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ON Z/3-GODEAUX SURFACES

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Abstract

We prove that Godeaux–Reid surfaces with torsion group $\mathbb{Z}/3$ have topological fundamental group $\mathbb{Z}/3$. For this purpose, we describe degenerations to stable KSBA surfaces with one $\frac{1}{4}(1,1)$ singularity, whose minimal resolution are elliptic fibrations with two multiplicity 3 fibres and one I_4 singular fibre. We study special such degenerations which have an involution, describing the corresponding Campedelli double plane construction. We also find some stable rational degenerations, some of which have more singularities, and one of which has a single $\frac{1}{9}(1,2)$ singularity, the minimal possible index for such a surface. Finally, we do the analogous study for the Godeaux surfaces with torsion $\mathbb{Z}/4$.

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1 Introduction

Godeaux surfaces are surfaces of general type with $p_g = q = 0$ and $K^2 = 1$. Such surfaces have been a classical object of study for a long time. We refer to [1] and [3] for surveys of differing vintage. In [14], Miyaoka showed among other things, that Godeaux surfaces have cyclic torsion group $H^2(X,\mathbb{Z})_{\text{tors}}$ of order at most five, and Reid [16] gave explicit constructions of the surfaces and their moduli spaces when the torsion has order $3, 4$ or 5. In the two latter cases, it is clear from Reid's construction that the topological fundamental group is isomorphic to the torsion group, but the fundamental group in the $\mathbb{Z}/3$ case has not been computed before. In this article, we refer to such surfaces as Godeaux–Reid surfaces, (or simply $\mathbb{Z}/3$-Godeaux surfaces). Our main result is that the topological fundamental group of the Godeaux–Reid surface is indeed $\mathbb{Z}/3$.

We consider KSBA stable degenerations [11] of certain Godeaux–Reid surfaces with an involution. Specifically, we study a boundary divisor corresponding to Godeaux surfaces with a single $\frac{1}{4}(1,1)$ singularity. The minimal resolution of those with an involution are elliptic fibrations with two multiplicity 3 fibres and one I_4 singular fibre, and we can compute the topological fundamental group explicitly.

Theorem 1.1. *The topological fundamental group of the Godeaux–Reid surface is* $\mathbb{Z}/3$.

The presence of an involution, which was only recently discovered by Reid [18], is crucial in the configuration of the singular fibres of the elliptic fibration, which in turn is important for the computation of the fundamental group. We also analyse the quotient, describing the Godeaux surface as a Campedelli double plane, see [2] and [10].

Theorem 1.2. *The Godeaux–Reid surface with an involution is a Campedelli double plane. That is, the quotient gives a double cover of* $\mathbb{P}^2$ *branched in a curve of degree 10 with five* $(3,3)$ *points and one 4-point.*

When the Godeaux surface is stable with a single $\frac{1}{4}(1,1)$ singularity and an involution, the branch curve breaks into two quintics, now with four $(3,3)$-points, one 4-point and a worse singularity of the form $\{(y^2-x^2)(y^2-x^4)=0\}$ in a neighbourhood of $0\in\mathbb{C}^2$; see Proposition 2.7. This splitting of the branch curve also occurs in degenerations of the Craighero–Gattazzo surface considered in [15].

We also find the quotient by this involution of the corresponding family of universal coverings, obtaining after a flip and a divisorial contraction a special family of K3 surfaces with nodes; see Subsection 3.2.

We then consider stable $\mathbb{Z}/3$-Godeaux surfaces with other types of Wahl singularity, through the phenomenon described by Kawamata in [8] of confluence of a multiple fibre and a non-multiple singular fibre in an elliptic fibration. We show (see Subsection 3.3)

Theorem 1.3. *There are KSBA stable rational* $\mathbb{Z}/3$*-Godeaux surfaces* X_1 *with a single* $\frac{1}{9}(1,2)$ *singularity,* X_2 *with one* $\frac{1}{4}(1,1)$ *and one* $\frac{1}{9}(1,2)$ *singularities, and* X_3 *with one* $\frac{1}{4}(1,1)$ *and two* $\frac{1}{9}(1,2)$ *singularities. The surfaces* X_3 *have a model as a blown up plane from a pencil of cubics.*

The minimal resolution of a stable Godeaux surface with a $\frac{1}{4}(1,1)$ must have Kodaira dimension 1 (see e.g. [20, Remark 5.3]), and so the smallest index of a Wahl singularity in a rational stable Godeaux surface is achieved by $\frac{1}{9}(1,2)$. We remark that for simply connected Godeaux surfaces, the singularity $\frac{1}{9}(1,2)$ appears in rational and non-rational stable surfaces; see [19, Table 1].

All stable degenerations mentioned above, have completely explicit descriptions in terms of Reid's original moduli parameters. In particular, we exhibit large subvarieties and even divisors in the boundary of the KSBA moduli space, and further degenerations can be studied simply by specialising the parameters.

In the last section, we make the corresponding study for $\mathbb{Z}/4$-Godeaux surfaces, where the covering surface is a complete intersection and so it is easier to understand. This also serves as a key to unlock the more complicated details of the $\mathbb{Z}/3$ case. We know of two different involutions on such surfaces, and we consider the one whose quotient is

an Enriques surface [10], [13]. We get similar results concerning stable degenerations of the double cover, including the phenomenon of the branch curve breaking into two pieces. We refer to Section 4 for details.

Gorenstein stable degenerations of Godeaux surfaces have been studied recently in a series of articles [4], [5]. Our work uses stable Godeaux surfaces of Gorenstein index two and three.

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2 Reid's construction of $\mathbb{Z}/3$-Godeaux surfaces

In [16], Reid showed that the moduli space of $\mathbb{Z}/3$-Godeaux surfaces is irreducible and 8-dimensional, by constructing surfaces Y' of general type with $p_g = 2$, $K^2 = 3$ and with fixed point free $\mathbb{Z}/3$-action σ, such that the quotient $Y'/\sigma = X'$ is the Godeaux surface. We refer to Y' as the covering surface of X'. More recently, Reid [17, §7], [18] clarified the construction of Y' using a parallel unprojection and a 4-dimensional key variety. We briefly describe this construction below.

2.1 The key variety $W \subset \mathbb{P}(1^3, 2^3, 3^3)$

Let $W \subset \mathbb{P}(1^3, 2^3, 3^3)$ be the 4-dimensional $\mathbb{Z}/3$-invariant variety defined by 9 equations R_0, R_1, R_2, S_0, S_1, S_2, T_0, T_1, T_2 formed by taking the orbits of

$$\begin{aligned} R_0\colon &\quad -x_0z_0 + y_1y_2 - r_0x_1x_2 = 0\\ S_0\colon &\quad -y_0z_0 + r_1x_2y_1 + r_2x_1y_2 + sx_1x_2 = 0\\ T_0\colon &\quad -z_1z_2 + r_0y_0^2 + sx_0y_0 + r_1r_2x_0^2 = 0 \end{aligned}$$

under the $\mathbb{Z}/3$-action σ. The coordinates x_i, y_i, z_i for $i = 0, 1, 2$ have respective weights $1, 2, 3$, and σ acts on them by cyclic permutation of indices (012):

$$\sigma\colon x_i \mapsto x_{\sigma(i)},\ y_i \mapsto y_{\sigma(i)},\ z_i \mapsto z_{\sigma(i)}.$$

Moreover, r_i (respectively s) are weighted homogeneous of degree 2 (resp. 3) and chosen so that σ permutes the indices of r_i (resp. fixes s). Let A denote the restriction of the ample generator of the ambient space to W, that is, $\mathcal{O}_W(A) = \mathcal{O}_W(1)$. Then according to [18], W is a Fano 4-fold with $K_W = -3A$, $A^3 = 1$ and $3 \times \frac{1}{3}(1, 1, 2, 2)$ points.

Theorem 2.1 (Reid [16, 18])**.** *Consider the intersection*

$$Y' = W \cap (x_0 + x_1 + x_2 = z_0 + z_1 + z_2 = 0)$$

of W with the σ-invariant weighted linear subspace $L\colon (\sum_i x_i = \sum_i z_i = 0)$. For general choices of r_i, s, Y' in $L \simeq \mathbb{P}(1^2, 2^3, 3^2)$ is the canonical model of a surface of general type

with $p_g = 2$, $K^2 = 3$. Moreover, σ acts freely on Y', and the quotient $Y'/\sigma = X'$ is a Godeaux surface X'.

Reid proves that for general choices of r_i and s, the surface Y' is smooth and irreducible (this is checked by computer), and the action of σ on Y' is fixed point free. Thus the quotient surface X' is a Godeaux surface with torsion $\mathbb{Z}/3$.

Proposition 2.2 (Reid [16, §3])**.** *The coarse moduli space $\mathbb{Z}/3$-Godeaux surfaces is irreducible, unirational of dimension 8, covered by the 9-dimensional parameter space given by the following forms for r_i and s*

$$\begin{aligned} r_0 = a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2 + b_0y_0 + b_1y_1,\ r_1 = \sigma(r_0),\ r_2 = \sigma(r_1),\\ s = c_2(x_0^2x_1 + x_1^2x_2 + x_2^2x_0) + c_3(x_0^2x_2 + x_1^2x_0 + x_2^2x_1)\\ +d_2(x_0y_1 + x_1y_2 + x_2y_0) + d_3(x_0y_2 + x_1y_0 + x_2y_1). \end{aligned}$$

Proof. Reid proves that every $\mathbb{Z}/3$-Godeaux surface is obtained by his original construction in [16]. This construction is equivalent to the key variety construction by a cyclotomic change of coordinates [17, 18]. We simply clean up the possible parameters appearing in r_i, s.

It is easy to write out the general forms for r_i and s:

$$r_0 = \sum_{i\le j} a_{ij}x_ix_j + \sum_i b_iy_i,\ r_1 = \sigma(r_0),\ r_2 = \sigma(r_1),$$

$$s = c_1\Sigma(x_0^3) + c_2\Sigma(x_0^2x_1) + c_3\Sigma(x_0^2x_2) + c_4x_0x_1x_2 + d_1\Sigma(x_0y_0) + d_2\Sigma(x_0y_1) + d_3\Sigma(x_0y_2),$$

where $\Sigma(m)$ denotes the orbit sum $m + \sigma(m) + \sigma^2(m)$ of m under the action of $\mathbb{Z}/3$. For s, there are only four independent sections of the invariant eigenspace $H^0(3K_{Y'})^{\mathrm{inv}}$ and we just choose them in a nice way.

Now, the terms of r_i involving $x_0 = -x_1 - x_2$ are redundant. Moreover, we can force $b_2 = 0$ by replacing y_i with $y_i + b_2x_ix_{i+1}$ and z_i with $z_i + b_2x_{i+2}y_{i+1}$ for $i = 0, 1, 2$. Indeed, it is clear that any extra terms arising in r_i and s as a result of this substitution can be absorbed by the other parameters in r_i and s. Since we have chosen σ-invariant substitutions, it suffices to show that b_2 no longer appears as a free parameter in R_0, S_0, T_0. Writing R_0' for R_0 after making the substitution, we get

$$\begin{aligned} R_0' &= -x_0(z_0 + b_2x_2y_1) + (y_1 + b_2x_1x_2)(y_2 + b_2x_2x_0) - r_0x_1x_2\\ &= -x_0z_0 + y_1y_2 - (r_0 - b_2y_2 - b_2^2x_0x_2)x_1x_2. \end{aligned}$$

So writing $r_0' = r_0 - b_2y_2 - b_2^2x_0x_2$, we see that R_0 no longer involves b_2 as a free parameter. We define r_1', r_2' to be permutations of r_0'. It is convenient to consider $S_0 - b_2x_1R_0$ rather than S_0 directly:

$$\begin{aligned} S_0' - b_2x_1R_0' &= -(y_0 + b_2x_0x_1)(z_0 + b_2x_2y_1) + r_1x_2(y_1 + b_2x_1x_2) + r_2x_1(y_2 + b_2x_2x_0)\\ &\quad + sx_1x_2 - b_2x_1(-x_0z_0 + y_1y_2 - r_0'x_1x_2)\\ &= -y_0z_0 + r_1'x_2y_1 + r_2'x_1y_2 + (s + b_2(x_0r_2 + x_1r_0 + x_2r_1 - b_2^2x_0x_1x_2))x_1x_2. \end{aligned}$$

Again, writing $s' = s + b_2(x_0r_2 + x_1r_0 + x_2r_1 - b_2^2x_0x_1x_2)$, we see that S_0 is now independent of b_2. Using the above substitutions, a direct computation shows that T_0 is independent of b_2.

