LIFSHITS TAILS FOR RANDOMLY TWISTED QUANTUM WAVEGUIDES

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Abstract. We consider the Dirichlet Laplacian H_γ on a 3D twisted waveguide with random Anderson-type twisting γ. We introduce the integrated density of states N_γ for the operator H_γ, and investigate the Lifshits tails of N_γ, i.e. the asymptotic behavior of $N_\gamma(E)$ as $E \downarrow \inf \operatorname{supp} dN_\gamma$. In particular, we study the dependence of the Lifshits exponent on the decay rate of the single-site twisting at infinity.

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1. Introduction

The spectral properties of quantum Hamiltonians on tubular domains (waveguides) have been actively studied for several decades (see the monograph [10], the survey [18], and the references cited there). Recently, there has been a particular interest in the so-called *twisted waveguides* (see [9, 8, 5, 21, 4, 3, 24]), whose general setting we are going to describe briefly below.
Let $m \subset \mathbb{R}^2$ be a bounded domain. Set $M := m \times \mathbb{R}$. Let $\theta \in C^1(\mathbb{R};\mathbb{R})$ have a bounded derivative $\dot\theta$. Define the twisted tube

$$\mathcal{M}_\theta := \{\mathcal{R}_\theta(x_3)x,\ x \in M\}$$

where

$$\mathcal{R}_\theta(x_3) := \begin{pmatrix} \cos\theta(x_3) & \sin\theta(x_3) & 0 \\ -\sin\theta(x_3) & \cos\theta(x_3) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_3 \in \mathbb{R}. \tag{1.1}$$

Let $\mathcal{H}_\theta$ be the self-adjoint operator generated in $\mathrm{L}^2(\mathcal{M}_\theta)$ by the closed quadratic form

$$\mathcal{Q}_\theta[u] := \int_{\mathcal{M}_\theta} |\nabla u|^2 dx, \quad u \in \mathrm{H}^1_0(\mathcal{M}_\theta),$$

where, as usual, $\mathrm{H}^1_0(\mathcal{M}_\theta)$ is the closure of $C_0^\infty(\mathcal{M}_\theta)$ in the first-order Sobolev space $\mathrm{H}^1(\mathcal{M}_\theta)$. Introduce the quadratic form

$$Q_{\dot\theta}[u] := \int_M \left(|\nabla_t u|^2 + |\dot\theta\partial_\tau u + \partial_3 u|^2\right) dx, \quad u \in \mathrm{H}^1_0(M),$$

where $\nabla_t := (\partial_1, \partial_2)$, and $\partial_\tau := x_1\partial_2 - x_2\partial_1$. Let $H_{\dot\theta}$ be the self-adjoint operator generated in $\mathrm{L}^2(M)$ by the closed quadratic form $Q_{\dot\theta}$. Define the unitary operator $U_\theta : \mathrm{L}^2(\mathcal{M}_\theta) \to \mathrm{L}^2(M)$ by

$$(U_\theta u)(x) := u(\mathcal{R}_\theta(x_3)\, x), \quad x \in M, \quad u \in \mathrm{L}^2(\mathcal{M}_\theta).$$

Then $H_{\dot\theta} = U_\theta \mathcal{H}_\theta U_\theta^{-1}$.

If $m \subset \mathbb{R}^2$ is a bounded domain with boundary $\partial m \in C^2$, and $\theta \in C^2(\mathbb{R};\mathbb{R})$ has bounded first and second derivatives, then

$$H_{\dot\theta} = -\partial_1^2 - \partial_2^2 - (\dot\theta\partial_\tau + \partial_3)^2, \quad \mathrm{Dom}(H_{\dot\theta}) = \mathrm{H}^2(M) \cap \mathrm{H}_0^1(M), \tag{1.2}$$

(see [4, Corollary 2.2]).
In this article we will consider the operator H_γ with random Anderson-type twisting $\dot\theta = \gamma$ (see (1.6) below). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Assume that $\lambda_k(\omega)$, $k \in \mathbb{Z}$, $\omega \in \Omega$, are independent, identically distributed random variables. Set

$$\lambda^- := \operatorname*{ess\,inf}_{\omega\in\Omega} \lambda_0(\omega), \quad \lambda^+ := \operatorname*{ess\,sup}_{\omega\in\Omega} \lambda_0(\omega).$$

Throughout the article we assume that

$$-\infty < \lambda^- < \lambda^+ < \infty. \tag{1.3}$$

Further, introduce the *single-site twisting* $w \in C(\mathbb{R};\mathbb{R})$ which is supposed to satisfy

$$|w(s)| \le C(1+|s|)^{-\alpha}, \quad s \in \mathbb{R}, \tag{1.4}$$

with some constants $C \in (0,\infty)$, and $\alpha \in (1,\infty)$. Moreover, we assume that

$$w \not\equiv 0 \quad \text{on} \quad \mathbb{R}. \tag{1.5}$$

Introduce the random twisting

$$\gamma(s;\omega) = \sum_{k\in\mathbb{Z}} \lambda_k(\omega) w(s-k), \quad s \in \mathbb{R}, \quad \omega \in \Omega. \tag{1.6}$$

Then γ is a $\mathbb{Z}$-ergodic random field, and the operator H_γ, self-adjoint in $\mathrm{L}^2(M)$, is ergodic with respect to the translations T_k, defined by

$$(T_k u)(x_t, x_3) = u(x_t, x_3 - k), \quad k \in \mathbb{Z}, \quad (x_t, x_3) \in M, \quad u \in \mathrm{L}^2(M).$$

By the general theory of ergodic operators (see e.g. [12, Section 4]), there exists a closed non-random subset Σ of $\mathbb{R}$ such that almost surely

$$\sigma(H_\gamma) = \Sigma. \tag{1.7}$$

Let us introduce *the integrated density of states* (IDS) of the operator H_γ. For a finite $\ell > 0$, set $M_\ell := m \times (-\ell/2, \ell/2)$, and define the operator $H_{\gamma,\ell}$ as the self-adjoint operator generated in $\mathrm{L}^2(M_\ell)$ by the closed quadratic form

$$Q_{\gamma,\ell}[u] = \int_{M_\ell} \left(|\nabla_t u|^2 + |\gamma(x_3;\omega)\partial_\tau u + \partial_3 u|^2\right) dx, \quad u \in \mathrm{H}_0^1(M_\ell).$$

Evidently, the spectrum of $H_{\gamma,\ell}$ is purely discrete. We will say that the non-increasing left-continuous function $N_\gamma : \mathbb{R} \to [0,\infty)$ is an IDS for the operator H_γ if almost surely we have

$$\lim_{\ell\to\infty} \ell^{-1} \operatorname{Tr} \mathbb{1}_{(-\infty,E)}(H_{\gamma,\ell}) = N_\gamma(E) \tag{1.8}$$

at the points of continuity $E \in \mathbb{R}$ of N_γ. Arguing as in [12, Theorem 6, Section 7] or [11], it is easy to show that there exists an IDS N_γ for H_γ, and $\operatorname{supp} dN_\gamma = \Sigma$ (see (1.7)).

