SOLUTIONS OF THE ALLEN-CAHN EQUATION INVARIANT UNDER SCREW-MOTION

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Abstract. We study entire solutions of the Allen-Cahn equation which are defined in 3-dimensional Euclidean space and which are invariant under screw-motion. In particular, we discuss the existence and non existence of nontrivial solutions whose nodal set is a helicoïd of $\mathbb{R}^3$.

1. Introduction and statement of the main results

In this short note, we are interested in entire solutions of the Allen-Cahn equation

$$\Delta u - F'(u) = 0, \tag{1.1}$$

which are defined in $\mathbb{R}^n$, with $n \geq 1$. Here F' is the derivative of the function F which is usually referred to as a *double well potential*. In this paper, we will assume that $t \mapsto F(t)$ is an even, positive function which is at least of class $\mathcal{C}^2$ and which has only two distinct nondegenerate absolute minima at the points $\pm t_* \in \mathbb{R}$, where $t_* > 0$. Hence, for all $t \in \mathbb{R}$,

$$F(t) \geq F(t_*),$$

with equality if and only if $t = t_*$. We further assume that

$$F''(0) < 0 \qquad \text{and} \qquad F''(0)\, t \leq F'(t), \tag{1.2}$$

for all $t \geq 0$. We define

$$\lambda_* := \frac{\pi}{\sqrt{-F''(0)}}.$$

Remark 1.1. *A typical example of such a double well potential is given by*

$$F(t) := \frac{1}{4}\,(1-t^2)^2, \tag{1.3}$$

in which case $t_* = 1$, $F''(t_*) = 1$ *and* $\lambda_* = \pi$.

In recent years, there has been many important results on the existence of nontrivial entire solutions of (1.1) and also trying to understand the classification of such solutions. As far as the existence of solutions is concerned, there are two completely different approaches : the use of the variational structure of (1.1) as in [5] or [1] ; or perturbation results based on the implementation of an infinite dimensional Liapunov-Schmidt reduction argument as in [7], [8], [9] or [10]. The former is usually simpler and takes advantage of the symmetries of the solutions constructed while the latter produces solutions with less (or even without any) symmetry but is technically more involved. Since we do not use this latter approach in this paper, we will not comment on it further and refer the interested reader to the above mentioned papers.

The variational method has already been successfully implemented to prove the existence of solutions of the Allen-Cahn equation whose nodal set is the minimal cone

$$C := \{(x,y) \in \mathbb{R}^{2m} \, : \, |x| = |y|\} \subset \mathbb{R}^{2m},$$

where $m \geq 1$. We refer to [1] and [2] for more information about these solutions which are usually referred to as the *saddle type solutions*. In dimension 2, the method extends to produce solutions which are invariant under dihedral symmetry [6]. In this short note, we use once more a variational argument to produce new solutions of (1.1). The arguments is by now standard so we only insist on the important points which are specific to our construction.

Since we will mainly be working in dimension 3, it will be convenient to identify $\mathbb{R}^3$ with $\mathbb{C} \times \mathbb{R}$. Given $\lambda > 0$, the helicoïd H_λ is defined to be the *minimal surface* which can be parameterized by

$$\mathbb{R} \times \mathbb{R} \ni (t, \theta) \longmapsto \left(t\, e^{i\theta}, \frac{\lambda}{\pi}\, \theta \right) \in \mathbb{C} \times \mathbb{R},$$

and, we define the screw motion of parameter λ, acting on $\mathbb{C} \times \mathbb{R}$, by

$$\sigma_\lambda^\alpha (z, t) = \left(e^{i\alpha}\, z, t + \frac{\lambda}{\pi} \alpha \right),$$

for all $\alpha \in \mathbb{R}$. Clearly, H_λ is invariant under the action of σ_λ^α, for all $\alpha \in \mathbb{R}$.

Our main result is the construction of nontrivial entire solutions of the Allen-Cahn equation (1.1) which are defined in $\mathbb{R}^3$ and whose zero set is equal to H_λ, provided λ is chosen large enough. More precisely we prove the:

Theorem 1.1. *Assume that (1.2) holds and that $\lambda > \lambda_*$. Then, there exists a solution of the Allen-Cahn equation (1.1) which is bounded and whose zero set is equal to H_λ. This solution is invariant under the screw motion of parameter λ, namely*

$$u \circ \sigma_\lambda^\alpha = u,$$

for all $\alpha \in \mathbb{R}$.

