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# QUANTUM WALKS WITH AN ANISOTROPIC COIN I: SPECTRAL THEORY

S Richard [1], A. Suzuki[2], R. Tiedra de Aldecoa[3]

[1] Graduate school of mathematics, Nagoya University, Chikusa-ku, Nagoya, Japan

[2] División of Mathematics and Physics, Faculty of Engineering, Shinshu University, Wakasato, Nagano, Japan

[2] Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Chile

# Quantum walks with an anisotropic coin I: spectral theory

S. Richard[1]*, A. Suzuki[2]†, R. Tiedra de Aldecoa[3]‡

[1] *Graduate school of mathematics, Nagoya University, Chikusa-ku, Nagoya 464-8602, Japan*

[2] *Division of Mathematics and Physics, Faculty of Engineering, Shinshu University, Wakasato, Nagano 380-8553, Japan*

[3] *Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Av. Vicuña Mackenna 4860, Santiago, Chile*

E-mails: *richard@math.nagoya-u.ac.jp, akito@shinshu-u.ac.jp, rtiedra@mat.puc.cl*

**Abstract**

We perform the spectral analysis of the evolution operator $U$ of quantum walks with an anisotropic coin, which include one-defect models, two-phase quantum walks, and topological phase quantum walks as special cases. In particular, we determine the essential spectrum of $U$, we show the existence of locally $U$-smooth operators, we prove the discreteness of the eigenvalues of $U$ outside the thresholds, and we prove the absence of singular continuous spectrum for $U$. Our analysis is based on new commutator methods for unitary operators in a two-Hilbert spaces setting, which are of independent interest.

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## 1 Introduction

The notion of discrete-time quantum walks appears in numerous contexts [1, 2, 16, 17, 29, 43]. Among them, Gudder [17], Meyer [29], and Ambainis et al. [2] introduced one-dimensional quantum walks as a quantum mechanical counterpart of classical random walks. Nowadays, these quantum walks and their generalisations have been physically implemented in various ways [27]. Versatile applications of quantum walks can be found in [8, 18, 32, 42] and references therein.

Recently, because of the controllability of their parameters, discrete-time quantum walks have attracted attention as promising candidates to realise topological insulators. In a series of papers [21, 22], Kitagawa et al. have shown that one and two dimensional quantum walks possess topological phases, and they experimentally observed a topologically protected bound state between two distinct phases. See [20] for an introductory review on the topological phenomena in quantum walks. Motivated by these studies, Endo et al. [11] (see also [9, 10]) have performed a thorough analysis of the asymptotic behaviour of two-phase quantum walks, whose evolution is given by a unitary operator $U_{\mathrm{TP}} = SC$ with $S$ a shift operator

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and $C$ a coin operator defined as a multiplication by unitary matrices $C(x) \in \mathsf{U}(2)$, $x \in \mathbb{Z}$. When $C(x)$ is given by

$$C(x) = \begin{cases} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{e}^{i\sigma_+} \\ \mathrm{e}^{-i\sigma_+} & -1 \end{pmatrix} & \text{if } x \geq 0 \\ \\ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{e}^{i\sigma_-} \\ \mathrm{e}^{-i\sigma_-} & -1 \end{pmatrix} & \text{if } x \leq -1 \end{cases} \tag{1.1}$$

with $\sigma_\pm \in [0, 2\pi)$, the two-phase quantum walk with evolution operator $U_{\mathrm{TP}}$ is called complete two-phase quantum walk, and when $C(x)$ satisfies the alternative condition at 0

$$C(0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.2}$$

the quantum walk is called two-phase quantum walk with one defect. In [10, 11], Endo et al. have proved a weak limit theorem [23, 24] similar to the de Moivre-Laplace theorem (or the Central limit theorem) for random walks, which describes the asymptotic behaviours of the two-phase quantum walk.

In the present paper and the companion paper [34], we consider one-dimensional quantum walks $U = SC$ with a coin operator $C$ exhibiting an anisotropic behaviour at infinity, with short-range convergence to the asymptotics. Namely, we assume that there exist matrices $C_\ell, C_\mathrm{r} \in \mathsf{U}(2)$ and positive constants $\varepsilon_\ell, \varepsilon_\mathrm{r} > 0$ such that

$$C(x) = \begin{cases} C_\ell + O\big(|x|^{-1-\varepsilon_\ell}\big) & \text{as } x \to -\infty \\ C_\mathrm{r} + O\big(|x|^{-1-\varepsilon_r}\big) & \text{as } x \to \infty. \end{cases} \tag{1.3}$$

We call this type of quantum walks quantum walks with an anisotropic coin or simply anisotropic quantum walks. They include two-phase quantum walks with coins defined by (1.1) and (1.2) and one-defect models [7, 25, 26, 45] as special cases. In the case $C_0 := C_\ell = C_\mathrm{r}$ and $\varepsilon_0 := \varepsilon_\ell = \varepsilon_\mathrm{r}$, quantum walks with an anisotropic coin reduce to one-dimensional quantum walks with a position dependent coin

$$C(x) = C_0 + O\big(|x|^{-1-\varepsilon_0}\big), \quad |x| \to \infty,$$

for which the absence of the singular continuous spectrum was proved in [4] and for which a weak limit theorem was derived in [40].

Quantum walks with an anisotropic coin are also related to Kitagawa's topological quantum walk model called a split-step quantum walk [20, 21, 22]. Indeed, if $R(\theta) \in \mathsf{U}(2)$ is a rotation matrix with rotation angle $\theta/2$, $R(\Theta_j)$ the multiplication operator by $R\big(\theta_j(\,\cdot\,)\big) \in \mathsf{U}(2)$ with $\theta_j : \mathbb{Z} \to [0, 2\pi)$, $j = 1, 2$, and $T_\downarrow, T_\uparrow$ shift operators satisfying $S = T_\downarrow T_\uparrow = T_\uparrow T_\downarrow$, then the evolution operator of the split-step quantum walk is defined as

$$U_{\mathrm{SS}}(\theta_1, \theta_2) := T_\downarrow R(\Theta_2) T_\uparrow R(\Theta_1).$$

Now, as mentioned in [20], $U_{\mathrm{SS}}(\theta_1, \theta_2)$ is unitarily equivalent to $T_\uparrow R(\Theta_1) T_\downarrow R(\Theta_2)$. Thus, our evolution operator $U$ describes a quantum walk unitarily equivalent to the one described by $U_{\mathrm{SS}}(\theta_1, \theta_2)$ if $\theta_1 \equiv 0$ and $C(\,\cdot\,) = R\big(\theta_2(\,\cdot\,)\big)$ (see [30, 39] for the definition of unitary equivalence between two quantum walks). In [20], Kitagawa dealt with the case

$$\theta_2(x) := \frac{1}{2}(\theta_{2-} + \theta_{2+}) + \frac{1}{2}(\theta_{2+} - \theta_{2-}) \tanh(x/3), \quad \theta_{2-}, \theta_{2+} \in [0, 2\pi),\ x \in \mathbb{Z},$$

which corresponds to taking the anisotropic coin (1.3) with $C_\ell = R(\theta_{2-})$ and $C_\mathrm{r} = R(\theta_{2+})$, and which cannot be covered by two-phase models.

The main goal of the present paper and [34] is to establish a weak limit theorem for the the evolution operator $U$ of the quantum walk with an anisotropic coin satisfying (1.3). As put into evidence in [40], in order to establish a weak limit theorem one has to prove along the way the following two important results:

(i) the absence of singular continuous spectrum,

(ii) the existence of the asymptotic velocity.

In the present paper, we perform the spectral analysis of the evolution operator $U$ of quantum walks with an anisotropic coin. We determine the essential spectrum of $U$, we show the existence of locally $U$-smooth operators, we prove the discreteness of the eigenvalues of $U$ outside the thresholds, and we prove the absence of singular continuous spectrum for $U$. In the companion paper [34], we will develop the scattering theory for the evolution operator $U$. We will prove the existence and the completeness of wave operators for $U$ and a free evolution operator $U_0$, we will show the existence of the asymptotic velocity for $U$, and we will finally establish a weak limit theorem for $U$. Other interesting related topics such as the existence and the robustness of a bound state localised around the phase boundary or a weak limit theorem for the split-step quantum walk with $\theta_1 \neq 0$ are considered in [14] and [13], respectively.

The rest of this paper is structured as follows. In Section 2, we give the precise definition of the evolution operator $U$ for the quantum walk with an anisotropic coin and we state our main results on the essential spectrum of $U$ (Theorem 2.2), the locally $U$-smooth operators (Theorem 2.3), and the eigenvalues and singular continuous spectrum of $U$ (Theorem 2.4). Section 3 is devoted to mathematical preliminaries. Here we develop new commutator methods for unitary operators in a two-Hilbert spaces setting, which are a key ingredient for our analysis and are of independent interest. In Section 4, we prove our main theorems as an application of the commutator methods developed in Section 3. In Subsection 4.2, we prove Theorem 2.2 and we define in Lemma 4.9 a conjugate operator $A$ for the evolution operator $U$ built from conjugate operators for the asymptotic evolution operators $U_\ell := SC_\ell$ and $U_{\mathrm{r}} := SC_{\mathrm{r}}$, where $C_\ell$ and $C_{\mathrm{r}}$ are the constant coin matrices given in (1.3). Finally, in Subsection 4.3 we prove Theorems 2.3 and 2.4.

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# 2 Model and main results

In this section, we give the definition of the model of anisotropic quantum walks that we consider, we state our main results on quantum walks, and we present the main tools we use for the proofs. These tools are results of independent interest on commutator methods for unitary operators in a two-Hilbert spaces setting. The proofs of our results on commutator methods are given in Section 3 and the proofs of our results on quantum walks are given in Section 4.

Let us consider the Hilbert space of square-summable $\mathbb{C}^2$-valued sequences

$$\mathcal{H} := \ell^2(\mathbb{Z}, \mathbb{C}^2) = \left\{ \Psi : \mathbb{Z} \to \mathbb{C}^2 \mid \sum_{x\in\mathbb{Z}} \|\Psi(x)\|_2^2 < \infty \right\},$$

where $\|\cdot\|_2$ is the usual norm on $\mathbb{C}^2$. The evolution operator of the one-dimensional quantum walk in $\mathcal{H}$ that we consider is defined by $U := SC$, with $S$ a shift operator and $C$ a coin operator defined as follows. The operator $S$ is given by

$$(S\Psi)(x) := \begin{pmatrix} \Psi^{(0)}(x+1) \\ \Psi^{(1)}(x-1) \end{pmatrix}, \quad \Psi = \begin{pmatrix} \Psi^{(0)} \\ \Psi^{(1)} \end{pmatrix} \in \mathcal{H},\ x \in \mathbb{Z},$$

and the operator $C$ is given by

$$(C\Psi)(x) := C(x)\Psi(x), \quad \Psi \in \mathcal{H},\ x \in \mathbb{Z},\ C(x) \in \mathrm{U}(2).$$

In particular, the evolution operator $U$ is unitary in $\mathcal{H}$ since both $S$ and $C$ are unitary in $\mathcal{H}$.

Throughout the paper, we assume that the coin operator $C$ exhibits an anisotropic behaviour at infinity. More precisely, we assume that $C$ converges with short-range rate to two asymptotic coin operators, one on the left and one on the right in the following way:

**Assumption 2.1** (Short-range assumption)**.** *There exist $C_\ell, C_r \in \mathrm{U}(2)$, $\kappa_\ell, \kappa_r > 0$, and $\varepsilon_\ell, \varepsilon_r > 0$ such that*

$$\begin{aligned} \big\|C(x) - C_\ell\big\|_{\mathscr{B}(\mathbb{C}^2)} &\le \kappa_\ell\,|x|^{-1-\varepsilon_\ell} \quad \text{if } x < 0 \\ \big\|C(x) - C_r\big\|_{\mathscr{B}(\mathbb{C}^2)} &\le \kappa_r\,|x|^{-1-\varepsilon_r} \quad \text{if } x > 0, \end{aligned}$$

*where the indexes $\ell$ and $\mathrm{r}$ stand for "left" and "right".*

This assumption provides us with two new unitary operators

$$U_\ell := SC_\ell \quad \text{and} \quad U_r := SC_r \tag{2.1}$$

describing the asymptotic behaviour of $U$ on the left and on the right. The precise sense (from the scattering point of view) in which the operators $U_\ell$ and $U_r$ describe the asymptotic behaviour of $U$ on the left and on the right will be given in [34], and the spectral properties of $U_\ell$ and $U_r$ are determined in Section 4.1. Here, we just introduce the set

$$\tau(U) := \partial\sigma(U_\ell) \cup \partial\sigma(U_r),$$

where $\partial\sigma(U_\ell), \partial\sigma(U_r)$ denote the boundaries in the unit circle $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ of the spectra $\sigma(U_\ell), \sigma(U_r)$ of $U_\ell, U_r$. In Section 4.1, we show that $\tau(U)$ is finite and can be interpreted as the set of thresholds in the spectrum of $U$.

Our main results on the operator $U$, proved in Sections 4.2 and 4.3, are the following three theorems on locally $U$-smooth operators and on the structure of the spectrum of $U$. The symbols $\sigma_{\mathrm{ess}}(U)$, $\sigma_{\mathrm{p}}(U)$ and $Q$ stand for the essential spectrum of $U$, the pure point spectrum of $U$, and the position operator in $\mathcal{H}$, respectively (see (4.9) for precise definition of $Q$).

**Theorem 2.2** (Essential spectrum of $U$)**.** *One has $\sigma_{\mathrm{ess}}(U) = \sigma(U_\ell) \cup \sigma(U_r)$.*

**Theorem 2.3** ($U$-smooth operators)**.** *Let $\mathcal{G}$ be an auxiliary Hilbert space and let $\Theta \subset \mathbb{T}$ be an open set with closure $\overline{\Theta} \subset \mathbb{T} \setminus \tau(U)$. Then each operator $T \in \mathscr{B}(\mathcal{H}, \mathcal{G})$ which extends continuously to an element of $\mathscr{B}\big(\mathcal{D}(\langle Q\rangle^{-s}), \mathcal{G}\big)$ for some $s > 1/2$ is locally $U$-smooth on $\Theta \setminus \sigma_{\mathrm{p}}(U)$.*

**Theorem 2.4** (Spectrum of $U$)**.** *For any closed set $\Theta \subset \mathbb{T} \setminus \tau(U)$, the operator $U$ has at most finitely many eigenvalues in $\Theta$, each one of finite multiplicity, and $U$ has no singular continuous spectrum in $\Theta$.*

To prove these theorems, we develop in Section 3 commutator methods for unitary operators in a two-Hilbert spaces setting: Given a triple $(\mathcal{H}, U, A)$ consisting in a Hilbert space $\mathcal{H}$, a unitary operator $U$, and a self-adjoint operator $A$, we determine how to obtain commutator results for $(\mathcal{H}, U, A)$ in terms of commutator results for a second triple $(\mathcal{H}_0, U_0, A_0)$ also consisting in a Hilbert space, a unitary operator, and a self-adjoint operator. In the process, a bounded identification operator $J : \mathcal{H}_0 \to \mathcal{H}$ must also be chosen. The intuition behind this approach comes from scattering theory which tells us that given a unitary operator $U$ describing some quantum system in a Hilbert space $\mathcal{H}$ there often exists a simpler unitary operator $U_0$ in a second Hilbert space $\mathcal{H}_0$ describing the same quantum system in some asymptotic regime.

