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LIFSHITS TAILS FOR SQUARED POTENTIALS

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ABSTRACT. We consider Schrödinger operators with a random potential which is the square of an alloy-type potential. We investigate their integrated density of states and prove Lifshits tails.
Our interest in this type of models is triggered by an investigation of randomly twisted waveguides.

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1. INTRODUCTION

In the 1960'ies Lifshits [12] discovered that the density of states for periodic systems and the one for random systems show very different behavior near the bottom of their spectra. While the integrated density of states $N(E)$ for a d-dimnesional periodic system behaves like $(E-E_0)^{\frac{d}{2}}$ near the ground state energy E_0, $N(E)$ it behaves like $e^{-C\,(E-E_0)^{-\frac{d}{2}}}$ for typical random systems.
Starting with the seminal work by Donsker and Varadhain [2] there has been a strong interest in this type of questions in the mathematical physics literature. For a review (as of 2006) see e.g. [8] (see also [1], [4]), some more recent developments are [3], [10], [11] and [15].
One of the most common random potentials and the one we are dealing with in this paper is the alloy-type potential

$$U_\omega(x) = \sum_{i\in\mathbb{Z}^d} q_i(\omega) f(x-i) \tag{1}$$

where $x \in \mathbb{R}^d$, q_i are independent, identically distributed random variables and f is a (say) bounded measurable function decaying sufficiently fast at infinity.
Lifshits tails for

$$H_\omega = -\Delta + U_\omega \tag{2}$$

are well known for alloy-type potentials as in (1) if both the q_i and the function f have definite sign.

Recently, there has been interest in the case that q_i and/or f change sign (see e.g. [3], [10], [11]). In these models the lack of monotonicity makes it much harder to prove Lifshits tails.
In our research on twisted waveguides [5] we came across a potential V_ω which is the *square* of an alloy-type potential, i.e. $V_\omega(x) = \big(U_\omega(x)\big)^2$.
In fact, Lifshits tails for the twisted waveguide correspond to Lifshits tails of the Schrödinger operator

$$H_\omega = -\Delta + V_\omega = -\Delta + {U_\omega}^2 \tag{3}$$

with an alloy-type potential U_ω as in (1).
In [5] we need only the one-dimensional case of (3) but here we will deal with this model in arbitrary dimension $d \geq 1$ as this will not cause additional complications.
Obviously the potential $V_\omega(x) = U_\omega(x)^2$ is non-negative. We will allow, however, that both q_i and f may change sign, so that we lose monotonicity.

2. Setting

We consider the random potential

$$U_\omega(x) = \sum_{i \in \mathbb{Z}^d} q_i(\omega) f(x-i) \ ; \ x \in \mathbb{R}^d, d \geq 1 \ , \tag{4}$$

on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The expectation with respect to $\mathbb{P}$ will be denoted by $\mathbb{E}$.
Throughout this paper we make the following assumptions.

Assumptions.

(1) The real valued random variables q_i are independent and identically distributed. Their common distribution is denoted by P_0.
(2) The support supp P_0 contains more than one point, $0 \in$ supp P_0 and supp $P_0 \subset [-Q, Q]$ for some $Q < \infty$.
(3) For some $K \geq 0, C > 0$ and all $\varepsilon > 0$ small enough
$$\mathbb{P}(|q_i| < \varepsilon) \geq C\varepsilon^K \ .$$
(4) f is a bounded (measurable) real valued function, $f \neq 0$, with
$$|f(x)| \ \leq \ \frac{C}{(1+|x|)^\alpha}$$
for some C and $\alpha > d$.

Finally, we set

$$V_\omega(x) = \Big(U_\omega(x)\Big)^2 = \Big(\sum q_i f(x-i)\Big)^2 . \tag{5}$$

and define the operator

$$H_\omega = H_0 + V_\omega \ . \tag{6}$$

with $H_0 = -\Delta$. Since V_ω is non-negative, it is clear that $\sigma(H_\omega) \subset [0, \infty)$. From general results we even have $\sigma(H_\omega) = [0, \infty)$ (see [6]. We remark that we made no assumption on the sign of the q_i or of f. In fact, unless otherwise stated, both may change sign.