Finally, a coordinate transformation of the form $x_i \mapsto \lambda x_i$, $y_i \mapsto y_i$, $z_i \mapsto \lambda^{-1}z_i$ for parameter λ reduces us to an 8-dimensional moduli space, because the transformation acts by scaling the parameters in r_i and s. □

We write $\mathcal{M}$ for the KSBA moduli space of $\mathbb{Z}/3$-Godeaux surfaces [11]. We do not distinguish between $\mathcal{M}$ and the moduli space of covering surfaces.

Recall from [16] that the blowup $\phi_{Y'}\colon \widetilde{Y}' \to Y'$ of the three basepoints of $|K_{Y'}|$ has a model inside the scroll $\mathbb{F} = \mathrm{Proj}_{\mathbb{P}^1}(\mathcal{O} \oplus 3\mathcal{O}(-2))$ with base and fibre coordinates (x_1, x_2) and (t, y_0, y_1, y_2) respectively. Ignoring the t variable for simplicity (see proof of Lemma 2.3 below and [16, §3]), the five defining equations of $\widetilde{Y}'$ are

$$\begin{aligned} f&\colon x_1x_2R_0 + x_2x_0R_1 + x_0x_1R_2 = 0\\ g&\colon x_0S_0 - y_0R_1 \equiv x_1S_1 - y_1R_2 \equiv x_2S_2 - y_2R_0 = 0\\ h_0&\colon x_1x_2S_0 + y_0x_2R_1 + y_0x_1R_2 = 0 \end{aligned}$$

and $h_1 = \sigma(h_0)$, $h_2 = \sigma(h_1)$. In fact, the general fibre of $\widetilde{Y}' \to \mathbb{P}^1$ is a weighted complete intersection $f_4 = g_6 = 0$ in $\mathbb{P}(1,2,2,2)$, and h_0 (respectively h_1, h_2) is used only to cut out the fibre over the distinguished point $P_0 = (0,1)$, (resp. $P_1 = (1,1)$, $P_2 = (0,1)$) of the base $\mathbb{P}^1$.

Notation For brevity, we denote a section $\alpha_0x_0 + \alpha_1x_1 + \alpha_2x_2$ of $|K'_Y|$ by the triple $(\alpha_0, \alpha_1, \alpha_2)$ such that $\alpha_0 + \alpha_1 + \alpha_2 = 0$.

Lemma 2.3. *Let Y' be a surface in $\mathcal{M}$ and suppose C is a curve in $|K_{Y'}|$ corresponding to the section $(\alpha_0, \alpha_1, \alpha_2)$ of $H^0(K_{Y'})$, where $\alpha_0 + \alpha_1 + \alpha_2 = 0$. Then C is isomorphic to the following complete intersection of a quadric and a cubic in $\mathbb{P}^3$ with coordinates t^2, y_0, y_1, y_2:*

(i) If the α_i are all nonzero,

$$\begin{gathered} \alpha_1\alpha_2(y_1y_2 - r_0x_1x_2) + \alpha_0\alpha_2(y_0y_2 - r_1x_0x_2) + \alpha_0\alpha_1(y_0y_1 - r_2x_0x_1) = 0,\\ -y_0y_1y_2 + r_0x_1x_2y_0 + r_1x_0x_2y_1 + r_2x_0x_1y_2 + sx_0x_1x_2 = 0 \end{gathered}$$

where x_i, r_i and s are evaluated at $x_i = \alpha_i t$.

(ii) If α_0 vanishes,

$$\begin{gathered} y_1y_2 - r_0x_1x_2 = 0,\\ y_0^2(y_1 - y_2) - x_1(r_1x_2y_1 + r_2x_1y_2 + sx_1x_2) = 0 \end{gathered}$$

where now x_i, r_i and s are evaluated at $x_0 = 0$, $x_1 = t$, $x_2 = -t$.

Curves corresponding to $\alpha_1 = 0$ and $\alpha_2 = 0$ are obtained in a similar manner, or from the action of σ on C_{α_0}.

Proof. Substitute $x_i = \alpha_i t$ for $i = 0, 1, 2$ in equations f, g. For reasons of bihomogeneity, we can divide f by t^2, and this gives the two relations in part (i). The h_i are redundant here because of the syzygies

$$y_0 f + x_1 x_2 g \equiv x_0 h_0,\ y_1 f + x_0 x_2 g \equiv x_1 h_1,\ y_2 f + x_0 x_1 g \equiv x_2 h_2.$$

For part (ii), examining the same syzygies, we see that if α_0 vanishes, then we must take f and h_0, because now g, h_1, h_2 are redundant. □

Remark 2.4. The automorphism σ acts on the base of $Y' \to \mathbb{P}^1$ and there are two σ-invariant curves F_ω and F_{ω^2} in $|K_{Y'}|$ corresponding to $(1, \omega, \omega^2)$ and $(1, \omega^2, \omega)$ respectively, where ω is a primitive cube root of unity. The action of σ on these curves is $(t, y_0, y_1, y_2) \mapsto (\omega^2 t, y_1, y_2, y_0)$. If we further assume that Y' has $\mathfrak{S}_3$ symmetry (see Section 2.2 below), then F_ω has equations

$$\begin{aligned} y_1y_2 + \omega^2 y_0y_2 + \omega y_0y_1 - b_0t(y_0 + \omega y_1 + \omega^2 y_2) + 3(a_{11} - a_{12})t^2 = 0 \\ -y_0y_1y_2 + b_0t(y_0^2 + \omega^2 y_1^2 + \omega y_2^2) + (a_{12} - a_{11} - d_2)t^2(y_0 + \omega y_1 + \omega^2 y_2) - 3c_2t^3 = 0 \end{aligned} \tag{2.1}$$

and F_{ω^2} is similar, with ω and ω^2 interchanged.

2.2 A family with $\mathfrak{S}_3$ symmetry and quotients

As observed in [18], there is a special subfamily $\mathcal{M}^s \subset \mathcal{M}$ for good choices of r_i and s, such that the general surface Y' in $\mathcal{M}^s$ has a larger automorphism group $\mathfrak{S}_3$ rather than just $\mathbb{Z}/3$. This family is defined by

$$\mathcal{M}^s\colon (a_{11} = a_{22},\ b_1 = 0,\ c_2 = c_3,\ d_2 = d_3) \subset \mathcal{M}.$$

and $\mathfrak{S}_3$ acts by permutation on the indices of x_i, y_i, z_i. The family $\mathcal{M}^s$ is irreducible and 4-dimensional. The action of $\mathfrak{S}_3$ on Y' descends to an involution on the $\mathbb{Z}/3$-Godeaux surface.

Remark 2.5. From now on, we will use the superscript s to denote the intersection $\mathcal{D}^s = \mathcal{D} \cap \mathcal{M}^s$ of a stratum $\mathcal{D} \subset \mathcal{M}$ with the $\mathfrak{S}_3$-symmetric family $\mathcal{M}^s$.

Let $\tau_0 = (12)$, $\tau_1 = (02)$, $\tau_2 = (01)$ be the involutions in $\mathfrak{S}_3$ acting on Y' in $\mathcal{M}^s$. They are conjugate under the action of σ, so we use $\tau = \tau_0$ for our computations. The action of τ on $\mathbb{P}(1^3, 2^3, 3^3)$ has eigenspace decomposition

Degree	+	−
1	$x_0,\ x_1 + x_2$	$x_1 - x_2$
2	$y_0,\ y_1 + y_2$	$y_1 - y_2$
3	$z_0,\ z_1 + z_2$	$z_1 - z_2$

The relevant part of the fixed locus of τ on the ambient space breaks into two pieces $\mathrm{Fix}(\tau) = \mathrm{Fix}_1 \cup \mathrm{Fix}_2$ and here we have to be a bit careful because of the weighted $\mathbb{C}^*$-action:

$$\begin{aligned} \mathrm{Fix}_1 &= (x_1 - x_2 = y_1 - y_2 = z_1 - z_2 = 0), \\ \mathrm{Fix}_2 &= (x_0 = x_1 + x_2 = y_1 - y_2 = z_1 + z_2 = z_0 = 0). \end{aligned}$$

The intersection $Y' \cap \mathrm{Fix}_1$ gives five isolated fixed points of τ_0 contained in the curve $(-2,1,1) \in |K_{Y'}|$. Indeed, according to Lemma 2.3, the intersection of $F_{(-2,1,1)}$ with $y_1 = y_2$ on $\widetilde{Y}'$ is six points, but one of these is the intersection of F with the (-1)-curve preimage of a basepoint of $|K_{Y'}|$, and this basepoint is actually contained in $Y' \cap \mathrm{Fix}_2$. The following proposition treats $Y' \cap \mathrm{Fix}_2$.

Proposition 2.6 (cf. [10])**.** *The fixed curve R of the involution τ on X' is a smooth rational curve. There are smooth curves R_0 and R_0' in Y' with $g(R_0) = 0$ and $g(R_0') = 1$ which intersect transversally at 4 points and $R_0 + R_0' \in |K_{Y'}|$. Moreover, $R_0^2 = -3$ and $R_0'^2 = -2$. The image of R_0 in X' is R and that of R_0' is R'.*

Proof. The preimage of R on Y' is the σ-orbit of the intersection $R_0 = \mathrm{Fix}_2 \cap Y'$. By the above eigenspace decomposition, we see that R_0 is a component of the curve C_0 in $|K_{Y'}|$ corresponding to $\alpha_0 = 0$. Since Y' is in $\mathcal{M}^s$, we see by Lemma 2.3 that C_0 is the intersection of the following quadric and cubic in $\mathbb{P}^3$:

$$C_0 \colon (2a_{11} - a_{12})t^4 + b_0 y_0 t^2 + y_1 y_2 = ((2a_{11} - a_{12} - d_2)t^4 + y_0^2 + b_0 t^2(y_1 + y_2))(y_1 - y_2) = 0.$$

The cubic is clearly reducible, so $C_0 = R_0 + R_0'$ has two components, where R_0 is a nonsingular conic, and R_0' is an intersection of two quadrics in $\mathbb{P}^3$ (an elliptic curve). The slice $\mathrm{Fix}_2 \cap Y'$ cuts out R_0. Note also that R_0 and R_0' intersect in four points, thus C_0 is a curve of arithmetic genus 4 as expected.

