GENERAL CRITERIA FOR CURVES TO BE SIMPLE

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Abstract. We extend previous results for parametrized curves in euclidean space to be simple. The new condition depends as before on Ahlfors' Schwarzian, and considers a *conformal metric* on a given interval and the new diameter. We derive some applications, among which we find Becker type conditions that depend on a pre-Schwarzian.

1. Introduction

The purpose of this paper is to extend results in [5], where the use of Sturm comparison and Ahlfors' Schwarzian for curves led to sufficient conditions for parametrized curves in euclidean space to be simple. In many cases, the condition was sharp. By considering a "conformal metric" on an interval, we derive here a more general condition of the same type that takes into account the modified diameter of the interval. The theorem fills in the gaps when the former condition was not optimal. In addition, suitable choices of the conformal factor give rise to criteria that depend on a pre-Schwarzian derivative, and analogues of criteria for holomorphic mappings in the disk due to Ahlfors, Becker, and Epstein [2], [3], [9].

We begin with a brief account on Ahlfors' Schwarzian for curves. In [1] the author generalizes the Schwarzian to cover $f : (a, b) \to \mathbb{R}^n$ by separately defining analogues of the real and imaginary parts $\mathrm{Re}\{Sf\}$ and $\mathrm{Im}\{Sf\}$ of the Schwarzian of a locally injective holomorphic mapping f. He defined

$$S_1 f = \frac{\langle f', f''' \rangle}{|f'|^2} - 3\frac{\langle f', f'' \rangle^2}{|f'|^4} + \frac{3}{2}\frac{|f''|^2}{|f'|^2}\,, \tag{1.1}$$

and

$$S_2 f = \frac{f' \wedge f'''}{|f'|^2} - 3\frac{\langle f', f'' \rangle}{|f'|^4} f' \wedge f''\,, \tag{1.2}$$

respectively. Here, for $\vec{a}, \vec{b} \in \mathbb{R}^n$, $\vec{a} \wedge \vec{b}$ is the antisymmetric bivector with components $(\vec{a} \wedge \vec{b})_{ij} = a_i b_j - a_j b_i$ and norm $(\sum_{i<j}(a_i b_j - a_j b_i)^2)^{1/2}$. Ahlfors indicated that he was led to these seemingly esoteric definitions by a direct identification of $\mathrm{Re}\{z\overline{\zeta}\}$ with the inner product $\langle z, \zeta \rangle$ of the 2-dimensional vectors z, ζ and the far

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from obvious identification of $\mathrm{Im}\{z\overline{\zeta}\}$ with the corresponding $\zeta \wedge z$ based on the fact that $(\mathrm{Im}\{z\overline{\zeta}\})^2 = |\zeta \wedge z|^2$. For the purposes of injectivity, this far only $S_1 f$ has played a role. Invariance In [5] a simpler form was obtained for $S_1 f$ in the form

$$S_1 f = \left(\frac{v'}{v}\right)' - \frac{1}{2}\left(\frac{v'}{v}\right)^2 + \frac{1}{2}v^2k^2 \,.$$

Thus, if $s(x)$ denotes arc length, then

$$S_1 f = Ss(x) + \frac{1}{2}v^2k^2 \,, \tag{1.3}$$

where $Sh = (h''/h')' - (1/2)(h''/h')^2$ is the usual Schwarzian.

Both of Ahlfors' operators are invariant under Möbius transformations of $\mathbb{R}^n$ (see [1], [5]). This allows to extend the definition to include curves into $\mathbb{R}^n \cup \{\infty\}$. Another important property that we will use, is a the chain rule, which states that under a change of parameter h,

$$S_1(f \circ h) = [(S_1 f) \circ h](h')^2 + Sh \,.$$

The combination of these properties together with comparison techniques from the Strum theory has given way to important applications of the S_1 operator in questions regarding the injectivity of the conformal immersion of planar domains into higher dimensional euclidean space [6],[7], [12], [4].

The following results were established in [5]:

Theorem A. Let $p = p(x)$ be a continuous function defined on an interval $I \subset \mathbb{R}$ with the property that no non-trivial solution of

$$u'' + pu = 0 \tag{1.4}$$

has more than one zero. Let $f : I \to \mathbb{R}^n \cup \{\infty\}$ be a curve of class C^3 with nowhere vanishing f'. If $S_1 f \leq 2p$ then f is injective on I.