We also prove that the above result is, in some sense, sharp. Indeed, we have the :

Theorem 1.2. *Assume that (1.2) holds and that $\lambda \in (0, \lambda_*]$. Then, there are no nontrivial bounded solution of the Allen-Cahn equation (1.1) whose zero set is equal to H_λ.*

Observe that, in this result, we do not assume that the solution is invariant under screw motion. In the last section of this note, we will derive some precise asymptotics for the solutions constructed in Theorem 1.1.

We briefly comment on the possibility to extend our result to higher dimensional Euclidean spaces. According to [4], there is an analogue of the helicoïd in any odd dimension. Given $m \geq 1$ and $\lambda > 0$, we can define the $(2m+2)$-dimensional helicoïd H_λ in $\mathbb{R}^{2m+3}$ to be the hypersurface parameterized by

(1.4)
$$\begin{array}{ccc} \mathbb{R} \times \mathbb{R} \times (S^m \times S^m) & \longrightarrow & \mathbb{R}^{2m+3} \\ (t, \theta, (\zeta_1, \zeta_2)) & \longmapsto & \left(t\, (\cos\theta\, \zeta_1 - \sin\theta\, \zeta_2), t\, (\sin\theta\, \zeta_1 + \cos\theta\, \zeta_2), \dfrac{\lambda}{\pi} \theta \right) \end{array}$$

The interested reader will check that, with this definition of the higher dimensional helicoïd, the above results extend in higher dimensions and lead to solutions of the Allen-Cahn equation whose zero set is H_λ provided $\lambda > \lambda_*$, while there is so such a solution when $\lambda \leq \lambda_*$.

As will become clear from the construction, the key point in the proof of Theorem 1.1 is the existence of a nontrivial minimizer for the one dimensional Allen-Cahn functional on the interval $[0, \lambda]$, for $\lambda > \lambda_*$, while the proof of Theorem 1.2 relies on the fact that 0 is the only minimizer of the one dimensional Allen-Cahn functional on the interval $[0, \lambda]$, for $\lambda \leq \lambda_*$.

2. Preliminary results

Given $\lambda > 0$, we consider the Allen-Cahn equation (1.1) defined in the interval $[0, \lambda]$, with 0 Dirichlet boundary values. This reduces to the study of the autonomous second order ordinary differential equation

$$\ddot{v} - F'(v) = 0, \tag{2.5}$$

with $v(0) = v(\lambda) = 0$, where $\cdot$ denotes the differentiation with respect to the variable $s \in [0, \lambda]$. The energy associated to this equation reads

$$\mathring{E}(v) := \int_0^\lambda \left(\frac{\dot{v}^2}{2} + F(v) \right) ds,$$

and is well defined for functions $v \in H_0^1([0, \lambda])$.

For the sake of completeness, we recall the proof of the following simple and standard result concerning minimizers of $\mathring{E}$. The arguments used in the proof of this result will be essential in our analysis :

Lemma 2.1. *Assume that $\lambda > \lambda_*$ is fixed. Then, there exists a nontrivial positive solution of (2.5) which is a minimizer of $\mathring{E}$ in $H_0^1([0, \lambda])$. Assume that $\lambda \leq \lambda_*$ is fixed. Then, there are no nontrivial* positive *solution of (2.5) and the trivial solution* 0 *is the unique minimizer of $\mathring{E}$ in $H_0^1([0, \lambda])$.*

Proof. Obviously 0 is always a solution of (2.5). So the question is whether it is a minimizer of the energy or not.