Our main results in this context are the following. First, we present in Theorem 3.7 conditions guaranteeing that $U$ and $A$ satisfy a Mourre estimate on a Borel set $\Theta \subset \mathbb{T}$ as soon as $U_0$ and $A_0$ satisfy a Mourre estimate on $\Theta$ (equivalently, we present conditions guaranteeing that $A$ is a conjugate operator for $U$ on $\Theta$ as soon as $A_0$ is a conjugate operator for $U_0$ on $\Theta$). Next, we present in Proposition 3.8 conditions guaranteeing that $U$ is regular with respect to $A$ (that is, $U \in C^1(A)$) as soon as $U_0$ is regular with respect to $A_0$ (that is, $U_0 \in C^1(A_0)$). Finally, we give in Assumption 3.10 and Corollaries 3.11-3.12 conditions guaranteeing that the most natural choice for the operator $A$, namely $A = JA_0J^*$, is indeed a conjugate operator for $U$ as soon as $A_0$ is a conjugate operator for $U_0$.

## 3 Unitary operators in a two-Hilbert spaces setting

In this section, we recall some facts on the spectral family of unitary operators, the Cayley transform of a unitary operator, locally smooth operators for unitary operators, and commutator methods for unitary operators in one Hilbert space. We also present new results on commutator methods for unitary operators in a two-Hilbert spaces setting.

### 3.1 Cayley transform

Let $\mathcal{H}$ be a Hilbert space with norm $\|\cdot\|_{\mathcal{H}}$ and scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{H}}$ linear in the second argument, let $\mathscr{B}(\mathcal{H})$ be the set of bounded linear operators in $\mathcal{H}$ with norm $\|\cdot\|_{\mathscr{B}(\mathcal{H})}$, and let $\mathscr{K}(\mathcal{H})$ be the set of compact linear operators in $\mathcal{H}$. A unitary operator $U$ in $\mathcal{H}$ is a surjective isometry, that is, an element $U\in\mathscr{B}(\mathcal{H})$ satisfying $U^*U=UU^*=1$. Since $U^*U=UU^*$, the spectral theorem for normal operators implies that $U$ admits exactly one complex spectral family $E_U$, with support $\operatorname{supp}(E_U)\subset\mathbb{T}$, such that $U=\int_{\mathbb{C}}z\,E_U(\mathrm{d}z)$. The support $\operatorname{supp}(E_U)$ is the set of points of non-constancy of $E_U$, which coincides with the spectrum $\sigma(U)$ of $U$ [44, Thm. 7.34(a)]. For each $s,t\in\mathbb{R}$, one has the factorization

$$E_U(s+it):=E_{\mathrm{Re}(U)}(s)\,E_{\mathrm{Im}(U)}(t),$$

where $E_{\mathrm{Re}(U)}$ and $E_{\mathrm{Im}(U)}$ are the real spectral families of the bounded self-adjoint operators

$$\mathrm{Re}(U):=\tfrac{1}{2}\,(U+U^*)\quad\text{and}\quad\mathrm{Im}(U):=\tfrac{1}{2i}\,(U-U^*).$$

One can associate in a canonical way a real spectral family $\widetilde{E}_U$, with support $\operatorname{supp}(\widetilde{E}_U)\subset[0,2\pi]$, to the complex spectral family $E_U$ by noting that

$$U=\int_{\mathbb{R}}\mathrm{e}^{i\lambda}\,\widetilde{E}_U(\mathrm{d}\lambda)\quad\text{with}\quad\widetilde{E}_U(\lambda):=\begin{cases}0&\text{if }\lambda<0\\E_U(\mathrm{e}^{i\lambda})&\text{if }\lambda\in[0,2\pi)\\1&\text{if }\lambda\ge2\pi.\end{cases}$$

Since $\widetilde{E}_U$ is a real spectral family, the corresponding real spectral measure $\widetilde{E}^U$ admits the decomposition

$$\widetilde{E}^U=\widetilde{E}^U_{\mathrm{p}}+\widetilde{E}^U_{\mathrm{sc}}+\widetilde{E}^U_{\mathrm{ac}},$$

with $\widetilde{E}^U_{\mathrm{p}}$, $\widetilde{E}^U_{\mathrm{sc}}$, $\widetilde{E}^U_{\mathrm{ac}}$ the pure point, the singular continuous, and the absolutely continuous components of $\widetilde{E}_U$, respectively. The corresponding subspaces $\mathcal{H}_{\mathrm{p}}(U):=\widetilde{E}^U_{\mathrm{p}}(\mathbb{R})\mathcal{H}$, $\mathcal{H}_{\mathrm{sc}}(U):=\widetilde{E}^U_{\mathrm{sc}}(\mathbb{R})\mathcal{H}$, $\mathcal{H}_{\mathrm{ac}}(U):=\widetilde{E}^U_{\mathrm{ac}}(\mathbb{R})\mathcal{H}$ provide an orthogonal decomposition

$$\mathcal{H}=\mathcal{H}_{\mathrm{p}}(U)\oplus\mathcal{H}_{\mathrm{sc}}(U)\oplus\mathcal{H}_{\mathrm{ac}}(U)$$

which reduces the operator $U$. The sets

$$\sigma_{\mathrm{p}}(U):=\sigma\big(U|_{\mathcal{H}_{\mathrm{p}}(U)}\big),\quad\sigma_{\mathrm{sc}}(U):=\sigma\big(U|_{\mathcal{H}_{\mathrm{sc}}(U)}\big),\quad\sigma_{\mathrm{ac}}(U):=\sigma\big(U|_{\mathcal{H}_{\mathrm{ac}}(U)}\big),$$

are called pure point spectrum, singular continuous spectrum, and absolutely spectrum continuous of $U$, respectively, and the set $\sigma_{\mathrm{c}}(U):=\sigma_{\mathrm{sc}}(U)\cup\sigma_{\mathrm{ac}}(U)$ is called the continuous spectrum of $U$.

If $1\notin\sigma_{\mathrm{p}}(U)$, then the subspace $(1-U)\mathcal{H}$ is dense in $\mathcal{H}$, and the Cayley transform of $U$ given by

$$H\varphi:=i(1+U)(1-U)^{-1}\varphi,\qquad\varphi\in\mathcal{D}(H):=(1-U)\mathcal{H},\tag{3.1}$$

is a self-adjoint operator in $\mathcal{H}$ [44, Thm. 8.4(b)]. Also, a simple calculation shows that

$$U=(H-i)(H+i)^{-1}=\mathrm{e}^{iL}\quad\text{with}\quad L:=2\arctan(H)+\pi.\tag{3.2}$$

Therefore, the points of the spectra $\sigma(L) \subset [0, 2\pi]$ of $L$ and $\sigma(U) \subset \mathbb{T}$ of $U$ are linked by the relation

$$\theta \in \sigma(U) \ \Leftrightarrow \ 2\arctan\left(i\frac{1+\theta}{1-\theta}\right) + \pi \in \sigma(L)$$

(in particular, the point $\theta = 1$ in $\sigma(U)$ corresponds to the points $\lambda = 0$ and $\lambda = 2\pi$ in $\sigma(L)$). In consequence, if $E^L$ denotes the real spectral measure of $L$, one has for any Borel set $\Theta \subset \mathbb{T}$ the equality

$$E^U(\Theta) = E^L(\Lambda) \quad \text{with} \quad \Lambda := \left\{2\arctan\left(i\frac{1+\theta}{1-\theta}\right) + \pi \mid \theta \in \Theta\right\}. \tag{3.3}$$

This implies for each Borel set $\Lambda \subset [0, 2\pi)$ that

$$\widetilde{E}^U(\Lambda) = E^U(\mathrm{e}^{i\Lambda}) = E^L\big(f(\Lambda)\big) \quad \text{with} \quad f(\lambda) := \begin{cases} 0 & \text{if } \ \lambda = 0 \\ 2\arctan\left(i\frac{1+\mathrm{e}^{i\lambda}}{1-\mathrm{e}^{i\lambda}}\right) + \pi & \text{if } \ \lambda \in (0, 2\pi). \end{cases}$$

But a simple calculation shows that $f(\lambda) = \lambda$ for each $\lambda \in [0, 2\pi)$. So, one has $\widetilde{E}^U(\Lambda) = E^L(\Lambda)$ for each Borel set $\Lambda \subset [0, 2\pi)$. Now, it is also clear from the definitions that $\widetilde{E}^U(\Lambda) = E^L(\Lambda)$ for each Borel set $\Lambda \subset \mathbb{R} \setminus [0, 2\pi)$. So, one concludes that $\widetilde{E}^U = E^L$, and thus that $U$ and $L$ possess the same spectral properties, up to the correspondence $U = \mathrm{e}^{iL}$.

## 3.2 Locally $U$-smooth operators

Let $U$ be a unitary operator in a Hilbert space $\mathcal{H}$, and let $\mathcal{G}$ be an auxiliary Hilbert space. Then, an operator $T \in \mathscr{B}(\mathcal{H}, \mathcal{G})$ is locally $U$-smooth on an open set $\Theta \subset \mathbb{T}$ if for each closed set $\Theta' \subset \Theta$ there exists $c_{\Theta'} \geq 0$ such that

$$\sum_{n \in \mathbb{Z}} \big\|T U^n E^U(\Theta')\varphi\big\|_{\mathcal{G}}^2 \leq c_{\Theta'} \|\varphi\|_{\mathcal{H}}^2 \quad \text{for each } \varphi \in \mathcal{H}, \tag{3.4}$$

and $T$ is (globally) $U$-smooth if (3.4) is satisfied with $\Theta' = \mathbb{T}$. The condition (3.4) is invariant under rotation by $\omega \in \mathbb{T}$ in the sense that if $T$ is $U$-smooth on $\Theta$, then $T$ is $(\omega U)$-smooth on $\omega\Theta$ since

$$\big\|T(\omega U)^n E^{\omega U}(\omega\Theta')\varphi\big\|_{\mathcal{G}} = \big\|T U^n E^U(\Theta')\varphi\big\|_{\mathcal{G}}$$

for each closed set $\Theta' \subset \Theta$ and each $\varphi \in \mathcal{H}$. An important consequence of the existence of a locally $U$-smooth operator $T$ on $\Theta$ is the inclusion $\overline{E^U(\Theta)T^*\mathcal{G}^*} \subset \mathcal{H}_{\mathrm{ac}}(U)$, with $\mathcal{G}^*$ the adjoint space of $\mathcal{G}$ (see [5, Thm. 2.1] for a proof).

Local smoothness with respect to a self-adjoint operator $H$ in $\mathcal{H}$ with domain $\mathcal{D}(H)$ is defined in a similar way. An operator $T \in \mathscr{B}\big(\mathcal{D}(H), \mathcal{G}\big)$ is locally $H$-smooth on an open set $\Lambda \subset \mathbb{R}$ if for each compact set $\Lambda' \subset \Lambda$ there exists $c_{\Lambda'} \geq 0$ such that

$$\int_{\mathbb{R}} \big\|T\,\mathrm{e}^{-itH}\,E^H(\Lambda')\varphi\big\|_{\mathcal{G}}^2 \,\mathrm{d}t \leq c_{\Lambda'} \|\varphi\|_{\mathcal{H}}^2 \quad \text{for each } \varphi \in \mathcal{H}, \tag{3.5}$$

and $T$ is (globally) $H$-smooth if (3.5) is satisfied with $\Lambda' = \mathbb{R}$. The condition (3.5) is invariant under translation by $s \in \mathbb{R}$ in the sense that if $T$ is $H$-smooth on $\Lambda$, then $T$ is $(H+s)$-smooth on $\Lambda + s$ since

$$\big\|T\,\mathrm{e}^{-it(H+s)}\,E^{H+s}(\Lambda'+s)\varphi\big\|_{\mathcal{G}} = \big\|T\,\mathrm{e}^{-itH}\,E^H(\Lambda')\varphi\big\|_{\mathcal{G}}$$

for each compact set $\Lambda' \subset \Lambda$ and each $\varphi \in \mathcal{H}$. Also, the existence of a locally $H$-smooth operator $T$ on $\Lambda \subset \mathbb{R}$ implies the inclusion $\overline{E^H(\Lambda)T^*\mathcal{G}^*} \subset \mathcal{H}_{\mathrm{ac}}(H)$ (see [3, Cor 7.1.2] for a proof).

If $1 \notin \sigma_{\mathrm{p}}(U)$, then the Cayley transform $H$ of $U$ and the operator $L = 2\arctan(H) + \pi$ are defined by (3.1) and (3.2), and the existence of locally $U$-smooth operators is equivalent to the existence of locally $H$-smooth operators and locally $L$-smooth operators:

**Lemma 3.1.** *Let $U$ be a unitary operator in a Hilbert space $\mathcal{H}$ with $1 \notin \sigma_{\rm p}(U)$, let $\mathcal{G}$ be an auxiliary Hilbert space, let $T \in \mathscr{B}(\mathcal{H},\mathcal{G})$, and let $\Theta \subset \mathbb{T}$ be an open set. Then, the following are equivalent:*

*(i) $T$ is locally $U$-smooth on $\Theta$,*

*(ii) $T$ is locally $L$-smooth on $\big\{2\arctan\big(i\,\frac{1+\theta}{1-\theta}\big)+\pi \mid \theta \in \Theta\big\}$,*

*(iii) $T(H+i)$ is locally $H$-smooth on $\big\{i\,\frac{1+\theta}{1-\theta} \mid \theta \in \Theta\big\}$.*

The equivalence (i) $\Leftrightarrow$ (ii) in the case $\Theta = \mathbb{T}$ is due to T. Kato (see [19, Sec. 7]).

*Proof.* Assume that $T$ is locally $U$-smooth on $\Theta$, take a closed set $\Theta' \subset \Theta$, and let

$$\Lambda' := \left\{2\arctan\left(i\,\frac{1+\theta}{1-\theta}\right)+\pi \mid \theta \in \Theta'\right\}.$$

Then, Equations (3.2)-(3.3) and Tonnelli's theorem imply for each $\varphi \in \mathcal{H}$ that

$$\begin{aligned}
\int_{\mathbb{R}} \big\|T\,\mathrm{e}^{-itL}\,E^L(\Lambda')\varphi\big\|_{\mathcal{G}}^2\,\mathrm{d}t &= \sum_{n\in\mathbb{Z}}\int_0^1 \big\|T\,\mathrm{e}^{-i(s+n)L}\,E^L(\Lambda')\varphi\big\|_{\mathcal{G}}^2\,\mathrm{d}s\\
&= \int_0^1 \sum_{n\in\mathbb{Z}} \big\|TU^{-n}E^U(\Theta')\,\mathrm{e}^{-isL}\,\varphi\big\|_{\mathcal{G}}^2\,\mathrm{d}s\\
&\le \int_0^1 c_{\Theta'}\,\big\|\,\mathrm{e}^{-isL}\,\varphi\big\|_{\mathcal{H}}^2\,\mathrm{d}s\\
&= c_{\Theta'}\,\|\varphi\|_{\mathcal{H}}^2\,.
\end{aligned}$$

This shows the implication (i) $\Rightarrow$ (ii). The implication (ii) $\Rightarrow$ (i) is shown in a similar way.