We are interested in the integrated density of states N of H_ω.

For $\Lambda = [-\frac{L}{2}, \frac{L}{2}]^d$ let H_Λ^N and H_Λ^D be the operator H_ω restricted to $L^2(\Lambda)$ with Neumann resp. Dirichlet boundary conditions. These operators have a purely discrete spectrum. By $\lambda_k(H_\Lambda^N)$ and $\lambda_k(H_\Lambda^D)$ we denote the eigenvalues of H_Λ^N respectively H_Λ^D in increasing order and counted according to multiplicity.

We define

$$N(H_\Lambda^N, E) := \#\{\lambda_k(H_\Lambda^N) \leq E\}\,, \tag{7}$$

$$N(H_\Lambda^D, E) := \#\{\lambda_k(H_\Lambda^D) \leq E\}\,. \tag{8}$$

The *integrated density of states* of H_ω is the limit

$$N(\lambda) = \lim_{L\to\infty} \frac{1}{L^d} N(H_\Lambda^N, E) \tag{9}$$

$$= \lim_{L\to\infty} \frac{1}{L^d} N(H_\Lambda^D, E)\,. \tag{10}$$

By *Lifshits tails* we mean that the integrated density of states N of H_ω behaves *roughly* like $e^{-C\,(E-E_0)^{-\gamma}}$ as $E \searrow E_0$ where E_0 is the bottom of the spectrum of H_ω, more precisely:

$$\lim_{E\searrow E_0} \frac{\ln|\ln N(E)|}{\ln E} = -\gamma \tag{11}$$

$\gamma > 0$ is called the *Lifshits exponent*. For *alloy-type potentials* as in (4) the Lifshits exponent depends on the behavior of f at infinity. If $|f(x)| \leq C|x|^{-(d+2)}$ for large $|x|$, then $\gamma = \frac{d}{2}$, the 'classical' value for γ. If $f(x) \sim C|x|^{-\alpha}$ for $d < \alpha < d+2$ then $\gamma = \frac{d}{\alpha-d}$ (see e.g. [9]).

3. Results

In this section we state our results for the squared random potential as in (5). As in the conventional case (i. e. for (4)) we obtain Lifshits behavior as in (11). Again, the Lifshits exponent depends on the behavior of f at infinity. This time, however, the threshold is $\alpha = d + 1$ rather than the 'conventional' $d + 2$.

Theorem 1. *Suppose q_i are independent random variables with common distribution P_0 satisfying Assumptions 2 and 3.*

(1) *If f satisfies Assumption 4 with some $\alpha \geq d+1$ then*

$$\lim_{E\searrow 0} \frac{\ln|\ln N(E)|}{\ln E} = -\frac{d}{2}\,.$$

(2) *If f satisfies*

$$\frac{C_1}{(1+|x|)^\alpha} \leq f(x) \leq \frac{C_2}{(1+|x|)^\alpha}$$

for some $d < \alpha < d+1$ and constants $C_1, C_2 > 0$, then

$$\lim_{E\searrow 0} \frac{\ln|\ln N(E)|}{\ln E} = -\frac{d}{2(\alpha-d)}\,.$$

Remarks 2.

(1) For 'non-squared' random potentials as in (4) the critical value of α is $d+2$, rather than $d+1$ for the squared case.
(2) We will proof Theorem 1 by proving corresponding upper and lower bounds on $N(\lambda)$. Assumption 2 is only needed for the lower bounds. Also, the lower bound on f in part 2 of the theorem is only used for the lower bound on N.

4. Strategy of the Proof

We use the technique of Dirichlet-Neumann-bracketing (see [7] and [9]). This method is based on the inequalities

$$\frac{1}{|\Lambda|}\,\mathbb{E}\big(N(H_\Lambda^D, E)\big) \;\leq N(E)\;\leq\; \frac{1}{|\Lambda|}\,\mathbb{E}\big(N(H_\Lambda^N, E)\big) \tag{12}$$

which are valid for any cube $\Lambda = [-\frac{L}{2}, +\frac{L}{2}]^d$ with $|\Lambda| = L^d$ being the volume of Λ.