For any given fibre of $\widetilde{Y}' \to \mathbb{P}^1$, the locus $(t = 0)$ gives the intersection of that fibre with the three (-1)-curves in $\widetilde{Y}'$. Thus we see from the above equations, that R_0 contains one basepoint and R_0' contains two. This implies that $K_{Y'} \cdot R_0 = 1$, $K_{Y'} \cdot R_0' = 2$ and the above stated self-intersection numbers follow from the adjunction formula. □

Proposition 2.7. *The surface $Z' = X'/\tau$ is rational with five A_1 singularities. If $\psi_{X'} \colon X'' \to X'$ is the blow-up of the five points fixed by τ, then the quotient $\pi_{Z''} \colon X'' \to Z''$ is such that Z'' is the minimal resolution of Z', $K_{Z''}^2 = -2$, and the divisor $3K_{Z''} + B$ is nef, where $B := \pi_{Z''}(\psi^*(R))$. The linear system $|3K_{Z''} + B|$ is a base point free pencil of rational curves. There is a sequence of 11 blowdowns from Z'' to $\mathbb{P}^2$ such that the image of B is an (irreducible) plane curve of degree 10 with five $(3,3)$-points and one 4-point (i.e., a Campedelli curve).*

Proof. Let $E_1, \ldots, E_5$ be the exceptional curves of $\psi_{X'}$, and let $C_i = \pi_{Z''}(E_i)$. The double cover formula reads

$$K_{X''} \equiv \pi_{Z''}^*\Big(K_{Z''} + \frac{1}{2}(B + C_1 + \cdots + C_5)\Big).$$

With this and the formula for the blow-up, one computes $\pi_{Z''}^*(3K_{Z''} + B) \equiv \psi^*(R')$, because $3K_{X'} \equiv R + R'$. Since $R'^2 = 0$, we have that $3K_{Z''} + B$ is nef. The surface Z'' is the minimal resolution of Z', and one computes $K_{Z''}^2 = -2$ with the canonical formula above.

Then we satisfy the requirements of [2, Prop. 3.9], and we belong to the surfaces analyzed in [2, §6]. In particular, by [2, Lemma 6.1], the linear system $|3K_{Z''} + B|$ is a base point free pencil of rational curves. The curves $C_1, \ldots, C_5$ are components in the

fibres. Since the canonical class of X' is ample and by the possibilities in [2, Cor. 6.2], the curves C_i are in distinct fibres. Moreover, each of these fibres is a chain of three rational curves: $A_{i,1} + C_i + A_{i,2}$ where the central curve is C_i, and the $A_{i,j}$ are (-1)-curves. One checks that B is a 6-section in this pencil, and intersects each $A_{i,j}$ in three points.

We notice that the image of the linear system $|K_{Y'}|$ in Z' defines a pencil of genus 4 curves with two base points (at the same point). Let Δ in Z'' be the image of $F_\omega + F_{\omega^2}$, where F_ω and F_{ω^2} are the two genus 4 curves in $|K_{Y'}|$ fixed by the action of σ, and permuted by τ. Then Δ is a smooth genus two curve. Let Γ in Z'' be the image of the curve in $|K_{Y'}|$ which contains the five fixed points of τ (see description of fixed loci by τ above). Then Γ is a nodal curve of arithmetic genus 2. Notice that Δ intersects each of the $A_{i,j}$ at one point transversally, and $\Delta \cdot C_i = 0$. Also, Γ intersects each $A_{i,j}$ at one point transversally, and $\Gamma \cdot C_i = 1$. Notice that $B^2 = -6$, $\Delta^2 = 2$, and $\Gamma^2 = 2$.

Let $Z'' \to \mathbb{F}_m$ be the blow-down of one of $A_{i,1}$ or $A_{i,2}$ and then C_i for each i, so blow-down ten times, where $\mathbb{F}_m$ is a Hirzebruch surface. If Γ_0 is the $(-m)$-curve in $\mathbb{F}_m$ and F is a fibre, then one verifies

$$\Delta' \sim 2\Gamma_0 + (m+3)F, \quad \Gamma' \sim 3\Gamma_0 + \frac{1}{2}(9+3m)F, \quad B' \sim 6\Gamma_0 + (3m+7)F$$

where Δ', Γ', B' are the images of Δ, Γ, B. Since none of these curves are equal to Γ_0, we have nonnegative intersection with Γ_0. Using that, one easily shows $m = 1$. This implies that $\Gamma_0 \cdot B' = 4$, $\Gamma_0 \cdot \Delta' = 2$, and $\Gamma_0 \cdot \Gamma' = 3$. Therefore, in $\mathbb{P}^2$, the image of B is an irreducible degree 10 curve with five $(3,3)$-points and one 4-point, the image of Δ is a degree 4 curve with a node, and the image of Γ is a degree 6 curve with six nodes and one triple point. □

Proposition 2.8. *The minimal model of the quotient Y'/τ is a K3 surface with five A_1 singularities.*

Proof. We write $\pi_{V'} \colon Y' \to V'$ for the quotient map. By the adjunction formula for double covers, $K_{Y'} = \pi_{V'}^* K_{V'} + R_0$, by Proposition 2.6 (see also [10]), we know that $K_{Y'} \cdot R_0 = 1$ and $R_0^2 = -3$. Thus $\pi_{V'}^* K_{V'}^2 = -2$, which implies that $K_{V'}^2 = -1$. The residual component R_0' of C_0 is also τ-invariant, and $\pi_{V'}|_{R_0'}$ is ramified in the four points where R_0' intersects R_0. Since $3 = C_0^2 = (R_0 + R_0')^2 = -3 + 2\cdot 4 + {R_0'}^2$, we see that ${R_0'}^2 = -2$, and hence $\pi_{V'}(R_0')$ is a (-1)-curve. Thus $\pi_{V'}(R_0')$ is contracted to get $V'_{\min}$. Since $R_0 + R_0' = C \in |K_{Y'}|$, it follows from the projection formula that $K_{V'_{\min}}$ is numerically trivial, and we also know that $p_g(V'_{\min}) = 1$, because we computed the invariants of τ above. Thus $V'_{\min}$ is a K3 surface. □

In the right column of Figure 1, we show the situation described in this section. In the next section we will describe and use the degenerated situation shown at the left side of Figure 1.

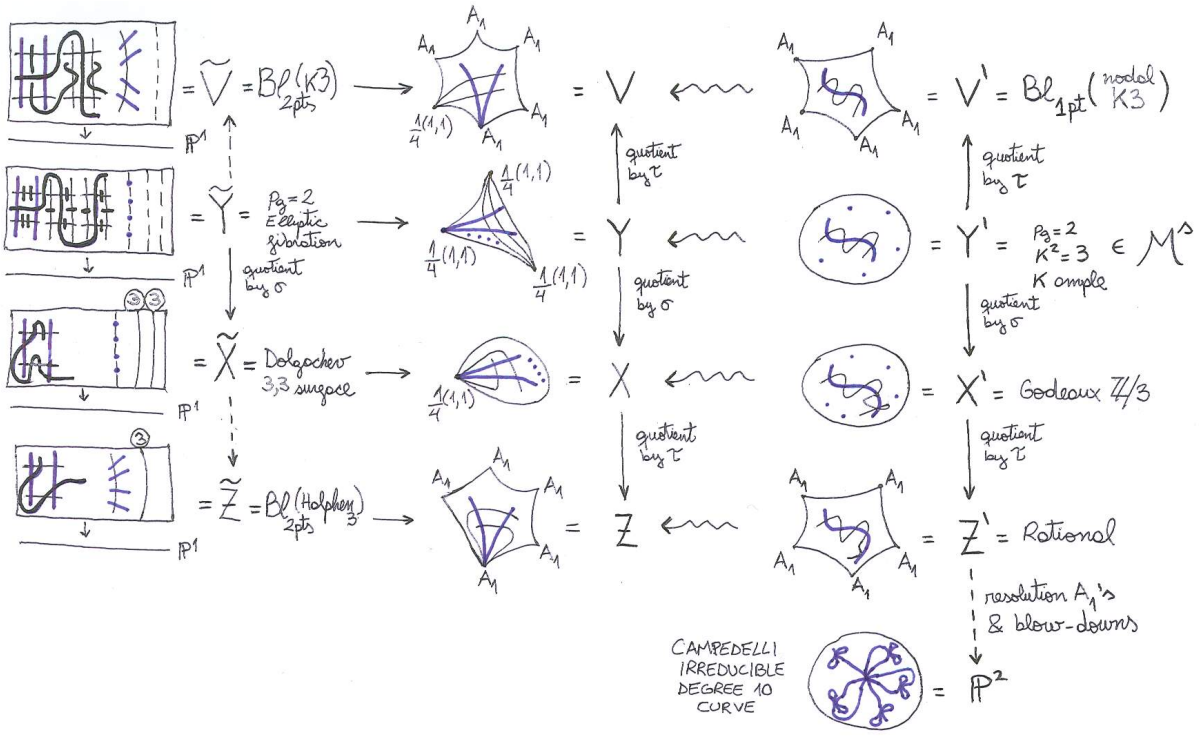

Figure 1: Big picture for Sections 2 and 3

3 A stable $\mathbb{Z}/3$-Godeaux with $\frac{1}{4}(1,1)$ singularity

3.1 Covering with three $\frac{1}{4}(1,1)$ and Dolgachev $3,3$ surface

We now consider the divisor $\mathcal{Y}$ in $\mathcal{M}$ defined by $b_0 = 0$, previously considered in [9]. According to Proposition 2.2, $\mathcal{Y}$ is covered by an 8-dimensional parameter space divided out by a $\mathbb{C}^*$-action. Moreover, the space of admissible parameters is connected and so $\mathcal{Y}$ is irreducible.

Lemma 3.1. *The general member Y in $\mathcal{Y}$ has a free σ-action and $3 \times \frac{1}{4}(1,1)$ points forming an orbit under σ, one at each of the coordinate points P_{y_i}. After taking the quotient by σ, we get a stable $\mathbb{Z}/3$-Godeaux surface with a single $\frac{1}{4}(1,1)$ singularity.*

Proof. We work in a neighbourhood of $P = P_{y_1}$. Relations R_0, R_2 and S_1 serve to eliminate variables y_0, y_2, z_1 in favour of local coordinates x_1, x_2, z_2. Thus P is a hypersurface singularity inside the $\frac{1}{2}(1,1,1)$ point induced by the $\mathbb{C}^*$-action on the ambient space. The tangent cone of $T_1 - b_1 R_0$ is a quadric of rank 3 in the local coordinates, and hence P is a $\frac{1}{4}(1,1)$ singularity.

The nonsingularity and irreducibility of Y follows by a computer calculation for choices of parameters. The fact that σ is fixed point free is a standard computation. □

Under the degeneration $Y' \rightsquigarrow Y$, the three basepoints of $|K_{Y'}|$ deform to the $\frac{1}{4}(1,1)$ singular points P_0, P_1, P_2 of Y. The resolution $\phi_Y \colon \widetilde{Y} \to Y$ is given by a single blow up

at each of P_0, P_1, P_2 with exceptional (-4)-curves E_0, E_1, E_2. Indeed,

$$K_{\widetilde{Y}} \equiv \phi_Y^*(K_Y) - \frac{1}{2}(E_0 + E_1 + E_2),$$

and $\widetilde{Y}$ has an elliptic fibration over $\mathbb{P}^1$ given by the vector space of global sections

$$H^0(K_{\widetilde{Y}}) = \langle x_0, x_1, x_2 \rangle / (\sum x_i = 0),$$

where we abuse notation to write $x_i = \phi_Y^*(x_i)$. The image of a fibre under ϕ_Y is a curve of arithmetic genus 4 with three nodes at basepoints P_0, P_1, P_2.

