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Unitary representations of affine Hecke algebras related to Macdonald spherical functions ☆

J.F. van Diejen [a], E. Emsiz [b,*]

[a] Instituto de Matemática y Física, Universidad de Talca, Casilla 747, Talca, Chile
[b] Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Casilla 306, Correo 22, Santiago, Chile

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ABSTRACT

For any reduced crystallographic root system, we introduce a unitary representation of the (extended) affine Hecke algebra given by discrete difference-reflection operators acting in a Hilbert space of complex functions on the weight lattice. It is shown that the action of the center under this representation is diagonal on the basis of Macdonald spherical functions. As an application, we compute an explicit Pieri formula for these spherical functions.

1. Introduction

It is well known that Macdonald's spherical functions (on p-adic symmetric spaces)—also referred to as generalized Hall–Littlewood polynomials associated with root systems—are intimately connected with the theory of affine Hecke algebras [M1,M2,NR]. In a nutshell, the Macdonald spherical functions form a canonical basis of the spherical subalgebra of the affine Hecke algebra obtained from a monomial basis via the so-called Satake isomorphism. For an overview of these and many other facts concerning Macdonald's spherical functions and their relations with affine Hecke algebras we refer the reader to the comprehensive survey [NR].

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* Corresponding author.
E-mail addresses: diejen@inst-mat.utalca.cl (J.F. van Diejen), eemsiz@mat.puc.cl (E. Emsiz).

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The interplay between affine Hecke algebras and Macdonald spherical functions has proven very fruitful. For instance, affine Hecke algebras turn out to be instrumental in obtaining explicit combinatorial formulas for the monomial expansion and for the structure constants (or Littlewood–Richardson type coefficients) of the Macdonald spherical functions [P,R,S]. Reversely, properties of Macdonald's spherical functions—in particular Macdonald's orthogonality relations and the (generalized) Kostka–Foulkes coefficients describing the transition between Macdonald's spherical functions and the basis of Weyl characters—are fundamental, respectively, in the harmonic analysis of the affine Hecke algebra [O] and for the explicit computation of the Kazhdan–Lusztig basis for the spherical Hecke algebra [NR,K].

The present paper studies the properties of a concrete difference-reflection representation of the affine Hecke algebra and its relations to the theory of Macdonald's spherical functions. Specifically, we introduce an explicit unitary representation of the (extended) affine Hecke algebra in terms of discrete difference-reflection operators acting in a Hilbert space of complex functions on the weight lattice and show that the action of its center under this representation is diagonal on the basis of Macdonald spherical functions. The main technical difficulty in the diagonalization proof is the verification of intertwining relations between our difference-reflection representation and a second auxiliary representation (in terms of discrete integral-reflection operators) that is dual to the standard induced polynomial representation of the affine Hecke algebra. As an application, we compute an explicit Pieri formula for the Macdonald spherical functions generalizing the Pieri formula for the Hall–Littlewood polynomials due to Morris (from root systems of type A to arbitrary type) [Mo].

Our results provide a link interpolating between the Hecke-algebraic techniques developed in the spectral theory of quantum integrable particle systems [HO,EOS] and those employed in Macdonald's theory of symmetric orthogonal polynomials [M4,C]. Indeed, it is known that the Macdonald spherical functions tend in an appropriate continuum limit to the eigenfunctions of the Laplacian perturbed by a delta potential supported on (the hyperplanes of) the corresponding root system [HO,D]. In this limiting situation the role of the affine Hecke algebra is played by the Drinfeld–Lusztig graded (or degenerate) affine Hecke algebra [HO,EOS]. Specifically, our difference-reflection representation gets replaced by a representation of the graded affine Hecke algebra built of Dunkl-type differential-reflection operators, the discrete integral-reflection (or polynomial) representation gets replaced by a representation of the graded affine Hecke algebra in terms of Gutkin–Sutherland continuous integral-reflection operators, and the intertwining operator relating both representations is given by the Gutkin–Sutherland propagation operator [GS,G,HO,EOS]. From this perspective, the present paper lifts this construction to the level of the affine Hecke algebra corresponding to the Macdonald spherical functions. On the other hand, it is well known that the Macdonald spherical functions are limiting cases of the celebrated Macdonald polynomials (corresponding to $q \to 0$) [M3]. The Macdonald polynomials in turn diagonalize a commuting algebra of Macdonald difference operators that can be constructed by means of Cherednik's extension of the polynomial representation of the affine Hecke algebra to the level of the double affine Hecke algebra [M4,C]. From this perspective, the difference-reflection representation of the affine Hecke algebra studied here provides the corresponding concrete construction of the commuting algebra of discrete difference operators that is diagonalized by the Macdonald spherical functions (and isomorphic to the Weyl-group invariant part of the group algebra over the weight lattice).

The paper is organized as follows. In Section 2 notational preliminaries concerning affine Weyl groups and affine Hecke algebras are recalled. In Section 3 our main representation of the affine Hecke algebra in terms of difference-reflection operators is introduced. The auxiliary representation of the affine Hecke algebra by integral-reflection operators and its relation to the standard polynomial representation are described in Section 4. Section 5 introduces an intertwining operator between the difference-reflection representation and the auxiliary integral-reflection representation, which is then used to show that the action of the center under the difference-reflection representation is diagonal on the Macdonald spherical functions. In Section 6 the appropriate Hilbert space structure is provided for which the difference-reflection representation is unitary. From this viewpoint the Macdonald spherical function constitutes the kernel of the Fourier transform—between the Weyl-group invariant sector of this Hilbert space and a closure of the Weyl-group invariant part of the group algebra of the weight lattice—diagonalizing the action of the center under our difference-reflection representation. Finally, in Section 7 we use the difference-reflection representation to compute the explicit Pieri formula for the Macdonald spherical functions. Some technical details pertaining to the proof of the braid relations in Section 3 and the intertwining relations in Section 5 are relegated to Appendices A and B, respectively. Moreover, in Appendix C a few illuminating explicit formulas are collected describing our principal objects of study in the important special case of a root system of type A_{N-1} (i.e., with the Weyl group being equal to the permutation group S_N).

2. Preliminaries

This section sets up the notation for affine Weyl groups and their Hecke algebras and recalls briefly some basic properties. A more thorough discussion with proofs can be found e.g. in the standard sources [B,M4].

2.1. *Affine Weyl group*

Let R be a crystallographic root system spanning a real (finite-dimensional) Euclidean vector space V with inner product $\langle\cdot,\cdot\rangle$. Throughout it will be assumed that R is both irreducible and reduced (unless explicitly stated otherwise). Following standard conventions, the dual root system is denoted by $R^\vee := \{\alpha^\vee \mid \alpha \in R\}$ with $\alpha^\vee := 2\alpha/\langle\alpha,\alpha\rangle$, the weight lattice by $P := \{\lambda \in V \mid \langle\lambda,\alpha^\vee\rangle \in \mathbb{Z}, \forall\alpha \in R\}$, and for a (fixed) choice of positive roots R^+ we write $P^+ := \{\lambda \in P \mid \langle\lambda,\alpha^\vee\rangle \geqslant 0,\ \forall\alpha \in R^+\}$ for the corresponding cone of dominant weights and $C := \{x \in V \mid \langle x,\alpha^\vee\rangle > 0,\ \forall\alpha \in R^+\}$ and $A := \{x \in V \mid 0 < \langle x,\alpha^\vee\rangle < 1,\ \forall\alpha \in R^+\}$ for the dominant Weyl chamber and Weyl alcove, respectively.

For $\alpha \in R^+$ and $k \in \mathbb{Z}$ let $s_{\alpha,k} : V \to V$ be the orthogonal reflection across the hyperplane $V_{\alpha,k} := \{x \in V \mid \langle x,\alpha^\vee\rangle = k\}$ and for $\lambda \in P$ let $t_\lambda : V \to V$ be the translation of the form $t_\lambda(x) := x + \lambda$ ($x \in V$). The (finite) *Weyl group* generated by the reflections $s_{\alpha,0}$, $\alpha \in R^+$, is denoted by W_0 and we write W for the (extended) *affine Weyl group* generated by the elements of W_0 and the translations t_λ, $\lambda \in P$. The length of a group element $w \in W$ is defined as the cardinality $\ell(w) := |S(w)|$ of the set $S(w) := \{V_{\alpha,k} \mid V_{\alpha,k}$ separates A and $wA\}$. (We say that a hyperplane $V_{\alpha,k}$ separates two (subsets of) points in V if these are contained in distinct connected components of $V \setminus V_{\alpha,k}$.) A useful explicit formula to compute the lengths of (affine) Weyl group elements is given by

$$\ell(vt_\lambda) = \sum_{\alpha\in R^+} \left|\langle\lambda,\alpha^\vee\rangle + \chi(v\alpha)\right| \quad (v \in W_0,\ \lambda \in P), \tag{2.1}$$

where $\chi : R \to \{0,1\}$ represents the characteristic function of $R^- := R \setminus R^+$ (so, in particular, for $v \in W_0$ and $\lambda,\mu \in P^+$ one has that $\ell(vt_\lambda) = \ell(v) + \ell(t_\lambda)$ and that $\ell(t_{v\mu}) = \ell(t_\mu)$, $\ell(t_{\lambda+\mu}) = \ell(t_\lambda) + \ell(t_\mu)$).

Let $\alpha_1,\ldots,\alpha_n$ ($n := \mathrm{rank}(R)$) be the basis of simple roots for R^+ and let α_0 be the positive root such that $\alpha_0^\vee$ is the highest root of $R^\vee$ (with respect to $(R^+)^\vee$). We set $s_0 := s_{\alpha_0,1}$ and $s_j := s_{\alpha_j,0}$ for $j = 1,\ldots,n$. The finite Weyl group W_0 is generated by the reflections across the boundary hyperplanes of C: $s_1,\ldots,s_n$, and the affine Weyl group W is generated by the (finite, Abelian) subgroup of elements of length zero $\Omega := \{u \in W \mid uA = A\}$ and the reflections across the boundary hyperplanes of A: $s_0,\ldots,s_n$.

It is instructive to detail the algebraic structure of these presentations of W_0 and W somewhat more explicitly. Let V_j denote the hyperplane fixed by s_j ($j = 0,\ldots,n$). The finite Weyl group W_0 amounts to the group generated by $s_1,\ldots,s_n$ subject to the relations

$$(s_js_k)^{m_{jk}} = 1, \tag{2.2}$$

with π/m_{jk} being the angle between V_j and V_k if $j \neq k$ (so, in particular, $m_{kj} = m_{jk}$) and $m_{jk} = 1$ when $j = k$. To characterize Ω it is convenient to associate with $\lambda \in P$ the affine Weyl group element $u_\lambda := t_\lambda v_\lambda^{-1}$, where v_λ refers to the shortest element of W_0 mapping λ to the closure of the antidominant Weyl chamber $-C$ (which implies that u_λ is the shortest element of the coset $t_\lambda W_0$ and

$\ell(t_\lambda)=\ell(u_\lambda)+\ell(v_\lambda)$). Upon setting $u_0:=1$ and $u_j:=u_{\omega_j}$ for $j=1,\ldots,n$, where $\omega_1,\ldots,\omega_n$ denote the basis of the fundamental weights, one has explicitly

$$\Omega=\{u_j \mid j=0 \text{ or } \langle\omega_j,\alpha_0^\vee\rangle=1\}. \tag{2.3}$$

It is clear from the definition that the elements of Ω permute the hyperplanes $V_0,\ldots,V_n$. Furthermore, for $u\in\Omega$ with $uV_j=V_k$ one has that

$$uu_j=u_ju=u_k \quad (u_j\in\Omega) \quad \text{and} \quad us_j=s_ku \quad (j=0,\ldots,n). \tag{2.4}$$

The affine Weyl group W can now be characterized as the group generated by $s_0,\ldots,s_n$ and the elements $u\in\Omega$ (2.3) subject to the relations (2.2), (2.4) (with the additional caveat that in the pathological case $n=1$ the order $m_{10}=m_{01}=\infty$).