The right hand side of (12) can further be estimated by

$$\frac{1}{|\Lambda|}\,\mathbb{E}\big(N(H_\Lambda^N, E)\big) \;\leq\; \frac{N(-\Delta_\Lambda^N, E)}{|\Lambda|}\;\mathbb{P}\big(\lambda_1(H_\Lambda^N)\;<\;E\big)$$

Consequently, we have to estimate

$$\mathbb{P}\big(\lambda_1(H_\Lambda^N)\;<\;E\big) \tag{13}$$

from above.

To do so, we use the McDiarmid inequality which we introduce in Section 5.2. The estimate of (13) using the McDiarmid inequality is done in Section 5.

In Section 6 we estimate the left hand side of (12) for a lower bound of N.

5. Upper Bound

5.1. Analytic estimate.

For the upper bound we use a perturbative approach following an idea of Stollmann [16].

We set $H_\Lambda^N(t) := -\Delta_\Lambda^N + t\,V_\omega$ on Λ with Neumann boundary conditions. By

$$E(t) = \lambda_1(H_\Lambda^N(t)) \tag{14}$$

we denote its lowest eigenvalue and by φ_0 is the normalized ground state of $H_\Lambda^N(0)$.

E is monotone increasing for $0 \leq t \leq 1$ and

$$E(0) = \lambda_1(-\Delta_\Lambda^N) = 0 \text{ and } E(1) = \lambda_1(H_\Lambda^N)\,.$$

Moreover, $E(\zeta)$ is a holomorphic function in $\{\zeta \in \mathbb{C} \mid \ |\zeta| \leq \nu\}$ for ν small, namely for $\nu \leq CL^{-2} = C\lambda_2(H_\Lambda^N(0))$. We have

$$E'(0) = \ <\varphi_0, V_\omega\varphi_0> \ = \frac{1}{|\Lambda|}\int_\Lambda V_\omega(x)\, dx$$

by the Hellmann-Feynman Theorem (see e. g. [14]).
By analytic pertubation theory:

Lemma 3. *(Stollmann* [17] *Lemma 2.1.2.) For* $0 \leq t \leq C_1L^{-2}$ *we have*

$$|E(t) - tE^{'}(0)| \leq C_2L^2t^2\,.$$

So, for $t \leq C_1L^{-2}$

$$\mathbb{P}(E(t) \leq bL^{-2}) \leq \mathbb{P}(E^{'}(0) \leq C_2L^2t + btL^{-2})\,. \tag{15}$$

The choice $t = t_0 \sim L^{-2}$ makes the right hand side of (15) smaller than $\mathbb{P}(E^{'}(0) \leq \tilde{C})$ where $\tilde{C} > 0$ can be made small if b is chosen appropriately. We obtain

$$\begin{aligned}\mathbb{P}(\lambda_1(H_\Lambda) \leq bL^{-2}) &\leq \mathbb{P}(E(t_0) \leq bL^{-2})\\ &\leq \mathbb{P}(E^{'}(0) \leq \tilde{C})\\ &\leq \mathbb{P}(|E^{'}(0) - \mathbb{E}(E^{'}(0))| > \lambda)\end{aligned}$$

by choosing $\tilde{C}$ small and λ appropriate. Thus, we are left with a large deviation estimate. For this estimate we employ the McDiarmid inequality which we introduce in the following section.

5.2. **McDiarmid inequality.**
To estimate $\mathbb{P}(\frac{1}{|\Lambda|}\int_\Lambda V_\omega(x)\,dx < \lambda)$ from above we will use a concentration inequality due to McDiarmid in a slightly extended form.