By the Kodaira formula for the canonical class, $K_{\widetilde{Y}} \sim (\chi(\mathcal{O}_{\widetilde{Y}}) - 2)F + \gamma F$, where F is a fibre and γ is a term which is zero if and only if there are no multiple fibres. Since $\chi(\mathcal{O}_{\widetilde{Y}}) = 3$ and the fibration is induced by $|K_{\widetilde{Y}}|$, we see that there are no multiple fibres, and the E_i are 2-sections of the fibration. The automorphism σ acts on the base of the fibration, and the fibres corresponding to $(1, \omega, \omega^2)$ and $(1, \omega^2, \omega)$ are σ-invariant. According to Lemma 2.3 above, each fibre has a birational model as the intersection of a quadric and a cubic in $\mathbb{P}^3$.

We consider now the special case when Y has an involution.

Proposition 3.2. *The general surface Y in $\mathcal{Y}^s$ with $\mathfrak{S}_3$-action has three fibres of type I_4 forming an orbit under σ, 24 I_1 fibres comprising eight orbits, and the two σ-invariant fibres F_ω and F_{ω^2} are nonsingular.*

Proof. Consider the fibre F_0 corresponding to $(0, 1, -1)$ in $|K_{\widetilde{Y}}|$. Since $b_0 = 0$ for Y in $\mathcal{Y}^s$, we use the proof of Proposition 2.6 to write $\psi_Y(F_0)$ as

$$Lt^4 + y_1 y_2 = 0, \ (Mt^4 - y_0^2)(y_1 - y_2) = 0,$$

where for brevity we write $L = 2a_{11} - a_{12}$ and $M = 2a_{11} - a_{12} - d_2$. A short calculation shows that by generality assumption, F_0 breaks into four components given by

$$\begin{aligned} &F_0^1 \colon \lambda t^2 + iy_1 = y_1 - y_2 = 0, && F_0^2 \colon \lambda t^2 - iy_1 = y_1 - y_2 = 0, \\ &F_0^3 \colon \mu t^2 + y_0 = Ly_0^2 + My_1y_2 = 0, && F_0^4 \colon \mu t^2 - y_0 = Ly_0^2 + My_1y_2 = 0 \end{aligned}$$

where $\lambda^2 = L$, $\mu^2 = M$. From the equations, we see that $P_0 \in F_0^1, F_0^2$, whereas $P_1, P_2 \in F_0^3, F_0^4$.

Thus F_0 is an I_4 fibre $F_0 = F_0^1 + F_0^3 + F_0^2 + F_0^4$ where F_0^1 and F_0^2 are opposite sides, and the 2-sections intersect F_0 as follows: E_0 intersects F_0^1 and F_0^2 transversally, while E_1 and E_2 intersect F_0^3 and F_0^4 transversally. See Figure 1.

Standard Euler number considerations predict the general $\widetilde{Y}$ has a further 24 nodes in fibres. We compute directly that for general Y, these correspond to 24 I_1 fibres. Indeed, let S be the disjoint union $S = \bigsqcup\{C : C \in |K_Y|^0\}$, where $|K_Y|^0$ means we ignore the three I_4 fibres where one of $\alpha_i = 0$. Then $S \subset \mathbb{P}^3 \times (\mathbb{P}^1 - \{3 \text{ points}\})$ is the relative sextic complete intersection of quadric and cubic described in Lemma 2.3(i). Consider the affine piece S_t where t is nonzero, to ignore the three lines of nodes in S. The Jacobian subscheme J of S is codimension 2 in the ambient space, defined by 2×2 minors of the Jacobian matrix

which has entries of degree $\left(\begin{smallmatrix}2&2&2\\1&1&1\end{smallmatrix}\right)$. Thus J has relative degree 7. Intersecting with S_t decreases this degree by 3 because of the three nodes on every curve, and so the expected degree of $S_t \cap J$ is $6 \times (7-3) = 24$. We check using a computer that this intersection is indeed reduced, and comprises 24 points on distinct fibres.

A direct computation using equations (2.1) shows that F_ω and F_{ω^2} are nonsingular for general Y. □

Corollary 3.3. *The quotient surface $\widetilde{X} = \widetilde{Y}/\sigma$ is a Dolgachev $(3,3)$-surface, that is an elliptic surface with two fibres of multiplicity 3. It also has one I_4 fibre, eight I_1 fibres and a 6-section E.*

Theorem 3.4. *The topological fundamental group of a $\mathbb{Z}/3$-Godeaux surface is $\mathbb{Z}/3$.*

Proof. The action of σ in a $\mathbb{Q}$-Gorenstein degeneration $Y' \rightsquigarrow Y$ over a disc is such that the quotient is a $\mathbb{Q}$-Gorenstein degeneration $X' \rightsquigarrow X$ over a disc. From X to X', we are $\mathbb{Q}$-Gorenstein smoothing the $\frac{1}{4}(1,1)$ singularity of X. On the other hand, the resolution of X is a Dolgachev $(3,3)$-surface $\widetilde{X}$, and so X has topological fundamental group $\pi_1(X)$ isomorphic to $\mathbb{Z}/3$ (see e.g. [3] or [6], Chapter II, Theorem 2.3).

We now follow the strategy in [12]. Using Seifert–van-Kampen theorem, we have that $\pi_1(X) \simeq \Big(\pi_1(\widetilde{X} \setminus E) * \pi_1(U)\Big)/\langle\gamma\rangle_n$ where U is a small neighborhood of the singularity in X, and so $\pi_1(U)$ is trivial, γ is a loop around E in $\widetilde{X} \setminus E$, and $\langle\gamma\rangle_n$ is the smallest normal subgroup in $\pi_1(\widetilde{X} \setminus E)$ containing γ. But by Proposition 3.2, there is a $\mathbb{P}^1$ from the I_4 fibre in $\widetilde{X}$, which intersects the 6-section E transversally at one point, and so γ is trivial in $\pi_1(\widetilde{X} \setminus E)$. Therefore $\pi_1(\widetilde{X} \setminus E) \simeq \mathbb{Z}/3$. Now, since $X' \rightsquigarrow X$ is $\mathbb{Q}$-Gorenstein smoothing, the surface X' is homeomorphic to $\widetilde{X} \setminus E$ union the Milnor fibre M_f of $\frac{1}{4}(1,1)$ corresponding to the smoothing, glued along the corresponding link L. So we apply Seifert–van-Kampen theorem again to X' using that decomposition. Since the generator γ of $\pi_1(L)$ is trivial in $\pi_1(\widetilde{X}\setminus E)$ and the inclusion induces $\pi_1(L) = \mathbb{Z}/4 \twoheadrightarrow \pi_1(M_f) = \mathbb{Z}/2$, we obtain that $\pi_1(X') \simeq \mathbb{Z}/3$. Similarly, one can use the same strategy with the $\mathbb{Q}$-Gorenstein degeneration $Y' \rightsquigarrow Y$ and the elliptic fibration $\widetilde{Y}$, to prove that Y' is simply-connected, and so $\pi_1(X') \simeq \mathbb{Z}/3$. □

3.2 Stable Campedelli double plane, K3 quotient and its flipped family

As before, since the involutions τ_i are conjugate under the action of σ, we use $\tau = \tau_0$ for our computations. The following explains how the involution descends to $\widetilde{X}$.

Proposition 3.5. *The action of τ on $\widetilde{X}$ has two invariant fibres, and swaps the two fibres of multiplicity three. One of the invariant fibres is the I_4 fibre, of which two components are fixed pointwise. The other invariant fibre is nonsingular and contains four isolated fixed points.*

Proof. From Proposition 3.2, the intersections of Fix_i with Y are:

$$Y \cap \mathrm{Fix}_1 = P_0 \cup \text{four points on the fibre } F_{(-2,1,1)},$$
$$Y \cap \mathrm{Fix}_2 = F_0^1 + F_0^2 =: R_0.$$

We can check that $F_{(-2,1,1)}$ is nonsingular directly using Lemma 2.3. Moreover, the two σ-invariant fibres are exchanged by τ. The involution must act nontrivially on the 6-section E, fixing two points which are the intersection points of E with Fix_2 from above.

The 2-sections E_1 and E_2 on $\widetilde{Y}$ form an orbit under τ, while E_0 is invariant with two fixed points at the intersection with the I_4 fibre. □

Corollary 3.6. *The quotient $\widetilde{X}/\tau$ has four A_1 singularities on one fibre. Its resolution is the blow-up at two points of a relatively minimal rational elliptic surface with one fibre of multiplicity three. The singular fibres are I_2, I_0^*, and four I_1. Thus a contraction of $\widetilde{X}/\tau$ to $\mathbb{P}^2$ gives a Halphen pencil of index 3.*

We now describe the corresponding stable Campedelli double plane which is an analogue of Proposition 2.7. We introduce some notation. Let $\widehat{X} \to \widetilde{X}$ be the blow-up at the four points fixed by τ, and let $\widetilde{Z}$ be the minimal resolution of $\widetilde{X}/\tau$. Let $\pi_{\widetilde{Z}}\colon \widehat{X} \to \widetilde{Z}$ be the quotient by τ. Let B_1, B_2 be the image by $\pi_{\widetilde{Z}}$ of the fixed curve by τ in $\widetilde{X}$, and let C_1, C_2, C_3, C_4 be the image of the exceptional curves of $\widehat{X} \to \widetilde{X}$. Thus, $B_1 + B_2 + C_1 + \ldots + C_4$ is the branch curve of $\pi_{\widetilde{Z}}$. Let $A_1 + A_2 + B_1 + B_2$ be the image of the I_4 fibre by $\pi_{\widetilde{Z}}$, and so A_1, A_2 and B_1, B_2 are opposite sides of the new I_4, and $A_i^2 = -1$, $B_i^2 = -4$. Let E_0 be the image by $\pi_{\widetilde{Z}}$ of the (-4)-curve from E in $\widetilde{X}$. Let Γ be the image by $\pi_{\widetilde{Z}}$ of proper transform of the fibre in $\widetilde{X}$ which contains the four fixed points by τ. Finally, let Δ be the image by $\pi_{\widetilde{Z}}$ of the two multiple fibres in $\widetilde{X}$.

Proposition 3.7. *The linear system defined by $A_1 + E_0 + A_2 \sim 3K_{\widetilde{Z}} + E_0 + B_1 + B_2$ defines a genus 0 fibration $f\colon \widetilde{Z} \to \mathbb{P}^1$, which pulls back to $\widetilde{X}$ as a genus 2 fibration. Each C_i belongs to one fibre of f which is formed by (-1)-curves $A_{i,1}$, $A_{i,2}$ and C_i. The blow-down of $A_1, A_{1,1}, \ldots, A_{4,1}$ and $E_0, C_1, \ldots, C_4$ is the blow-up at one point of $\mathbb{P}^2$. After blowing-down to $\mathbb{P}^2$ the curves B_1, B_2 become two quintics such that $B_1 + B_2$ has four $(3,3)$-points, one 4-point, and one singular point of the form $\{(y^2 - x^2)(y^2 - x^4) = 0\}$ locally at $(0,0) \in \mathbb{C}^2_{x,y}$. We also realize Δ as a quartic with two nodes, and Γ as a sextic with two triple points and four double points.*

Proof. The linear system $|A_1 + E_0 + A_2|$ defines a rational fibration in $\widetilde{Z}$. The curve $B_1 + B_2$ is a 6-section, and so the pull back is a fibration of genus 2 curves. Notice that the curves C_i are in fibres of $|A_1 + E_0 + A_2|$. The options for the irreducible components of the fibre containing C_i are as in [2, Cor. 6.2]. In $\widetilde{X}$ a (-2)-curve has to intersect the 6-section E since K_X is ample. Therefore, as in Proposition 2.7, the only possible irreducible components for a fibre containing C_i are two (-1)-curves $A_{i,1}$, $A_{i,2}$, so that they form a chain with C_i as central curve. Since after blowing down A_1, A_2 we obtain a relatively minimal Halphen fibration of index 3, one can check that Γ intersects each $A_{i,j}$ transversally at one point. Notice that $\Gamma \cdot C_i = 1$ as well, and Γ is a 3-section of $|A_1 + E_0 + A_2|$. Also, $\Delta \cdot A_{i,j} = 1$ for all i, j, and Δ is a 2-section of $|A_1 + E_0 + A_2|$. We also have $B_j \cdot A_i = 1$, $B_i \cdot E_0 = 1$, $B_1 \cdot B_2 = 0$, $\Delta^2 = 0$, and $\Gamma^2 = -2$. Since $K^2_{\widetilde{Z}} = -2$, after blowing down $A_1, A_{1,1}, \ldots, A_{4,1}$ and $E_0, C_1, \ldots, C_4$ we arrive to a Hirzebruch surface $\mathbb{F}_m$. Since we know the self-intersections of the images of Δ, Γ, and $B_1 + B_2$, we obtain as in Proposition 2.7 that $m = 1$.