Suitable choices of function p on on the interval $I = (-1, 1)$ were shown to render analogues of the classical injectivity criteria due to Nehari [11]. By considering $p = \pi^2/\delta^2$ on an interval I of length δ we obtain from Theorem A the important corollary that if

$$S_1 f \leq \frac{2\pi^2}{\delta^2} \tag{1.5}$$

on I, then f is injective. The fact that this choice of function p satisfies the hypothesis of the theorem is readily verified by observing that one can arrange for a suitable trigonometric function to be a non-vanishing solution of $u'' + pu = 0$ on I.

In Theorem B below, the interval I was normalized to be $(-1, 1)$. The analysis was simplified with the additional assumption of symmetry of the function p. The function $F : I \to \mathbb{R}$ was defined to be the unique function with $SF = 2p$ and $F(0) = 0, F'(0) = 1, F''(0) = 0$.

Theorem B. Let $f : (-1, 1) \to \mathbb{R}^n \cup \{\infty\}$ satisfy $f(0) = 0$, $|f'(0)| = 1, f''(0) = 0$ and suppose that $S_1 f \le 2p$ for some even function for which (1.4) is disconjugate. Then

(a) $|f'(x)| \le F'(|x|)$ on $(-1, 1)$ and f admits a (spherically) continuous extension to $[-1, 1]$.

(b) If $F(1) < \infty$, then f is one-to-one on $[-1, 1]$ and $f([-1, 1])$ has finite length.

(c) If $F(1) = \infty$, then either f is one-to-one on $[-1, 1]$ or, up to rotation, $f = F$.

2. Main Result

We first prove an injectivity condition for parametrized curves that parallels Theorem A.

Theorem 2.1. *Let $\varphi = \varphi(x)$ be a C^2 function defined on an interval $I \subset \mathbb{R}$, and let*

$$\delta = \int_I e^{\varphi} dx \,.$$

Let $f : I \to \mathbb{R}^n \cup \{\infty\}$ be a curve of class C^3 with nowhere vanishing f'. If

$$S_1 f \le \varphi'' - \frac{1}{2} (\varphi')^2 + e^{2\varphi} \frac{2\pi^2}{\delta^2} \,, \tag{2.1}$$

then f is injective on I.

Proof. Fix $x_0 \in I$ and let

$$F = F(x) = \int_{x_0}^{x} e^{\varphi(t)} dt \,. \tag{2.2}$$

The function F maps $I = (a, b)$ in a one-to-one manner onto an interval J of total length δ. We will show that (2.1) implies (1.4) for the curve $g = f \circ h$, where $h = F^{-1}$. The chain rule implies that

$$S_1 g = (S_1 f \circ h) \, (h')^2 + Sh \,,$$

which together with the relation

$$Sh = -(SF)(h')^2 = -[\varphi'' - \frac{1}{2}(\varphi')^2](h')^2$$

gives that

$$S_1 g \le \frac{2\pi^2}{\delta^2} \,.$$

This proves that g, and hence f, are both injective.

□

We claim that Theorem A follows from Theorem 2.1 by choosing φ adequately. It is well known that a locally injective function $G : I \to \mathbb{R} \cup \{\infty\}$ has $SG = 2p$ on I if and only if

$$G = \frac{u_1}{u_2}\,,$$

for a pair u_1, u_2 of linearly independent solutions of (1.4) [8]. On the other hand, given a solution u, variation of parameters gives a second, linearly independent solution of the form $u(x)\int^x u^{-2}(t)dt$. Consequently,

$$G(x) = \int^x u^{-2}(t)dt$$

is always a function with $SG = 2p$. The zeros of u will be mapped under F to the point at infinity. The action of the group of Möbius transformations $T(x) = (Ax+B)/(Cx+D)$, $AD - BC \neq 0$, gives rise to all other functions $H = T(G)$ with $SG = 2p$ and all other solutions of (1.4), $v = (H')^{-1/2}$.