2.2. Affine Hecke algebra

Let $q:W\to\mathbb{R}\setminus\{0\}$ be a length multiplicative function, viz. (i) $q_{ww'}=q_wq_{w'}$ if $\ell(ww')=\ell(w)+\ell(w')$ and (ii) $q_w=1$ if $\ell(w)=0$. This implies that q_{s_j} depends only on the conjugacy class of s_j $(j=0,\ldots,n)$, whence the value of q_w is determined by the number of reflections (in the short roots and in the long roots, respectively) appearing in a reduced expression $w=us_{j_1}\cdots s_{j_\ell}$ (with $u\in\Omega$ and $\ell=\ell(w)$). Following customary habits, the multiplicity function associated with the length multiplicative function will also be denoted by q. This is the function $q:R^+\times\mathbb{Z}\to\mathbb{R}\setminus\{0\}$ such that $q_{\alpha,k}=q_{s_j}$ if $V_{\alpha,k}=V_j$ $(0\leqslant j\leqslant n)$ and $q_{\alpha',k'}=q_{\alpha,k}$ if $V_{\alpha',k'}=wV_{\alpha,k}$ for some $w\in W$. This implies that $q_{\alpha,k}=q_{\alpha,0}$ depends only on the length of α. Reversely, the length multiplicative function can be reconstructed from the multiplicity function via the formula

$$q_w=\prod_{\substack{\alpha\in R^+,\,k\in\mathbb{Z}\\ V_{\alpha,k}\in S(w)}} q_{\alpha,k}. \tag{2.5}$$

We will write $\mathcal{H}$ for the (extended) *affine Hecke algebra* associated with W and q. This algebra can be characterized as the complex associative algebra with basis T_w, $w\in W$, satisfying the *quadratic relations*

$$(T_j-q_j)\big(T_j+q_j^{-1}\big)=0, \quad j=0,\ldots,n, \tag{2.6a}$$

where $T_j:=T_{s_j}$ and $q_j:=q_{s_j}$, and the *braid relations*

$$T_{ww'}=T_wT_{w'} \quad \text{if } \ell\big(ww'\big)=\ell(w)+\ell\big(w'\big). \tag{2.6b}$$

The assignment $T_w\to T_w^*$ with

$$T_w^*:=T_{w^{-1}} \tag{2.7}$$

extends to an antilinear anti-involution of $\mathcal{H}$ thus turning the affine Hecke algebra into an involutive or $*$-algebra. The subalgebra of $\mathcal{H}$ spanned by the basis T_w, $w\in W_0$, is referred to as the finite *Hecke algebra* $\mathcal{H}_0$ (associated with W_0 and q).

The affine Hecke algebra $\mathcal{H}$ admits a simple presentation as the algebra generated by $T_0,\ldots,T_n$ and T_u, $u\in\Omega$, (2.3) subject to the quadratic relations (2.6a) (cf. Eq. (2.2) with $j=k$), the braid relations

$$\underbrace{T_jT_kT_j\cdots}_{m_{jk}\text{ factors}}=\underbrace{T_kT_jT_k\cdots}_{m_{jk}\text{ factors}}, \quad j\neq k \tag{2.8}$$

(cf. Eq. (2.2) with $j \neq k$), and the relations

$$T_u T_{u_j} = T_{uu_j} = T_{u_k} \quad (u_j \in \Omega) \quad \text{and} \quad T_u T_j = T_k T_u \quad (j = 0, \ldots, n), \tag{2.9}$$

with $V_k = uV_j$ (cf. Eq. (2.4)). The finite Hecke algebra $\mathcal{H}_0$ in turn amounts to the (sub)algebra generated by $T_1, \ldots, T_n$ subject to the quadratic relations (2.6a) and the braid relations (2.8).

For $\lambda \in P$, the element

$$Y^\lambda := T_{t_\mu} T_{t_\nu}^{-1} \quad \text{with } \mu, \nu \in P^+ \text{ such that } \lambda = \mu - \nu \tag{2.10}$$

is well defined in the sense that it does not depend on the particular choice of the decomposition of λ as a difference of dominant weights μ and ν. Furthermore, the elements Y^λ, $\lambda \in P$, form a basis of a subalgebra of $\mathcal{H}$ isomorphic to the group algebra of the weight lattice $\mathbb{C}[P]$:

$$Y^\lambda Y^\mu = Y^{\lambda+\mu} \quad (\lambda, \mu \in P) \quad \text{and} \quad Y^0 = 1, \tag{2.11a}$$

satisfying in addition the relations

$$T_j Y^\lambda - Y^{s_j\lambda} T_j = \left(q_j - q_j^{-1}\right) \frac{Y^\lambda - Y^{s_j\lambda}}{1 - Y^{-\alpha_j}} \quad (j = 1, \ldots, n). \tag{2.11b}$$

The elements $T_w Y^\lambda$, $w \in W_0$, $\lambda \in P$, constitute a basis of $\mathcal{H}$, which gives rise to a second (very useful) presentation of the affine Hecke algebra (due to Bernstein, Lusztig, and Zelevinsky) as the algebra generated by $T_1, \ldots, T_n$ and Y^λ, $\lambda \in P$, subject to the relations (2.6a), (2.8), (2.11a) and

$$\begin{aligned} T_j Y^\lambda &= Y^\lambda T_j \quad \text{if } \langle \lambda, \alpha_j^\vee \rangle = 0, \\ T_j Y^\lambda &= Y^{s_j\lambda} T_j + \left(q_j - q_j^{-1}\right) Y^\lambda \quad \text{if } \langle \lambda, \alpha_j^\vee \rangle = 1 \end{aligned} \tag{2.12}$$

(cf. Eq. (2.11b) with $\lambda \in V_{\alpha_j,0} \cup V_{\alpha_j,1}$). In other words, the affine Hecke algebra $\mathcal{H}$ is a merger of the finite Hecke algebra $\mathcal{H}_0$ and the group algebra $\mathbb{C}[P]$ with the cross relations (2.12).

It can be seen with the aid of the latter presentation that the center $\mathcal{Z}$ of $\mathcal{H}$ is spanned by

$$m_\lambda(Y) := \sum_{\mu \in W_0\lambda} Y^\mu, \quad \lambda \in P^+ \tag{2.13}$$

(and thus isomorphic to the W_0-invariant part $\mathbb{C}[P]^{W_0}$ of the group algebra of the weight lattice). Moreover, since

$$\left(Y^\lambda\right)^* = T_{w_o} Y^{-w_o\lambda} T_{w_o}^{-1} \tag{2.14}$$

(where w_o denotes the longest element of W_0), one has that

$$m_\lambda(Y)^* = m_{\lambda^*}(Y), \quad \text{with } \lambda^* := -w_o\lambda. \tag{2.15}$$

3. Difference-reflection operators

In this section we introduce our main representation of the affine Hecke algebra in terms of difference-reflection operators.

The action of the affine Weyl group on $P \subset V$ induces a representation of W on the space $C(P) := \{f \mid f : P \to \mathbb{C}\}$

$$(wf)(\lambda) := f\big(w^{-1}\lambda\big) \quad (w \in W,\ \lambda \in P). \tag{3.1}$$

We consider the following difference-reflection operators on $C(P)$

$$\widehat{T}_j := q_j + \chi_j(s_j - 1), \quad j = 0, \ldots, n, \tag{3.2a}$$

where q_j and χ_j act by multiplication with

$$\chi_j(\lambda) := \begin{cases} q_j & \text{if } V_j \text{ separates } \lambda \text{ and } A, \\ 1 & \text{if } \lambda \in V_j, \\ q_j^{-1} & \text{otherwise.} \end{cases} \tag{3.2b}$$

Theorem 3.1 *(Difference-Reflection Representation $\widehat{T}(\mathcal{H})$). The assignment $T_j \to \widehat{T}_j$ $(j = 0, \ldots, n)$ and $T_u \to u$ $(u \in \Omega)$ extends (uniquely) to a representation $h \to \widehat{T}(h)$ $(h \in \mathcal{H})$ of the affine Hecke algebra on $C(P)$.*

Inferring this theorem amounts to verifying the relations

$$(\widehat{T}_j - q_j)\big(\widehat{T}_j + q_j^{-1}\big) = 0 \quad (0 \leqslant j \leqslant n), \tag{3.3a}$$

$$\underbrace{\widehat{T}_j\widehat{T}_k\widehat{T}_j\cdots}_{m_{jk}\text{ factors}} = \underbrace{\widehat{T}_k\widehat{T}_j\widehat{T}_k\cdots}_{m_{jk}\text{ factors}} \quad (0 \leqslant j \neq k \leqslant n), \tag{3.3b}$$

$$u\widehat{T}_j = \widehat{T}_k u \quad \text{if } uV_j = V_k\ (u \in \Omega,\ 0 \leqslant j \leqslant n). \tag{3.3c}$$

The quadratic relations in Eq. (3.3a) follow from a short computation:

$$\begin{aligned} \widehat{T}_j^2 &= q_j^2 + \big(2q_j\chi_j - 1 - \chi_j^2\big)(s_j - 1) \\ &= q_j^2 + \big(q_j - q_j^{-1}\big)\chi_j(s_j - 1) = \big(q_j - q_j^{-1}\big)\widehat{T}_j + 1, \end{aligned}$$

where we used (in the second identity) that for $\lambda \notin V_j$

$$2q_j\chi_j - 1 - \chi_j^2 = \big(q_j - q_j^{-1}\big)\chi_j = \begin{cases} q_j^2 - 1 & \text{if } \chi_j = q_j, \\ 1 - q_j^{-2} & \text{if } \chi_j = q_j^{-1}, \end{cases}$$

together with the observation that for any $f \in C(P)$ the difference $(s_j f)(\lambda) - f(\lambda)$ vanishes when $\lambda \in V_j$. The commutation relations in Eq. (3.3c) are in turn immediate from the definition of $\widehat{T}_j$ and the corresponding affine Weyl group relations in Eq. (2.4). The proof of the braid relations in Eq. (3.3b) is a bit more intricate and hinges on two lemmas that require some additional notation. For $x \in V$ let $W_{0,x} \subset W_0$ denote the stabilizer subgroup $\{w \in W_0 \mid wx = x\}$. We will consider the following equivalence relation on V: $x \sim y$ iff $W_{0,x} = W_{0,y}$ and both points lie on the closure of the same Weyl chamber wC (for some $w \in W_0$). The finite number of equivalence classes of V with respect to the relation $\sim$ are called facets and constitute the so-called Coxeter complex $\mathcal{C}$ of W_0.

Lemma 3.2. *Let $\widehat{D}$ be an operator in $\mathbb{C}\langle\widehat{T}_1,\dots,\widehat{T}_n\rangle$ and let $\lambda,\mu\in P$ with $\lambda\sim\mu$. Then*

$$(\widehat{D}f)(\lambda)=0\quad\forall f\in C(P)\quad\Longrightarrow\quad(\widehat{D}f)(\mu)=0\quad\forall f\in C(P).\tag{3.4}$$

Proof. Given $f\in C(P)$ and $\lambda,\mu\in P$ with $\lambda\sim\mu$, pick an $\tilde{f}\in C(P)$ such that $\tilde{f}(w\lambda)=f(w\mu)$ for all $w\in W_0$. (Such a function $\tilde{f}$ exists, since $w\lambda=w'\lambda\Rightarrow w^{-1}w'\in W_{0,\lambda}=W_{0,\mu}\Rightarrow w\mu=w'\mu$.) From the definition of the difference-reflection operators $\widehat{T}_1,\dots,\widehat{T}_n$ it is then immediate that $(\widehat{D}f)(\mu)=(\widehat{D}\tilde{f})(\lambda)$ (because $w\mu$ and $w\lambda$ ($w\in W_0$) cannot be separated by the hyperplanes V_j, $1\leqslant j\leqslant n$, as both weights lie on the same facet). Hence, the hypothesis on the LHS of formula (3.4) implies that for *any* $f\in C(P)$: $(\widehat{D}f)(\mu)=(\widehat{D}\tilde{f})(\lambda)=0$. □

Lemma 3.3. *For any affine Weyl group W, the braid relations in Eq.* (3.3b) *follow from the braid relations corresponding to the* finite *Weyl groups W_0 associated with the* (*not necessarily irreducible*) *root systems of rank two.*

Proof. Without restriction we may assume that $n\geqslant 2$ (as for $n=1$ there is no braid relation to check since then $m_{01}=m_{10}=\infty$). For any pair $1\leqslant j\neq k\leqslant n$, the reflections s_j,s_k generate a finite Weyl group corresponding to the rank-two root subsystem R_{jk} with basis α_j,α_k. The Weyl group in question acts trivially on the orthogonal complement $V_{jk}^{\perp}$ of $V_{jk}:=\mathrm{Span}_{\mathbb{R}}(\alpha_j,\alpha_k)$ in V. It follows that the action of $\widehat{T}_j$ and $\widehat{T}_k$ on $C(P)$ extends to a decomposition of the form $\widehat{T}_j(R_{jk})\otimes 1$ and $\widehat{T}_k(R_{jk})\otimes 1$ on $F(P_{jk})\otimes F(V_{jk}^{\perp})\supset C(P)$, where P_{jk} denotes the image of the orthogonal projection of P onto V_{jk} (and $F(P_{jk})$, $F(V_{jk}^{\perp})$ are the spaces of complex functions on P_{jk} and $V_{jk}^{\perp}$, respectively). Here $\widehat{T}_j(R_{jk})$ and $\widehat{T}_k(R_{jk})$ refer to the corresponding operators on $F(P_{jk})$ associated with the simple reflections of R_{jk}. (Notice in this connection that P_{jk} amounts to the weight lattice $P(R_{jk})$ associated with R_{jk} and that the image A_{jk} of the alcove A under the orthogonal projection onto V_{jk} is contained in the Weyl alcove $A(R_{jk})$ associated with the basis α_j,α_k of R_{jk}.) The upshot is that the braid relations for $\widehat{T}_j$ and $\widehat{T}_k$ follow from the braid relations for $\widehat{T}_j(R_{jk})$ and $\widehat{T}_k(R_{jk})$. If one of the two indices (j say) takes the value 0, then the above arguments apply verbatim upon picking for R_{0k} the translated rank-two root system with basis $-\alpha_0,\alpha_k$ relative to the origin at $V_{0k}\cap V_0\cap V_k$ (where $V_{0k}=\mathrm{Span}_{\mathbb{R}}(\alpha_0,\alpha_k)$). (Now the projection P_{0k} of P onto V_{0k} amounts rather to the weight lattice of the untranslated rank-two root subsystem with basis $-\alpha_0,\alpha_k$, but this is no obstacle in view of Remark 3.4 below.) □