Let

$$\phi(s) := \sin\left(\frac{\pi}{\lambda} s\right),$$

be an eigenfunction associated to the first eigenvalue of $-\partial_s^2$ over $[0, \lambda]$, with 0 boundary conditions. We just use a small multiple of ϕ as a test function to prove that 0 is not a minimizer when $\lambda > \lambda_*$. Indeed, we have

$$\mathring{E}(0) = \lambda\, F(0),$$

while, Taylor's expansion of F implies

$$\begin{aligned} \mathring{E}(\epsilon\,\phi) &= \lambda\, F(0) + \frac{\epsilon^2}{2} \int_0^\lambda (\dot{\phi}^2 + F''(0)\,\phi^2)\, ds + \mathcal{O}(\epsilon^4) \\ &= \lambda\, F(0) + \frac{\epsilon^2}{4}\, \lambda \left(\left(\frac{\pi}{\lambda}\right)^2 + F''(0) \right) + \mathcal{O}(\epsilon^4) \\ &< \mathring{E}(0), \end{aligned}$$

for $\epsilon > 0$ small provided $\lambda > \lambda_*$. Therefore, when $\lambda > \lambda_*$, we get a nontrivial minimizer of the energy $\mathring{E}$, which by standard arguments can be chosen to be positive.

To prove that there are no nontrivial solutions when $\lambda \leq \lambda_*$, we just multiply (2.5) by ϕ and integrate by parts the result over $[0, \lambda]$. We get

$$\int_0^\lambda \phi \left(F'(v) + \left(\frac{\pi}{\lambda}\right)^2 v \right) ds = 0,$$

which immediately implies that $v \equiv 0$ when $\lambda \leq \lambda_*$ since we have assumed that $F'(u) \geq F''(0)\, u$ for all $u \geq 0$. □

As a by product of the proof of this result, if v denotes the positive minimizer of $\mathring{E}$, we have the inequality

$$\mathring{E}(v) < \mathring{E}(0) = \lambda\, F(0), \tag{2.6}$$

when $\lambda > \lambda_*$. Observe that v depends on λ but we have chosen not to make this apparent in the notations.

3. The existence of a solution when $\lambda > \lambda_*$

In this section, it is convenient to use cylindrical coordinates $(r, \theta, t) \in [0, \infty) \times S^1 \times \mathbb{R}$ to parameterize $\mathbb{R}^3$. We look for a solution of (1.1) which is defined in $\mathbb{R}^3$ and which is invariant under the action of the screw motion σ_λ^α for all $\alpha \in \mathbb{R}$. That is, given $\lambda > 0$, we assume that

$$u(r, \theta, t) = u \left(r, \theta + \alpha, t + \frac{\lambda}{\pi}\, \alpha \right),$$

for all $\alpha \in \mathbb{R}$. Observe that this implies that

$$u(r, \theta, t) = u(r, \theta, t + 2\,\lambda),$$

and also that

$$u(r, \theta, t) = u \left(r, 0, t - \frac{\lambda}{\pi}\, \theta \right).$$

We further assume that

$$u(r, \theta, t + \lambda) = -u(r, \theta, t).$$

Assuming that the solution u we are looking for satisfies all these invariance, in order to construct u, it is enough to know the function U defined in $[0, \infty) \times [0, \lambda]$ by

$$U(r, s) := u(r, 0, s).$$

Observe that, if the function U is positive in $(0, \infty) \times (0, \lambda)$ and vanishes on the boundary of $[0, \infty) \times [0, \lambda]$, then the zero set of the function u is exactly the helicoïd H_λ.

In terms of the function U, the Allen-Cahn equation reduces to

$$\partial_r^2 U + \frac{1}{r}\, \partial_r U + \left(1 + \frac{\lambda^2}{\pi^2\, r^2} \right) \partial_s^2 U - F'(U) = 0, \tag{3.7}$$

and the Allen-Cahn energy reads

$$E(U) := \frac{1}{2} \int \left(|\partial_r U|^2 + \left(1 + \frac{\lambda^2}{\pi^2\, r^2} \right) |\partial_s U|^2 \right) r\, dr ds + \int F(U)\, r\, dr ds,$$

where the domain of integration is the infinite half strip $[0, \infty) \times [0, \lambda]$.

