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LOCAL EIGENVALUE ASYMPTOTICS OF THE PERTURBED KREIN LAPLACIAN

Vincent Bruneau [1], Georgi Raikov[2]

[1] UMR CNRS 5251, Institut de Mathématiques de Bordeaux, Université de Bordeaux, Talence, France

[2] Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Chile

Local Eigenvalue Asymptotics of the Perturbed Krein Laplacian

Vincent Bruneau and Georgi Raikov

Abstract. We consider the Krein Laplacian on a regular bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, perturbed by a real-valued multiplier V vanishing on the boundary. Assuming that V has a definite sign, we investigate the asymptotics of the functions counting the eigenvalues of $K+V$ which converge to the origin from below or from above. We show that the effective Hamiltonian that governs the main asymptotic term of these functions is the harmonic Toeplitz operator T_V with symbol V, unitarily equivalent to a pseudodifferential operator on the boundary. In the cases where V admits a power-like decay at $\partial\Omega$, or V is compactly supported in Ω, and Ω and $\operatorname{supp} V$ are radially symmetric, we obtain the main asymptotic term of the eigenvalue counting functions.

1. Introduction

In this article we study the spectral properties of the perturbed Krein Laplacian $K+V$ in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, with boundary $\partial\Omega \in C^\infty$. The Krein Laplacian $K := -\Delta$ is defined on sufficiently regular functions $u : \Omega \to \mathbb{C}$ which satisfy the boundary condition

$$\frac{\partial u}{\partial \nu} = \mathcal{D}u \quad \text{on} \quad \partial\Omega,$$

where ν is the unit outer normal vector at $\partial\Omega$, and $\mathcal{D}$ is the Dirichlet-to-Neumann operator, a first-order elliptic pseudodifferential operator (ΨDO), self-adjoint in $L^2(\partial\Omega)$. Then $K \geq 0$ is self-adjoint in $L^2(\Omega)$, and one of its remarkable properties is that its essential spectrum is not empty. Namely, $\sigma_{\rm ess}(K) = \{0\}$, and the zero is an isolated eigenvalue of K of infinite multiplicity. Further, we assume that the perturbation of K is the multiplier by the function $V \in C(\overline{\Omega}; \mathbb{R})$. Then, evidently the operator $K+V$, on the domain of K, is self-adjoint in $L^2(\Omega)$. Moreover,

$$\sigma_{\rm ess}(K+V) = V(\partial\Omega)$$

(see Theorem 2.1 below). Assuming that V vanishes identically on $\partial\Omega$, we get

$$\sigma_{\rm ess}(K+V) = \sigma_{\rm ess}(K) = \{0\}.$$

However, in contrast to the unperturbed operator K, the zero is an accumulation point of the discrete eigenvalues of the perturbed operator $K+V$. We suppose

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in addition that V has a definite (negative or positive) sign and investigate the asymptotic distribution of the eigenvalues of $K+V$ adjoining the origin. First, in Theorem 2.2 we show that the effective Hamiltonian governing the eigenvalue counting functions for $K+V$ is the Toeplitz operator $T_V := PV_{|\mathrm{Ker}\,K}$, where P is the orthogonal projection onto $\mathrm{Ker}\,K$. That is why, in Section 3 we discuss the general spectral properties of T_V. Further, in Section 4 we assume that V admits a power-like decay at $\partial\Omega$, and examine the eigenvalue asymptotics for the compact operator T_V, unitarily equivalent to a classical ΨDO on the boundary. We obtain the main asymptotic term and a sharp remainder estimate of the eigenvalue counting function for T_V. Finally, in Section 5, we analyze the case where V is compactly supported in Ω. More precisely, we suppose that Ω is the unit ball in $\mathbb{R}^d$ while $\mathrm{supp}\,V$ is the concentric ball of radius $c \in (0,\infty)$, and obtain the main asymptotic term of the eigenvalue counting function for T_V.

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A more detailed exposition of some of the results of this paper can be found in the preprint [**13**].

2. The Krein Laplacian and its perturbations

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with smooth boundary $\partial\Omega$. For $s \in \mathbb{R}$ denote by $H^s(\Omega)$ and $H^s(\partial\Omega)$ the Sobolev spaces on Ω and $\partial\Omega$ respectively. Moreover, as usual, we denote by $H^s_0(\Omega)$, $s > 1/2$, the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$. Set also $H^2_D(\Omega) := H^2(\Omega) \cap H^1_0(\Omega)$. Define the minimal Laplacian

$$\Delta_{\min} := \Delta, \quad \mathrm{Dom}\,\Delta_{\min} = H^2_0(\Omega).$$

The operator $\Delta_{\min}$ is symmetric but not self-adjoint in $L^2(\Omega)$, since we have

$$\Delta^*_{\min} =: \Delta_{\max} = \Delta, \quad \mathrm{Dom}\,\Delta_{\max} = \left\{u \in L^2(\Omega) \,|\, \Delta\, u \in L^2(\Omega)\right\}, \tag{2.1}$$

Δu being the distributional Laplacian of $u \in L^2(\Omega)$. Note that

$$\mathrm{Ker}\,\Delta_{\max} = \mathcal{H}(\Omega) := \left\{u \in L^2(\Omega) \,|\, \Delta u = 0 \text{ in } \Omega\right\}. \tag{2.2}$$

It is well known that $\mathcal{H}(\Omega)$ is a closed subspace of $L^2(\Omega)$ (see e.g. [**23**]). The Laplace equation $\Delta u = 0$ in (2.2) is understood *a priori* in the distributional sense. However, by the Weyl lemma, if u belongs to $\mathcal{D}'(\Omega)$, the class of distributions over $C_0^\infty(\Omega)$, and $\Delta u = 0$, then $u \in C^\infty(\Omega)$ (see the original work [**30**] for a proof in the case $u \in L^1_{\mathrm{loc}}(\Omega)$, and the monograph [**17**] whose Chapter 10 contains an extension to general $u \in \mathcal{D}'(\Omega)$).

Lemma 2.1. *The domain* $\mathrm{Dom}\,\Delta_{\max}$ *admits the direct-sum decomposition*

$$\mathrm{Dom}\,\Delta_{\max} = \mathcal{H}(\Omega) \dotplus H^2_D(\Omega). \tag{2.3}$$

Proof. Let us first show that the sum at the r.h.s. of (2.3) is direct. Assume that $u_1 \in \mathcal{H}(\Omega)$, $u_2 \in H^2_D(\Omega)$, and $u_1 + u_2 = 0$. Then u_2 satisfies the homogeneous boundary-value problem

$$\begin{cases} \Delta u_2 = 0 & \text{in} \quad \Omega, \\ u_2 = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Hence, $u_2 = 0$, and $u_1 = 0$. Evidently, if $u_1 \in \mathcal{H}(\Omega)$, $u_2 \in H^2_D(\Omega)$, then $u_1 + u_2 \in \operatorname{Dom} \Delta_{\max}$. Pick now $u \in \operatorname{Dom} \Delta_{\max}$, and define the Dirichlet Laplacian

$$\Delta_D := \Delta, \quad \operatorname{Dom} \Delta_D := H^2_D(\Omega).$$

Then u_1 and u_2 defined by $u_2 := \Delta_D^{-1} \Delta u$, $u_1 := u - u_2$ clearly satisfy

$$u_1 \in \mathcal{H}(\Omega), \quad u_2 \in H^2_D(\Omega), \quad u = u_1 + u_2.$$