By Lemma 3.3, it is sufficient to verify the braid relations $\widehat{T}_1\widehat{T}_2\widehat{T}_1\cdots=\widehat{T}_2\widehat{T}_1\widehat{T}_2\cdots$ (with m_{12} factors on both sides) associated with the simple reflections s_1 and s_2 for the root systems $A_1\times A_1$, A_2, B_2, and G_2 (for which $m_{12}=2$, 3, 4, and 6, respectively). Moreover by Lemma 3.2—upon acting with both sides on an arbitrary lattice function in $C(P)$—it is only needed to verify these braid relations on a finite W_0-invariant set of weights representing the facets of the Coxeter complex $\mathcal{C}$. This reduces the verification of Eq. (3.3b) to a routine case-by-case computation that is somewhat tedious by hand for the three root systems other than $A_1\times A_1$ (and particularly so for the root systems B_2 and G_2) but completely straightforward to perform in all four cases with the aid of symbolic computer algebra. To illustrate the idea of the computation in question we have outlined the details for the root system A_2 in Appendix A.

Remark 3.4. The action of the affine Weyl group in Eq. (3.1) and the operators $\widehat{T}_j$ (3.2a), (3.2b) make in fact sense on the space $F(\mathcal{A})$ of complex functions on the Coxeter complex $\mathcal{A}$ of the affine Weyl group (which may also be seen as the space of functions on V that are piecewise constant on the affine facets). (Here the affine facets are the equivalence classes of V with points being equivalent if they belong to the closure of the same Weyl alcove wA ($w\in W$) and have the same stabilizer inside the affine Weyl group.) The space $C(P)$ can be naturally embedded into $F(\mathcal{A})$ as the space of functions with support in the affine facets containing a weight (since points differing by a nonzero weight necessarily belong to distinct affine facets). With this extension of the domain, the Hecke-algebra relations in Eqs. (3.3a)–(3.3c) remain valid. Indeed, Lemma 3.2 and its proof generalize verbatim from P

to $\mathcal{A}$. In other words, the representation in Theorem 3.1 extends naturally to a representation of the affine Hecke algebra on the space $F(\mathcal{A})$.

4. Integral-reflection operators

In this section we describe the auxiliary representation of the affine Hecke algebra in terms of integral-reflection operators. The representation in question is dual to a standard polynomial representation of the affine Hecke algebra on the group algebra of the weight lattice.

We consider the following integral-reflection operators on $C(P)$ associated with the simple reflections $s_1, \ldots, s_n$:

$$I_j := q_j s_j + \big(q_j - q_j^{-1}\big) J_j, \quad j = 1, \ldots, n, \tag{4.1a}$$

where $J_j : C(P) \to C(P)$ denotes a discrete integral operator which—grosso modo—integrates the lattice function $f(\lambda)$ over the α_j-string from λ to $s_j\lambda$:

$$(J_j f)(\lambda) := \begin{cases} -f(\lambda - \alpha_j) - f(\lambda - 2\alpha_j) - \cdots - f(s_j\lambda) & \text{if } \langle \lambda, \alpha_j^\vee \rangle > 0, \\ 0 & \text{if } \langle \lambda, \alpha_j^\vee \rangle = 0, \\ f(\lambda) + f(\lambda + \alpha_j) + \cdots + f(s_j\lambda - \alpha_j) & \text{if } \langle \lambda, \alpha_j^\vee \rangle < 0. \end{cases} \tag{4.1b}$$

Proposition 4.1 *(Integral-Reflection Representation $I(\mathcal{H})$). The assignment $T_j \to I_j$ ($j = 1, \ldots, n$) and $Y^\lambda \to t_\lambda$ ($\lambda \in P$) extends (uniquely) to a representation $h \to I(h)$ ($h \in \mathcal{H}$) of the affine Hecke algebra on $C(P)$.*

In the remainder of this section the proposition is proved by exploiting that $I(\mathcal{H})$ may be seen as the dual of a standard representation of the affine Hecke algebra in terms of Demazure–Lusztig operators.

Let us denote by e^λ, $\lambda \in P$, the standard basis of the group algebra $\mathbb{C}[P]$ (so $e^\lambda e^\mu = e^{\lambda+\mu}$ and $e^0 = 1$) and consider the following nondegenerate sesquilinear pairing $(\cdot,\cdot) : C(P) \times \mathbb{C}[P] \to \mathbb{C}$

$$(f, p) := (\bar{p} f)(0) \quad \big(f \in C(P),\ p \in \mathbb{C}[P]\big), \tag{4.2}$$

where $\bar{p}$ refers to the complex conjugate $\sum_\lambda \bar{c}_\lambda e^\lambda$ of $p = \sum_\lambda c_\lambda e^\lambda$ ($c_\lambda \in \mathbb{C}$), and the action of $\mathbb{C}[P]$ on f is determined by $e^\lambda f := t_\lambda f$ (so, in particular, $(f, e^\lambda) = (t_\lambda f)(0) = f(-\lambda)$). We will use the notational convention $(p, f) := \overline{(f, p)}$. The action of W on P lifts to an action of the affine Weyl group on $\mathbb{C}[P]$ via $we^\lambda := e^{w\lambda}$ ($w \in W$, $\lambda \in P$). Notice that with these conventions $(vt_\lambda f, p) = (f, t_\lambda v^{-1} p)$ ($v \in W_0$, $\lambda \in P$, $f \in C(P)$, $p \in \mathbb{C}[P]$), i.e. the action of W_0 is 'unitary' and the action of P is 'symmetric' with respect to the above pairing.

It is well known (cf. e.g. Ref. [M4]) that the trivial one-dimensional representation $T_j \to q_j$ ($j = 1, \ldots, n$) of $\mathcal{H}_0$ on $\mathbb{C}$ immediately induces a representation of the finite Hecke algebra on the group algebra through the relations in Eq. (2.11b). Indeed, the latter representation $h \to \check{T}(h)$ of $\mathcal{H}_0$ on $\mathbb{C}[P]$ is generated by the Demazure–Lusztig operators:

$$\check{T}_j := q_j s_j + \big(q_j - q_j^{-1}\big)\big(1 - e^{-\alpha_j}\big)^{-1}(1 - s_j), \quad j = 1, \ldots, n. \tag{4.3}$$

Proposition 4.1 is now a direct consequence of the two subsequent lemmas and the Bernstein–Lusztig–Zelevinsky presentation of the affine Hecke algebra with the relations in Eq. (2.12).

Lemma 4.2. *The assignment $T_j \to I_j$ ($j = 1, \ldots, n$) extends (uniquely) to a representation $h \to I(h)$ ($h \in \mathcal{H}_0$) of the finite Hecke algebra on $C(P)$, i.e.*

$$(I_j - q_j)\big(I_j + q_j^{-1}\big) = 0 \quad (1 \leqslant j \leqslant n), \tag{4.4a}$$

$$\underbrace{I_j I_k I_j \cdots}_{m_{jk}\text{ factors}} = \underbrace{I_k I_j I_k \cdots}_{m_{jk}\text{ factors}} \quad (1 \leqslant j \neq k \leqslant n). \tag{4.4b}$$

Proof. By acting with the Demazure–Lusztig operator $\check{T}_j$ (4.3) on the basis element e^λ it is seen that

$$\begin{aligned}
\check{T}_j e^\lambda &= q_j e^{s_j\lambda} + \big(q_j - q_j^{-1}\big)\frac{e^\lambda - e^{\lambda - \langle\lambda,\alpha_j^\vee\rangle\alpha_j}}{1 - e^{-\alpha_j}} \\
&= q_j e^{s_j\lambda} + \big(q_j - q_j^{-1}\big) \times \begin{cases} e^\lambda + e^{\lambda-\alpha_j} + \cdots + e^{s_j\lambda+\alpha_j} & \text{if } \langle\lambda,\alpha_j^\vee\rangle > 0, \\ 0 & \text{if } \langle\lambda,\alpha_j^\vee\rangle = 0, \\ -e^{\lambda+\alpha_j} - e^{\lambda+2\alpha_j} - \cdots - e^{s_j\lambda} & \text{if } \langle\lambda,\alpha_j^\vee\rangle < 0, \end{cases}
\end{aligned}$$

whence $(I_j f, e^\lambda) = (f, \check{T}_j e^\lambda)$ $(f \in C(P),\ \lambda \in P)$. The quadratic relations and braid relations for $I_1, \ldots, I_n$ thus follow from those for $\check{T}_1, \ldots, \check{T}_n$ (and $(I(h)f, p) = (f, \check{T}(h^*)p)$, $h \in \mathcal{H}_0$, $f \in C(P)$, $p \in \mathbb{C}[P]$). $\square$

Lemma 4.3. *The operators I_j $(j = 1, \ldots, n)$ and t_λ $(\lambda \in P)$ on $C(P)$ satisfy the cross relations*

$$\begin{aligned}
I_j t_\lambda &= t_\lambda I_j \quad \textit{if } \langle\lambda, \alpha_j^\vee\rangle = 0, \\
I_j t_\lambda &= t_{s_j\lambda} I_j + \big(q_j - q_j^{-1}\big) t_\lambda \quad \textit{if } \langle\lambda, \alpha_j^\vee\rangle = 1.
\end{aligned} \tag{4.5}$$

Proof. Since $s_j t_\lambda = t_{s_j\lambda} s_j$, it is sufficient to infer that $J_j t_\lambda = t_\lambda J_j$ if $\langle\lambda,\alpha_j^\vee\rangle = 0$ and that $J_j t_\lambda = t_{s_j\lambda} J_j + t_\lambda$ if $\langle\lambda,\alpha_j^\vee\rangle = 1$. Both identities are seen to hold manifestly upon acting on an arbitrary function in $C(P)$ and comparing the terms on both sides (taking into account that $s_j\lambda = \lambda - \langle\lambda,\alpha_j^\vee\rangle\alpha_j$). $\square$

Remark 4.4. By Eqs. (2.11a), (2.11b), the Demazure–Lusztig operators $\check{T}_j$ $(j = 1, \ldots, n)$ together with the multiplicative action of the basis elements e^λ $(\lambda \in P)$ in fact determine a representation $h \to \check{T}(h)$ $(h \in \mathcal{H})$ of the affine Hecke algebra on $\mathbb{C}[P]$ (extending the assignment $T_j \to \check{T}_j$ $(j = 1, \ldots, n)$, $Y^\lambda \to e^\lambda$ $(\lambda \in P)$). Furthermore, the mapping $T_w Y^\lambda \to Y^\lambda T_{w^{-1}}$ $(w \in W_0,\ \lambda \in P)$ extends to an antilinear anti-involution $\star$ of $\mathcal{H}$ (agreeing with the previous $*$-anti-involution on the subalgebra $\mathcal{H}_0$). With respect to the new $\star$-anti-involution and the pairing in Eq. (4.2) the integral-reflection representation in Proposition 4.1 is dual to the polynomial representation $\check{T}(\mathcal{H})$ in the sense that $(I(h)f, p) = (f, \check{T}(h^\star)p)$ $(h \in \mathcal{H},\ f \in C(P),\ p \in \mathbb{C}[P])$.

Remark 4.5. The integral-reflection operators I_j (4.1a), (4.1b) constitute a discrete counterpart of integral-reflection operators introduced by Gutkin and Sutherland in the context of their study of the spectral problem for the Laplacian perturbed by a delta potential supported on the reflection hyperplanes of the root system R [GS,G]. Proposition 4.1 is the corresponding analog of the observation in Ref. [HO] that these Gutkin–Sutherland integral-reflection operators determine a representation of the Drinfeld–Lusztig graded affine Hecke algebra.

