On the quadratic Fock functor*

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Abstract

We prove that the quadratic second quantization of an operator p on $L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ is an orthogonal projection on the quadratic Fock space if and only if $p = \mathcal{M}_{\chi_I}$, where $\mathcal{M}_{\chi_I}$ is a multiplication operator by a characteristic function χ_I.

1 Introduction

The renormalized square of white noise (RSWN) was firstly introduced by Accardi-Lu–Volovich in [AcLuVo]. Later, Sniady studied the connection between the RSWN and the free white noise (cf [Sn]). Subsequently, its relation with the Lévy processes on real Lie algebras was established in [AcFrSk].

Recently, in [AcDh1]–[AcDh2], the authors obtained the Fock representation of the RSWN. They started defining the quadratic Fock space and the quadratic second quantization. After doing that, they characterized the operators on the one-particle Hilbert algebra whose quadratic second quantization is isometric (resp. unitary). A sufficient condition for the contractivity of the quadratic second quantization was derived too.

It is well known that the first order second quantization $\Gamma_1(p)$ of an operator p, defined on the usual Fock space, is an orthogonal projection if and only if p is an orthogonal projection (cf [Par]). Within this paper, it is shown that the set of orthogonal projections p, for which its quadratic

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second quantization $\Gamma_2(p)$ is an orthogonal projection, is quite reduced. More precisely, we prove that $\Gamma_2(p)$ is an orthogonal projection if and only if p is a multiplication operator by a characteristic function χ_I, $I \subset \mathbb{R}^d$.

This paper is organized as follows. In section 2, we recall some main properties of the quadratic Fock space and the quadratic second quantization. The main result is proved in section 3.

2 Quadratic Fock functor

The algebra of the renormalized square of white noise (RSWN) with test function Hilbert algebra

$$\mathcal{A} := L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$$

is the $*$-Lie-algebra, with central element denoted 1, generators $B_f^+, B_h, N_g,\ f, g, h \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\}$, involution

$$(B_f^+)^* = B_f \qquad , \qquad N_f^* = N_{\bar{f}}$$

and commutation relations

$$[B_f, B_g^+] = 2c\langle f, g\rangle + 4N_{\bar{f}g},\ \ [N_a, B_f^+] = 2B_{af}^+ \tag{1}$$

$$[B_f^+, B_g^+] = [B_f, B_g] = [N_a, N_{a'}] = 0,$$

for all a, a', f, $g \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

The Fock representation of the RSWN is characterized by a cyclic vector Φ satisfying

$$B_f\Phi = N_g\Phi = 0$$

for all $f, g \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ (cf [AcAmFr], [AcFrSk]).

2.1 Quadratic Fock space

In this subsection, we recall some basic definitions and properties of the quadratic exponential vectors and the quadratic Fock space. We refer the interested reader to [AcDh1]–[AcDh2] for more details.

The quadratic Fock space $\Gamma_2(L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ is the closed linear span of $\{B_f^{+n}\Phi,\ n \in \mathbb{N},\ f \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\}$, where $B_f^{+0}\Phi = \Phi$, for all $f \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. From [AcDh2] it follows that $\Gamma_2(L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ is an interacting Fock space. Moreover, the scalar product between two n-particle vectors is given by the following (cf [AcDh1]).

Proposition 1 *For all f, $g \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, one has*

$$\langle B_f^{+n}\Phi, B_g^{+n}\Phi\rangle = c\sum_{k=0}^{n-1} 2^{2k+1}\frac{n!(n-1)!}{((n-k-1)!)^2}\langle f^{k+1}, g^{k+1}\rangle \langle B_f^{+(n-k-1)}\Phi, B_g^{+(n-k-1)}\Phi\rangle.$$

The quadratic exponential vector of an element $f \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, if it exists, is given by

$$\Psi(f) = \sum_{n\geq 0}\frac{B_f^{+n}\Phi}{n!}$$

where by definition

$$\Psi(0) = B_f^{+0}\Phi = \Phi. \tag{2}$$

It is proved in [AcDh1] that the quadratic exponential vector $\Psi(f)$ exists if and only if $\|f\|_\infty < \frac{1}{2}$. Furthermore, the scalar product between two exponential vectors, $\Psi(f)$ and $\Psi(g)$, is given by

$$\langle\Psi(f), \Psi(g)\rangle = e^{-\frac{c}{2}\int_{\mathbb{R}^d}\ln(1-4\bar{f}(s)g(s))ds}. \tag{3}$$

Now, we refer to [AcDh1] for the proof of the following theorem.