□

Introduce the Krein Laplacian

$$K := -\Delta, \quad \operatorname{Dom} K = \mathcal{H}(\Omega) \dotplus H^2_0(\Omega).$$

The operator $K \geq 0$, self-adjoint in $L^2(\Omega)$, is the von Neumann - Krein "soft" extension of $-\Delta_{\min}$, remarkable for the fact that any other self-adjoint extension $S \geq 0$ of $-\Delta_{\min}$ satisfies

$$(S + I)^{-1} \leq (K + I)^{-1}$$

(see [**29, 24**]). Evidently, $\operatorname{Ker} K = \mathcal{H}(\Omega)$. The domain $\operatorname{Dom} K$ admits a more explicit description in the terms of the Dirichlet-to-Neumann operator $\mathcal{D}$. For $f \in C^\infty(\partial\Omega)$, $\mathcal{D}f$ is defined by

$$\mathcal{D}f = \frac{\partial u}{\partial \nu}_{|\partial\Omega},$$

where u is the solution of the boundary-value problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ u = f & \text{on} \quad \partial\Omega. \end{cases}$$

The operator $\mathcal{D}$ is a first-order elliptic operator; by the elliptic regularity, it extends to a bounded operator form $H^s(\partial\Omega)$ into $H^{s-1}(\partial\Omega)$, $s \in \mathbb{R}$. Then we have

$$\operatorname{Dom} K = \left\{ u \in \operatorname{Dom} \Delta_{\max} \;\middle|\; \frac{\partial u}{\partial \nu}_{|\partial\Omega} = \mathcal{D}\left(u_{|\partial\Omega}\right) \right\}$$

(see [**18**, Theorem III.1.2]).

Denote by L the restriction of K onto $\operatorname{Dom} K \cap \mathcal{H}(\Omega)^\perp$ where $\mathcal{H}(\Omega)^\perp := L^2(\Omega) \ominus \mathcal{H}(\Omega)$. Then, L is self-adjoint in the Hilbert space $\mathcal{H}(\Omega)^\perp$.

Proposition 2.1. ([**24**], [**4**, Theorem 5.1]) *The spectrum of L is purely discrete and positive, and, hence, L^{-1} is a compact operator on $\mathcal{H}(\Omega)^\perp$. As a consequence, $\sigma_{\mathrm{ess}}(K) = \{0\}$, and the zero is an isolated eigenvalue of K of infinite multiplicity.*

The Krein Laplacian K arises naturally in the so called *abstract buckling problem*

$$\begin{cases} \Delta^2 u = -\lambda \Delta u, \\ u_{|\partial\Omega} = \frac{\partial u}{\partial \nu}_{|\partial\Omega} = 0, \\ u \in \operatorname{Dom} \Delta_{\max}. \end{cases}$$

Then the admissible values of $\lambda \neq 0$ for which the buckling problem has a non-trivial solution, coincide with the non-zero eigenvalues of K (see e.g. [**19, 5**]).

Let $V \in C(\overline{\Omega}; \mathbb{R})$. Then the operator $K+V$ with domain $\operatorname{Dom} K$ is self-adjoint in $L^2(\Omega)$. In this article, we investigate the spectral properties of $K+V$.

Remark: The perturbations K_V of the Krein Laplacian K discussed in [**6**] are of different nature than the perturbations $K+V$ considered here. Namely, the authors of [**6**] assume that $V \geq 0$, define the maximal operator $K_{V,\max}$ as

$$K_{V,\max} := -\Delta + V, \quad \operatorname{Dom} K_{V,\max} := \operatorname{Dom} \Delta_{\max},$$

and set

$$K_V := -\Delta + V, \quad \operatorname{Dom} K_V := \operatorname{Ker} K_{V,\max} \dot{+} H_0^2(\Omega).$$

Thus, if $V \neq 0$, then the operators K_V and $K_0 = K$ are self-adjoint on different domains, while the operators $K+V$ introduced here are self-adjoint on the same domain $\operatorname{Dom} K$. It is shown in [**6**] that for any $0 \leq V \in L^\infty(\Omega)$ we have $K_V \geq 0$, $\sigma_{\rm ess}(K_V) = \{0\}$, and the zero is an isolated eigenvalue of K_V of infinite multiplicity. As we will see in what follows, the spectral properties of $K+V$ could be quite different.

In Theorem 2.1 below we locate the essential spectrum of the operator $K+V$. For its proof we need some additional notations and definitions. Let $P : L^2(\Omega) \to L^2(\Omega)$ be the orthogonal projection onto $\mathcal{H}(\Omega)$. Assume that $V \in C(\overline{\Omega})$, and introduce *the harmonic Toeplitz operator* $T_V := PV : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$. Evidently, T_V is bounded, and if V is real-valued, then T_V is self-adjoint. Note that T_V could be well defined as a bounded and even compact operator for a much wider class of symbols V which are locally integrable in Ω and satisfy certain regularity properties near $\partial\Omega$.

We start our analysis with the location of the essential spectrum $\sigma_{\rm ess}(T_V)$ of T_V, and a criterion for the compactness of T_V in the case $V \in C(\overline{\Omega})$.

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with boundary $\partial\Omega \in C^\infty$. Let $V \in C(\overline{\Omega})$.*
(i) [**14**, Theorem 4.5] *We have $\sigma_{\rm ess}(T_V) = V(\partial\Omega)$.*
(ii) [**14**, Corollary 4.7] *The operator T_V is compact in $\mathcal{H}(\Omega)$ if and only if $V = 0$ on $\partial\Omega$.*

Now we are in position to locate $\sigma_{\rm ess}(K+V)$.

Theorem 2.1. *Let $V \in C(\overline{\Omega}; \mathbb{R})$. Then we have*

$$\sigma_{\rm ess}(K+V) = V(\partial\Omega). \tag{2.4}$$

In particular, $\sigma_{\rm ess}(K+V) = \{0\}$ if and only if $V_{|\partial\Omega} = 0$.

Proof. First, we will show that the operator

$$(K+V-i)^{-1} - (K+PVP-i)^{-1} \tag{2.5}$$

is compact. Set $Q := I - P$. Then

$$(K+V-i)^{-1} - (K+PVP-i)^{-1} =$$

$$-(K+V-i)^{-1}(K-i)(K-i)^{-1}(QVQ+PVQ+QVP)(K-i)^{-1}(K-i)(K+PVP-i)^{-1}. \tag{2.6}$$

Evidently, the operators $(K+V-i)^{-1}(K-i)$, $(K-i)(K+PVP-i)^{-1}$, P, and V, are bounded in $L^2(\Omega)$. Moreover, using the orthogonal decomposition $L^2(\Omega) =$

$\mathcal{H}(\Omega) \oplus \mathcal{H}(\Omega)^\perp$, and bearing in mind Proposition 2.1, we find that the operators $Q(K-i)^{-1}$ and $(K-i)^{-1}Q$ are compact in $L^2(\Omega)$. Now the compactness of the operator defined in (2.5) follows from (2.6). Therefore,

$$\sigma_{\rm ess}(K+V) = \sigma_{\rm ess}(K+PVP). \tag{2.7}$$

Further, we have $K+PVP = T_V \oplus L$ in $L^2(\Omega) = \mathcal{H}(\Omega) \oplus \mathcal{H}(\Omega)^\perp$, and, hence,

$$\sigma_{\rm ess}(K+PVP) = \sigma_{\rm ess}(T_V) \cup \sigma_{\rm ess}(L). \tag{2.8}$$

By Proposition 2.2 (i), we have $\sigma_{\rm ess}(T_V) = V(\partial\Omega)$, and by Proposition 2.1, $\sigma_{\rm ess}(L) = \emptyset$. Thus, (2.7) and (2.8) imply (2.4). $\square$

In the rest of the section we assume that $0 \leq V \in C(\overline{\Omega})$ with $V_{|\partial\Omega} = 0$, and investigate the asymptotic distribution of the discrete spectrum of the operators $K \pm V$, adjoining the origin. In particular, we show that the harmonic Toeplitz operator T_V is the effective Hamiltonian governing the main asymptotic term of the corresponding eigenvalue counting functions (see (2.11) – (2.12) below).