5. Diagonalization of $\widehat{T}(\mathcal{Z})$

In this section we diagonalize the action of the center of $\mathcal{H}$ under our difference-reflection representation by means of Macdonald's spherical functions. Our main tool is an intertwining operator relating the difference-reflection representation to the auxiliary integral-reflection (or polynomial) representation.

We will employ the shorthand notation $\widehat{T}_w := \widehat{T}(T_w)$ and $I_w := I(T_w)$ $(w \in W)$.

5.1. Intertwining operator

Let $\mathcal{J} : C(P) \to C(P)$ be the operator defined by

$$(\mathcal{J} f)(\lambda) := q_{\bar{u}_\lambda}\big(I_{\bar{u}_\lambda}^{-1} f, 1\big) \quad \big(f \in C(P),\ \lambda \in P\big), \tag{5.1}$$

where $\bar{u}_\lambda := w_o u_{w_o\lambda} w_o = t_\lambda w_\lambda^{-1}$ with

$$w_\lambda := w_o v_{w_o\lambda} w_o = v_{-\lambda}$$

(i.e. w_λ is the shortest element of W_0 mapping λ into the dominant cone P^+) and $(\cdot,\cdot)$ refers to the pairing in Eq. (4.2). Notice that

$$(\mathcal{J} f)(\lambda) = q_{t_\lambda} q_{w_\lambda}\big(I_{w_\lambda^{-1}}^{-1} f\big)(\lambda_+) \quad \text{with } \lambda_+ := w_\lambda \lambda. \tag{5.2}$$

So in particular, on the dominant cone $\mathcal{J}$ acts simply as a multiplication operator: $(\mathcal{J} f)(\lambda) = q_{t_\lambda} f(\lambda)$ for $\lambda \in P^+$.

Theorem 5.1 *(Intertwining Property). The operator $\mathcal{J} : C(P) \to C(P)$ (5.1) enjoys the following intertwining property, connecting the difference-reflection representation $\widehat{T}(\mathcal{H})$ with the integral-reflection representation $I(\mathcal{H})$*

$$\widehat{T}_w \mathcal{J} = \mathcal{J} I_w \tag{5.3}$$

(*for all $w \in W$*).

For $w \in W_0$, the intertwining property in Eq. (5.3) is an immediate consequence of the next lemma, whose proof boils down to some straightforward computations based on the well-known elementary Hecke algebra relations (cf. e.g. [M4, (4.1.2)])

$$T_j T_w = T_{s_j w} + \chi\big(w^{-1}\alpha_j\big)\big(q_j - q_j^{-1}\big) T_w, \tag{5.4a}$$

$$T_w^{-1} T_j^{-1} = T_{s_j w}^{-1} - \chi\big(w^{-1}\alpha_j\big)\big(q_j - q_j^{-1}\big) T_w^{-1}, \tag{5.4b}$$

$j = 1, \ldots, n$ (where χ is in accordance with Eq. (2.1) and the second relation follows from the first one by applying the anti-involution $T_w \to T_w^{-1}$, $q \to q^{-1}$) together with the observation that for $w, w' \in W_0$ and λ dominant

$$q_w\big(I_w^{-1} f\big)(\lambda) = q_{w'}\big(I_{w'}^{-1} f\big)(\lambda) \quad \text{if } w^{-1} w' \in W_{0,\lambda} \tag{5.5}$$

(which is readily seen by induction on $\ell(w^{-1}w')$).

Lemma 5.2. *The representations $\widehat{T}(\mathcal{H}_0)$ and $I(\mathcal{H}_0)$ satisfy the finite intertwining relations*

$$\widehat{T}_j \mathcal{J} = \mathcal{J} I_j \quad (j = 1, \ldots, n). \tag{5.6}$$

Proof. Let $f \in C(P)$. Elementary manipulations reveal that

$$\begin{aligned}
q_{t_\lambda}^{-1}\big(\widehat{T}_j^{-1}\mathcal{J}f\big)(\lambda) &\overset{(5.5)}{=} q_j^{-1}q_{w_\lambda}\big(I_{w_\lambda^{-1}}^{-1}f\big)(\lambda_+) \\
&\quad + q_j^{-\operatorname{sign}(w_\lambda\alpha_j)}\big(q_{w_\lambda s_j}\big(I_{s_jw_\lambda^{-1}}^{-1}f\big)(\lambda_+) - q_{w_\lambda}\big(I_{w_\lambda^{-1}}^{-1}f\big)(\lambda_+)\big) \\
&= q_{w_\lambda}\big(\big(I_{s_jw_\lambda^{-1}}^{-1}f\big)(\lambda_+) - \chi(w_\lambda\alpha_j)\big(q_j - q_j^{-1}\big)\big(I_{w_\lambda^{-1}}^{-1}f\big)(\lambda_+)\big) \\
&\overset{(5.4b)}{=} q_{w_\lambda}\big(I_{w_\lambda^{-1}}^{-1}I_j^{-1}f\big)(\lambda_+) = q_{t_\lambda}^{-1}\big(\mathcal{J}I_j^{-1}f\big)(\lambda),
\end{aligned}$$

whence $\widehat{T}_j^{-1}\mathcal{J} = \mathcal{J}I_j^{-1}$. $\square$

The extension of the intertwining property in Eq. (5.3) from W_0 to W hinges on a second lemma, whose proof in contrast is technically somewhat more involved and therefore being relegated to Appendix B.

Lemma 5.3. *The representation $\widehat{T}(\mathcal{H})$ and $I(\mathcal{H})$ satisfy the affine intertwining relations*

$$\widehat{T}_0\mathcal{J} = \mathcal{J}I_0 \quad (I_0 := I_{s_0}), \tag{5.7a}$$

$$u\mathcal{J} = \mathcal{J}I_u \quad (u \in \Omega). \tag{5.7b}$$

Remark 5.4. By the duality in Remark 4.4, the action of the intertwining operator can be rewritten in terms of the polynomial representation $\check{T}(\mathcal{H})$ as

$$(\mathcal{J}f)(\lambda) = q_{\bar{u}_\lambda}\big(f, \big(\check{T}^\star_{\bar{u}_\lambda}\big)^{-1}1\big) \quad \big(f \in C(P),\ \lambda \in P\big), \tag{5.8}$$

where we have used the shorthand notation $\check{T}^\star_w := \check{T}(T^\star_w)$.

Remark 5.5. The intertwining operator $\mathcal{J}$ (5.1) is a discrete counterpart of the Gutkin–Sutherland propagation operator, which relates the spectral problem for the Laplacian with a delta potential in Remark 4.5 to that of the free Laplacian [GS,G]. From this perspective, Theorem 5.1 yields the corresponding generalization of the fact that the propagation operator in question intertwines the integral-reflection representation and the Dunkl-type differential-reflection representation of the Drinfeld–Lusztig graded affine Hecke algebra of Refs. [HO] and [EOS], respectively. In fact, our difference-reflection representation $\widehat{T}(\mathcal{H})$, which was obtained by pushing the integral-reflection representation $I(\mathcal{H})$ through the intertwining operator $\mathcal{J}$, provides us with the Dunkl-type difference-reflection operators for a discretization of the Laplacian with a delta potential on root hyperplanes that was introduced and studied in Ref. [D] (cf. also Remark 7.6 below).

5.2. Bijectivity of the intertwining operator

We will now show that the intertwining operator $\mathcal{J}: C(P) \to C(P)$ is bijective. The existence of this bijection intertwining the difference-reflection representation $\widehat{T}(\mathcal{H})$ and the integral-reflection representation $I(\mathcal{H})$ reveals that these two representations of the affine Hecke algebra in $C(P)$ are in fact equivalent. Moreover, it provides an alternative (indirect) proof of Theorem 3.1 as a consequence of Proposition 4.1. Indeed, Lemmas 5.2 and 5.3—together with the bijectivity of $\mathcal{J}$—disclose that the affine Hecke-algebra relations in Eqs. (3.3a)–(3.3c) may be seen as a consequence of the corresponding relations for $I_0, \ldots, I_n$ and I_u, $u \in \Omega$ (which follow in turn from Proposition 4.1).

To prove now the bijectivity in question some further notation is needed. Let $\preccurlyeq$ represent the *dominance order* on the cone of dominant weights P^+ and let $\leqslant$ denote the *Bruhat order* on the finite Weyl group W_0 [B,M4]. Specifically,

$$\forall \lambda, \mu \in P^+: \quad \mu \preccurlyeq \lambda \quad \text{iff} \quad \lambda - \mu \in Q^+$$

with $Q^+ := \operatorname{Span}_{\mathbb{Z}_{\geqslant 0}}(R^+)$, and $\forall v, v' \in W_0$: $v' \leqslant v$ iff $v' = s_{i_1} \cdots s_{i_p}$ for a certain subsequence $(i_1, \ldots, i_p)$ of $(j_1, \ldots, j_\ell)$ with $v = s_{j_1} \cdots s_{j_\ell}$ a reduced expression (i.e. $\ell = \ell(v)$). The dominance order can be conveniently extended from P^+ to P with the aid of the Bruhat order (cf. Ref. [M4, Sec. 2.1])

$$\forall \lambda, \mu \in P: \quad \mu \preccurlyeq \lambda \quad \text{iff} \quad \begin{cases} \mu_+ \prec \lambda_+ & \text{(i)}, \\ \text{or} & \\ \mu_+ = \lambda_+ \text{ and } w_\mu \leqslant w_\lambda & \text{(ii)}. \end{cases}$$

Theorem 5.6 *(Automorphism). The operator $\mathcal{J}$ (5.1) constitutes a linear automorphism of the space $C(P)$.*

Corollary 5.7 *(Equivalence). The difference-reflection representation $\widehat{T}(\mathcal{H})$ and the integral-reflection representation $I(\mathcal{H})$ of the affine Hecke algebra in $C(P)$ are equivalent*:

$$\widehat{T}(h) = \mathcal{J} I(h) \mathcal{J}^{-1} \quad \forall h \in \mathcal{H}.$$

Proof. It is clear that the intertwining property in Theorem 5.1 and the invertibility of $\mathcal{J}$ ensure that $\widehat{T}(\mathcal{H})$ and $I(\mathcal{H})$ are equivalent representations of the affine Hecke algebra in $C(P)$, i.e. the corollary is in effect a direct consequence of the theorem. The proof of the theorem—which amounts to showing that the linear operator $\mathcal{J} : C(P) \to C(P)$ is bijective—is in turn immediate from the following triangularity property:

$$\big(I_{w_\lambda^{-1}}^{-1} f\big)(\lambda_+) = q_{w_\lambda}^{-1} f(\lambda) + \sum_{\mu \in P,\, \mu \prec \lambda} * f(\mu) \quad \big(f \in C(P),\ \lambda \in P\big). \tag{5.9}$$

Here and below the star symbols $*$ refer to the expansion coefficients of lower terms (with respect to the partial order $\preccurlyeq$) whose precise values are not relevant for the argument of the proof. Indeed, it is clear from the triangularity in Eq. (5.9) that for any $g \in C(P)$ the linear equation $(\mathcal{J} f)(\lambda) = g(\lambda)$ ($\lambda \in P$) can be uniquely solved inductively in λ with respect to the partial order $\preccurlyeq$.

The triangularity in Eq. (5.9) hinges on well-known saturation properties of the convex hull of the orbit of a weight with respect to the action of the finite Weyl group [B,M4]. For our purposes it is enough to recall that for any $\lambda \in P$ the weights in the convex hull of $W_0\lambda$ are given by the saturated set $P(\lambda) := \{\mu \in P \mid \mu_+ \preccurlyeq \lambda_+\}$. For $\lambda, \mu \in P$ one has that: (i) if $\mu \preccurlyeq \lambda$ then $\mu \in P(\lambda)$, and (ii) if $\mu \in P(\lambda)$ then $[\mu, s_\alpha \mu] \subset P(\lambda)$ for any $\alpha \in R$, where $s_\alpha := s_{\alpha,0}$ and $[\mu, s_\alpha\mu]$ refers to the α-string from μ to $s_\alpha\mu$, i.e. $[\mu, s_\alpha\mu] := \{\mu - k\alpha \mid k = 0, \ldots, \langle \mu, \alpha^\vee \rangle\}$.

