STABLE LIMIT LAWS AND STRUCTURE OF THE SCALING FUNCTION FOR REACTION-DIFFUSION IN RANDOM ENVIRONMENT

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Abstract. We prove the emergence of stable fluctuations for reaction-diffusion in random environment with Weibull tails. This completes our work around the quenched to annealed transition phenomenon in this context of reaction diffusion. In [9], we had already considered the model treated here and had studied fully the regimes where the law of large numbers is satisfied and where the fluctuations are Gaussian, but we had left open the regime of stable fluctuations. Our work is based on a spectral approach centered on the classical theory of rank-one perturbations. It illustrates the gradual emergence of the role of the higher peaks of the environments. This approach also allows us to give the delicate exact asymptotics of the normalizing constants needed in the stable limit law.

1. Introduction

We establish the emergence of stable fluctuations for branching random walks in a random environment. More precisely, we consider branching random walks on the lattice $\mathbb{Z}^d$, and denote by $v(x)$ the rate of branching at site $x \in \mathbb{Z}^d$. We assume that the branching is binary (each particle gives birth to two offsprings) and that the rates $(v(x))_{x\in\mathbb{Z}^d}$ are i.i.d random variables, which we call here the random environment.

We are interested in the spatial fluctuations of the number $N_x(t)$ of particles at time $t > 0$, whose ancestor at time 0 was at site $x \in \mathbb{Z}^d$, or rather in the behavior of its mean $m(t, x)$, as a function of the "quenched" random environment.

This question can be formulated as an equivalent problem about reaction-diffusion in random environment, since the random function $m(x, t)$ is the solution of the reaction-diffusion equation

$$\frac{\partial m(x,t)}{\partial t} = \kappa\Delta m(x,t) + v(x)m(x,t), \qquad t \geq 0, x \in \mathbb{Z}^d, \tag{1}$$

with initial condition,

$$m(x, 0) = 1, \qquad x \in \mathbb{Z}^d, \tag{2}$$

where Δ is the discrete Laplacian on $\mathbb{Z}^d$, defined for every function $f \in l^1(\mathbb{Z}^d)$ as

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$$\Delta f(x) := \sum_{e \in E} (f(x+e) - f(x)),$$

where $E := \{e \in \mathbb{Z}^d : |e|_1 = 1\}$ and $|\cdot|_1$ is the $l^1(\mathbb{Z}^d)$ norm.

Following [9], we will study the mean number of particles at time t, if one starts with a particle at time 0, whose position is picked uniformly at random in the box Λ_L of size L. More precisely we will consider the spatial average

$$m_L(t) = \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} m(x,t,v) \tag{3}$$

This quantity exhibits a very rich dynamical transition, when t increases, as a function of L. This transition was introduced a decade ago, as a mechanism for the "transition from annealed to quenched asymptotics" for Markovian dynamics, in [8] in the context of random walks on random obstacles, and in [9] in the current context of reaction-diffusion in random environment. The basic intuition behind this rich picture can in fact be understood in the much simpler context of sums of i.i.d random exponentials, as was done in [7]. In this simple context of i.i.d random variables, the transition boils down to the graduate emergence of the role of extreme values, which gradually impose a breakdown of the CLT first and eventually of the LLN, and induce stable fluctuations.

In the present context of branching random walks in random environment, we have proved in [9], that the extreme values of the random environment play a similar role, and established the precise breakdown of the Central Limit Theorem and of the Law of Large Numbers. But we had left open the much more delicate question of stable fluctuations. This is the purpose of this work.

Our main result in this work establishes that, when the tails of the branching rate are Weibull distributed (as in [9]) and in the regime where the Central Limit Theorem fails, $m_L(t)$, once properly centered and scaled, converges to a stable distribution. Before establishing these stable limit laws, one major difficulty is to understand the needed exact asymptotics of the centering and scaling constants, as functions of t. This asymptotic behavior is unusually delicate, as we will see below. The validity of these stable limits was until now only proved in two much simpler situations: either the simple case treated in [7] of sums of i.i.d random variables, or in the one dimensional case of random walks among random obstacles treated in [8]. In a future work, we plan to extend our results to branching rates with double exponentially decaying tails.

Our result was also claimed in the recent work of Gartner and Schnitzler in [13]. Nevertheless we believe that the proof given in [13] is incomplete. In fact the statement of Theorem 1 in [13], giving the stable limit law, is ambiguous. The problem lays with the understanding of the normalizing function. In fact, this is one of the crucial places where in this article we use decisively the rank-one perturbation theory. We believe that the estimate of the normalizing function (called here $e^{th(t)}$ and there $B_\alpha(t)$) is flawed.

The main tool here is naturally based on spectral theory, and more precisely on the classical theory of rank-one perturbations for self-adjoint operators, which we

recall briefly in the Appendix. Using rank-one perturbation theory in this context of random media is quite natural, as for instance in [10] (Biskup-Konig), where the case of faster tails (doubly exponential) is studied in great depth for the parabolic Anderson model. More broadly, spectral tools have been pushed quite far for the understanding of the extreme balues of the spectrum of the Anderson operator in the beautiful series of work [2, 3, 4, 5, 6] (Astrauskas).

Let us now describe more precisely the content of this article. We state precisely our results in Section2. First, in Section 2.1, we begin by giving our notations, and then state, in Section 2.2 our results for the exact behavior of the centering constant. In Section 2.3 we give our results about the scaling function, and finally in Section 2.4 we give our main result, i.e the convergence to a stable distribution for $m_L(t)$, once properly centered and scaled. In Section 3, we establish the needed spectral results about our random Schrodinger operator, using the rank-one perturbation theory recalled in Appendix A. In Section 4, we recall the basic facts of extreme value theory for i.i.d Weibull distributions. And finally in Section 5, we establish our Theorem 1 about the behavior of the centering. In Section 6, we establish our Theorem 2 re the behavior of the scaling function. And finally in Section 7 we prove the main result, Theorem 3, establishing stable limit laws.

2. Notations and results

Let $v := \{v(x) : x \in \mathbb{Z}^d\}$ where $v(x) \in [0, \infty)$. Consider the space $W := [0, \infty)^{\mathbb{Z}^d}$, endowed with its natural σ-algebra. Let μ be the probability measure on W such that the coordinates of v are i.i.d. Let us now consider a simple symmetric random walk of rate $\kappa > 0$, which branches at a site x at rate $v(x)$ giving birth to two particles. Let us call $m(x, t, v)$ the expectation of the total number of random walks at time t given that initially there was only one random walk at site x, in the environment v. We will frequently write $m(x, t)$ instead of $m(x, t, v)$. In Propositions 1 and 8 of [9] it is shown that μ-a.s. this expectation is finite and that it satisfies the parabolic Anderson equations (1) and (2).

Throughout, we will call v the *potential* of the equation (1) and we will assume that it has a Weibull law of parameter $\rho > 1$ so that

$$\mu(v(0) > y) = \exp\left\{-\frac{y^\rho}{\rho}\right\}. \tag{4}$$

For any function f of the potential, we will use the notations $\langle f \rangle := \int f d\mu$, and $Var_\mu(f) := \langle (f - \langle f \rangle)^2 \rangle$ whenever they are well defined. Let us also introduce the *conjugate exponent* ρ' of $\rho > 1$, defined by the equation

$$\frac{1}{\rho'} + \frac{1}{\rho} = 1,$$

and the notation $f \sim g$ to indicate that $\lim_{t\to\infty} f(t)/g(t) = 1$. Our first result gives the precise asymptotic behavior of the average of the expectation of the total number of random walks.

Theorem 1. *Consider the solution of (1) and (2). Then,*

$$\langle m(0,t)\rangle \sim \left(\frac{\pi}{\rho-1}\right)^{1/2} t^{1-\frac{\rho'}{2}} e^{\frac{t^{\rho'}}{\rho'}-2d(\kappa t-t^{2-\rho'})}. \tag{5}$$

The average $\langle m(0,t)\rangle$ of Theorem 1 will turn out to be the adequate centering of the empirical average of the field $\{m(x,t : x \in \mathbb{Z}^d\}$ in certain regimes, leading to the appearance of stable laws. To state the corresponding results we still need to introduce additional notation.

Consider on $\mathbb{Z}^d$ the norm $||x|| := \sup\{|x_i| : 1 \leq i \leq d\}$. For each $r \geq 0$ and $x \in \mathbb{Z}^d$ consider the subset $\Lambda(x,r) := \{y \in \mathbb{Z}^d : ||y|| \leq r\}$. For $L > 0$, we will use the notation Λ_L instead of $\Lambda(0,L)$. We now define the averaged first moment at scale L as

$$m_L(t) := \frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L} m(x,t).$$

We will say that a sequence $\gamma = \{z_1, z_2, \ldots, z_p\}$ in $\mathbb{Z}^d$ is a *path* if for each $1 \leq j \leq p-1$, the sites z_j and z_{j+1} are nearest neighbors. The *length* of a path γ, denoted by $|\gamma|$, is equal to the number p of sites defining it. Furthermore, we will say that γ connects sites x and y if $z_1 = x$ and $z_p = y$. Denote the set of paths contained in a set $U \subset \mathbb{Z}^d$ and connecting x to y, by $\mathbb{P}_U(x,y)$; the set of paths starting from x and contained in U by $\mathbb{P}_U(x)$. For each $n \geq 1$, define $\mathbb{P}_U^n(x,y) := \{\gamma \in \mathbb{P}_U : |\gamma| = n\}$. Furthermore, to each path $\gamma \in \mathbb{P}_U^n(0)$ we can associate a set $\{y_1, \ldots, y_k\}$, which represents the different sites visited by the path, and a set $\{n_1, \ldots, n_j\}$ such that $\sum_{i=1}^k n_i = n$, such that n_i represents the number of times the site y_i was visited by γ. Finally, whenever $U = \mathbb{Z}^d$ we will use the notation $\mathbb{P}(x)$, $\mathbb{P}(x,y)$, $\mathbb{P}^n(0)$, instead of $\mathbb{P}_U(x)$, $\mathbb{P}_U(x,y)$, and $\mathbb{P}^n(0)$, respectively. For each $v \in W$ and natural N, consider the function defined for $s \geq 0$ as

$$B_N(s,v) := \frac{s+2d\kappa}{1+\sum_{j=1}^N \sum_{\gamma\in\mathbb{P}^{2j+1}(0,0)} \prod_{z\in\gamma, z\neq z_1} \frac{\kappa}{2d\kappa+(s-v_0(z))_+}}, \tag{6}$$

where $v_0(z) := v(z)$ for $z \neq 0$ while $v_0(0) = 0$. Also, define the constants

$$M := \min\{j \geq 1 : 2j\rho' > 2(j+1)\}, \tag{7}$$

$$\mathcal{B}_N := \frac{2d\kappa}{1+\sum_{j=1}^N \sum_{\gamma\in\mathbb{P}^{2j+1}(0,0)} \left(\frac{1}{2d}\right)^{2j}}, \tag{8}$$

$$\mathcal{A}_0 = \mathcal{A}_0(\gamma,\rho) := \left(\frac{\gamma\rho}{\rho'}\right)^{1/\rho},$$

$$\alpha = \alpha(\gamma,\rho) := \left(\frac{\gamma\rho}{\rho'}\right)^{1/\rho'} \tag{9}$$

and

$$\gamma_1 := \frac{\rho'}{\rho} \qquad \text{and} \qquad \gamma_2 := \frac{\rho'}{\rho} 2^{1/\rho'}. \tag{10}$$

We also will need to define for each natural N the function $\zeta_N(s) : [0,\infty) \to \left(0, e^{-\frac{1}{\rho}\mathcal{B}_N^\rho}\right)$ by

$$\zeta_N(s) := E_\mu \left[e^{-\frac{1}{\rho} B_N(s,v)^\rho} \right]. \tag{11}$$

Our second result establishes the existence of a function $h(t)$, which we will call the *scaling function* and which provides the adequate scaling factor $e^{th(t)}$ which will give the limiting stable laws. To state it we need to define the function $\tau : [1,\infty) \to \mathbb{R}$ by

$$\tau(L) := \frac{d\rho'}{\gamma} \log L. \tag{12}$$

Theorem 2. *Let $0 < \gamma < \gamma_2$, $L(t)$ an increasing function and $\tau(t) = \tau(L(t))$ defined in (12). Then, the following statements are satisfied.*

(i) *There exists a unique function $h(t) : [t_0,\infty) \to [0,\infty)$ where t_0 is defined by $\frac{1}{L(t_0)^d} = e^{-\frac{1}{\rho}\mathcal{B}_M^\rho}$ [c.f. (8)] and M in (7), which satisfies the equation*

$$\zeta_M(h(t)) = \frac{1}{L(t)^d}. \tag{13}$$

(ii) *The solution $h(t)$ of (13) of part (i) admits the expansion*

$$h(t) = \mathcal{A}_0 \tau^{\rho'-1}(t) - 2d\kappa + \sum_{1\le j\le M} h_j(t) + O\left(\frac{1}{t^{(2M+1)\frac{\rho'}{\rho}}}\right), \tag{14}$$

where $h_j(t)$, $1 \le j \le M$ are functions recursively defined as

$$h_0(t) := \mathcal{A}_0 \tau^{\rho'-1}(t) - 2d\kappa, \tag{15}$$

and for $1 \le j \le M$

$$h_j(t) := \frac{1}{\mathcal{A}_0^{\rho-1}} \left(\frac{\gamma}{\rho'} \tau^{\rho'-1}(t) + \frac{1}{\tau(t)} \log E_\mu \left[e^{-\frac{1}{\rho} B_j(h_0+\cdots+h_{j-1},v)^\rho} \right] \right) \tag{16}$$

and

$$h_j(t) = O\left(\frac{1}{t^{(2j-1)\frac{\rho'}{\rho}}}\right). \tag{17}$$

Throughout, for $\alpha \in (0,2)$ we will call S_α the distribution of the totally asymmetric stable law of exponent α (and skewness parameter 1) with characteristic function

$$\phi_\alpha(u) := \begin{cases} \exp\left\{-\Gamma(1-\alpha)|u|^\alpha e^{-\frac{i\pi\alpha}{2}\mathrm{sgn}(u)}\right\} & \text{if } 0<\alpha<1 \\ \exp\left\{iu(1-\bar{\gamma}) - \frac{\pi}{2}|u|(1+i\ \mathrm{sgn}(u)\cdot\frac{2}{\pi}\log|u|)\right\} & \text{if } \alpha=1 \\ \exp\left\{\frac{\Gamma(2-\alpha)}{\alpha-1}|u|^\alpha e^{-\frac{i\pi\alpha}{2}\mathrm{sgn}(u)}\right\} & \text{if } 1<\alpha<2, \end{cases} \tag{18}$$

where $\Gamma(s) := \int_0^\infty x^{s-1}e^{-x}dx$ is the gamma function, $\mathrm{sgn}(u) := \frac{u}{|u|}$ for $u \neq 0$ and $\mathrm{sgn}(u) := 0$ for $u = 0$ and $\gamma = 0.5772...$ is the Euler constant.

As the following result shows, the scaling function of Theorem 1 appears as the right choice to rescale the empirical average m_L of the potential field, in order to obtain stable limiting laws.

Theorem 3. *Consider a potential v having a Weibull distribution of parameter $\rho > 1$. Consider an increasing scale L and let $\tau(t) = \tau(L(t))$ as defined in (12). We then have that*

$$\lim_{t\to\infty} \frac{m_L(t) - A(t)}{e^{th(t)}|\Lambda_L|^{-1}} = S_\alpha,$$

where

$$A(t) := \begin{cases} 0 & \text{if} \quad 0<\gamma<\gamma_1 \\ \left\langle m(0,t), \sum_{x\in\Lambda_t} m(x,t) \leq e^{th(t)} \right\rangle & \text{if} \quad \gamma=\gamma_1 \\ \langle m(0,t)\rangle & \text{if} \quad \gamma_1<\gamma<\gamma_2, \end{cases} \tag{19}$$

and where the convergence is in distribution and $\alpha = \alpha(\gamma,\rho)$ is given by (9).

An immediate consequence of Theorem 3, is the appearance of a transition mechanism in the asymptotic behavior of the quenched-annealed transition, where as the value of ρ increases in $(1,\infty)$, the number of terms in the asymptotic expansion of the scaling function h which are relevant also increase. for even values of ρ. Indeed, we have the following corollary of Theorem 3, which also shows that the first terms in the expansion of the scaling function can be computed explicitly.

Corollary 1. *Under the assumptions of Theorem 2, for $0 < \gamma < \gamma_2$, the scaling function $h(t)$ defined in (13) admits the expansion:*

(i) *for $1 < \rho < 2$, $h(t) = \mathcal{A}_o\tau^{\rho'-1} - 2d\kappa$,*
(ii) *for $2 \leq \rho < 3$, $h(t) = \mathcal{A}_o\tau^{\rho'-1} - 2d\kappa + 2d\kappa^2\mathcal{A}_0^3\tau^{1-\rho'}$ and*
(iii) *for $3 \leq \rho < \frac{3+\sqrt{17}}{2}$,*

$$h(t) = \mathcal{A}_o\tau^{\rho'-1} - 2d\kappa + K_2\tau^{1-\rho'} + K_3\tau^{\frac{\rho}{\rho-1}(3-2\rho')-1} + K_4\tfrac{1}{\tau} + K_5\log\tau + o\left(\tfrac{1}{t}\right),$$

where

$$K_2 := \mathcal{A}_0^{2-\rho} K_1, \quad K_3 := \frac{K_1 (\mathcal{A}_0 K_1)^{\frac{1}{\rho-1}}}{\mathcal{A}_0^{\rho-1}} \left(1 - \frac{1}{\rho}\mathcal{A}_0\right),$$

$$K_4 := (3 - 2\rho') \log \mathcal{A}_0 + \frac{1}{2} \log \pi(\rho - 1) - \mathcal{A}_0^{3-\rho} K_1^2 \quad \text{and} \quad K_5 := \frac{\rho}{2(\rho - 1)}.$$

For ρ larger than $(3 + \sqrt{17})/2$ extra terms in the expansion of h have to be computed in Corollary 1. In order to keep the length of the computations limited, we have decided to stop there. There seems to be no straightforward interpretation on the appearance of this number.