To prove the existence of U, solution of (3.7) which vanishes on the boundary of the infinite half strip $[0,\infty)\times[0,\lambda]$, we use a variational argument which has already been used in [5] and [1]. Given $R>0$, we define

$$S_R := [0,R]\times[0,\lambda],$$

and we denote by E_R the corresponding energy of a function U defined on S_R. For all $R>0$, we minimize E_R in $H^1_0(S_R)$ (the measure used to define $H^1_0(S_R)$ is $r\,dr\,ds$). Classical results in the calculus of variations imply that the minimum is achieved by a function $U_R \in H^1_0(S_R)$ whose energy is finite. Moreover, we can assume without loss of generality that U_R takes values in $[0,1]$. One easily checks that U_R is a solution of (3.7). Next, elliptic estimates allows one to pass to the limit in the sequence U_R for a sequence of R tending to ∞. Let us call U the limit function. The function $U\geq 0$ and is a solution of (3.7) which extends to a solution of (2.5). Therefore, it remains to prove that U is positive in the infinite strip $(0,\infty)\times(0,\lambda)$.

Proposition 3.1. *Assume that (1.2) holds and that $\lambda>\lambda_*$, then U is not identically equal to 0 and in fact $U>0$ in $(0,\infty)\times(0,\lambda)$.*

Proof. We argue by contradiction. Assume that $\lambda>\lambda_*$ and that $U\equiv 0$ then, given $R_0>0$, U_R converges uniformly to 0 on S_{R_0}, for a sequence of R tending to infinity.

We compute

$$E_R(0) = \frac{\lambda}{2}\, F(0)\, R^2.$$

Since U_R minimizes the energy and converges uniformly to 0 in S_{R_0}, we have

$$E_{R_0}(V) \geq E_{R_0}(0),$$

for any test function V which vanishes on the boundary of S_{R_0}. To define an appropriate test function, we first cook up some cutoff function η which is identically equal to 0 when $r\in[0,1/2]\cup[R_0-1/2,R_0]$ and identically equal to 1 when $r\in[1,R_0-1]$. We can also assume that the gradient of η remains bounded uniformly in R_0, as R_0 tends to infinity. We define

$$V(r,s) = \eta(r)\, v(s),$$

where v is the unique solution of the Allen-Cahn equation in $[0,\lambda]$ which minimizes $\mathring{E}$.

A simple computation yields

$$E_{R_0}(V) = \frac{R_0^2}{2}\int_0^\lambda \left(\frac{\dot v^2}{2} + F(v)\right) ds + \mathcal{O}(R_0).$$

Hence we have the inequality

$$\frac{\lambda}{2}\,\mathring{E}(v)\, R_0^2 + \mathcal{O}(R_0) \geq \frac{\lambda}{2}\, F(0)\, R_0^2.$$

Therefore, choosing R_0 large enough, we reach a contradiction if

$$\lambda\, F(0) > \mathring{E}(v),$$

but this is precisely the inequality (2.6) which holds when $\lambda>\lambda_*$. □

Remark 3.1. *In general, we can write*

$$u(r,\theta,s) = U\left(r,\theta,s+\frac{\lambda}{\pi}\theta\right)$$

where U is a function of $r \in \mathbb{R}^*$ (observe that here $r \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and not $r \in (0,\infty)$ as usual), $\theta \in S^1$ and $s \in \mathbb{R}$. In which case the Allen Cahn equation becomes

$$\partial_r^2 U + \frac{1}{r}\,\partial_r U + \frac{1}{r^2}\,\partial_\theta^2 U + \frac{2\lambda}{\pi r^2}\,\partial_s\partial_\theta U + \left(1+\frac{\lambda^2}{\pi^2 r^2}\right)\partial_s^2 U - F'(U) = 0.$$

Let us comment on the modifications which are needed to handle the higher dimensional cases. When $m \geq 1$, the construction of a nontrivial bounded solution of the Allen Cahn equation (1.1) in dimension $\mathbb{R}^{2m+3}$ whose level set is equal to the $(2m+2)$-dimensional helicoïd H_λ which has been defined in (1.4), follows very closely the line of the above construction, after taking into account the following facts.

Let $(X,Y,Z) \in \mathbb{R}^{m+1}\times\mathbb{R}^{m+1}\times\mathbb{R} = \mathbb{R}^{2m+3}$ and parametrize this space as follows

$$X = t(\cos\theta\, z_1 - \sin\theta\, z_2), \quad Y = t(\sin\theta\, z_1 + \cos\theta\, z_2), \quad Z = s,$$

where $(t,\theta,z_1,z_2,s) \in \mathbb{R}\times\mathbb{R}\times S^m\times S^m\times\mathbb{R}$.