Theorem 1 *The quadratic exponential vectors are linearly independents. Moreover, the set of quadratic exponential vectors is a total set in $\Gamma_2(L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d))$.*

2.2 Quadratic second quantization

For all linear operator T on $L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, we define its quadratic second quantization, if it is well defined, by

$$\Gamma_2(T)\Psi(f) = \Psi(Tf)$$

for all $f \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Note that in [AcDh2], the authors have proved that $\Gamma_2(T)$ is well defined if and only if T is a contraction on $L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ with respect to the norm $||.||_\infty$. Moreover, they have given a characterization of operators T on $L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ whose quadratic second quantization is isometric (resp. unitary). The contractivity of $\Gamma_2(T)$ was also investigated.

3 Main result

Given a contraction p on $L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with respect to $\|.\|_\infty$, the aim of this section is to prove under which condition $\Gamma_2(p)$ is an orthogonal projection on $\Gamma_2(L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$.

Lemma 1 *For all $f, g \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, one has*

$$\langle B_f^{+n}\Phi, B_g^{+n}\Phi\rangle = n!\frac{d^n}{dt^n}\Big|_{t=0}\langle \Psi(\sqrt{t}f), \Psi(\sqrt{t}g)\rangle.$$

Proof. Let f, $g \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ such that $\|f\|_\infty > 0$ and $\|g\|_\infty > 0$. Consider $0 \le t \le \delta$, where

$$\delta < \frac{1}{4}\inf\Big(\frac{1}{\|f\|_\infty^2}, \frac{1}{\|g\|_\infty^2}\Big).$$

It is clear that $\|\sqrt{t}f\|_\infty < \frac{1}{2}$ and $\|\sqrt{t}g\|_\infty < \frac{1}{2}$. Moreover, one has

$$\langle \Psi(\sqrt{t}f), \Psi(\sqrt{t}g)\rangle = \sum_{m\ge 0}\frac{t^m}{(m!)^2}\langle B_f^{+m}\Phi, B_g^{+m}\Phi\rangle.$$

Note that for all $m \ge n$, one has

$$\begin{aligned}
&\frac{d^n}{dt^n}\Big(\frac{t^m}{(m!)^2}\langle B_f^{+m}\Phi, B_g^{+m}\Phi\rangle\Big)\\
&= \frac{m!t^{m-n}}{(m!)^2(m-n)!}\langle B_f^{+m}\Phi, B_g^{+m}\Phi\rangle\\
&= \frac{t^{m-n}}{m!(m-n)!}\langle B_f^{+m}\Phi, B_g^{+m}\Phi\rangle.
\end{aligned}$$

Put

$$K_m = \frac{\delta^{m-n}}{m!(m-n)!}\|B_f^{+m}\Phi\|\|B_g^{+m}\Phi\|.$$

Then, from Proposition 1, it follows that

$$\begin{aligned}
||B_f^{+m}\Phi||^2 &= c\sum_{k=0}^{m-1}2^{2k+1}\frac{m!(m-1)!}{((m-k-1)!)^2}\||f^{k+1}\|_2^2\|B_f^{+(m-k-1)}\Phi\|^2\\
&= c\sum_{k=1}^{m-1}2^{2k+1}\frac{m!(m-1)!}{((m-k-1)!)^2}\||f^{k+1}\|_2^2\|B_f^{+(m-k-1)}\Phi\|^2
\end{aligned}$$

$$
\begin{aligned}
& +2mc\|f\|_2^2\|B_f^{+(m-1)}\Phi\|^2 \\
= \;& c\sum_{k=0}^{m-2} 2^{2k+3}\frac{m!(m-1)!}{(((m-1)-k-1)!)^2}|\|f^{k+2}\|_2^2\|B_f^{+((m-1)-k-1)}\Phi\|^2 \\
& +2mc\|f\|_2^2\|B_f^{+(m-1)}\Phi\|^2 \\
\leq \;& \Big(4m(m-1)\|f\|_\infty^2\Big)\Big[c\sum_{k=0}^{m-2} 2^{2k+1}\frac{(m-1)!(m-2)!}{(((m-1)-k-1)!)^2}\|f^{k+1}\|_2^2 \\
& \quad \|B_f^{+((m-1)-k-1)}\Phi\|^2\Big].
\end{aligned}
$$