Let $\lambda_0 := \inf \sigma(L)$. By Proposition 2.1, we have $\lambda_0 > 0$. Introduce the eigenvalue counting functions

$$\mathcal{N}_-(\lambda) := {\rm Tr}\, \mathbb{1}_{(-\infty,-\lambda)}(K-V), \quad \lambda > 0,$$
$$\mathcal{N}_+(\lambda) := {\rm Tr}\, \mathbb{1}_{(\lambda,\lambda_0)}(K+V), \quad \lambda \in (0,\lambda_0).$$

Here and in the sequel $\mathbb{1}_S$ denotes the characteristic function of the set S; thus $\mathbb{1}_{\mathcal{I}}(T)$ is the spectral projection of the operator $T = T^*$ corresponding to the interval $\mathcal{I} \subset \mathbb{R}$. Let $T = T^*$ be a compact operator in a Hilbert space, and $s > 0$. Set

$$n_\pm(s;T) := {\rm Tr}\, \mathbb{1}_{(s,\infty)}(\pm T). \tag{2.9}$$

Thus, $n_+(s;T)$ (resp., $n_-(s;T)$) is just the number of the eigenvalues of T larger than s (resp., smaller than $-s$), counted with their multiplicities.

If $T_j = T_j^*$, $j = 1,2$, are two compact operators, then the Weyl inequalities

$$n_\pm(s_1+s_2; T_1+T_2) \leq n_\pm(s_1;T_1) + n_\pm(s_2;T_2) \tag{2.10}$$

hold for $s_j > 0$, $j=1,2$, (see e.g. [**10**, Theorem 9, Section 9.2]).

Theorem 2.2. *Assume that* $0 \leq V \in C(\overline{\Omega})$ *and* $V_{|\partial\Omega} = 0$. *Then for any* $\varepsilon \in (0,1)$ *we have*

$$n_+(\lambda;T_V) \leq \mathcal{N}_-(\lambda) \leq n_+((1-\varepsilon)\lambda;T_V) + O(1), \tag{2.11}$$

$$n_+((1+\varepsilon)\lambda;T_V) + O(1) \leq \mathcal{N}_+(\lambda) \leq n_+((1-\varepsilon)\lambda;T_V) + O(1), \tag{2.12}$$

as $\lambda \downarrow 0$.

Proof. By the Birman-Schwinger principle [**7**, Lemma 1.1], we have

$$\mathcal{N}_-(\lambda) = n_+(1;(K+\lambda)^{-1/2}V(K+\lambda)^{-1/2}) = n_+(1;V^{1/2}(K+\lambda)^{-1}V^{1/2}), \ \lambda > 0. \tag{2.13}$$

It follows from the mini-max principle that

$$n_+(1;(K+\lambda)^{-1/2}V(K+\lambda)^{-1/2}) \geq$$
$$n_+(1;P(K+\lambda)^{-1/2}V(K+\lambda)^{-1/2}P) = n_+(\lambda;PVP) = n_+(\lambda;T_V),$$

which, combined with the first equality in (2.13), implies the lower bound in (2.11). Further, by the Weyl inequalities (2.10), we have

$$n_+(1; V^{1/2}(K+\lambda)^{-1}V^{1/2}) \leq$$

$$n_+((1-\varepsilon)\lambda; V^{1/2}PV^{1/2}) + n_+(\varepsilon; V^{1/2}Q(K+\lambda)^{-1}V^{1/2}), \ \lambda > 0, \ \varepsilon \in (0,1), \tag{2.14}$$

where, as above, $Q = I - P$. Evidently,

$$n_+(s; V^{1/2}PV^{1/2}) = n_+(s; PVP) = n_+(s; T_V), \quad s > 0, \tag{2.15}$$

while Proposition 2.1 easily implies that for any $\varepsilon > 0$ we have

$$n_\pm(\varepsilon; V^{1/2}Q(K+\lambda)^{-1}V^{1/2}) = O(1), \quad \lambda \to 0. \tag{2.16}$$

Putting together (2.13) and (2.14) – (2.16), we obtain the upper bound in (2.11). In order to prove (2.12), we recall that the generalized Birman-Schwinger principle (see e.g. [**2**, Theorem 1.3]), easily entails

$$\mathcal{N}_+(\lambda) = n_-(1; V^{1/2}(K-\lambda)^{-1}V^{1/2}) + O(1), \quad \lambda \downarrow 0. \tag{2.17}$$

By the Weyl inequalities, the estimates

$$n_+((1+\varepsilon)\lambda; V^{1/2}PV^{1/2}) - n_+(\varepsilon; V^{1/2}Q(K-\lambda)^{-1}V^{1/2}) \leq$$

$$n_-(1; V^{1/2}(K-\lambda)^{-1}V^{1/2}) \leq$$

$$n_+((1-\varepsilon)\lambda; V^{1/2}PV^{1/2}) + n_-(\varepsilon; V^{1/2}Q(K-\lambda)^{-1}V^{1/2}) \tag{2.18}$$

hold true for every $\varepsilon \in (0,1)$. Now (2.17), (2.18), (2.15), and (2.16), imply (2.12). $\square$

3. General properties of harmonic Toeplitz operators

In this section we establish sufficient conditions which guarantee $T_V \in S_p$, the pth Schatten-von Neumann class, or $T_V \in S_{p,\mathrm{w}}$, the weak counterpart of S_p.

We first introduce the notations we need. Let X and Y be separable Hilbert spaces. We denote by $\mathcal{L}(X,Y)$ (resp., $S_\infty(X,Y)$) the class of linear bounded (resp., compact) operators $T : X \to Y$. Let $T \in S_\infty(X,Y)$. Next, $S_p(X,Y)$, $p \in (0,\infty)$, is the class of compact operators $T : X \to Y$ for which the functional

$$\|T\|_p := \left(\mathrm{Tr}\ (T^*T)^{p/2}\right)^{1/p}$$

is finite.

Let $T \in S_\infty(X,Y)$. Pick $s > 0$, and bearing in mind notation (2.9), set

$$n_*(s;T) := n_+(s^2; T^*T). \tag{3.1}$$

Thus, $n_*(s;T)$ is the number of the singular values of the operator T, larger than s, and counted with their multiplicities. Evidently, if $T = T^* \in S_\infty(X,X)$, then

$$n_\pm(s;T) \leq n_*(s;T), \quad s > 0. \tag{3.2}$$

Further, $S_{p,\mathrm{w}}(X,Y)$, $p \in (0,\infty)$, is the class of operators $T \in S_\infty(X,Y)$ for which the functional

$$\|T\|_{p,\mathrm{w}} := \sup_{s>0} s\, n_*(s;T)^{1/p} \tag{3.3}$$

is finite.