Suppose now that p satisfies the hypothesis of Theorem A, that is, that (1.4) is disconjugate. This implies that any function G with $SG = 2p$ will be injective on I (see, *e.g.*, [8]). Thus $J = G(I)$ is a non-overlapping interval on $\mathbb{R} \cup \{\infty\}$, in other words, $\mathbb{R} \cup \{\infty\} \backslash J$ contains at least one point. By choosing a suitable Möbius shift $F = T(G)$, we may assume that $\infty \notin J$. Hence $u = (F')^{-1/2}$ is a solution of (1.4) that is non-vanishing on I. We now choose φ so that

$$e^{\varphi} = u^{-2}\,,$$

which gives that

$$SF = 2p = \varphi'' - \frac{1}{2}\left(\varphi'\right)^2\,,$$

showing the claim.

We now analyze under what circumstances Theorem 2.1 improves Theorem A, that is, whether it is possible to choose $\varphi = -2\log u$ so that $\delta < \infty$. Let p be a given function on I for which (1.4) is disconjugate. We distinguish two cases. Suppose first that the interval J as chosen above, is the entire real line $\mathbb{R}$. This means that

$$\int_a u^{-2}dx = \int^b u^{-2}dx = \infty\,,$$

which is equivalent to saying that u is principal at both endpoints of the interval $I = (a, b)$ [10]. Any other solution of (1.4) that does not vanish on I will be a constant multiple of u. In this case, Theorem 2.1 will not improve Theorem A (and corresponds to the case (c) treated in Theorem B).

Suppose now that J is a proper subset of $\mathbb{R}$. This allows for a second Möbius shift so that J is a bounded interval. In other words, there exists a nowhere vanishing solution u of (1.4) that is not principal at either a nor b. The choice $\varphi = -2\log u$

produces a finite diameter, and Theorem 2.1 improves Theorem A exactly by the last term on (2.1).

An interesting point is whether in this last case, there exists an optimal choice of φ. For this, we need to analyze how the term

$$\Lambda = \Lambda_F = e^{2\varphi}\frac{2\pi^2}{\delta^2}$$

is affected under Möbius changes $G = T(F)$ that map the bounded interval $J = F(I) = (\alpha, \beta)$ to another bounded interval. If T is affine, it is readily seen that $\Lambda_F = \Lambda_G$. If T is not affine, then up to an affine change, T is an inversion of the form

$$T(y) = \frac{1}{y - y_0}\,,$$

for some $y_0 \notin \overline{J}$. A direct calculation shows that

$$\Lambda_G = \mu^2 \Lambda_F\,,$$

where

$$\mu = \frac{(y_0-\alpha)(y_0-\beta)}{(y_0 - F(x))^2}\,.$$

The extreme values of μ are the reciprocal quantities

$$\left|\frac{y_0-\alpha}{y_0-\beta}\right| \quad , \quad \left|\frac{y_0-\beta}{y_0-\alpha}\right|\,,$$

and are attained when $F(x)$ is an endpoint of J. The values of μ stay close to 1 for relatively large y_0, but can vary significantly when y_0 is close to an endpoint of J. In summary, among all functions φ for which $\varphi'' - (1/2)(\varphi')^2 = 2p$, there is no optimal choice that maximizes the term Λ.

We analyze the sharpness of Theorem 2.1. For given φ we let $2p = \varphi'' - (1/2)(\varphi')^2$. Since $u = e^{-\varphi/2}$ is a non-vanishing solution of $u'' + pu = 0$, we see from the Sturm theory that any solution of this equation can vanish at most once on I. We now distinguish two cases. We say that Theorem 2.1 is of *infinite diameter type* if $\delta = \infty$ and

$$\int_a e^{\varphi} dx = \int^b e^{\varphi} dx = \infty\,. \tag{2.3}$$

Equivalently, the equation $u'' + pu = 0$ admits a non-vanishing solution that is principal at both endpoints of I.

We say that Theorem 2.1 is of *finite diameter type* if at least one of the integrals in (2.3) is finite. As we saw, in this case it is possible to modify φ without changing $2p = \varphi'' - (1/2)(\varphi')^2$ so that both resulting integrals in (2.3) are finite.