Our main results concern the asymptotic behavior of the IDS N_γ near $\Sigma_0 := \inf \Sigma$. This behavior is usually characterized by a very fast decay of the IDS, and is known as a *Lifshits-tail behavior*. More precisely, we show that under suitable assumptions

$$\lim_{E\downarrow 0} \frac{\ln|\ln N_\gamma(\Sigma_0 + E)|}{\ln E} = -\varkappa \tag{1.9}$$

with a constant $\varkappa > 0$ called *the Lifshits exponent*, which depends, as we will see, on the decay rate of w. Namely, if w satisfies (1.4) with $\alpha \in [2,\infty)$, then $\varkappa = \frac{1}{2}$ (see Theorems 3.2 (i), 3.4, and 3.5 below), while if $w(s) \sim |s|^{-\alpha}$ as $|s| \to \infty$, with $\alpha \in (1,2)$, then $\varkappa = \frac{1}{2(\alpha-1)}$ (see Theorem 3.2 (ii)).

One of the important assumptions of geometric nature we impose in order that (1.9) hold true, implies that the cross section m is *not* rotationally symmetric with respect to the origin. Otherwise, the operator H_γ would be unitarily equivalent to $\mathcal{H}_0$, the IDS N_γ would be independent of γ, and can be calculated explicitly (see (2.4) below). Note that in this case N_γ has at Σ_0 a van Hove singularity, i.e. a non smooth power-like decay, instead of a Lifshits tail (see e.g. [6] and the references cited there for a general discussion of the van Hove singularities).

Lifshits tails concerning various random 2D waveguides were considered in [16, 22]. Related spectral properties were studied in [1, 2].

The paper is organized as follows. In the next section we estimate $N_\gamma(\Sigma_0+E)$ with small $E > 0$ in terms of the IDS for suitable 1D Schrödinger operators $h_{\gamma,\epsilon}$ (see (2.7) below) whose potential depends on the random twisting γ and on the real parameter ϵ. In Section 3, we formulate and prove our main results on the Lifshits tails for the IDS N_γ, applying the estimates obtained in Section 2, as well as certain results on the Lifshits tails for the operator $h_{\gamma,\epsilon}$. Some of these necessary results turned out to be available in the literature (see [17, 26]) and some of them are borrowed from our companion paper [14] where Lifshits tails for Schrödinger operators with squared Anderson-type potentials are investigated in any dimension $d \geq 1$.

2. Estimates of N_γ in terms of the IDS for 1D random Schrödinger operators

In this section we show that if $\operatorname{ess\,inf}_{\omega\in\Omega} \lambda_0(\omega)^2 = 0$, then almost surely $\inf \sigma(H_\gamma)$ coincides with μ_1, the lowest eigenvalue of the transversal Dirichlet Laplacian, and obtain suitable two-sided estimates of $N(\mu_1 + E)$ for sufficiently small $E > 0$, in terms of the IDS for appropriate 1D random Schrödinger operators $h_{\gamma,\epsilon}$ (see (2.7) below).

Let $\{\mu_j\}_{j\in\mathbb{N}}$ be the non-decreasing sequence of the eigenvalues of the transversal Dirichlet Laplacian $-\Delta_t^D$, generated in $\mathrm{L}^2(m)$ by the closed quadratic form

$$\int_m |\nabla_t u|^2 dx_t, \quad u \in \mathrm{H}_0^1(m),$$

with $x_t := (x_1, x_2)$. We have

$$0 < \mu_1 < \mu_2. \tag{2.1}$$

Let $\{\varphi_j\}_{j\in\mathbb{N}}$ be an orthonormal basis in $\mathrm{L}^2(m)$ consisting of real-valued eigenfunctions of $-\Delta_t^D$ which satisfy

$$-\Delta_t^D \varphi_j = \mu_j \varphi_j, \quad j \in \mathbb{N}.$$

It is well known that φ_1 could be chosen so that

$$\varphi_1(x_t) > 0, \quad x_t \in m.$$

Set

$$\mathcal{T} := \|\partial_\tau \varphi_1\|_{\mathrm{L}^2(m)}. \tag{2.2}$$

Arguing as in the proof of [5, Proposition 2.2], we can show that if $\partial m \in C^2$, then the inequality

$$\mathcal{T} \neq 0 \tag{2.3}$$

holds true if and only if m is not rotationally symmetric with respect to the origin. On the other hand, if m is any bounded rotationally symmetric domain, then $\mathcal{T} = 0$. Moreover, in this case the operator $H_{\dot\theta}$ is unitarily equivalent to H_0, the spectrum $\sigma(H_{\dot\theta}) = [\mu_1, \infty)$ is absolutely continuous, the IDS $N_{\dot\theta} = N_0$, independent of $\dot\theta$, is well defined by analogy with (1.8), and we have

$$N_0(E) = \frac{1}{\pi} \sum_{j=1}^{\infty} (E - \mu_j)_+^{1/2}, \quad E \in \mathbb{R}. \tag{2.4}$$

In particular,

$$N_0(\mu_1 + E) = \frac{1}{\pi} E_+^{1/2}, \quad E \in (-\infty, \mu_2 - \mu_1). \tag{2.5}$$

Assume (1.3), (1.4), and

$$w \in C^1(\mathbb{R}; \mathbb{R}), \quad |\dot w(s)| \leq C(1 + |s|)^{-\alpha}, \quad s \in \mathbb{R}. \tag{2.6}$$

For $\epsilon \in \mathbb{R}$ introduce the operator $h_{\gamma,\epsilon}$ as the self-adjoint operator generated in $\mathrm{L}^2(\mathbb{R})$ by the closed quadratic form

$$q_{\gamma,\epsilon}[f] := \int_{\mathbb{R}} \left(|\dot f|^2 + \left(\mathcal{T}^2 \gamma(s;\omega)^2 - \epsilon \dot\gamma(s;\omega)^2 \right) |f|^2 \right) ds, \quad f \in \mathrm{H}^1(\mathbb{R}).$$

Remark: If $\epsilon = 0$, then we can omit assumption (2.6) in the definition of the operator $h_{\gamma,\epsilon}$.