But, one has

$$
||B_f^{+(m-1)}\Phi||^2 = c\sum_{k=0}^{m-2} 2^{2k+1}\frac{(m-1)!(m-2)!}{((m-1)-k-1)!)^2}|\|f^{k+1}\|_2^2\|B_f^{+((m-1)-k-1)}\Phi\|^2.
$$

Therefore, one gets

$$
||B_f^{+m}\Phi||^2 \leq \Big[4m(m-1)\|f\|_\infty^2 + 2m\|f\|_2^2\Big]\|B_f^{+(m-1)}\Phi\|^2.
$$

This proves that

$$
\frac{K_m}{K_{m-1}} \leq \frac{\sqrt{4m(m-1)\|f\|_\infty^2 + 2m\|f\|_2^2}\sqrt{4m(m-1)\|g\|_\infty^2 + 2m\|g\|_2^2}}{m(m-n)}\,\delta.
$$

It follows that

$$
\lim_{m\to\infty}\frac{K_m}{K_{m-1}} \leq 4\|f\|_\infty\|g\|_\infty\delta < 1.
$$

Hence, the series $\sum_m K_m$ converges. Finally, we have proved that

$$
\frac{d^n}{dt^n}\langle\Psi(\sqrt{t}f),\Psi(\sqrt{t}g)\rangle = \sum_{m\geq n}\frac{t^{m-n}}{m!(m-n)!}\langle B_f^{+m}\Phi, B_g^{+m}\Phi\rangle. \tag{4}
$$

Thus, by taking $t=0$ in the right hand side of (4), the result of the above lemma holds. □

As a consequence of the above lemma, we prove the following.

Lemma 2 *Let p be a contraction on $L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ with respect to the norm $\|.\|_\infty$. If $\Gamma_2(p)$ is an orthogonal projection on $\Gamma_2(L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d))$, then one has*

$$
\langle B_{p(f)}^{+n}\Phi, B_{p(g)}^{+n}\Phi\rangle = \langle B_{p(f)}^{+n}\Phi, B_g^{+n}\Phi\rangle = \langle B_f^{+n}\Phi, B_{p(g)}^{+n}\Phi\rangle
$$

for all f, $g\in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ and all $n\geq 1$.

Proof. It $\Gamma_2(p)$ is an orthogonal projection on $\Gamma_2(L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d))$, then

$$\langle\Psi(\sqrt{t}p(f)),\Psi(\sqrt{t}p(g))\rangle=\langle\Psi(\sqrt{t}p(f)),\Psi(\sqrt{t}g)\rangle=\langle\Psi(\sqrt{t}f),\Psi(\sqrt{t}p(g))\rangle \quad (5)$$

for all $f,\, g\in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ (with $\|f\|_\infty>0$ and $\|g\|_\infty>0$) and all $0\leq t\leq\delta$ such that

$$\delta<\frac{1}{4}\inf\Big(\frac{1}{\|f\|_\infty^2},\frac{1}{\|g\|_\infty^2}\Big).$$

Therefore, the result of the above lemma follows from Lemma 1 and identity (5). □

Lemma 2 ensures that the following result holds true.

Lemma 3 *Let p be a contraction on $L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ with respect to the norm $\|.\|_\infty$. If $\Gamma_2(p)$ is an orthogonal projection on $\Gamma_2(L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d))$, then one has*

$$\langle(p(f))^n,(p(g))^n\rangle=\langle f^n,(p(g))^n\rangle=\langle(p(f))^n,g^n\rangle \quad (6)$$

for all $f,\, g\in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ and all $n\geq1$.

Proof. Suppose that $\Gamma_2(p)$ is an orthogonal projection on $\Gamma_2(L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d))$. Then, in order to prove the above lemma we have to use induction.

- For $n=1$: Lemma 2 implies that

$$\langle B^+_{p(f)}\Phi,B^+_{p(g)}\Phi\rangle=\langle B^+_{p(f)}\Phi,B^+_{g}\Phi\rangle=\langle B^+_{f}\Phi,B^+_{p(g)}\Phi\rangle$$

for all $f,\, g\in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. Using the fact that $B_f\Phi=0$ for all $f\in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ and the commutation relations in (1) to get

$$\begin{aligned}\langle B^+_{p(f)}\Phi,B^+_{p(g)}\Phi\rangle &= \langle\Phi,B_{p(f)}B^+_{p(g)}\Phi\rangle=2c\langle p(f),p(g)\rangle\\ \langle B^+_{p(f)}\Phi,B^+_{g}\Phi\rangle &= \langle\Phi,B_{p(f)}B^+_{g}\Phi\rangle=2c\langle p(f),g\rangle\\ \langle B^+_{f}\Phi,B^+_{p(g)}\Phi\rangle &= \langle\Phi,B_{f}B^+_{p(g)}\Phi\rangle=2c\langle f,p(g)\rangle.\end{aligned}$$

This proves that identity (6) holds true for $n=1$.