To show the equivalence (i) $\Leftrightarrow$ (iii) we observe that (3.4) is equivalent to

$$\sup_{z\in\mathbb{D},\,\psi\in\mathcal{G},\,\|\psi\|_{\mathcal{G}}=1}\Big|\big\langle\psi,T\,\delta(U,z)E^U(\Theta')T^*\psi\big\rangle_{\mathcal{G}}\Big| < \infty,$$

with $\mathbb{D} := \{z\in\mathbb{C} \mid |z|<1\}$ and $\delta(U,z) := (1-zU^{-1})^{-1}-(1-\overline{z}^{-1}U^{-1})^{-1}$ (this follows from the proof of [5, Thm. 2.2]), and we observe that (3.5) is equivalent to

$$\sup_{\omega\in\mathbb{H},\,\psi\in\mathcal{G},\,\|\psi\|_{\mathcal{G}}=1}\Big|\big\langle\psi,T\,\mathrm{Im}\,\big((H-\omega)^{-1}\big)E^H(\Lambda')T^*\psi\big\rangle_{\mathcal{G}}\Big| < \infty$$

with $\mathbb{H} := \{z\in\mathbb{C} \mid \mathrm{Im}(z)>0\}$ (this follows from [3, Prop. 7.1.1]). Also, we note that

$$\delta(U,z) = \big(H^2+1\big)\,\mathrm{Im}\left(\left(H-i\,\frac{1+z}{1-z}\right)^{-1}\right),\quad z\in\mathbb{D},$$

and we recall that the map $\mathbb{D}\ni z\mapsto i\,\frac{1+z}{1-z}\in\mathbb{H}$ (the Cayley transform) is a bijection. So, for any closed set $\Theta'\subset\Theta$, we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D},\,\psi\in\mathcal{G},\,\|\psi\|_{\mathcal{G}}=1}\Big|\big\langle\psi,T\,\delta(U,z)E^U(\Theta')T^*\psi\big\rangle_{\mathcal{G}}\Big|\\
&= \sup_{\omega\in\mathbb{H},\,\psi\in\mathcal{G},\,\|\psi\|_{\mathcal{G}}=1}\Big|\big\langle\psi,T(H+i)\,\mathrm{Im}\,\big((H-\omega)^{-1}\big)E^H(\Lambda')\big(T(H+i)\big)^*\psi\big\rangle_{\mathcal{G}}\Big|
\end{aligned}$$

with $\Lambda' = \big\{i\,\frac{1+\theta}{1-\theta} \mid \theta\in\Theta'\big\}$, and thus (i) and (iii) are equivalent (note that the operator

$$T(H+i)\,\mathrm{Im}\,\big((H-\omega)^{-1}\big)E^H(\Lambda')\big(T(H+i)\big)^*$$

belongs to $\mathscr{B}(\mathcal{G})$ for each $\omega\in\mathbb{H}$ even if $1\in\Theta'$, that is, even if $\Lambda'$ is not bounded). $\square$

## 3.3 Commutator methods in one Hilbert space

In this section, we present some results on commutator methods for unitary operators in one Hilbert space $\mathcal{H}$. We start by recalling definitions and results borrowed from [3, 12, 38]. Let $S \in \mathscr{B}(\mathcal{H})$ and let $A$ be a self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(A)$. For any $k \in \mathbb{N}$, we say that $S$ belongs to $C^k(A)$, with notation $S \in C^k(A)$, if the map

$$\mathbb{R} \ni t \mapsto \mathrm{e}^{-itA} S \, \mathrm{e}^{itA} \in \mathscr{B}(\mathcal{H})$$

is strongly of class $C^k$. In the case $k = 1$, one has $S \in C^1(A)$ if and only if the quadratic form

$$\mathcal{D}(A) \ni \varphi \mapsto \langle A\varphi, S\varphi \rangle_{\mathcal{H}} - \langle \varphi, SA\varphi \rangle_{\mathcal{H}} \in \mathbb{C}$$

is continuous for the topology induced by $\mathcal{H}$ on $\mathcal{D}(A)$. The operator corresponding to the continuous extension of the form is denoted by $[A, S] \in \mathscr{B}(\mathcal{H})$, and it verifies

$$[A, S] = \operatorname*{s-lim}_{\tau \to 0} [A_\tau, S] \quad \text{with} \quad A_\tau := (i\tau)^{-1}\big(\mathrm{e}^{i\tau A} - 1\big) \in \mathscr{B}(\mathcal{H}), \quad \tau \in \mathbb{R} \setminus \{0\}.$$

Three regularity conditions slightly stronger than $S \in C^1(A)$ are defined as follows: $S$ belongs to $C^{1,1}(A)$, with notation $S \in C^{1,1}(A)$, if

$$\int_0^1 \big\| \mathrm{e}^{-itA} S \, \mathrm{e}^{itA} + \mathrm{e}^{itA} S \, \mathrm{e}^{-itA} - 2S \big\|_{\mathscr{B}(\mathcal{H})} \frac{\mathrm{d}t}{t^2} < \infty.$$

$S$ belongs to $C^{1+0}(A)$, with notation $S \in C^{1+0}(A)$, if $S \in C^1(A)$ and

$$\int_0^1 \big\| \mathrm{e}^{-itA}[A, S] \, \mathrm{e}^{itA} - [A, S] \big\|_{\mathscr{B}(\mathcal{H})} \frac{\mathrm{d}t}{t} < \infty.$$

$S$ belongs to $C^{1+\varepsilon}(A)$ for some $\varepsilon \in (0, 1)$, with notation $S \in C^{1+\varepsilon}(A)$, if $S \in C^1(A)$ and

$$\big\| \mathrm{e}^{-itA}[A, S] \, \mathrm{e}^{itA} - [A, S] \big\|_{\mathscr{B}(\mathcal{H})} \leq \text{Const.}\, t^\varepsilon \quad \text{for all } t \in (0, 1).$$

As banachisable topological vector spaces, the sets $C^2(A)$, $C^{1+\varepsilon}(A)$, $C^{1+0}(A)$, $C^{1,1}(A)$, $C^1(A)$, and $C^0(A) = \mathscr{B}(\mathcal{H})$, satisfy the continuous inclusions [3, Sec. 5.2.4]

$$C^2(A) \subset C^{1+\varepsilon}(A) \subset C^{1+0}(A) \subset C^{1,1}(A) \subset C^1(A) \subset C^0(A).$$

Now, we adapt to the case of unitary operators the definition of two useful functions introduced in [3, Sec. 7.2] in the case of self-adjoint operators. For that purpose, we let $U$ be a unitary operator with $U \in C^1(A)$, for $S, T \in \mathscr{B}(\mathcal{H})$ we write $T \gtrsim S$ if there exists an operator $K \in \mathscr{K}(\mathcal{H})$ such that $T + K \geq S$, and for $\theta \in \mathbb{T}$ and $\varepsilon > 0$ we set

$$\Theta(\theta; \varepsilon) := \big\{\theta' \in \mathbb{T} \mid |\arg(\theta - \theta')| < \varepsilon\big\} \quad \text{and} \quad E^U(\theta; \varepsilon) := E^U\big(\Theta(\theta; \varepsilon)\big).$$

With these notations at hand, we define the functions $\varrho_U^A : \mathbb{T} \to (-\infty, \infty]$ and $\widetilde{\varrho}_U^A : \mathbb{T} \to (-\infty, \infty]$ by

$$\varrho_U^A(\theta) := \sup\big\{a \in \mathbb{R} \mid \exists \varepsilon > 0 \text{ such that } E^U(\theta; \varepsilon) U^{-1}[A, U] E^U(\theta; \varepsilon) \geq a \, E^U(\theta; \varepsilon)\big\}$$

and

$$\widetilde{\varrho}_U^A(\theta) := \sup\big\{a \in \mathbb{R} \mid \exists \varepsilon > 0 \text{ such that } E^U(\theta; \varepsilon) U^{-1}[A, U] E^U(\theta; \varepsilon) \gtrsim a \, E^U(\theta; \varepsilon)\big\}.$$

In applications, the function $\widetilde{\varrho}_U^A$ is more convenient than the function $\varrho_U^A$ since it is defined in terms of a weaker positivity condition (positivity up to compact terms). A simple argument shows that $\widetilde{\varrho}_U^A(\theta)$ can be defined in an equivalent way by

$$\widetilde{\varrho}_U^A(\theta) = \sup\big\{a \in \mathbb{R} \mid \exists \eta \in C^\infty(\mathbb{T}, \mathbb{R}) \text{ such that } \eta(\theta) \neq 0 \text{ and } \eta(U) U^{-1}[A, U] \eta(U) \gtrsim a \eta(U)^2\big\}. \tag{3.6}$$

Further properties of the functions $\widetilde{\varrho}_U^A$ and $\varrho_U^A$ are collected in the following lemmas. The first one corresponds to [12, Prop. 2.3].

**Lemma 3.2** (Virial Theorem for $U$)**.** *Let $U$ be a unitary operator in $\mathcal{H}$ and let $A$ be a self-adjoint operator in $\mathcal{H}$ with $U \in C^1(A)$. Then,*

$$E^U(\{\theta\})U^{-1}[A,U]E^U(\{\theta\}) = 0$$

*for each $\theta \in \mathbb{T}$. In particular, one has $\langle \varphi, U^{-1}[A,U]\varphi \rangle = 0$ for each eigenvector $\varphi \in \mathcal{H}$ of $U$.*

**Lemma 3.3.** *Let $U$ be a unitary operator in $\mathcal{H}$ and let $A$ be a self-adjoint operator in $\mathcal{H}$ with $U \in C^1(A)$. Assume there exist an open set $\Theta \subset \mathbb{T}$ and $a \in \mathbb{R}$ such that $E^U(\Theta)U^{-1}[A,U]E^U(\Theta) \gtrsim a\,E^U(\Theta)$. Then, for each $\theta \in \Theta$ and each $\eta > 0$ there exist $\varepsilon > 0$ and a finite rank orthogonal projection $F$ with $E^U(\{\theta\}) \geq F$ such that*

$$E^U(\theta;\varepsilon)U^{-1}[A,U]E^U(\theta;\varepsilon) \geq (a-\eta)\big(E^U(\theta;\varepsilon) - F\big) - \eta F.$$

*In particular, if $\theta$ is not an eigenvalue of $U$, then*

$$E^U(\theta;\varepsilon)U^{-1}[A,U]E^U(\theta;\varepsilon) \geq (a-\eta)E^U(\theta;\varepsilon),$$

*while if $\theta$ is an eigenvalue of $U$, one has only*

$$E^U(\theta;\varepsilon)U^{-1}[A,U]E^U(\theta;\varepsilon) \geq \min\{a-\eta,-\eta\}E^U(\theta;\varepsilon).$$

*Proof.* The proof uses Virial Theorem for $U$ and is analogous to the proof of [3, Lemma 7.2.12] in the self-adjoint case. One just needs to replace in the proof of [3, Lemma 7.2.12] $[iH,A]$ by $U^{-1}[A,U]$, $E(J)$ by $E^U(\Theta)$, $E(\{\lambda\})$ by $E^U(\{\theta\})$, and $E(\lambda;1/k)$ by $E^U(\theta;1/k)$. □

**Lemma 3.4.** *Let $U$ be a unitary operator in $\mathcal{H}$ and let $A$ be a self-adjoint operator in $\mathcal{H}$ with $U \in C^1(A)$.*

*(a) The function $\varrho_U^A : \mathbb{T} \to (-\infty,\infty]$ is lower semicontinuous, and $\varrho_U^A(\theta) < \infty$ if and only if $\theta \in \sigma(U)$.*

*(b) The function $\widetilde{\varrho}_U^A : \mathbb{T} \to (-\infty,\infty]$ is lower semicontinuous, and $\widetilde{\varrho}_U^A(\theta) < \infty$ if and only if $\theta \in \sigma_{\rm ess}(U)$.*

*(c) $\widetilde{\varrho}_U^A \geq \varrho_U^A$.*

*(d) If $\theta \in \mathbb{T}$ is an eigenvalue of $U$ and $\widetilde{\varrho}_U^A(\theta) > 0$, then $\varrho_U^A(\theta) = 0$. Otherwise, $\varrho_U^A(\theta) = \widetilde{\varrho}_U^A(\theta)$.*

*Proof.* The proof is an adaptation of the proofs of Lemma 7.2.1, Proposition 7.2.3(a), Proposition 7.2.6 and Theorem 7.2.13 of [3] to the case of unitary operators.

(a) The fact that $\varrho_U^A(\theta) < \infty$ if and only if $\theta \in \sigma(U)$ follows from the definition of $\varrho_U^A$ and the closedness of $\sigma(U)$. Let $\theta_0 \in \mathbb{T}$ and let $r \in \mathbb{R}$ be such that $\varrho_U^A(\theta_0) > r$. To show the lower semicontinuity of $\varrho_U^A$ we must show that there is a neighbourhood of $\theta_0$ on which $\varrho_U^A > r$. Since $\varrho_U^A(\theta_0) > r$, there exist $a > r$ and $\varepsilon_0 > 0$ such that

$$E^U(\theta_0;\varepsilon_0)U^{-1}[A,U]E^U(\theta_0;\varepsilon_0) \geq a\,E^U(\theta_0;\varepsilon_0).$$

Let $\varepsilon := \varepsilon_0/2$ and $\theta \in \Theta(\theta_0;\varepsilon_0)$. By multiplying on the left and on the right the preceding inequality by $E^U(\theta;\varepsilon)$ and by using the fact that $E^U(\theta;\varepsilon)E^U(\theta_0;\varepsilon_0) = E^U(\theta;\varepsilon)$, one obtains

$$E^U(\theta;\varepsilon)U^{-1}[A,U]E^U(\theta;\varepsilon) \geq a\,E^U(\theta;\varepsilon).$$

This implies that $\varrho_U^A(\theta) \geq a > r$ for all $\theta \in \Theta(\theta_0;\varepsilon_0)$.

(b)-(c) The lower semicontinuity of $\widetilde{\varrho}_U^A$ is obtained similarly to that of $\varrho_U^A$ in point (a), and the inequality $\widetilde{\varrho}_U^A \geq \varrho_U^A$ is immediate from the definitions. For the last claim, we use the fact that $\theta \notin \sigma_{\rm ess}(U)$ if and only if $E^U(\theta;\varepsilon) \in \mathscr{K}(\mathcal{H})$ for some $\varepsilon > 0$. So $\theta \notin \sigma_{\rm ess}(U)$ implies that $\widetilde{\varrho}_U^A(\theta) = \infty$. Conversely, if $\widetilde{\varrho}_U^A(\theta) = \infty$, let $m := \|E^U(\theta;1)U^{-1}[A,U]E^U(\theta;1)\|_{\mathscr{B}(\mathcal{H})}$ and $a > m$. Then, there is $\varepsilon \in (0,1)$ such that

$$E^U(\theta;\varepsilon)U^{-1}[A,U]E^U(\theta;\varepsilon) \gtrsim a\,E^U(\theta;\varepsilon).$$

On another hand, the inequality $m \geq E^U(\theta;1)U^{-1}[A,U]E^U(\theta;1)$ and the fact that $E^U(\theta;\varepsilon)E^U(\theta;1) = E^U(\theta;\varepsilon)$ imply that

$$m\,E^U(\theta;\varepsilon) \geq E^U(\theta;\varepsilon)U^{-1}[A,U]E^U(\theta;\varepsilon).$$

Thus $m\,E^U(\theta;\varepsilon) \gtrsim a\,E^U(\theta;\varepsilon)$, and there exists $K \in \mathscr{K}(\mathcal{H})$ such that $K \geq (a-m)\,E^U(\theta;\varepsilon)$. This implies by heredity of compactness that $E^U(\theta;\varepsilon) \in \mathscr{K}(\mathcal{H})$.