Thus,

$$h_{\gamma,\epsilon} = -\frac{d^2}{ds^2} + \mathcal{T}^2\gamma^2 - \epsilon\dot{\gamma}^2 \tag{2.7}$$

is a 1D Schrödinger operator with random potential $\mathcal{T}^2\gamma(s;\omega)^2 - \epsilon\dot{\gamma}(s;\omega)^2$, $s \in \mathbb{R}$, $\omega \in \Omega$. This operator is $\mathbb{Z}$-ergodic, and its spectrum is almost surely independent of $\omega \in \Omega$. Introduce the IDS for the operator $h_{\gamma,\epsilon}$ as the non-decreasing function $\nu_{\gamma,\epsilon} : \mathbb{R} \to \mathbb{R}$ which almost surely satisfies

$$\lim_{\ell\to\infty} \ell^{-1} \operatorname{Tr} \mathbb{1}_{(-\infty,E)}(h_{\gamma,\epsilon,\ell}) = \nu_{\gamma,\epsilon}(E), \quad E \in \mathbb{R}, \tag{2.8}$$

$h_{\gamma,\epsilon,\ell}$ being the self-adjoint operator generated in $\mathrm{L}^2(-\ell/2,\ell/2)$ by the closed quadratic form

$$q_{\gamma,\epsilon,\ell}[f] := \int_{-\ell/2}^{\ell/2} \left(|\dot{f}|^2 + \left(\mathcal{T}^2\gamma(s;\omega)^2 - \epsilon\dot{\gamma}(s;\omega)^2\right)|f|^2 \right) ds, \quad f \in \mathrm{H}_0^1(-\ell/2,\ell/2). \tag{2.9}$$

The IDS $\nu_{\gamma,\epsilon}$ exists and is continuous (see [23, Theorem 3.2]). Moreover, in the definition (2.8) of $\nu_{\gamma,\epsilon}$ we can replace the operator $h_{\gamma,\epsilon,\ell}$ equipped with Dirichlet boundary conditions, by the operator generated by the quadratic form (2.9) with domain $\mathrm{H}^1(-\ell/2,\ell/2)$, equipped with Neumann boundary conditions. Further, it follows from (1.3) that

$$\tilde{\lambda}^+ := \operatorname*{ess\,sup}_{\omega\in\Omega} \lambda_0(\omega)^2 > 0.$$

In what follows, we assume that

$$\tilde{\lambda}^- := \operatorname*{ess\,inf}_{\omega\in\Omega} \lambda_0(\omega)^2 = 0. \tag{2.10}$$

Note that (2.10) implies that almost surely

$$\sigma(h_{\gamma,0}) = [0,\infty) \tag{2.11}$$

(see [13]).

Proposition 2.1. *Assume* (1.3), (1.4), *and* (2.10). *Then almost surely we have*

$$\sigma(H_\gamma) = [\mu_1,\infty). \tag{2.12}$$

Proof. We have

$$\inf \sigma(H_\gamma) = \inf_{0\neq u\in \mathrm{H}_0^1(M)} \frac{Q_\gamma[u]}{\|u\|^2_{\mathrm{L}^2(M)}}. \tag{2.13}$$

Since

$$Q_\gamma[u] \geq \int_M |\nabla_t u|^2\, dx, \quad u \in \mathrm{H}_0^1(M),$$

it follows from (2.13) and

$$\mu_1 = \inf_{0\neq u\in \mathrm{H}_0^1(M)} \frac{\int_M |\nabla_t u|^2\, dx}{\int_M |u|^2\, dx},$$

that

$$\inf \sigma(H_\gamma) \geq \mu_1. \tag{2.14}$$

Let us now prove the almost sure inclusion

$$\sigma(H_\gamma) \supset [\mu_1, \infty). \tag{2.15}$$

Fix $E \geq 0$. Arguing along the lines of the proof of (2.11) in [13], we can construct a sequence $\{f_n\}_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R})$, normalized to one in $\mathrm{L}^2(\mathbb{R})$, such that, almost surely

$$\|-\ddot{f}_n - E f_n\|_{\mathrm{L}^2(\mathbb{R})} \xrightarrow[n\to\infty]{} 0 \qquad \text{and} \qquad \|\gamma\|_{\mathrm{L}^\infty(\mathrm{supp} f_n)} \xrightarrow[n\to\infty]{} 0. \tag{2.16}$$

Notice that, by writing $\|\dot{f}_n\|^2_{\mathrm{L}^2(\mathbb{R})} = -(\ddot{f}_n, f_n)_{\mathrm{L}^2(\mathbb{R})} \leq \|\ddot{f}_n\|_{\mathrm{L}^2(\mathbb{R})}$, it follows from the first limit in (2.16) that the sequence $\{\dot{f}_n\}_{n\in\mathbb{N}}$ is almost surely bounded in $\mathrm{L}^2(\mathbb{R})$. The sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathrm{H}^1(M)$ defined by

$$u_n := \varphi_1 \otimes f_n$$

is normalized to one in $\mathrm{L}^2(M)$. By the Weyl criterion adapted to quadratic forms (see [19, Theorem 5]), the desired inclusion (2.15) will hold if we show that, almost surely,

$$\sup_{0\neq\phi\in\mathrm{H}_0^1(M)} \frac{|Q_\gamma(u_n,\phi) - (\mu_1+E)(u_n,\phi)_{\mathrm{L}^2(M)}|}{\|\phi\|_{\mathrm{H}^1(M)}} \xrightarrow[n\to\infty]{} 0, \tag{2.17}$$

where $Q_\gamma(\cdot,\cdot)$ is the sesquilinear form generated by the quadratic form $Q_\gamma[u]$, $u \in \mathrm{H}_0^1(M)$, and $(\cdot,\cdot)_{\mathrm{L}^2(M)}$ is the scalar product in $\mathrm{L}^2(M)$.
Integrating by parts, using the normalizations of f_n and φ_1, and applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned} |Q_\gamma(u_n,\phi) - (\mu_1+E)(u_n,\phi)_{\mathrm{L}^2(M)}| \; &\leq \|\phi\|_{\mathrm{L}^2(M)}\, \|-\ddot{f}_n - E f_n\|_{\mathrm{L}^2(\mathbb{R})} \\ &\quad + \|\partial_3\phi\|_{\mathrm{L}^2(M)}\, \|\gamma\|_{\mathrm{L}^\infty(\mathrm{supp} f_n)}\, \mathcal{T} \\ &\quad + \|\partial_\tau\phi\|_{\mathrm{L}^2(M)}\, \|\gamma\|_{\mathrm{L}^\infty(\mathrm{supp} f_n)}\, \|\dot{f}_n\|_{\mathrm{L}^2(\mathbb{R})} \\ &\quad + \|\partial_\tau\phi\|_{\mathrm{L}^2(M)}\, \|\gamma^2\|_{\mathrm{L}^\infty(\mathrm{supp} f_n)}\, \mathcal{T}. \end{aligned} \tag{2.18}$$