The proof of Theorem 3 is based on rank-one perturbation methods to obtain asymptotic expansions of the largest eigenvalues of the Laplacian operator in a potential having high peaks.

Throughout this article, a constant C will always denote a non-random number, independent of time. We will use the letters C or $C_1, C_2, \ldots$ to denote them. We will use the notation $O(s) : [0, \infty) \to [0, \infty)$ to denote a function (possibly random) which satisfies for all $s \geq 0$,

$$|O(s)| \leq Cs$$

for some constant C.

3. Rank-one perturbation for Schrödinger operators

Here we will apply perturbation theory to study the asymptotic behavior of the principal Dirichlet eigenvalue and eigenfunction of the Laplacian operator plus a potential with a rank-one perturbation. The results we will present are deterministic, in the sense that we do not assume that the potential is random, and are not particularly original, since they correspond to a standard application of rank-one perturbation theory (see for example [16]). With the aim of giving a self-contained presentation, the basic tools of rank-one perturbation theory that will be used are presented in Appendix A. In subsection 3.1, we give a precise statement about the principal Dirichlet eigenvalue and eigenfunction of the perturbed operator in Theorem 4. In subsection 3.2 we derive some estimates and formulas about the spectrum of the unperturbed Schrödinger operator and its Green function. In subsection 3.3 we relate the results of subsection 3.2 to the spectrum of the perturbed Schrödinger operator through the rank-one perturbation theory presented in Appendix A, to prove Theorem 4.

3.1. **Principal eigenvalue and eigenfunction.**

Let U be a finite connected subset of $\mathbb{Z}^d$. Consider the *Schrödinger operator*

$$H_{U,w}^0 := \kappa \Delta_U + w,$$

on U, with Dirichlet boundary conditions, where $\kappa > 0$ and w is a *non-negative potential* on U: a set $w = \{w(x) : x \in U\}$, where $w(x) \geq 0$. In other words, $H^0_{U,w}$ is the operator defined on $l^2(U)$ acting as,

$$H^0_{U,w} f(x) = \kappa \sum_{j=1}^{2d} (f(x + e_j) - f(x)) + w(x) f(x), \qquad x \in U,$$

with the convention that $f(y) = 0$ if $y \notin U$, and where $\{e_j : 1 \leq j \leq 2d\}$ are the canonical generators of $\mathbb{Z}^d$ and their corresponding inverses. Note that $H^0_{U,w}$ is a bounded symmetric operator on $l^2(U)$. Define

$$\bar{w}_U := \max\{w(x) : x \in U\}.$$

Theorem 4. *Let $U \subset \mathbb{Z}^d$ and w a potential on U and $x_0 \in U$, and assume that $w(x_0) = 0$. Consider the Schrödinger operator $H^0_{U,w}$. Then, whenever $h > \bar{w}_U$,*

$$H_{U,w,h} := H^0_{U,w} + h\delta_{x_0}$$

has a simple principal Dirichlet eigenvalue λ_0 and a principal Dirichlet eigenfunction ψ_0. Furthermore, the following are satisfied.

(i) *The principal Dirichlet eigenvalue has the expansion*

$$\lambda_0 = h - 2d\kappa + h \sum_{j=1}^{\infty} \sum_{\gamma \in \mathbb{P}_U^{2j+1}(x_0,x_0)} \prod_{z \in \gamma, z \neq z_1} \frac{\kappa}{\lambda_0 + 2d\kappa - w(z)}. \tag{20}$$

(ii) *There is a constant K such that for all $x \in U$ one has that*

$$\psi_0(x) = K \sum_{\gamma \in \mathbb{P}_U(x,x_0)} \prod_{z \in \gamma} \frac{\kappa}{\lambda_0 + 2d\kappa - w(z)}.$$

(iii) *Whenever $h \geq 4d\kappa$ we have that*

$$\psi_0(x) = \mathbb{1}_{x_0}(x) + \varepsilon(x),$$

where $\varepsilon(x)$ satisfies for all $x \in U$

$$|\varepsilon(x)| \leq C \frac{1}{(h - \bar{w}_U)^{|x - x_0|_1 + 1}},$$

for some constant C that does not depend on h nor w.

(iv) *We have that*

$$\sup\{\lambda \in \sigma(H_{U,\omega,h}) : \lambda < \lambda_0\} \leq \bar{w}_U.$$

Theorem 4 will be proved in subsection 3.3.

3.2. **Green function of the unperturbed operator.** Throughout, given any self-adjoint operator A defined on $l^2(U)$, we will denote by $res(A)$ and $\sigma(A)$ the resolvent set and the spectrum of A respectively. For $\lambda \in res(H^0_{U,w})$, let us introduce on $U \times U$ the function

$$g^{U,w}_\lambda(x,y) := (\delta_x, (\lambda I - H^0_{U,w})^{-1}\delta_y).$$

Note that $g^{U,w}_\lambda$ is the *Green function* of the operator $H^0_{U,w} - \lambda I$.

Note that,

$$\mathbb{P}_U(x,y) = \cup_{k=1}^\infty \mathbb{P}^k_U(x,y), \tag{21}$$

where the union is disjoint. Let us now introduce the norm $|x|_1 := \sum_{j=1}^d |x_j|$ on $\mathbb{Z}^d$. Note that if $\gamma \in \mathbb{P}_U(x,y)$, then

$$|\gamma| \geq |x-y|_1 + 1. \tag{22}$$

Furthermore

$$|\mathbb{P}^k_U(x,y)| \leq (2d)^{k-1}. \tag{23}$$

Lemma 1. *Let U be a bounded connected subset of $\mathbb{Z}^d$, w a non-negative potential on U and $\kappa > 0$. Let $x, y \in U$. Then, the Green function $g^{U,w}_\lambda(x,y)$ is analytic if $\lambda \in res(H^0_{U,w})$ and*

$$g^{U,w}_\lambda(x,y) = \frac{1}{\kappa} \sum_{\gamma \in \mathbb{P}_U(x,y)} \prod_{z \in \gamma} \frac{\kappa}{\lambda + 2d\kappa - w(z)}. \tag{24}$$

Proof. Now, let us remark that for $x, y \in \mathbb{Z}^d$,

$$\begin{aligned} g^{U,w}_\lambda(x,y) &= \int_0^\infty E_x \left[e^{\int_0^t (w(X_s)-\lambda)ds} \delta_y(X_t) \right] dt \\ &= \sum_{\gamma \in \mathbb{P}_U(x,y)} \int_0^\infty E_x \left[e^{\int_0^t (w(X_s)-\lambda)ds} C_\gamma \right] dt, \end{aligned} \tag{25}$$

where $\{X_t : t \geq 0\}$ is a simple symmetric random walk of total jump rate $2d\kappa$ starting from x and E_x the corresponding expectation. For each $\gamma = \{x_1 = x, \ldots, x_n = y\} \in \mathbb{P}_U(x,y)$ define $B_\gamma := E_x \left[e^{\int_0^t (w(X_s)-\lambda)ds} C_\gamma \right]$. Then, noting that the probability density on the path γ is given by $e^{-2d\kappa s_1 - \cdots - 2d\kappa s_{n-1} - 2d\kappa(t - s_1 - \cdots - s_{n-1})}$, where s_i is the time spent on x_i, we see that,

$$\begin{aligned} B_\gamma &= \frac{(2d\kappa)^{n-1}}{(2d)^{n-1}} e^{-2d\kappa t} \int_{S_{n-1}} e^{\bar{w}(x_1)s_1 + \cdots + \bar{w}(x_{n-1})s_{n-1} + \bar{w}(x_n)(t - s_1 - \cdots - s_{n-1})} ds_1 \cdots ds_{n-1} \\ &= \kappa^{n-1} e^{-2d\kappa t + \bar{w}_n t} \int_{S_{n-1}} e^{(\bar{w}_1 - \bar{w}_n)s_1 + \cdots (\bar{w}_{n-1} - \bar{w}_n)s_{n-1}} ds_1 \cdots ds_{n-1}, \end{aligned}$$

where $\bar{w}_i = w(x_i) - \lambda$ and $S_{n-1} = \{s_1 + \cdots + s_{n-1} < t\}$. Using induction on n we can compute the above integral to obtain,

$$B_\gamma = \frac{1}{\kappa} \prod_{z \in \gamma} \frac{\kappa}{\lambda + 2d\kappa - w(z)}.$$

Substituting this expression back in (25) finishes the proof. □

Note that the largest eigenvalue of $H_{U,w}$ can be expressed as $\lambda_+^{U,w} := \sup\{\lambda \in \sigma(H_{U,w})\}$.

Lemma 2. *For every finite connected set U and non-negative potential w on U.*

$$\bar{w}_U - 2d\kappa \leq \lambda_+^{U,w} \leq \bar{w}_U. \tag{26}$$

Proof. Let x_m be some site where $\bar{w}_U = w(x_m)$. The first inequality of (26) follows from the computation $(f, H_{U,w} f) \geq w(x_m) - 2d\kappa$ for $f = \delta_{x_m}$. The second from the estimate $(f, H_{U,w} f) \leq w(x_m)$, for arbitrary $f \in l^2(U)$ with unit norm. □

Corollary 2. *Let $x, y \in U$. If $\lambda - \lambda_+^{U,w} > 2d\kappa$, then*

$$\frac{1}{\kappa} \left(\frac{\kappa}{\lambda + 2d\kappa} \right)^{|x-y|_1+1} \leq g_\lambda^{U,w}(x,y) \leq \frac{(2d\kappa)^{|x-y|_1}}{(\lambda - \lambda_+^{U,w})^{|x-y|_1}} \frac{1}{\lambda - \lambda_+^{U,w} - 2d\kappa}. \tag{27}$$

Proof. For every $z \in U$, $\lambda + 2d\kappa \geq \lambda + 2d\kappa - w(z)$. By part (i) of lemma 1 and the fact that the shortest path between x and y has length $|x-y|_1 + 1$, we prove the first inequality of display (27). Also, using again part (i) of lemma 1, inequality (22) and the decomposition (21) we see that,

$$g_\lambda^{U,w}(x,y) \leq \frac{1}{\kappa} \sum_{k=|x-y|_1+1}^{\infty} \sum_{\gamma \in \mathbb{P}_U^k(x,y)} \left(\frac{\kappa}{\lambda - \lambda_+^{U,w}} \right)^k.$$

Now, from (23), we conclude the proof.

□

We can now derive the following lemma.

Lemma 3. *Assume that $h > \bar{w}_U + 2d\kappa$. Then, there exists a unique $\lambda_0 > \lambda_+^{U,w}$, which satisfies the equation,*

$$h g_{\lambda_0}^{U,w}(x,x) = 1. \tag{28}$$

Proof. By Lemma 12, note that equation (28) has a unique solution $\lambda_0 > \lambda_+^{U,w}$ if,

$$h > \frac{1}{\lim_{\lambda \searrow \lambda_+^{U,w}} g_\lambda^{U,w}(x,x)}.$$

Now, by the first inequality of (27) of part (i) of corollary 2 and by lemma 2 the right-hand side of the above inequality is larger than $\bar{w}_U + 2d\kappa$. □

3.3. **Proof of Theorem 4.** Let us now prove Theorem 4. To prove part (i) note that the unique λ_0 which satisfies (28) of Lemma 3 has to be the principal Dirichlet eigenvalue by Theorem 6 of Appendix A. Applying the expansion (24) of Lemma 1 of the Green function we obtain part (i). To prove part (ii) let us first note that by Lemma 3, λ_0 is an isolated point of the spectrum of $H_{U,\omega}$. It follows that there is simple closed curve Γ in the complex plane which contains λ_0 in its interior and the rest of the spectrum of $H_{U,\omega}$ in its exterior. Therefore

$$P := \int_\Gamma R_\lambda d\lambda,$$

where R_λ is the resolvent of $H_{U,\omega}$, is the orthogonal projection onto the eigenspace of λ_0. Now, from Theorem 5 of Appendix A, we can see that

$$P = C_3(q_{\lambda_0}, \cdot)q_{\lambda_0},$$

for some constant C_3. From here we can deduce part (ii). Part (iii) follows immediately from part (ii). Part (iv) follows from Corollary 3 of Appendix A and Lemma 2.

4. High peak statistics of a Weibull potential

In this section we will derive several results describing the asymptotic behavior as $l \to \infty$ of quantities defined in terms of the order statistics $v_{(1)} \geq v_{(2)} \geq \cdots \geq v_{(N)}$ of an i.i.d. potential v on the box Λ_l with Weibull distribution. We will ocasionally also use the notation $v(x_{(j)})$ instead of $v_{(j)}$ to indicate explicitly the site $x_{(j)}$ where the value $v_{(j)}$ is attained.

Lemma 4. *Consider the order statistics $\{v_{(1)}, \ldots, v_{(N)}\}$ of the potential v. Let*

$$a_l := (\rho \log |\Lambda_l|)^{1/\rho}.$$

(i) *For every $x \in \mathbb{R}$,*

$$\lim_{l\to\infty} \mu[v_{(1)} < a_l + b_l x] = \exp\left\{-e^{-x}\right\},$$

where $b_l := \frac{1}{(\rho \log |\Lambda_l|)^{1-1/\rho}}$. In particular, μ-a.s.,

$$\lim_{l\to\infty} \frac{v_{(1)}}{a_l} = 1.$$

(ii) *For every sequence $\{c_l\}$ such that $c_l \geq a_l$, we have that*

$$\mu[v_{(1)} \geq c_l] \leq |\Lambda_l| e^{-\frac{1}{\rho}c_l^\rho} + o\left(|\Lambda_l| e^{-\frac{1}{\rho}c_l^\rho}\right). \tag{29}$$

and that

$$\mu[v_{(2)} \geq c_l] \leq |\Lambda_l|^4 e^{-\frac{2}{\rho}c_l^\rho} \tag{30}$$

Proof. Consider a sequence $\{c_l : l \geq 0\}$ and note that

$$\mu(v_{(1)} < c_l) = (1 - \mu(v_1 \geq c_l))^{|\Lambda_l|} = \left(1 - \frac{1}{|\Lambda_l|^{\left(\frac{c_l}{a_l}\right)^\rho}}\right)^{|\Lambda_l|}.$$

Therefore, using the fact the for all natural n, $e^{-1}(1-1/n) \leq (1-1/n)^n \leq e^{-1}$, we have that

$$\exp\left\{-|\Lambda_l|^{1-\left(\frac{c_l}{a_l}\right)^\rho}\right\}\left(1 - \frac{1}{|\Lambda_l|^{\left(\frac{c_l}{a_l}\right)^\rho}}\right)^{|\Lambda_l|^{1-\left(\frac{c_l}{a_l}\right)^\rho}} \leq \mu(v_{(1)} < c_l) \leq \exp\left\{-|\Lambda_l|^{1-\left(\frac{c_l}{a_l}\right)^\rho}\right\}. \tag{31}$$

Choosing $c_l = a_l + b_l x$ in the above inequalities and taking the limit when $l \to \infty$, we prove part (i).

Part (ii). Note that for all $x \geq 0$, $e^{-x} \leq 1 - x + \frac{x^2}{2}$. Therefore, by (31), we have that

$$\begin{aligned}\mu(v_{(1)} \geq c_l) &\leq \left(\frac{1}{|\Lambda_l|^{\left(\frac{c_l}{a_l}\right)^\rho - 1}} - \frac{1}{|\Lambda_l|^{2\left(\frac{c_l}{a_l}\right)^\rho - 2}}\right)\left(1 - \frac{1}{|\Lambda_l|^{\left(\frac{c_l}{a_l}\right)^\rho}}\right)^{|\Lambda_l|^{1-\left(\frac{c_l}{a_l}\right)^\rho}} \\ &= |\Lambda_l| e^{-\frac{1}{\rho}c_l^\rho} + o\left(|\Lambda_l| e^{-\frac{1}{\rho}c_l^\rho}\right),\end{aligned}$$

which gives (29). The proof of (30) is completely similar. □

5. Stable limit laws and structure of the scaling

Here we will prove Theorem 2. Part (i) will be proved in subsection 5.1 and part (ii) in subsection 5.2.

5.1. **Existence of the scaling function.**

Here we will prove part (i) of Theorem 2, showing the existence of a function $h(t)$ which satisfies the equality (13).

Lemma 5. *For every potential $v \in W$ and natural N the function $\zeta_N(s) : [0, \infty) \to (0, e^{-\frac{1}{\rho}}\mathcal{B}_M^\rho]$ [c.f. (8)] defined by (11) is a homeomorphism.*

Proof. Note that for every fixed potential $v \in W$ and natural N the function $B_N(s, v)$ defined in (6) is strictly increasing in $[0, \infty)$. By the bounded convergence theorem, this implies that ϕ is strictly decreasing and continuous with range $(0, e^{-\frac{1}{\rho}}\mathcal{B}_M^\rho]$, which proves the lemma. □

Now note that by the definition of t_0 given in part (i) of Theorem 2, since the scale $L(t)$ is an increasing function of t, we have that whenever $t \geq t_0$

$$0 < e^{-\gamma \frac{\tau(t)^{\rho'}}{\rho'}} \leq e^{-\frac{1}{\rho}\mathcal{B}_M^{\rho}}.$$

By Lemma 5 it is clear that for each $t \geq t_0$ there exists a $h(t) \in [0, \infty)$ such that (13) is satisfied. We define now as in the statement of Theorem 13 for $t \geq 0$,

$$g(t) := th(t). \tag{32}$$

5.2. **Properties of the scaling function.** Here we will prove part (ii) of Theorem 2, which states that the scaling function $h(t)$ defined in (13) has the expansion specified by (14), (16) and (17).