- Let $n\geq1$ and suppose that identity (6) is satisfied. Then, from Lemma 2, it follows that for all $f,\, g\in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ one has

$$\langle B^{+(n+1)}_{p(f)}\Phi,B^{+(n+1)}_{p(g)}\Phi\rangle=\langle B^{+(n+1)}_{p(f)}\Phi,B^{+(n+1)}_{g}\Phi\rangle=\langle B^{+(n+1)}_{f}\Phi,B^{+(n+1)}_{p(g)}\Phi\rangle. \quad (7)$$

Identity (7) and Proposition 1 imply that

$$\begin{aligned}\langle B^{+(n+1)}_{p(f)}\Phi, B^{+(n+1)}_{p(g)}\Phi\rangle &= 2^{2n+3}cn!(n+1)!\langle (p(f))^{n+1}, (p(g))^{n+1}\rangle \\ &\quad +c\sum_{k=0}^{n-1} 2^{2k+1}\frac{n!(n+1)!}{((n-k)!)^2}\langle (p(f))^{k+1}, (p(g))^{k+1}\rangle \\ &\qquad \langle B^{+(n-k)}_{p(f)}\Phi, B^{+(n-k)}_{p(g)}\Phi\rangle \\ &= 2^{2n+3}cn!(n+1)!\langle f^{n+1}, (p(g))^{n+1}\rangle \\ &\quad +c\sum_{k=0}^{n-1} 2^{2k+1}\frac{n!(n+1)!}{((n-k)!)^2}\langle f^{k+1}, (p(g))^{k+1}\rangle \\ &\qquad \langle B^{+(n-k)}_{f}\Phi, B^{+(n-k)}_{p(g)}\Phi\rangle \\ &= 2^{2n+3}cn!(n+1)!\langle (p(f))^{n+1}, g^{n+1}\rangle \\ &\quad +c\sum_{k=0}^{n-1} 2^{2k+1}\frac{n!(n+1)!}{((n-k)!)^2}\langle (p(f))^{k+1}, g^{k+1}\rangle \\ &\qquad \langle B^{+(n-k)}_{p(f)}\Phi, B^{+(n-k)}_{g}\Phi\rangle.\end{aligned} \tag{8}$$

Note that by induction assumption, one has

$$\langle (p(f))^{k+1}, (p(g))^{k+1}\rangle = \langle f^{k+1}, (p(g))^{k+1}\rangle = \langle (p(f))^{k+1}, g^{k+1}\rangle \tag{9}$$

for all $k = 0, \dots, n-1$. Therefore, from Lemma 2 and identity (9), one gets

$$\begin{aligned}& c\sum_{k=0}^{n-1} 2^{2k+1}\frac{n!(n+1)!}{((n-k)!)^2}\langle (p(f))^{k+1}, (p(g))^{k+1}\rangle\langle B^{+(n-k)}_{p(f)}\Phi, B^{+(n-k)}_{p(g)}\Phi\rangle \\ &= c\sum_{k=0}^{n-1} 2^{2k+1}\frac{n!(n+1)!}{((n-k)!)^2}\langle f^{k+1}, (p(g))^{k+1}\rangle\langle B^{+(n-k)}_{f}\Phi, B^{+(n-k)}_{p(g)}\Phi\rangle \\ &= c\sum_{k=0}^{n-1} 2^{2k+1}\frac{n!(n+1)!}{((n-k)!)^2}\langle (p(f))^{k+1}, g^{k+1}\rangle\langle B^{+(n-k)}_{p(f)}\Phi, B^{+(n-k)}_{g}\Phi\rangle.\end{aligned}$$

Finally, from (8) one can conclude. □

Note that the set of contractions p on $L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ with respect to $\|.\|_\infty$, such that $\Gamma_2(p)$ is an orthogonal projection on $\Gamma_2(L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d))$, is reduced to the following.

Lemma 4 *Let p be a contraction on $L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ with respect to the norm $\|.\|_\infty$. If $\Gamma_2(p)$ is an orthogonal projection on $\Gamma_2(L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d))$, then*

$$p(\bar{f})=\overline{p(f)}.$$

for all $f\in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$.