Thus, (2.18) and (2.16) imply (2.17), and hence (2.15).
Now (2.12) follows from (2.14) and (2.15). □

Further, we need several notations which will allow us to formulate certain assumptions of geometric nature. Assume (1.3), (1.4), and set

$$D_1 := \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{s\in\mathbb{R}} (5\gamma(s;\omega)^2 + 1).$$

Then $D_1 < \infty$.
Further, assume (1.3), (1.4), (2.6), and (2.3). Suppose in addition that the logarithmic derivative $\dot{\gamma}/\gamma$ is well defined and

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{s\in\mathbb{R}} \left| \frac{\dot{\gamma}(s;\omega)}{\gamma(s,\omega)} \right| < \infty. \tag{2.19}$$

Set

$$D_2 := \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{s\in\mathbb{R}} \left(6\gamma(s;\omega)^2 + \frac{2\dot{\gamma}(s;\omega)^2}{\mathcal{T}^2\gamma(s;\omega)^2} \right).$$

Then $D_2 < \infty$.

Remark: Assumption (2.19) holds true if w does not vanish at any $s \in \mathbb{R}$ and admits a regular power-like decay at infinity, but it is false if w has a compact support.

Finally, put
$$a := \sup_{x_t \in m} |x_t|.$$

Theorem 2.2. *Assume* (1.3) *and* (1.4).
(i) *We have*

$$\nu_{\gamma,0}(E) \leq N_\gamma(\mu_1 + E), \quad E \in \mathbb{R}. \tag{2.20}$$

(ii) *Let* $\delta_0 \in (0,1)$. *Suppose in addition that* (2.6) *holds true, and*

$$a^2 \left(1 - \frac{\mu_1}{\mu_2}\right)^{-1} D_1 < \delta_0. \tag{2.21}$$

Then we have

$$N_\gamma(\mu_1 + E) \leq \nu_{\gamma,\delta/(1-\delta)^{-1}}((1-\delta)^{-1}E) \tag{2.22}$$

for any $\delta \in \left(a^2 \left(1 - \frac{\mu_1}{\mu_2}\right)^{-1} D_1, \delta_0\right)$ *and* $E \in (0, \mu_2(1 - \delta^{-1} D_1 a^2) - \mu_1)$.
(iii) *Suppose in addition that* (2.6), (2.3), *and* (2.19) *hold true, and*

$$a^2 \left(1 - \frac{\mu_1}{\mu_2}\right)^{-1} D_2 < 1. \tag{2.23}$$

Then we have

$$N_\gamma(\mu_1 + E) \leq \nu_{\gamma,0}((1-\delta)^{-1}E) \tag{2.24}$$

for any $\delta \in \left(a^2 \left(1 - \frac{\mu_1}{\mu_2}\right)^{-1} D_1, 1\right)$ *and* $E \in (0, \mu_2(1 - \delta^{-1} D_2 a^2) - \mu_1)$.

Remark: If γ is fixed and $D_1 < \infty$ (resp., $D_2 < \infty$), then (2.21) (resp., (2.23)) holds true if a is small enough. Note that it follows from the results of [7, 20] that the operator $H_\gamma - \mu_1$ converges in an appropriate sense to $h_{\gamma,0}$ as $a \downarrow 0$.

Proof of Theorem 2.2. If we restrict the quadratic form $Q_{\gamma,\ell}$ to functions of the form
$$u_1 = \varphi_1 \otimes f, \quad f \in \mathrm{H}_0^1(-\ell/2, \ell/2),$$
then

$$Q_{\gamma,\ell}[u_1] = q_{\gamma,0,\ell}[f] + \mu_1 \|f\|^2_{\mathrm{L}^2(-\ell/2,\ell/2)}, \quad \|u_1\|^2_{\mathrm{L}^2(M)} = \|f\|^2_{\mathrm{L}^2(\mathbb{R})}, \tag{2.25}$$

the quadratic form $q_{\gamma,\epsilon,\ell}$ being defined in (2.9). Hence, the mini-max principle implies

$$\mathrm{Tr}\, \mathbb{1}_{(-\infty,\mu_1+E)}(H_{\gamma,\ell}) \geq \mathrm{Tr}\, \mathbb{1}_{(-\infty,E)}(h_{\gamma,0,\ell}), \quad E \in \mathbb{R}. \tag{2.26}$$

Combining (1.8), (2.8), and (2.26), we get (2.20).
Next, set

$$\mathcal{D}_1 := \left\{u_1 = \varphi_1 \otimes f \mid f \in \mathrm{H}_0^1(-\ell/2, \ell/2)\right\},$$

$$\mathcal{D}_2 := \left\{u_2 \in \mathrm{H}_0^1(M_\ell) \mid \int_{M_\ell} u_2(x)\overline{u_1(x)}dx = 0,\ \forall u_1 \in \mathcal{D}_1)\right\}.$$

Then, for $u = u_1 + u_2$ with $u_1 = \varphi_1 \otimes f \in \mathcal{D}_1$ and $u_2 \in \mathcal{D}_2$, we have

$$\|u\|^2_{\mathrm{L}^2(M_\ell)} = \|u_1 + u_2\|^2_{\mathrm{L}^2(M_\ell)} = \|f\|^2_{\mathrm{L}^2(-\ell/2,\ell/2)} + \|u_2\|^2_{\mathrm{L}^2(M_\ell)}.$$

Moreover, integrating by parts, we get

$$Q_{\gamma,\ell}[u] = Q_{\gamma,\ell}[u_1 + u_2] =$$

$$Q_{\gamma,\ell}[u_1] + Q_{\gamma,\ell}[u_2] + 2\mathrm{Re}\int_{M_\ell}\left(\gamma^2\partial_\tau u_1\overline{\partial_\tau u_2} + \gamma\partial_3 u_1\overline{\partial_\tau u_2} + \gamma\partial_\tau u_1\overline{\partial_3 u_2}\right)dx =$$

$$Q_{\gamma,\ell}[u_1] + Q_{\gamma,\ell}[u_2] + 2\mathrm{Re}\int_{M_\ell}\left(\gamma^2\partial_\tau u_1 + 2\gamma\partial_3 u_1 + \dot{\gamma}u_1\right)\overline{\partial_\tau u_2}\,dx. \tag{2.27}$$