If $X = Y$, we write $\mathcal{L}(X)$, $S_p(X)$, and $S_{p,\mathrm{w}}(X)$, instead of $\mathcal{L}(X,X)$, $S_p(X,X)$, and $S_{p,\mathrm{w}}(X,X)$, respectively. Moreover, whenever appropriate, we omit X and Y in the notations $\mathcal{L}$, S_p, and $S_{p,\mathrm{w}}$.

Let us now turn to the study of the spectral properties of the harmonic Toeplitz operators $T_V = PV$. It is well known that the projection P onto $\mathcal{H}(\Omega)$ (see (2.2)) admits an integral kernel $\mathcal{R} \in C^\infty(\Omega \times \Omega)$, called *the reproducing kernel* of P (see e.g. [**23, 14**]). Thus

$$(Pu)(x) = \int_\Omega \mathcal{R}(x,y)u(y)dy, \quad x \in \Omega, \quad u \in L^2(\Omega).$$

Let $\{\varphi_j\}_{j\in\mathbb{N}}$ be an orthogonal basis in $\mathcal{H}(\Omega)$. Then

$$\mathcal{R}(x,y) = \sum_{j\in\mathbb{N}} \varphi_j(x)\overline{\varphi_j(y)}, \quad x, y \in \Omega, \tag{3.4}$$

the series being locally uniformly convergent in $\Omega \times \Omega$. Evidently, $\mathcal{R}(x,y)$ is independent of the choice of the basis $\{\varphi_j\}_{j\in\mathbb{N}}$. Moreover, the kernel $\mathcal{R}$ is real-valued and symmetric. For $x \in \Omega$ put

$$\varrho(x) := \mathcal{R}(x,x).$$

Then, (3.4) implies that

$$|\mathcal{R}(x,y)| \le \varrho(x)^{1/2}\, \varrho(y)^{1/2}, \quad x, y \in \Omega.$$

For $x, y \in \Omega$, set

$$r(x) := \mathrm{dist}(x, \partial\Omega), \quad \delta(x,y) := |x-y| + r(x) + r(y). \tag{3.5}$$

LEMMA 3.1. [**23**, Theorem 1.1] *For any multiindices $\alpha, \beta \in \mathbb{Z}_+^d$ there exists a constant $C_{\alpha,\beta} \in (0,\infty)$ such that*

$$\left|D_x^\alpha D_y^\beta \mathcal{R}(x,y)\right| \le \frac{C_{\alpha,\beta}}{\delta(x,y)^{d+|\alpha|+|\beta|}}, \quad x, y \in \Omega. \tag{3.6}$$

Moreover, there exists a constant $C \in (0,\infty)$ such that

$$\varrho(x) \ge C r(x)^{-d}, \quad x \in \Omega. \tag{3.7}$$

For a Borel set $\mathcal{A} \subset \Omega$ set $\rho(\mathcal{A}) := \int_{\mathcal{A}} \varrho(x)dx$. By (3.6) with $\alpha = \beta = 0$, and (3.7), ρ is an infinite σ-finite measure on Ω which is absolutely continuous with respect to the Lebesgue measure.

Our next goal is to establish conditions which guarantee $T_V \in S_p(\mathcal{H}(\Omega))$, $p \in [1,\infty)$, or $T_V \in S_{p,\mathrm{w}}(\mathcal{H}(\Omega))$, $p \in (1,\infty)$. For $p \in (0,\infty)$ define $L^p_{\mathrm{w}}(\Omega; d\rho)$ as the class of ρ-measurable functions $u : \Omega \to \mathbb{C}$ for which the quasinorm

$$\|u\|_{L^p_{\mathrm{w}}(\Omega;d\rho)} := \sup_{t>0} t\rho\left(\{x \in \Omega \,|\, |u(x)| > t\}\right)^{1/p}$$

is finite.

PROPOSITION 3.1. *Let $\Omega \subset \mathbb{R}^d$, $d \ge 2$ be a bounded domain with boundary $\partial\Omega \in C^\infty$.*

(i) *Assume $V \in L^p(\Omega; d\rho)$, $p \in [1,\infty)$. Then $T_V \in S_p(\mathcal{H}(\Omega))$ and*

$$\|T_V\|_p \le \|V\|_{L^p(\Omega;d\rho)}. \tag{3.8}$$

(ii) *Assume $V \in L^p_{\mathrm{w}}(\Omega; d\rho)$, $p \in (1,\infty)$. Then $T_V \in S_{p,\mathrm{w}}(\mathcal{H}(\Omega))$ and*

$$\|T_V\|_{p,\mathrm{w}} \le \|V\|_{L^p_{\mathrm{w}}(\Omega;d\rho)}. \tag{3.9}$$

Proof. Let us consider the operator PVP as defined on $L^2(\Omega)$. Evidently,

$$\|T_V\|_p = \|PVP\|_p, \quad \|T_V\|_{p,\mathrm{w}} = \|PVP\|_{p,\mathrm{w}}, \quad p \in (0,\infty). \tag{3.10}$$

We have $PVP = F^* e^{i \arg V} F$ where $F : L^2(\Omega) \to L^2(\Omega)$ is the operator with integral kernel

$$|V(x)|^{1/2}\mathcal{R}(x,y), \quad x,y \in \Omega.$$

Assume $V \in L^1(\Omega; d\rho)$. Then

$$\|PVP\|_1 \leq \|F^*\|_2 \|e^{i \arg V}\| \|F\|_2 = \|F\|_2^2 = \|V\|_{L^1(\Omega;d\rho)}. \tag{3.11}$$

Assume now $V \in L^\infty(\Omega; d\rho)$. Since $\|P\| = 1$ and $d\rho$ is absolutely continuous with respect to the Lebesgue measure,

$$\|PVP\| \leq \|V\|_{L^\infty(\Omega)} = \|V\|_{L^\infty(\Omega;d\rho)}. \tag{3.12}$$

Interpolating between (3.11) and (3.12), and applying [**8**, Theorem 3.1], we find that

$$\|PVP\|_p \leq \|V\|_{L^p(\Omega;d\rho)}, \quad p \in [1,\infty),$$
$$\|PVP\|_{p,\mathrm{w}} \leq \|V\|_{L^p_{\mathrm{w}}(\Omega;d\rho)}, \quad p \in (1,\infty),$$

which combined with (3.10), implies (3.8) and (3.9). □

Remark: Let $\mu \geq 0$ be a finite Borel measure on Ω. In this case, the harmonic Toeplitz operator T_μ is defined by

$$(T_\mu u)(x) := \int_\Omega \mathcal{R}(x,y) u(y) d\mu(y), \quad u \in \mathcal{H}(\Omega), \quad x \in \Omega.$$

If $d\mu(x) = V(x)dx$ with $0 \leq V \in L^1(\Omega)$, then, of course, $T_\mu = T_V$. Criteria on μ which guarantee the boundedness of T_μ, the compactness of T_μ, or the inclusion $T_\mu \in S_p(\mathcal{H}(\Omega))$, $p \in [1,\infty)$, are contained in [**14**]. These criteria are formulated in terms of the Berezin transform $\tilde{\mu}$ of the measure μ, defined by

$$\tilde{\mu}(x) := \varrho(x)^{-1} \int_\Omega \mathcal{R}(x,y)^2 d\mu(y), \quad x \in \Omega.$$

The combination of Theorem 2.2 with Proposition 3.1 entails the following

Corollary 3.1. *Let Ω be a bounded domain with boundary $\partial\Omega \in C^\infty$. Assume that $V \in C(\overline{\Omega}; \mathbb{R}) \cap L^p_{\mathrm{w}}(\Omega; d\rho)$, $p \in (1,\infty)$. Then for any $\varepsilon \in (0,1)$ we have*

$$\mathcal{N}_\pm(\lambda) \leq (1-\varepsilon)^{-p} \lambda^{-p} \|V\|^p_{L^p_{\mathrm{w}}(\Omega;d\rho)} + O(1), \tag{3.13}$$

for sufficiently small $\lambda > 0$.