(d) If $\theta$ is not an eigenvalue of $U$, then Lemma 3.3 implies that $\widetilde{\varrho}_U^A(\theta) \leq \varrho_U^A(\theta)$, and so these two numbers must be equal by point (c). Now assume that $\theta$ is an eigenvalue of $U$. If $\widetilde{\varrho}_U^A(\theta) \leq 0$, then $a \leq 0$ in Lemma 3.3, hence $\min\{a-\eta, -\eta\} = a-\eta$ and we have the same result as before. If $\widetilde{\varrho}_U^A(\theta) > 0$, we may take $a > 0$ in Lemma 3.3, which leads to the inequality $\varrho_U^A(\theta) \geq 0$; the opposite inequality $\varrho_U^A(\theta) \leq 0$ follows by using Virial theorem for $U$ : if $a < \varrho_U^A(\theta)$, there is $\varepsilon > 0$ such that $E^U(\theta;\varepsilon)U^{-1}[A,U]E^U(\theta;\varepsilon) \geq a\,E^U(\theta;\varepsilon)$; hence $0 = E^U(\{\theta\})U^{-1}[A,U]E^U(\{\theta\}) \geq a\,E^U(\{\theta\})$. Since $E^U(\{\theta\}) \neq 0$, we must have $a \leq 0$. ☐

By analogy with the self-adjoint case, we say that $A$ is conjugate to $U$ at the point $\theta \in \mathbb{T}$ if $\widetilde{\varrho}_U^A(\theta) > 0$, and that $A$ is strictly conjugate to $U$ at $\theta$ if $\varrho_U^A(\theta) > 0$. Since $\widetilde{\varrho}_U^A(\theta) \geq \varrho_U^A(\theta)$ for each $\theta \in \mathbb{T}$ by Lemma 3.4(c), strict conjugation is a property stronger than conjugation.

**Theorem 3.5** ($U$-smooth operators)**.** *Let $U$ be a unitary operator in $\mathcal{H}$, let $A$ be a self-adjoint operator in $\mathcal{H}$, and let $\mathcal{G}$ be an auxiliary Hilbert space. Assume either that $U$ has a spectral gap and $U \in C^{1,1}(A)$, or that $U \in C^{1+0}(A)$. Suppose also there exist an open set $\Theta \subset \mathbb{T}$, a number $a > 0$ and an operator $K \in \mathscr{K}(\mathcal{H})$ such that*

$$E^U(\Theta)\,U^{-1}[A,U]\,E^U(\Theta) \geq aE^U(\Theta) + K.$$

*Then, each operator $T \in \mathscr{B}(\mathcal{H},\mathcal{G})$ which extends continuously to an element of $\mathscr{B}\big(\mathcal{D}(\langle A\rangle^s)^*,\mathcal{G}\big)$ for some $s > 1/2$ is locally $U$-smooth on $\Theta \setminus \sigma_{\rm p}(U)$.*

*Proof.* The claim follows by adapting the proof of [12, Prop. 2.9] to locally $U$-smooth operators $T$ with values in the auxiliary Hilbert space $\mathcal{G}$, taking into account the results of Section 3.2. ☐

The last theorem of this section corresponds to [12, Thm. 2.7]:

**Theorem 3.6** (Spectrum of $U$)**.** *Let $U$ be a unitary operator in $\mathcal{H}$ and let $A$ be a self-adjoint operator in $\mathcal{H}$. Assume either that $U$ has a spectral gap and $U \in C^{1,1}(A)$, or that $U \in C^{1+0}(A)$. Suppose also there exist an open set $\Theta \subset \mathbb{T}$, a number $a > 0$ and an operator $K \in \mathscr{K}(\mathcal{H})$ such that*

$$E^U(\Theta)\,U^{-1}[A,U]\,E^U(\Theta) \geq aE^U(\Theta) + K.$$

*Then, $U$ has at most finitely many eigenvalues in $\Theta$, each one of finite multiplicity, and $U$ has no singular continuous spectrum in $\Theta$.*

## 3.4 Commutator methods in a two-Hilbert spaces setting

From now on, in addition to the triple $(\mathcal{H},U,A)$, we consider a second triple $(\mathcal{H}_0,U_0,A_0)$ with $\mathcal{H}_0$ a Hilbert space, $U_0$ a unitary operator in $\mathcal{H}_0$, and $A_0$ a self-adjoint operator in $\mathcal{H}_0$. We also consider an identification operator $J \in \mathscr{B}(\mathcal{H}_0,\mathcal{H})$. The existence of two such triples with an identification operator is quite standard in scattering theory of unitary operators, at least for the pairs $(\mathcal{H},U)$ and $(\mathcal{H}_0,U_0)$ (see for instance the books [6, 46]). Part of our goal in this section is to show that the existence of the conjugate operators $A$ and $A_0$ is also natural, in the same way it is in the self-adjoint case [35].

In the one-Hilbert space setting, the unitary operator $U$ is usually a multiplicative perturbation of the unitary operator $U_0$. In this case, if $U - U_0$ is compact, the stability of the function $\widetilde{\varrho}_{U_0}^{A_0}$ under compact perturbations allows one to infer information on $U$ from similar information on $U_0$ (see [12, Cor. 2.10]). In the two-Hilbert spaces setting, we are not aware of any general result relating the functions $\widetilde{\varrho}_U^A$ and $\widetilde{\varrho}_{U_0}^{A_0}$. The obvious reason for this being the impossibility to consider $U$ as a direct perturbation of $U_0$ since these operators do not act in the same Hilbert space. Nonetheless, the next theorem provides a result in that direction. For two arbitrary Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ and two operators $S, T \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$, we use the notation $T \approx S$ if $(T - S) \in \mathscr{K}(\mathcal{H}_1, \mathcal{H}_2)$.

**Theorem 3.7.** *Let $(\mathcal{H}_0, U_0, A_0)$ and $(\mathcal{H}, U, A)$ be as above, let $J \in \mathscr{B}(\mathcal{H}_0, \mathcal{H})$, and assume that*

*(i) $U_0 \in C^1(A_0)$ and $U \in C^1(A)$,*

*(ii) $JU_0^{-1}[A_0, U_0]J^* - U^{-1}[A, U] \in \mathscr{K}(\mathcal{H})$,*

*(iii) $JU_0 - UJ \in \mathscr{K}(\mathcal{H}_0, \mathcal{H})$,*

*(iv) For each $\eta \in C(\mathbb{C}, \mathbb{R})$, $\eta(U)(JJ^* - 1)\eta(U) \in \mathscr{K}(\mathcal{H})$.*

*Then, one has $\widetilde{\varrho}_U^A \geq \widetilde{\varrho}_{U_0}^{A_0}$.*

An induction argument together with a Stone-Weierstrass density argument shows that (iii) is equivalent to the apparently stronger condition

(iii') For each $\eta \in C(\mathbb{C}, \mathbb{R})$, $J\eta(U_0) - \eta(U)J \in \mathscr{K}(\mathcal{H}_0, \mathcal{H})$.

Therefore, in the sequel, we will sometimes use the condition (iii') instead of (iii).

*Proof.* For each $\eta \in C(\mathbb{C}, \mathbb{R})$, we have

$$\eta(U)U^{-1}[A, U]\eta(U) \approx \eta(U)JU_0^{-1}[A_0, U_0]J^*\eta(U) \approx J\eta(U_0)U_0^{-1}[A_0, U_0]\eta(U_0)J^* \tag{3.7}$$

due to Assumption (i)-(iii). Furthermore, if there exists $a \in \mathbb{R}$ such that

$$\eta(U_0)U_0^{-1}[A_0, U_0]\eta(U_0) \gtrsim a\eta(U_0)^2,$$

then Assumptions (iii)-(iv) imply that

$$J\eta(U_0)U_0^{-1}[A_0, U_0]\eta(U_0)J^* \gtrsim a\,J\eta(U_0)^2J^* \approx a\eta(U)JJ^*\eta(U) \approx a\eta(U)^2. \tag{3.8}$$

Thus, we obtain $\eta(U)U^{-1}[A, U]\eta(U) \gtrsim a\eta(U)^2$ by combining (3.7) and (3.8). This last estimate, together with the definition (3.6) of the functions $\widetilde{\varrho}_{U_0}^{A_0}$ and $\widetilde{\varrho}_U^A$, implies the claim. ◻

The regularity of $U_0$ with respect to $A_0$ is usually easy to check, while the regularity of $U$ with respect to $A$ is in general difficult to establish. For that purpose, various perturbative criteria have been developed for self-adjoint operators in one Hilbert space, and often a distinction is made between so-called short-range and long-range perturbations. Roughly speaking, the two terms of the formal commutator $[A, U] = AU - UA$ are treated separately in the short-range case, while the commutator $[A, U]$ is really computed in the long-range case. In the sequel, we discuss the case of short-range type perturbations for unitary operators in a two-Hilbert spaces setting. The results we obtain are analogous to the ones obtained in [35, Sec. 3.1] for self-adjoint operators in a two-Hilbert spaces setting.

We start by showing how the condition $U \in C^1(A)$ and the assumptions (ii)-(iii) of Theorem 3.7 can be verified for a class of short-range type perturbations. Our approach is to infer the desired information on $U$ from equivalent information on $U_0$, which are usually easier to obtain. Accordingly, our results exhibit some perturbative flavor. The price one has to pay is to impose some compatibility conditions between $A_0$ and $A$. For brevity, we set

$$B := JU_0 - UJ \in \mathscr{B}(\mathcal{H}_0, \mathcal{H}) \quad \text{and} \quad B_* := JU_0^* - U^*J \in \mathscr{B}(\mathcal{H}_0, \mathcal{H}).$$

**Proposition 3.8.** *Let $U_0 \in C^1(A_0)$, assume that $\mathscr{D} \subset \mathcal{H}$ is a core for $A$ such that $J^*\mathscr{D} \subset \mathcal{D}(A_0)$, and suppose that*

$$\overline{BA_0 \upharpoonright \mathcal{D}(A_0)} \in \mathscr{B}(\mathcal{H}_0, \mathcal{H}), \quad \overline{B_*A_0 \upharpoonright \mathcal{D}(A_0)} \in \mathscr{B}(\mathcal{H}_0, \mathcal{H}) \quad \text{and} \quad \overline{(JA_0J^* - A) \upharpoonright \mathscr{D}} \in \mathscr{B}(\mathcal{H}). \tag{3.9}$$

*Then, $U \in C^1(A)$.*

*Proof.* For $\varphi \in \mathscr{D}$, a direct calculation gives

$$\begin{aligned}
&\langle A\varphi, U\varphi\rangle_{\mathcal{H}} - \langle \varphi, UA\varphi\rangle_{\mathcal{H}} \\
&= \langle A\varphi, U\varphi\rangle_{\mathcal{H}} - \langle \varphi, UA\varphi\rangle_{\mathcal{H}} - \langle \varphi, J[A_0, U_0]J^*\varphi\rangle_{\mathcal{H}} + \langle \varphi, J[A_0, U_0]J^*\varphi\rangle_{\mathcal{H}} \\
&= \langle \varphi, BA_0J^*\varphi\rangle_{\mathcal{H}} - \langle B_*A_0J^*\varphi, \varphi\rangle_{\mathcal{H}} + \langle U^*\varphi, (JA_0J^* - A)\varphi\rangle_{\mathcal{H}} - \langle (JA_0J^* - A)\varphi, U\varphi\rangle_{\mathcal{H}} \\
&\quad + \langle \varphi, J[A_0, U_0]J^*\varphi\rangle_{\mathcal{H}}.
\end{aligned}$$

Furthermore, we have

$$\left|\langle \varphi, BA_0J^*\varphi\rangle_{\mathcal{H}} - \langle B_*A_0J^*\varphi, \varphi\rangle_{\mathcal{H}}\right| \leq \mathrm{Const.}\, \|\varphi\|_{\mathcal{H}}^2$$

due to the first two conditions in (3.9), and we have

$$\left|\langle U^*\varphi, (JA_0J^* - A)\varphi\rangle_{\mathcal{H}} - \langle (JA_0J^* - A)\varphi, U\varphi\rangle_{\mathcal{H}}\right| \leq \mathrm{Const.}\, \|\varphi\|_{\mathcal{H}}^2$$

due to the third condition in (3.9). Finally, since $U_0 \in C^1(A_0)$ and $J \in \mathscr{B}(\mathcal{H}_0, \mathcal{H})$ we also have

$$\left|\langle \varphi, J[A_0, U_0]J^*\varphi\rangle_{\mathcal{H}}\right| \leq \mathrm{Const.}\, \|\varphi\|_{\mathcal{H}}^2.$$

Since $\mathscr{D}$ is a core for $A$, this implies that $U \in C^1(A)$. □

We now show how the assumption (ii) of Theorem 3.7 is verified for a short-range type perturbation. Note that the hypotheses of the following proposition are slightly stronger than the ones of Proposition 3.8. Thus, $U$ automatically belongs to $C^1(A)$.

**Proposition 3.9.** *Let $U_0 \in C^1(A_0)$, assume that $\mathscr{D} \subset \mathcal{H}$ is a core for $A$ such that $J^*\mathscr{D} \subset \mathcal{D}(A_0)$, and suppose that*

$$\overline{BA_0 \upharpoonright \mathcal{D}(A_0)} \in \mathscr{B}(\mathcal{H}_0, \mathcal{H}), \quad \overline{B_*A_0 \upharpoonright \mathcal{D}(A_0)} \in \mathscr{K}(\mathcal{H}_0, \mathcal{H}) \quad \text{and} \quad \overline{(JA_0J^* - A) \upharpoonright \mathscr{D}} \in \mathscr{K}(\mathcal{H}). \tag{3.10}$$

*Then, the difference of bounded operators $JU_0^{-1}[A_0, U_0]J^* - U^{-1}[A, U]$ belongs to $\mathscr{K}(\mathcal{H})$.*

*Proof.* The facts that $U_0 \in C^1(A_0)$ and $J^*\mathscr{D} \subset \mathcal{D}(A_0)$ imply the inclusions

$$U_0J^*\mathscr{D} \subset U_0\mathcal{D}(A_0) \subset \mathcal{D}(A_0).$$

Using this and the last two conditions of (3.10), we obtain for $\varphi \in \mathscr{D}$ and $\psi \in U^{-1}\mathscr{D}$ that

$$\begin{aligned}
&\langle \psi, \left(JU_0^{-1}[A_0, U_0]J^* - U^{-1}[A, U]\right)\varphi\rangle_{\mathcal{H}} \\
&= \langle \psi, B_*A_0U_0J^*\varphi\rangle_{\mathcal{H}} + \langle B_*A_0J^*U\psi, \varphi\rangle_{\mathcal{H}} + \langle (JA_0J^* - A)U\psi, U\varphi\rangle_{\mathcal{H}} - \langle \psi, (JA_0J^* - A)\varphi\rangle_{\mathcal{H}} \\
&= \langle \psi, K_1U_0J^*\varphi\rangle_{\mathcal{H}} + \langle K_1J^*U\psi, \varphi\rangle_{\mathcal{H}} + \langle K_2U\psi, U\varphi\rangle_{\mathcal{H}} - \langle \psi, K_2\varphi\rangle_{\mathcal{H}}
\end{aligned}$$

with $K_1 \in \mathscr{K}(\mathcal{H}_0, \mathcal{H})$ and $K_2 \in \mathscr{K}(\mathcal{H})$. Since $\mathscr{D}$ and $U^{-1}\mathscr{D}$ are dense in $\mathcal{H}$, it follows that the operator $JU_0^{-1}[A_0, U_0]J^* - U^{-1}[A, U]$ belongs to $\mathscr{K}(\mathcal{H})$. □

In the rest of the section, we particularize the previous results to the case where $A = JA_0J^*$. This case deserves a special attention since it represents the most natural choice of a conjugate operator $A$ for $U$ when a conjugate operator $A_0$ for $U_0$ is given. However, one needs in this case the following assumption to guarantee the self-adjointness of the operator $A$ :

**Assumption 3.10.** *There exists a set $\mathscr{D} \subset \mathcal{D}(A_0J^*) \subset \mathcal{H}$ such that $JA_0J^* \upharpoonright \mathscr{D}$ is essentially self-adjoint, with corresponding self-adjoint extension denoted by $A$.*

Assumption 3.10 might be difficult to check in general, but in concrete situations the choice of the set $\mathscr{D}$ can be quite natural (see for example Lemma 4.9 for the case of quantum walks or [36, Rem. 4.3] for the case of manifolds with asymptotically cylindrical ends). The following two corollaries follow directly from Propositions 3.8-3.9 in the case Assumption 3.10 is satisfied.