Assume (2.21) and pick $\delta \in \left(a^2\left(1 - \frac{\mu_1}{\mu_2}\right)^{-1} D_1, \delta_0\right)$. We have

$$2\mathrm{Re}\int_{M_\ell}\left(\gamma^2\partial_\tau u_1 + 2\gamma\partial_3 u_1 + \dot{\gamma}u_1\right)\overline{\partial_\tau u_2}\,dx \geq$$

$$-\delta\int_{M_\ell}\left(\gamma^2|\partial_\tau u_1|^2 + |\partial_3 u_1|^2 + \dot{\gamma}^2|u_1|^2\right)dx - \delta^{-1}\int_{M_\ell}(5\gamma^2+1)|\partial_\tau u_2|^2 dx =$$

$$-\delta\int_{-\ell/2}^{\ell/2}\left(|\dot{f}|^2 + (\mathcal{T}^2\gamma^2 + \dot{\gamma}^2)|f|^2\right)dx_3 - \delta^{-1}\int_{M_\ell}(5\gamma^2+1)|\partial_\tau u_2|^2 dx. \tag{2.28}$$

Then, (2.25), (2.27), and (2.28) easily imply

$$Q_{\gamma,\ell}[u] \geq (1-\delta)q_{\gamma,\delta/(1-\delta),\ell}[f] + \mu_1\|f\|^2_{\mathrm{L}^2(-\ell/2,\ell/2)} + \tilde{Q}_{\gamma,\ell}[u_2] \tag{2.29}$$

where

$$\tilde{Q}_{\gamma,\ell}[u_2] :=$$

$$\int_{M_\ell}\left(|\nabla_t u_2|^2 - \delta^{-1}(5\gamma^2+1)|\partial_\tau u_2|^2 + |\gamma\partial_\tau u_2 + \partial_3 u_2|^2\right)dx, \quad u_2 \in \mathcal{D}_2.$$

Let $\tilde{H}_{\gamma,\ell}$ be the operator generated by the closed quadratic form $\tilde{Q}_{\gamma,\ell}$ in the Hilbert space $\mathcal{D}_1^\perp$, the orthogonal complement of $\mathcal{D}_1$ in $\mathrm{L}^2(M_\ell)$. Then the mini-max principle implies

$$\mathrm{Tr}\,\mathbb{1}_{(-\infty,\mu_1+E)}(H_{\gamma,\ell}) \leq \mathrm{Tr}\,\mathbb{1}_{(-\infty,E)}((1-\delta)h_{\gamma,\delta/(1-\delta),\ell}) + \mathrm{Tr}\,\mathbb{1}_{(-\infty,\mu_1+E)}(\tilde{H}_{\gamma,\ell}), \quad E \in \mathbb{R}. \tag{2.30}$$

Since $|\partial_\tau u_2| \leq |x_t||\nabla_t u_2|$, we have

$$\tilde{Q}_{\gamma,\ell}[u_2] \geq \mu_2\left(1 - \delta^{-1}a^2 D_1\right)\int_{M_\ell}|u_2|^2 dx. \tag{2.31}$$

Therefore, if $E \in (0, \mu_2 (1 - \delta^{-1} a^2 D_1) - \mu_1)$, we have

$$\operatorname{Tr} \mathbb{1}_{(-\infty, \mu_1 + E)}(\tilde{H}_{\gamma,\ell}) = 0,$$

and by (2.30),

$$\operatorname{Tr} \mathbb{1}_{(-\infty,\mu_1+E)}(H_{\gamma,\ell}) \leq \operatorname{Tr} \mathbb{1}_{(-\infty,E)}((1-\delta)h_{\gamma,\delta/(1-\delta),\ell}) = \operatorname{Tr} \mathbb{1}_{(-\infty,(1-\delta)^{-1}E)}(h_{\gamma,\delta/(1-\delta),\ell}). \tag{2.32}$$

Now (1.8), (2.8), and (2.32), imply (2.22).

Finally, assume (2.23) and pick $\delta \in \left(a^2 \left(1 - \frac{\mu_1}{\mu_2}\right)^{-1} D_2, 1\right)$. Similarly to (2.28), we have

$$2\mathrm{Re} \int_{M_\ell} \left(\gamma^2 \partial_\tau u_1 + 2\gamma \partial_3 u_1 + \dot{\gamma} u_1\right) \overline{\partial_\tau u_2}\, dx \geq$$

$$-\delta \int_{M_\ell} \left(\frac{\gamma^2}{2} |\partial_\tau u_1|^2 + |\partial_3 u_1|^2 + \frac{\mathcal{T}^2 \gamma^2}{2} |u_1|^2\right) dx - \delta^{-1} \int_{M_\ell} \left(6\gamma^2 + \frac{2\dot{\gamma}^2}{\mathcal{T}^2 \gamma^2}\right) |\partial_\tau u_2|^2 dx =$$

$$-\delta \int_{-\ell/2}^{\ell/2} \left(|\dot{f}|^2 + \mathcal{T}^2 \gamma^2 |f|^2\right) dx_3 - \delta^{-1} \int_{M_\ell} \left(6\gamma^2 + \frac{2\dot{\gamma}^2}{\mathcal{T}^2 \gamma^2}\right) |\partial_\tau u_2|^2 dx.$$

Hence, by analogy with (2.29) and (2.31), we have

$$Q_{\gamma,\ell}[u] \geq$$

$$(1-\delta) q_{\gamma,0,\ell}[f] + \mu_1 \|f\|^2_{\mathrm{L}^2(-\ell/2,\ell/2)} +$$

$$\int_{M_\ell} \left(|\nabla_t u_2|^2 - \delta^{-1} \left(6\gamma^2 + \frac{2\dot{\gamma}^2}{\mathcal{T}^2 \gamma^2}\right) |\partial_\tau u_2|^2 + |\gamma \partial_\tau u_2 + \partial_3 u_2|^2\right) dx \geq$$

$$(1-\delta) q_{\gamma,0,\ell}[f] + \mu_1 \|f\|^2_{\mathrm{L}^2(-\ell/2,\ell/2)} + \mu_2 \left(1 - \delta^{-1} a^2 D_2\right) \int_{M_\ell} |u_2|^2 dx.$$

Therefore, if $E \in (0, \mu_2 (1 - \delta^{-1} a^2 D_2) - \mu_1)$, we have

$$\operatorname{Tr} \mathbb{1}_{(-\infty,\mu_1+E)}(H_{\gamma,\ell}) \leq \operatorname{Tr} \mathbb{1}_{(-\infty,(1-\delta)^{-1}E)}(h_{\gamma,0,\ell}). \tag{2.33}$$

Now (1.8), (2.8), and (2.33), imply (2.24).

□

3. Lifshits tails for the operator H_γ

In this section we formulate and prove our main results concerning the asymptotic behavior of $N_\gamma(\mu_1 + E)$ as $E \downarrow 0$. In Subsection 3.1 we consider single-site twisting w of power-like decay while in Subsection 3.2 we handle the case of compactly supported w.