Proof. Estimate (3.13) follows immediately from (2.11) - (2.12), combined with (3.2) and (3.3). □

At the end of this section we show that if the symbol V is compactly supported in Ω, then $T_V \in S_p$ for any $p \in (0,\infty)$, even if the behavior of V is quite irregular. In fact, we will replace in this case V by $\phi \in \mathcal{E}'(\Omega)$, the class of distributions over $\mathcal{E}(\Omega) := C^\infty(\Omega)$. We recall that $\phi \in \mathcal{D}'(\Omega)$ is in $\mathcal{E}'(\Omega)$, if and only if $\operatorname{supp}\phi$ is compact in Ω. If $\phi \in \mathcal{E}'(\Omega)$, we define $T_\phi : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$ as the operator with integral kernel

$$K_\phi(x,y) := (\phi, \mathcal{R}(x,\cdot)\mathcal{R}(\cdot,y)), \quad x,y \in \Omega,$$

where $(\cdot,\cdot)$ denotes the pairing between $\mathcal{E}'(\Omega)$ and $\mathcal{E}(\Omega)$. Of course, if $\phi = \mu$ and $\mu \geq 0$ is a finite Borel measure such that $\operatorname{supp}\mu$ is compact in Ω, then $T_\phi = T_\mu$.

PROPOSITION 3.2. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with boundary $\partial\Omega \in C^\infty$. Assume that $\phi \in \mathcal{E}'(\Omega)$. Then we have $T_\phi \in S_p(\mathcal{H}(\Omega))$ for any $p \in (0,\infty)$, and, hence,*

$$n_*(\lambda; T_\phi) = O(\lambda^{-\alpha}), \quad \lambda \downarrow 0, \tag{3.14}$$

for any $\alpha \in (0,\infty)$.

PROOF. Since $\operatorname{supp}\phi$ is compact in Ω, we have $K_\phi \in C^\infty(\overline{\Omega} \times \overline{\Omega})$. Therefore, (3.14) follows immediately from, say, [**8**, Proposition 2.1]. □

Remarks: (i) In Section 5 we will show that if Ω is the unit ball in $\mathbb{R}^d$, and $V \geq 0$ is compactly supported, and possesses a partial radial symmetry, then the eigenvalues of T_V decay exponentially fast.
(ii) Harmonic Toeplitz operators T_ϕ with $\phi \in \mathcal{E}'(\Omega)$ were considered in [**3**] where, in particular, it was proved that $\operatorname{rank} T_\phi < \infty$, if and only if $\operatorname{supp}\phi$ is finite.

4. Spectral asymptotics of T_V for V of power-like decay at $\partial\Omega$

4.1. Statement of the main results. In this section we assume that V is smooth and positive near the boundary, and admits a power-like decay at $\partial\Omega$, and investigate the asymptotic behavior of the discrete spectrum of T_V near the origin. We obtain the main asymptotic term of $n_+(\lambda; T_V)$ as $\lambda \downarrow 0$, and give a sharp estimate of the remainder (see Theorems 4.1 and 4.2 below).

In what follows we consider $\partial\Omega$ as a compact $(d-1)$-dimensional Riemannian manifold with metric tensor $g(y) := \{g_{jk}(y)\}_{j,k=1}^{d-1}$, $y \in \partial\Omega$, generated by the Euclidean metrics in $\mathbb{R}^d$, and denote by dS the measure induced by g on $\partial\Omega$.

Let $a, \tau \in C^\infty(\overline{\Omega})$ satisfy $a > 0$ on $\overline{\Omega}$, $\tau > 0$ on Ω, and $\tau = r := \operatorname{dist}(\cdot, \partial\Omega)$ (see (3.5)) in a neighborhood of $\partial\Omega$. Assume that

$$V(x) = \tau(x)^\gamma a(x), \quad \gamma \geq 0, \quad x \in \Omega. \tag{4.1}$$

Set $a_0 := a_{|\partial\Omega}$.

THEOREM 4.1. *Assume that V satisfies (4.1) with $\gamma > 0$. Then we have*

$$n_+(\lambda; T_V) = \mathcal{C}\, \lambda^{-\frac{d-1}{\gamma}} \left(1 + O(\lambda^{\frac{1}{\gamma}})\right), \quad \lambda \downarrow 0, \tag{4.2}$$

where

$$\mathcal{C} := \omega_{d-1} \left(\frac{\Gamma(\gamma+1)^{\frac{1}{\gamma}}}{4\pi}\right)^{d-1} \int_{\partial\Omega} a_0(y)^{\frac{d-1}{\gamma}}\, dS(y), \tag{4.3}$$

and $\omega_n = \pi^{n/2}/\Gamma(1+n/2)$ is the Lebesgue measure of the unit ball in $\mathbb{R}^n$, $n \geq 1$.

The proof of Theorem 4.1 can be found in Subsection 4.2.

The assumption of Theorem 4.1 that V satisfies (4.1) *in the whole domain* Ω is rather restrictive. In the following theorem we consider Toeplitz operators with symbols which satisfy (4.1) with $\gamma > 0$ only in a neighborhood of $\partial\Omega$ while they can have quite an irregular behavior on a compact subset of Ω; in particular, away from the boundary, these symbols are not obliged to be smooth and positive.

Theorem 4.2. *Let V satisfy the assumptions of Theorem 4.1, and $\phi \in \mathcal{E}'(\Omega;\mathbb{R})$. Then we have*

$$n_+(\lambda; T_{V+\phi}) = \mathcal{C}\,\lambda^{-\frac{d-1}{\gamma}}\left(1+O(\lambda^{\frac{\varepsilon}{\gamma}})\right), \quad \lambda \downarrow 0, \tag{4.4}$$

where $T_{V+\phi} := T_V + T_\phi$, $\mathcal{C}$ is the constant defined in (4.3), $\varepsilon = 1$ if $d \geq 3$, and $\varepsilon < 1$ is arbitrary if $d = 2$.

The proof of Theorem 4.2 is contained in Subsection 4.3.

Remark: Let $d \geq 3$. Then Theorem 4.2 implies that (4.2) remains true if we replace T_V by $T_{V+\phi}$ with $\phi \in \mathcal{E}'(\Omega;\mathbb{R})$. In particular, (4.2) is valid if $V \in L^1_{\rm loc}(\Omega;\mathbb{R})$ satisfies (4.1) with $\gamma > 0$ only in a neighborhood of $\partial\Omega$. If $d = 2$, the prize we have to pay for the substitution of T_V by $T_{V+\phi}$ is that the remainder estimate in (4.2) is better than in (4.4). However, as mentioned in the remark at the end of Subsection 4.3, if $d = 2$ and the distribution $\phi \in \mathcal{E}'(\Omega)$ is non-negative, then (4.4) holds true also with $\varepsilon = 1$.

Combining Theorems 2.1 and 4.2, we obtain the following

Corollary 4.1. *Assume that $0 \leq V \in C(\overline{\Omega})$, and (4.1) with $\gamma > 0$ holds true in a neighborhood of $\partial\Omega$. Then we have*

$$\lim_{\lambda \downarrow 0} \lambda^{\frac{d-1}{\gamma}}\,\mathcal{N}_\pm(\lambda) = \mathcal{C}, \tag{4.5}$$

$\mathcal{C}$ being the constant defined in (4.3).