We will now prove that the functions defined recursively by (15) and (16) of Theorem 3 are such that (14) and (17) are satisfied. Define for $1 \leq j \leq M$, $\mathcal{H}_j := h_0 + \cdots + h_j$. We will need the following lemma.

Lemma 6. *For every $\tau > 0$ the following are satisfied.*

(i) *For all $\epsilon > 0$ and $\max_{e \in E} v(e) \leq (1-\epsilon)\mathcal{A}_0 \tau^{\rho'-1}$, we have that*

$$B_1(h_0, v)^{\rho} = \mathcal{A}_0^{\rho} \tau^{\rho'} + O\left(\frac{1}{\tau^{\rho'-2}}\right), \tag{33}$$

and for $j \geq 1$ that

$$B_{j+1}(\mathcal{H}_j, v)^{\rho} = B_j(\mathcal{H}_{j-1}, v)^{\rho} + \rho \mathcal{A}_0 \tau h_j + O\left(\frac{1}{\tau^{(2j+1)\rho'-(2j+2)}}\right). \tag{34}$$

where both in (33) and (34) the error term satisfies for all real x,

$$|O(x)| \leq C_8 |x|.$$

(ii) *There is a constant $C_{6,1}$ such that*

$$e^{-\left(\gamma \frac{\tau^{\rho'}}{\rho'} + \frac{C_{6,1}}{\tau^{\rho'-2}}\right)} \leq E_{\mu}\left[e^{-\frac{1}{\rho}B_1(h_0,v)^{\rho}}\right] \leq e^{-\left(\gamma \frac{\tau^{\rho'}}{\rho'} - \frac{C_{6,1}}{\tau^{\rho'-2}}\right)}. \tag{35}$$

Similarly, for each $j \geq 1$ there is a constant $C_{6,j}$ such that

$$E_{\mu}\left[e^{-\left(\frac{1}{\rho}B_j(\mathcal{H}_{j-1},v)^{\rho} + \rho\tau h_j + \frac{C_{6,j}}{\tau^{(2j+1)\rho'-(2j+2)}}\right)}\right] \leq E_{\mu}\left[e^{-\frac{1}{\rho}B_{j+1}(\mathcal{H}_j,v)^{\rho}}\right]$$
$$\leq E_{\mu}\left[e^{-\left(\frac{1}{\rho}B_j(\mathcal{H}_{j-1},v)^{\rho} + \rho\tau h_j - \frac{C_{6,j}}{\tau^{(2j+1)\rho'-(2j+2)}}\right)}\right]. \tag{36}$$

Proof. Part (i) follows from a standard Taylor expansion. We will now prove (35) of part (ii). Define for $\epsilon > 0$ the event

$$A := \left\{\max_{e \in U} v(e) \leq (1-\epsilon)\mathcal{A}_0 \tau^{\rho'}\right\}.$$

Then

$$E_\mu\left[e^{-\frac{1}{\rho}B_1(h_0,v)^\rho}\right] = E_\mu\left[e^{-\frac{1}{\rho}B_1(h_0,v)^\rho}, A\right] + E_\mu\left[e^{-\frac{1}{\rho}B_1(h_0,v)^\rho}, A^c\right].$$

But by part (i) we have that

$$E_\mu\left[e^{-\frac{1}{\rho}B_1(h_0,v)^\rho}, A\right] \leq e^{-\left(\gamma\frac{\tau^{\rho'}}{\rho'} - \frac{C_8}{\tau^{\rho'-2}}\right)}. \tag{37}$$

On the other hand, since

$$B_1(h_0,v)^\rho \geq \tau^{\rho'}(\mathcal{A}_0\mathcal{B}_1)^\rho,$$

we have that for ϵ small enough

$$\begin{aligned} E_\mu\left[e^{-\frac{1}{\rho}B_1(h_0,v)^\rho}, A^c\right] &\leq 2de^{-\tau^{\rho'}\left((\mathcal{A}_0\mathcal{B}_1)^\rho + \frac{1}{\rho}(1-\epsilon)^\rho\mathcal{A}_0^\rho\right)} \\ &= 2de^{-\tau^{\rho'}\left((\mathcal{A}_0\mathcal{B}_1)^\rho + \frac{\gamma}{\rho'}(1-\epsilon)^\rho\right)} = O\left(e^{-(1+\epsilon)\gamma\frac{\tau^{\rho'}}{\rho'}}\right). \end{aligned} \tag{38}$$

Combining (37) with (38) we see that there is a constant C_6 such that

$$E_\mu\left[e^{-\frac{1}{\rho}B_1(h_0,v)^\rho}\right] \leq e^{-\left(\gamma\frac{\tau^{\rho'}}{\rho'} - \frac{C_6}{\tau^{\rho'-2}}\right)}.$$

On the other hand, from $B_1(h_0,v)^\rho \leq \gamma\frac{\tau^{\rho'}}{\rho'}$, we immediately get that

$$E_\mu\left[e^{-\frac{1}{\rho}B_1(h_0,v)^\rho}\right] \geq e^{-\gamma\frac{\tau^{\rho'}}{\rho'}},$$

which finishes the proof of (35). The proof of (36) is analogous.

□

Let us now prove (17) of part (ii) of Theorem 2. By the definition of h_1 in (16) and by (35) of part (ii) of Lemma 6, we see that

$$\begin{aligned} -\tfrac{C_{6,1}}{t^{\rho'-1}} &\leq \tfrac{1}{\mathcal{A}_0^{\rho-1}}\left(\gamma\tfrac{\tau^{\rho'-1}}{\rho'} + \tfrac{1}{\tau}\log e^{-\left(\gamma\frac{\tau^{\rho'}}{\rho'} + \frac{C_{6,1}}{\tau^{\rho'-2}}\right)}\right) \\ &\leq \tfrac{1}{\mathcal{A}_0^{\rho-1}}\left(\gamma\tfrac{\tau^{\rho'-1}}{\rho'} + \tfrac{1}{\tau}\log E_\mu\left[e^{-\frac{1}{\rho}B_1(h_0,v)^\rho}\right]\right) = h_1(t) \\ &\leq \tfrac{1}{\mathcal{A}_0^{\rho-1}}\left(\gamma\tfrac{\tau^{\rho'-1}}{\rho'} + \tfrac{1}{\tau}\log e^{-\left(\gamma\frac{\tau^{\rho'}}{\rho'} - \frac{C_{6,1}}{\tau^{\rho'-2}}\right)}\right) \leq \tfrac{C_{6,1}}{t^{\rho'-1}}. \end{aligned}$$

Hence,

$$h_1(t) = O\left(\frac{1}{t^{\rho'-1}}\right).$$

We will now prove that for $j \geq 1$

$$h_{j+1}(t) = O\left(\frac{1}{t^{(2j+1)(\rho'-1)}}\right). \tag{39}$$

Using the definition (16) for h_j and h_{j+1}, by (36) of part (ii) of Lemma 6 we conclude that

$$\begin{aligned}
-\tfrac{C_{6,j-1}}{t^{(2j+1)(\rho'-1)}} &\leq \tfrac{1}{\mathcal{A}_0^{\rho-1}}\left(\gamma\tfrac{\tau^{\rho'-1}}{\rho'} + \tfrac{1}{\tau}\log E_\mu\left[e^{-\left(\frac{1}{\rho}B_j(\mathcal{H}_{j-1},v)^\rho+\rho\mathcal{A}_o\tau h_j+\frac{C_{6,j-1}}{\tau^{(2j+1)\rho'-(2j+2)}}\right)}\right]\right)\\
&\leq \tfrac{1}{\mathcal{A}_0^{\rho-1}}\left(\gamma\tfrac{\tau^{\rho'-1}}{\rho'} + \tfrac{1}{\tau}\log E_\mu\left[e^{-\frac{1}{\rho}B_{j+1}(\mathcal{H}_j,v)^\rho}\right]\right) = h_{j+1}(t)\\
\leq \tfrac{1}{\mathcal{A}_0^{\rho-1}}\left(\gamma\tfrac{t^{\rho'-1}}{\rho'} + \tfrac{1}{t}\log E_\mu\left[e^{-\left(\frac{1}{\rho}B_j(\mathcal{H}_{j-1},v)^\rho+\rho\mathcal{A}_o\tau h_j-\frac{C_{6,j-1}}{\tau^{(2j+1)\rho'-(2j+2)}}\right)}\right]\right) &\leq \tfrac{C_{6,j-1}}{t^{(2j+1)(\rho'-1)}},
\end{aligned}$$

which proves (39) and hence (17) of part (ii) of Theorem 2. It now follows that

$$e^{-\gamma\frac{\tau^{\rho'}}{\rho'}} = E_\mu\left[e^{-\frac{1}{\rho}B_M(\mathcal{H}_{M-1},v)^\rho-\mathcal{A}_0^{\rho-1}\tau h_M}\right] = E_\mu\left[e^{-\frac{1}{\rho}B_M(\mathcal{H}_M,v)^\rho+O\left(\frac{1}{\tau^{(2M+1)\rho'-(2M+2)}}\right)}\right]$$

which implies that $h(t) - \mathcal{H}_M(t) = O\left(\frac{1}{t^{(2M+1)(\rho'-1)}}\right)$, which proves (14) of part (ii) of Theorem 2.

6. Convergence to stable laws

We will now prove Theorem 3. Some of the computations will be similar to those done by Ben Arous, Bogachev and Molchanov in [7], within the context of sums of i.i.d. random exponential variables. As a first step, we will first recall the coarse graining methods introduced in [8], which will enable us to reduce the problem to a sum of approximately independent random variables in subsection 6.0.1. In subsection 6.0.2, we will show how to reduce the problem to an sum of independent random variables. In subsection 6.0.3 we recall a classical criteria in Theorem 5 for convergence to infinite divisible distributions. These criteria will be verified in subsections 6.0.4, 6.0.5 and 6.0.6.

6.0.1. *Mesoscopic scales.* Let us now recall the coarse graining methods introduced in [8]. Let $L \geq 0$ and consider a box Λ_L. Here we will need the *strip-box partition* of [8]. We introduce a parameter l smaller than or equal to L, called the *mesoscopic scale*. Then, there exist natural numbers q and $\bar{q}$ such that $2L+1 = ql + \bar{q}$, with $0 \leq \bar{q} \leq q$. Hence,

$$2L+1 = \sum_{i=1}^{q} l_i,$$

where $l_i = l + \theta_{\bar{q}}(i)$ and $\theta_{\bar{q}}(i) = 1$ for $i \leq \bar{q}$ and $\theta_{\bar{q}}(i) = 0$ for $i > \bar{q}$. Given a pair of real numbers a, b, we will use the notation $[[a,b]]$ for $[a,b]\cap\mathbb{Z}$. Now define $I_1 := [[-L, -L + l_1 - 1]]$ and for $1 < i \leq q$ let $I_i := \left[\left[-L + \sum_{j=1}^{i-1} l_i, -L + \sum_{j=1}^{i} l_i - 1\right]\right]$. Now, we

introduce a second parameter r which is a natural number smaller than or equal to l, called the *fine scale*. Let $r_i := r + \theta_{\bar{q}}(i)$. Define $J_1 := [[-L + r_1, -L + l_1 - 1 - r_1]]$ and for $1 < i \leq q$ let $J_i := \left[\left[-L + \sum_{j=1}^{i-1} l_i + r_i, -L + \sum_{j=1}^{i} l_i - 1 - r_i\right]\right]$.

Now, let $\mathcal{I} := \{1, 2, \ldots, q\}^d$. For a given element $\mathbf{i} \in \mathcal{I}$, of the form $\mathbf{i} = (i_1, \ldots, i_d)$ with $1 \leq i_k \leq q$, $1 \leq k \leq d$, we define,

$$\Lambda''_{\mathbf{i}} := J_{i_1} \times J_{i_2} \times \cdots \times J_{i_d},$$

called a *main box*. Its cardinality is $|\Lambda''_{\mathbf{i}}| = (l - 2r)^d$. Now let,

$$S_L := \Lambda_L - \bigcup_{\mathbf{i} \in \mathcal{I}} \Lambda''_{\mathbf{i}},$$

called the *strip set*. The sets S_L and $\{\Lambda''_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}\}$ define a partition of Λ_L called the *strip-box partition at scale l* of Λ_L.

Let us now write,

$$\sum_{x \in \Lambda_L} m(x,t) = \sum_{\mathbf{i} \in \mathcal{I}} m_{\mathbf{i}} + \sum_{x \in S_L} m(x,t), \tag{40}$$

where

$$m_{\mathbf{i}} := \sum_{x \in \Lambda''_{\mathbf{i}}} m(x,t).$$

We will use the notation $\mathbf{0} := \{0, \ldots, 0\}$. Throughout we will make the following choices

$$L := e^{\gamma \frac{t^{\rho'}}{\rho'}}, \tag{41}$$

$$l := e^t \tag{42}$$

and

$$r := t^2. \tag{43}$$

6.0.2. *Dirichlet boundary conditions.* Throughout, we will denote by $m(x,t,v)$ the solution $m(x,t)$ of the parabolic Anderson problem (1), emphasizing the dependence on the potential v of it. For each finite set $U \subset \mathbb{Z}^d$ and environment $v \in W$, we define $\tilde{m}_U(x,t) = m_U(x,t,v_U)$ as the solution of the parabolic Anderson equation with potential v with Dirichlet boundary conditions on U and initial condition $\mathbf{1}_U$, so that

$$\frac{\partial \tilde{m}_U(x,t)}{\partial t} = \kappa \Delta \tilde{m}_U(x,t) + v(x)\tilde{m}_U(x,t), \quad \text{for all } t > 0, x \in \mathbb{Z}^d,$$
$$\tilde{m}_U(x,0) = 1_U(x),$$

where the Laplacian Δ has Dirichlet boundary conditions on U. In other words, $v_U(x) = v(x)$ for $x \in U$ while $V_U(x) = -\infty$ for $x \notin U$, with $m(x,t,v_U)$ the solution of

(1). Throughout, for $r > 0$, we will use the notation $\tilde{m}_r(x,t)$ instead of $\tilde{m}_{\Lambda(0,r)}(x,t)$. The following lemma can be proved in the same way as part (*iii*) of Proposition 9 of [9].

Lemma 7. *Consider a finite subset $U \in \mathbb{Z}^d$. Then, for each $\beta > 0$, $\gamma > 0$, there is a constant $C > 0$ such that for all $R \geq 2\kappa t$,*

$$\left\langle |m(x,t) - \tilde{m}_R(x,t)|^\beta \right\rangle \leq C(R+1)^d e^{-2\beta\kappa t I\left(\frac{R}{2\kappa t}\right)} e^{H(\beta t)},$$

where $I := y \sinh^{-1} y - \sqrt{1+y^2} + 1$.

We will also define for $\mathbf{i} \in \mathcal{I}$

$$\tilde{m}_{\mathbf{i}}(t) := \sum_{x \in \Lambda''_{\mathbf{i}}} \tilde{m}_{\Lambda''_{\mathbf{i}}}(x,t). \tag{44}$$

Let

$$s(t) := e^{g(t)}, \tag{45}$$

where $g(t)$ is defined in (32). Theorem 3 states that

$$\lim_{t\to\infty} \frac{|\Lambda_L|}{s(t)}(m_L(t) - A(t)) = S_\alpha,$$

in distribution. By Lemma 7, the decomposition (40) and the choice of scales (41), (42) and (43), it is enough to prove that

$$\lim_{t\to\infty} \frac{1}{s(t)} \left(\sum_{\mathbf{i}\in\mathcal{I}} \tilde{m}_{\mathbf{i}} - |\Lambda_L| A(t) \right) = S_\alpha. \tag{46}$$

6.0.3. *Criteria for convergence to stable laws.* In order to prove part (*iii*) of Theorem 3, we recall the following result which gives conditions for a triangular array of independent random variables to converge to a given infinite divisible distribution (see for example Theorems 7 and 8 of Chapter 4 of Petrov [15] or [7]).

A random variable is infinite divisible if and only if its characteristic function $\phi(t)$ admits the expression (see for example Theorem 5 of Chapter 2 of Petrov [15])

$$\phi(t) = \exp\left\{ i\nu t - \frac{\sigma^2 t^2}{2} + \int_{-\infty}^{\infty} \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) d\mathcal{L}(x) \right\}, \tag{47}$$

where ν is a real constant, σ^2 a non-negative constant, the function $\mathcal{L}(x)$ is non-decreasing in $(-\infty, 0)$ and $(0, \infty)$ and satisfies $\lim_{x\to\infty} \mathcal{L}(x) = \lim_{x\to-\infty} \mathcal{L}(x) = 0$ and for every $\delta > 0$, $\int_{x:0<|x|\leq\delta} x^2 d\mathcal{L}(x) < \infty$. $\mathcal{L}$ is called the *Lévy-Khintchine spectral function*. We will denote by $X_{\nu,\sigma,\mathcal{L}}$ the infinite divisible random variable with characteristic function (47).

Theorem 5. *For each $t \geq 0$ let $\mathcal{S}(t)$ be a growing set of indexes and consider a set $\{Y_{\mathbf{i}}(t) : \mathbf{i} \in \mathcal{S}(t)\}$ of independent i.i.d. random variables. Call P_t the law common law of these random variables, say of $Y_{\mathbf{0}}(t)$, where we just call $\mathbf{0}$ an arbitrary element of $\mathcal{S}(t)$. Assume that for every $\epsilon > 0$ it is true that,*

$$\lim_{t\to\infty} P_t(Y_{\mathbf{0}}(t) > \epsilon) = 0.$$

Now let $\mathcal{L}(x) : \mathbb{R}/\{0\} \to \mathbb{R}$ be a Lévy-Khintchine spectral function, $\nu \in \mathbb{R}$ and $\sigma > 0$, and let $A(t) : [0,\infty) \to \mathbb{R}$ be some function. Then, if $n(t) := |\mathcal{S}(t)|$ the following statements are equivalent,

(i) *We have that*

$$\lim_{t\to\infty} \left(\sum_{\mathbf{i}\in\mathcal{S}(t)} Y_{\mathbf{i}}(t) - A(t) \right) = X_{\nu,\sigma,\mathcal{L}},$$

where the convergence is in distribution.