3.1. **Single-site twisting w of power-like decay.** The following proposition contains results from [14] on the Lifshits tails for 1D Schrödinger operators with squared random Anderson-type potentials.

Proposition 3.1 ([14, Theorem 1])**.** *Assume* (2.3)*. Suppose that w satisfies* (1.4) *with $\alpha \in (1, \infty)$, and* (1.5)*, while λ_0 satisfies* (1.3) *and* (2.10)*. Suppose moreover that*

$$\mathbb{P}(\{\omega \in \Omega \,|\, |\lambda_0(\omega)| < \varepsilon\}) \geq C\varepsilon^\kappa, \tag{3.1}$$

for some $\kappa > 0$, $C > 0$, and any sufficiently small $\varepsilon > 0$.
(i) *If $\alpha \geq 2$, then*

$$\lim_{E \downarrow 0} \frac{\ln |\ln \nu_{\gamma,0}(E)|}{\ln E} = -\frac{1}{2}. \tag{3.2}$$

(ii) *Let $1 < \alpha < 2$. Assume that*

$$w(s) \geq C(1 + |s|)^{-\alpha}, \quad s \in \mathbb{R}, \quad C > 0, \tag{3.3}$$

and

$$\lambda^- = 0. \tag{3.4}$$

Then

$$\lim_{E \downarrow 0} \frac{\ln |\ln \nu_{\gamma,0}(E)|}{\ln E} = -\frac{1}{2(\alpha - 1)}.$$

Remark: Evidently, we may replace the assumptions (3.3) and (3.4), by $w(s) \leq -C(1 + |s|)^{-\alpha}$, $s \in \mathbb{R}$, with $C > 0$, and $\lambda^+ = 0$ respectively. A similar remark applies to Theorems 3.2 (ii) and 3.5.

Combining Theorem 2.2 with Proposition 3.1, we obtain the following theorem concerning the Lifshits tails of the IDS N_γ for the randomly twisted waveguide:

Theorem 3.2. *Let $m \subset \mathbb{R}^2$ be a bounded domain such that $\mathcal{T} \neq 0$. Assume that:*

- *$w \in C^1(\mathbb{R}; \mathbb{R})$ does not vanish identically on $\mathbb{R}$ and satisfies the upper bound* (1.4) *with $\alpha \in (1, \infty)$;*
- *λ_0 satisfies* (1.3)*,* (2.10)*, and* (3.1)*;*
- *the logarithmic derivative $\dot{\gamma}/\gamma$ satisfies the boundedness condition* (2.19)*;*
- *the waveguide satisfies "the thinness condition"* (2.23)*.*

(i) *Let $\alpha \in [2, \infty)$. Then we have*

$$\lim_{E \downarrow 0} \frac{\ln |\ln N_\gamma(\mu_1 + E)|}{\ln E} = -\frac{1}{2}. \tag{3.5}$$

(ii) *Let $\alpha \in (1, 2)$. Suppose moreover that the lower bounds* (3.3) *and* (3.4) *hold true. Then we have*

$$\lim_{E \downarrow 0} \frac{\ln |\ln N_\gamma(\mu_1 + E)|}{\ln E} = -\frac{1}{2(\alpha - 1)}.$$

Remark: If $\mathcal{T} = 0$, then

$$\nu_{\gamma,0}(E) = \nu_{0,0}(E) = \frac{1}{\pi}E_+^{1/2}, \quad E \in \mathbb{R}.$$

Therefore, (2.20) implies

$$\liminf_{E\downarrow 0} \frac{\ln|\ln N_\gamma(\mu_1 + E)|}{\ln E} \geq 0,$$

i.e. N_γ does not exhibit a Lifshits tail near μ_1. As mentioned in Section 2, if $\partial m \in C^2$, then $\mathcal{T} = 0$ is equivalent to the fact that m is rotationally invariant with respect to the origin, and (2.4) and (2.5) hold true, i.e. H_γ exhibits near μ_1 a van Hove singularity instead of a Lifshits tail. A similar remark applies to Theorems 3.4 and 3.5 below.

3.2. **Single-site twisting w of compact support.** In this subsection we assume that (1.5) holds true, and

$$w \in C^1(\mathbb{R};\mathbb{R}), \quad \operatorname{supp} w \subset [-\beta/2, \beta/2], \tag{3.6}$$

with $\beta \in (0,\infty)$.

First, we consider the case where the support of w is small, i.e. (3.6) holds with $\beta \in (0,1]$. Then the multiplier by $\mathcal{T}^2\gamma(s;\omega)^2 - \epsilon\dot{\gamma}(s;\omega)^2$ coincides with the multiplier by

$$\sum_{k\in\mathbb{Z}} \lambda_k(\omega)^2 v_\epsilon(s-k), \quad s \in \mathbb{R},$$

where

$$v_\epsilon(s) := \mathcal{T}^2 w(s)^2 - \epsilon \dot{w}(s)^2, \quad s \in \mathbb{R}. \tag{3.7}$$

For $\epsilon \in \mathbb{R}$ denote by $\mathcal{E}^\pm(\epsilon)$ the lowest eigenvalue of the operator

$$h_\epsilon^\pm := -\frac{d^2}{ds^2} + \tilde{\lambda}^\pm v_\epsilon, \tag{3.8}$$

acting in $L^2(-1/2,1/2)$, and equipped with Neumann boundary conditions. If (2.10) is fulfilled, then, evidently,

$$\mathcal{E}^-(\epsilon) = 0, \quad \epsilon \in \mathbb{R}. \tag{3.9}$$

Put

$$\epsilon_0 := \sup\{\epsilon \in \mathbb{R} \,|\, \mathcal{E}^+(\epsilon) > 0\}. \tag{3.10}$$

It follows from (1.5) and (1.3) that if (2.3) is valid, then $\epsilon_0 > 0$ since $\mathcal{E}^+(0) > 0$, and $\mathcal{E}^+$ is a continuous (as a matter of fact, real analytic) non-increasing function of $\epsilon \in \mathbb{R}$. Thus,

$$\mathcal{E}^+(\epsilon) > 0, \quad \epsilon \in (-\infty, \epsilon_0). \tag{3.11}$$

Proposition 3.3. *Assume that w satisfies* (1.5), (3.6) *with $\beta \in (0,1]$, while λ_0 satisfies* (1.3) *and* (2.10). *Let $\epsilon \in (-\infty, \epsilon_0)$.*
(i) *We have almost surely*

$$\inf \sigma(h_{\gamma,\epsilon}) = 0. \tag{3.12}$$

(ii) *Moreover,*

$$\limsup_{E\downarrow 0} \frac{\ln|\ln \nu_{\gamma,\epsilon}(E)|}{\ln E} \le -\frac{1}{2}. \tag{3.13}$$