Theorem 4. *(Mc Diarmid) Suppose* $\{X_n\}_{n\in\mathbb{Z}}$ *is a sequence of independent real valued random variables such that* X_n *takes values in* $R_n \subset \mathbb{R}$.
Let $F : \prod_{n\in\mathbb{Z}} R_n \to \mathbb{R}$ *be a measurable function (with respect to the product σ-algebra) with the following property:*
Suppose $\underline{X} = \{X_n\}, \underline{X}^{'} = \{X_n^{'}\} \in \prod R_n$ *differ only in the* j*-component, i.e.* $X_n = X_n^{'}$ *for* $n \neq j$*, then*

$$|F(\underline{X}) - F(\underline{X}^{'})| \leq \sigma_j\,. \tag{16}$$

uniformly in $\underline{X}, \underline{X}^{'}$*. Set* $\sigma^2 := \sum \sigma_j^2$*.*
If $\sum \sigma_j < \infty$*, then for all* $\lambda > 0$

$$\mathbb{P}(|F(X) - \mathbb{E}(F(X))| > \lambda) \leq 2e^{-2\frac{\lambda^2}{\sigma^2}}$$

Proof: This theorem original from [13] can be found in various sources, for example in [18], but only for finite collections $\{X_i\}_{i=1}^M$ of random variables.
The 'limit $M \to \infty$' can be taken in the following way:

Consider the vector $\underline{X}_M = (X_1, \cdots, X_M)$ of random variables and the non random vector

$$\underline{Y}^M := (Y_{M+1}, Y_{M+2}, \cdots) \in \prod_{n \leq M+1}^{\infty} R_n$$

Set $F_M(\underline{X}) := F(X_1, \cdots, X_M, Y_{M+1}, \cdots)$ and $\mathbb{E}_M = \mathbb{E}(F_M(\underline{X}))$.
(Both F_M and $\mathbb{E}_M$ depend on Y^M.)
By (16)

$$|F(\underline{X}) - F_M(\underline{X})| \leq \sum_{n=M+1}^{\infty} \sigma_n \to 0$$

and

$$|\mathbb{E}\big(F(\underline{X})\big) - \mathbb{E}_M| \leq \sum_{n=M+1}^{\infty} \sigma_n \to 0$$

uniformly in $\underline{Y}^M$.
Now,

$$\begin{aligned}
&\mathbb{P}\left(|F(\underline{X}) - \mathbb{E}(F(\underline{X}))| > \lambda\right) \\
&\leq \mathbb{P}\left(|F(\underline{X}) - F_N(\underline{X})| + |F_M(\underline{X}) - E_M| + |E_M - \mathbb{E}(F(\underline{X})) > \lambda\right) \\
&\leq \mathbb{P}\left(|F_M(\underline{X}) - E_M| > \lambda - 2\sum_{n=M+1}^{\infty} \sigma_N\right) \;=\; (\natural)
\end{aligned}$$

$F_M(\underline{X})$ depends only on finitely many random variables (namely $X_1, \cdots, X_M$), so we may apply (the known version of) the McDiarmid inequality to obtain:

$$(\natural) \;\leq\; 2e^{-\frac{2(\lambda - 2\sum_{i=M+1}^{\infty} \sigma_i)^2}{\sum_{i=1}^{M} \sigma_i^2}} \leq 2e^{-\frac{2\lambda^2}{\sum_{i=1}^{\infty} \sigma_i^2}}.$$

$\square$

5.3. **Probabilistic estimate.**
Now, we estimate the probability that $F(q_i) := \int_\Lambda (\sum q_i f(x-i))^2 \, dx$ is small.