Theorem 2.1 is always sharp, in the following two senses. First, there exists a curve satisfying the hypothesis which is not injective in the closed interval $\overline{I}$. It is also sharp in the sense that for any $\epsilon = \epsilon(x) \gneq 0$ there exists a non-injective $f : I \to \mathbb{R}^n \cup \{\infty\}$ with

$$S_1 f \leq \varphi'' - \frac{1}{2}(\varphi')^2 + e^{2\varphi}\frac{2\pi^2}{\delta^2} + \epsilon\,.$$

To establish these claims, we consider as before

$$F(x) = \int_{x_0}^{x} e^{\varphi(t)} dt\,.$$

If we are in the infinite diameter case, then $F(a) = F(b)$ are the point at infinity, hence F fails to be injective on $\overline{I}$.

If we are in the finite diameter case, we can assume that φ produces $\delta < \infty$. Choose $x_0 \in I$ so that $F(b) = -F(a) = \delta/2$, and let

$$\phi(x) = \tan\left(\frac{\pi}{2}F(x)\right)\,.$$

Then ϕ is an increasing function mapping I onto $\mathbb{R}$, with $\phi(a) = \phi(b)$ equal to the point at infinity. Furthermore

$$S\phi = \varphi'' - \frac{1}{2}(\varphi')^2 + e^{2\varphi}\frac{2\pi^2}{\delta^2}\,.$$

The functions F, ϕ are called the *extremal* functions for Theorem 2.1 for the finite and infinite diameter cases. In the last section, they will be shown to be unique up to Möbius transformations.

We show the second claim for both cases with the following theorem.

Theorem 2.2. *Let $H : I \to \mathbb{R}$ be a C^3 function with $H' > 0$ and $SH = 2q$. Suppose that $H(a) = H(b)$ are the point at infinity. If $\epsilon = \epsilon(x) \gneq 0$ then the differential equation*

$$v'' + (q + \epsilon)v = 0 \tag{2.4}$$

admits a non-trivial solution with two zeros.

Proof. Fix $x_0 \in I$ and consider the solution v of (2.4) with

$$v(x_0) = u(x_0) \quad , \quad v'(x_0) = u'(x_0)\,,$$

where $u = (H')^{-1/2}$. For $y \in \mathbb{R}$ let

$$w(y) = \frac{v}{u}(H^{-1}(y))\,.$$

Then

$$w''(y) = -(u^4\epsilon)w(y)\,,$$

where $u^4\epsilon$ is evaluated at $H^{-1}(y)$. Furthermore, $w(y_0) = 1, w'(y_0) = 0$, $y_0 = H^{-1}(x_0)$. Thus w is a non-constant concave function with a maximum at y_0. If w is non-constant to the right and to the left of y_0, then by concavity, it will vanish at y_1, y_2 for some $y_1 < y_0 < y_2$. Then v will vanish at $x_1 = H^{-1}(y_1)$ and $x_2 = H^{-1}(y_2)$.

If not, then w is constant, say, to the right of y_0 and non-constant to the left. Then v has a zero at some $x_1 < x_0$ and

$$\int^b v^{-2} dx = \int^b u^{-2} dx = \infty\,.$$

Thus

$$G(x) = \int_{x_0}^{x} v^{-2}(t)dt$$

is a function with $SG = 2(q+\epsilon)$ with $G(x_1) = G(b)$ equal to the point at infinity. Because $G : I \to \mathbb{R} \cup \{\infty\}$ is locally injective, it follows that $G(x_2) = G(x_3) = c < \infty$ for some $x_2 < x_1$ and $x_3 < b$. The function

$$H = \frac{1}{G-c}$$

has $SH = SG$ and $\tilde{v} = (H')^{-1/2}$ will be the desired solution.

□

3. Other Corollaries

In this section we derive a few other corollaries that we find to be of particular interest. In all cases, the proof is based on considering a particular choice of function φ in Theorem 2.1.

Corollary 3.1. *Let $f : I \to \mathbb{R}^n \cup \{\infty\}$ be a curve of class C^3 with nowhere vanishing f' and finite length δ. If*

$$k \le \frac{2\pi}{\delta}$$

then f is injective on I.

Proof. We choose $\varphi = \log|f'| = \log v$, so that $Ss(x) = \varphi'' - (1/2)(\varphi')^2$. The corollary follows at once.