Proof. Let p be a contraction on $L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ with respect to the norm $\|.\|_\infty$ such that $\Gamma_2(p)$ is an orthogonal projection on $\Gamma_2(L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d))$. Then, from Lemma 3 it is clear that $p=p^*=p^2$ (taking $n=1$ in (6)). Moreover, for all $f_1,f_2,g_1,g_2\in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$, one has

$$\langle (p(f_1+f_2))^2,\,(g_1+g_2)^2\rangle=\langle (p(f_1+f_2))^2,\,(p(g_1+g_2))^2\rangle.$$

It follows that

$$\begin{aligned}&\langle (p(f_1))^2,g_1^2+g_2^2\rangle+\langle (p(f_2))^2,g_1^2+g_2^2\rangle+4\langle p(f_1)p(f_2),g_1g_2\rangle\\&=\langle (p(f_1))^2,(p(g_1))^2+(p(g_2))^2\rangle+\langle (p(f_2))^2,(p(g_1))^2+(p(g_2))^2\rangle\\&+4\langle p(f_1)p(f_2),p(g_1)p(g_2)\rangle.\end{aligned}\tag{10}$$

Then, using (6) and (10) to obtain

$$\langle p(f_1)p(f_2),g_1g_2\rangle=\langle p(f_1)p(f_2),p(g_1)p(g_2)\rangle\tag{11}$$

for all $f_1,f_2,g_1,g_2\in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. Now, denote by $\mathcal{M}_a$ the multiplication operator by the function $a\in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. Then, identity (11) implies that

$$\langle \mathcal{M}_{p(f_2)\bar{g_2}}p(f_1),g_1\rangle=\langle \mathcal{M}_{p(f_2)\overline{p(g_2)}}p(f_1),p(g_1)\rangle$$

for all $f_1,f_2,g_1,g_2\in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. This gives that

$$\mathcal{M}_{p(f_2)\bar{g_2}}p=p\mathcal{M}_{p(f_2)\overline{p(g_2)}}p\tag{12}$$

for all $f_2,g_2\in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. Taking the adjoint in (12), one gets

$$p\mathcal{M}_{\overline{p(f_2)}\,g_2}=p\mathcal{M}_{\overline{p(f_2)}p(g_2)}p.\tag{13}$$

Note that, for all $f,g\in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$, identity (12) implies that

$$\mathcal{M}_{p(f)\bar{g}}p=p\mathcal{M}_{p(f)\,\overline{p(g)}}p.\tag{14}$$

Moreover, from (13), one has

$$p\mathcal{M}_{p(f)\overline{p(g)}}p = p\mathcal{M}_{f\,\overline{p(g)}}. \tag{15}$$

Therefore, identities (14) and (15) yield

$$\mathcal{M}_{p(f)\bar{g}}p = p\mathcal{M}_{f\,\overline{p(g)}} \tag{16}$$

for all $f, g \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Hence, for all $f, g, h \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, identity (16) gives

$$p(f\,\overline{p(g)}h) = p(f)\bar{g}p(h). \tag{17}$$

Taking $f = g = p(h)$ in (16) to get

$$\mathcal{M}_{|p(h)|^2}p = p\mathcal{M}_{|p(h)|^2}. \tag{18}$$

Then, if we put $f = h = p(g)$ in (17), one has

$$p(\overline{p(g)}\,p(g)^2) = p(|p(g)|^2p(g)) = (p(g))^2\bar{g}.$$

But, from (18), one has

$$p(|p(g)|^2p(g)) = (p\mathcal{M}_{|p(g)|^2})(p(g)) = \mathcal{M}_{|p(g)|^2}p(p(g)) = |p(g)|^2p(g).$$

Hence, one obtains

$$|p(g)|^2p(g) = (p(g))^2\bar{g} \tag{19}$$

for all $g \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Now, let g be a real function in $L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. So, the polar decomposition of $p(g)$ is given by

$$p(g) = |p(g)|e^{i\theta_{p(g)}}.$$

Thus, identity (19) implies that

$$|p(g)|^3e^{-i\theta_{p(g)}} = |p(g)|^2g.$$

This proves that for all $x \in \mathbb{R}^d$, $\theta_{p(g)}(x) = k_x\pi$, $k_x \in \mathbb{Z}$. Therefore, $p(g)$ is a real function. Now, taking $f = f_1 + if_2 \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, where f_1, f_2 are real functions on $\mathbb{R}^d$. It is clear that

$$p(\bar{f}) = p(f_1 - if_2) = \overline{p(f_1) + ip(f_2)} = \overline{p(f)}.$$

This completes the proof of the above lemma. □

As a consequence of Lemmas 3 and 4 we prove the following theorem.