Now the point is that $B_1 + B_2$ has degree 10 in $\mathbb{P}^2$, but there are possibilities for the degrees of B_1 and B_2. Since A_1, E_0 gives nodes to B_1 and B_2, the degree cannot be smaller than 3. The possibilities of $B_k \cdot A_{i,1}$ are $0, 1, 2, 3$. We check that the only possibility that works is B_k intersects two $A_{i,1}$ at two points each (possible infinitely near) and the other two at one point each. Therefore, each B_k becomes a quintic in $\mathbb{P}^2$. The image in $\mathbb{P}^2$ of Δ is a quartic with two nodes, and the image of Γ is a sextic with two triple points and four double points. □

Proposition 3.8. *The minimal model of the quotient $\widetilde{Y}/\tau$ is a K3 surface, which has an elliptic fibration with singular fibres I_2, I_4 and I_0^*, a (-2)-curve which is a 2-section, and a (-4)-curve which is a 4-section.*

Proof. Let us consider the I_4 fibre $F_0 = \sum_{i=1}^4 F_0^i$ that is τ-invariant ($= \tau_0$). Under the quotient by τ, the two curves F_0^1 and F_0^2 that are pointwise fixed go to (-4)-curves, while F_0^3 and F_0^4 of go to (-1)-curves. After contracting the (-1)-curves, we get an I_2 fibre. The other two I_4 fibres are identified to give a single I_4 fibre. The nonsingular invariant fibre maps to an I_0^* after resolving the four A_1 singularities. The two (-4)-curves E_1 and E_2 which form a τ-orbit are identified to give a (-4)-curve which is a 4-section. After blowing down F_0^3 and F_0^4, it passes through the two nodes of the I_2-fibre. The image of the single (-4)-curve E_0 is a (-2)-curve which is a 2-section. The multiplicity 2 of the (-1)-components disappears after contracting and stays in the central curve of the I_0^*, so there are no multiple fibres. Notice that $p_g = 1$, and so the minimal model of $\widetilde{Y}/\tau_0$ is a K3 surface. □

Let $\pi_V \colon Y \to V$ be the quotient by τ. Notice that V has five A_1 singularities (four from the four fixed points, and one from two of the $\frac{1}{4}(1,1)$ singularities), and one $\frac{1}{4}(1,1)$ singularity. The minimal resolution $\widetilde{V}$ of V is the minimal resolution of $\widetilde{Y}/\tau$. Using the notation in the proof of Proposition 3.8, let D_1 and D_2 be the images of F_0^3 and F_0^4 in V. Notice that in $\widetilde{V}$, the proper transforms of D_1 and D_2 are (-1)-curves intersecting transversally at one point the (-4)-curve which is the exceptional divisor of the $\frac{1}{4}(1,1)$ singularity. See Figure 1 for this and what follows.

We have $V' \rightsquigarrow V$, where $V' := Y'/\tau$ is the blow-up at one point of a nodal K3 surface, as in Proposition 2.8. Let us consider that degeneration over a disc $\mathbb{D}$. This is, we consider the $\mathbb{Q}$-Gorenstein deformation $(V \subset \mathcal{V}) \to (0 \in \mathbb{D})$ where V' is a fibre for some $t \in \mathbb{D} \setminus \{0\}$. Let $V \to U$ be the contraction of the curve D_1. Then, there is a blow-down deformation $(U \subset \mathcal{U}) \to (0 \in \mathbb{D})$ which commutes with $\mathcal{V} \to \mathbb{D}$ over $\mathbb{D}$. This and what follows is explained in [7], see also [20, §2]. In this way, we have an extremal neighborhood $(D_1 \subset \mathcal{V}) \to (Q \in \mathcal{U})$ of type k1A. Notice that $(Q \in U)$ is a $\frac{1}{3}(1,1)$ singularity. This extremal neighborhood is of flipping type, and it is the simplest case among all k1A (see [20, Prop.2.15]). After we perform the flip, we obtain a Gorenstein deformation $(V_1 \subset \mathcal{V}_1) \to (0 \in \mathbb{D})$ where V_1 is the minimal resolution of U. Hence the flipping curve is the (-3)-exceptional curve. Since this was a flip, the fibre V' in $\mathcal{V}$ appears also in $\mathcal{V}_1$.

After that, the proper transform of D_2 together with the (-1)-curve in V' generate a contractible divisor in $\mathcal{V}_1$. After this divisorial contraction, we obtain a Gorenstein

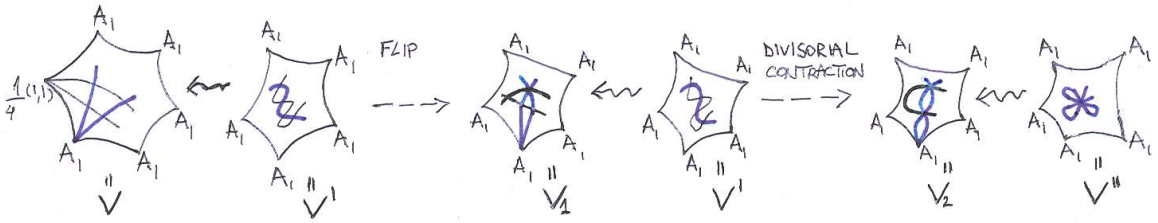

Figure 2: Birational transformations on K3 family

deformation $(V_2 \subset \mathcal{V}_2) \to (0 \in \mathbb{D})$ where V_2 is the blow-down of $D_2 \subset V_1$, and the general fibre is a nodal K3 surface V_1'. Notice that the branch curve of $Y' \to V'$ in V_1' has a 4-point, and degenerates nontrivially to a nonreduced curve formed by three irreducible components (one with multiplicity 2), two passing through one A_1, and the three together having two 3-points in the smooth locus of V_2, as shown in Figure 2. We have proved:

Proposition 3.9. *The $\mathbb{Q}$-Gorenstein deformation $V' \rightsquigarrow V$ over a disk $\mathbb{D}$ is birationally equivalent over $\mathbb{D}$ to a Gorenstein deformation $V'' \rightsquigarrow V_2$, where V'' is the blow-down of a (-1)-curve in V', and V_2 is the K3 surface with five nodes obtained after the flip and divisorial contraction explained above.*

3.3 Stable rational $\mathbb{Z}/3$-Godeaux surfaces

A known method for producing $\frac{1}{n^2}(1, n-1)$ singularities on an elliptic fibration is to merge a nodal fibre with a multiple fibre of multiplicity n; see e.g. [8], [20, §4]. We apply this method with $n = 3$ to the Dolgachev $(3, 3)$ surfaces $\widetilde{X}$. To achieve that, we first merge a nodal fibre in $\widetilde{Y}$, the universal cover of $\widetilde{X}$, with a σ-fixed fibre so that $\widetilde{Y}$ acquires an A_2 singularity. Then the quotient by σ gives what we want. We can do this independently with each of the two fixed fibres. In fact, this approach extends to Y' in $\mathcal{M}$ with a single A_2-singularity that is fixed by σ. For such Y', one of the σ-invariant genus 4 fibres of the canonical pencil $\widetilde{Y}' \to \mathbb{P}^1$ acquires a node.

Proposition 3.10. *There are two divisors $\mathcal{B}_\omega$ and $\mathcal{B}_{\omega^2}$ each defined by hypersurfaces in $\mathcal{M}$ corresponding to surfaces Y with an A_2 singularity that is fixed by σ. The quotient X of such a surface is a Godeaux surface with a $\frac{1}{9}(1, 2)$ singularity.*

Proof. We consider the σ-invariant fibre F_ω on Y, corresponding to $(1, \omega, \omega^2)$ in $H^0(K_Y)$. The computation for F_{ω^2} is similar. Suppose F_ω has a node Q. Since Q is a distinguished point of F_ω, it is fixed by σ. Thus by Lemma 2.3 and the discussion following, $Q = (1, \lambda, \omega^2\lambda, \omega\lambda)$ in $\mathbb{P}^3$ for some λ. We substitute Q into the equations for F_ω and rearrange a little to get the following two conditions:

$$\begin{aligned} \lambda^2 &= (b_0 + \omega^2 b_1)\lambda + \omega^2 a_{11} + a_{12} + \omega a_{22} \\ 2\lambda^3 &= -3((\omega^2 d_2 + \omega d_3)\lambda + \omega c_2 + \omega^2 c_3). \end{aligned} \tag{3.1}$$

Eliminating λ gives a single hypersurface in $\mathcal{M}$ defining $\mathcal{B}_\omega$. The other divisor $\mathcal{B}_{\omega^2}$ is obtained by interchanging ω and ω^2.

It remains to check that the general surface in $\mathcal{B}_\omega$ has a single A_2 singularity and no others. Consider a neighbourhood U of $Q \in F_\omega$ in Y. We can write U as the complete intersection in $\mathbb{A}^3 \times \Delta_s$ given by the equations of Lemma 2.3(i) with $\alpha_1 = \omega + s$ and $\alpha_2 = \omega^2 - s$, where s is the coordinate on the disc Δ and we assume $t \neq 0$. A careful (but tedious) analysis of the two equations confirms that Q is an A_2 singularity for the general surface in $\mathcal{B}_\omega$. □

Remark 3.11. The hypersurface defining $\mathcal{B}_\omega$ is too large to reproduce here, so we consider $\mathcal{B}^s_\omega$, the restriction of $\mathcal{B}_\omega$ to $\mathcal{M}^s$. In fact, we have $\mathcal{B}^s_\omega = \mathcal{B}^s_{\omega^2} = (\mathcal{B}_\omega \cap \mathcal{B}_{\omega^2})^s$, because the involution maps the first A_2 singularity to a second A_2 singularity. The hypersurface in $\mathcal{M}^s$ defining $(\mathcal{B}_\omega \cap \mathcal{B}_{\omega^2})^s$ is

$$\frac{2}{3}(a_{11}-a_{12})^3 + 2(a_{11}-a_{12})^2 d_2 + (a_{11}-a_{12})(3b_0c_2 - b_0^2 d_2 + \frac{3}{2}d_2^2) + \frac{1}{2}c_2(3c_2 + 3b_0 d_2 - 2b_0^3).$$

Lemma 3.12. *All intersections between $\mathcal{Y}$, $\mathcal{B}_\omega$, and $\mathcal{B}_{\omega^2}$ are transversal. Thus, for a general Y in any of these intersections, each singularity of Y can be $\mathbb{Q}$-Gorenstein smoothed independently, and the same applies to the Godeaux quotient X, see Figure 3 below.*

Proof. Let Y be a general surface in $\mathcal{Y} \cap \mathcal{B}_\omega$. For simplicity, we work analytically on $\mathcal{B}_\omega$, but the Lemma also holds in the algebraic setting. From equation (3.1) above, we can solve for λ using the quadratic, and substitute into the cubic to get an analytic hypersurface defining $\mathcal{B}_\omega$.