(ii) *Define for $y > 0$ the truncated random variable at level y as $Z_y(t) := Y_{\mathbf{0}}(t)\mathbf{1}_{|Y_1(t)|\leq y}$. Also, let $E_t(\cdot)$ and $Var_t(\cdot)$ denote the expectation and variance corresponding to the law P_t. Then if x is a continuity point of $\mathcal{L}$,*

$$\mathcal{L}(x) = \begin{cases} \lim\limits_{t\to\infty} n(t)P_t(Y_{\mathbf{0}}(t) \leq x) & \text{for } \ x < 0, \\ -\lim\limits_{t\to\infty} n(t)P_t(Y_{\mathbf{0}}(t) > x) & \text{for } \ x > 0, \end{cases} \tag{48}$$

$$\sigma^2 = \lim_{y\to 0}\lim_{t\to\infty} n(t)Var_t(Z_y(t)), \tag{49}$$

and for any $y > 0$ which is a continuity point of $\mathcal{L}(x)$,

$$\nu = \lim_{n\to\infty} (n(t)E_t(Z_y(t)) - A(t)) + \int_{|x|>y} \frac{x}{1+x^2} d\mathcal{L}(x) - \int_{y\geq|x|>0} \frac{x^3}{1+x^2} d\mathcal{L}(x). \tag{50}$$

We will apply Theorem 5, to the set of i.i.d. random variables $\{Y_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}\}$ with

$$Y_{\mathbf{i}} := \frac{1}{s(t)} \tilde{m}_{\mathbf{i}}, \tag{51}$$

$$n(t) := \left(\frac{L(t)}{l(t)} \right)^d = \frac{e^{\gamma \frac{\tau(t)\rho'}{\rho'}}}{e^{dt}}. \tag{52}$$

and

$$\tilde{A}(t) := \begin{cases} 0 & \text{if} \quad 0 < \gamma < \gamma_1 \\ E_\mu[Y_{\mathbf{0}}, Y_{\mathbf{0}} \leq 1] & \text{if} \quad \gamma = \gamma_1 \\ E_\mu[Y_{\mathbf{0}}] & \text{if} \quad \gamma_1 < \gamma < \gamma_2 \end{cases}$$

The first, second and third cases correspond to the definition of $A(t)$ in (19) of Theorem 3.

6.0.4. *Lévy-Khintchine spectral function.* Here we will verify that the set of i.i.d. random variables $\{Y_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}\}$ defined in (51) satisfy condition (48) of part (ii) of Theorem 5, with the Lévy-Khintchine spectral function

$$\mathcal{L}(x) := \begin{cases} 0 & \text{for } x \leq 0 \\ -\frac{1}{x^{\alpha(\gamma,\rho)}} & \text{for } x > 0, \end{cases} \tag{53}$$

where $\alpha(\gamma, \rho)$ is defined in (9). To prove that condition (48) is satisfied with the Lévy-Khintchine spectral function (53) it will be enough to show that the following proposition is satisfied.

Proposition 1. *Consider $\tilde{m}_{\mathbf{0}}$ defined in (44), with v an i.i.d. potential with Weibull law μ. Then, for $s(t)$ defined in (45) and $n(t)$ in (52) we have that for all $u > 0$ it is true that*

$$\lim_{t\to\infty} n(t)\mu\left(\frac{\tilde{m}_{\mathbf{0}}}{s(t)} > u\right) = \frac{1}{u^{\alpha(\gamma,\rho)}}.$$

Proof. We will use the expansion

$$\tilde{m}_{\Lambda_0''}(x,t) = \sum_{n=0}^{N} e^{t\lambda_n}\psi_n(x)(\psi_n, 1(\Lambda_{\mathbf{0}}'')), \tag{54}$$

where $\{\lambda_n : 0 \leq n \leq N\}$ and $\{\psi_n : 0 \leq n \leq N\}$ are the eigenvalues in decreasing order and eigenfunctions, respectively, of the Schrödinger operator $H_{\Lambda_{\mathbf{0}}'',v}$ on Λ_0'' with Dirichlet boundary conditions. Furthermore, we will need to choose δ_1 and δ_2 so that $0 < \delta_1 < \delta_2$ and

$$2(1-\delta_2)^\rho > 1 \tag{55}$$

and to consider the events

$$A_1 := \left\{v_{(1)} \geq (1-\delta_1)\mathcal{A}_0 t^{\rho'-1}\right\},$$

and

$$A_2 := \left\{v_{(2)} \leq (1-\delta_2)\mathcal{A}_0 t^{\rho'-1}\right\}.$$

Step 1. Note that

$$\begin{aligned} \mu\left(\tfrac{\tilde{m}_{\mathbf{0}}}{s(t)} > u\right) &= \mu\left(A_1, A_2, \tfrac{\tilde{m}_{\mathbf{0}}}{s(t)} > u\right) \\ +\mu\left(A_1^c, A_2, \tfrac{\tilde{m}_{\mathbf{0}}}{s(t)} > u\right) &+ \mu\left(A_1, A_2^c, \tfrac{\tilde{m}_{\mathbf{0}}}{s(t)} > u\right). \end{aligned} \tag{56}$$

By inequality (30) of part (ii) of Lemma 4, we have that

$$\mu(A_2^c) = \mu\left(v_{(2)} > (1-\delta_2)\left(\frac{\gamma\rho}{\rho'}\right)^{1/\rho} t^{\rho'-1}\right) \leq e^{-2(1-\delta_2)^\rho \frac{\gamma}{\rho'} t^{\rho'}}. \tag{57}$$

But from (55) we have that

$$\lim_{t\to\infty} n(t)\mu(A_2^c) = 0. \tag{58}$$

It follows from (56) and from (58) that

$$\begin{aligned}\lim_{t\to\infty} n(t)\mu\left(\tfrac{\tilde{m}_{\mathbf{0}}}{s(t)} > u\right) = \lim_{t\to\infty} n(t)\mu\left(A_1, A_2, \tfrac{\tilde{m}_{\mathbf{0}}}{s(t)} > u\right)\\ + \lim_{t\to\infty} n(t)\mu\left(A_1^c, A_2, \tfrac{\tilde{m}_{\mathbf{0}}}{s(t)} > u\right).\end{aligned}$$

Step 2. From the expansion (54) note that on the event A_1^c we have that

$$\begin{aligned}\tilde{m}_{\mathbf{0}} = e^{t\lambda_0}\textstyle\sum_{n=0}^{N} e^{-t(\lambda_0-\lambda_n)}(\psi_n, 1(\Lambda_{\mathbf{0}}''))^2\\ \leq e^{t\lambda_0} l^{2d} \leq e^{t(\lambda_0+2d)} \leq e^{t(v_{(1)}+2d)} \leq e^{(1-\delta_1)\mathcal{A}_0 t^{\rho'}+2dt},\end{aligned}$$

where in the second to last inequality we have used (26). Using the fact that $\lim_{t\to\infty} e^{-g(t)}e^{(1-\delta_1)\mathcal{A}_0 t^{\rho'}+2dt} = 0$, we therefore see that $\lim_{t\to\infty} \tilde{m}_{\mathbf{0}} e^{-g(t)} = 0$ and hence

$$\lim_{t\to\infty} n(t)\mu(A_1^c, \tilde{m}_{\mathbf{0}} e^{-g(t)} > u) = 0.$$

It follows from (56) that

$$\lim_{t\to\infty} n(t)\mu\left(\frac{\tilde{m}_{\mathbf{0}}}{s(t)} > u\right) = \lim_{t\to\infty} n(t)\mu\left(A_1, A_2, \frac{\tilde{m}_{\mathbf{0}}}{s(t)} > u\right). \tag{59}$$

Step 3. By part (iii) of Theorem 4, on the event $A_1 \cap A_2$, the normalized principal Dirichlet eigenfunction ψ_0 is such that for some $x_0 \in \Lambda_{\mathbf{0}}''$ one has that

$$\psi_0(x) = 1 + \varepsilon(x), \tag{60}$$

where

$$|\varepsilon(x)| \leq \frac{C_4}{t^{(|x-x_0|_1+1)(\rho'-1)}}.$$

for some constant C_4. Therefore, from (60) and the expansion (54) we can see that on the event $A_1 \cap A_2$,

$$\tilde{m}_{\mathbf{0}} = e^{t\lambda_0}\left(1 + O\left(\frac{1}{t^{\rho'-1}}\right)\right) + \sum_{n=1}^{N} e^{t\lambda_n}(\psi_n, 1(\Lambda_{\mathbf{0}}''))^2.$$

Therefore, from the identity (20) of part (i) of Theorem 4 and part (iv) of the same theorem, we see that on $A_1 \cap A_2$ it is true that

$$e^{t\lambda_0}\left(1+O\left(\tfrac{1}{t^{\rho'-1}}\right)\right) \le \tilde{m}_{\mathbf{0}}$$
$$\le e^{t\lambda_0}\left(1+O\left(\tfrac{1}{t^{\rho'-1}}\right)+(2l+1)^2 e^{-(\delta_2-\delta_1)\left(\frac{\gamma\rho}{\rho'}\right)^{1/\rho}t^{\rho'}}\right)$$
$$= e^{t\lambda_0}\left(1+O\left(\tfrac{1}{t^{\rho'-1}}\right)\right).$$

Hence, on $A_1 \cap A_2$,

$$\tilde{m}_{\mathbf{0}} = e^{t\lambda_0}\left(1+O\left(\frac{1}{t^{\rho'-1}}\right)\right). \tag{61}$$

Step 4. Using (61) in display (59) we see that for any sequence $\{t_k : n \ge 1\}$ such that $\lim_{k\to\infty} n(t_k)\mu\left(\frac{\tilde{m}_{\mathbf{0}}(t_k)}{s(t_k)}\right)$ exists (possibly being equal to ∞), one has that

$$\lim_{k\to\infty} n(t_k)\mu\left(\tfrac{\tilde{m}_{\mathbf{0}}(t_k)}{s(t_k)} > u\right)$$
$$= \lim_{k\to\infty} n(t_k)\mu\left(A_1, A_2, e^{t_k\lambda_0}\left(1+O\left(\tfrac{1}{t_k^{\rho'-1}}\right)\right) \ge ue^{g(t_k)}\right)$$
$$= \lim_{k\to\infty} n(t_k)\mu\left(e^{t_k\lambda_0}\left(1+O\left(\tfrac{1}{t_k^{\rho'-1}}\right)\right) \ge ue^{g(t_k)}\right), \tag{62}$$

where in the last step we used (58) of Step 2 and the fact that since on A_1^c it is true that $\lambda_0 \le (1-\delta_1)g(t)$, eventually in t,

$$e^{t\lambda_0} \le xe^{(1-\delta_1)g(t)}. \tag{63}$$

From (62) we now get that there is a constant C_1 such that

$$\liminf_{t\to\infty} n(t)\mu\left(\lambda_0 \ge h(t)+\tfrac{1}{t}\log u+\tfrac{C_1}{t^{\rho'}}\right) \le \liminf_{t\to\infty} n(t)\mu\left(\tfrac{\tilde{m}_{\mathbf{0}}}{s(t)} > u\right)$$
$$\le \limsup_{t\to\infty} n(t)\mu\left(\tfrac{\tilde{m}_{\mathbf{0}}}{s(t)} > u\right) \le \limsup_{t\to\infty} n(t)\mu\left(\lambda_0 \ge h(t)+\tfrac{1}{t}\log u-\tfrac{C_1}{t^{\rho'}}\right). \tag{64}$$

Step 5. For each $t > 0$, define W_t as the set of potentials v such that $v(y) = 0$ for $y \notin \Lambda_{\mathbf{0}}''$. For each $v \in W_t$ and $x \in \Lambda_{\mathbf{0}}''$, consider the function

$$B(s,v,x) := \frac{s+2d\kappa}{1+\sum_{j=1}^{\infty}\sum_{\gamma\in\mathbb{P}^{2j+1}(x,x)}\prod_{z\in\gamma,z\neq z_1}\frac{\kappa}{s+2d\kappa-v_0(z)}},$$

which is well defined whenever $s > \bar{v} := \max\{v(x) : x \in \Lambda_{\mathbf{0}}''\}$. Using the fact that the function $B(s,v)$ is increasing on $[\bar{v},\infty)$ and part (i) of Theorem 4, note that on the event $A_1 \cap A_2$, the inequality $\lambda_0 \ge s$ is equivalent to $v_{(1)} \ge B(s,v,x_0)$, where $x_0 \in \Lambda_{\mathbf{0}}''$ is such that $v_{(1)} = v(x_0)$. It follows from (58) of Step 2 and (63) of Step 4 that

$$\begin{aligned}
&\liminf_{t\to\infty} n(t)\mu\left(\lambda_0 \geq h(t) + \tfrac{1}{t}\log u + \tfrac{C_1}{t^{\rho'}}\right) \\
&= \liminf_{t\to\infty} n(t)\mu\left(A_1, A_2, \lambda_0 \geq h(t) + \tfrac{1}{t}\log u + \tfrac{C_1}{t^{\rho'}}\right) \\
&= \liminf_{t\to\infty} n(t)\mu\left(A_1, A_2, v_{(1)} \geq B\left(h(t) + \tfrac{1}{t}\log u + \tfrac{C_1}{t^{\rho'}}, v, x_0\right)\right). \qquad (65)
\end{aligned}$$

Now, note that

$$A_1 \cap A_2 = \bigcup_{x\in\Lambda_{\mathbf{0}}''}^{\cdot} \left\{v(x) \geq (1-\delta_1)\mathcal{A}_0 t^{\rho'-1}\right\} \cap A_2,$$

where the symbol $\cup$ denotes a disjoint union. Therefore, the probability appearing in the right-most hand side of display (65) can be written as

$$\begin{aligned}
&\mu\left(A_1, A_2, v_{(1)} \geq B\left(h(t) + \tfrac{1}{t}\log u + \tfrac{C_1}{t^{\rho'}}, v, x_0\right)\right) \\
&= \sum_{x\in\Lambda_{\mathbf{0}}''} \mu\Big(A_2, v(x) \geq (1-\delta_1)\mathcal{A}_0 t^{\rho'-1}, v(x) \geq B\left(h(t) + \tfrac{1}{t}\log u + \tfrac{C_1}{t^{\rho'}}, v, x\right)\Big). \qquad (66)
\end{aligned}$$

Furthermore, for each $x \in \Lambda_{\mathbf{0}}''$, on the event $A_2 \cap \{v(x) \geq (1-\delta_1)\mathcal{A}_0 t^{\rho'-1}\}$ whenever $s \geq (1-\delta_1)t^{\rho'-1}$, we have that

$$\sum_{j=1}^{\infty} \sum_{\gamma\in\mathbb{P}^{2j+1}(x,x)} \prod_{z\in\gamma, z\neq z_1} \frac{\kappa}{s + 2d\kappa - v_0(z)} = O\left(\frac{1}{t^{\rho'-1}}\right),$$

Now, by part (ii) of Theorem 2, we have that

$$h(t) = h_0(t) + O\left(\frac{1}{t^{\rho'-1}}\right). \qquad (67)$$

Hence, $h(t) = \mathcal{A}_0\tau^{\rho'-1} + O(1)$ and on the event $A_2 \cap \{v(x) \geq (1-\delta_1)\mathcal{A}_0 t^{\rho'-1}\}$ we have

$$\begin{aligned}
B\left(h(t) + \tfrac{1}{t}\log u + \tfrac{C_1}{t^{\rho'}}, v, x\right) &= h(t) + \tfrac{1}{t}\log u + \tfrac{C_1}{t^{\rho'}} + 2d\kappa + O\left(\tfrac{h(t)}{t^{\rho'-1}}\right) \\
&\geq \mathcal{A}_0\tau^{\rho'-1} + O(1) \qquad (68)
\end{aligned}$$

and also that

$$\begin{aligned}
B\left(h(t) + \tfrac{1}{t}\log u + \tfrac{C_1}{t^{\rho'}}, v, x\right) &= B_M\left(h(t) + \tfrac{1}{t}\log u, \theta_x v\right) + O\left(\tfrac{1}{t^{\rho'}}\right) + O\left(\tfrac{1}{t^{2M(\rho'-1)}}\right) \\
&= B_M\left(h(t), \theta_x v\right) + \tfrac{1}{t}\log u + O\left(\tfrac{1}{t^{\rho'}}\right) + O\left(\tfrac{1}{t^{2M(\rho'-1)}}\right). \qquad (69)
\end{aligned}$$

where $\{\theta_x : x \in \mathbb{Z}^d\}$ is the canonical set of translations acting on the potentials $v \in W_t$ as $\theta_x v(y) := v(x+y)$. It follows from (68) that eventually in t for all $x \in \Lambda_{\mathbf{0}}''$ one has the inclusion

$$\{v(x) \geq (1-\delta_1)\mathcal{A}_0 t^{\rho'-1}\} \subset \left\{ v(x) \geq B\left(h(t) + \frac{1}{t}\log u + \frac{C_1}{t^{\rho'}}, \theta_x v \right) \right\}. \tag{70}$$

Hence, going back to (66) we see after considering (69) and (70) that eventually in t it is true that

$$\begin{aligned} &\mu\left(A_1, A_2, v_{(1)} \geq B\left(h(t) + \tfrac{1}{t}\log u + \tfrac{C_1}{t^{\rho'}}, v, x_0 \right) \right) \\ =&\sum_{x\in\Lambda_{\mathbf{0}}''} \mu\left(A_2, v(x) \geq B_M\left(h(t), \theta_x v\right) + \tfrac{1}{t}\log u + O\left(\tfrac{1}{t^{\rho'}}\right) + O\left(\tfrac{1}{t^{2M(\rho'-1)}}\right) \right). \end{aligned} \tag{71}$$