Given $\lambda > 0$, we look for solutions to

$$\Delta u - F'(u) = 0,$$

in $\mathbb{R}^{2m+3}$ which are rotationally invariant in the z_1 and z_2 variables, that is

$$u(t,\theta,\tau z_1,z_2,s) = u(t,\theta,z_1,z_2,s), \quad u(t,\theta,z_1,\tau z_2,s) = u(t,\theta,z_1,z_2,s),$$

for any rotation $\tau \in O(m)$. Thus in particular we get that

$$u(t,\theta,z_1,z_2,s) = u(t,\theta,p_*,p_*,s),$$

where p_* denotes the north pole in S^m. Furthermore, as in the case of the 2-dimensional helicoïd, we assume that

$$u(t,\theta,z_1,z_2,s) = u\left(t,\theta+\alpha,z_1,z_2,s+\frac{\lambda}{\pi}\,\alpha\right),$$

for all $\alpha \in \mathbb{R}$ and

$$u(t,\theta,z_1,z_2,s+\lambda) = -u(t,\theta,z_1,z_2,s).$$

If u satisfies these invariances, in order to construct u, it is enough to look for a function U which is defined in $[0,\infty)\times[0,\lambda]$ by

$$U(r,s) := u(r,0,p_*,p_*,s).$$

If in addition we assume that the function U is positive in $(0,\infty)\times(0,\lambda)$ and vanishes on the boundary of $[0,\infty)\times[0,\lambda]$, we get a solution of the Allen-Cahn equation whose zero set is exactly the helicoïd H_λ. If $\lambda > \lambda_*$, the proof of the existence of U follows to the one already performed for the 2-dimensional helicoïd H_λ. When $\lambda \leq \lambda_*$ the nonexistence follows as in the two dimensional case. We leave the details to the reader.

4. The nonexistence result

In this section, we obtain a nonexistence result for solutions of the Allen-Cahn equation whose zero set is a helicoïd. The proof of the following result is very reminiscent of the proof of the famous *de Giorgi conjecture* in dimension 2 :

Theorem 4.1. *Assume that (1.2) holds and that $\lambda \leq \lambda_*$. Then, there are no non-trivial, bounded solution to the Allen-Cahn equation whose zero set is the helicoïd H_λ.*

Proof. We use Remark 3.1 and write $u(r,\theta,t) = U(r,\theta,t+\lambda\theta)$. We test the equation satisfied by U against the function $U\,\eta^2$ where η is a cutoff function which only depends on r and which is defined so that $\eta \equiv 1$ for $r \leq R$ and $\eta \equiv 0$ for $r \geq 2R$. Moreover, we assume that $|\nabla \eta| \leq C\,R^{-1}$ for some constant $C > 0$ independent of $R \geq 2$. We obtain

$$\int \left(|\partial_r U|^2 + \frac{1}{r^2}\left(\partial_\theta + \frac{\lambda}{\pi}\,\partial_s U\right)^2 + |\partial_s U|^2 + F'(U)\,U \right) \eta^2\, r\, dr\, d\theta\, ds \\ = \int U\,\partial_r U\,\eta\,\partial_r \eta\, r dr\, d\theta\, ds,$$

where this time, the integrals are understood over $[0,\infty) \times S^1 \times [0,\lambda]$.