Theorem 2 *Let p be a contraction on $L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with respect to the norm $\|.\|_\infty$. Then, $\Gamma(p)$ is an orthogonal projection on $\Gamma_2(L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ if and only if $p = \mathcal{M}_{\chi_I}$, where $\mathcal{M}_{\chi_I}$ is a multiplication operator by a characteristic function χ_I, $I \subset \mathbb{R}^d$.*

Proof. Note that if $p = \mathcal{M}_{\chi_I}$, $I \subset \mathbb{R}^d$, then from identity (3) it is clear that

$$\begin{aligned} e^{-\frac{c}{2}\int_I \ln(1-4\bar{f}(s)g(s))ds} &= \langle \Psi_2(p(f)), \Psi_2(g)\rangle \\ &= \langle \Psi_2(f), \Psi_2(p(g))\rangle \\ &= \langle \Psi_2(p(f)), \Psi_2(p(g))\rangle \end{aligned}$$

for all f, $g \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ such that $\|f\|_\infty < \frac{1}{2}$ and $\|g\|_\infty < \frac{1}{2}$. Hence, $\Gamma_2(p)$ is an orthogonal projection on $\Gamma_2(L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$.

Now, suppose that $\Gamma_2(p)$ is an orthogonal projection on $\Gamma_2(L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$. Then, Lemma 3 implies that

$$\langle (p(f))^n, (p(g))^n\rangle = \langle f^n, (p(g))^n\rangle = \langle (p(f))^n, g^n\rangle$$

for all f, $g \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and all $n \geq 1$. In particular, if $n = 2$ one has

$$\langle (p(f_1 + \bar{f}_2))^2, g^2\rangle = \langle (f_1 + \bar{f}_2)^2, (p(g))^2\rangle$$

for all f_1, f_2, $g \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. This gives

$$\begin{aligned} &\langle (p(f_1))^2, g^2\rangle + 2\langle p(f_1)p(\bar{f}_2), g^2\rangle + \langle (p(\bar{f}_2))^2, g^2\rangle \\ &= \langle f_1^2, (p(g))^2\rangle + 2\langle f_1\bar{f}_2, (p(g))^2\rangle + \langle (\bar{f}_2)^2, (p(g))^2\rangle. \end{aligned} \tag{20}$$

Using identity (20) and Lemma 3 to get

$$\langle p(f_1)p(\bar{f}_2), g^2\rangle = \langle f_1\bar{f}_2, (p(g))^2\rangle.$$

This yields

$$\int_{\mathbb{R}^d} \bar{f}_1(x)f_2(x)(p(g))^2(x)dx = \int_{\mathbb{R}^d} \overline{p(f_1)}(x)p(f_2)(x)g^2(x)dx. \tag{21}$$

But, from Lemma 4, one has $\overline{p(f)} = p(\bar{f})$, for all $f \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then, identity (21) implies that

$$\langle f_1, M_{(p(g))^2} f_2\rangle = \langle f_1, (pM_{g^2}p)f_2\rangle$$

for all f, $g \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Hence, one obtains

$$M_{(p(g))^2} = pM_{g^2}p. \tag{22}$$

In particular, for $g = \chi_I$ where $I \subset \mathbb{R}^d$, one has

$$\mathcal{M}_{(p(\chi_I))^2} = p\mathcal{M}_{\chi_I}p. \tag{23}$$

If I tends to $\mathbb{R}^d$, the operator $\mathcal{M}_{\chi_I}$ converges to id (identity of $L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$) for the strong topology. From (23), it follows that

$$p(f) = p^2(f) = \lim_{I \uparrow \mathbb{R}^d} \mathcal{M}_{(p(\chi_I))^2} f$$

for all $f \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. But, the set of multiplication operators is a closed set for the strong topology. This proves that $p = \mathcal{M}_a$, where $a \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Note that $p = p^2$ is a positive operator. This implies that a is a positive function. Moreover, one has $p^n = p$ for all $n \in \mathbb{N}^*$. This gives $\mathcal{M}_{a^n} = \mathcal{M}_a$ for all $n \in \mathbb{N}^*$. It follows that $a^n = a$ for all $n \in \mathbb{N}^*$. Therefore, the operator a is necessarily a characteristic function on $\mathbb{R}^d$. □

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