilde{\mu}_\ell^2(T) = h_T\|(1-\pi_\ell)\nabla_\Gamma\widehat{U}_\ell\|_{L_2(T)}^2 =: h_T\|\widehat{\Psi}\|_{L_2(T)}^2, \tag{206}$$

we proceed as in Section 9.5.1 to compute the L_2-projection

$$\pi_\ell\nabla_\Gamma\widehat{U}_\ell = \frac{1}{2}\Big((\nabla_\Gamma\widehat{U}_\ell)|_{T^+} + (\nabla_\Gamma\widehat{U}_\ell)|_{T^-}\Big). \tag{207}$$

With α resp. β given by $\widehat{\Psi} = \sum_{j=1}^2\alpha_j\widehat{\Psi}_j$ resp. $\widehat{U}_\ell = \sum_{j=1}^3\beta_j\eta_j$, a straightforward computation shows that

$$\alpha = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\widehat{\mathbf{G}}_T\beta, \tag{208}$$

Putting this into (190), we get

$$h_T\|\widehat{\Psi}\|_{L_2(T)}^2 = (2\beta_3 - \beta_1 - \beta_2)^2. \tag{209}$$

Using $\beta_j = \widehat{U}_\ell(x_j)$, the local refinement indicators become

$$\widetilde{\mu}_\ell^2(T) = (2\widehat{U}_\ell(x_3) - \widehat{U}_\ell(x_1) - \widehat{U}_\ell(x_2))^2. \tag{210}$$

Again, the computation of $\widetilde{\mu}_\ell$ is more attractive compared to the computation μ_ℓ, since only the solution $\widehat{U}_\ell$ on the fine mesh is needed.

Let $p=1$ and let $\{\widehat{\Psi}_j\}_{j=1}^4$ be given as in (196). We choose the basis $\{\eta_j\}_{j=1}^5$, where η_1,η_2,η_3 are the (linear) nodal basis functions with respect to x_1,x_2,x_3, and $\eta_4 := \eta_1\eta_3$ as well as $\eta_5 := \eta_2\eta_3$. Arguing as for $p=0$, we obtain

$$\widehat{\mathbf{D}}_T = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{3} \end{pmatrix}. \tag{211}$$

Using (203) with $\widehat{\Psi} := \nabla_\Gamma(\widehat{U}_\ell - U_\ell)|_T = \sum_{j=1}^5\beta_j\nabla_\Gamma\eta_j$, we infer

$$\mu_\ell^2(T) = 2((\beta_1-\beta_3)^2 + (\beta_2-\beta_3)^2 + \frac{1}{3}(\beta_4^2+\beta_5^2)). \tag{212}$$

For the computation of the local error indicators of $\widetilde{\mu}_\ell$, let $\beta\in\mathbb{R}^5$ be given by $\widehat{U}_\ell = \sum_{j=1}^5\beta_j\eta_j$. We proceed as in the case $p=0$ and get

$$\widetilde{\mu}_\ell^2(T) = (2\beta_3 - \beta_1 - \beta_2)^2 + \frac{1}{3}(\beta_4-\beta_5)^2. \tag{213}$$

Similar results hold for $d=3$ resp. $p>1$.

9.6 Weighted-residual error estimator

9.6.1 2D weakly singular integral equation

We consider the computation of the local error indicators

$$\eta_\ell^2(T) = h_T \|\nabla_\Gamma(V\Phi_\ell - f)\|^2_{L_2(T)} =: h_T\|\nabla_\Gamma R_\ell\|^2_{L_2(T)} \quad (214)$$

of the weighted-residual error estimator for $d = 2$. By the definition of $\nabla_\Gamma(\cdot)$, we have

$$(\nabla_\Gamma R_\ell)\circ F_T = 2h_T^{-1}(\nabla_\Gamma(R_\ell \circ F_T)), \quad (215)$$

where F_T denotes the affine mapping from $T_{\rm ref} = (-1,1)$ to $T = \mathrm{conv}\{x_1,x_2\}$. For (214), this yields

$$\begin{aligned} h_T\|\nabla_\Gamma R_\ell\|^2_{L_2(T)} &= h_T\int_T (\nabla_\Gamma R_\ell)^2\,dx \\ &= 2\int_{T_{\rm ref}} (\nabla_\Gamma(R_\ell\circ F_T))^2\,dy. \end{aligned} \quad (216)$$

Recall that $\nabla_\Gamma(\cdot) = (\cdot)'$, where $(\cdot)'$ denotes the arclength derivative. We approximate $R_\ell \circ F_T$ by some polynomial $\Psi \in \mathscr{P}^{2q}(T_{\rm ref})$ with $q \geq 1$, i.e.

$$\int_{T_{\rm ref}} (\nabla_\Gamma(R_\ell\circ F_T))^2\,dy \approx \int_{T_{\rm ref}} (\Psi')^2\,dy. \quad (217)$$