After these preliminaries we are now in a position to prove the triangularity in question by induction on $\ell(w_\lambda)$ starting from the straightforward case that $\ell(w_\lambda) \leqslant 1$. (The case $\ell(w_\lambda) = 0$ is in fact trivial since then $\lambda \in P^+$ and $(I_{w_\lambda^{-1}}^{-1} f)(\lambda_+) = f(\lambda)$.) It is manifest from the explicit formula for the action of I_j (cf. Eqs. (4.1a), (4.1b)) and the above properties of the saturated set $P(\lambda)$ that for $j = 1, \ldots, n$:

$$\big(I_j^{-1} f\big)(\lambda) = \begin{cases} q_j^{-1} f(s_j\lambda) + \sum_{\mu \in P,\, \mu \prec s_j\lambda} * f(\mu) & \text{if } s_j\lambda \succ \lambda, \\ \sum_{\mu \in P,\, \mu \preccurlyeq s_j\lambda} * f(\mu) & \text{if } s_j\lambda \preccurlyeq \lambda, \end{cases} \tag{5.10}$$

which implies Eq. (5.9) for $\lambda \in P$ such that $\ell(w_\lambda) = 1$. Upon picking $\lambda \in P$ such that the triangularity in Eq. (5.9) holds for all $\mu \in P$ with $\ell(w_\mu) \leqslant \ell(w_\lambda)$, it is readily seen that for any $j \in \{1, \ldots, n\}$ such that $\lambda \prec s_j\lambda$ (or equivalently $\ell(w_{s_j\lambda}) = \ell(w_\lambda) + 1$) one has that

$$\begin{aligned}
\big(I^{-1}_{w^{-1}_{s_j\lambda}} f\big)\big((s_j\lambda)_+\big) &\overset{\text{(i)}}{=} \big(I^{-1}_{w^{-1}_\lambda} I_j^{-1} f\big)(\lambda_+), \\
&\overset{\text{(ii)}}{=} q^{-1}_{w_\lambda}\big(I_j^{-1} f\big)(\lambda) + \sum_{\nu \in P,\, \nu \prec \lambda} *\big(I_j^{-1} f\big)(\nu), \\
&\overset{\text{(iii)}}{=} q^{-1}_{w_{s_j\lambda}} f(s_j\lambda) + \sum_{\mu \in P,\, \mu \prec s_j\lambda} *f(\mu)
\end{aligned} \tag{5.11}$$

(thus completing the induction). Here step (i) of the derivation exploits that $w_{s_j\lambda} = w_\lambda s_j$ (with $\ell(w_\lambda s_j) = \ell(w_\lambda) + 1$) and step (ii) relies on invoking of the induction hypothesis that the triangularity holds for w_λ. Step (iii) follows in turn upon applying Eq. (5.10) to all terms on the second line of Eq. (5.11). Indeed, if $\nu \prec \lambda \prec s_j\lambda$ then $s_j\nu \prec s_j\lambda$ (which is immediate from the definitions if $\nu_+ \prec \lambda_+$ and which follows from the elementary estimates $w_{s_j\nu} \leqslant w_\nu s_j < w_\lambda s_j = w_{s_j\lambda}$ if $\nu_+ = \lambda_+$ and $w_\nu < w_\lambda$). $\square$

5.3. *Macdonald spherical functions*

Let $\xi \in V$. By $e^{i\xi} \in C(P)$ we denote the plane wave $e^{i\xi}(\lambda) := e^{i\langle\lambda,\xi\rangle} = (e^{i\xi}, e^{-\lambda}) = (e^\lambda, e^{i\xi})$, $\lambda \in P$. By definition, the *Macdonald spherical function* Φ_ξ, $\xi \in V$, is the function in $C(P)$ of the form

$$\Phi_\xi := \mathcal{J}\phi_\xi \quad \text{with } \phi_\xi := I(\mathbf{1}_0)e^{i\xi}, \tag{5.12a}$$

where

$$\mathbf{1}_0 := \sum_{w \in W_0} q_w T_w. \tag{5.12b}$$

The Macdonald spherical function is W_0-invariant in the sense that

$$\begin{aligned}
\Phi_\xi \in C(P)^{W_0} &:= \big\{ f \in C(P) \,\big|\, wf = f,\ w \in W_0 \big\} \\
&= \big\{ f \in C(P) \,\big|\, \widehat{T}_w f = q_w f,\ w \in W_0 \big\}.
\end{aligned}$$

Indeed, since $T_j\mathbf{1}_0 = q_j\mathbf{1}_0$ for $1 \leqslant j \leqslant n$ in view of Eq. (5.4a), it is clear that $\widehat{T}_j\Phi_\xi = \widehat{T}_j\mathcal{J}\phi_\xi = \mathcal{J}I_j\phi_\xi = \mathcal{J}q_j\phi_\xi = q_j\Phi_\xi$, $j = 1, \ldots, n$.

The symmetric monomials $m_\lambda := \sum_{\mu \in W_0\lambda} e^\mu$, $\lambda \in P^+$, form a basis of $\mathbb{C}[P]^{W_0}$. For $p = \sum_{\lambda \in P^+} c_\lambda m_\lambda \in \mathbb{C}[P]^{W_0}$ ($c_\lambda \in \mathbb{C}$), we define $\widehat{p(Y)} := \widehat{T}(p(Y))$ where $p(Y) := \sum_{\lambda \in P^+} c_\lambda m_\lambda(Y)$. The center of the affine Hecke algebra is then given by $\mathcal{Z} = \{p(Y) \mid p \in \mathbb{C}[P]^{W_0}\}$ and moreover $\widehat{T}(\mathcal{Z}) = \{\widehat{p(Y)} \mid p \in \mathbb{C}[P]^{W_0}\}$. Clearly the space $C(P)^{W_0}$ is stable under the action of $\widehat{T}(\mathcal{Z})$.

Theorem 5.8 *(Diagonalization). The commuting subalgebra $\widehat{T}(\mathcal{Z}) \subset \widehat{T}(\mathcal{H})$ is diagonalized by the Macdonald spherical function*:

$$\widehat{p(Y)}\Phi_\xi = E_p(\xi)\Phi_\xi \quad \text{with } E_p(\xi) = \big(p, e^{-i\xi}\big), \tag{5.13}$$

for $\xi \in V$ and $p \in \mathbb{C}[P]^{W_0}$.

The proof of this theorem hinges on the intertwining operator and an explicit formula for $\phi_\xi = I(\mathbf{1}_0)e^{i\xi}$ following from the work of Macdonald [M1,M2].

Proposition 5.9. *The function ϕ_ξ (5.12a), (5.12b) is given explicitly by*

$$\phi_\xi(\lambda) = \big(e^{i\xi}, P_\lambda\big), \quad \lambda \in P, \tag{5.14a}$$

where

$$P_\lambda := \sum_{w\in W_0} e^{-w\lambda} \prod_{\alpha\in R^+} \frac{1-q_\alpha^2 e^{w\alpha}}{1-e^{w\alpha}} \tag{5.14b}$$

and $q_\alpha := q_{\alpha,0}$.

Proof. Let us recall from Remark 4.4 that $(I(h)f, p) = (f, \check{T}(h^*)p)$ for $h \in \mathcal{H}_0$, $f \in C(P)$, $p \in \mathbb{C}[P]$, where $\check{T}(\mathcal{H}_0)$ refers to the standard polynomial representation of the finite Hecke algebra generated by the Demazure–Lusztig operators in Eq. (4.3). The lemma is now an immediate consequence of Macdonald's celebrated formula $\mathbf{1}_0 Y^{-\lambda}\mathbf{1}_0 = P_\lambda(Y)\mathbf{1}_0$ ($\lambda \in P$) [M1, Thm. 1] and (with more details) [M2, (4.1.2)] (see also e.g. [NR, Thm. 2.9(a)] and [P, Thm. 6.9]). Indeed, Macdonald's formula implies that $\check{T}(\mathbf{1}_0)e^{-\lambda} = P_\lambda$, whence $\phi_\xi(\lambda) = (\phi_\xi, e^{-\lambda}) = (I(\mathbf{1}_0)e^{i\xi}, e^{-\lambda}) = (e^{i\xi}, \check{T}(\mathbf{1}_0)e^{-\lambda}) = (e^{i\xi}, P_\lambda)$. $\square$

Proposition 5.9 reveals that ϕ_ξ decomposes as a linear combination of plane waves $e^{iw\xi}$, $w \in W_0$ (with coefficients $\prod_{\alpha\in R^+} \frac{1-q_\alpha^2 e^{-i\langle w\xi,\alpha\rangle}}{1-e^{-i\langle w\xi,\alpha\rangle}}$). With this information the proof of Theorem 5.8 reduces to an elementary computation:

$$\widehat{p(Y)}\Phi_\xi = \widehat{p(Y)}\mathcal{J}\phi_\xi = \mathcal{J}I\big(p(Y)\big)\phi_\xi = \mathcal{J}p\phi_\xi = \mathcal{J}\big(p, e^{-i\xi}\big)\phi_\xi = \big(p, e^{-i\xi}\big)\Phi_\xi,$$

where we have used that $pe^{iw\xi} = (p, e^{-i\xi})e^{iw\xi}$ for $w \in W_0$, since $p \in \mathbb{C}[P]^{W_0}$ and $e^\lambda e^{i\xi} = t_\lambda e^{i\xi} = e^{-i\langle\lambda,\xi\rangle}e^{i\xi} = (e^\lambda, e^{-i\xi})e^{i\xi}$.

Remark 5.10. It is immediate from Proposition 5.9 and the W_0-invariance of the Macdonald spherical function Φ_ξ that

$$\Phi_\xi(\lambda) = q_{t_\lambda} \sum_{w\in W_0} e^{i\langle w\xi,\lambda_+\rangle} \prod_{\alpha\in R^+} \frac{1-q_\alpha^2 e^{-i\langle w\xi,\alpha\rangle}}{1-e^{-i\langle w\xi,\alpha\rangle}}, \quad \lambda \in P. \tag{5.15}$$

6. Unitarity

In this section we describe a Hilbert space structure for which our difference-reflection representation becomes unitary.

Here it is always assumed that $q : W \to (0,1)$. We will employ the shorthand notation $X(q^2) := \sum_{w\in X} q_w^2$ for $X \subset W_0$. So in particular, $W_0(q^2)$ and $W_{0,x}(q^2)$ ($x \in V$) represent the (generalized) Poincaré series of W_0 and $W_{0,x}$ associated with q^2, respectively. Let $l^2(P,\delta)$ be the Hilbert space of functions $\{f \in C(P) \mid \langle f, f\rangle_\delta < \infty\}$, where

$$\langle f, g\rangle_\delta := \sum_{\lambda\in P} f(\lambda)\overline{g(\lambda)}\delta_\lambda \quad \big(f, g \in l^2(P,\delta)\big), \tag{6.1a}$$

with

$$\delta_\lambda := \mathcal{N}_0^{-1} q_{u_\lambda}^{-2} = \mathcal{N}_0^{-1} \prod_{\substack{\alpha\in R^+,\, k\in\mathbb{Z}\\ V_{\alpha,k}\in S(\lambda)}} q_{\alpha,k}^{-2}, \quad \mathcal{N}_0 := W_0\big(q^2\big) \tag{6.1b}$$

and $S(\lambda) := S(u_\lambda) = \{V_{\alpha,k} \mid V_{\alpha,k} \text{ separates } \lambda \text{ and } A\}$ (cf. [M4, (2.4.8)]).