From the bound (57) of Step 3, using the equality (71) we conclude that in fact there is a constant C_2 such that

$$\begin{aligned} &\liminf_{t\to\infty} n(t)\mu\left(A_1, A_2, v_{(1)} \geq B\left(h(t) + \tfrac{1}{t}\log u + \tfrac{C_1}{t^{\rho'}}, v, x_0 \right) \right) \\ \geq &\liminf_{t\to\infty} n(t) \textstyle\sum_{x\in\Lambda_{\mathbf{0}}''} \mu\left(v(x) \geq B_M\left(h(t), \theta_x v\right) + \tfrac{1}{t}\log u + \tfrac{C_2}{t^{\rho'}} + \tfrac{C_2}{t^{2M(\rho'-1)}} \right) \\ = &\liminf_{t\to\infty} n(t) |\Lambda_{\mathbf{0}}''| \mu\left(v(0) \geq B_M\left(h(t), v\right) + \tfrac{1}{t}\log u + \tfrac{C_2}{t^{\rho'}} + \tfrac{C_2}{t^{2M(\rho'-1)}} \right), \end{aligned} \tag{72}$$

where in the last equality we have used the fact that the terms under the summation in (72) are translation invariant when $x \in \Lambda_{\mathbf{0}}$ is far enough of the boundary, and that the points which do not have this property have a negligible cardinality with respect to $|\Lambda_{\mathbf{0}}''|$. Combining this with (65) we get that

$$\begin{aligned} &\liminf_{t\to\infty} n(t)\mu\left(\lambda_0 \geq h(t) + \tfrac{1}{t}\log u + \tfrac{C_1}{t^{\rho'}} \right) \\ \geq &\liminf_{t\to\infty} e^{-\gamma\frac{\tau^{\rho'}}{\rho'}} E_\mu\left[e^{-\frac{1}{\rho}\left(B_M(h(t),v) + \frac{1}{t}\log u + \frac{C_2}{t^{\rho'}} + \frac{C_2}{t^{2M(\rho'-1)}} \right)^\rho} \right]. \end{aligned} \tag{73}$$

Now, for $x > 1$ and $\epsilon < 1$ one has that

$$(x+\epsilon)^\rho = x^\rho + \rho x^{\rho-1}\epsilon + O\left(\frac{\epsilon^2}{x^{2-\rho}}\right).$$

Calling for the moment $B_M = B_M(h(t), v)$, we see from the lower bound (67) of Step 1 that this implies that

$$\begin{aligned} &\tfrac{1}{\rho}\left(B_M + \tfrac{1}{t}\log u + O\left(\tfrac{1}{t^{\rho'}}\right) + O\left(\tfrac{1}{t^{2M(\rho'-1)}}\right) \right)^\rho \\ =&\tfrac{1}{\rho}B_M^\rho + \tfrac{B_M^{\rho-1}}{t}\log u + + O\left(\tfrac{B_M^{\rho-1}}{t^{2M(\rho'-1)}}\right) + \tfrac{1}{B_M^{2-\rho}}\left(O\left(\tfrac{1}{t^2}\right) + O\left(\tfrac{1}{t^{4M(\rho'-1)}}\right) \right). \end{aligned} \tag{74}$$

Now, by the bounds (67) and (74), and the fact that $B_M \geq h(t) + 2d\kappa$, on the event A_2 we have that

$$\mathcal{A}_0\tau^{\rho'-1} \leq B_M \leq \mathcal{A}_0\tau^{\rho'-1} + \frac{C_5}{t^{\rho'-1}}. \tag{75}$$

Combining (75) with (74) we conclude that on the event A_2 one has that

$$\begin{aligned}&\tfrac{1}{\rho}\left(B_M+\tfrac{1}{t}\log u+O\left(\tfrac{1}{t^{\rho'}}\right)+O\left(\tfrac{1}{t^{2M(\rho'-1)}}\right)\right)^{\rho}\\&=\tfrac{1}{\rho}B_M^{\rho}+\alpha\log u+O\left(\tfrac{1}{t^{\rho'}}\right)+O\left(\tfrac{1}{t^{2M\rho'-(2M+1)}}\right).\end{aligned}$$

Inserting this in (73) we conclude that

$$\begin{aligned}&\liminf_{t\to\infty} n(t)\mu\left(\lambda_0\geq h(t)+\tfrac{1}{t}\log u+\tfrac{C_1}{t^{\rho'}}\right)\\&\geq\liminf_{t\to\infty} e^{-\gamma\frac{\tau^{\rho'}}{\rho'}}\tfrac{1}{u^{\alpha}}E_{\mu}\left[e^{-\frac{1}{\rho}B_M(h(t),v)+\frac{C_5}{t^{\rho'}}+\frac{C_5}{t^{2M\rho'-(2M+1)}}}\right]=\tfrac{1}{u^{\alpha}},\end{aligned}\tag{76}$$

where in the last equality we have used the definition of M given in (7), which implies that

$$2M\rho'-(2M+1)>1,$$

and the definition of B_N given in (6). By a similar argument, where in order to control the terms close to the boundary we have to use the fact that for all $x\in\Lambda_{\mathbf{0}}''$ it is true that for some constant C_6

$$\begin{aligned}&\mu\left(v(x)\geq B_M\left(h(t)+\tfrac{1}{t}\log u,\theta_x v\right)-\tfrac{C_6}{t^{\rho'}}-\tfrac{C_6}{t^{2M\rho'-(2M+1)}}\right)\\&\leq\mu\left(v(0)\geq B_M\left(h(t)+\tfrac{1}{t}\log u,v\right)-\tfrac{C_6}{t^{\rho'}}-\tfrac{C_6}{t^{2M\rho'-(2M+1)}}\right),\end{aligned}$$

we can get that

$$\limsup_{t\to\infty} n(t)\mu\left(\lambda_0\geq h(t)+\frac{1}{t}\log u-\frac{C_1}{t^{\rho'}}\right)\leq\frac{1}{u^{\alpha}}.\tag{77}$$

Inserting (76) (77) in (64) we finish the proof of the proposition.

□

6.0.5. *The truncated moments.* Here we will compute some quantities related to the moments of the random variable $Y_{\mathbf{0}}$ defined in (51) which will be later used to prove that conditions (49) and (50) are satisfied.

Lemma 8. *Consider the random variable $Y_{\mathbf{0}}$ given in (51). Assume that $0<\gamma<\gamma_2$. Then the following statements are satisfied.*

(i) *For $y>0$ we have that*

$$\lim_{t\to\infty} n(t)\left(E_{\mu}\left[Y_{\mathbf{0}}(t),Y_{\mathbf{0}}(t)\leq y\right]-\tilde{A}(t)\right)=\begin{cases}\frac{\alpha}{1-\alpha}y^{1-\alpha} & \text{if}\gamma\in(0,\gamma_1)\cup(\gamma_1,\gamma_2)\\ \log y & \text{if }\gamma=\gamma_1.\end{cases}\tag{78}$$

(ii) *For $y > 0$ we have that*

$$\lim_{t\to\infty} n(t)E_\mu\left[Y_{\mathbf{0}}^2(t), Y_{\mathbf{0}}(t) \le y\right] = \frac{\alpha}{2-\alpha}y^{2-\alpha}.$$

Here $\alpha = \alpha(\gamma,\rho)$ is defined in (9).

Proof. Part (i). We will first prove (78) for the case $0 < \gamma < \gamma_1$. Note that for each N we have that

$$E_\mu\left[Y_{\mathbf{0}}(t)1(Y_{\mathbf{0}}(t) \le y)\right] = \sum_{i=0}^{N-1} E_\mu\left[Y_{\mathbf{0}}(t)1\left(\frac{i}{N}y \le Y_{\mathbf{0}}(t) \le \frac{i+1}{N}y\right)\right]. \tag{79}$$

Now, for each $1 \le i \le N-1$ we have by Proposition 1 that

$$\begin{aligned}&\overline{\lim}_{t\to\infty} n(t)E_\mu\left[Y_{\mathbf{0}}(t)1\left(\tfrac{i}{N}y \le Y_{\mathbf{0}}(t) \le \tfrac{i+1}{N}y\right)\right]\\ &\le \tfrac{i+1}{N}y\,\overline{\lim}_{t\to\infty} n(t)\mu\left(\tfrac{i}{N}y \le Y_{\mathbf{0}}(t) \le \tfrac{i+1}{N}y\right) = \tfrac{i+1}{N}y\left(\tfrac{1}{(iy/N)^\alpha} - \tfrac{1}{((i+1)y/N)^\alpha}\right).\end{aligned} \tag{80}$$

Similarly, we can conclude that

$$\underline{\lim}_{t\to\infty} n(t)E_\mu\left[Y_{\mathbf{0}}(t)1\left(\frac{i}{N}y \le Y_{\mathbf{0}}(t) \le \frac{i+1}{N}y\right)\right] \ge \frac{i}{N}y\left(\frac{1}{(iy/N)^\alpha} - \frac{1}{((i+1)y/N)^\alpha}\right). \tag{81}$$

Combining (80) and (81) with (79) we get

$$\begin{aligned}&\textstyle\sum_{i=0}^{N-1} \frac{i}{N}y\left(\frac{1}{(iy/N)^\alpha} - \frac{1}{((i+1)y/N)^\alpha}\right) \le \underline{\lim}_{t\to\infty} n(t)E_\mu\left[Y_{\mathbf{0}}(t)1(Y_{\mathbf{0}}(t) \le y)\right]\\ &\le \overline{\lim}_{t\to\infty} n(t)E_\mu\left[Y_{\mathbf{0}}(t)1(Y_{\mathbf{0}}(t) \le y)\right] \le \textstyle\sum_{i=0}^{N-1} \frac{i+1}{N}y\left(\frac{1}{(iy/N)^\alpha} - \frac{1}{((i+1)y/N)^\alpha}\right).\end{aligned} \tag{82}$$

Now note that for $0 < \gamma < \gamma_1$, one has that $0 < \alpha < 1$. We can hence take the limit when $N \to \infty$ in (82) to deduce that

$$\lim_{t\to\infty} n(t)E_\mu\left[Y_{\mathbf{0}}(t)1(Y_{\mathbf{0}}(t) \le y)\right] = \int_0^y \frac{1}{x^\alpha}dx - y^{1-\alpha} = \frac{\alpha}{1-\alpha}y^{1-\alpha},$$

which proves (78) for the case $0 < \gamma < \gamma_1$. The proof of (78) for the case $\gamma_1 < \gamma < \gamma_2$ has to take into account that

$$E_\mu[Y_{\mathbf{0}}(t), Y_{\mathbf{0}}(t) \le y] - \tilde{A}(t) = -E_\mu[Y_{\mathbf{0}}(t), Y_{\mathbf{0}}(t) > y],$$

and then follows an analysis similar to the previous case. The proof of (78) in the case $\gamma = \gamma_1$ uses the fact that

$$E_\mu[Y_{\mathbf{0}}(t), Y_{\mathbf{0}}(t) \le y] - \tilde{A}(t) = \begin{cases} E_\mu[Y_{\mathbf{0}}(t), 1 < Y_{\mathbf{0}}(t) \le y] & \text{if} \qquad y \ge 1\\ -E_\mu[Y_{\mathbf{0}}(t), y < Y_{\mathbf{0}}(t) < 1] & \text{if} \qquad y < 1,\end{cases}$$

which then enables one to prove that

$$\lim_{N\to\infty}\Big(E_\mu[Y_{\mathbf{0}}(t), Y_{\mathbf{0}}(t)\le y]-\tilde{A}(t)\Big)=\int_1^y\frac{1}{x}dx=\log y.$$

Part (ii). In analogy to the inequalities (82), we can conclude that

$$\begin{aligned}&\textstyle\sum_{i=0}^{N-1}\left(\frac{i}{N}y\right)^2\left(\frac{1}{(iy/N)^\alpha}-\frac{1}{((i+1)y/N)^\alpha}\right)\le\underline{\lim}_{t\to\infty}n(t)E_\mu\left[Y_{\mathbf{0}}^2(t)1(Y_{\mathbf{0}}(t)\le y)\right]\\&\le\overline{\lim}_{t\to\infty}n(t)E_\mu\left[Y_{\mathbf{0}}^2(t)1(Y_{\mathbf{0}}(t)\le y)\right]\le\textstyle\sum_{i=0}^{N-1}\left(\frac{i+1}{N}y\right)^2\left(\frac{1}{(iy/N)^\alpha}-\frac{1}{((i+1)y/N)^\alpha}\right).\end{aligned}$$

As in (82), since $0<\gamma<\gamma_2$ implies that $0<\alpha<2$, we can take the limit as $N\to\infty$ above to deduce that

$$\lim_{t\to\infty}n(t)E_\mu\left[Y_{\mathbf{0}}^2(t)1(Y_{\mathbf{0}}(t)\le y)\right]=\int_0^y\frac{1}{x^{\alpha-1}}dx-y^{2-\alpha}=\frac{\alpha}{2-\alpha}y^{2-\alpha},$$

□

6.0.6. *The parameters of the infinite divisible law.* Here we will show that the set of i.i.d. random variables $\{Y_{\mathbf{i}}:\mathbf{i}\in\mathcal{I}\}$ satisfy conditions (49) and (50) of part (ii) of Theorem 5 with

$$\sigma^2=\lim_{y\to0}\lim_{t\to\infty}n(t)Var_t(Z_y(t))=0,\tag{83}$$

and

$$\nu:=\begin{cases}\frac{\alpha\pi}{2\cos\frac{\alpha\pi}{2}} & \text{if}\quad\gamma\in(0,\gamma_1)\cup(\gamma_1,\gamma_2)\\ 0 & \text{if}\quad\gamma=\gamma_1.\end{cases}\tag{84}$$

The proof of (83) is a direct consequence of part (ii) of Lemma 8. Since the proof of (84) is completely analogous to the proofs of Propositions 6.4 and 6.5 of [7], we will just give an outline here. Note that by (50), part (i) of Lemma 8 and (53), we should have

$$\nu=\frac{\alpha}{1-\alpha}y^{1-\alpha}+\alpha\int_y^\infty\frac{x^{-\alpha}}{1+x^2}dx-\alpha\int_0^y\frac{x^{2-\alpha}}{1+x^2}dx.$$

Using the identity $\int_0^\infty\frac{x^{-\alpha}}{1+x^2}dx=\frac{\pi}{2\cos\frac{\alpha\pi}{2}}$ in the case $\gamma\neq\gamma_1$, we can obtain (84) for that case. A similar analysis gives the case $\gamma=\gamma_1$.

6.0.7. *Conclusion.* Gathering (53), (83) and (84), into Theorem 5, and the observation that it is enough to find the limiting law in (46), we conclude that for $0 < \gamma < \gamma_2$,

$$\lim_{t\to\infty} \frac{1}{s(t)} m_L(t) = Z,$$

where the convergence is in distribution and Z is an infinite divisible distribution with characteristic function

$$\phi(t) = \exp\left\{ i\nu u + \alpha \int_0^\infty \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) \frac{dx}{x^{\alpha+1}} \right\}, \tag{85}$$

with ν defined in (84). Now, by Theorem 6.6 of [7], we know that any infinite divisible distribution corresponding to (85) has the canonical representation given in (18). This finishes the proof of Theorem 3.

7. Annealed asymptotics

In this section we will prove prove (5) of Theorem 1.

Proposition 2. *Consider the solution $\{m(x,t) : x \in \mathbb{Z}^d, t \geq 0\}$ of the parabolic Anderson equation (1) with Weibull potential of parameter $\rho > 1$. Then*

$$\langle m(0,t) \rangle \sim \left(\frac{\pi}{\rho - 1} \right)^{1/2} t^{1-\frac{\rho'}{2}} e^{\frac{t^{\rho'}}{\rho'} - 2d(\kappa t - t^{2-\rho'})}.$$

Define the *cumulant generating function* by,

$$H(t) = \log \left\langle e^{v(0)t} \right\rangle, \qquad t \geq 0.$$

Using the independence of the coordinates of the potential v, note that,

$$\langle m(0,t) \rangle = E_0 \left[e^{\sum_{x\in\mathbb{Z}^d} H(\mathcal{L}(t,x))} \right],$$

where E_0 is the expectation defined by the law P_0 of a simple symmetric random walk on $\mathbb{Z}^d$, starting from 0, of total jump rate 1 and for each $t \geq 0$ and site $x \in \mathbb{Z}^d$, $\mathcal{L}(t,x)$ is the total time spent by the random walk in the time interval $[0,t]$ at x.

The basis of our proof of Theorem 1 will be the following result.

Proposition 3. *Consider the solution $\{m(x,t) : x \in \mathbb{Z}^d, t \geq 0\}$ of the parabolic Anderson equation (1) with Weibull potential of parameter $\rho > 1$. Then*

$$\langle m(0,t) \rangle \sim \left(\frac{\pi}{\rho - 1} \right)^{1/2} e^{\frac{t^{\rho'}}{\rho'} - 2d\kappa t} \sum_{n=1}^{\infty} \frac{\kappa^{n-1}}{t^{(\rho'-1)n}} \sum_{\gamma\in\mathbb{P}^n(0)} \sum_{i=1}^{k} \frac{t^{\rho'\left(n_i - \frac{1}{2}\right)}}{(n_i - 1)!}. \tag{86}$$

7.0.8. *Preliminary estimates.* To prove Proposition 3, we will need the precise asymptotics of the cumulant generating function, given by the following lemma.

Lemma 9. *For $t \geq 0$,*

$$H(t) = \frac{t^{\rho'}}{\rho'} + \frac{\rho'}{2}\log t + \frac{1}{2}\log\frac{\pi}{\rho - 1} + \varepsilon(t),$$

where $\lim_{t\to\infty} \varepsilon(t) = 0$.