**Corollary 3.11.** *Let $U_0 \in C^1(A_0)$, suppose that Assumption 3.10 holds for some set $\mathscr{D} \subset \mathcal{H}$, and assume that*

$$\overline{BA_0 \upharpoonright \mathcal{D}(A_0)} \in \mathscr{B}(\mathcal{H}_0, \mathcal{H}) \quad \text{and} \quad \overline{B_* A_0 \upharpoonright \mathcal{D}(A_0)} \in \mathscr{B}(\mathcal{H}_0, \mathcal{H}).$$

*Then, $U$ belongs to $C^1(A)$.*

**Corollary 3.12.** *Let $U_0 \in C^1(A_0)$, suppose that Assumption 3.10 holds for some set $\mathscr{D} \subset \mathcal{H}$, and assume that*

$$\overline{BA_0 \upharpoonright \mathcal{D}(A_0)} \in \mathscr{B}(\mathcal{H}_0, \mathcal{H}) \quad \text{and} \quad \overline{B_* A_0 \upharpoonright \mathcal{D}(A_0)} \in \mathscr{K}(\mathcal{H}_0, \mathcal{H}).$$

*Then, the difference of bounded operators $JU_0^{-1}[A_0, U_0]J^* - U^{-1}[A, U]$ belongs to $\mathscr{K}(\mathcal{H})$.*

# 4 Quantum walks with an anisotropic coin

In this section, we apply the abstract theory of Section 3 to prove our results on the spectrum of the evolution operator $U$ of the quantum walk with an anisotropic coin defined in Section 2. For this, we first determine in Section 4.1 the spectral properties and prove a Mourre estimate for the asymptotic operators $U_\ell$ and $U_{\mathrm{r}}$. Then, in Section 4.2, we use the Mourre estimate for $U_\ell$ and $U_{\mathrm{r}}$ to derive a Mourre estimate for $U$. Finally, in Section 4.3, we use the Mourre estimate for $U$ to prove our results on $U$. We recall that the behaviour of the coin operator $C$ at infinity is determined by Assumption 2.1.

## 4.1 Asymptotic operators $U_\ell$ and $U_{\mathrm{r}}$

For the study of the asymptotic operators $U_\ell$ and $U_{\mathrm{r}}$, we use the symbol $\star$ to denote either the index $\ell$ or the index r. Also, we introduce the subspace $\mathcal{H}_{\mathrm{fin}} \subset \mathcal{H}$ of elements with finite support

$$\mathcal{H}_{\mathrm{fin}} := \bigcup_{n \in \mathbb{N}} \big\{\Psi \in \mathcal{H} \mid \Psi(x) = 0 \text{ if } |x| \geq n\big\},$$

the Hilbert space $\mathcal{K} := \mathrm{L}^2\left([0, 2\pi), \frac{\mathrm{d}k}{2\pi}, \mathbb{C}^2\right)$, and the discrete Fourier transform $\mathscr{F} : \mathcal{H} \to \mathcal{K}$, which is the unitary operator defined as the unique continuous extension of the operator

$$(\mathscr{F}\Psi)(k) := \sum_{x \in \mathbb{Z}} \mathrm{e}^{-ikx}\, \Psi(x), \quad \Psi \in \mathcal{H}_{\mathrm{fin}},\ k \in [0, 2\pi).$$

A direct computation shows that the operator $U_\star$ is decomposable in the Fourier representation, namely, for all $f \in \mathcal{K}$ and almost every $k \in [0, 2\pi)$ we have

$$(\mathscr{F}\, U_\star \mathscr{F}^* f)(k) = \widehat{U_\star}(k) f(k) \quad \text{with} \quad \widehat{U_\star}(k) := \begin{pmatrix} \mathrm{e}^{ik} & 0 \\ 0 & \mathrm{e}^{-ik} \end{pmatrix} C_\star \in \mathrm{U}(2).$$

Moreover, since $\widehat{U_\star}(k) \in \mathrm{U}(2)$ the spectral theorem implies that $\widehat{U_\star}(k)$ can be written as

$$\widehat{U_\star}(k) = \sum_{j=1}^{2} \lambda_{\star,j}(k)\, \Pi_{\star,j}(k),$$

with $\lambda_{\star,j}(k)$ the eigenvalues of $\widehat{U_\star}(k)$ and $\Pi_{\star,j}(k)$ the corresponding orthogonal projections.

The next lemma furnishes some information on the spectrum of $U_\star$. To state it, we use the following parametrisation for the matrices $C_\star$ :

$$C_\star = \mathrm{e}^{i\delta_\star/2}\begin{pmatrix} a_\star\,\mathrm{e}^{i(\alpha_\star-\delta_\star/2)} & b_\star\,\mathrm{e}^{i(\beta_\star-\delta_\star/2)} \\ -b_\star\,\mathrm{e}^{-i(\beta_\star-\delta_\star/2)} & a_\star\,\mathrm{e}^{-i(\alpha_\star-\delta_\star/2)} \end{pmatrix} \tag{4.1}$$

with $a_\star, b_\star \in [0,1]$ satisfying $a_\star^2 + b_\star^2 = 1$, and $\alpha_\star, \beta_\star, \delta_\star \in (-\pi, \pi]$. The determinant $\det(C_\star)$ of $C_\star$ is equal to $\mathrm{e}^{i\delta_\star}$. For brevity, we also set

$$\begin{aligned} \tau_\star(k) &:= a_\star \cos(k+\alpha_\star-\delta_\star/2),\\ \eta_\star(k) &:= \sqrt{1-\tau_\star(k)^2},\\ \varsigma_\star(k) &:= a_\star \sin(k+\alpha_\star-\delta_\star/2),\\ \theta_\star &:= \arccos(a_\star). \end{aligned}$$

**Lemma 4.1** (Spectrum of $U_\star$)**.** *(a) If $a_\star = 0$, then $U_\star$ has pure point spectrum*

$$\sigma(U_\star) = \sigma_{\mathrm{p}}(U_\star) = \big\{ i\,\mathrm{e}^{i\delta_\star/2}, -i\,\mathrm{e}^{i\delta_\star/2} \big\}$$

*with each point an eigenvalue of $U_\star$ of infinite multiplicity.*

*(b) If $a_\star \in (0,1)$, then $\sigma_{\mathrm{p}}(U_\star) = \varnothing$ and*

$$\sigma(U_\star) = \sigma_{\mathrm{c}}(U_\star) = \big\{ \mathrm{e}^{i\gamma} \mid \gamma \in [\delta_\star/2+\theta_\star, \pi+\delta_\star/2-\theta_\star] \cup [\pi+\delta_\star/2+\theta_\star, 2\pi+\delta_\star/2-\theta_\star] \big\}.$$

*(c) If $a_\star = 1$, then $\sigma_{\mathrm{p}}(U_\star) = \varnothing$ and $\sigma(U_\star) = \sigma_{\mathrm{c}}(U_\star) = \mathbb{T}$.*

*Proof.* Using the parametrisation for $C_\star$ given in (4.1), one gets

$$\widehat{U_\star}(k) = \mathrm{e}^{i\delta_\star/2}\begin{pmatrix} a_\star(k) & b_\star(k) \\ -\overline{b_\star(k)} & \overline{a_\star(k)} \end{pmatrix}$$

with

$$a_\star(k) := a_\star\,\mathrm{e}^{i(k+\alpha_\star-\delta_\star/2)} \quad \text{and} \quad b_\star(k) := b_\star\,\mathrm{e}^{i(k+\beta_\star-\delta_\star/2)}.$$

Therefore, the spectrum of $U_\star$ is given by

$$\sigma(U_\star) = \big\{ \lambda_{\star,j}(k) \mid j = 1,2,\ k \in [0,2\pi) \big\}$$

with $\lambda_{\star,j}(k)$ the solution of the characteristic equation

$$\det\big(\widehat{U_\star}(k) - \lambda_{\star,j}(k)\big) = 0, \quad j = 1,2,\ k \in [0,2\pi).$$

In case (a), we obtain

$$\lambda_{\star,1}(k) = i\,\mathrm{e}^{i\delta_\star/2} \quad \text{and} \quad \lambda_{\star,2}(k) = -i\,\mathrm{e}^{i\delta_\star/2}.$$

In case (b), we obtain

$$\lambda_{\star,j}(k) = \mathrm{e}^{i\delta_\star/2}\big(\tau_\star(k) + i\,(-1)^{j-1}\eta_\star(k)\big), \quad j = 1,2.$$

Finally, in case (c) we obtain

$$\lambda_{\star,1}(k) = \mathrm{e}^{i(k+\alpha_\star)} \quad \text{and} \quad \lambda_{\star,2}(k) = \mathrm{e}^{-i(k+\alpha_\star-\delta_\star)}.$$

□

We now exhibit normalised eigenvectors $u_{\star,j}(k)$ of $\widehat{U_\star}(k)$ associated with the eigenvalues $\lambda_{\star,j}(k)$ which are $C^\infty$ in the variable $k$ :

$$\begin{cases} u_{\star,j}(k) := \frac{\sqrt{\eta_\star(k)+(-1)^{j-1}\varsigma_\star(k)}}{b_\star\sqrt{2\eta_\star(k)}}\begin{pmatrix} ib_\star(k) \\ \varsigma_\star(k)+(-1)^j\eta_\star(k)\end{pmatrix} & \text{if } a_\star \in [0,1) \\ u_{\star,1}(k) := \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad u_{\star,2}(k) := \begin{pmatrix}0\\1\end{pmatrix} & \text{if } a_\star = 1. \end{cases}$$

We leave the reader check that $u_{\star,j}(k)$ are indeed normalised eigenvectors of $\widehat{U_\star}(k)$ with eigenvalues $\lambda_{\star,j}(k)$. In addition, since for $a_\star \in [0,1)$ one has $\eta_\star(k) > 0$ and $\eta_\star(k)+(-1)^{j-1}\varsigma_\star(k) > 0$, we note that the $2\pi$-periodic map $\mathbb{R} \ni k \mapsto u_{\star,j}(k) \in \mathbb{C}^2$ is of class $C^\infty$.

Our next goal is to construct a suitable conjugate operator for the operator $U_\star$. For this, a few preliminaries are necessary. First, we equip the interval $[0,2\pi)$ with the addition modulo $2\pi$, and for any $n \in \mathbb{N}$ we define the space $C^n\big([0,2\pi),\mathbb{C}^2\big) \subset \mathcal{K}$ as the set of functions $[0,2\pi) \to \mathbb{C}^2$ of class $C^n$. In particular, we have $u_{\star,j} \in C^\infty\big([0,2\pi),\mathbb{C}^2\big)$, and the space $\mathscr{F}\mathcal{H}_{\rm fin} \subset C^\infty\big([0,2\pi),\mathbb{C}^2\big)$ is the set of $\mathbb{C}^2$-valued trigonometric polynomials.

Next, we define the asymptotic velocity operator for the operator $U_\star$. For $j = 1,2$, we let $v_{\star,j} : [0,2\pi) \to \mathbb{R}$ be the bounded function given by

$$v_{\star,j}(k) := \frac{i\,\lambda'_{\star,j}(k)}{\lambda_{\star,j}(k)}, \quad k \in [0,2\pi). \tag{4.2}$$

Here, $(\,\cdot\,)'$ stands for the derivative with respect to $k$, and $v_{\star,j}$ is real valued because $\lambda_{\star,j}$ takes values in the complex unit circle. Finally, for all $f \in \mathcal{K}$ and almost every $k \in [0,2\pi)$, we define the decomposable operator $\widehat{V_\star} \in \mathscr{B}(\mathcal{K})$ by

$$\big(\widehat{V_\star}f\big)(k) := \widehat{V_\star}(k)f(k) \quad \text{where} \quad \widehat{V_\star}(k) := \sum_{j=1}^{2} v_{\star,j}(k)\,\Pi_{\star,j}(k) \in \mathscr{B}(\mathbb{C}^2), \tag{4.3}$$

and we call asymptotic velocity operator the operator $V_\star$ given as inverse Fourier transform of $\widehat{V_\star}$, namely,

$$V_\star := \mathscr{F}^*\widehat{V_\star}\mathscr{F}.$$

The basic spectral properties of $V_\star$ are collected in the following lemma.

**Lemma 4.2** (Spectrum of $V_\star$)**.** *Let $C_\star$ be parameterised as in* (4.1).

*(a) If $a_\star = 0$, then $v_{\star,j} = 0$ for $j = 1,2$, and $V_\star = 0$.*

*(b) If $a_\star \in (0,1)$, then $v_{\star,j}(k) = \frac{(-1)^j\varsigma_\star(k)}{\eta_\star(k)}$ for $j = 1,2$ and $k \in [0,2\pi)$, $\sigma_{\rm p}(V_\star) = \varnothing$ and*

$$\sigma(V_\star) = \sigma_{\rm c}(V_\star) = [-a_\star, a_\star].$$

*(c) If $a_\star = 1$, then $v_{\star,j} = (-1)^j$ for $j = 1,2$, and $V_\star$ has pure point spectrum*

$$\sigma(V_\star) = \sigma_{\rm p}(V_\star) = \{-1,1\}$$

*with each point an eigenvalue of $V_\star$ of infinite multiplicity.*

*Proof.* The claims follow from simple calculations using the formulas for $\lambda_{\star,j}(k)$ in the proof of Lemma 4.1 and the definition (4.2) of $v_{\star,j}(k)$. $\square$

For any $\xi,\zeta \in C\big([0,2\pi),\mathbb{C}^2\big)$, we define the operator $|\xi\rangle\langle\zeta| : C\big([0,2\pi),\mathbb{C}^2\big) \to C\big([0,2\pi),\mathbb{C}^2\big)$ by

$$\big(|\xi\rangle\langle\zeta|f\big)(k) := \big\langle\zeta(k),f(k)\big\rangle_2\,\xi(k), \quad f \in C\big([0,2\pi),\mathbb{C}^2\big),\ k\in[0,2\pi),$$

where $\langle\,\cdot\,,\,\cdot\,\rangle_2$ is the usual scalar product on $\mathbb{C}^2$. This operator extends continuously to an element of $\mathscr{B}(\mathcal{K})$, with norm satisfying the bound

$$\big\||\xi\rangle\langle\zeta|\big\|_{\mathscr{B}(\mathcal{K})} \le \|\xi\|_{\mathrm{L}^\infty([0,2\pi),\frac{dk}{2\pi},\mathbb{C}^2)}\,\|\zeta\|_{\mathrm{L}^\infty([0,2\pi),\frac{dk}{2\pi},\mathbb{C}^2)}. \tag{4.4}$$

We also define the self-adjoint operator $P$ in $\mathcal{K}$

$$Pf := -if', \quad f\in\mathcal{D}(P) := \big\{f\in\mathcal{K}\mid f \text{ is absolutely continuous, } f'\in\mathcal{K}, \text{ and } f(0)=f(2\pi)\big\}.$$

With these definitions at hand, we can prove the self-adjointness of an operator useful for the definition of our future the conjugate operator for $U$ :

**Lemma 4.3.** *The operator*

$$\widehat{X_\star}f := -\sum_{j=1}^2\big(|u_{\star,j}\rangle\langle u_{\star,j}|P - i\,|u_{\star,j}\rangle\langle u'_{\star,j}|\big)f, \quad f\in\mathscr{F}\mathcal{H}_{\rm fin},$$

*is essentially self-adjoint in $\mathcal{K}$, with closure denoted by the same symbol. In particular, the Fourier transform $X_\star := \mathscr{F}^*\widehat{X_\star}\mathscr{F}$ of $\widehat{X_\star}$ is essentially self-adjoint on $\mathcal{H}_{\rm fin}$ in $\mathcal{H}$.*

*Proof.* The proof consists in checking the assumptions of Nelson's commutator theorem [33, Thm. X.37] applied with the comparison operator $N := P^2+1$.