Theorem 6.1 *(Unitarity of $\widehat{T}(\mathcal{H})$). The difference-reflection representation $h \to \widehat{T}(h)$ ($h \in \mathcal{H}$) on $C(P)$ restricts to a unitary representation of the affine Hecke algebra into the space of bounded operators on $l^2(P,\delta)$, i.e.*

$$\big\langle \widehat{T}(h) f, g\big\rangle_\delta = \big\langle f, \widehat{T}\big(h^*\big) g\big\rangle_\delta \quad \big(h \in \mathcal{H},\ f, g \in l^2(P,\delta)\big). \tag{6.2}$$

Proof. Let $f, g \in l^2(P,\delta)$. It suffices to show that the actions of $\widehat{T}_j$ ($0 \leqslant j \leqslant n$) and u ($u \in \Omega$) determine bounded operators on $l^2(P,\delta)$ satisfying (i) $\langle \widehat{T}_j f, g\rangle_\delta = \langle f, \widehat{T}_j g\rangle_\delta$ and (ii) $\langle uf, g\rangle_\delta = \langle f, u^{-1} g\rangle_\delta$. Property (ii) follows by performing the change of coordinates $\lambda \to u\lambda$ to the (discrete) integral $\langle uf, g\rangle_\delta$. Indeed, invoking of the symmetry $\delta_{u\lambda} = \delta_\lambda$ (as $S(u\lambda) = S(\lambda)$) then produces the integral $\langle f, u^{-1} g\rangle_\delta$. Property (i) follows in turn by performing the change of coordinates $\lambda \to s_j\lambda$ to the integral $\langle \chi_j s_j f, g\rangle_\delta$, which entails the integral $\langle f, \chi_j s_j g\rangle_\delta$. Here one uses the symmetries $s_j\chi_j = \chi_j^{-1} s_j$ and $\delta_{s_j\lambda} = \chi_j^2(\lambda)\delta_\lambda$ (as $S(s_j\lambda) = S(\lambda) \setminus \{V_j\}$ if $V_j \in S(\lambda)$, $S(s_j\lambda) = S(\lambda)$ if $\lambda \in V_j$, and $S(s_j\lambda) = S(\lambda) \cup \{V_j\}$ otherwise). The computations in question also reveal that the actions of u and s_j (and thus that of $\widehat{T}_j$) are indeed bounded in $l^2(P,\delta)$ (as $\langle uf, uf\rangle_\delta = \langle f, f\rangle_\delta$ and $\langle s_j f, s_j f\rangle_\delta = \langle \chi_j s_j f, \chi_j^{-1} s_j f\rangle_\delta = \langle f, \chi_j s_j \chi_j^{-1} s_j f\rangle_\delta = \langle f, \chi_j^2 f\rangle_\delta$, and χ_j is a bounded function on P). $\square$

Since P^+ is a fundamental domain for the action of W_0 on P, the symmetric subspace $l^2(P,\delta)^{W_0} := l^2(P,\delta) \cap C(P)^{W_0}$ can be identified with the Hilbert space $l^2(P^+,\Delta)$ of functions $\{f : P^+ \to \mathbb{C} \mid \langle f, f\rangle_\Delta < \infty\}$, where

$$\langle f, g\rangle_\Delta := \sum_{\lambda\in P^+} f(\lambda)\overline{g(\lambda)}\Delta_\lambda \quad \big(f, g \in l^2\big(P^+,\Delta\big)\big), \tag{6.3a}$$

with

$$\Delta_\lambda := \sum_{\mu\in W_0\lambda} \delta_\mu = q_{t_\lambda}^{-2} \frac{W_0^\lambda(q^2)}{W_0(q^2)} = \frac{q_{t_\lambda}^{-2}}{W_{0,\lambda}(q^2)} \quad \big(\lambda \in P^+\big) \tag{6.3b}$$

and $W_0^\lambda := \{w_\mu \mid \mu \in W_0\lambda\}$. The first equality in Eq. (6.3b) follows from the relations $q_{u_\mu} = q_{t_\mu} q_{v_\mu}^{-1} = q_{t_{\mu_+}} q_{v_\mu}^{-1}$ and $q_{v_\mu} = q_{w_{w_0\mu}}$ ($\mu \in P$); the second equality is readily inferred upon observing that the mapping $(w, w') \to ww'$ determines a bijection of $W_{0,\lambda} \times W_0^\lambda$ onto W_0 satisfying $\ell(ww') = \ell(w) + \ell(w')$, whence $q_{ww'} = q_w q_{w'}$ and thus $W_0(q^2) = W_{0,\lambda}(q^2) W_0^\lambda(q^2)$.

The following adjointess relations for the basis elements $\widehat{m_\lambda(Y)}$ are an immediate consequence of the unitarity in Theorem 6.1 (recall in this connection also the last paragraph of Section 2).

Corollary 6.2 *(Adjointness Relations in $\widehat{T}(\mathcal{Z})$). The basis operators $\widehat{m_\lambda(Y)}$, $\lambda \in P^+$, spanning $\widehat{T}(\mathcal{Z})$ satisfy the adjointness relations*

$$\big\langle \widehat{m_\lambda(Y)} f, g\big\rangle_\Delta = \big\langle f, \widehat{m_{\lambda^*}(Y)} g\big\rangle_\Delta \quad \big(\lambda \in P^+,\ f, g \in l^2\big(P^+,\Delta\big)\big). \tag{6.4}$$

In particular, it is evident from Corollary 6.2 that the symmetrized operators $(\widehat{m_\lambda(Y)} + \widehat{m_{\lambda^*}(Y)})$ and $i(\widehat{m_\lambda(Y)} - \widehat{m_{\lambda^*}(Y)})$ are self-adjoint in $l^2(P^+,\Delta)$.

Remark 6.3. Let

$$e_q(\lambda) := \prod_{\alpha\in R^+} q_\alpha^{\langle\lambda,\alpha^\vee\rangle}, \quad \lambda\in P.$$

It is instructive to recall to mind that q_{t_λ} and $W_{0,\lambda}(q^2)$ can be conveniently written explicitly in terms of the multiplicity function via the evaluation formula $q_{t_\lambda} = e_q(\lambda_+)$ (by Eq. (6.1b) with the symmetries $q_{t_\lambda} = q_{t_{w_o(\lambda_+)}} = q_{u_{w_o(\lambda_+)}}$, $q_{\alpha,k} = q_\alpha$) and Macdonald's classic product formula

$$W_{0,\lambda}\big(q^2\big) = \prod_{\substack{\alpha\in R^+\\ \langle\lambda,\alpha^\vee\rangle=0}} \frac{1-q_\alpha^2 e_q(\alpha)}{1-e_q(\alpha)}, \tag{6.5}$$

respectively. In particular, evaluation of the RHS of Δ (6.3b) produces

$$\Delta_\lambda = e_q(-2\lambda) \prod_{\substack{\alpha\in R^+\\ \langle\lambda,\alpha^\vee\rangle=0}} \frac{1-e_q(\alpha)}{1-q_\alpha^2 e_q(\alpha)} \quad \big(\lambda\in P^+\big). \tag{6.6}$$

Remark 6.4. Let $\mathrm{Vol}(A) := \int_A \mathrm{d}\xi$, where $\mathrm{d}\xi$ denotes the Lebesgue measure on V, and let

$$\breve{\Delta}(\xi) := \breve{\mathcal{N}}_0^{-1} \prod_{\alpha\in R^+} \left| \frac{1-e^{i\langle\alpha,\xi\rangle}}{1-q_\alpha^2 e^{i\langle\alpha,\xi\rangle}} \right|^2, \quad \breve{\mathcal{N}}_0 := (2\pi)^n |W_0| \,\mathrm{Vol}(A). \tag{6.7}$$

For $\breve{f}, \breve{g}$ in the Hilbert space $L^2(2\pi A, \breve{\Delta}(\xi)\,\mathrm{d}\xi)$, their inner product is written as

$$\langle \breve{f}, \breve{g}\rangle_{\breve{\Delta}} := \int_{2\pi A} \breve{f}(\xi)\overline{\breve{g}(\xi)}\breve{\Delta}(\xi)\,\mathrm{d}\xi. \tag{6.8}$$

It is well known from Macdonald's theory [M1,M2] (cf. also [M3, §10]) that the measure Δ (6.6) turns the Fourier–Macdonald pairing

$$f = \mathcal{F}_q(\breve{f}) := \big\langle \breve{f}, \Phi(\cdot)\big\rangle_{\breve{\Delta}} = \int_{2\pi A} \breve{f}(\xi)\overline{\Phi_\xi(\cdot)}\breve{\Delta}(\xi)\,\mathrm{d}\xi, \tag{6.9a}$$

with the kernel function (cf. Remark 5.10)

$$\Phi_\xi(\lambda) = e_q(\lambda)\big(e^{i\xi}, P_\lambda\big) = e_q(\lambda) \sum_{w\in W_0} e^{i\langle w\xi,\lambda\rangle} \prod_{\alpha\in R^+} \frac{1-q_\alpha^2 e^{-i\langle w\xi,\alpha\rangle}}{1-e^{-i\langle w\xi,\alpha\rangle}} \tag{6.9b}$$

($\xi\in 2\pi A$, $\lambda\in P^+$), into a Hilbert space isomorphism $\mathcal{F}_q : L^2(2\pi A, \breve{\Delta}(\xi)\,\mathrm{d}\xi) \to l^2(P^+, \Delta)$ with the inversion formula given by

$$\breve{f} = \mathcal{F}_q^{-1}(f) = \langle f, \Phi.\rangle_\Delta = \sum_{\lambda\in P^+} f(\lambda)\overline{\Phi.(\lambda)}\Delta_\lambda \tag{6.9c}$$

(where the dot $\cdot$ refers to the suppressed argument). From this perspective, Theorem 5.8 (with $\xi\in 2\pi A$) provides the spectral decomposition $\mathcal{F}_q \circ E_p \circ \mathcal{F}_q^{-1}$ of the bounded normal discrete difference operator $\widehat{p(Y)}$ in the Hilbert space $l^2(P^+, \Delta) \cong l^2(P, \delta)^{W_0}$ (where E_p refers to the multiplication operator $(E_p \check{f})(\xi) := E_p(\xi)\check{f}(\xi)$ on $L^2(2\pi A, \check{\Delta}(\xi)\,d\xi))$. For ω a (quasi-)minuscule weight, the explicit action of the corresponding difference operator $\widehat{m_\omega(Y)}$ in $l^2(P^+, \Delta)$ is provided by Corollary 7.2 below.

7. The explicit action of $\widehat{m_\omega(Y)}$ and associated Pieri formulas

Throughout this section it will be assumed that $\omega \in P^+$ is (quasi-)minuscule (cf. Appendix B below). By computing the action of $\widehat{m_\omega(Y)}$ on $C(P)$ in closed form, Theorem 5.8 gives rise to an explicit Pieri formula for the Macdonald spherical functions. To describe the action in question let us introduce a similarity transformation $\epsilon : C(P) \to C(P)$ and a difference operator $M_\omega : C(P) \to C(P)$ of the form $(\epsilon f)(\lambda) := q_{t_\lambda} f(w_o\lambda)$ $(f \in C(P),\ \lambda \in P)$ and

$$(M_\omega f)(\lambda) := \sum_{\nu \in W_0\omega} \bigl(a_{\lambda,\nu} f(\lambda-\nu) + b_{\lambda,\nu} f(\lambda)\bigr) \quad \bigl(f \in C(P), \lambda \in P\bigr), \tag{7.1a}$$

with

$$a_{\lambda,\nu} := q_{w_{w_\lambda(\lambda-\nu)}} q_{w_{w_\lambda(\lambda-\nu)}w_\lambda} q_{w_\lambda}^{-1} \quad \text{and} \quad b_{\lambda,\nu} := \varepsilon_{\lambda,\nu}\bigl(1-q_0^{-2}\bigr)e_q(w_\lambda\nu), \tag{7.1b}$$

where $e_q(\cdot)$ is as defined in Remark 6.3,

$$\varepsilon_{\lambda,\nu} := \begin{cases} \theta(w_\lambda(\lambda-\nu)) & \text{if } (\lambda-\nu)_+ \neq \lambda_+, \\ \chi(\nu) & \text{if } (\lambda-\nu)_+ = \lambda_+, \end{cases} \tag{7.1c}$$

$\theta(\mu) := \langle \mu_+ - \mu, \rho^\vee\rangle - \ell(w_\mu)$, and $\rho^\vee := \frac{1}{2}\sum_{\alpha\in R^+}\alpha^\vee$.

Theorem 7.1. *One has that* $\widehat{m_\omega(Y)} = \epsilon M_\omega \epsilon^{-1}$.

Corollary 7.2. *The restriction of the action of* $\widehat{m_\omega(Y)}$ *to* $C(P)^{W_0} \cong C(P^+)$ *is given by*

$$\bigl(\widehat{m_\omega(Y)} f\bigr)(\lambda) = U_{\lambda,-\omega}\bigl(q^2\bigr) f(\lambda) + \sum_{\substack{\nu\in W_0\omega \\ \lambda-\nu\in P^+}} V_{\lambda,-\nu}\bigl(q^2\bigr) f(\lambda-\nu) \quad \bigl(f \in C\bigl(P^+\bigr), \lambda \in P^+\bigr), \tag{7.2a}$$

with

$$V_{\lambda,\nu}\bigl(q^2\bigr) := e_q(-\nu) \prod_{\substack{\alpha\in R^+ \\ \langle\lambda,\alpha^\vee\rangle=0 \\ \langle\nu,\alpha^\vee\rangle>0}} \frac{1-q_\alpha^2 e_q(\alpha)}{1-e_q(\alpha)} \tag{7.2b}$$

and

$$U_{\lambda,\mu}\bigl(q^2\bigr) := \begin{cases} 0 & \text{for } \mu_+ \text{ minuscule}, \\ \sum_{\nu\in W_0\mu} e_q(\nu) - \sum_{\substack{\nu\in W_0\mu \\ \lambda+\nu\in P^+}} V_{\lambda,\nu}\bigl(q^2\bigr) & \text{for } \mu_+ \text{ quasi-minuscule}. \end{cases} \tag{7.2c}$$

The diagonalization Theorem 5.8 combined with the symmetric reduction (Corollary 7.2) of the explicit action of $\widehat{m_\omega(Y)}$ (Theorem 7.1), immediately produces the following Pieri formula expressing the multiplicative action of m_ω in $\mathbb{C}[P]^{W_0}$ in terms of the Macdonald spherical basis $p_\lambda := e_q(\lambda)P_{\lambda^*}$, $\lambda \in P^+$.