Remark: Assume the hypotheses of Corollary 4.1. Then, recalling (3.6) with $\alpha = \beta = 0$, and (3.7), we find that $V \in L^p_{\rm w}(\Omega; d\rho)$ if and only if $p = \frac{d-1}{\gamma}$. Thus, if $\gamma < d-1$, then (4.5) implies that the order of our estimate (3.13) is sharp.

4.2. Proof of Theorem 4.1. Assume that $f \in H^s(\partial\Omega)$, $s \in \mathbb{R}$. Then the boundary-value problem

$$\left\{\begin{array}{lll} \Delta u = 0 & \text{in} & \Omega, \\ u = f & \text{on} & \partial\Omega, \end{array}\right. \tag{4.6}$$

admits a unique solution $u \in H^{s+1/2}(\Omega)$, we have

$$\|u\|_{H^{s+1/2}(\Omega)} \asymp \|f\|_{H^s(\partial\Omega)}, \tag{4.7}$$

and, therefore, the mapping $f \mapsto u$ defines an isomorphism between $H^s(\partial\Omega)$ and $H^{s+1/2}(\Omega)$ (see [**25**, Sections 5, 6, 7, Chapter 2]).

If $s = 0$, we set

$$u = Gf. \tag{4.8}$$

By (4.7) with $s = 0$, and the compactness of the embedding of $H^{1/2}(\Omega)$ into $L^2(\Omega)$, we find that the operator $G : L^2(\partial\Omega) \to L^2(\Omega)$ is compact. By [**16**, Theorem 12, Section 2.2], we have

$$u(x) = \int_{\partial\Omega} \mathcal{K}(x,y) f(y) dS(y), \quad x \in \Omega, \tag{4.9}$$

where

$$\mathcal{K}(x,y) := -\frac{\partial \mathcal{G}}{\partial \nu_y}(x,y), \quad x \in \Omega, \quad y \in \partial\Omega, \tag{4.10}$$

$\mathcal{G}$ is the Dirichlet Green function associated with Ω. Note that

$$\mathcal{K} \in C^\infty(\Omega \times \partial\Omega). \tag{4.11}$$

LEMMA 4.1. *We have*

$$\operatorname{Ker} G = \{0\}, \tag{4.12}$$

$$\overline{\operatorname{Ran} G} = \mathcal{H}(\Omega). \tag{4.13}$$

PROOF. Relation (4.12) follows from (4.7) with $s = 0$. Let us check (4.13). Pick $u \in \mathcal{H}(\Omega)$. Then, by (4.6) with $s = -1/2$, we have $f := u_{|\partial\Omega} \in H^{-1/2}(\partial\Omega)$. Let $f_n \in L^2(\partial\Omega)$, $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \|f_n - f\|_{H^{-1/2}(\partial\Omega)} = 0. \tag{4.14}$$

Set $u_n := Gf_n$. Then $u_n \in \operatorname{Ran} G$, $n \in \mathbb{N}$, and by (4.7) with $s = -1/2$, and (4.14), we have $\lim_{n\to\infty} \|u_n - u\|_{L^2(\Omega)} = 0$ which implies (4.13). $\square$

Set $J := G^*G$. Then the operator $J = J^* \geq 0$ is compact in $L^2(\partial\Omega)$. Due to (4.12), we have $\operatorname{Ker} J = \{0\}$. Let $\{\lambda_j\}_{j\in\mathbb{N}}$ be the non-increasing sequence of the eigenvalues $\lambda_j > 0$ of J, and let $\{\phi_j\}_{j\in\mathbb{N}}$ be the corresponding orthonormal eigenbasis in $L^2(\partial\Omega)$ with $J\phi_j = \lambda_j\phi_j$, $j \in \mathbb{N}$. Define the operator J^{-1}, self-adjoint in $L^2(\partial\Omega)$, by

(4.15)

$$J^{-1}u := \sum_{j\in\mathbb{N}} \lambda_j^{-1}\langle u, \phi_j\rangle\phi_j, \quad \operatorname{Dom} J^{-1} := \left\{ u \in L^2(\partial\Omega) \mid \sum_{j\in\mathbb{N}} \lambda_j^{-2} \left|\langle u, \phi_j\rangle\right|^2 < \infty \right\},$$

$\langle \cdot, \cdot \rangle$ being the scalar product in $L^2(\partial\Omega)$. Evidently, $\overline{JJ^{-1}} = J^{-1}J = I$.

Further, introduce the polar decomposition of the operator $G = U|G| = UJ^{1/2}$ where $U : L^2(\partial\Omega) \to L^2(\Omega)$ is an isometric operator. By Lemma 4.1, we have $\operatorname{Ker} U = \{0\}$ and $\operatorname{Ran} U = \mathcal{H}(\Omega)$. Thus, we obtain the following

PROPOSITION 4.1. *The orthogonal projection P onto $\mathcal{H}(\Omega)$ satisfies*

$$P = GJ^{-1}G^* = UU^*. \tag{4.16}$$

Assume that V satisfies (4.1) with $\gamma \geq 0$, and set $J_V := G^*VG$; from this point of view, we have $J = J_1$.

PROPOSITION 4.2. *Let V satisfy (4.1) with $\gamma \geq 0$. Then the operator T_V is unitarily equivalent to (the closure of) the operator $J^{-1/2}J_VJ^{-1/2}$.*

PROOF. By (4.16), we have

$$PVP = UJ^{-1/2}G^*VGJ^{-1/2}U^* = UJ^{-1/2}J_VJ^{-1/2}U^*,$$

and the operator U maps unitarily $L^2(\partial\Omega)$ onto $\mathcal{H}(\Omega)$. $\square$

Let $y \in \partial\Omega$ and $\eta \in T_y^*\partial\Omega$. Set

$$|\eta| = |\eta|_y := \left(\sum_{j,k=1}^{d-1} g^{jk}(y)\eta_j\eta_k \right)^{1/2},$$

where $\left\{g^{jk}(y)\right\}_{j,k=1}^{d-1}$ is the matrix inverse to $g(y)$.

PROPOSITION 4.3. *Under the assumptions of Proposition 4.2 the operator* $J^{-1/2}J_VJ^{-1/2}$ *is a* ΨDO *with principal symbol*

$$2^{-\gamma}\Gamma(\gamma+1)|\eta|^{-\gamma}a_0(y), \quad (y,\eta) \in T^*\partial\Omega. \tag{4.17}$$

PROOF. Using the pseudo-differential calculus due to L. Boutet de Monvel (see [**11, 12**]), M. Engliš showed recently in [**15**, Sections 6, 7] that if V satisfies (4.1) with $\gamma \geq 0$, then the operator J_V is a ΨDO with principal symbol

$$2^{-\gamma-1}\Gamma(\gamma+1)|\eta|^{-\gamma-1}a_0(y), \quad (y,\eta) \in T^*\partial\Omega.$$

In particular, $J = J_1$ is a ΨDO with principal symbol $2^{-1}|\eta|^{-1}$. Then the pseudo-differential calculus (see e.g. [**28**, Chapters I, II]) easily implies that $J^{-1/2}$ is a ΨDO with principal symbol $2^{1/2}|\eta|^{1/2}$, and $J^{-1/2}J_VJ^{-1/2}$ is a ΨDO with principal symbol defined in (4.17). □