Proof. Note that

$$\left\langle e^{tv(0)} \right\rangle = \int_0^\infty u^{\rho-1} e^{-u^\rho/\rho} e^{ut} du.$$

Making the variable change $u = wt^{1/(\rho-1)}$, this becomes,

$$t^{\rho'} \int_0^\infty w^{\rho-1} e^{-t^{\rho'}\left(\frac{w^\rho}{\rho} - w\right)} dw. \tag{87}$$

The function $f(w) := \frac{w^\rho}{\rho} - w$, attains its minimum value $-1/\alpha'$ at $w = 1$, having the expansion,

$$f(w) = -\frac{1}{\rho'} + (w-1)^2(\rho - 1) + (w-1)^3 f^{(3)}(\bar{w}), \tag{88}$$

where $\bar{w}$ is between 1 and w. Now, let $0 < \epsilon < 1$, and make the decomposition,

$$\int_0^\infty w^{\rho-1} e^{-t^{\rho'}\left(\frac{w^\rho}{\rho} - w\right)} dw = A_1 + A_2 + A_3, \tag{89}$$

where $A_1 = \int_0^{1-\epsilon} w^{\rho-1} e^{-t^{\rho'}\left(\frac{w^\rho}{\rho} - w\right)} dw$, $A_2 = \int_{1-\epsilon}^{1+\epsilon} w^{\rho-1} e^{-t^{\rho'}\left(\frac{w^\rho}{\rho} - w\right)} dw$ and $A_3 = \int_{1+\epsilon}^\infty w^{\rho-1} e^{-t^{\rho'}\left(\frac{w^\rho}{\rho} - w\right)} dw$. It is easy to check that there exists a constant $C > 0$ such that, $A_1 \leq Ce^{-t^{\rho'} f(1+\epsilon)}$ and $A_2 \leq Ce^{-t^{\rho'} f(1+\epsilon)}$. Furthermore, from the expansion (88), we see that,

$$A_3 \leq (1+\epsilon)^{\rho-1} e^{c\epsilon^3} e^{\frac{t^{\rho'}}{\rho'}} \int_{-\infty}^\infty e^{-(\rho-1)t^{\rho'}x^2} dx = (1+\epsilon)^{\rho-1} e^{c\epsilon^3} e^{\frac{t^{\rho'}}{\rho'}} \sqrt{\frac{\pi}{(\rho-1)t^{\rho'}}},$$

for $c = |f^{(3)}(2)|$. Similarly we have that,

$$A_3 \geq (1-\epsilon)^{\rho-1} e^{-c\epsilon^3} e^{\frac{t^{\rho'}}{\rho'}} \left(\sqrt{\frac{\pi}{(\rho-1)t^{\rho'}}} - O(e^{-t^{\rho'}\epsilon^2}) \right).$$

Substituting these estimates for A_1, A_2 and A_3 in (89) and this in (87), and choosing $\epsilon = t^{-\gamma}$ for $\gamma > 0$ small enough, we finish the proof of the lemma. □

Let us finish this section with the following elementary computation.

Lemma 10. *Let $k \geq 1$ and $n_1, n_2, \ldots, n_k$ be natural numbers larger than 0. For $u \geq 0$, let $J_{n_1,\ldots,n_k}(u) = \int_{\sum_{i=1}^k x_k < u} x_1^{n_1-1} \cdots x_k^{n_k-1} dx_1 \cdots dx_k$. Then,*

$$\int_0^\infty J_{n_1,\ldots,n_k}(u) e^{-u} du = \prod_{i=1}^k (n_i - 1)!.$$

7.0.9. *Path decomposition of annealed first moment.* For $j \geq 1$, call τ_j the time of the j-th jump of the random walk. Note that these random times are independent exponential random variables of rate $2d\kappa$. Also,

$$\langle m(0,t) \rangle = \sum_{\gamma \in \mathbb{P}(0)} E_0 \left[e^{\sum_{x \in \mathbb{Z}^d} H(\mathcal{L}(t,x))} C_\gamma \right], \tag{90}$$

where C_γ is the event that in the time interval $[0,t]$, the random walk follows the path γ. Let us examine a single term A_γ, of the summation in (90) corresponding to a path $\gamma = \{x_1 = 0, x_2, \ldots, x_n\}$, visiting sites $\{y_1 = 0, y_2, \ldots, y_k\}$. Without loss of generality, we assume that $y_k = x_n$, so that y_k is the last visited site. For each $1 \leq i \leq k$, let us call n_i, the number of times the path visits site y_i. Furthermore, let $i_1, \ldots, i_{n_i}$ be the set of indices of the set $\{1, 2, \ldots, n-1\}$ such that $x_{1_1} = \cdots = x_{i_{n_i}} = y_i$. Furthermore, note that $C_\gamma = \bar{C}_\gamma \cap \{\tau_1 + \cdots + \tau_{n-1} < t\} \cap \{\tau_n \geq t - \tau_1 - \cdots \tau_{n-1}\}$, where $\bar{C}_\gamma = \cap_{j=1}^{n-1} \{X_{\tau_j} = x_{j+1}\}$. Therefore, if $S_{n-1} := \{s_1 + \cdots + s_{n-1} < t\} \subset \mathbb{R}^{n-1}$ then,

$$A_\gamma = \frac{(2d\kappa)^{n-1}}{(2d)^{n-1}} \int_{S_{n-1}} e^{-2d\kappa(s_1+\cdots+s_{n-1})} e^{-2d\kappa(t-s_1-\cdots-s_{n-1})} e^{H_1+\cdots+H_k} ds_1 \cdots ds_{n-1},$$

where $H_i := H(s_{i_1} + \cdots + s_{i_{n_i}})$, for $1 \leq i \leq k-1$, while $H_k := H(t - v_1 - \cdots - v_{k-1})$, with $v_i = s_{i_1} + \cdots + s_{i_{n_i}}$. Thus,

$$\begin{aligned} A_\gamma &= \kappa^{n-1} e^{-2d\kappa t} \int_{T_k} \frac{v_1^{n_1-1}}{(n_1-1)!} \cdots \frac{v_k^{n_k-1}}{(n_k-1)!} e^{H(v_1)+\cdots+H(v_{k-1})+H_k} dv_1 \cdots dv_k \\ &= \kappa^{n-1} t^n e^{-2d\kappa t} \frac{1}{\prod_{i=1}^k (n_i-1)!} \int_{T_k'} u_1^{n_1-1} \cdots u_k^{n_k-1} e^{H(tu_1)+\cdots+H(tu_{k-1})+H(t\bar{u}_k)} du_1 \cdots du_k, \end{aligned} \tag{91}$$

where $T_k := \{v_1 + \cdots + v_k < t\}$, $T_k' := \{u_1 + \cdots + u_k < 1\}$, $\bar{u}_k := 1 - u_1 - \cdots - u_{k-1}$, in the second equality we made the variable change $u_i = tv_i$ and we have used the fact that $n_1 + \cdots + n_k = n$. Let now $0 < \delta < 1/2$, and define $W_k = \{u_1 + \cdots + u_k < 1, \max_{1 \leq i \leq k} u_i < 1 - \delta\}$ and $V_k = T_k' - W_k$.

7.0.10. *Asymptotic lower bound.* Here we compute an asymptotic lower bound for A_γ. Note that $H(t\bar{u}_k) \geq H(tu_k)$. Hence,

$$A_\gamma \geq \kappa^{n-1} t^n e^{-2d\kappa t} \frac{1}{\prod_{i=1}^k (n_i-1)!} \int_{V_k} u_1^{n_1-1} \cdots u_k^{n_k-1} e^{H(tu_1)+\cdots+H(tu_k)} du_1 \cdots du_k.$$

Now, note that $V_k = \cup_{j=1}^k V_{k,j}$, where the union is disjoint and $V_{k,j} := \{u_1 + \cdots + u_k < 1, u_j \geq 1 - \delta\}$. Therefore, $A_\gamma \geq t^n e^{-2dt} \frac{1}{\prod_{i=1}^k (n_i-1)!} \sum_{j=1}^k I_{k,j}$ where,

$$I_{k,j} := \int_{V_{k,j}} u_1^{n_1-1} \cdots u_k^{n_k-1} e^{H(tu_1)+\cdots+H(tu_k)} du_1 \cdots du_k.$$

By symmetry it is enough to examine $I_{k,1}$. From lemma 9 and (91) we obtain,

$$I_{k,1} \geq t^{\frac{\rho'}{2}} \left(\frac{\pi}{\rho-1}\right)^{1/2} \int_{V_{k,1}} u_1^{n_1-1+\frac{\rho'}{2}} u_2^{n_1-1} \cdots u_k^{n_k-1} e^{\frac{t^{\rho'}}{\rho'} u_1^{\rho'} + \varepsilon(tu_1)} du_1 \cdots du_k.$$

Now, the variable change $u_1' := 1-u_1$ transforms the integral in the above expression to,

$$I'_{k,1} := \int_0^\delta \int_{T_{k,1}} (1-u_1')^{n_1-1+\frac{\rho'}{2}} u_2^{n_2-1} \cdots u_k^{n_k-1} e^{\frac{t^{\rho'}}{\rho'}(1-u_1')^{\rho'} + \varepsilon(t(1-u_1'))} du_1' du_2 \cdots du_k.$$

where $T_{k,1} := \{u_2 + \cdots u_k < u_1'\}$. Note that, $(1-u_1')^{\rho'} = 1 - \rho' u_1' + \rho'(\rho'-1)\bar{u}^2/2$, for some $0 \leq \bar{u} \leq u_1'$. Therefore,

$$1 - \rho' u_1' \leq (1-u_1')^{\rho'} \leq 1 - \rho' u_1' + \rho'(\rho'-1) u_1'^2/2. \tag{92}$$

Then, we get a lower bound,

$$\begin{aligned} I'_{k,1} &\geq e^{\frac{t^{\rho'}}{\rho'}} \int_0^\delta J_{\gamma,1}(x)(1-x)^{n_1-1+\frac{\rho'}{2}} e^{-t^{\rho'}x + \varepsilon(t(1-x))} dx \\ &= \tfrac{1}{t^{\rho'}} e^{\frac{t^{\rho'}}{\rho'}} \int_0^{\delta t^{\rho'}} J_{\gamma,1}\left(\tfrac{y}{t^{\rho'}}\right) \left(1 - \tfrac{y}{t^{\rho'}}\right)^{n_1-1+\frac{\rho'}{2}} e^{-y + \varepsilon(t(1-y/t^{\rho'}))} dy. \end{aligned} \tag{93}$$

where for $1 \leq j \leq k$, we define $J_{\gamma,j}(x) := \int_{\sum_{i\neq j} u_i < x} \prod_{i\neq j} u_i^{n_i-1} du_i$. Now,

$$J_{\gamma,1}\left(\frac{y}{t^{\rho'}}\right) = t^{-\rho'(n-n_1)} J_\gamma(y). \tag{94}$$

Therefore, by the dominated convergence theorem, we see that the right-hand side of (93) is asymptotically equal to, $\frac{1}{t^{\rho'(n+1-n_1)}} e^{\frac{t^{\rho'}}{\rho'}} \int_0^\infty J_{\gamma,1}(x) e^{-x} dx$. We therefore, conclude that $I_{k,1}$ is asymptotically lower bounded by,

$$\left(\frac{\pi}{\rho-1}\right)^{1/2} \frac{1}{t^{\rho'\left(n+\frac{1}{2}-n_1\right)}} e^{\frac{t^{\rho'}}{\rho'}} \int_0^\infty J_{\gamma,1}(x) e^{-x} dx.$$

and that A_γ is asymptotically lower bounded by,

$$\frac{\left(\frac{\pi}{\rho-1}\right)^{1/2}}{\prod_{i=1}^k (n_i-1)!} t^n e^{-2dt} \sum_{j=1}^k \frac{1}{t^{\rho'\left(n+\frac{1}{2}-n_j\right)}} e^{\frac{t^{\rho'}}{\rho'}} K_{\gamma,j},$$

where $K_{\gamma,j} := \int_0^\infty J_{\gamma,j}(u) e^{-u} du$. Using lemma 10, we obtain the desired asymptotic lower bound.

7.0.11. *Asymptotic upper bound.* Here we will obtain an asymptotic upper bound for A_γ. First, let us examine the integral,

$$\bar{I}_k := \int_{\bar{W}_k} u_1^{n_1-1} \cdots u_k^{n_k-1} e^{H(tu_1)+\cdots+H(tu_{k-1})+H(t\bar{u}_k)} du_1 \cdots du_k,$$

where $\bar{W}_k := \{u_1 + \cdots + u_k < 1, \max_{1\le i\le k-1} u_i < 1-\delta, \bar{u}_k < 1-\delta\}$. Since on $\bar{W}_k$ we have $\max_{1\le j\le k-1} u_j < 1-\delta$ and $\bar{u}_k < 1-\delta$, the following inequality is satisfied, $\frac{1}{\rho'} t^{\rho'}(u_1^{\rho'} + \cdots + u_{k-1}^{\rho'} + \bar{u}_k^{\rho'}) \le \frac{1}{\rho'}(1-\delta)^{\rho'-1} t^{\rho'}$. Hence, the integral $\bar{I}_k$ is upper bounded by,

$$(t(1-\delta))^{k\frac{\rho'}{2}} \left(\frac{\pi}{\rho-1}\right)^{k/2} e^{(1-\delta)^{\rho'-1}\frac{t^{\rho'}}{\rho'} + k\varepsilon(t(1-\delta))} \int_{W_k} u_1^{n_1-1} \cdots u_k^{n_k-1} du_1 \cdots du_k.$$

Therefore, $\bar{I}_k \le c(k) e^{(1-\delta)^{\rho'-1}\frac{t^{\rho'}}{\rho'}}$, for some constant $c(k)$. Let us now define $\bar{V}_k := T_k' - \bar{W}_k$ and note that $\bar{V}_k = \cup_{j=1}^k \bar{V}_{k,j}$ where the union is disjoint and $\bar{V}_{k,j} := \{u_1 + \cdots + u_k < 1, u_j \ge 1-\delta\}$ for $1 \le j \le k-1$ while $\bar{V}_{k,k} := \{u_1 + \cdots + u_k < 1, \bar{u}_k \ge 1-\delta\}$. Then if,

$$\bar{I}_{k,j} := \int_{\bar{V}_{k,j}} u_1^{n_1-1} \cdots u_k^{n_k-1} e^{H(tu_1)+\cdots+H(tu_{k-1})+H(t\bar{u}_k)} du_1 \cdots du_k,$$

it follows that,

$$A_\gamma \le c(k) e^{(1-\delta)^{\rho'-1}\frac{t^{\rho'}}{\rho'}} + t^n e^{-2dt} \frac{1}{\prod_{i=1}^k (n_i-1)!} \sum_{j=1}^k \bar{I}_{k,j}. \tag{95}$$

Then, we need to upper bound the integrals $\bar{I}_{k,j}$. Define $u_1' = 1 - u_1$. By the subadditivity of the cumulant generating function, note that $H(tu_2) + \cdots + H(tu_{k-1}) + H(t\bar{u}_k) \le H(t(u_2 + \cdots + u_{k-1} + \bar{u}_k)) = H(t(1-u_1))$. Hence, $\bar{I}_{k,1}$ is upper bounded by,

$$t^{\frac{\rho'}{2}} \left(\frac{\pi}{\rho-1}\right)^{1/2} \int_{\bar{V}_{k,1}} u_1^{n_1-1+\frac{\rho'}{2}} u_2^{n_1-1} \cdots u_k^{n_k-1} e^{\frac{t^{\rho'}}{\rho'} u_1^{\rho'} + H(t(1-u_1)) + \varepsilon(tu_1)} du_1 \cdots du_k.$$

But by the second inequality of display (92), the integral in the above expression is upper bounded by,

$$\begin{aligned} &e^{\frac{t^{\rho'}}{\rho'}} \int_0^\delta J_{\gamma,1}(x)(1-x)^{n_1-1+\frac{\rho'}{2}} e^{-t^{\rho'}x + t^{\rho'}\rho'(\rho'-1)\frac{x^2}{2} + H(tx) + \varepsilon(t(1-x))} dx \\ &= \tfrac{1}{t^{\rho'}} e^{\frac{t^{\rho'}}{\rho'}} \int_0^{\delta t^{\rho'}} J_{\gamma,1}\left(\tfrac{y}{t^{\rho'}}\right) e^{-y + \rho'(\rho'-1)\frac{y^2}{2t^{\rho'}} + H(t^{1-\rho'}) + \varepsilon(t(1-y/t^{\rho'}))} dy. \end{aligned}$$

By (94) and the dominated convergence theorem, this is asymptotically equivalent as $t \to \infty$ to $\frac{1}{t^{\rho'(n+1-n_1)}} e^{\frac{t^{\rho'}}{\rho'}} K_{\gamma,1}$. This provides an asymptotic upper bound for $\bar{I}_{k,1}$. A

similar argument gives us the asymptotic upper bounds $\frac{1}{t^{\rho'(n+1-n_1)}} e^{\frac{t^{\rho'}}{\rho'}} K_{\gamma,j}$, for $\bar{I}_{k,j}$, $2 \leq j \leq k$. Combining these estimates with (95), and using lemma 10, finishes the proof of the asymptotic upper bound.