We make use of the assumption that $F'(t)t \geq F''(0)\,t^2$ for any $t \in \mathbb{R}$, together with the fact that $\lambda \leq \lambda_*$, to get

$$\int_0^\lambda (|\partial_s U|^2 + F'(U)\,U)\, ds \geq \int_0^\lambda (|\partial_s U|^2 + F''(0)\,U^2)\, ds \geq 0.$$

Therefore, we can write

$$\int |\partial_r U|^2\,\eta^2\, r\, dr\, d\theta\, ds \leq \int U\,\partial_r U\,\eta\,\partial_r\eta\, r dr\, d\theta\, ds.$$

Using Cauchy-Schwarz inequality, we conclude that

$$\begin{aligned} \int |\partial_r U|^2\,\eta^2\, r\, dr\, d\theta\, ds \quad &\leq \quad \left(\int_{r\in[R,2R]} |\partial_r U|^2\,\eta^2\, r\, dr\, d\theta\, ds \right)^{1/2} \\ &\qquad \times \left(\int_{r\in[R,2R]} |\partial_r \eta|^2\, U^2\, r\, dr\, d\theta\, ds \right)^{1/2}. \end{aligned} \tag{4.8}$$

Observe that the second integral is bounded (this is where we use the fact that our domain is two dimensional). This immediately implies that

$$\int_{r\in[0,R]} |\partial_r U|^2\,\eta^2\, r\, dr\, d\theta\, ds,$$

is bounded independently of R. Letting R tend to infinity, we conclude that

$$\int |\partial_r U|^2\, r\, dr\, d\theta\, ds \leq C,$$

in particular, there exists a sequence R_j tending to $+\infty$ such that

$$\lim_{j\to+\infty} \int_{r\in[R_j,2R_j]} |\partial_r U|^2\,\eta^2\, r\, dr\, d\theta\, ds = 0.$$

Using once more (4.8), we conclude that

$$\int |\partial_r U|^2 \, r \, dr \, d\theta \, ds \leq 0,$$

which completes the proof of the result. □

Remark 4.1. *A similar non existence result for nontrivial bounded solutions to the Allen-Cahn equation (1.1) in* $\mathbb{R}^{2m+3}$ *whose zero level set is the* $(2m+2)$*-dimensional helicoïd* H_λ *readily follows using similar arguments.*

5. Asymptotic behavior of the solution U

In this last section, we derive some precise asymptotics for the solutions whose existence is guarantied by Theorem 1.1. We know from Lemma 2.1 that for all $\lambda > \lambda_*$, there exists v a non trivial minimizer of $\mathring{E}$ on $[0, \lambda]$. We further assume from now on that

$$v \text{ is the only positive minimizer of } \mathring{E} \text{ on } [0, \lambda]. \tag{5.9}$$

Example 5.1. *In the case where*

$$F(t) := \frac{1}{4} \, (1 - t^2)^2,$$

we have already seen that $\lambda_* = \pi$*. It follows from* [3] *that the period function for solutions of (1.1) is monotone. More precisely, for all* $t \in (0, 1)$ *there exists a unique positive solution of*

$$\ddot{v} - F''(v) = 0$$

such that $v(0) = t$ *and* $\dot{v}(0) = 0$ *and the result of* [3] *implies that the first positive zero of this solution, is a strictly monotone function of* t *which tends to* 0 *as* t *tends to* 0 *and which tends to* $+\infty$ *as* t *tends to* 1*. In particular, this implies that for each* $\lambda > 0$ *there exists a unique positive solution of*

$$\ddot{v} - F''(v) = 0,$$

such that $v(0) = v(\lambda) = 0$*.*

We now assume that $\lambda > \lambda_* = \pi$ and that U is the solution defined in the previous section and we derive some precise asymptotics for the function U as r tends to infinity.

First, we prove the following general result :

Lemma 5.1. *Assume that (1.2) and (5.9) hold. Any positive, bounded solution* $\bar{U}$ *of the Allen-Cahn equation which is defined in a strip* $\mathbb{R} \times [0, \lambda]$*, vanishes on the boundary of this strip and which is minimizing depends only on one variable. Hence* $\bar{U} \equiv 0$ *when* $\lambda \leq \lambda_*$ *and* $\bar{U} \equiv v$ *when* $\lambda > \lambda_*$*.*

Proof. We choose $L > 0$. On the piece of strip

$$S_L := [-L, L] \times [0, \lambda],$$

the function $\bar{U}$ is a minimizer of the energy with respect to functions which have the same boundary values as $\bar{U}$ on ∂S_L.