For this, we first note that the operator $N$ is essentially self-adjoint on $\mathscr{F}\mathcal{H}_{\rm fin}$ because it is the Fourier transform of a multiplication operator acting on functions with finite support (see [31, Ex. 5.1.15]). Next, by performing an integration by parts with boundary terms canceling each other out, we verify that $\widehat{X_\star}$ is symmetric on $\mathscr{F}\mathcal{H}_{\rm fin}$. Then, by using the definition of $\widehat{X_\star}$ and the estimate (4.4), we check that the inequality $\|\widehat{X_\star}f\|_{\mathcal{K}} \le \mathrm{Const.}\,\|Nf\|_{\mathcal{K}}$ holds for each $f\in\mathscr{F}\mathcal{H}_{\rm fin}$. Finally, a direct calculation shows that for all $\xi,\zeta\in C^2\big([0,2\pi),\mathbb{C}^2\big)$ and $f\in\mathscr{F}\mathcal{H}_{\rm fin}$

$$\begin{aligned}&\big\langle Nf,|\xi\rangle\langle\zeta|f\big\rangle_{\mathcal{K}} - \big\langle f,|\xi\rangle\langle\zeta|Nf\big\rangle_{\mathcal{K}}\\ &= \big\langle f,\big(|\xi''\rangle\langle\zeta| - |\xi\rangle\langle\zeta''| - 2\,|\xi'\rangle\langle\zeta'| - 2i\,|\xi'\rangle\langle\zeta|\,P - 2i\,|\xi\rangle\langle\zeta'|\,P\big)f\big\rangle_{\mathcal{K}}.\end{aligned}$$

This, together with the definition of $\widehat{X_\star}$, implies that

$$\big|\big\langle\widehat{X_\star}f,Nf\big\rangle_{\mathcal{K}} - \big\langle Nf,\widehat{X_\star}f\big\rangle_{\mathcal{K}}\big| \le \mathrm{Const.}\,\langle f,Nf\rangle_{\mathcal{K}}.$$

Thus, all the assumptions of Nelson's commutator theorem are verified, and the claim is proved. □

The main relations between the operators introduced so far are summarized in the following proposition. To state it, we need one more decomposable operator $\widehat{H_\star}\in\mathscr{B}(\mathcal{K})$ defined for all $f\in\mathcal{K}$ and almost every $k\in[0,2\pi)$ by

$$\big(\widehat{H_\star}f\big)(k) := \widehat{H_\star}(k)f(k) \quad \text{where} \quad \widehat{H_\star}(k) := -\sum_{j=1}^2 v'_{\star,j}(k)\,\Pi_{\star,j}(k)\in\mathscr{B}(\mathbb{C}^2).$$

We also need the inverse Fourier transform $H_\star := \mathscr{F}^*\widehat{H_\star}\mathscr{F}$ of $\widehat{H_\star}$.

**Proposition 4.4.** *(a) One has the equality $[iX_\star,V_\star] = H_\star$ in the form sense on $\mathcal{H}_{\rm fin}$.*

*(b) $U_\star$, $V_\star$ and $H_\star$ are mutually commuting.*

*(c) One has the equality $[X_\star, U_\star] = U_\star V_\star$ in the form sense on $\mathcal{H}_{\rm fin}$.*

*Proof.* (a) Let $f, g \in \mathscr{F}\mathcal{H}_{\rm fin}$. Then, a direct calculation using an integration by parts (with boundary terms canceling each other out) implies that

$$\big\langle \widehat{X_\star} f, i\widehat{V_\star} g\big\rangle_{\mathcal{K}} - \big\langle f, i\widehat{V_\star}\widehat{X_\star} g\big\rangle_{\mathcal{K}} = \big\langle f, \widehat{H_\star} g\big\rangle_{\mathcal{K}}.$$

Therefore, the claim follows by an application of the Fourier transform $\mathscr{F}$.

(b) The mutual commutativity of the operators $U_\star$, $V_\star$ and $H_\star$ is a direct consequence of their boundedness and their definition in terms of the orthogonal projections $\Pi_{\star,j}(k)$, $k \in [0, 2\pi)$.

(c) As in point (a), the proof consists in computing for $f, g \in \mathscr{F}\mathcal{H}_{\rm fin}$ the difference

$$\big\langle \widehat{X_\star} f, \widehat{U_\star} g\big\rangle_{\mathcal{K}} - \big\langle f, \widehat{U_\star}\widehat{X_\star} g\big\rangle_{\mathcal{K}}$$

with an integration by parts, checking that this difference is equal to $\big\langle g, \widehat{U_\star}\widehat{V_\star} f\big\rangle_{\mathcal{K}}$, and applying the Fourier transform $\mathscr{F}$. $\square$

Since $X_\star$ is essentially self-adjoint on $\mathcal{H}_{\rm fin}$, Proposition 4.4(a) implies that $V_\star \in C^1(X_\star)$. Therefore, the operator

$$A_\star\Psi := \frac{1}{2}\big(X_\star V_\star + V_\star X_\star\big)\Psi, \quad \Psi \in \mathcal{D}(A_\star) := \big\{\Psi \in \mathcal{H} \mid V_\star\Psi \in \mathcal{D}(X_\star)\big\},$$

is self-adjoint in $\mathcal{H}$, and essentially self-adjoint on $\mathcal{H}_{\rm fin}$ (see [41, Lemma 2.4]). We can now state and prove the main results of this section. We recall that $\mathrm{Int}(\Theta)$ and $\partial\Theta$ denote the interior and the boundary of a set $\Theta \subset \mathbb{T}$. We also recall that the functions $\varrho^{A_\star}_{U_\star}$ and $\widetilde{\varrho}^{A_\star}_{U_\star}$ have been defined in Section 3.3.

**Proposition 4.5.** *(a) $U_\star \in C^1(A_\star)$ with $U_\star^{-1}[A_\star, U_\star] = V_\star^2$.*

*(b) $\varrho^{A_\star}_{U_\star} = \widetilde{\varrho}^{A_\star}_{U_\star}$, and*

- *(i) if $a_\star = 0$, then $\widetilde{\varrho}^{A_\star}_{U_\star}(\theta) = 0$ for $\theta \in \big\{i\,\mathrm{e}^{i\delta_\star/2}, -i\,\mathrm{e}^{i\delta_\star/2}\big\}$ and $\widetilde{\varrho}^{A_\star}_{U_\star}(\theta) = \infty$ otherwise,*
- *(ii) if $a_\star \in (0,1)$, then $\widetilde{\varrho}^{A_\star}_{U_\star}(\theta) > 0$ for $\theta \in \mathrm{Int}\big(\sigma(U_\star)\big)$, $\widetilde{\varrho}^{A_\star}_{U_\star}(\theta) = 0$ for $\theta \in \partial\sigma(U_\star)$, and $\widetilde{\varrho}^{A_\star}_{U_\star}(\theta) = \infty$ otherwise,*
- *(iii) if $a_\star = 1$, then $\widetilde{\varrho}^{A_\star}_{U_\star}(\theta) = 1$ for all $\theta \in \mathbb{T}$.*

*(c) (i) If $a_\star \in (0,1)$, then $U_\star$ has purely absolutely continuous spectrum*

$$\sigma(U_\star) = \sigma_{\rm ac}(U_\star) = \big\{\mathrm{e}^{i\gamma} \mid \gamma \in [\delta_\star/2 + \theta_\star, \pi + \delta_\star/2 - \theta_\star] \cup [\pi + \delta_\star/2 + \theta_\star, 2\pi + \delta_\star/2 - \theta_\star]\big\}.$$

*(ii) If $a_\star = 1$, then $U_\star$ has purely absolutely continuous spectrum $\sigma(U_\star) = \sigma_{\rm ac}(U_\star) = \mathbb{T}$.*

*Proof.* (a) A calculation in the forme sense on $\mathcal{H}_{\rm fin}$ using points (b) and (c) of Proposition 4.4 gives

$$[A_\star, U_\star] = \frac{1}{2}\big(V_\star[X_\star, U_\star] + [X_\star, U_\star]V_\star\big) = U_\star V_\star^2.$$

Since $A_\star$ is essentially self-adjoint on $\mathcal{H}_{\rm fin}$, this implies that $U_\star \in C^1(A_\star)$ with $U_\star^{-1}[A_\star, U_\star] = V_\star^2$.

(b) Take $\theta \in \mathbb{T}$ and $\varepsilon > 0$. Then, using the result of point (a) and (4.3), we obtain for almost every $k \in [0, 2\pi)$

$$\begin{aligned}\big(\mathscr{F}E^{U_\star}(\theta;\varepsilon)U_\star^{-1}[A_\star, U_\star]E^{U_\star}(\theta;\varepsilon)\mathscr{F}^*\big)(k) &= \big(\mathscr{F}E^{U_\star}(\theta;\varepsilon)V_\star^2E^{U_\star}(\theta;\varepsilon)\mathscr{F}^*\big)(k)\\ &= E^{\widehat{U_\star}(k)}(\theta;\varepsilon)\widehat{V_\star}(k)^2E^{\widehat{U_\star}(k)}(\theta;\varepsilon)\\ &\geq \min\big\{v_{\star,1}(k)^2, v_{\star,2}(k)^2\big\}E^{\widehat{U_\star}(k)}(\theta;\varepsilon).\end{aligned}$$

Then, the definition (4.2) of $v_{\star,j}(k)$ shows that $v_{\star,j}(k) = 0$ if and only if $\lambda'_{\star,j}(k) = 0$, which occurs when $\lambda_{\star,j}(k) \in \partial\sigma(U_\star)$. Therefore, one gets $\varrho^{A_\star}_{U_\star} = \widetilde{\varrho}^{A_\star}_{U_\star}$ by Lemma 3.4(d), and to conclude one just has to take into account the form of the boundary sets $\sigma(U_\star)$ given in Lemma 4.1.

(c) We know from point (a) that $U_\star \in C^1(A_\star)$ with $U_\star^{-1}[A_\star, U_\star] = V_\star^2$, and Proposition 4.4(a) implies that $V_\star \in C^1(A_\star)$. Thus, $U_\star \in C^2(A_\star)$. Therefore, if $a_\star \in (0,1)$, we infer from point (b.ii) and Theorem 3.6 that $U_\star$ has no singular continuous spectrum in $\operatorname{Int}\big(\sigma(U_\star)\big)$. This, together with Lemma 4.1(b), implies the claim in the case $a_\star \in (0,1)$. The claim in the case $a_\star = 1$ is proved in a similar way. ☐

## 4.2 Mourre estimate for $U$

In this section, we use the Mourre estimate for the asymptotic operators $U_\ell$ and $U_{\rm r}$ to derive a Mourre estimate for $U$. To achieve this, we apply the abstract construction introduced in Section 3.4, starting by choosing $\mathcal{H}_0 := \mathcal{H} \oplus \mathcal{H}$ as second Hilbert space and $U_0 := U_\ell \oplus U_{\rm r}$ as second unitary operator in $\mathcal{H}_0$.

The spectral properties of $U_0$ are obtained as a consequence of Lemma 4.1(a), Proposition 4.5(c) and the direct sum decomposition of $U_0$ :

**Lemma 4.6** (Spectrum of $U_0$)**.** *One has $\sigma(U_0) = \sigma(U_\ell) \cup \sigma(U_{\rm r})$ and $\sigma_{\rm sc}(U_0) = \varnothing$. Furthermore,*

*(a) if $a_\ell = a_{\rm r} = 0$, then $U_0$ has pure point spectrum*

$$\sigma(U_0) = \sigma_{\rm p}(U_0) = \sigma_{\rm p}(U_\ell) \cup \sigma_{\rm p}(U_{\rm r}) = \big\{ i\,\mathrm{e}^{i\delta_\ell/2}, -i\,\mathrm{e}^{i\delta_\ell/2}, i\,\mathrm{e}^{i\delta_{\rm r}/2}, -i\,\mathrm{e}^{i\delta_{\rm r}/2} \big\}$$

*with each point an eigenvalue of $U_0$ of infinite multiplicity,*

*(b) if $a_\ell = 0$ and $a_{\rm r} \in (0,1]$, then $\sigma_{\rm ac}(U_0) = \sigma_{\rm ac}(U_{\rm r})$ with $\sigma_{\rm ac}(U_{\rm r})$ as in Proposition 4.5(c), and*

$$\sigma_{\rm p}(U_0) = \sigma_{\rm p}(U_\ell) = \big\{ i\,\mathrm{e}^{i\delta_\ell/2}, -i\,\mathrm{e}^{i\delta_\ell/2} \big\}$$

*with each point an eigenvalue of $U_0$ of infinite multiplicity,*

*(c) if $a_\ell \in (0,1]$ and $a_{\rm r} = 0$, then $\sigma_{\rm ac}(U_0) = \sigma_{\rm ac}(U_\ell)$ with $\sigma_{\rm ac}(U_\ell)$ as in Proposition 4.5(c), and*

$$\sigma_{\rm p}(U_0) = \sigma_{\rm p}(U_{\rm r}) = \big\{ i\,\mathrm{e}^{i\delta_{\rm r}/2}, -i\,\mathrm{e}^{i\delta_{\rm r}/2} \big\}$$

*with each point an eigenvalue of $U_0$ of infinite multiplicity,*

*(d) if $a_\ell, a_{\rm r} \in (0,1]$, then $U_0$ has purely absolutely continuous spectrum*

$$\sigma(U_0) = \sigma_{\rm ac}(U_0) = \sigma_{\rm ac}(U_\ell) \cup \sigma_{\rm ac}(U_{\rm r})$$

*with $\sigma_{\rm ac}(U_\ell)$ and $\sigma_{\rm ac}(U_{\rm r})$ as in Proposition 4.5(c).*

Also, as intuition suggests and as already stated in Theorem 2.2, the spectrum of $U_0$ coincides with the essential spectrum of $U$, namely,

$$\sigma_{\rm ess}(U) = \sigma(U_\ell) \cup \sigma(U_{\rm r}) = \sigma(U_0).$$

*Proof of Theorem 2.2.* The proof is based on an argument using crossed product $C^*$-algebras inspired from [15, 28].

Let $\mathcal{A}$ be the algebra of functions $\mathbb{Z} \to \mathscr{B}(\mathbb{C}^2)$ admitting limits at $\pm\infty$, and let $\mathcal{A}_0$ be the ideal of $\mathcal{A}$ consisting in functions $\mathbb{Z} \to \mathscr{B}(\mathbb{C}^2)$ vanishing at $\pm\infty$. Since $\mathcal{A}$ is equipped with an action of $\mathbb{Z}$ by translation, namely,

$$\big(T_y\varphi\big)(x) := \varphi(x+y), \quad x, y \in \mathbb{Z}, \ \varphi \in \mathcal{A},$$

we can consider the crossed product algebra $\mathcal{A} \rtimes \mathbb{Z}$, and the functoriality of the crossed product implies the identities

$$(\mathcal{A} \rtimes \mathbb{Z})/(\mathcal{A}_0 \rtimes \mathbb{Z}) \cong (\mathcal{A}/\mathcal{A}_0) \rtimes \mathbb{Z} = \big(\mathscr{B}(\mathbb{C}^2) \oplus \mathscr{B}(\mathbb{C}^2)\big) \rtimes \mathbb{Z} = \big(\mathscr{B}(\mathbb{C}^2) \rtimes \mathbb{Z}\big) \oplus \big(\mathscr{B}(\mathbb{C}^2) \rtimes \mathbb{Z}\big), \quad (4.5)$$

where the equality $\mathcal{A}/\mathcal{A}_0 = \mathscr{B}(\mathbb{C}^2) \oplus \mathscr{B}(\mathbb{C}^2)$ is obtained by evaluation of the functions $\varphi \in \mathcal{A}$ at $\pm\infty$.