7.0.12. *Proof of Proposition 2.* Define $\mathbb{Q}^n(0,0)$ as the set of paths $\gamma \in \mathbb{P}^n(0,0)$ such that 0 is visited n_1 times with $n_1 > n/2$. Note that if n is even this set is empty, whereas for $n > 1$ odd all of these paths start and end at 0, and they are of the form $\gamma = \{x_1 = 0, x_2, x_3, \ldots, x_{n-1}, x_n = 0\}$, with $x_i = 0$ for i odd, $1 \leq i \leq n$. Let us express the series

$$\sum_{n=1}^{\infty} \frac{1}{t^{(\rho'-1)n}} \sum_{\gamma \in \mathbb{P}^n(0)} \sum_{i=1}^{k} \frac{t^{\rho'\left(n_i - \frac{1}{2}\right)}}{(n_i - 1)!},$$

of the right-hand side of display (86) as, $S_1 + S_2 + S_3$, where

$$\begin{aligned} S_1 &:= \textstyle\sum_{n=1}^{\infty} \frac{1}{t^{(\rho'-1)n}} \sum_{\gamma \in \mathbb{Q}^n(0)} \frac{t^{\rho'\left(n_1 - \frac{1}{2}\right)}}{(n_1-1)!} \\ S_2 &:= \textstyle\sum_{n=2}^{\infty} \frac{1}{t^{(\rho'-1)n}} \sum_{\gamma \in \mathbb{Q}^n(0)} \sum_{i=2}^{k} \frac{t^{\rho'\left(n_i - \frac{1}{2}\right)}}{(n_i-1)!} \\ S_3 &:= \textstyle\sum_{n=2}^{\infty} \frac{1}{t^{(\rho'-1)n}} \sum_{\gamma \in \mathbb{R}^n(0)} \sum_{i=1}^{k} \frac{t^{\rho'\left(n_i - \frac{1}{2}\right)}}{(n_i-1)!}, \end{aligned}$$

where $\mathbb{R}^n(0) := \mathbb{P}^n(0) - \mathbb{Q}^n(0)$. We will show that only S_1 contributes to the final result.

By our previous remarks, note that the summation in S_1 runs only over odd values of $n = 2m + 1$, $m \geq 0$. Furthermore, $n_1 = m + 1$ and $|\mathbb{Q}^n(0)| = (2d)^m$. Therefore,

$$S_1 = \sum_{m=0}^{\infty} \frac{1}{t^{(\rho'-1)(2m+1)}} (2d)^m \frac{t^{\rho'\left(m+\frac{1}{2}\right)}}{m!} = t^{1-\frac{\rho'}{2}} \sum_{m=0}^{\infty} \frac{(2dt^{2-\rho'})^m}{m!} = t^{1-\frac{\rho'}{2}} e^{2dt^{2-\rho'}}.$$

Let us next examine S_2. Note that for $\gamma \in \mathbb{Q}^n(0)$, we have $n_i = 1$ for $2 \leq i \leq k$. Again, only the odd terms in the series count, and we have,

$$S_2 = t^{1-\frac{\rho'}{2}} \sum_{m=1}^{\infty} \left(\frac{2d}{t^{2(\rho'-1)}} \right)^m = t^{1-\frac{\rho'}{2}} \frac{2d}{t^{2(\rho'-1)} - 2d} \ll S_1.$$

Now, let $c < 1/\rho'$. Let us write $S_3 = S_3' + S_3''$, where

$$\begin{aligned} S_3' &:= \textstyle\sum_{n=2}^{\infty} \frac{1}{t^{(\rho'-1)n}} \sum_{\gamma \in \mathbb{R}^n(0)} \sum_{i=1}^{k} 1(n_i \geq cn) \frac{t^{\rho'\left(n_i - \frac{1}{2}\right)}}{(n_i-1)!} \\ S_3'' &:= \textstyle\sum_{n=2}^{\infty} \frac{1}{t^{(\rho'-1)n}} \sum_{\gamma \in \mathbb{R}^n(0)} \sum_{i=1}^{k} 1(n_i < cn) \frac{t^{\rho'\left(n_i - \frac{1}{2}\right)}}{(n_i-1)!}. \end{aligned}$$

Using the bounds $k \leq n$ and $|\mathbb{Q}^n(0)| \leq (2d)^n$, note that,

$$S_3'' \leq \sum_{n=2}^{\infty} \frac{1}{t^{(\rho'-1)n}} n(2d)^n t^{\rho'\left(cn-\frac{1}{2}\right)} = t^{-\frac{\rho'}{2}} \sum_{n=2}^{\infty} n \left(2dt^{2(1+\rho(c-1))}\right)^n \ll S_1,$$

since $1+\rho(c-1)<0$. To estimate S_3', note that all the paths in $\mathbb{R}^n(0)$, are such that $n_i \leq n/2$, for $1 \leq i \leq k$. Then,

$$S_3' \leq t^{-\frac{\rho'}{2}} \sum_{n=2}^{\infty} \frac{(2d)^n}{t^{(\rho'-1)n}} \sum_{m=\lfloor cn \rfloor}^{\lfloor n/2 \rfloor} \frac{t^{\rho' m}}{(m-1)!} \leq t^{-\frac{\rho'}{2}} \sum_{m=0}^{\infty} \sum_{n=f(m)}^{\infty} \frac{(2d)^n}{t^{(\rho'-1)n}} \frac{t^{\rho' m}}{(m-1)!}.$$

Now, performing summation by parts, we see that,

$$\begin{aligned} \textstyle\sum_{m=0}^{\infty} \sum_{n=f(m)}^{\infty} \frac{(2d)^n}{t^{(\rho'-1)n}} \frac{t^{\rho' m}}{(m-1)!} &= \left(\tfrac{2d}{t^{\rho'-1}}\right)^2 + \textstyle\sum_{m=1}^{\infty} \left(\tfrac{2d}{t^{\rho'-1}}\right)^{2m} \frac{t^{\rho' m}}{(m-1)!} \\ &= (2d)^2 t^{2-2\rho'} + (2d)^2 t^{2-\rho'} e^{2dt^{2-\rho'}}. \end{aligned}$$

Hence,

$$S_3' \leq (2d)^2 t^{2-5\rho'/2} + (2d)^2 t^{2-3\rho'/2} e^{2dt^{2-\rho'}} \ll t^{1-\rho'/2} e^{2dt^{2-\rho'}}.$$

8. Asymptotic expansion of the scaling function for $1 < \rho < (3+\sqrt{17})/2$

Here we will prove Corollary 1, giving an expansion of the scaling function h defined in (13) for $1 < \rho < (3+\sqrt{17})/2$. We make the calculations only up to the value $(3+\sqrt{17})/2$ because for ρ larger than this number extra terms in the expansion of h have to be computed, and in order to keep the length of this section limited, we have decided to stop there. There seems to be no straightforward interpretation on the appearance of this number. On the other hand, as it will be shown, transitions in the behavior of h occur for $\rho = 2, 3$ and the number $(3+\sqrt{17})/2$. Above this last value, transitions should appear at integer values of ρ, and additionally we expect that for some other non-integer values of ρ. To prove part (i) of Corollary 1, note that when $1 < \rho < 2$, we have that $\rho' > 2$. Therefore, $M = 1$ [c.f. (7)]. It follows from (17) of Theorem 2, that $h_1(t) = O\left(\frac{1}{t}\right)$. This together with (14) of Theorem 2, shows that

$$h(t) = \mathcal{A}_0 \tau^{\rho'-1} - 2d\kappa + O\left(\frac{1}{t^{\rho'-1}}\right),$$

from where using that $\rho' > 2$, part (i) of Corollary 1 follows. Let us now prove parts (ii) and (iii) of Corollary 1. Note that for ϵ small enough

$$E_\mu\left[e^{-\frac{1}{\rho}B_1(h_0,v)^\rho}1(\max_{e\in E}v(e)>(1-\epsilon)\mathcal{A}_0\tau^{\rho'-1})\right]$$
$$\leq e^{-\frac{\mathcal{A}_0^\rho\tau^{\rho'}}{1+\frac{1}{2d}}}\mu(\max_{e\in E}v(e)>(1-\epsilon)\mathcal{A}_0\tau^{\rho'-1}) = e^{-\frac{\mathcal{A}_0^\rho\tau^{\rho'}}{1+\frac{1}{2d}}}e^{-\frac{\gamma}{\rho'}(1-\epsilon)^\rho\tau^{\rho'}}$$
$$= o\left(e^{-\frac{\gamma}{\rho'}\tau^{\rho'}}\right), \tag{96}$$

where in the inequality we have used the fact that $B_1(h_0,v)\geq\frac{1}{1+\frac{1}{2d}}\mathcal{A}_0\tau^{\rho'-1}$ and in the last equality that ϵ is small enough.

It follows from (14) that

$$h(t)=\mathcal{A}_0\tau^{\rho'-1}-2d\kappa+h_1(t)+O\left(\frac{1}{t^{3(\rho'-1)}}\right).$$

Now,

$$h_1(t):=\frac{1}{\mathcal{A}_0^{\rho-1}}\left(\frac{\gamma}{\rho'}\tau^{\rho'-1}(t)+\frac{1}{\tau(t)}\log E_\mu\left[e^{-\frac{1}{\rho}B_1(h_0,v)^\rho}\right]\right) \tag{97}$$

and for $s\geq 0$,

$$B_1(s,v)=\frac{s+2d\kappa}{1+\frac{\kappa}{2d\kappa+s}\sum_{e:|e|_1=1}\frac{\kappa}{2d\kappa+(s-v(e))_+}}.$$

In analogy with the proof of part (i) of Lemma 6, but going to a higher order Taylor expansion, we see that when $\sup_{e\in E}v_0(e)\leq\mathcal{A}_0\tau^{\rho'-1}(1-\epsilon)$ for some $\epsilon>0$, we have

$$\frac{1}{\rho}B_1^\rho(h_0(t),v)=\frac{\gamma}{\rho'}\tau^{\rho'}-\frac{\gamma\rho}{\rho'\mathcal{A}_0}\kappa^2\sum_{e\in E}\frac{\tau}{\mathcal{A}_0\tau^{\rho'-1}-v_0(e)}+O\left(\frac{1}{\tau^{3\rho'-4}}\right).$$

Now, note that for all $x>0$ one has that

$$\frac{\tau}{\mathcal{A}_0\tau^{\rho'-1}-x}=\mathcal{A}_0^{-1}\tau^{2-\rho'}+\mathcal{A}_0^{-2}x\tau^{3-2\rho'}+\mathcal{A}_0^{-3}x^2\frac{\tau^{4-3\rho'}}{1-\frac{x}{\tau^{\rho'-1}}}$$
$$=\mathcal{A}_0^{-1}\tau^{2-\rho'}+\mathcal{A}_0^{-2}x\tau^{3-2\rho'}+x^2O\left(\tau^{4-3\rho'}\right)$$

It follows that

$$E_\mu\left[e^{-\frac{1}{\rho}B_1(h_0,v)^\rho}1(\max_{e\in E}v(e)\leq(1-\epsilon)\mathcal{A}_0\tau^{\rho'-1})\right]=e^{-\frac{\gamma}{\rho'}\tau^{\rho'}+2d\frac{\gamma\rho}{\rho'}\mathcal{A}_0^{-2}\kappa^2\tau^{2-\rho'}}$$
$$\times\left(\int_0^{(1-\epsilon)\mathcal{A}_0\tau^{\rho'-1}}e^{K_1x\tau^{3-2\rho'}+x^2O\left(\tau^{4-3\rho'}\right)}e^{-\frac{1}{\rho}x^\rho}x^{\rho-1}dx\right)^{2d}, \tag{98}$$

where

$$K_1:=\frac{\gamma\rho}{\rho'}\kappa^2\mathcal{A}_0^{-3}$$

Now, for $2\leq\rho<3$ we claim that

$$A := \int_0^{(1-\epsilon)\mathcal{A}_0\tau^{\rho'-1}} e^{K_1 x\tau^{3-2\rho'}+x^2O\left(\tau^{4-3\rho'}\right)} e^{-\frac{1}{\rho}x^\rho} x^{\rho-1}dx = 1+o(1). \tag{99}$$

To prove (99), for $\delta < 2\rho' - 3$ and t large enough, write

$$A = \int_0^{\tau^\delta} g(x)dx + \int_{(1-\epsilon)\mathcal{A}_0\tau^{\rho'-1}}^{\tau^\delta} g(x)dx \tag{100}$$

where

$$g(x) := e^{K_1 x\tau^{3-2\rho'}+x^2O\left(\tau^{4-3\rho'}\right)} e^{-\frac{1}{\rho}x^\rho} x^{\rho-1}dx.$$

Note that for $0 \le x \le \tau^\delta$, since $x\tau^{3-2\rho'} = o(1)$ and also $x^2\tau^{4-3\rho'} = o(1)$, with $\lim_{t\to\infty}\sup_{0\le x\le\tau^\delta} o(1) = 0$, we have that

$$g(x) = e^{o(1)}e^{-\frac{1}{\rho}x^\rho}x^{\rho-1} = e^{-\frac{1}{\rho}x^\rho}x^{\rho-1} + o(1).$$

Therefore the first integral in (100) satisfies

$$\int_0^{\tau^\delta} g(x)dx = 1+o(1).$$

For the second integral in (100), remark that

$$\sup_{\tau^\delta\le x\le(1-\epsilon)\mathcal{A}_0\tau^{\rho'-1}} x^2O\left(\tau^{4-3\rho'}\right) = o(1).$$

On the other hand, the function

$$u(x) := K_1 x\tau^{3-2\rho'} - \frac{1}{\rho}x^\rho,$$

is decreasing in the interval $[\tau^\delta, (1-\epsilon)\mathcal{A}_0\tau^{\rho'-1}]$, so that

$$\int_{(1-\epsilon)\mathcal{A}_0\tau^{\rho'-1}}^{\tau^\delta} g(x)dx = O\left(e^{-C\tau^{\delta\rho}}\right) = o(1),$$

for some constant $C > 0$. This finishes the proof of (99). Substituting now (99) into (98) we conclude that

$$E_\mu\left[e^{-\frac{1}{\rho}B_1(h_0,v)^\rho}1(\max_{e\in E} v(e) \le (1-\epsilon)\mathcal{A}_0\tau^{\rho'-1})\right] = e^{-\frac{\gamma}{\rho'}\tau^{\rho'}+2d\frac{\gamma\rho}{\rho'}\mathcal{A}_0^{-2}\kappa^2\tau^{2-\rho'}}(1+o(1)). \tag{101}$$

Combining (101) with (96) we conclude that

$$E_\mu\left[e^{-\frac{1}{\rho}B_1(h_0,v)^\rho}\right] = e^{-\frac{\gamma}{\rho'}\tau^{\rho'}+2d\frac{\gamma\rho}{\rho'}\mathcal{A}_0^{-2}\kappa^2\tau^{2-\rho'}}(1+o(1)). \tag{102}$$

Substituting (102) back into (97) we see that for $2 \le \rho < 3$,

$$h_1(t) = 2d\kappa^2 \mathcal{A}_0^3 \tau^{1-\rho'}(t) + o\left(\frac{1}{t}\right),$$

which proves part (ii).

To prove part (iii), consider the function

$$r(x) := \mathcal{A}_0^2 K_1 \frac{\tau}{\mathcal{A}_0 \tau^{\rho'-1} - x} - \frac{1}{\rho} x^\rho + (\rho - 1) \log x, \tag{103}$$

definde for $x > 0$. We will establish the following lemma.

Lemma 11. *Let $3 < \rho \leq 4$. Then, the following are satisfied.*

(i) *The function r defined in (103) has a global maximum x_t on the interval $[0, (1-\epsilon)\mathcal{A}_0 \tau^{\rho'-1}]$ where its derivative vanishes and such that*

$$x_t = (A_0 K_1)^{\frac{1}{\rho-1}} \tau^{\frac{1}{\rho-1}(3-2\rho')} - \frac{1}{\mathcal{A}_0 K_1} \tau^{-(3-2\rho')} + o\left(\tau^{-(3-2\rho')}\right). \tag{104}$$

(ii) *The function r satisfies*

$$\begin{aligned} r(x_t) = \mathcal{A}_0 K_1 \tau^{2-\rho'} + K_1 (\mathcal{A}_0 K_1)^{\frac{1}{\rho-1}} \left(1 - \tfrac{1}{\rho}\mathcal{A}_0\right) \tau^{\frac{\rho}{\rho-1}(3-2\rho')} \\ + (3-2\rho') \log \mathcal{A}_o \tau - \mathcal{A}_0^2 K_1^2 + o(1). \end{aligned}$$

(iii) *For every $t > 0$ we have that*

$$E_\mu \left[e^{-\frac{1}{\rho} B_1(h_0, v)^\rho} \right] = e^{r(x_t)} \sqrt{\frac{\pi(\rho-1)}{\tau^{\frac{\rho-2}{\rho-1}(3-2\rho')}}} \left(1 + o(1)\right). \tag{105}$$

Proof. Proof of parts (i) and (ii). Note that

$$\begin{aligned} r'(x) &= \mathcal{A}_0^2 K_1 \tfrac{\tau}{(\mathcal{A}_0 \tau^{\rho'-1} - x)^2} - x^{\rho-1} + (\rho-1)\tfrac{1}{x} \quad \text{and} \\ r''(x) &= 2\mathcal{A}_0^2 K_1 \tfrac{\tau}{(\mathcal{A}_0 \tau^{\rho'-1} - x)^3} - (\rho-1)x^{\rho-2} - (\rho-1)\tfrac{1}{x^2}. \end{aligned}$$

Now, for every $\epsilon' > 0$ we have that $r'(x) > 0$ whenever $x \leq \bar{x} := \tau^{\frac{1-\epsilon'}{\rho-1}(3-2\rho')}$, while $r'(x) < 0$ whenever $x \geq (1-\epsilon)\mathcal{A}_0 \tau^{\rho'-1}$. On the other hand, it is easy to check that $r''(x) < 0$ for $\bar{x} \leq x \leq (1-\epsilon)\mathcal{A}_0 \tau^{\rho'-1}$. It follows that there exists only one root x_t of the equation $r'(x) = 0$ on the interval $[0, (1-\epsilon)\mathcal{A}_0 \tau^{\rho'-1})$. To prove (104), as a first step, we note that

$$x_t = (\mathcal{A}_0 K_1)^{\frac{1}{\rho-1}} \tau^{\frac{1}{\rho-1}(3-2\rho')} + y_t,$$

where $y_t = o\left(\tau^{\frac{1}{\rho-1}(3-2\rho')}\right)$. Furthermore

$$2K_1 \tau^{4-3\rho'} x_t - (\rho-1) x_t^{\rho-2} y_t - (\rho-1)\frac{1}{x_t} = u(t), \tag{106}$$

where $u(t)$ is of smaller order in t than the three terms of the left-hand side of (106). Now, for $\rho < \frac{3+\sqrt{17}}{2}$, the last term of the left-hand side of (106) has a higher order than the first term. This implies that

$$y_t = -\frac{1}{x_t^{\rho-1}} + o\left(\frac{1}{x_t^{\rho-1}}\right) = -\frac{1}{\mathcal{A}_0 K_1 \tau^{(3-2\rho')}} + o\left(\frac{1}{\tau^{(3-2\rho')}}\right),$$

which proves (104) of part (i). The proof of part (ii) now follows using the expansion (104) of x_t of part (i).