Now we are in position to prove Theorem 4.1. It is easy to see that under its assumptions we have Ker $J^{-1/2}J_VJ^{-1/2} = \{0\}$. Using the spectral theorem, define the operator

$$A := \left(J^{-1/2}J_VJ^{-1/2}\right)^{-1/\gamma}$$

(cf. (4.15)). Then, by the pseudo-differential calculus, A is a ΨDO with principal symbol

$$2\Gamma(\gamma+1)^{-1/\gamma}|\eta|a_0(y)^{-1/\gamma}, \quad (y,\eta) \in T^*\partial\Omega.$$

By Proposition 4.2 and the spectral theorem, we have

$$n_+(\lambda; T_V) = n_+(\lambda; J^{-1/2}J_VJ^{-1/2}) = \mathrm{Tr}\, \mathbb{1}_{\left(-\infty,\lambda^{-1/\gamma}\right)}(A), \quad \lambda > 0. \tag{4.18}$$

A classical result of L. Hörmander [**22**] easily implies that

$$\mathrm{Tr}\, \mathbb{1}_{(-\infty,E)}(A) =$$
$$(2\pi)^{-d+1}\left|\left\{(y,\eta) \in T^*\partial\Omega \,\middle|\, 2\Gamma(\gamma+1)^{-1/\gamma}|\eta|a_0(y)^{-1/\gamma} < E\right\}\right| + O(E^{-(d-2)}) =$$
$$\mathcal{C}E^{d-1}(1 + O(E^{-1})), \quad E \to \infty, \tag{4.19}$$

where $|\cdot|$ is the Lebesgue measure on $T^*\partial\Omega$, and $\mathcal{C}$ is the constant defined in (4.3). Combining (4.18) and (4.19), we arrive at (4.2).

Remark: The natural idea to parametrize the functions $u \in \mathcal{H}(\Omega)$ by their restrictions on $\partial\Omega$ has been used in the theory of harmonic Toeplitz operators and related areas by various authors; it could be traced back at least to the classical work [**11**], and has been recently applied in [**15**] in order to obtain a suitable representation of the operator J_V. We would like also to mention here the article [**9**] where the authors consider the operator generated by the ratio of two quadratic differential forms defined on the solutions of a homogeneous elliptic equation. The order of the numerator is lower than the order of the denominator, and, since the

domain considered is supposed to be bounded and to have a regular boundary, the operator generated by the ratio is compact.
The harmonic Toeplitz operator T_V could be interpreted as the operator generated by the quadratic-form ratio

$$\frac{\int_\Omega V|u|^2\,dx}{\int_\Omega |u|^2\,dx}, \quad u \in \mathcal{H}(\Omega). \tag{4.20}$$

Note that both the numerator and the denominator in (4.20) are of zeroth order, and the compactness of T_V is now due to the fact that V vanishes at $\partial\Omega$.
In spite of the differences between the operators considered in [**9**], and the harmonic Toeplitz operators studied here, the unitary equivalence of T_V and $J^{-1/2}J_VJ^{-1/2}$ established in our Proposition 4.2 has much in common with the reduction to a ΨDO on $\partial\Omega$, performed in [**9**].

4.3. Proof of Theorem 4.2. The Weyl inequalities (2.10) imply

$$n_+(\lambda(1+\lambda^\theta);T_V) - n_-(\lambda^{1+\theta};T_\phi) \leq$$
$$n_+(\lambda;T_{V+\phi}) \leq$$
$$n_+(\lambda(1-\lambda^\theta);T_V) + n_+(\lambda^{1+\theta};T_\phi), \tag{4.21}$$

for $\lambda \in (0,1)$ and $\theta > 0$. By (4.2),

$$n_+(\lambda(1\pm\lambda^\theta);T_V) =$$
$$\mathcal{C}\left(\lambda(1\pm\lambda^\theta)\right)^{-\frac{d-1}{\gamma}} + O\left(\lambda^{-\frac{d-2}{\gamma}}\right) = \mathcal{C}\lambda^{-\frac{d-1}{\gamma}} + O\left(\lambda^{-\frac{d-2}{\gamma}}\right), \quad \lambda\in(0,1), \tag{4.22}$$

provided that $\theta > 1/\gamma$. Next, by estimate (3.14), we have

$$n_\pm(\lambda^{1+\theta};T_\phi) = O(\lambda^{-\alpha(1+\theta)}), \lambda > 0, \tag{4.23}$$

for any $\alpha \in (0,\infty)$. Assume $d \geq 3$ and choose $\alpha \in \left(0, \frac{d-2}{\gamma(1+\theta)}\right)$. Then (4.4) follows from (4.21) - (4.23). If $d = 2$, then we can pick any $\varepsilon > 0$ and choose $\alpha \in \left(0, \frac{1-\varepsilon}{\gamma(1+\theta)}\right)$, in order to check that in this case (4.21) – (4.23) again imply (4.4).

Remark: Arguing as in the proof of Theorem 4.1 (see Propositions 4.2 and 4.3), we can show that $T_{V+\phi}$ with $\phi \in \mathcal{E}'(\Omega;\mathbb{R})$ is unitarily equivalent to a self-adjoint ΨDO with principal symbol defined in (4.17). The only problem to extend in a straightforward manner our proof of Theorem 4.1 to $T_{V+\phi}$ is that this operator may have a non trivial kernel unless, for example, $\phi \geq 0$. In particular, if $d = 2$ and $\phi \in \mathcal{E}'(\Omega;\mathbb{R})$ satisfies $\phi \geq 0$, then (4.4) holds also for $\varepsilon = 1$.

5. Spectral asymptotics of T_V for compactly supported V

In this section we consider the asymptotics of $n_+(\lambda;T_V)$ as $\lambda \downarrow 0$ in the case where V is compactly supported in Ω, i.e. when V vanishes identically in a neighborhood of $\partial\Omega$. In this case T_V admits an integral kernel which is in the class $C^\infty(\overline{\Omega}\times\overline{\Omega})$, and T_V can be considered as a ΨDO of order $-\infty$.

Set

$$B_R := \left\{x \in \mathbb{R}^d \mid |x| < R\right\}, \quad d \geq 2, \quad R \in (0,\infty).$$

Since we are still unable to handle the case of general bounded Ω and compactly supported V, we suppose that Ω is the unit ball B_1 in $\mathbb{R}^d$ while $\operatorname{supp} V$ coincides with B_c with $c \in (0,1)$. Using the known fact that if V is proportional to $\mathbb{1}_{B_c}$, then the eigenvalues of T_V can be calculated explicitly, we obtain the main asymptotic term of $n_+(\lambda; T_V)$ as $\lambda \downarrow 0$, for generic T_V such that $\operatorname{supp} V = B_c$.