□

Corollary 3.2. *Let $f : (-1,1) \to \mathbb{R}^n \cup \{\infty\}$ be a curve of class C^3 with nowhere vanishing f'. If*

$$S_1 f \le 2t\frac{1+(1-t)x^2}{(1-x^2)^2}\ ,\ t \ge 1 \tag{3.1}$$

or

$$S_1 f \le 2t\frac{1+(1-t)}{(1-x^2)^2} + \frac{2\pi}{(1-x^2)^{2t}}\left(\frac{\Gamma(\frac{3}{2}-t)}{\Gamma(1-t)}\right)^2\ ,\ 0 \le t < 1 \tag{3.2}$$

then f is injective on I.

Proof. We choose $\varphi = -t\log(1-x^2)$. For $t \geq 1$, $\delta = \infty$, and for $0 \leq t < 1$ the diameter is finite and is given by

$$\delta = \sqrt{\pi}\frac{\Gamma(1-t)}{\Gamma(\frac{3}{2}-t)}\,.$$

□

Inequalities (3.1) and (3.2) represent analogues of criteria for holomorphic mappings by Ahlfors [Ah1].

Corollary 3.3. *Let $f : I \to \mathbb{R}^n \cup \{\infty\}$ be a curve of class C^3 with nowhere vanishing f'. If σ is a C^2 function on I and*

$$S_1 f + \frac{2x}{1-x^2} \leq \sigma'' - \frac{1}{2}(\sigma')^2 + \frac{2}{(1-x^2)^2}$$

then f is injective on I.

Proof. We let $\varphi = \sigma - \log(1-x^2)$ in Theorem 2.1.

□

Condition (3.3) can be considered an analogue of the Epstein criterion [9].

Corollary 3.4. *Let $f : I \to \mathbb{R}^n \cup \{\infty\}$ be a curve of class C^3 with nowhere vanishing f'. If σ is a C^2 function on I and*

$$\frac{v'}{v}\sigma' + \frac{1}{2}v^2k^2 \leq \sigma'' - \frac{1}{2}(\sigma')^2$$

then f is injective on I.

Proof. We let $\varphi = \log v + \sigma$ in Theorem 2.1.

□

Corollary 3.5. *Let $f : (-1,1) \to \mathbb{R}^n \cup \{\infty\}$ be a curve of class C^3 with nowhere vanishing f'. If*

$$\frac{2x}{1-x^2}\frac{v'}{v} + \frac{1}{2}v^2k^2 \leq \frac{2}{(1-x^2)^2}$$

then f is injective on I.

This final criterion represents an analogue of the condition be Becker [3].

Proof. We let $\varphi = \log v - \log(1-x^2)$ in Theorem 2.1.

□

4. Distortion and Extensions

The purpose of this final section is to derive the corresponding version of Theorem B for the main result here.

Theorem 4.1. *Let $f : (-1,1) \to \mathbb{R}^n \cup \{\infty\}$ be a curve of class C^3 with nowhere vanishing f', satisfying (2.1). Let $H = F$ or $H = \phi$ be the extremal function depending on whether condition (2.1) is of infinite or finite type. For $x_0 \in I$ fixed, suppose that*

$$|f'(x_0)| = H'(x_0)\ ,\ |f'|'(x_0) = H''(x_0)\,. \tag{4.1}$$

Then

(i) $|f(x_1) - f(x_2)| \leq |H(x_1) - H(x_2)|$ and $|f'(x)| \leq H'(x)$;

(ii) f admits a spherically continuous extension to $\overline{I}$. If f is not injective in the closed interval then, up to a Möbius transformation, $f = H$.

Proof. The normalizations (4.1) are no restriction, in the sense that for given f, they can be achieved by postcomposition with a suitable Möbius transformation of $\mathbb{R}^n \cup \{\infty\}$.

To show (i), we consider $u = -2\log|f'|$. Then

$$u'' + qu = 0\,,$$

where $2q = Ss$. Because of (2.1), we have that $q \leq (1/2)SH$. The inequality $|f'(x)| \leq H'(x)$ follows now from Sturm comparison, while the first inequality in (i) follows after integration.

We show the continuous extension at, say, $x = b$. Fix x_0 near b and let T be a Möbius transformation of $\mathbb{R}^n \cup \{\infty\}$ so that the curve $g = T(f)$ satisfies

$$|g'(x_0)| = G'(x_0)\ ,\ |g'|'(x_0) = G''(x_0)\,,$$

for $G = -1/H$. If x_0 is close to b then G is regular on (x_0, b). The previous argument implies that

$$|g'(x)| \leq G'(x)\,,$$

and hence for $x_0 \leq x_1, x_2 < b$

$$|g(x_1) - g(x_2)| \leq |G(x_1) - G(x_2)|\,.$$

This shows that the modulus of continuity of g is controlled by that of G, and the continuous extension for g follows.