To smooth the $\frac{1}{4}(1,1)$ singularity, let Y_ε be the surface with parameters $b_0 = \varepsilon$, $b_1 = b_1 - \varepsilon\omega$ and all others the same as those for Y. Thus Y_ε remains in $\mathcal{B}_\omega$, but is no longer in $\mathcal{Y}$. To smooth the $\frac{1}{9}(1,2)$ singularity, simply vary one of a_{ij} or b_1 while fixing $b_0 = 0$ and the other parameters, so that any root of the quadratic is no longer a root of the cubic in (3.1). □

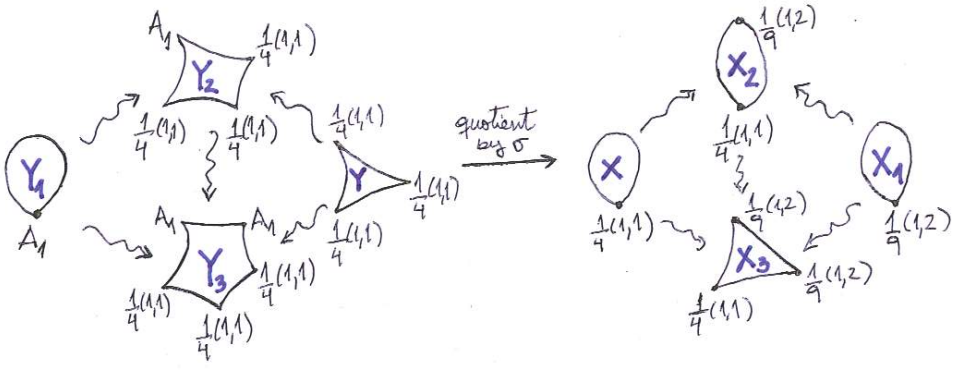

Figure 3: Stable $\mathbb{Z}/3$-degenerations of Y and X

Let X_i for $i = 1, 2, 3$ be stable Godeaux surfaces such that X_1 has one $\frac{1}{9}(1,2)$ singularity, X_2 has one $\frac{1}{4}(1,1)$ and one $\frac{1}{9}(1,2)$, and X_3 has one $\frac{1}{4}(1,1)$ and two $\frac{1}{9}(1,2)$ singularities. Let Y_i be the corresponding cyclic cover of X_i. All of them relate under $\mathbb{Q}$-Gorenstein degenerations as shown in Figure 3.

Proposition 3.13. *The surfaces X_2 and X_3 are rational.*

Proof. Let $i = 2, 3$. By construction, if $\widetilde{Y}_i \to Y_i$ is the minimal resolution of the three $\frac{1}{4}(1,1)$ singularities, then there is elliptic fibration $\widetilde{Y}_i \to \mathbb{P}^1$ such that the A_2 singularity(ies) is(are) in the node of an(two) I_1 fibre(s) F_ω and/or F_{ω^2} that are σ-invariant. The quotient $\widetilde{X}_i := \widetilde{Y}_i/\sigma$ is the minimal resolution of X_i at the $\frac{1}{4}(1,1)$ singularity. Therefore, there is an elliptic fibration $X_i \to \mathbb{P}^1$ with an(two) I_1 fibre(s) such that the $\frac{1}{9}(1,2)$ of the surface is(are) at the node(s), and that (those) fibre(s) is(are) multiple with multiplicity 3. Take one $\frac{1}{9}(1,2)$ singularity and minimally resolve. Then the multiple fibre becomes a I_3 fibre in the induced elliptic fibration. Notice that $K^2_{\widetilde{X}_i} = 0$, and so the self-intersection of the canonical class in the resolution is -2. Therefore the proper transform of the I_1 is a (-1)-curve.

In the case $i = 2$, we obtain an elliptic fibration with $p_g = q = 0$ and one multiple fibre of multiplicity 3. This corresponds to a Halphen pencil of index 3 in $\mathbb{P}^2$. In the case $i = 3$, we obtain an elliptic fibration with $p_g = q = 0$ and no multiple fibre. This corresponds to a pencil of cubics in $\mathbb{P}^2$. So in both cases they are rational surfaces.

□

Proposition 3.14. *The surface X_1 is rational.*

Proof. Assume X_1 is not rational. Let $\widetilde{X}_1$ be its minimal resolution, and let S be the minimal model of $\widetilde{X}_1$. We know that S cannot be of general type since $K^2_{X_1} = 1$ and K_{X_1} is ample; see e.g. [20, Prop. 3.6]. Therefore $K^2_S = 0$. Also, we have that $K^2_{\widetilde{X}_1} = 1-2 = -1$, and so S is the blow down of one (-1)-curve $E \subset \widetilde{X}_1$. Let Γ_1 be the (-2)-curve in the exceptional divisor of $\widetilde{X}_1 \to X_1$, and let Γ_2 be the (-5)-curve. Since K_S is nef, we have that E cannot touch Γ_1. Let $m = E \cdot \Gamma_2$. Since K_{X_1} is ample, we have $m \geq 2$. Since K_S is nef, we have $m \leq 3$. If $m = 3$, then S cannot have Kodaira dimension 1 since $K_S \cdot \Gamma'_5 = 0$ and Γ'_2 is not a fibre of an elliptic fibration (image of Γ_2). Therefore S must be an Enriques surface since $p_g(X_1) = q(X_1) = 0$. Then $\pi_1(X_1) = \mathbb{Z}/2$. On the other hand, we have a $\mathbb{Q}$-Gorenstein smoothing $X' \rightsquigarrow X_1$ over a disk $\mathbb{D}$, where X' is a nonsingular $\mathbb{Z}/3$-Godeaux surface, and so we have a surjective morphism $\pi_1(X') = \mathbb{Z}/3 \to \pi_1(X_1) = \mathbb{Z}/2$, which is a contradiction.

Thus $m = 2$. Let us again consider the $\mathbb{Q}$-Gorenstein smoothing $X' \rightsquigarrow X_1$ over $\mathbb{D}$ above. We notice that a loop α around Γ_2 will satisfy $\alpha^2 = 1$ in $\pi_1(\widetilde{X}_1 \setminus \{\Gamma_1 \cup \Gamma_2\})$, because $\Gamma_2 \cdot E = 2$, $\Gamma_1 \cdot E = 0$, and $E = \mathbb{P}^1$. But $\alpha^9 = 1$ as well, and so α is trivial in $\pi_1(\widetilde{X}_1 \setminus \{\Gamma_1 \cup \Gamma_2\})$. By the same Seifert–Van-Kampen argument in Theorem 3.4, this would imply $\pi_1(X_1) = \pi_1(X') = \mathbb{Z}/3$. Then S is a Dolgachev surface with two multiple fibres of multiplicities $3a$ and $3b$ where $\gcd(a, b) = 1$. Using the canonical class formula and $K_S \cdot \Gamma'_2 = 1$, we get $\frac{1}{3a} + \frac{1}{3b} + \frac{1}{3at} = 1$ where $3at$ is the degree of the multi-section Γ'_2 in the elliptic fibration of S. Then $a = b = t = 1$, and Γ'_2 is a 3-section.

We now consider the $\mathbb{Q}$-Gorenstein degeneration $X_1 \rightsquigarrow X_2$ over $\mathbb{D}$ above. Since around the singularity $1/9(1,2)$ this deformation is trivial, we resolve simultaneously to obtain a $\mathbb{Q}$-Gorenstein smoothing $X'_1 \rightsquigarrow X'_2$ over $\mathbb{D}$ (of the singularity $1/4(1,1)$). Since $K^2_{X'_1} = -1$, the canonical class of the 3-fold is not nef. As in [21, §2], we either have a divisorial contraction or a flip for the family (the other "ending options" are not possible). In case of a flip, the central fiber would become smooth, and then there would be a contradiction

since the Kodaira dimension must be constant in a smooth deformation (and X_2 is rational by the previous proposition). So we have a divisorial contraction, and it must correspond to a (-1)-curve in X_2' which does not touch the singularity $1/4(1,1)$. But then in the general fiber this (-1)-curve propagates as the curve E above. So we contract this divisor in the family, to obtain another $\mathbb{Q}$-Gorenstein smoothing $X_1'' \rightsquigarrow X_2''$ over $\mathbb{D}$, where X_1'' is a Dolgachev $3,3$ surface. But now the canonical class of the 3-fold must be nef (again flips are not allowed by the above argument) and so, by [8, Theorem 5.1], the canonical class of this new 3-fold has index 3 (or 1), but this is locally a $\mathbb{Q}$-Gorenstein smoothing of $1/4(1,1)$, which has index 2, a contradiction.

□

4 On $\mathbb{Z}/4$-Godeaux surfaces

4.1 Setup and involution

Start with $\mathbb{P}(1,1,1,2,2)$ with coordinates x_1, x_2, x_3, y_1, y_3 and a $\mathbb{Z}/4$-action

$$\sigma\colon (x_1, x_2, x_3, y_1, y_3) \mapsto (ix_1, -x_2, -ix_3, iy_1, -iy_3).$$

Now let Y' be the intersection of two quartics q_0 and q_2 in eigenspaces 0, 2 of the form

$$\begin{aligned} q_0\colon & x_1^4 + x_2^4 + x_3^4 + x_1^2x_3^2 + a_1x_1x_2^2x_3 + a_2x_1x_2y_1 + a_3x_2x_3y_3 + a_4y_1y_3, \\ q_2\colon & x_1^2x_2^2 + x_2^2x_3^2 + y_1^2 + y_3^2 + b_1x_1^3x_3 + b_2x_1x_3^3 + b_3x_1x_2y_3 + b_4x_2x_3y_1. \end{aligned}$$

Reid [16] (see also [14]) showed that for general choices of parameters a_i, b_i, the surface Y' is the nonsingular canonical model of general type with $p_g = 3$, $K^2 = 4$ and a free $\mathbb{Z}/4$-action, whose quotient X' is a Godeaux surface with $\pi_1 = \mathbb{Z}/4$. Their coarse moduli space is 8-dimensional, irreducible and unirational. As before, we use $\mathcal{M}$ for the KSBA moduli space of stable $\mathbb{Z}/4$-Godeaux surfaces.

Remark 4.1. For future use, we want to allow certain monomials to appear with coefficient zero, thus we do not use the above displayed parameters.

Following Keum–Lee [10, §4.3], there is a 4-dimensional family $\mathcal{M}^s \subset \mathcal{M}$ with an involution τ on X' induced by the following action on Y':

$$\tau\colon (x_1, x_2, x_3, y_1, y_3) \mapsto (-x_1, x_2, -x_3, y_1, y_3). \tag{4.1}$$

The group generated by τ and σ is $\mathbb{Z}/2 \times \mathbb{Z}/4$. For surfaces in $\mathcal{M}^s$, the allowed monomials for q_0 and q_2 are

$$\begin{aligned} q_0\colon & x_1^4,\ x_2^4,\ x_3^4,\ x_1^2x_3^2,\ x_1x_2^2x_3,\ y_1y_3, \\ q_2\colon & x_1^2x_2^2,\ x_2^2x_3^2,\ x_1^3x_3,\ x_1x_3^3,\ y_1^2,\ y_3^2. \end{aligned}$$

Let Y' in $\mathcal{M}^s$ be the nonsingular cover of a $\mathbb{Z}/4$-Godeaux $X' = Y'/\sigma$. The following Lemma is an easy version of Proposition 2.6.