Corollary 7.3 *(Pieri formula). One has that*

$$m_\omega p_\lambda = U_{\lambda,\omega}\big(q^2\big)p_\lambda + \sum_{\substack{\nu\in W_0\omega\\ \lambda+\nu\in P^+}} V_{\lambda,\nu}\big(q^2\big)p_{\lambda+\nu} \quad \big(\lambda\in P^+\big). \tag{7.3}$$

The Pieri formula in Corollary 7.3 is a special case of Pieri formulas for the Macdonald spherical functions obtained via degeneration descending from the level of the Macdonald polynomials [DE]. For root systems of type A and ω minuscule the Pieri formula under consideration amounts to a classic Pieri formula for the Hall–Littlewood polynomials due to Morris [Mo] (cf. Appendix C below).

Remark 7.4. The factor $\varepsilon_{\lambda,\nu}$ in the coefficients of M_ω (7.1a), (7.1c) takes values in $\{0,1\}$. For ω minuscule the factor in question vanishes (so $b_{\lambda,\nu}=0$) and the coefficient $a_{\lambda,\nu}$ simplifies to $q^2_{w_{w_\lambda(\lambda-\nu)}}$.

Remark 7.5. It is manifest from the relations in Corollary 6.2 that the adjoint of $\widehat{m_\omega(Y)}$ (7.2a)–(7.2c) in the Hilbert space $\ell^2(P^+,\Delta)$ is given by the action of $\widehat{m_{\omega^*}(Y)}$ on $\ell^2(P^+,\Delta)$. More generally, it follows from the unitarity in Theorem 6.1 that the adjoint of $\widehat{m_\omega(Y)}$ (7.1a), (7.1c) in the Hilbert space $\ell^2(P,\delta)$ is given by the action of $\widehat{m_{\omega^*}(Y)}$ on $\ell^2(P,\delta)$. Since ω^* is (quasi-)minuscule if (and only if) ω is (quasi-)minuscule, this means that these adjoints are given by the same formulas of Corollary 7.2 and Theorem 7.1, respectively, with ω being replaced by ω^*. In particular, for ω quasi-minuscule the operators in question are self-adjoint (as in this situation $\omega^*=\omega$).

Remark 7.6. Corollary 7.2 provides an explicit formula for the discretization of the Laplacian with delta potential associated with R from Ref. [D] (cf. Remark 5.5).

Remark 7.7. The standard polynomial representation of the affine Hecke algebra in terms of Demazure–Lusztig operators (dual to our integral-reflection representation $I(\mathcal{H})$) was extended by Cherednik to a representation of the double affine Hecke algebra [C,M4]. The representation in question contains Dunkl-type q-difference-reflection operators that were used for the construction of Macdonald's commuting q-difference operators diagonalized by the Macdonald polynomials [C,M4]. Since Macdonald's polynomials are a q-deformation of the Macdonald spherical functions [M3], our difference-reflection representation $\widehat{T}(\mathcal{H})$ is expected to correspond to a suitable degeneration of Cherednik's representation of the double affine Hecke algebra (therewith linking the latter representation to the differential-reflection representation of the graded affine Hecke algebra in Ref. [EOS]).

7.1. Proof of Theorem 7.1

Since $\ell(w_o w)=\ell(w_o)-\ell(w)$ for any $w\in W_0$, it follows that $q_{w_o w^{-1}}=q_{w_o}q_w^{-1}$ and $T^{-1}_{w_o w^{-1}}T_{w_o}=T_w$, whence

$$q_{w_o w^{-1}}\big(I^{-1}_{w_o w^{-1}}I_{w_o}f\big)(\lambda)=q_{w_o}q_w^{-1}(I_w f)(\lambda) \quad \big(f\in C(P),\ \lambda\in P,\ w\in W_0\big). \tag{7.4}$$

Combined with Eq. (5.5), this yields the following stability property for $w, w'\in W_0$ and $\lambda\in P^+$

$$q_w^{-1}(I_w f)(\lambda)=q_{w'}^{-1}(I_{w'}f)(\lambda) \quad \text{if } w\big(w'\big)^{-1}\in W_{0,\lambda}. \tag{7.5}$$

Let us now abbreviate $I(m_\omega(Y))=\sum_{\nu\in W_0\omega}t_\nu$ as $m_\omega(t)$. In view of the intertwining relations (Theorem 5.1) and the bijectivity of the intertwining operator $\mathcal{J}$ (Theorem 5.6), it is sufficient for proving the theorem to show that

$$\mathcal{J}m_\omega(t)=\epsilon M_\omega\epsilon^{-1}\mathcal{J},$$

or equivalently (since $m_\omega(Y)\in\mathcal{Z}(\mathcal{H})$), that

$$\epsilon^{-1}\mathcal{J}I_{w_o}m_\omega(t)=M_\omega\epsilon^{-1}\mathcal{J}I_{w_o}.$$

Relation (7.4) and stability properties in Eqs. (5.5), (7.5) imply that

$$\big(\epsilon^{-1}\mathcal{J}I_{w_o}f\big)(\lambda)=q_{w_o}q_{w_\lambda}^{-1}(I_{w_\lambda}f)(\lambda_+)\quad\big(f\in C(P),\ \lambda\in P\big),\tag{7.6}$$

and thus (using again that $m_\omega(Y)\in\mathcal{Z}(\mathcal{H})$)

$$\big(\epsilon^{-1}\mathcal{J}I_{w_o}m_\omega(t)f\big)(\lambda)=q_{w_o}q_{w_\lambda}^{-1}\sum_{\nu\in W_0\omega}(I_{w_\lambda}f)\big(w_\lambda(\lambda-\nu)\big).$$

That this expression is equal to $(M_\omega\epsilon^{-1}\mathcal{J}I_{w_o}f)(\lambda)$ hinges on the identity

$$a_{\lambda,\nu}\big(\epsilon^{-1}\mathcal{J}I_{w_o}f\big)(\lambda-\nu)=q_{w_o}q_{w_\lambda}^{-1}\big((I_{w_\lambda}f)\big(w_\lambda(\lambda-\nu)\big)-\varepsilon_{\lambda,\nu}\big(1-q_0^{-2}\big)e_q(w_\lambda\nu)(I_{w_\lambda}f)(\lambda_+)\big)\tag{7.7}$$

(combined with Eq. (7.6)). To infer the identity in Eq. (7.7) the following lemmas are instrumental.

Lemma 7.8. *For $\lambda\in P^+$ and $\nu\in W_0\omega$, we are in either one of the following two situations*: (i) *if $(\lambda-\nu)_+\neq\lambda$ then $w_{\lambda-\nu}\in W_{0,\lambda}$ and*

$$\theta(\lambda-\nu)=\begin{cases}1 & \text{for } \nu\in R(w_{\lambda-\nu}),\\ 0 & \text{for } \nu\notin R(w_{\lambda-\nu}),\end{cases}\quad \text{where } R(w):=R^+\cap w^{-1}\big(R^-\big),$$

or (ii) *if $(\lambda-\nu)_+=\lambda$ then $w_{\lambda-\nu}\nu=-\alpha_j$ for some $j\in\{1,\dots,n\}$, moreover, $s_jw_{\lambda-\nu}\in W_{0,\lambda}$, $\theta(\lambda-\nu)=0$, $R(w_{\lambda-\nu})=R(s_jw_{\lambda-\nu})\cup\{\nu\}$ and $q_j=q_0$.*

Before embarking on the proof of this lemma, let us first highlight some crucial (though elementary) observations. For any $\lambda\in P$ the set $R(w_\lambda)$ is given by

$$R(w_\lambda)=\big\{\alpha\in R^+\ \big|\ \langle\lambda,\alpha^\vee\rangle<0\big\}$$

(cf. [M4, Eq. (2.4.4)]) and for any simple root $\alpha_j\in R(w_\lambda)$ we have that $w_\lambda s_j=w_{s_j\lambda}$ with $\ell(w_{s_j\lambda})=\ell(w_\lambda)-1$. Given $\lambda\in P^+$, $\nu\in W_0\omega$ and $\mu=s_j(\lambda-\nu)$ with $\alpha_j\in R(w_{\lambda-\nu})$, we are in one of the following three cases:

(A) $\langle\lambda,\alpha_j^\vee\rangle=0$ and $\langle\nu,\alpha_j^\vee\rangle=1$. Then $\mu=\lambda-s_j\nu\in\lambda-W_0\omega$ and $\theta(\lambda-\nu)=\theta(\mu)$. (Notice that $s_j\in W_{0,\lambda}$.)
(B) $\langle\lambda,\alpha_j^\vee\rangle=0$ and $\langle\nu,\alpha_j^\vee\rangle=2$. Then $\mu=\lambda-s_j\nu=\lambda+\alpha_j\in\lambda-W_0\omega$ and $\theta(\lambda-\nu)=\theta(\mu)+1$. (Notice that $s_j\in W_{0,\lambda}$ and $\nu=\alpha_j$.)
(C) $\langle\lambda,\alpha_j^\vee\rangle=1$ and $\langle\nu,\alpha_j^\vee\rangle=2$. Then $\mu=\lambda$ and $\theta(\lambda-\nu)=\theta(\mu)=0$. (Notice that $w_{\lambda-\nu}=s_j$ and $\nu=\alpha_j$.)

It is moreover evident that in the cases (B) and (C), which occur only when ω is quasi-minuscule, one has that $q_j=q_0$ (since $\alpha_j\in W_0\omega$ with $\omega=\alpha_0$).

Proof of Lemma 7.8. It is sufficient to restrict attention to the case that $\lambda-\nu\notin P^+$ (as for $\lambda-\nu\in P^+$ the lemma is trivial). For a reduced decomposition $w_{\lambda-\nu}=s_{j_\ell}\cdots s_{j_1}$ with $\ell=\ell(w_{\lambda-\nu})\geqslant 1$, we write

$$\nu_k:=s_{j_k}\cdots s_{j_1}\nu\quad\text{for } k=0,\dots,\ell$$

and

$$\beta_k := s_{j_1} \cdots s_{j_k} \alpha_{j_{k+1}} \quad \text{for } k = 0, \ldots, \ell - 1$$

(with the conventions that $\nu_0 := \nu$ and $\beta_0 := \alpha_{j_1}$). This means that

$$R(w_{\lambda-\nu}) = \{\beta_0, \ldots, \beta_{\ell-1}\}$$

(cf. [M4, (2.2.9)]). It is immediate from the observations (A)–(C) above that the minimal sequence of weights taking $\lambda - \nu$ to $(\lambda - \nu)_+$ by successive application of the simple reflections in our reduced decomposition of $w_{\lambda-\nu}$ is either of the form (situation (i)):

$$\lambda - \nu = \lambda - \nu_0 \xrightarrow{s_{j_1}} \lambda - \nu_1 \xrightarrow{s_{j_2}} \cdots \xrightarrow{s_{j_{\ell-1}}} \lambda - \nu_{\ell-1} \xrightarrow{s_{j_\ell}} \lambda - \nu_\ell = (\lambda - \nu)_+, \tag{7.8}$$

or of the form (situation (ii)):

$$\lambda - \nu = \lambda - \nu_0 \xrightarrow{s_{j_1}} \lambda - \nu_1 \xrightarrow{s_{j_2}} \cdots \xrightarrow{s_{j_{\ell-1}}} \lambda - \nu_{\ell-1} \xrightarrow{s_{j_\ell}} \lambda = (\lambda - \nu)_+, \tag{7.9}$$

because case (C) can at most occur at the last step: $\lambda - \nu_{\ell-1} \xrightarrow{s_{j_\ell}} \longrightarrow (\lambda - \nu)_+$ (as this case takes us back to P^+). In situation (i) (i.e. case (C) does not occur at the last step) we have that

$$w_{\lambda-\nu} \in W_{0,\lambda} \quad \text{and} \quad (\lambda - \nu)_+ \neq \lambda,$$

whereas in situation (ii) (i.e. case (C) does occur at the last step) we have that

$$s_{j_\ell} w_{\lambda-\nu} = s_{j_{\ell-1}} \cdots s_{j_1} \in W_{0,\lambda}, \qquad q_{j_\ell} = q_0 \quad \text{and} \quad (\lambda - \nu)_+ = \lambda.$$

Moreover, in the latter situation $\nu_{\ell-1} = \alpha_{j_\ell}$, i.e. $w_{\lambda-\nu}\nu = -\alpha_{j_\ell}$ and

$$\nu = (s_{j_\ell} w_{\lambda-\nu})^{-1} \alpha_{j_\ell} = s_{j_1} \cdots s_{j_{\ell-1}} \alpha_{j_\ell} = \beta_{\ell-1} \in R(w_{\lambda-\nu}) \setminus R(s_{j_\ell} w_{\lambda-\nu}).$$