Proof of part (iii). By a standard Taylor expansion, we see that for every real y such that $x_t - |y| > 0$, there is a $\vartheta \in [x_t - |y|, x_t + |y|]$ such that

$$r(x_t + y) = r(x_t) + \frac{y^2}{2} r''(x_t) + \frac{y^3}{6} r'''(\vartheta).$$

Note that

$$r'''(x) = 6\mathcal{A}_0^2 K_1 \frac{\tau}{(\mathcal{A}_0 \tau^{\rho'-1} - x)^4} - (\rho-2)(\rho-1)x^{\rho-3} + 2(\rho-1)\frac{1}{x^3}.$$

Therefore,

$$r''(x_t) = -(\rho-1)x_t^{\rho-2} + O\left(t^{4-3\rho'}\right)$$

and for $|\vartheta| \le 2x_t$,

$$r'''(\vartheta) = -(\rho-1)(\rho-2)\vartheta^{\rho-3} + O\left(t^{5-4\rho'}\right). \tag{107}$$

It follows that

$$\begin{aligned} E_\mu\left[e^{-\frac{1}{\rho}B_1(h_0,v)^\rho} 1(\max_{e\in E} v(e) \le (1-\epsilon)\mathcal{A}_0\tau^{\rho'-1})\right] &= \int_0^{(1-\epsilon)\mathcal{A}_o\tau^{\rho'-1}} e^{r(x)}dx \\ &= e^{r(x_t)} \int_{-x_t}^{(1-\epsilon)\mathcal{A}_o\tau^{\rho'-1}-x_t} e^{\frac{1}{2}y^2 r''(x_t) + \frac{1}{6}y^3 r'''(\vartheta)}dy \\ &= e^{r(x_t)} \int_{-x_t}^{(1-\epsilon)\mathcal{A}_o\tau^{\rho'-1}-x_t} e^{-\frac{\rho-1}{2}y^2 x_t^{\rho-2} + + y^2 O\left(t^{4-3\rho'}\right) + \frac{1}{6}y^3 r'''(\vartheta)}dy. \end{aligned} \tag{108}$$

For δ such that $\tau^\delta \le x_t$ write

$$\begin{aligned} &\int_{-x_t}^{(1-\epsilon)\mathcal{A}_o\tau^{\rho'-1}-x_t} e^{-\frac{\rho-1}{2}y^2 x_t^{\rho-2} + + y^2 O\left(t^{4-3\rho'}\right) + \frac{1}{6}y^3 r'''(\vartheta)}dy \\ &= \int_{-\tau^\delta}^{\tau^\delta} e^{-\frac{\rho-1}{2}y^2 x_t^{\rho-2} + + y^2 O\left(t^{4-3\rho'}\right) + \frac{1}{6}y^3 r'''(\vartheta)}dy + \int_{B_\delta} e^{-\frac{\rho-1}{2}y^2 x_t^{\rho-2} + + y^2 O\left(t^{4-3\rho'}\right) + \frac{1}{6}y^3 r'''(\vartheta)}dy, \end{aligned} \tag{109}$$

where $B_\delta := \{y : |y| \ge \tau^\delta, -x_t \le y \le (1-\epsilon)\mathcal{A}_o\tau^{\rho'-1} - x_t\}$. For the first integral in the right-hand side of (109), we have that

$$\int_{-\tau^\delta}^{\tau^\delta} e^{-\frac{\rho-1}{2}y^2x_t^{\rho-2}++y^2O\left(t^{4-3\rho'}\right)+\frac{1}{6}y^3r'''(\vartheta)}dy$$

$$= \frac{1}{\sqrt{x_t^{\rho-2}}}\int_{-\tau^\delta\sqrt{x_t^{\rho-2}}}^{\tau^\delta\sqrt{x_t^{\rho-2}}} e^{-\frac{\rho-1}{2}y^2+y^2O\left(\frac{t^{4-3\rho'}}{x_t^{\rho-2}}\right)+\frac{1}{6x_t^{3(\rho-2)/2}}y^3r'''(\vartheta)}dy \tag{110}$$

$$= (1+o(1))\frac{1}{\sqrt{x_t^{\rho-2}}}\int_{-\tau^\delta\sqrt{x_t^{\rho-2}}}^{\tau^\delta\sqrt{x_t^{\rho-2}}} e^{-\frac{\rho-1}{2}y^2+\frac{1}{6x_t^{3(\rho-2)/2}}y^3r'''(\vartheta)}dy. \tag{111}$$

where we have used the fact that when $\rho > 3$, one has that $y^2O\left(t^{4-3\rho'}/x_t^{\rho-2}\right) \leq Ct^{2\delta+4-3\rho'} = o(1)$. Now

$$\begin{aligned}&\frac{1}{\sqrt{x_t^{\rho-2}}}\int_{-\tau^\delta\sqrt{x_t^{\rho-2}}}^{\tau^\delta\sqrt{x_t^{\rho-2}}} e^{-\frac{\rho-1}{2}y^2+\frac{1}{6x_t^{3(\rho-2)/2}}y^3r'''(\vartheta)}dy\\ &= \frac{1}{\sqrt{x_t^{\rho-2}}}\int_{-\tau^\delta}^{\tau^\delta} e^{-\frac{\rho-1}{2}y^2+\frac{1}{6x_t^{3(\rho-2)/2}}y^3r'''(\vartheta)}dy\\ &+\frac{1}{\sqrt{x_t^{\rho-2}}}\int_{D_\delta} e^{-\frac{\rho-1}{2}y^2+\frac{1}{6x_t^{3(\rho-2)/2}}y^3r'''(\vartheta)}dy,\end{aligned} \tag{112}$$

where $D_\delta := \{y : |y| \geq \tau^\delta, -\tau^\delta\sqrt{x_t^{\rho-2}} \leq y \leq \tau^\delta\sqrt{x_t^{\rho-2}}\}$. For the first integral of the right-hand side of (112), we have that

$$\frac{1}{\sqrt{x_t^{\rho-2}}}\int_{-\tau^\delta}^{\tau^\delta} e^{-\frac{\rho-1}{2}y^2+\frac{1}{6x_t^{3(\rho-2)/2}}y^3r'''(\vartheta)}dy = \sqrt{\frac{\pi(\rho-1)}{x_t^{\rho-2}}} + o\left(x_t^{-(\rho-2)/2}\right), \tag{113}$$

where we have used the fact that by (107) we have that $|y^3|x_t^{-3(\rho-2)/2}r'''(\vartheta) \leq C\tau^{3\delta}x_t^{\rho-3-3(\rho-2)/2} \leq Ct^{3\delta-1/2} = o(1)$ for $\delta < 1/6$. For the second integral on the right-hand side of (112), note that since $\frac{1}{x_t^{3(\rho-2)/2}}y^3r'''(\vartheta) \leq y^2o(t)$ uniformly for $y \in D_\delta$, we have

$$\frac{1}{\sqrt{x_t^{\rho-2}}}\int_{D_\delta} e^{-\frac{\rho-1}{2}y^2+\frac{1}{6x_t^{3(\rho-2)/2}}y^3r'''(\vartheta)}dy = \frac{1}{\sqrt{x_t^{\rho-2}}}O\left(e^{-\tau^{2\delta}}\right) = o\left(x_t^{-(\rho-2)/2}\right). \tag{114}$$

Substituting (113) and (114) into (112), (111), (109) and (108), and using (96) together with (104) we conclude that (105) of part (ii) of Lemma 11 is satisfied.

□

Appendix A. Abstract rank-one perturbation theory

For the sake of completeness, we review here the standard rank-one perturbation theory (see [14, 16] for an overview). Let H_0 be a bounded self-adjoint operator in a Hilbert space $\mathcal{H}$. Here we want to establish some cases under which a rank-one self-adjoint perturbation H of H_0, has a principal eigenvalue and eigenfunction possibly with a series expansion on some small parameter. We will need the resolvents $R_\lambda := (\lambda I - H)^{-1}$ and $R^0_\lambda := (\lambda I - H_0)^{-1}$, of H and H_0 respectively, defined for λ not in the corresponding spectrums $\sigma(H)$ and $\sigma(H_0)$. Let us denote by $res(H_0)$ and $res(H)$ the respective resolvent sets. The top of the spectrum of H_0, will be denoted by

$$\lambda^0_+ := \sup\{\lambda : \lambda \in \sigma(H_0)\}.$$

A.1. **Definitions.** Let us consider a rank one perturbation of H_0 depending on a *large* parameter $h > 0$:

$$H := H_0 + hB, \qquad B := (\phi, \cdot)\phi$$

for some normalized $\phi \in \mathcal{H}$. Note that H is also bounded and self-adjoint. We will show that if h is large enough, H has a principal eigenvalue and eigenfunction with a Laurent series expansion on h. Define then the following two families of elements of $\mathcal{H}$,

$$r_\lambda := (\lambda I - H)^{-1}\phi, \qquad \lambda \notin \sigma(H),$$

$$q_\lambda := (\lambda I - H_0)^{-1}\phi, \qquad \lambda \notin \sigma(H_0).$$

A.2. **The Aronszajn-Krein formula.** Here we will state and prove the famous Aronzajn-Krein formula (see for example [16]), in our particular context. Let us first define the following set,

$$\mathcal{S} := \{\lambda \in res(H_0) : h(\phi, q_\lambda) = 1\}.$$

and the quantity,

$$h_0 := \frac{1}{\lim_{\lambda \searrow \lambda^0_+} (\phi, q_\lambda)}. \tag{115}$$

Note that (ϕ, q_λ) is decreasing in λ for $\lambda > \lambda^0_+$. Indeed, $\frac{d(\phi,q_\lambda)}{d\lambda} = -||q_\lambda||^2 < 0$, since $\phi \neq 0$. Hence, the limit in display (115) exists, possibly having the value ∞. In the sequel, we will interpret the quantity h_0 as 0 when the limit in the denominator of the right hand side of (115) is ∞.

Lemma 12. *$\mathcal{S} \subset \mathbb{R}$, and has only isolated points. Furthermore, there is a $\lambda \in \mathcal{S}$ such that $\lambda > \lambda^0_+$ if and only if $h > h_0$. In this case it is unique.*

Proof. Note that $h(\phi, q_\lambda) - 1$ is an analytic function on the open set $res(H_0)$. Therefore, its zeros are isolated. On the other hand, since H is self-adjoint, they have to be real. The last statement follows from the fact that (ϕ, q_λ) is decreasing if $\lambda > \lambda_+^0$ and $\lim_{\lambda\to\infty}(\phi, q_\lambda) = 0$. □

Theorem 6. *Consider the bounded selfadjoint operators H_0 and H.*

(i) **Aronszajn-Krein formula.** *If $\lambda \notin \sigma(H_0) \cup \sigma(H)$ then,*

$$R_\lambda = R_\lambda^0 + \frac{h}{1 - (\phi, q_\lambda)h}(q_\lambda, \cdot)q_\lambda. \tag{116}$$

(ii) **Spectrum of H.**

$$\mathcal{S} \subset \sigma(H) \subset \mathcal{S} \cup \sigma(H_0). \tag{117}$$

Proof. Let us first prove part (i) and the first inclusion of part (ii). Assume that $\lambda \in res(H_0) \cap res(H)$. By definition we have, $(\lambda I - H)r_\lambda = \phi$. Hence,

$$(\lambda I - H_0)r_\lambda = (1 + (\phi, r_\lambda)h)\phi.$$

Making the resolvent $(\lambda I - H_0)^{-1}$ act on both sides of this equality, we get,

$$r_\lambda = (1 + (\phi, r_\lambda)h)q_\lambda. \tag{118}$$

Taking the scalar product with ϕ, we see that $(1 - (\phi, q_\lambda)h)(\phi, r_\lambda) = (\phi, q_\lambda)$. This shows that $(\phi, q_\lambda)h \neq 1$ and hence,

$$(\phi, r_\lambda) = \frac{(\phi, q_\lambda)}{1 - (\phi, q_\lambda)h}. \tag{119}$$

Therefore, $\mathcal{S} \subset \sigma(H)$. Substituting (119) back in the identity (118) and using $R_\lambda = R_\lambda^0 + hr_\lambda(q_\lambda, \cdot)$ proves (116). Now, assume that $\lambda \notin \mathcal{S} \cap \sigma(H_0)$. Then the right hand side of (116) is well defined as a bounded selfadjoint operator in $\mathcal{H}$. A simple computation shows that it is the inverse of the operator $(\lambda I - H)$. □

From theorem 6 we can now deduce the following corollary.

Corollary 3. *Either of the following is true:*

(i) *If $h > h_0$, H has a unique simple eigenvalue $\lambda_{max} > \lambda_+^0$ and $\sigma(H)/\{\lambda_{max}\} \subset (-\infty, \lambda_+^0]$.*
(ii) *If $h \leq h_0$, then $\sigma(H) \subset (-\infty, \lambda_+^0]$.*

Furthermore, if (i) is satisfied the eigenfunction of λ_{max} is proportional to $q_{\lambda_{max}}$ and there exist an $r_0 > h_0$ such that λ_{max} admits a Laurent series expansion for $h > r_0$,

$$\lambda_{max} = h + \sum_{k=0}^{\infty} \frac{b_k}{h^k}.$$

Proof. If $h \leq h_0$, by lemma 12 the equation $h(\phi, q_\lambda) = 1$ does not have any solution $\lambda > \lambda_+^0$. By theorem 6, there is no $\lambda \in \sigma(H)$ such that $\lambda > \lambda_+^0$. On the other hand, by lemma 12, if $h > h_0$, there is a unique $\lambda_{max} > \lambda_+^0$ such that $h(\phi, q_{\lambda_{max}}) = 1$. By theorem 6, $\lambda_{max} \in \sigma(H_0)$ and the spectral projector of H on λ_{max} is given by,

$$P = \frac{1}{||q_{\lambda_{max}}||^2}(q_{\lambda_{max}}, \cdot)q_{\lambda_{max}}.$$

This shows that the eigenfunction of λ_{max} is proportional to q_λ. Finally, defining $u := 1/h$, we see that if $\lambda(u)$ satisfies $(\phi, q_{\lambda(u)}) = u$, then

$$\frac{d(\phi, q_{\lambda(u)})}{du} = 1.$$

By the implicit function theorem, this implies that there is a neighborhood of the point $u = 0$, where the function $1/\lambda(u)$ is analytic. □

References

[1] P. Antal. *Enlargement of obstacles for the simple random walk.* Ann. Probab. **23** 1061-1101 (1995).

[2] A. Astrauskas. *Poisson-type limit theorems for eigenvalues of finite-volume Anderson Hamiltonians*, Acta Appl. Math. 96, no. 1-3, 3-15 (2007).

[3] A. Astrauskas. *Extremal theory for spectrum of random discrete Schrödinger operator. I. Asymptotic expansion formulas*, J. Stat. Phys. 131, no. 5, 867-916 (2008).

[4] A. Astrauskas. *Extremal theory for spectrum of random discrete Schrödinger operator. II. Distributions with heavy tails*, J. Stat. Phys. 146, no. 1, 98-117 (2012).

[5] A. Astrauskas. *Extremal theory for spectrum of random discrete Schrödinger operator. III. Localization properties*, J. Stat. Phys. 150, no. 5, 889-907 (2013).

[6] A. Astrauskas. *From extreme values of i.i.d. random fields to extreme eigenvalues of finite-volume Anderson Hamiltonian*, Probab. Surv. 13, 156-244 (2016).

[7] G. Ben Arous, L. Bogachev, and S. Molchanov. *Limit theorems for sums of random exponentials*, Probab. Theory Relat. Fields 132, no. 4, 579-612 (2005).

[8] G. Ben Arous, S. Molchanov and A.F. Ramírez, *Transition from the annealed to the quenched asymptotics for a random walk on random obstacles*, Ann. of Probab. **33** 6, 2149-2187 (2005).

[9] G. Ben Arous, S. Molchanov and A.F. Ramírez, *Transition asymptotics for reaction-diffusion in random media. Probability and mathematical physics*, 1-40, CRM Proc. Lecture Notes, 42, Amer. Math. Soc., Providence, RI, (2007).

[10] M. Biskup and W. König. *Eigenvalue order statistics for random Schrödinger operators with doubly-exponential tails*, Comm. Math. Phys. 341, 179-218 (2016).

[11] M. Donsker and S.R.S. Varadhan. *Asymptotics for the Wiener sausage*, Commun. Pure Appl. Math. **28** 525-565 (1975).

[12] M. Donsker and S.R.S. Varadhan. *On the number of distinct sites visited by a random walk*, Commun. Pure Appl. Math. **32** 721-747 (1979).

[13] J. Gärtner and A. Schnitzler, *Stable limit laws for the parabolic Anderson model between quenched and annealed behaviour.* Ann. Inst. Henri Poincaré Probab. Stat. 51, no. 1, 194.206 (2015).

[14] T. Kato, *Perturbation Theory for Linear Operators*, Springer Verlag (1995).

[15] V. V. Petrov, *Sums of independent random variables*, Springer Berlin (1975).

[16] B. Simon, *Spectral Analysis of Rank One Perturbations and Applications*, CRM Proceeding and Lecture Notes, Volume **8**, 109-149, (1995).

[17] A.S. Sznitman. *Brownian motion obstacles and random media*, Springer-Verlag, Berlin (1998).

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