Let $\Omega = B_1$. Thus, $\partial\Omega = \mathbb{S}^{d-1} := \left\{x \in \mathbb{R}^d \,|\, |x| = 1\right\}$. The space $\mathcal{H}(B_1)$ admits an explicit orthonormal eigenbasis which we are now going to describe. Recall that $k(k+d-2)$, $k \in \mathbb{Z}_+$, are the eigenvalues of the Beltrami-Laplace operator $-\Delta_{\mathbb{S}^{d-1}}$, self-adjoint in $L^2(\mathbb{S}^{d-1})$ (see e.g. [**28**, Section 22]). Moreover,

$$\dim \operatorname{Ker}\,(-\Delta_{\mathbb{S}^{d-1}} - k(k+d-2)I) =: m_k = \binom{d+k-1}{d-1} - \binom{d+k-3}{d-1}$$

where $\binom{m}{n} = \frac{m!}{(m-n)!\,n!}$ if $m \geq n$, and $\binom{m}{n} = 0$ if $m < n$ (see e.g. [**28**, Theorem 22.1]). Set

$$M_k := \binom{d+k-1}{d-1} + \binom{d+k-2}{d-1}, \quad k \in \mathbb{Z}.$$

Evidently,

$$M_k = \frac{2k^{d-1}}{(d-1)!}\left(1 + O\left(k^{-1}\right)\right), \quad k \to \infty, \tag{5.1}$$

(see e.g. [**1**, Eq. 6.1.47]). By induction, we easily find that

$$\sum_{j=0}^{k} m_j = M_k, \quad k \in \mathbb{Z}_+. \tag{5.2}$$

Let $\psi_{k,\ell}$, $\ell = 1, \dots, m_k$, be an orthonormal basis in $\operatorname{Ker}\,(-\Delta_{\mathbb{S}^{d-1}} - k(k+d-2)I)$, $k \in \mathbb{Z}_+$. It is well known that $\psi_{k,\ell}$ are restrictions on $\mathbb{S}^{d-1}$ of homogeneous polynomials of degree k, harmonic in $\mathbb{R}^d$ (see e.g [**28**, Section 22]). Then the functions

$$\phi_{k,\ell}(x) := \sqrt{2k+d}\,|x|^k \psi_{k,\ell}(x/|x|), \quad x \in B_1, \quad \ell = 1, \dots, m_k, \quad k \in \mathbb{Z}_+,$$

form an orthonormal basis in $\mathcal{H}(B_1)$. Let $\mathcal{H}_k(B_1)$, $k \in \mathbb{Z}_+$, be the subspace of $\mathcal{H}(B_1)$ generated by $\phi_{k,\ell}$, $\ell = 1, \dots, m_k$.

Further, let $V(x) = v(|x|)$, $x \in B_1$, and let $v : [0,1) \to \mathbb{R}$ satisfy $\lim_{r\uparrow 1} v(r) = 0$, $v \in L^1((0,1); r^{d-1}dr)$. Then T_V is self-adjoint and compact in $\mathcal{H}(B_1)$, and

$$T_V u = \mu_k u, \quad u \in \mathcal{H}_k(B_1), \tag{5.3}$$

where

$$\mu_k(v) := (2k+d)\int_0^1 v(r) r^{2k+d-1}\,dr, \quad k \in \mathbb{Z}_+. \tag{5.4}$$

Set

$$\xi(s; v) = \#\left\{k \in \mathbb{Z}_+ \,|\, \mu_k(v) > s\right\}, \quad s > 0.$$

Let us calculate the eigenvalues of T_V in the simple model situation where $v(r) = b\,\mathbb{1}_{[0,c]}(r)$, $r \in [0,1)$, with $b > 0$ and $c \in (0,1)$. Then (5.4) implies

$$\mu_k(v) = b\,c^{2k+d}, \quad k \in \mathbb{Z}_+. \tag{5.5}$$

Evidently, the sequence $\{\mu_k(v)\}_{k\in\mathbb{Z}_+}$ is decreasing. Setting $V(x) := v(|x|)$, $x \in \mathbb{R}^d$, we get

$$n_+(\lambda; T_V) = M_{\xi(\lambda; v)-1}, \quad \lambda > 0. \tag{5.6}$$

Let us discuss the asymptotics of $n_+(\lambda; T_V)$ as $\lambda \downarrow 0$. By (5.5),

$$\xi(\lambda; v) = \frac{1}{2}\frac{|\ln \lambda|}{|\ln c|} + O(1), \quad \lambda \downarrow 0. \tag{5.7}$$

By (5.6), (5.1), and (5.7), we get

$$n_+(\lambda; T_V) = \frac{2^{-d+2}}{(d-1)!|\ln c|^{d-1}} |\ln \lambda|^{d-1} + O\left(|\ln \lambda|^{-d+2}\right), \quad \lambda \downarrow 0. \tag{5.8}$$

Remark: The fact that the basis $\{\phi_{k,\ell}\}$ diagonalizes the operator T_V with radially symmetric symbol V, acting in $\mathcal{H}(B_1)$, was noted in [**27**, Part 2.3.2], and was used there, in particular, to obtain asymptotic relations of type (5.8). The fact that the Toeplitz operators with radially symmetric symbols, acting in the holomorphic *Fock-Segal-Bargmann* space, are diagonalized in a certain canonic basis, was utilized already in [**26, 21**]. A similar result concerning Toeplitz operators with radially symmetric symbols, acting in the holomorphic *Bergman* space, can be found in [**20**].

Next, we use (5.8) in order to study the spectral asymptotics for Toeplitz operators with symbols V which possess partial radial symmetry.

Theorem 5.1. *Let $\Omega = B_1$. Assume that $V : B_1 \to [0, \infty)$ satisfies $V \in L^\infty(B_1)$ and $\operatorname{supp} V = \overline{B_c}$ for some $c \in (0, 1)$. Suppose moreover that for any $\delta \in (0, c)$ we have $\operatorname{ess\,inf}_{x \in B_\delta} V(x) > 0$. Then*

$$\lim_{\lambda \downarrow 0} |\ln \lambda|^{-d+1}\, n_+(\lambda; T_V) = \frac{2^{-d+2}}{(d-1)!|\ln c|^{d-1}}. \tag{5.9}$$

Proof. Pick $\delta \in (0, c)$. Then for almost every $x \in B_1$ we have

$$b_- \mathbb{1}_{B_\delta}(x) \leq V(x) \leq b_+ \mathbb{1}_{B_c}(x),$$

where

$$b_- := \operatorname{ess\,inf}_{x \in B_\delta} V(x), \quad b_+ := \operatorname{ess\,sup}_{x \in B_1} V(x).$$

Then the mini-max principle and (5.8) imply

$$\frac{2^{-d+2}}{(d-1)!|\ln \delta|^{d-1}} \leq$$
$$\liminf_{\lambda \downarrow 0} |\ln \lambda|^{-d+1}\, n_+(\lambda; T_V) \leq \limsup_{\lambda \downarrow 0} |\ln \lambda|^{-d+1}\, n_+(\lambda; T_V) \leq$$
$$\frac{2^{-d+2}}{(d-1)!|\ln c|^{d-1}}.$$

Letting $\delta \uparrow c$, we obtain (5.9). □

Remark: Hopefully, in a future work we will extend the result of Theorem 5.1 to more general domains Ω, and more general compactly supported V.

Putting together Theorems 2.2 and 5.1, we obtain the following

Corollary 5.1. *Let $\Omega = B_1 \subset \mathbb{R}^d$, $d \geq 2$, $0 \leq V \in C(\overline{B}_1)$. Assume that $\operatorname{supp} V = \overline{B_c}$ for some $c \in (0, 1)$, and that for any $\delta \in (0, c)$ we have $\inf_{x \in B_\delta} V(x) > 0$. Then*

$$\lim_{\lambda \downarrow 0} |\ln \lambda|^{-d+1}\, \mathcal{N}_\pm(\lambda) = \frac{2^{-d+2}}{(d-1)!|\ln c|^{d-1}}.$$

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UMR CNRS 5251, Institut de Mathématiques de Bordeaux, Université de Bordeaux, 351, Cours de la Libération, 33405 Talence, France
E-mail address: vbruneau@math.u-bordeaux.fr

Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Vicuña Mackenna 4860, Santiago de Chile
E-mail address: graikov@mat.uc.cl