Lemma 4.2. *The fixed curve R_0 of τ on Y' is the curve of genus 5 defined by $(x_2 = 0)$. The five σ-orbits $Q_1, \dots, Q_5$ on Y' corresponding to the five isolated fixed points $P_1, \dots, P_5$ of τ on $X' = Y'/\sigma$ are given by $Q_1 \colon (x_1 = x_3 = 0)$ and $Q_2, \dots, Q_5 \colon (y_1 = y_3 = 0)$. The σ-orbit Q_1 is pointwise fixed by τ, whereas $Q_2, \dots, Q_5$ are only τ-invariant.*

Remark 4.3. The Classification Theorem of [2] implies that the quotient X'/τ must be an Enriques surface, and by [13, Thm 3.2], this is the unique involution on any Godeaux surface such that the quotient is an Enriques surface. We work out this quotient in detail below.

First note that $H^0(3K_{Y'})^{\mathrm{inv}} = \langle x_1^2 x_2, x_2 x_3^2 \rangle$, and so $|3K_{Y'}^{\mathrm{inv}} - R_0|$ is the pencil in $|2K_{Y'}|$ spanned by x_1^2, x_3^2; cf. [10, §4]. Next consider the pencil Λ in $|K_{Y'}|$ spanned by x_1, x_3, of genus 5 curves on Y' with four base points forming the σ-orbit Q_1. By construction of Y', we see that σ acts as an involution on Λ. Thus each curve in $|3K_{Y'}^{\mathrm{inv}} - R_0|$ is a reducible curve $C + \sigma(C)$ for C in Λ, with a node at each of the four basepoints.

Lemma 4.4. *The image $\Lambda_{X'}$ of Λ on X' is a pencil of curves of geometric genus 3 with one node at the only basepoint P_1. Each curve in $\Lambda_{X'}$ is the image of a σ-orbit of two curves in Λ. The other four isolated fixed points $P_2, \dots, P_5$ of τ appear as second nodes on four distinct curves in $\Lambda_{X'}$. Finally, there are two τ-invariant curves in $\Lambda_{X'}$.*

Proof. The description of the general curve in $\Lambda_{X'}$ follows from the discussion preceding the Lemma. Using an easier version of the same computation as in the proof of Proposition 3.2, we see that for $i = 1, \dots, 4$, the σ-orbit $C_i + \sigma(C_i)$ in Λ passing through Q_i has a node at each of the four points of Q_i. Thus the image of this orbit under quotient by σ is a curve with two nodes, one at P_1 and the other at P_i. The two τ-invariant curves in Λ are the images of $F_1 \colon (x_1 = 0)$ and $F_3 \colon (x_3 = 0)$. □

Let R denote the image of R_0 in X'. Then R is a smooth genus 2 curve with $R^2 = 1$, and the above Lemma shows that $\Lambda_{X'} = |3K_{X'} - R|$.

Corollary 4.5. *The pencil $\Lambda_{X'}$ descends to an elliptic half pencil on the Enriques surface given by the minimal resolution of the five A_1 points in $Z' = X'/\tau$. The image of R is a 2-section of the pencil on Z'.*

Corollary 4.6. *For general Y' in $\mathcal{M}^s$, the quotient $V' := Y'/\tau$ is a nodal K3 surface $(2,2,2) \subset \mathbb{P}^5$ with four A_1 singularities. The minimal resolution of V' is a K3 surface with the elliptic fibration image of the pencil $\Lambda = \langle x_1, x_3 \rangle \subset |K_{Y'}|$, and so the four (-2)-curves coming from the A_1 are sections.*

Proof. The invariant monomials of the τ-action on Y' are x_2 , $v_1 = x_1^2$, $v_2 = x_1 x_3$, $v_3 = x_3^2$, y_1, y_3. There is one monomial relation between these generators: $v_1 v_3 = v_2^2$ and the two quartics q_0 and q_2 can also be expressed in terms of the invariants. This gives a complete intersection $(4,4,4) \subset \mathbb{P}(1,2,2,2,2,2)$ which is however, not *well formed*. The variable x_2 appears only as a square, so we can divide all weights by two and set $v_0 = x_2^2$ to get the K3 surface $(2,2,2) \subset \mathbb{P}^5$.

By Lemma 4.2, the σ-orbit Q_1 descends to four A_1 singularities on Y/τ, while the other orbits $Q_2, \dots, Q_5$ descend to pairs of nonsingular points on Y/τ. After blowing up the four A_1 singularities, the image of Λ is an elliptic fibration with ramification curve comprising the 4-section R_0 together with the four (-2)-curves. □

As we did for $\mathbb{Z}/3$-Godeaux, in the right column of Figure 3 we show the situation described in this subsection. In the next subsections we will describe and use the degenerate situation shown at the left side of Figure 3.

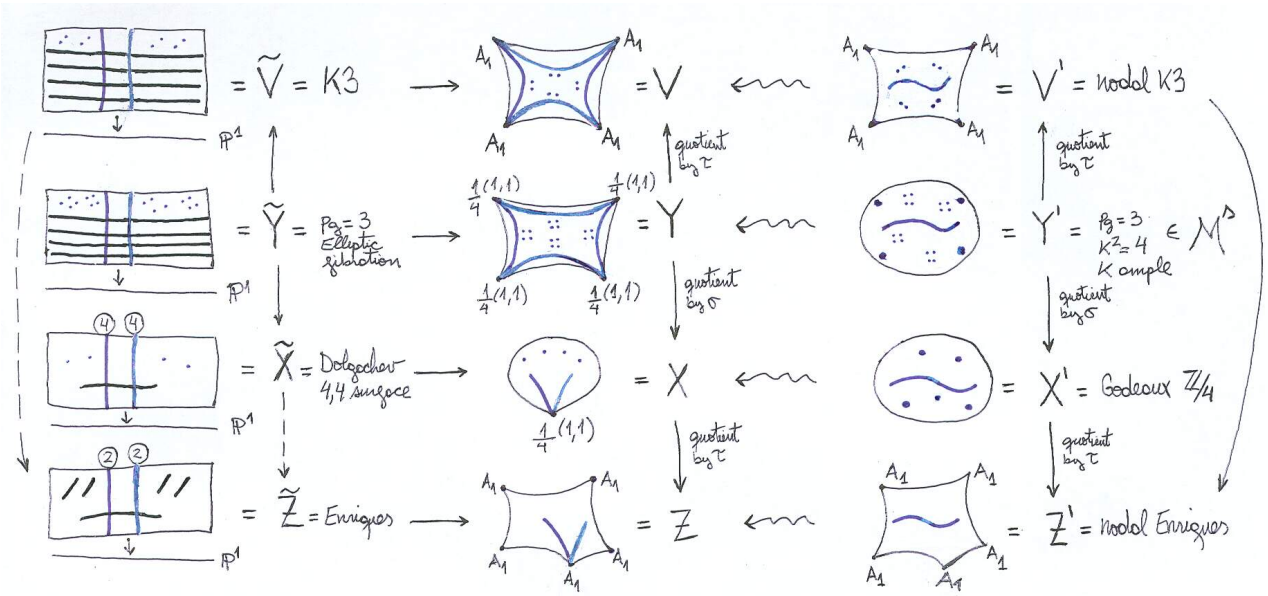

Figure 4: Big picture for Section 4

4.2 A stable $\mathbb{Z}/4$-Godeaux with $\frac{1}{4}(1,1)$ singularity

We exhibit a divisor $\mathcal{Y} \subset \mathcal{M}$ in the moduli space corresponding to $\mathbb{Z}/4$-Godeaux surfaces with a single $\frac{1}{4}(1,1)$ singularity, cf. [9]. Take $Y_{8,8} \subset \mathbb{P}(1,1,4,4,4)$ with $\mathbb{Z}/4$-action

$$\sigma\colon (u_1, u_3, y_1, y_2, y_3) \mapsto (\varepsilon u_1, \varepsilon^3 u_3, iy_1, y_2, -iy_3),$$

where $\varepsilon = \exp(\frac{2\pi i}{8})$. Although ε is a primitive 8th root of unity, σ has order four because of the $\mathbb{C}^*$-action on the weighted projective space. The defining equations q_0, q_2 of Y are in eigenspaces 0 and 2 respectively, and are combinations of the following monomials:

$$q_0\colon u_1^8,\ u_1^4u_3^4,\ u_3^8,\ u_1^3u_3y_1,\ u_1^2u_3^2y_2,\ u_1u_3^3y_3,\ y_2^2,\ y_1y_3,$$
$$q_2\colon u_1^6u_3^2,\ u_1^2u_3^6,\ u_1u_3^3y_1,\ u_1^4y_2,\ u_3^4y_2,\ u_1^3u_3y_3,\ y_1^2,\ y_3^2.$$

Lemma 4.7. *The general member Y in $\mathcal{Y}$ is a complete intersection*

$$Y_{2,4,4} \subset \mathbb{P}(1,1,1,2,2,2)$$

defined by equations $x_1x_3 - x_2^2 = 0$ and $q_i(x_1, x_2, x_3, y_1, y_2, y_3) = 0$ for $i = 0, 2$ where $x_1 = u_1^2, x_2 = u_1u_3, x_3 = u_3^2$ and the q_i are quartics. The four $\frac{1}{4}(1,1)$ singularities $Y \cap (x_1 = x_2 = x_3 = 0)$ form a σ-orbit. The quotient $X = Y/\sigma$ is a stable $\mathbb{Z}/4$-Godeaux surface with a single $\frac{1}{4}(1,1)$ singularity, and there is a $\mathbb{Q}$-Gorenstein smoothing $X \rightsquigarrow X'$ to a smooth $\mathbb{Z}/4$-Godeaux surface.

Proof. The general $Y_{8,8} \subset \mathbb{P}(1,1,4,4,4)$ is quasismooth with four $\frac{1}{4}(1,1)$ singularities at $Y \cap (u_1 = u_3 = 0)$. Using adjunction, $\omega_Y = \mathcal{O}_Y(2)$, and we compute $p_g(Y) = 3$, $K_Y^2 = 4$. The graded ring $R(Y, K_Y) = \bigoplus_{k\geq 0} H^0(\mathcal{O}_Y(2k))$ is generated by $x_1 = u_1^2$, $x_2 = u_1u_3$, $x_2 = u_3^2$ in degree 1 and y_1, y_2, y_3 in degree 2, and the surface described in the statement of the Lemma is precisely that given by $\operatorname{Proj} R$. The σ-action extends that of Section 4.1 with y_2 invariant, and this is still fixed point free for general Y. The $\mathbb{Q}$-Gorenstein smoothing of the four singularities can be done simultaneously and $\mathbb{Z}/4$-equivariantly by varying the quadric equation to $x_1x_3 - x_2^2 = \lambda y_2$ to eliminate y_2 for $\lambda \neq 0$. □

Remark 4.8. We use the above Lemma to interchange between $Y_{8,8} \subset \mathbb{P}(1,1,4,4,4)$ and $Y_{2,4,4} \subset \mathbb{P}(1,1,1,2,2,2)$ without further comment.

As before, there is a family $\mathcal{Y}^s = \mathcal{M}^s \cap \mathcal{Y}$ of stable $\mathbb{Z}/4$-Godeaux surfaces with an involution. The action of τ on the ambient space $\mathbb{P}(1,1,4,4,4)$ is

$$\tau\colon (u_1, u_3, y_1, y_2, y_3) \mapsto (iu_1, -iu_3, y_1, y_2, y_3),$$

and the defining equations of Y are linear combinations of the following monomials

$$q_0\colon u_1^8,\ u_1^4u_3^4,\ u_3^8,\ u_1^2u_3^2y_2,\ y_2^2,\ y_1y_3,$$
$$q_2\colon u_1^6u_3^2,\ u_1^2u_3^6,\ u_1^4y_2,\ u_3^4y_2,\ y_1^2,\ y_3^2.$$

Again, we can check that τ extends the smooth case of Section 4.1.