Suppose now that f is not injective on $\overline{I}$. Since it is injective on I, then either $f(a) = f(b)$ or $f(c) \in \{f(a), f(b)\}$, for some $c \in I$. We claim that the latter cannot occur. Suppose, by way of contradiction, that say $f(c) = f(b)$ for some $c \in I$. We may assume that the common point lies at infinity. For $y \in [\gamma, \infty)$, $H(c) = \gamma$, consider the function

$$w(y) = \frac{u}{v}(H^{-1}(y))\,,$$

where $u = |f'|^{-1/2}$ and $v = (H')^{-1/2}$. Then

$$w''(y) = \frac{1}{2} w^4 \left(SH - Ss \right) \geq 0 \, .$$

If w were constant on $[c, b)$, then $|f'|$ would be a constant multiple of H' on $[c, b)$, which would contradict that $f(c)$ is the point at infinity. Hence w is not constant, and is therefore bounded below by some line $my + n$, $m \neq 0$. If $m > 0$ we analyze the inequality $w(y) \geq my + n > 0$ for large values of y, which leads to

$$|f'| \leq \frac{H'}{(mH + n)^2}$$

for x close to b. This last estimate implies that

$$\int^b |f'| dx < \infty \, ,$$

a contradiction to the fact that $f(b) = \infty$.

If $m < 0$ we analyze the inequality $w(y) \geq my + n > 0$ for values of y close to γ, to reach the contradicting conclusion that $f(c)$ is a finite point. This shows that if f is not injective on $\overline{I}$ then $f(a) = f(b)$. Again, we may assume that the common point lies at infinity, and follow the previous argument to conclude that w must be constant, and hence that $|f'| = H'$ (up to a constant factor). But if w is constant then $SH \equiv Ss$, which implies that the curvature $k \equiv 0$. Therefore the image of f is a straight line, traced at the same speed as the extremal H. We conclude that $f = H$ up to an affine change. This finishes the proof.

□

References

[1] Ahlfors, L.V., *Cross-ratios and Schwarzian derivatives in* $\mathbb{R}^n$, Complex Analysis: Articles dedicated to Albert Pfluger on the occasion of his 80th birthday, Birkhäuser Verlag, Basel, 1989, 1-15.

[2] Ahlfors, L.V., *Sufficient conditions for quasi-conformal extension.* Discontinuous groups and Riemann surfaces, Annals of Math. Studies 79 (1974), 23-29.
Trans. Amer. Math. Soc. 324 (1991), 823-842.

[3] Becker, J., *Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen*, J. Reine Angew. Math. 255 (1972), 23-43.

[4] Chuaqui, M., *Injectivity of minimal immersions and homeomorphic extensions to space*, Israel J. Math., to appear (arXiv:1508.00432v2[math.DG])

[5] Chuaqui, M. and Gevirtz, J., *Simple curves in* $\mathbb{R}^n$ *and Ahlfors' Schwarzian derivative*, Proc. Amer. Math. Soc. 132 (2004), 223-230.

[6] Chuaqui, M., Duren, P. and Osgood, B., *Univalence criteria for lifts of harmonic mappings to minimal surfaces*, J. Geom. Anal. 17 (2007), 49-74.

[7] Chuaqui, M., Duren, P. and Osgood, B., *Injectivity criteria for holomorphic curves in* $\mathbb{C}^n$, Pure Appl. Math. Quarterly 7 (2011), 223-251.

[8] Duren, P., *Univalent Functions*, Springer-Verlag, 1983.

[9] Epstein, Ch., *The hyperbolic Gauss map and quasiconformal reflections*, J. Reine Angew. Math. 380 (1987), 196-214
[10] P. Hartman, *Ordinary Differential Equations*, 2nd Edition, Birkhäuser, 1982.
[11] Nehari, Z., *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. 55 (1949), 545-551.
[12] Stowe, D., *An Ahlfors derivative for conformal immersions*, J. Geom. Anal. 25 (2015), 592-615.

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