Recall that v is defined to be the minimizer of $\mathring{E}$ on $[0, \lambda]$. We compare the energy of $\bar{U}$ on S_L with the energy of a test function $\bar{v}$ which is identically equal to v on $[-L+1, L-1] \times [0, \lambda]$ (namely $\bar{v}(t, s) = v(s)$ on this set) and which interpolates between v and $\bar{U}$ in the sets $[-L, -L+1] \times [0, \lambda]$ and $[L-1, L] \times [0, \lambda]$, so that $\bar{v}$

has the same boundary data as $\bar U$ on ∂S_L. It is easy to check that one can define such a test function $\bar v$ so that its energy on S_L is bounded by $2\,L\,\mathring{E}(v)+C$, for some constant $C>0$ independent of $L\geq 2$. Since $\bar U$ is a minimizer, we conclude that

$$\int_{S_L}\frac{1}{2}|\partial_r\bar U|^2 dsdr+\int_{S_L}\left(\frac{1}{2}|\partial_s\bar U|+F(U)\right)ds\,dr\leq 2\,L\,\mathring{E}(v)+C,$$

where $C>0$ does not depend on $L\geq 2$.

Now, since v is the minimizer of $\mathring{E}$, we have

$$\mathring{E}(v)\leq\int_0^\lambda\left(\frac{1}{2}|\partial_s\bar U|+F(U)\right)ds.$$

Integrating this inequality over $r\in[-L,L]$, we conclude that

$$2\,L\,\mathring{E}(v)\leq\int_{S_L}\left(\frac{1}{2}|\partial_s\bar U|+F(U)\right)ds\,dr.$$

Therefore,

$$\int_{S_L}|\partial_r\bar U|^2 dsdr\leq 2\,C,$$

for some constant $C>0$ which does not depend on $L\geq 2$. Letting L tend to infinity, we conclude that

$$\int_{\mathbb{R}\times[0,\lambda]}|\partial_r\bar U|^2 dsdr\leq 2\,C.$$

Now, we make use of the fact that $\partial_r\bar U$ satisfies

$$\Delta\partial_r\bar U-F''(U)\,\partial_r\bar U=0,$$

and vanishes on the boundary of the infinite strip. In particular, this implies that

$$\int_{\mathbb{R}\times[0,\lambda]}(|\nabla\partial_r\bar U|^2+F''(U)\,|\partial_r\bar U|^2)\,dr\,ds=0.$$

Moreover, since $\bar U$ is a minimizer we have

$$Q(W):=\int_{\mathbb{R}\times[0,\lambda]}(|\nabla W|^2+F''(U)\,W^2)\,dr\,ds\geq 0,$$

for any function W having compact support in $\mathbb{R}\times(0,\lambda)$. Classical arguments imply that $\partial_r\bar U$ does not change sign in $\mathbb{R}\times(0,\lambda)$ and in fact is either identically equal to 0 or does not vanish in $\mathbb{R}\times(0,\lambda)$. Indeed, from the above, we see that $\partial_r U$ is a minimizer of the quadratic form Q and hence so is $|\partial_r U|$. Therefore, $|\partial_r U|$ is a solution of $(-\Delta+F''(U))\,|\partial_r U|=0$ and elliptic regularity then implies that $|\partial_r U|$ is a smooth function. Finally, Hopf maximum principle implies that $|\partial_r U|$ does not vanish in $\mathbb{R}\times(0,\lambda)$ and hence $\partial_r U$ does not change sign.

This analysis implies that $(r,s)\mapsto\bar U(r,s)$ is a monotone function of r. Now, since $\bar U$ is bounded, we conclude that, as r tends to $+\infty$, the function $r\longmapsto\bar U(r,\cdot)$ converges uniformly (in $\mathcal{C}^2$ topology) to some function V which only depends on s over $[-1,1]\times[0,\lambda]$. Certainly V is a solution of (2.5) and hence since we have assumed that this equation has only one positive solutions either $V=v$ or $V=0$. A similar arguments holds as r tends to $-\infty$.