Idea of the proof of Proposition 3.3: Taking into account (2.10), (3.9), and (3.11), we find that (3.12) follows from [17, Proposition 0.1]. Note that the hypotheses of [17, Proposition 0.1] contain also the condition that v_ϵ be an even function of $s \in \mathbb{R}$. However, this condition is needed to guarantee that the eigenfunction of the operator h_ϵ^- is even, which in our setting is immediately implied by (2.10).
Further, bearing in mind (3.12), (3.9), and (3.11), we easily conclude that (3.13) follows from [17, Theorem 0.1].
It should be noted here that the assumptions of Proposition 0.1 and Theorem 0.1 of [17] require that $\operatorname{supp} v_\epsilon \subset (-1/2, 1/2)$ which may formally exclude the case $\beta = 1$ in (3.6). A careful analysis of the proofs of Proposition 0.1 and Theorem 0.1 of [17] however shows that these proofs extend without any problem to the case $\operatorname{supp} v_\epsilon \subset [-1/2, 1/2]$. □

Remarks: (i) Proposition 3.3 also follows from the results of the article [26] which extends [17]. More precisely, (3.12) follows from [26, Theorem 1.1], while (3.13) follows from [26, Theorem 1.2].
(ii) If $\epsilon \le 0$ and hence v_ϵ does not change sign, (3.12) and (3.13) have been known since long ago (see [13] and [15] respectively). However, the case $\epsilon \le 0$ is not appropriate for our purposes.

Theorem 3.4. *Let $m \subset \mathbb{R}^2$ be a bounded domain such that $\mathcal{T} \neq 0$. Assume that:*

- *w does not vanish identically on $\mathbb{R}$ and satisfies* (3.6) *with $\beta \in (0,1]$;*
- *λ_0 satisfies* (1.3), (2.10), *and* (3.1)*;*
- *the waveguide satisfies "the thinness condition"* (2.21) *with $\delta_0 = \frac{\epsilon_0}{1+\epsilon_0}$, ϵ_0 being defined in* (3.10).

Then (3.5) *is valid again.*

Proof. If $\delta < \frac{\epsilon_0}{1+\epsilon_0}$, then $\delta/(1-\delta) < \epsilon_0$. Therefore, (3.5) follows from (2.20), (2.22), (3.2) and (3.13). □

Further, we consider the case where the support of w may be large, i.e. (1.5), and (3.6) with $\beta \in (1, \infty)$ hold true; then the supports of the translates of w may have a substantial overlap. Without any loss of generality, we assume that $\beta = 2p + 1$ with $p \in \mathbb{N}$. Set $\mathcal{J} := \{-p, \dots, p\}$, and

$$\mathcal{J}_1 := \left\{ j \in \mathcal{J} \,|\, w \not\equiv 0 \quad \text{on} \left[-\frac{1}{2} + j, \frac{1}{2} + j\right] \right\},$$

$$\mathcal{J}_2 := \left\{ j \in \mathcal{J} \,|\, \dot{w} \not\equiv 0 \quad \text{on} \left[-\frac{1}{2} + j, \frac{1}{2} + j \right] \right\}.$$

$$n_k := \#\mathcal{J}_k, \quad k = 1, 2.$$

Evidently, $\mathcal{J}_2 \subset \mathcal{J}_1$, and $n_1 \geq n_2 \geq 1$. By analogy with (3.7), set

(3.14)
$$v_{j,\epsilon}(s) := \left(\mathcal{T}^2 w(s+j)^2 - n_2 \epsilon \dot{w}(s+j)^2 \right) \mathbb{1}_{[-1/2,1/2)}(s), \quad s \in \mathbb{R}, \quad \epsilon \in \mathbb{R}, \quad j \in \mathcal{J},$$

so that $\operatorname{supp} v_{j,\epsilon} \subset [-1/2, 1/2]$. By analogy with (3.8), for $\epsilon \in \mathbb{R}$, consider the Neumann realization of the operators

$$h^{\pm}_{j,\epsilon} := -\frac{d^2}{ds^2} + n_1 \tilde{\lambda}^{\pm} v_{j,\epsilon}, \quad j \in \mathcal{J}_1, \tag{3.15}$$

restricted on $(-1/2, 1/2)$. Denote by $\mathcal{E}^{\pm}_j(\epsilon)$, $j \in \mathcal{J}_1$, the lowest eigenvalue of the operator $h^{\pm}_{j,\epsilon}$. Put

$$\epsilon_0^{\min} := \min_{j \in \mathcal{J}_1} \sup \left\{ \epsilon \in \mathbb{R} \,|\, \mathcal{E}^+_j(\epsilon) > 0 \right\}.$$

By analogy with (3.9), we have

$$\mathcal{E}^-_j(\epsilon) = 0, \quad \epsilon \in \mathbb{R}, \quad j \in \mathcal{J}_1, \tag{3.16}$$

if (2.10) holds true. Moreover, if (1.5), (1.3), and (2.3) are valid, we have $\epsilon_0^{\min} > 0$, and

$$\mathcal{E}^+_j(\epsilon) > 0, \quad \epsilon \in (-\infty, \epsilon_0^{\min}), \quad j \in \mathcal{J}_1, \tag{3.17}$$

by analogy with (3.11).

Theorem 3.5. *Let $m \subset \mathbb{R}^2$ be a bounded domain such that $\mathcal{T} \neq 0$. Assume that:*

- *w does not vanish identically on $\mathbb{R}$, and satisfies* (3.6) *with $\beta = 2p + 1$, $p \in \mathbb{N}$, and*

$$w(s) \geq 0, \quad s \in \mathbb{R}, \tag{3.18}$$

- *λ_0 satisfies* (1.3), (2.10), (3.1), *and* (3.4)*;*
- *the waveguide satisfies "the thinness condition"* (2.21) *with $\delta_0 = \frac{\epsilon_0^{\min}}{1+\epsilon_0^{\min}}$.*

Then, again, we have (3.5).

For the proof of Theorem 3.5, we will need Lemma 3.6 below. Let us recall that by (2.7), $h_{0,0}$ is simply the operator $-\frac{d^2}{ds^2}$, self-adjoint in $\mathrm{L}^2(\mathbb{R})$, while $h_{0,0,\ell}$ is the Dirichlet realization of its restriction onto $(-\ell/2, \ell/2)$, $\ell \in (0, \infty)$.