Proposition 4.9. *Let X be a stable Godeaux surface with a single $\frac{1}{4}(1,1)$ point P and involution τ, arising as the quotient $X = Y/\sigma$ for general Y in $\mathcal{Y}^s$. The fixed curve of τ is $R = C_1 + C_2$, where the intersection $C_1 \cap C_2$ is the singular point P. There are four isolated fixed points of τ on X.*

Proof. This is similar to Lemma 4.2. The fixed curve on Y is defined by $(x_2 = 0)$, and since $x_2 = u_1u_3$, R splits into two components $C_1 + C_2$ on X. Moreover, the locus $Y \cap (u_1 = u_3 = 0)$ is the σ-orbit Q_1 of $\frac{1}{4}(1,1)$ singularities, so the C_i intersect at a single $\frac{1}{4}(1,1)$ point on X. The isolated fixed points of τ are the images of four σ-orbits $Q_2, \dots, Q_5$ defined by $(y_1 = y_3 = 0)$ on Y. □

Let $\phi_Y\colon \widetilde{Y} \to Y$ be the minimal resolution of the four $\frac{1}{4}(1,1)$ singularities on Y. Then $K_{\widetilde{Y}} = \phi_Y^*(K_Y) - \frac{1}{2}(E_1 + \dots + E_4)$, where the E_i are the (-4)-exceptional curves. Since $\widetilde{Y}$ is an elliptic surface with $p_g = 3$, by the Kodaira bundle formula, we also have $K_{\widetilde{Y}} \sim 2F$ where F is a fibre. The curves E_i are therefore sections of the fibration.

Thus the elliptic fibration $\widetilde{Y} \to \mathbb{P}^1$ is the resolution of the pencil $\langle u_1, u_3\rangle$, and all fibres are complete intersection curves $F_{8,8} \subset \mathbb{P}(1,4,4,4)$. Unlike in the $\mathbb{Z}/3$ case, these fibres are really just hyperplane sections of Y, and because $K_{\widetilde{Y}} = 2F$, they are nonsingular at the $\frac{1}{4}(1,1)$ points.

Lemma 4.10. *The elliptic surface $\widetilde{Y}$ associated to a general surface Y in $\mathcal{Y}$ has two nonsingular fibres F_1, F_3 that are invariant under the action of σ and 48 I_1 fibres. If Y is in $\mathcal{Y}^s$, then 16 of the nodes in I_1 fibres form four σ-orbits of isolated τ-fixed points.*

Proof. The automorphism σ acts on the base of the fibration only, and the two fibres $F_1\colon (u_1 = 0)$ and $F_3\colon (u_3 = 0)$ are invariant. A direct computation shows that for general Y, F_i are nonsingular, and the computation of singular fibres and fixed points is again an easier version of the argument in the last paragraph of the proof of Proposition 3.2. □

Let $\widetilde{X}$ be the elliptic surface obtained as the quotient $\widetilde{Y}/\sigma$.

Corollary 4.11. *The minimal resolution $\widetilde{X}$ of X is a $(4,4)$-Dolgachev surface, and the (-4)-curve E is a 4-section. The involution τ fixes the two multiple fibres pointwise, and E is invariant under τ, with two fixed points at the intersection with the multiple fibres. The quotient $\widetilde{X}/\tau$ is a nodal Enriques surface with a (-2)-curve which is a 2-section of the elliptic fibration, and four A_1-singularities lying on distinct fibres.*

Proof. The two σ-invariant fibres F_i on $\widetilde{Y}$ give rise to two fibres of multiplicity 4 on the quotient $\widetilde{X}$. From Proposition 4.9, these two fibres are pointwise fixed by τ, and so their multiplicity on X/τ drops to two. The image of E is a (-2)-curve because E is τ-invariant. □

Corollary 4.12. *For general Y in $\mathcal{Y}^s$ with a $\frac{1}{4}(1,1)$ singularity, the quotient surface Y/τ is a nodal K3 surface with four A_1 singularities. The ramification curve breaks into two components, each of which passes through all four A_1 singularities.*

Proof. The fact that the quotient is an intersection of three quadrics is similar to Corollary 4.6, starting from the description of Y in Lemma 4.7.

The singularities are again induced by the σ-orbit Q_1 on Y, but this time Q_1 comprises four $\frac{1}{4}(1,1)$ singularities. The local orbifold coordinates near a $\frac{1}{4}(1,1)$ point P of Q_1 are u_1, u_3, and the action of τ on these coordinates is $\frac{1}{4}(1,3)$. Thus the invariant monomials of the composite action are $w_1 = u_1^4$, $w_2 = u_1^2u_3^2$, and $w_3 = u_3^4$ with single relation $w_1w_3 = w_2^2$, and the image of P on Y/τ is an A_1 singularity. □

4.3 Stable rational $\mathbb{Z}/4$-Godeaux surfaces

Following the strategy explained in Section 3.3, we look for stable $\mathbb{Z}/4$-Godeaux surfaces $\widetilde{X}$ with an elliptic fibration where the multiple fibres become nodal. As in Section 3.3, this behaviour is induced by Y having a single σ-fixed point in the hyperplane $(x_1 = 0)$ (or $(x_3 = 0)$). Such a fixed point must be an A_3 singularity or worse. This condition is independent of the presence of $\frac{1}{4}(1,1)$ singularities.

Lemma 4.13. *The following divisors $\mathcal{B}_1, \mathcal{B}_3 \subset \mathcal{M}$ defined by*

$$\mathcal{B}_1\colon \text{no } x_1^4 \text{ monomial in } q_0$$
$$\mathcal{B}_3\colon \text{no } x_3^4 \text{ monomial in } q_0$$

parametrise surfaces Y with an A_3 singularity at $P_1 = (1,0,0,0,0)$ (respectively $P_3 = (0,0,1,0,0)$) which is fixed under the action of σ. Moreover, the quotient $X = Y/\sigma$ of a general surface in $\mathcal{B}_i$ is a Godeaux surface with a $\frac{1}{16}(1,3)$ singularity.

Proof. Consider a general surface Y in $\mathcal{B}_1$. Since q_0 does not contain the monomial x_1^4, the coordinate point $P = P_{x_1}$ is contained in Y. Ignoring coefficients, the leading terms of q_0, q_2 at P are

$$y_1y_3 + x_2y_1 + x_3^2 + h.o.t.,\ x_3 + x_2^2 + h.o.t.$$

Thus eliminating x_3 with x_2^2 and a coordinates change shows that Y has an A_3 singularity at P, and clearly P is fixed under the action of σ. Computing with local coordinates near P, we see that the action of σ on (y_1, y_3, x_2) is $(-iy_1, iy_3, ix_2)$ and so P induces a $\frac{1}{16}(1,3)$ point on X. □

Lemma 4.14. *All intersections between $\mathcal{Y}$, $\mathcal{B}_1$ and $\mathcal{B}_3$ are transversal.*

Proof. The proof is similar to that of Lemma 3.12, and we do not give full details. We only remark that a surface Y in $\mathcal{Y} \cap \mathcal{B}_1$ is realised as a complete intersection $Y_{8,8} \subset \mathbb{P}(1,1,4,4,4)$ for which q_0 does not contain the monomial u_1^8. The $\mathbb{Q}$-Gorenstein smoothing of the $\frac{1}{4}(1,1)$ point is in Lemma 4.7, and the $\frac{1}{16}(1,3)$ point is smoothed by allowing u_1^8 to appear again. □

In this case, the A_3 singularities on Y are fixed under τ, so it is not true that $\mathcal{B}_i^s = (\mathcal{B}_1 \cap \mathcal{B}_3)^s$ here, unlike Section 3.3.

Let X_1 be the stable Godeaux surface with one $\frac{1}{16}(1,3)$, X_2 with singularities $\frac{1}{4}(1,1)$ and $\frac{1}{16}(1,3)$, and X_3 with one $\frac{1}{4}(1,1)$ and two $\frac{1}{16}(1,3)$ singularities. Let Y_i be the corresponding cyclic cover of X_i. According to Lemma 4.14, the surfaces are related under $\mathbb{Q}$-Gorenstein degenerations as shown in Figure 5.

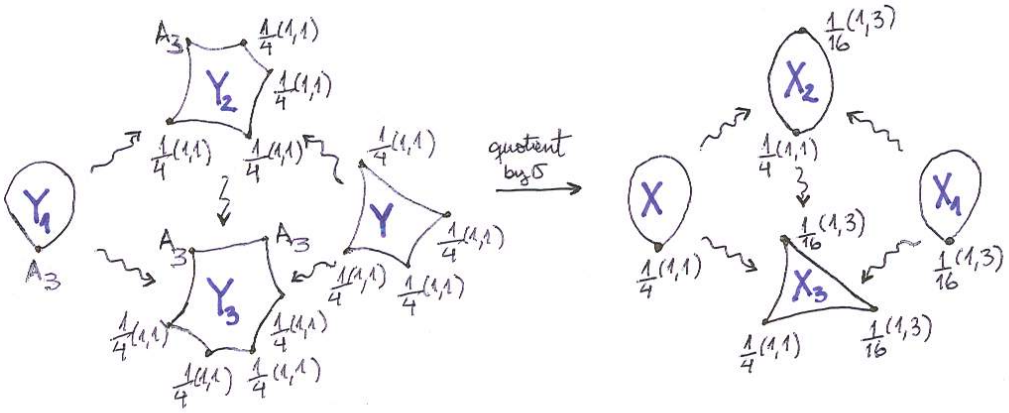

Figure 5: Stable $\mathbb{Z}/4$-degenerations of Y and X

Proposition 4.15. *The surfaces X_i are rational for $i = 2, 3$.*

Proof. This is the same proof as in the $\mathbb{Z}/3$ case, Proposition 3.13.

□

Proposition 4.16. *The surface X_1 is rational.*

Proof. Let us consider a $\mathbb{Q}$-Gorenstein deformation $X_1 \rightsquigarrow X_2$ over a disk $\mathbb{D}$, so that the deformation around the singularity $\frac{1}{16}(1,3)$ is constant. Then we resolve it simultaneously (and minimally) to obtain a $\mathbb{Q}$-Gorenstein smoothing $X_1' \rightsquigarrow X_2'$. By the previous proposition, in the central fibre X_2' we have a (-1)-curve between the (-6)-curve and the last (-2)-curve of the exceptional divisor. This (-1)-curve intersects the (-4)-curve of the exceptional divisor of $\widetilde{X}_2 \to X_2$ at one point and transversally. Therefore, we can apply the flip used in Proposition 3.9. After the flip, the deformation $X_1' \rightsquigarrow X_2'$ becomes smooth, and so the Kodaira dimension is preserved. By the previous proposition, the surface X_2 is rational, and so X_1 is rational as well. □

Remark 4.17. We know of one other involution on the $\mathbb{Z}/4$-Godeaux surface, from [10, §4.2]. The automorphism $\tau\colon (x_1, x_2, x_3, y_1, y_3) \mapsto (x_3, x_2, x_1, y_3, y_1)$ acts on a subfamily of $\mathcal{M}$, and the group generated by σ and τ is D_4. The fixed curve is genus 1 and there are five isolated fixed points. According to [2, §8.7], the quotient X/τ is a double plane of du Val type. It would be interesting to study the corresponding degenerations for this involution, and understand the behaviour of the double plane.

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