Now, the algebras $\mathcal{A} \rtimes \mathbb{Z}$ and $\mathcal{A}_0 \rtimes \mathbb{Z}$ can be faithfully represented in $\mathcal{H}$ by mapping the elements of $\mathcal{A}$ and $\mathcal{A}_0$ to multiplication operators in $\mathcal{H}$ and the elements of $\mathbb{Z}$ to the shifts $T_z$. Writing $\mathfrak{A}$ and $\mathfrak{A}_0$ for these representations of $\mathcal{A} \rtimes \mathbb{Z}$ and $\mathcal{A}_0 \rtimes \mathbb{Z}$ in $\mathcal{H}$, we can note three facts. First, $\mathfrak{A}_0$ is equal to the ideal of compact operators $\mathscr{K}(\mathcal{H})$. Secondly, the operator $U$ belongs to $\mathfrak{A}$, since

$$U = SC = T_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} C + T_{-1} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} C$$

with $T_1, T_{-1}$ shifts and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) C, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) C$ multiplication operators in $\mathcal{H}$. Thirdly, the essential spectrum of $U$ in $\mathfrak{A}$ is equal to the spectrum of the image of $U$ in the quotient algebra $\mathfrak{A}/\mathscr{K}(\mathcal{H}) = \mathfrak{A}/\mathfrak{A}_0$. These facts, together with (4.5) and Lemma 4.6, imply the equalities

$$\sigma_{\rm ess}(U) = \sigma\big(SC(-\infty) \oplus SC(+\infty)\big) = \sigma\big(SC_\ell \oplus SC_{\rm r}\big) = \sigma(U_\ell) \cup \sigma(U_{\rm r}) = \sigma(U_0),$$

which prove the claim. $\square$

Next, we define the identification operator $J \in \mathscr{B}(\mathcal{H}_0, \mathcal{H})$ by

$$J(\Psi_\ell, \Psi_{\rm r}) := j_\ell \Psi_\ell + j_{\rm r} \Psi_{\rm r}, \quad (\Psi_\ell, \Psi_{\rm r}) \in \mathcal{H}_0,$$

where

$$j_{\rm r}(x) := \begin{cases} 1 & \text{if } x \ge 0 \\ 0 & \text{if } x \le -1 \end{cases} \quad \text{and} \quad j_\ell := 1 - j_{\rm r}.$$

The adjoint operator $J^* \in \mathscr{B}(\mathcal{H}, \mathcal{H}_0)$ satisfies

$$J^*\Psi = (j_\ell \Psi, j_{\rm r} \Psi), \quad \Psi \in \mathcal{H}.$$

Moreover, using the same notation for the functions $j_\ell, j_{\rm r}$ and the associated multiplication operators in $\mathcal{H}$, one directly gets:

**Lemma 4.7.** *$J^*J = j_\ell \oplus j_{\rm r}$ is an orthogonal projection on $\mathcal{H}_0$, and $JJ^* = 1_{\mathcal{H}}$.*

The first result of the next lemma is an analogue of Proposition 4.5(a) in the Hilbert space $\mathcal{H}_0$. To state it, we need to introduce the operator $A_0 := A_\ell \oplus A_{\rm r}$ (which will be used as a conjugate operator for $U_0$) and the operator $V_0 := V_\ell \oplus V_{\rm r}$.

**Lemma 4.8.** *(a) $U_0 \in C^1(A_0)$ with $U_0^{-1}[A_0, U_0] = V_0^2$.*

*(b) $B := JU_0 - UJ \in \mathscr{K}(\mathcal{H}_0, \mathcal{H})$ and $B_* := JU_0^* - U^*J \in \mathscr{K}(\mathcal{H}_0, \mathcal{H})$.*

*Proof.* The proof of point (a) is similar to the proof of Proposition 4.5(a); one just has to replace the operators $U_\star, A_\star, V_\star$ in $\mathcal{H}$ by the operators $U_0, A_0, V_0$ in $\mathcal{H}_0$. For point (b), a direct computation with $(\Psi_\ell, \Psi_{\rm r}) \in \mathcal{H}_0$ gives

$$\begin{aligned} B(\Psi_\ell, \Psi_{\rm r}) &= \big(j_\ell U_\ell \Psi_\ell + j_{\rm r} U_{\rm r} \Psi_{\rm r}\big) - U\big(j_\ell \Psi_\ell + j_{\rm r} \Psi_{\rm r}\big) \\ &= \big([j_\ell, U_\ell] - (U - U_\ell) j_\ell\big)\Psi_\ell + \big([j_{\rm r}, U_{\rm r}] - (U - U_{\rm r}) j_{\rm r}\big)\Psi_{\rm r} \\ &= \big([j_\ell, S]C_\ell - S(C - C_\ell) j_\ell\big)\Psi_\ell + \big([j_{\rm r}, S]C_{\rm r} - S(C - C_{\rm r}) j_{\rm r}\big)\Psi_{\rm r}. \end{aligned} \quad (4.6)$$

Since we have $[j_\star, S] \in \mathscr{K}(\mathcal{H})$ and $(C - C_\star) j_\star \in \mathscr{K}(\mathcal{H})$ as a consequence of Assumption 2.1, it follows that $B \in \mathscr{K}(\mathcal{H}_0, \mathcal{H})$. The inclusion $B_* \in \mathscr{K}(\mathcal{H}_0, \mathcal{H})$ is proved in a similar way. $\square$

The next step is to define a conjugate operator $A$ for $U$ by using the conjugate operator $A_0$ for $U_0$. For this, we consider the operator $JA_0J^*$ which is well-defined and symmetric on $\mathcal{H}_{\rm fin}$. We have the equality

$$JA_0J^* = j_\ell A_\ell j_\ell + j_{\rm r} A_{\rm r} j_{\rm r} \quad \text{on} \quad \mathcal{H}_{\rm fin}, \tag{4.7}$$

and $JA_0J^*$ is essentially self-adjoint on $\mathcal{H}_{\rm fin}$ :

**Lemma 4.9** (Conjugate operator for $U$)**.** *The operator $JA_0J^*$ is essentially self-adjoint on $\mathcal{H}_{\rm fin}$, with corresponding self-adjoint extension denoted by $A$.*

*Proof.* The operator $\widehat{j_\star} := \mathscr{F} j_\star \mathscr{F}^* \in \mathscr{B}(\mathcal{K})$ satisfies $\widehat{j_\star}\,\mathcal{D}(P) \subset \mathcal{D}(P)$ and $[\widehat{j_\star}, P] = 0$ on $\mathcal{D}(P)$. Therefore, we have the following equalities on $\mathscr{F}\mathcal{H}_{\rm fin}$

$$\begin{aligned}
\mathscr{F} j_\star A_\star j_\star \mathscr{F}^* &= \tfrac{1}{2}\mathscr{F} j_\star\big(X_\star V_\star + V_\star X_\star\big) j_\star \mathscr{F}^* \\
&= \tfrac{1}{2}\widehat{j_\star}\big(\widehat{X_\star}\widehat{V_\star} + \widehat{V_\star}\widehat{X_\star}\big)\widehat{j_\star} \\
&= \widehat{j_\star}\big(\widehat{V_\star}\widehat{X_\star} - \tfrac{i}{2}\widehat{H_\star}\big)\widehat{j_\star} \\
&= -\sum_{j=1}^{2}\Big(\widehat{j_\star}\big|v_{\star,j}u_{\star,j}\big\rangle\big\langle u_{\star,j}\big|\widehat{j_\star}\,P - i\,\widehat{j_\star}\big|v_{\star,j}u_{\star,j}\big\rangle\big\langle u_{\star,j}'\big|\widehat{j_\star}\Big) - \tfrac{i}{2}\widehat{j_\star}\widehat{H_\star}\widehat{j_\star}.
\end{aligned}$$

which give on $\mathscr{F}\mathcal{H}_{\rm fin}$

$$\mathscr{F} JA_0J^*\mathscr{F}^* = -\sum_{j=1}^{2}\sum_{\star\in\{\ell,{\rm r}\}}\widehat{j_\star}\big|v_{\star,j}u_{\star,j}\big\rangle\big\langle u_{\star,j}\big|\widehat{j_\star}\,P + i\sum_{j=1}^{2}\sum_{\star\in\{\ell,{\rm r}\}}\widehat{j_\star}\big|v_{\star,j}u_{\star,j}\big\rangle\big\langle u_{\star,j}'\big|\widehat{j_\star} - \tfrac{i}{2}\sum_{\star\in\{\ell,{\rm r}\}}\widehat{j_\star}\widehat{H_\star}\widehat{j_\star}.$$

The rest of the proof consists in an application of Nelson's commutator theorem [33, Thm. X.37] with the comparison operator $N := P^2 + 1$. The estimates necessary to apply the theorem are similar to the ones mentioned in the proof of Lemma 4.3. As a consequence, it follows that $\mathscr{F} JA_0J^*\mathscr{F}^*$ is essentially self-adjoint on $\mathscr{F}\mathcal{H}_{\rm fin}$, and thus that $JA_0J^*$ is essentially self-adjoint on $\mathcal{H}_{\rm fin}$. ☐

We are thus in the setup of Assumption 3.10 with the set $\mathscr{D} = \mathcal{H}_{\rm fin}$. So, the next step is to show the inclusion $U \in C^1(A)$. For this, we use Corollary 3.11. Using Corollary 3.12, we also get an additional compacity result:

**Lemma 4.10.** *$U \in C^1(A)$ and $JU_0^{-1}[A_0, U_0]J^* - U^{-1}[A, U] \in \mathscr{K}(\mathcal{H})$.*

*Proof.* First, we recall that $U_0 \in C^1(A_0)$ due to Lemma 4.8(a), and that Assumption 3.10 holds with $\mathscr{D} = \mathcal{H}_{\rm fin}$. Next, we note that the expression for $B(\Psi_\ell, \Psi_{\rm r})$ with $(\Psi_\ell, \Psi_{\rm r}) \in \mathcal{H}_0$ is given in (4.6), and that

$$B_*(\Psi_\ell, \Psi_{\rm r}) = \big(C^*[j_\ell, S^*] - (C^* - C_\ell^*)j_\ell S^*\big)\Psi_\ell + \big(C^*[j_{\rm r}, S^*] - (C^* - C_{\rm r}^*)j_{\rm r} S^*\big)\Psi_{\rm r}.$$

Furthermore, we know from Lemma 4.8(b) that $B, B_* \in \mathscr{K}(\mathcal{H}_0, \mathcal{H})$. In consequence, due to Corollaries 3.11-3.12, the claims will follow if we show that $\overline{BA_0 \upharpoonright \mathcal{D}(A_0)} \in \mathscr{B}(\mathcal{H}_0, \mathcal{H})$ and $\overline{B_*A_0 \upharpoonright \mathcal{D}(A_0)} \in \mathscr{K}(\mathcal{H}_0, \mathcal{H})$. For this, we first note that computations as in the proof of Lemma 4.9 imply on $\mathcal{H}_{\rm fin}$ the equalities

$$\begin{aligned}
A_\star &= X_\star V_\star + \tfrac{i}{2}H_\star \\
&= -\mathscr{F}^*\left\{P\sum_{j=1}^{2}\Big(\big|u_{\star,j}\big\rangle\big\langle v_{\star,j}u_{\star,j}\big| + i\,\big|u_{\star,j}'\big\rangle\big\langle v_{\star,j}u_{\star,j}\big|\Big)\right\}\mathscr{F} + \tfrac{i}{2}H_\star \\
&= Q\,\mathscr{F}^*\left\{\sum_{j=1}^{2}\Big(\big|u_{\star,j}\big\rangle\big\langle v_{\star,j}u_{\star,j}\big| + i\,\big|u_{\star,j}'\big\rangle\big\langle v_{\star,j}u_{\star,j}\big|\Big)\right\}\mathscr{F} + \tfrac{i}{2}H_\star
\end{aligned} \tag{4.8}$$

with $Q$ the self-adjoint multiplication operator defined by

$$\big(Q\Psi\big)(x) = x\Psi(x), \quad x \in \mathbb{Z}, \ \Psi \in \mathcal{D}(Q) := \big\{\Psi \in \mathcal{H} \mid \|Q\Psi\|_{\mathcal{H}} < \infty\big\}. \tag{4.9}$$

Therefore, since all the operators on the right of $Q$ in (4.8) are bounded, it is sufficient to show that

$$\overline{B(Q \oplus Q) \upharpoonright \mathcal{D}(Q) \oplus \mathcal{D}(Q)} \in \mathscr{B}(\mathcal{H}_0, \mathcal{H}) \quad \text{and} \quad \overline{B_*(Q \oplus Q) \upharpoonright \mathcal{D}(Q) \oplus \mathcal{D}(Q)} \in \mathscr{K}(\mathcal{H}_0, \mathcal{H}).$$

However, this can be deduced from the Assumption 2.1 once the following observations are made: $\big[j_\star, S\big] = Sm_\star$ with $m_\star : \mathbb{Z} \to \mathscr{B}(\mathbb{C}^2)$ a function with compact support, $[j_\star, S^*] = S^* n_\star$ with $n_\star : \mathbb{Z} \to \mathscr{B}(\mathbb{C}^2)$ a function with compact support, and $S^*Q = QS^* + b$ with $b \in \mathsf{L}^\infty\big(\mathbb{Z}, \mathscr{B}(\mathbb{C}^2)\big)$. ☐

We now recall that the set

$$\tau(U) := \partial\sigma(U_\ell) \cup \partial\sigma(U_{\rm r}).$$

has been introduced in Section 2. Due to Lemma 4.1, $\tau(U)$ contains at most 8 values. Moreover, since we show in the next proposition that a Mourre estimate holds on the set $\{\sigma(U_\ell) \cup \sigma(U_{\rm r})\} \setminus \tau(U)$, it is natural to interpret $\tau(U)$ as the set of thresholds in the spectrum of $U$.

**Proposition 4.11** (Mourre estimate for $U$)**.** *We have $\widetilde{\varrho}_U^A \ge \widetilde{\varrho}_{U_0}^{A_0}$ with $\widetilde{\varrho}_{U_0}^{A_0} = \min\big\{\widetilde{\varrho}_{U_\ell}^{A_\ell}, \widetilde{\varrho}_{U_{\rm r}}^{A_{\rm r}}\big\}$ and $\widetilde{\varrho}_{U_\ell}^{A_\ell}, \widetilde{\varrho}_{U_{\rm r}}^{A_{\rm r}}$ given in Proposition 4.5. In particular, $\widetilde{\varrho}_{U_0}^{A_0}(\theta) > 0$ if $\theta \in \{\sigma(U_\ell) \cup \sigma(U_{\rm r})\} \setminus \tau(U)$.*

*Proof.* The first claim follows from Theorem 3.7, with the assumptions of this theorem verified in Lemmas 4.7-4.10. The second claim follows from Proposition 4.5 and standard results on the function $\widetilde{\varrho}_{U_0}^{A_0}$ when $A_0$ and $U_0$ are direct sums of operators (see [3, Prop. 8.3.5] for a proof in the case of direct sums of self-adjoint operators). ☐

## 4.3 Spectral properties of $U$

In order to go one step further in the study of $U$, a regularity property of $U$ with respect to $A$ stronger than $U \in C^1(A)$ has to be established. This regularity property will be obtained by considering first the operator $JU_0J^*$, and then by analysing the difference $U - JU_0J^*$. We note that $JU_0J^*$ and $U - JU_0J^*$ satisfy the equalities

$$JU_0J^* = j_\ell U_\ell j_\ell + j_{\rm r} U_{\rm r} j_{\rm r} \tag{4.10}$$

and

$$U - JU_0J^* = j_\ell(U - U_\ell)j_\ell + j_{\rm r}(U - U_{\rm r})j_{\rm r} + j_\ell U j_{\rm r} + j_{\rm r} U j_\ell. \tag{4.11}$$

**Lemma 4.12.** *$JU_0J^* \in C^2(A)$.*

*Proof.* The proof is based on standard results from toroidal pseudodifferential calculus, as presented for example in [37, Chap. 4]. The normalisation we use for the Fourier transform differs from the one used in [37], but the difference is harmless.