We have already proven that $\bar U$ is monotone in r (say for example that $\bar U$ is increasing), then either $\bar U\equiv v$, or $\bar U\equiv 0$ or $\bar U$ is monotone increasing in r and tends to 0, as r tends to $-\infty$ and tends to v as r tends to $+\infty$. In particular, since

$v \equiv 0$ when $\lambda \leq \lambda_*$, we have proven that $\bar{U} \equiv 0$ in this case. When $\lambda > \lambda_*$, thanks to (2.6) we see that U cannot be a minimizer if $\bar{U}$ is close to 0 on some long enough piece of strip. Therefore, the last two cases do not occur and this also completes the proof of the result when $\lambda > \lambda_*$. □

As a corollary, we get

Proposition 5.1. *Assume that (1.2) and (5.9) hold. As r tends to ∞, $\bar{U}(r, \cdot)$ tends to v uniformly on $[0, \lambda]$.*

Proof. We argue by contradiction. If, for some $\epsilon > 0$ and for some sequence r_j tending to $+\infty$ we have $\sup_{s \in [0,\lambda]} |\bar{U}(r_j, \cdot) - v| \geq \epsilon$, then extracting subsequences one concludes that there exists a function $\tilde{U}$ which is defined on an infinite strip and which is a positive solution of the Allen-Cahn equation. Moreover, $\sup_{s \in [0,\lambda]} |\tilde{U}(0, \cdot) - v| \geq \epsilon$ and $\tilde{U}$ is a minimizer (since it is the limit of a family of minimizers). This certainly contradicts the result of the previous Lemma. □

Let us now give further details about the asymptotic behavior of the solution U. This requires yet some extra assumption which we now describe. Let us denote by ψ_1 a positive eigenfunction of $-\partial_s^2 + F''(v)$ on $[0, \lambda]$ (here v is the positive solution obtained in Lemma 2.1 which is assumed to be unique), which associated to the first eigenvalue λ_1, i.e.

$$(-\partial_s^2 + F''(v))\, \psi_1 = \lambda_1\, \psi_1,$$

and $\psi_1(0) = \psi_1(\lambda) = 0$. We know that $\lambda_1 \geq 0$ since v is a minimizer of the energy. We further assume that

$$\lambda_1 > 0. \tag{5.10}$$

Example 5.2. *Assume that*

$$F''(t)\, t < F'(t), \tag{5.11}$$

for $t > 0$, then $\lambda_1 > 0$. Indeed, multiply the equation satisfied by ψ_1 by v and the equation satisfied by v by ψ_1 and integrate the difference between 0 and λ. We find

$$\lambda_1 \int_0^\lambda \psi_1\, v\, ds = \int_0^\lambda \left(F''(v)\, v - F'(v)\right)\, \psi_1\, ds$$

and hence we conclude that $\lambda_1 > 0$. Observe that (1.2) and (5.11) are compatible and in fact the double well potential defined in (1.3) is an example of potential which satisfies all our assumptions.

We write

$$U(r, s) = v(s) + \phi(r, s).$$

We now prove the

Proposition 5.2. *Assume that (1.2), (5.10) and (5.11) hold. Then, there exists a constant $C > 0$ such that*

$$|\phi(r, s)| \leq \frac{C}{r^2}\, \psi_1(s).$$

Proof. Using both the equation satisfied by U and the ordinary differential equation satisfied by v, we can rewrite the equation satisfied by ϕ as

$$\left(\partial_r^2 + \frac{1}{r}\, \partial_r + \left(1 + \frac{\lambda^2}{\pi^2 r^2}\right)\, \partial_s^2\right)\, \phi - \phi + (v + \phi)^3 - v^3 = -\frac{\lambda^2}{\pi^2 r^2}\, \partial_s v. \tag{5.12}$$

Is is easy to check that, provided $C > 0$ is chosen large enough and $r \geq R$, where R is fixed large enough, the function

$$(r, s) \longmapsto C\,\psi_1(s)\,r^{-2} + \epsilon\,\psi_1(s)\,r,$$

is a supersolution for our equation, for all $\epsilon > 0$. Moreover the operator which appears on the left hand side of (5.12) satisfies the maximum principle in $[R, +\infty) \times [0, \lambda]$, provided R is chosen large enough. This implies that

$$|\phi(r, s)| \leq \frac{C}{r^2}\,\psi_1(s) + \epsilon\,\psi_1(s)\,r.$$

The estimate in the Proposition follows at once by letting ϵ tend to 0. □

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