First, we compute $\mathbb{E}(F(q_i))$: Let $C_0 = [-\frac{1}{2}, \frac{1}{2}]^d$

$$\mathbb{E}\Big(\int_{C_0} \big(\sum q_i f(x-i) \big)^2 \, dx \Big) = \sum_{i,j} \mathbb{E}(q_i q_j) \int_{C_0} f(x-i) f(x-j) \, dx$$

$$= \sum_i \mathbb{V}(q_i) \int_{C_0} f(x-i)^2 \, dx + \sum_{i,j} \mathbb{E}(q_i)\mathbb{E}(q_j) \int_{C_0} f(x-i) f(x-j) \, dx$$

$$= \mathbb{V}(q_0) ||f||_2^2 + \mathbb{E}(q_0)^2 \int_{C_0} \big(\sum f(x-i) \big)^2 \, dx$$

$$=: \underline{\rho} \qquad \text{where } \|f\|_2^2 := \int_{\mathbb{R}} |f(x)|^2 \, d\,x \,.$$

Consequently, for integer L

$$\mathbb{E}\Big(\int_\Lambda V(x) \, dx \Big) = \rho^{|\Lambda|} \,.$$

To apply Mc Diarmid's inequality we have to compute the σ_j's.
$Q = \{q_i\}, Q' = \{q_i'\}$ with $q_i = q_i'$, for $i \neq j$ $q_j = a$ $q_j' = b$.

$$|F(Q') - F(Q)| \leq \int_\Lambda \Big| \big(\sum q_i' f(x-i) \big)^2 - \big(\sum q_i f(x-i) \big)^2 \Big| \, dx$$

$$= \int_\Lambda |(b-a) f(x-j)| \;\; |\sum (q_i' + q_i) f(x-i)| \, dx$$

$$\leq C \int_\Lambda |f(x-j)| \, dx$$

where $C = 4 \Big(\sup \ \mathrm{supp}(P_0) \Big)^2 \sup_{x \in \mathbb{R}} \ \sum |f(x-i)|$.

So, we got to estimate $\int_\Lambda |f(x-j)| \, dx$.

Case 1: $|j| \leq ML$ with $M \geq 3$ to be chosen later.
Then we estimate

$$\int_\Lambda |f(x-j)| \, dx \leq \int_{\mathbb{R}^d} |f(x)| \, dx = ||f||_1 =: \sigma_j \,.$$

Case 2: $|j| > ML$, so dist $(j, \Lambda) \geq (M-1)L$.
Then

$$\int_\Lambda |f(x-j)| \, dx \leq C \int_\Lambda \frac{1}{|x-j|^\alpha} \, dx$$

$$\leq C L^d \frac{1}{\inf\limits_{x \in \Lambda} |x-j|^\alpha} \leq C' \frac{L^\alpha}{|j|^\alpha} \,.$$

So

$$\sum_i \sigma_i^2 \ \leq \ C_1 \Big(\sum_{|i| \leq ML} ||f||_1^2 + \sum_{|i| \geq ML} \big(\frac{L^\alpha}{|j|^\alpha} \big)^2 \Big)$$

$$\leq \ C_2 L^d (1 + L^d L^{-2\alpha + d}) \leq C_3 L^d \qquad \text{as } \alpha > d \,.$$

Thus $\sum_i \sigma_i^2 \leq CL^d$, so Mc Diarmid's inequality gives:

$$\mathbb{P}\Big(\Big|\int_\Lambda V(x)\,dx - \mathbb{E}\big(\int_\Lambda V(x)\,dx\big)\Big| \geq \lambda L^d\Big)$$
$$\leq C' e^{-C'' \frac{\lambda^2 L^{2d}}{\sum \sigma_i^2}} \leq C' e^{-C''' \lambda^2 L^d}\,.$$

In particular

$$\mathbb{P}\big(\frac{1}{L^d}\int_\Lambda V(x)\,dx < \varepsilon\big) \leq C e^{-\tilde{C} L^d}$$

whenever $\varepsilon < \mathbb{E}(\int_{C_0} V(x)\,dx)$.

5.4. **Upper Bound 2.**

The general upper bound turns out to be correct (i.e. to agree with the lower bound) in the case $\alpha \geq d+1$. For the long range case ($d < \alpha < d+1$) we need another estimate and stronger assumptions.What we need (at least for our proof) is that both q_i and f have a definite sign.