(i) First, we note that $\widehat{j_\star}$ is a toroidal pseudodifferential operator on $\mathscr{F}\mathcal{H}_{\rm fin}$ with symbol in $S^0_{\rho,0}(\mathbb{T}\times\mathbb{Z})$ for each $\rho > 0$ (see the definitions 4.1.7 and 4.1.9 of [37] for details). Similarly, the equation (4.8) shows that $\widehat{A_\star}$ is a first order differential operator on $\mathscr{F}\mathcal{H}_{\rm fin}$ with matrix coefficients in $\mathsf{M}\big(2, C^\infty(\mathbb{T})\big) \subset \mathsf{M}\big(2, S^0_{\rho,0}(\mathbb{T}\times\mathbb{Z})\big)$ for each $\rho > 0$. In consequence, it follows from [37, Thm. 4.7.10] that the commutator $\big[\widehat{j_\star}, \widehat{A_\star}\big]$ on $\mathscr{F}\mathcal{H}_{\rm fin}$ is well-defined and equal to a toroidal pseudodifferential operator with matrix coefficients in $\mathsf{M}\big(2, S^{1-\rho}_{\rho,0}(\mathbb{T}\times\mathbb{Z})\big)$ for each $\rho > 0$. This implies that $\big[\widehat{j_\star}, \widehat{A_\star}\big]$ is bounded on $\mathscr{F}\mathcal{H}_{\rm fin}$, and thus that $\widehat{j_\star} \in C^1(\widehat{A_\star})$ since $\mathscr{F}\mathcal{H}_{\rm fin}$ is dense in $\mathcal{D}(\widehat{A_\star})$. By Fourier transform, it follows that $j_\star \in C^1(A_\star)$.

(ii) A calculation in the form sense on $\mathcal{H}_{\rm fin}$ using equations (4.7) and (4.10) and the identities $j_\ell j_{\rm r} = 0 = j_{\rm r} j_\ell$ gives

$$\begin{aligned}\big[JU_0J^*, A\big] &= \big[j_\ell U_\ell j_\ell, j_\ell A_\ell j_\ell\big] + \big[j_{\rm r} U_{\rm r} j_{\rm r}, j_{\rm r} A_{\rm r} j_{\rm r}\big]\\ &= \sum_{\star\in\{\ell,{\rm r}\}} j_\star\big(U_\star j_\star A_\star - A_\star j_\star U_\star\big) j_\star\\ &= \sum_{\star\in\{\ell,{\rm r}\}} j_\star\big(\big[U_\star, j_\star\big]A_\star + \big[j_\star U_\star, A_\star\big]\big) j_\star. \end{aligned}\tag{4.12}$$

Since $j_\star U_\star \in C^1(A_\star)$ by Proposition 4.5(a), point (i) and [3, Prop. 5.1.5], the second term on the r.h.s. of (4.12) belongs to $\mathscr{B}(\mathcal{H})$. Furthermore, a calculation using the definition of the shift operator $S$ shows that

$$\big[U_\star, j_\star\big] = \big[S, j_\star\big]C_\star = B_\star m_\star$$

with $B_\star \in \mathscr{B}(\mathcal{H})$ and $m_\star : \mathbb{Z} \to \mathscr{B}(\mathbb{C}^2)$ a function with compact support. It follows from (4.8) that $\big[U_\star, j_\star\big]A_\star$ is bounded on $\mathcal{H}_{\rm fin}$. Therefore, both terms on the r.h.s. of (4.12) are bounded on $\mathcal{H}_{\rm fin}$, and thus we infer from the density of $\mathcal{H}_{\rm fin}$ in $\mathcal{D}(A)$ that $JU_0J^* \in C^1(A)$.

(iii) To show that $JU_0J^* \in C^2(A)$, one has to commute the r.h.s. of (4.12) once more with $A$. Doing this in the form sense on $\mathcal{H}_{\rm fin}$ with the notation $\sum_{\star\in\{\ell,{\rm r}\}} j_\star D_\star j_\star$ with $D_\star := [U_\star, j_\star]A_\star + [j_\star U_\star, A_\star]$ for the r.h.s. of (4.12), one gets that $JU_0J^* \in C^2(A)$ if the operators $[D_\star, A_\star]$, $[D_\star, j_\star]A_\star$ and $A_\star[D_\star, j_\star]$ defined in the form sense on $\mathcal{H}_{\rm fin}$ extend continuously to elements of $\mathscr{B}(\mathcal{H})$.

For the first operator, we have in the form sense on $\mathcal{H}_{\rm fin}$ the equalities

$$\begin{aligned}[D_\star, A_\star] &= \big[[U_\star, j_\star]A_\star + j_\star[U_\star, A_\star] + [j_\star, A_\star]U_\star, A_\star\big]\\ &= \big[[U_\star, j_\star]A_\star, A_\star\big] + j_\star\big[[U_\star, A_\star], A_\star\big] + [j_\star, A_\star][U_\star, A_\star] + [j_\star, A_\star][U_\star, A_\star] + \big[[j_\star, A_\star], A_\star\big]U_\star.\end{aligned}\tag{4.13}$$

Then, simple adaptations of the arguments presented in points (i) and (ii) show that the operators $[j_\star, A_\star], [U_\star, j_\star] \in \mathscr{B}(\mathcal{H})$ can be multiplied in the form sense on $\mathcal{H}_{\rm fin}$ by several operators $A_\star$ on the left and/or on the right and that the resultant operators extend continuously to elements of $\mathscr{B}(\mathcal{H})$. Therefore, the first, the third, the fourth and the fifth terms in (4.13) extend continuously to elements of $\mathscr{B}(\mathcal{H})$. For the second term, we note from Propositions 4.4(a) and 4.5(a) that $U_\star, V_\star \in C^1(A_\star)$ with $[U_\star, A_\star] = -U_\star V_\star^2$. In consequence, we have $U_\star V_\star^2 \in C^1(A_\star)$ by [3, Prop. 5.1.5] and

$$j_\star\big[[U_\star, A_\star], A_\star\big] = -j_\star\big[U_\star V_\star^2, A_\star\big] \in \mathscr{B}(\mathcal{H}).$$

The proof that the operators $[D_\star, j_\star]A_\star$ and $A_\star[D_\star, j_\star]$ defined in the form sense on $\mathcal{H}_{\rm fin}$ extend continuously to elements of $\mathscr{B}(\mathcal{H})$ is similar. The only noticeable difference is the appearance of terms $[U_\star V_\star^2, j_\star]A_\star$ and $A_\star[U_\star V_\star^2, j_\star]$. However, by observing that $V_\star^2 \in C^1(A_\star)$ and that $[V_\star^2, j_\star]$ is a toroidal pseudodifferential operator with matrix coefficients in $\mathrm{M}\big(2, S^{-\rho}_{\rho,0}(\mathbb{T}\times\mathbb{Z})\big)$ for each $\rho > 0$, one infers that $[U_\star V_\star^2, j_\star]A_\star$ and $A_\star[U_\star V_\star^2, j_\star]$ extend continuously to elements of $\mathscr{B}(\mathcal{H})$. □

In the next lemma, we prove that $U$ satisfies sufficient regularity with respect to $A$, namely that $U \in C^{1+\varepsilon}(A)$ for some $\varepsilon \in (0,1)$. We recall from Section 3.3 that the sets $C^2(A)$, $C^{1+\varepsilon}(A)$, $C^{1+0}(A)$ and $C^{1,1}(A)$ satisfy the continuous inclusions

$$C^2(A) \subset C^{1+\varepsilon}(A) \subset C^{1+0}(A) \subset C^{1,1}(A).$$

**Lemma 4.13.** *$U \in C^{1+\varepsilon}(A)$ for each $\varepsilon \in (0,1)$ with $\varepsilon \le \min\{\varepsilon_\ell, \varepsilon_{\rm r}\}$.*

*Proof.* (i) Since $JU_0J^* \in C^2(A)$ by Lemma 4.12 and $C^2(A) \subset C^{1+\varepsilon}(A)$, it is sufficient to show that $U - JU_0J^* \in C^{1+\varepsilon}(A)$, with $U - JU_0J^*$ given by (4.11). Moreover, calculations as in the proof of Lemma

4.12 show that the last two terms $j_\ell U j_\mathrm{r}$ and $j_\mathrm{r} U j_\ell$ of (4.11) belong to $C^2(A)$. So, it only remains to show that $j_\ell(U-U_\ell)j_\ell + j_\mathrm{r}(U-U_\mathrm{r})j_\mathrm{r} \in C^{1+\varepsilon}(A)$.

(ii) In order to show the mentioned inclusion, we first observe from (2.1) and (4.7) that we have in the form sense on $\mathcal{H}_\mathrm{fin}$ the equalities

$$\begin{aligned}\big[j_\ell(U-U_\ell)j_\ell + j_\mathrm{r}(U-U_\mathrm{r})j_\mathrm{r}, A\big] &= \sum_{\star\in\{\ell,\mathrm{r}\}} \big[j_\star(U-U_\star)j_\star, j_\star A_\star j_\star\big] \\ &= \sum_{\star\in\{\ell,\mathrm{r}\}} \big(j_\star S(C-C_\star)j_\star A_\star j_\star - j_\star A_\star j_\star S(C-C_\star)j_\star\big). \qquad (4.14)\end{aligned}$$

Then, using Assumption 2.1, the formula (4.8) for $A_\star$ on $\mathcal{H}_\mathrm{fin}$, and a similar formula with the operator $Q$ on the right (recall that $Q$ is the position operator defined in (4.9)), one obtains that the operator on the r.h.s. of (4.14) defined as

$$D_\star := j_\star S(C-C_\star)j_\star A_\star j_\star - j_\star A_\star j_\star S(C-C_\star)j_\star$$

extends continuously to an element of $\mathscr{B}(\mathcal{H})$. Since $\mathcal{H}_\mathrm{fin}$ is dense in $\mathcal{D}(A)$, this implies that $j_\ell(U-U_\ell)j_\ell + j_\mathrm{r}(U-U_\mathrm{r})j_\mathrm{r} \in C^1(A)$.

(iii) To show that $j_\ell(U-U_\ell)j_\ell + j_\mathrm{r}(U-U_\mathrm{r})j_\mathrm{r} \in C^{1+\varepsilon}(A)$, it remains to check that

$$\big\| \mathrm{e}^{-itA} D_\star \mathrm{e}^{itA} - D_\star\big\|_{\mathscr{B}(\mathcal{H})} \le \mathrm{Const.}\, t^\varepsilon \quad \text{for all } t\in(0,1).$$

But, algebraic manipulations as presented in [3, p. 325-326] show that for all $t\in(0,1)$

$$\begin{aligned}\big\| \mathrm{e}^{-itA} D_\star \mathrm{e}^{itA} - D_\star\big\|_{\mathscr{B}(\mathcal{H})} &\le \mathrm{Const.}\big(\|\sin(tA)D_\star\|_{\mathscr{B}(\mathcal{H})} + \|\sin(tA)(D_\star)^*\|_{\mathscr{B}(\mathcal{H})}\big) \\ &\le \mathrm{Const.}\big(\|tA(tA+i)^{-1}D_\star\|_{\mathscr{B}(\mathcal{H})} + \|tA(tA+i)^{-1}(D_\star)^*\|_{\mathscr{B}(\mathcal{H})}\big).\end{aligned}$$

Furthermore, if we set $A_t := tA(tA+i)^{-1}$ and $\Lambda_t := t\langle Q\rangle(t\langle Q\rangle + i)^{-1}$, we obtain that

$$A_t = \big(A_t + i(tA+i)^{-1}A\langle Q\rangle^{-1}\big)\Lambda_t$$

with $A\langle Q\rangle^{-1}\in\mathscr{B}(\mathcal{H})$ due to (4.7)-(4.8). Thus, since $\big\|A_t + i(tA+i)^{-1}A\langle Q\rangle^{-1}\big\|_{\mathscr{B}(\mathcal{H})}$ is bounded by a constant independent of $t\in(0,1)$, it is sufficient to prove that

$$\|\Lambda_t D_\star\|_{\mathscr{B}(\mathcal{H})} + \|\Lambda_t(D_\star)^*\|_{\mathscr{B}(\mathcal{H})} \le \mathrm{Const.}\, t^\varepsilon \quad \text{for all } t\in(0,1).$$

Now, this estimate will hold if we show that the operators $\langle Q\rangle^\varepsilon D_\star$ and $\langle Q\rangle^\varepsilon(D_\star)^*$ defined in the form sense on $\mathcal{H}_\mathrm{fin}$ extend continuously to elements of $\mathscr{B}(\mathcal{H})$. For this, we fix $\varepsilon\in(0,1)$ with $\varepsilon\le\min\{\varepsilon_\ell,\varepsilon_\mathrm{r}\}$, and note that $\langle Q\rangle^{1+\varepsilon}(C-C_\star)\,j_\star\in\mathscr{B}(\mathcal{H})$. With this inclusion and the fact that $\langle Q\rangle^{-1}A_\star$ defined in the form sense on $\mathcal{H}_\mathrm{fin}$ extend continuously to elements of $\mathscr{B}(\mathcal{H})$, one readily obtains that $\langle Q\rangle^\varepsilon D_\star$ and $\langle Q\rangle^\varepsilon(D_\star)^*$ defined in the form sense on $\mathcal{H}_\mathrm{fin}$ extend continuously to elements of $\mathscr{B}(\mathcal{H})$, as desired. □

With what precedes, we can now prove our last two main results on $U$ which have been stated in Section 2.

*Proof of Theorem 2.3.* Theorem 3.5, whose assumptions are verified in Proposition 4.11 and Lemma 4.13, implies that each $T\in\mathscr{B}(\mathcal{H},\mathcal{G})$ which extends continuously to an element of $\mathscr{B}\big(\mathcal{D}(\langle A\rangle^s)^*,\mathcal{G}\big)$ for some $s>1/2$ is locally $U$-smooth on $\Theta\setminus\sigma_\mathrm{p}(U)$. Moreover, we know from the proof of of Lemma 4.13 that $\mathcal{D}(Q)\subset\mathcal{D}(A)$. Therefore, we have $\mathcal{D}(\langle Q\rangle^s)\subset\mathcal{D}(\langle A\rangle^s)$ for each $s>1/2$, and it follows by duality that $\mathcal{D}(\langle A\rangle^s)^*\subset\mathcal{D}(\langle Q\rangle^s)^*\equiv\mathcal{D}(\langle Q\rangle^{-s})$ for each $s>1/2$. In consequence, any operator $T\in\mathscr{B}(\mathcal{H},\mathcal{G})$ which extends continuously to an element of $\mathscr{B}\big(\mathcal{D}(\langle Q\rangle^{-s}),\mathcal{G}\big)$ some $s>1/2$ also extends continuously to an element of $\mathscr{B}\big(\mathcal{D}(\langle A\rangle^s)^*,\mathcal{G}\big)$. This concludes the proof. □

*Proof of Theorem 2.4.* The claim follows from Theorem 3.6, whose hypotheses are verified in Lemma 4.13 and Proposition 4.11. □

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