Note hat $\Psi' \in \mathscr{P}^{2q-1}(T_{\rm ref})$, whence $(\Psi')^2 \in \mathscr{P}^{4q-2}$. To compute the integral on the right-hand side of (217), we use a $2q$-point Gaussian quadrature rule, which is exact for polynomials of order $2(2q)-1$, and thus for $(\Psi')^2$. Let $\{y_i\}_{i=1}^{2q}$ denote the quadrature nodes on $T_{\rm ref}$ with corresponding weights $\{w_i\}_{i=1}^{2q}$. This leads for the local error indicators

$$h_T\|\nabla_\Gamma R_\ell\|^2_{L_2(T)} \approx 2\int_{T_{\rm ref}} (\Psi')^2\,dy = 2\sum_{i=1}^{2q} w_i(\Psi'(y_i))^2. \quad (218)$$

For the construction of $\Psi \in \mathscr{P}^{2q}(T_{\rm ref})$ by interpolation, we use the $2q$ points $\{y_i\}_{i=1}^{2q}$ from the Gaussian quadrature rule plus the midpoint of the reference element $y_{2q+1} = 0$. According to (218), we need to evaluate Ψ' at the points y_i for $i = 1,\dots,2q$. To that end, let $\{L_i\}_{i=1}^{2q+1}$ denote the Lagrange basis with respect to the points $\{y_i\}_{i=1}^{2q+1}$. It holds

$$\Psi = \sum_{k=1}^{2q+1} \beta_k L_k$$

with $\beta_k := \Psi(y_k)$. Define the matrix $\mathbf{L}' \in \mathbb{R}^{2q\times(2q+1)}$ by

$$(\mathbf{L}')_{jk} := L_k'(y_j). \quad (219)$$

Then, $\alpha_j := \Psi'(y_j)$ can be obtained by the matrix-vector multiplication

$$\alpha = \mathbf{L}'\beta. \quad (220)$$

We consider the lowest-order case $p = 0$, where we use a 2-point Gaussian quadrature rule ($q = 1$) on $T_{\rm ref}$ with $y_1 = -1/\sqrt{3}$, $y_2 = 1/\sqrt{3}$ and weights $w_1 = 1 = w_2$. The derivatives of the Lagrange basis L_1, L_2, L_3 with respect to the points $y_1, y_2, y_3 := 0$ are given by

$$L_1'(y) = 3y - \frac{\sqrt{3}}{2}, \quad L_2'(y) = 3y + \frac{\sqrt{3}}{2}, \quad L_3' = 1 - 3y^2. \quad (221)$$

For $\alpha_1 = \Psi'(y_1), \alpha_2 = \Psi'(y_2)$, we get

$$\begin{aligned} \alpha &= \begin{pmatrix} L_1'(y_1) & L_2'(y_1) & L_3'(y_1) \\ L_1'(y_2) & L_2'(y_2) & L_3'(y_2) \end{pmatrix}\beta \\ &= \sqrt{\frac{3}{4}}\begin{pmatrix} -3 & -1 & 4 \\ 1 & 3 & -4 \end{pmatrix}\beta. \end{aligned} \quad (222)$$

Altogether, we approximate the local error indicators $\eta_\ell^2(T)$ by

$$\rho_\ell^2(T) \approx \sum_{i=1}^{2} w_i\alpha_i^2 = \frac{3}{4}\beta^T \begin{pmatrix} 10 & 6 & -16 \\ 6 & 10 & -16 \\ -16 & -16 & 32 \end{pmatrix}\beta, \quad (223)$$

where $\beta_j := (R_\ell \circ F_T)(y_j)$ for $j = 1,2,3$. The advantage of this approach is that we do not need to evaluate the arclength derivative R_ℓ' of the function R_ℓ numerically, but instead evaluate the function at specific points directly.

In practice, it suffices to use a small number of quadrature points q. For $p = 0$, we stress that the choice $q = 1$ is sufficient. Define $\eta_\ell^{(q)}$ as the residual error estimator η_ℓ with the difference that the local indicators $\rho_\ell(T)^2$ are approximated as given above, and $2q$ denotes the number of points in the chosen Gaussian quadrature rule. On uniformly refined meshes, we stress that for a sufficiently smooth function R_ℓ it holds

$$|\rho_\ell - \rho_\ell^{(q)}| \lesssim N^{-(q+1/2)} \quad \text{with } N = \#\mathscr{T}_\ell. \quad (224)$$

Thus for $p = 1$, we use $q = 2$, i.e., four quadrature points y_1, y_2, y_3, y_4 and the additional evaluation point $y_5 := 0$.