Lemma 3.6. *Let $n \in \mathbb{N}$, $V_j : \mathbb{R} \times \Omega \to \mathbb{R}$, $j = 1, \ldots, n$, be almost surely bounded ergodic potentials. Let ρ_j be the IDS for the operator $h_{0,0} + nV_j$, $j = 1, \ldots, n$, and ρ be the IDS for the operator $h_{0,0} + \sum_{j=1}^n V_j$. Then we have*

$$\rho(E) \leq \sum_{j=1}^n \rho_j(E), \quad E \in \mathbb{R}. \tag{3.19}$$

Remark: Lemma 3.6 admits an immediate extension to general multi-dimensional ergodic Schrödinger operators. The above formulation of the lemma is both convenient and sufficient for our purposes.

Proof of Lemma 3.6. Let $E \in \mathbb{R}$. Then

$$\rho_j(E) = \lim_{\ell\to\infty} \ell^{-1}\,\mathrm{Tr}\,\mathbb{1}_{(-\infty,E)}(h_{0,0,\ell} + nV_j), \quad j = 1,\ldots,n, \tag{3.20}$$

$$\rho(E) = \lim_{\ell\to\infty} \ell^{-1}\,\mathrm{Tr}\,\mathbb{1}_{(-\infty,E)}\left(h_{0,0,\ell} + \sum_{j=1}^{n} V_j\right). \tag{3.21}$$

On the other hand, a suitable version of the Weyl inequalities (see e.g. [25, Eq.(125)]) implies

$$\mathrm{Tr}\,\mathbb{1}_{(-\infty,E)}\left(h_{0,0,\ell} + \sum_{j=1}^{n} V_j\right) = \mathrm{Tr}\,\mathbb{1}_{(-\infty,0)}\left(\sum_{j=1}^{n}\left(\frac{1}{n}h_{0,0,\ell} + V_j - \frac{1}{n}E\right)\right) \leq$$

$$\sum_{j=1}^{n} \mathrm{Tr}\,\mathbb{1}_{(-\infty,0)}\left(\frac{1}{n}h_{0,0,\ell} + V_j - \frac{1}{n}E\right) = \sum_{j=1}^{n} \mathrm{Tr}\,\mathbb{1}_{(-\infty,E)}\left(h_{0,0,\ell} + nV_j\right). \tag{3.22}$$

Combining (3.20), (3.21), and (3.22), we arrive at (3.19). □

Proof of Theorem 3.5. By (2.20) and (3.2), we immediately get

$$\liminf_{E\downarrow 0} \frac{\ln|\ln N_\gamma(\mu_1 + E)|}{\ln E} \geq -\frac{1}{2}. \tag{3.23}$$

Let us obtain the corresponding upper bound. By (3.18) and (3.4), we have

$$\gamma(s;\omega)^2 \geq \sum_{k\in\mathbb{Z}} \lambda_k(\omega)^2 w(s-k)^2, \quad s \in \mathbb{R}. \tag{3.24}$$

Applying the Cauchy-Schwarz inequality, we easily find that

$$\dot\gamma(s;\omega)^2 = \left(\sum_{k\in\mathbb{Z}} \lambda_k(\omega)\dot w(s-k)\right)^2 =$$

$$= \left(\sum_{k\in\mathbb{Z}} \lambda_k(\omega)\dot w(s-k) \sum_{j\in\mathcal{J}_2} \mathbb{1}_{[-1/2,1/2)}(s-k-j)\right)^2 \leq n_2 \sum_{k\in\mathbb{Z}} \lambda_k(\omega)^2 \dot w(s-k)^2, \quad s \in \mathbb{R}. \tag{3.25}$$

Putting together (3.24) and (3.25), we find that if $\epsilon \geq 0$, then

$$\mathcal{T}^2\gamma(s;\omega)^2 - \epsilon\dot\gamma(s;\omega)^2 \geq \sum_{k\in\mathbb{Z}} \lambda_k(\omega)^2 \left(\mathcal{T}^2 w(s-k)^2 - n_2\epsilon\dot w(s-k)^2\right), \quad s \in \mathbb{R}. \tag{3.26}$$

Introduce the operator

$$\tilde h_{\gamma,\epsilon} := h_{0,0} + \sum_{k\in\mathbb{Z}} \lambda_k(\omega)^2 \left(\mathcal{T}^2 w(s-k)^2 - n_2\epsilon\dot w(s-k)^2\right),$$

self-adjoint and $\mathbb{Z}$-ergodic in $\mathrm{L}^2(\mathbb{R})$, and denote by $\tilde{\nu}_{\gamma,\epsilon}$ its IDS. Then (3.26) implies

$$\nu_{\gamma,\epsilon}(E) \leq \tilde{\nu}_{\gamma,\epsilon}(E), \quad E \in \mathbb{R}, \quad \epsilon \geq 0. \tag{3.27}$$

Next,

$$\sum_{k\in\mathbb{Z}} \lambda_k(\omega)^2 \left(\mathcal{T}^2 w(s-k)^2 - n_2\epsilon \dot{w}(s-k)^2\right) = \sum_{j\in\mathcal{J}_1}\sum_{k\in\mathbb{Z}} \lambda_{k-j}(\omega)^2 v_{j,\epsilon}(s-k), \quad s \in \mathbb{R}, \tag{3.28}$$

the potentials $v_{j,\epsilon}$ being defined in (3.14). Denote by $\tilde{\nu}_{\gamma,\epsilon.j}$, $j \in \mathcal{J}_1$, the IDS for the operator

$$h_{0,0} + \sum_{k\in\mathbb{Z}} \lambda_{k-j}(\omega)^2 v_{j,\epsilon}(s-k),$$

self-adjoint and $\mathbb{Z}$-ergodic in $\mathrm{L}^2(\mathbb{R})$. By (3.28), and Lemma 3.6,

$$\tilde{\nu}_{\gamma,\epsilon}(E) \leq \sum_{j\in\mathcal{J}_1} \tilde{\nu}_{\gamma,\epsilon,j}(E), \quad E \in \mathbb{R}, \quad \epsilon \in \mathbb{R}. \tag{3.29}$$

Arguing as in the proof of (3.13), we can show that (2.10), (3.16), and (3.17), imply

$$\limsup_{E\downarrow 0} \frac{\ln|\ln \tilde{\nu}_{\gamma,\epsilon,j}(E)|}{\ln E} \leq -\frac{1}{2}, \quad j \in \mathcal{J}_1, \quad \epsilon < \epsilon_0^{\min}. \tag{3.30}$$

Combining (2.22), (3.27), (3.29), and (3.30), we get

$$\limsup_{E\downarrow 0} \frac{\ln|\ln N_\gamma(\mu_1 + E)|}{\ln E} \leq -\frac{1}{2}. \tag{3.31}$$

Putting together (3.23) and (3.31), we arrive at (3.5).

□

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