For definiteness, we assume supp $P_0 \subset [0, Q]$ and:

$$c_1(1+|x|)^{-\alpha} \leq f(x) \leq c_2(1+|x|)^{-\alpha}$$

with $d < \alpha < d+1$ and $c_1, c_2 > 0$. We estimate:

$$\mathbb{P}\big(\lambda_1(H_\Lambda^N) < \lambda\big) \leq \mathbb{P}\big(\inf_{x\in\Lambda} V_\omega(x) < \lambda\big)$$
$$\leq \mathbb{P}\big(\inf_{x\in\Lambda} U_\omega(x) < \lambda^{\frac{1}{2}}\big)\,. \tag{17}$$

An estimate for (17) can be found in [9]. For the reader's convenience we give here an alternative proof using Mc Diarmid's inequality.

So, we estimate

$$\mathbb{P}\left(\inf_{x\in\Lambda} \sum q_i f(x-i) < \rho\right)$$
$$\leq \mathbb{P}\left(C \sum q_i \frac{1}{1+\sup\limits_{x\in\Lambda}|x-i|^\alpha} < \rho\right)$$
$$\leq \mathbb{P}\left(C' \sum q_i \frac{1}{(L+|i|)^\alpha}\right).$$

We apply McDiarmid's inequality to $\sum q_i \frac{1}{(L+|i|)^\alpha}$, then

$$\sigma_j = 2Q\frac{1}{(L+|j|)^\alpha} \leq \begin{cases} \frac{C}{L^\alpha} & \text{for } |j| \leq ML\,, \\ \frac{C}{|j|^\alpha} & \text{for } |j| > ML\,. \end{cases} \tag{18}$$

So

$$\sum \sigma_j^2 \leq C' L^{d-2\alpha}\,. \tag{19}$$

Moreover,

$$\mathbb{E}\big(\sum q_i \frac{1}{(L+|i|)^\alpha}\big) = C \sum \frac{1}{(L+|i|)^\alpha} \sim L^{d-\alpha}\,.$$

So we have to take $\rho \sim L^{d-\alpha}$.
With this choice, McDiarmid's inequality gives

$$\mathbb{P}\Big(\sum q_i \frac{1}{(L+|i|)^\alpha} < C L^{d-\alpha} \Big) \;\leq\; e^{-C' L^d}$$

$$\leq e^{-C'' \rho^{-\frac{d}{\alpha-d}}} \;=\; e^{-C'' \lambda^{-\frac{d}{2(\alpha-d)}}} \,. \tag{20}$$

This estimate is better than the general estimate $(\lambda^{-\frac{d}{2}})$ if

$$2(\alpha - d) < 2 \quad \text{i.e. } \alpha < d+1\,.$$

6. Lower bound

We assume that $\mathbb{P}(|q_i| \leq \varepsilon) \geq C\varepsilon^K$ for some $C > 0$ and K and all $\varepsilon > 0$ small enough.
Now, we consider H_Λ^D with Dirichlet boundary conditions.
The ground state energy λ_0 of $-\Delta|_\Lambda^D$ is given by:

$$\lambda_0 = d(\frac{\pi}{L})^2$$

and the ground state is

$$\varphi_0(x) = \big(\frac{2}{L}\big)^{\frac{d}{2}} \prod_{i=1}^{d} \cos(\frac{\pi}{L} x_i)\,.$$

We consider the set $\Omega_L^\varepsilon \subset \Omega$ with

$$\Omega_L^\varepsilon = \{\varepsilon \mid \; |q_i| \leq \varepsilon \text{ for } |i| \leq L+R\}, R \geq L$$

We choose R later, namely as $R = L^\beta, \beta \geq 1$. Then

$$\mathbb{P}(\Omega_L^\varepsilon) \geq (\varepsilon^K)^{(L+R)^d} = e^{K(\ln\;\varepsilon)(L+R)^d}\,.$$

We will show that $\lambda_1(H_\Lambda^D)$ is small on Ω_L^ε. We have

$$\lambda_1(H_\Lambda^D) \leq \; < \varphi_0, H_\Lambda^D \varphi_0 > = \lambda_0 + \int_\Lambda V \varphi_0^2 \, dx$$
$$\leq \; \lambda_0 + \frac{C}{|\Lambda'|} \int_\Lambda V(x)\, dx$$

where $\lambda_0 = \lambda_1(H_0|_\Lambda^D) \approx L^{-2}$.