9.6.2 2D hypersingular integral equation

We consider the computation of the local refinement indicators

$$\eta_\ell^2(T) = h_T\|WU_\ell - (1/2 - K')\phi\|^2_{L_2(T)} =: h_T\|R_\ell\|^2_{L_2(T)} \quad (225)$$

of the weighted residual error estimator η_ℓ for $d = 2$. Let F_T denote the affine mapping from $T_{\rm ref} = (-1,1)$ to T. We use a q-point Gaussian quadrature rule with nodes $\{y_i\}_{i=1}^q$ and weights $\{w_i\}_{i=1}^q$ on the reference element $T_{\rm ref}$, which is exact for polynomials in $\mathscr{P}^{2q-1}(T_{\mathrm{ref}})$. The local refinement indicator $\eta_\ell^2(T)$ is approximated by

$$\begin{aligned} h_T\|R_\ell\|_{L_2(T)}^2 &= \frac{h_T}{2}\int_{T_{\mathrm{ref}}}(R_\ell\circ F_T)^2\,dx \\ &\approx \frac{h_T}{2}\sum_{i=1}^{q} w_i(R_\ell\circ F_T)^2(y_i). \end{aligned} \tag{226}$$

Define $\eta_\ell^{(q)}$ as η_ℓ with the difference that the local refinement indicators $\eta_\ell^2(T)$ are approximated as given above. On uniformly refined meshes and for sufficiently smooth residual R_ℓ, we stress that

$$|\eta_\ell - \eta_\ell^{(q)}| \lesssim N^{-(2q+1/2)} \quad \text{with } N = \#\mathscr{T}_\ell. \tag{227}$$

For the lowest order case $p=0$, it is sufficient to use $q=2$ quadrature points, whereas for $p=1$ it is sufficient to use $q=3$ quadrature points.

10 Conclusion

In this work, we presented all a posteriori error estimators for Galerkin BEM that are available in the mathematical literature. Up to now, it is known how to apply contemporary convergence and optimality analysis only to several of them, and an overview of this analysis and its application to these estimators was given.

Although all estimators behave well in numerical experiments (cf. the references given above, where the respective estimators haven been introduced), they differ with respect to overhead in implementation, computational expense, and mathematically guaranteed convergence as well as optimality of the related ABEM algorithms. The choice is also affected by the regularity of the right-hand side data.

With respect to these requirements, a short summary of the advantages and disadvantages of the estimators follows. We consider only estimators that have been mathematically analyzed on locally refined meshes.

The $(\boldsymbol{(h-h/2)})$**-type estimators** (Section 4.2.2) are, in general, structurally easy. They require nearly no overhead for their implementation and can be used to steer anisotropic mesh refinement in a straightforward manner (Section 6.10), which are their biggest advantages. Also, there is no additional requirement on the data. They are always efficient, but reliability is equivalent to the saturation assumption, which is still an open problem. They are proven to converge, but the convergence is not proven to be optimal.

Averaging estimators (Section 4.3) are globally equivalent to $(h-h/2)$-type estimators, and the implementationally and computationally interesting variants are even locally equivalent to their $(h-h/2)$ counterparts.

Advantages of the $\boldsymbol{ZZ}$**-type estimators** (Section 4.4) are that the implementation is also very easy, and as they avoid artificial mesh refinement, they are much cheaper than $(h-h/2)$-type estimators. While reliability relies on an appropriate saturation assumption, efficiency involves higher-order terms and is therefore weaker than for $(h-h/2)$-type estimators. In addition, it is not clear how to steer anisotropic refinement. Convergence is proven, but optimality remains open.

The **two-level estimators** (Section 4.2.1) also need the assembly of the Galerkin data on a finer mesh. Moreover, their reliability depends also on the saturation assumption. Up to now, nothing is known about convergence or optimality. However, their great strength is that they are analyzed with respect to hp-methods and that they are reliable on anisotropic refinement in the case of weakly singular integral operators (assuming saturation).

The **weighted residual estimators** (Section 4.1.3) are the only ones which are proven to converge with optimal rates. As they are also reliable, they are preferred in theory. However, their disadvantage is that their requirements on the data is quite strong and that the implementation requires a certain amount of overhead. Anisotropic meshes can be used, but there is no analysis in this respect.

The **local double norm estimators** (Section 4.1.4) are the only estimators that are known to be efficient *and* reliable without any further assumption on data or saturation. However, their stable implementation is non-trivial, and nothing is known with respect to convergence or optimality.

Approximation of the given right-hand side data renders an important aspect for BEM implementations and can be used and mathematically controlled in adaptive boundary element methods on isotropic meshes. Convergence and optimality is proven as long as the data satisfy additional regularity and appropriate approximation operators are used. There is no analysis on anisotropic meshes.

Several important **open problems** in this field center around anisotropic mesh refinement. There is a need for estimators which are reliable on anisotropic meshes (two-level estimators for weakly singular operators are reliable on anisotropic meshes under the saturation assumption), and optimality theory needs to be extended in this respect. To that end, suitable mesh refinement strategies need to be developed and analyzed. Furthermore, the approximation of smooth geometries in isoparametric BEM algorithms, its incorporation into ABEM, as well as a mathematical analysis with respect to convergence and optimality remains an interesting open problem. Also, as stated in the introduction, competitive BEM algorithms need to employ fast methods for matrix compression. Hence, it is an inevitable task to control the introduced error in adaptive BEM algorithms.

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