Now for $w \in \Omega_L^\varepsilon$

$$
\begin{aligned}
&\frac{1}{|\Lambda|}\int_\Lambda V(x)\,dx\\
&\leq \frac{1}{|\Lambda|}\int_\Lambda \Big(\varepsilon \sum_{|i|\leq L+R} f(x-i) + Q\sum_{|i|>L+R}\frac{1}{|x-i|^\alpha}\Big)^2\,dx\\
&\leq \frac{2\varepsilon}{|\Lambda|}\int_\Lambda \big(\sum_{|i|\leq L+R} f(x-i)\big)^2\,dx + \frac{2Q}{|\Lambda|}\int_\Lambda \big(\sum_{|i|>L+R}\frac{1}{|x-i|^\alpha}\big)^2\,dx\\
&\leq C\Big(\varepsilon + \sup_{x\in\Lambda}\big(\sum_{|i|>L+R}\frac{1}{|x-i|^\alpha}\big)^2\Big)\\
&\leq C'\big(\varepsilon + \big(\sum_{|i|>L+R}\frac{1}{|i|^\alpha}\big)^2\big)\\
&\leq C''\big(\varepsilon + R^{-2\alpha+2d}\big)\,.
\end{aligned}
$$

Let us choose $R = L^\beta$.
If $\alpha \geq d+1$ we take $\beta = 1$, then

$$\frac{1}{|\Lambda|}\int_\Lambda V\,dx \leq C''\big(\varepsilon + L^{-2\alpha+2d}\big) \leq C''\varepsilon + L^{-2}$$

thus for $\omega \in \Omega_L^{L^{-2}}$ we have

$$\lambda_1(H_\Lambda^D) \leq CL^{-2}\,.$$

So

$$
\begin{aligned}
\mathbb{P}\big(\lambda_1(H_\Lambda^D) \leq CL^{-2}\big) &\geq \mathbb{P}(\Omega_L^{L^{-2}})\\
&\geq e^{+C(\ln L^{-2})L^d}
\end{aligned}
$$

and, with $\lambda = L^{-2}$,

$$\ln\,\mathbb{P}\big(\lambda_1(H_\Lambda^D) \leq C\lambda\big) \geq -C(\ln\lambda)\lambda^{-\frac{d}{2}}\,,$$

and

$$
\begin{aligned}
\frac{\ln\,|\ln\,\mathbb{P}(\lambda_1(H_\Lambda^D) \leq C\lambda)|}{\ln\lambda} &\geq -\frac{d}{2} + \frac{\ln|\ln\lambda|}{\ln\lambda} + \frac{\ln\,C}{\ln\,\lambda}\\
&\to -\frac{d}{2}\,.
\end{aligned}
$$

Now, we turn to the case $d < \alpha < d+1$. In this case we take $\beta = \frac{1}{\alpha-d} > 1$.
Then again

$$\lambda_1(H_\Lambda^D) \leq \tilde{C}L^{-2} \quad (\text{ for } \omega \in \Omega_L^{L^{-2}})$$

and so

$$
\begin{aligned}
&\mathbb{P}(\lambda_1(H_\Lambda^D) \leq \tilde{C}L^{-2}) \geq P(\Omega_L^{L^{-2}})\\
&\geq e^{+C(\ln L^{-2})R^d} \leq e^{+C'(\ln L^{-2})L^{\frac{d}{\alpha-d}}}\,.
\end{aligned}
$$

With $\lambda = L^{-2}$ we get again

$$\mathbb{P}(\lambda_1(H_\Lambda^D) \leq \tilde{C}\lambda) \geq e^{+C'(\ln\lambda)\lambda^{\frac{d}{2(\alpha-d)}}}\ .$$

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