It remains to compute $\theta(\lambda - \nu)$. Since $\theta((\lambda - \nu)_+) = 0$, it is clear from the observations (A)–(C) that $\theta(\lambda - \nu)$ is equal to the number of times case (B) occurs in the above sequences, i.e. the number of times that

$$\langle \nu_k, \alpha_{j_{k+1}}^\vee \rangle = 2 \quad \text{for } k = 0, \ldots, \ell' - 1,$$

with $\ell' = \ell$ in situation (i) and $\ell' = \ell - 1$ in situation (ii). Since for $k = 0, \ldots, \ell' - 1$:

$$\langle \nu_k, \alpha_{j_{k+1}}^\vee \rangle = 2 \quad \Leftrightarrow \quad \langle \nu, \beta_k^\vee \rangle = 2 \quad \Leftrightarrow \quad \nu = \beta_k,$$

it is clear that in situation (i) $\theta(\lambda - \nu)$ is equal to 0 or 1 depending whether $\nu \notin R(w_{\lambda-\nu})$ or $\nu \in R(w_{\lambda-\nu})$, respectively, and in situation (ii) $\theta(\lambda - \nu) = 0$ (because now $\nu = \beta_{\ell'}$). $\square$

Lemma 7.9. *For $\lambda \in P^+$ and $\nu \in W_0\omega$, the following explicit formula holds*

$$q_{w_{\lambda-\nu}} (I_{w_{\lambda-\nu}} f)\big((\lambda - \nu)_+\big) = f(\lambda - \nu) - \theta(\lambda - \nu)\big(1 - q_0^{-2}\big) e_q(\nu) f(\lambda). \tag{7.10}$$

The proof exploits the elementary identities (for $f \in C(P)$, $\lambda \in P$, $j = 1, \ldots, n$)

$$q_j(I_j f)(\lambda) = \begin{cases} f(s_j\lambda) = f(\lambda - \alpha_j) & \text{if } \langle \lambda, \alpha_j^\vee \rangle = 1, \\ f(\lambda - 2\alpha_j) + (1 - q_j^2) f(\lambda - \alpha_j) & \text{if } \langle \lambda, \alpha_j^\vee \rangle = 2 \end{cases} \tag{7.11}$$

and $q_j^{-1}(I_j)(s_j\lambda) = f(\lambda)$ if $\langle \lambda, \alpha_j^\vee \rangle = 0$ (cf. Eq. (7.5)).

Proof of Lemma 7.9. The proof of the lemma employs induction on $\ell(w_{\lambda-\nu})$ starting from the trivial base $\lambda - \nu \in P^+$. Let $\ell(w_{\lambda-\nu}) > 1$ and s_j ($1 \leqslant j \leqslant n$) be such that

$$\ell(w_{\lambda-\nu} s_j) = \ell(w_{\lambda-\nu}) - 1$$

(i.e. $\alpha_j \in R(w_{\lambda-\nu})$). From the observations following the statement of Lemma 7.8 it is clear that $w_{\lambda-\nu} s_j = w_{s_j(\lambda-\nu)}$ with either $s_j(\lambda - \nu) = \lambda - s_j\nu$ (cases (A) and (B)) or $s_j(\lambda - \nu) = \lambda (\in P^+)$ (case (C)). In the latter case $w_{\lambda-\nu} = s_j$ and the statement of the lemma reduces to the first case of Eq. (7.11) (with λ replaced by $\lambda - \nu$). Moreover, in the cases (A) and (B) invoking of the induction hypothesis yields

$$\begin{aligned} q_{w_{\lambda-\nu}}(I_{w_{\lambda-\nu}} f)\big((\lambda - \nu)_+\big) &= q_{w_{\lambda-s_j\nu}} q_j (I_{w_{\lambda-s_j\nu}} I_j f)\big((\lambda - s_j\nu)_+\big) \\ &= q_j(I_j f)(\lambda - s_j\nu) \\ &\quad - q_j \theta(\lambda - s_j\nu)\big(1 - q_0^{-2}\big) e_q(s_j\nu)(I_j f)(\lambda) \end{aligned} \tag{7.12}$$

(where we have used that $(\lambda - s_j\nu)_+ = (\lambda - \nu)_+$). In case (A), one has that

$$q_j(I_j f)(\lambda - s_j\nu) = f(\lambda - \nu)$$

(by the first case of Eq. (7.11) with λ replaced by $\lambda - s_j\nu$) and

$$(I_j f)(\lambda) = q_j f(\lambda)$$

(as $s_j \in W_{0,\lambda}$), which completes the induction step for this situation upon observing that $\theta(\lambda - s_j\nu) = \theta(\lambda - \nu)$, $e_q(s_j\nu) = e_q(\nu) q_j^{-2\langle \nu, \alpha_j^\vee \rangle} = e_q(\nu) q_j^{-2}$. In case (B) we have that $\theta(\lambda - s_j\nu) = 0$ (since $0 \leqslant \theta(\lambda - s_j\nu) < \theta(\lambda - \nu) \leqslant 1$ (cf. Lemma 7.8)) and

$$q_j(I_j f)(\lambda - s_j\nu) = f(\lambda - \nu) - q_0^2\big(1 - q_0^{-2}\big) f(\lambda)$$

(by the second case of Eq. (7.11) with λ replaced by $\lambda - s_j\nu$ and the fact that $q_j = q_0$), which completes the induction step for this situation upon observing that $\theta(\lambda - \nu) = 1$ and $e_q(\nu) = e_q(\alpha_j) = q_j^2 = q_0^2$ (as $e_q(\alpha_j) = e_q(-\alpha_j) q_j^{2\langle \alpha_j, \alpha_j^\vee \rangle} = e_q(-\alpha_j) q_j^4$). $\square$

We are now in a position to verify Eq. (7.7) by making the action of the operator on the LHS explicit:

$$\begin{aligned} \big(\epsilon^{-1} \mathcal{J} I_{w_o} f\big)(\lambda - \nu) &\overset{\text{Eq. (7.6)}}{=} q_{w_o} q_{w_{\lambda-\nu}}^{-1} (I_{w_{\lambda-\nu}} f)\big((\lambda - \nu)_+\big) \\ &\overset{\text{Eq. (7.5)}}{=} q_{w_o} q_{w_{w_\lambda(\lambda-\nu)} w_\lambda}^{-1} (I_{w_{w_\lambda(\lambda-\nu)} w_\lambda} f)\big((\lambda - \nu)_+\big). \end{aligned}$$

For $(\lambda-\nu)_+ \neq \lambda_+$, Lemma 7.8 (with λ and ν replaced by λ_+ and $w_\lambda \nu$) ensures that $w_{w_\lambda(\lambda-\nu)} \in W_{0,\lambda_+}$, whence

$$\ell(w_{w_\lambda(\lambda-\nu)} w_\lambda) = \ell(w_{w_\lambda(\lambda-\nu)}) + \ell(w_\lambda)$$

and we may rewrite the expression in question as

$$\begin{aligned}
&q_{w_o} q^{-1}_{w_{w_\lambda(\lambda-\nu)} w_\lambda} (I_{w_{w_\lambda(\lambda-\nu)}} I_{w_\lambda} f)\big((\lambda-\nu)_+\big) \\
&\overset{\text{Lem. 7.9}}{=} a^{-1}_{\lambda,\nu} q_{w_o} q^{-1}_{w_\lambda} \big((I_{w_\lambda} f)\big(w_\lambda(\lambda-\nu)\big) - \theta\big(w_\lambda(\lambda-\nu)\big)\big(1-q_0^{-2}\big) e_q(w_\lambda \nu)(I_{w_\lambda} f)(\lambda_+)\big),
\end{aligned}$$

which proves Eq. (7.7) when $(\lambda-\nu)_+ \neq \lambda_+$. Similarly, for $(\lambda-\nu)_+ = \lambda_+$ we rewrite the expression under consideration as

$$\begin{aligned}
&q_{w_o} q^{-1}_{w_{w_\lambda(\lambda-\nu)} w_\lambda} (I_{w_{w_\lambda(\lambda-\nu)} w_\lambda} f)(\lambda_+) \\
&\overset{\text{Eq. (5.4a)}}{=} q_{w_o} q^{-1}_{w_{w_\lambda(\lambda-\nu)} w_\lambda} \big((I_j I_{s_j w_{w_\lambda(\lambda-\nu)} w_\lambda} f)(\lambda_+) \\
&\qquad - \chi\big((s_j w_{w_\lambda(\lambda-\nu)} w_\lambda)^{-1} \alpha_j\big)\big(q_j - q_j^{-1}\big)(I_{s_j w_{w_\lambda(\lambda-\nu)} w_\lambda} f)(\lambda_+)\big) \\
&= a^{-1}_{\lambda,\nu} q_{w_o} q^{-1}_{w_\lambda} \big((I_{w_\lambda} f)\big(w_\lambda(\lambda-\nu)\big) - \chi(\nu)\big(1-q_0^{-2}\big) q^2_{w_{w_\lambda(\lambda-\nu)}} (I_{w_\lambda} f)(\lambda_+)\big).
\end{aligned}$$

In the last step it was used that for j chosen as in Lemma 7.8 (with λ and ν replaced by λ_+ and $w_\lambda \nu$), one has that

$$(s_j w_{w_\lambda(\lambda-\nu)} w_\lambda)^{-1} \alpha_j = \nu, \quad s_j w_{w_\lambda(\lambda-\nu)} \in W_{0,\lambda_+}, \qquad \theta\big(w_\lambda(\lambda-\nu)\big) = 0,$$

and $q_j = q_0$. It thus follows for the first term that

$$(I_j I_{s_j w_{w_\lambda(\lambda-\nu)} w_\lambda} f)(\lambda_+) = (I_{w_{w_\lambda(\lambda-\nu)}} I_{w_\lambda} f)\big((\lambda-\nu)_+\big) \overset{\text{Lem. 7.9}}{=} q^{-1}_{w_{w_\lambda(\lambda-\nu)}} (I_{w_\lambda} f)\big(w_\lambda(\lambda-\nu)\big)$$

and for the second term that

$$(I_{s_j w_{w_\lambda(\lambda-\nu)} w_\lambda} f)(\lambda_+) \overset{\text{Eq. (7.5)}}{=} q_{w_{w_\lambda(\lambda-\nu)}} q_0^{-1} (I_{w_\lambda} f)(\lambda_+),$$

where we have exploited that

$$\ell(s_j w_{w_\lambda(\lambda-\nu)} w_\lambda) = \ell(s_j w_{w_\lambda(\lambda-\nu)}) + \ell(w_\lambda) = \ell(w_{w_\lambda(\lambda-\nu)}) + \ell(w_\lambda) - 1.$$

The case $(\lambda-\nu)_+ = \lambda_+$ of the identity in Eq. (7.7) now follows from the fact that $q^2_{w_{w_\lambda(\lambda-\nu)}} = e_q(w_\lambda \nu)$. Indeed, for any $w \in W_0$ and $\mu \in P$ one has that

$$\langle w^{-1}\mu, \rho^\vee \rangle = \langle \mu, \rho^\vee \rangle + \sum_{\alpha \in R(w)} \langle w^{-1}\mu, \alpha^\vee \rangle \tag{7.13a}$$

and

$$e_q\big(w^{-1}\mu\big) = e_q(\mu) \prod_{\alpha \in R(w)} q_\alpha^{2\langle w^{-1}\mu, \alpha^\vee \rangle} \tag{7.13b}$$

(cf. [M4, Eq. (1.5.3)]), and

$$q_w = \prod_{\alpha \in R(w)} q_\alpha \tag{7.14}$$

(cf. Eq. (2.5)). Lemma 7.8 (with λ and ν replaced by λ_+ and $w_\lambda \nu$) and properties (7.13a), (7.13b) with $\mu = \alpha_j = -w_{w_\lambda(\lambda-\nu)} w_\lambda \nu$ and $w = s_j w_{w_\lambda(\lambda-\nu)}$ entail that

$$e_q(w_\lambda \nu) = q_0^2 \prod_{\alpha \in R(s_j w_{w_\lambda(\lambda-\nu)})} q_\alpha^{2\langle w_\lambda \nu, \alpha^\vee \rangle} \tag{7.15}$$

and

$$\begin{aligned} \ell(w_{w_\lambda(\lambda-\nu)}) &\overset{\theta(w_\lambda(\lambda-\nu))=0}{=} \langle w_\lambda \nu, \rho^\vee \rangle \\ &= 1 + \sum_{\alpha \in R(s_j w_{w_\lambda(\lambda-\nu)})} \langle w_\lambda \nu, \alpha^\vee \rangle = 1 + \ell(s_j w_{w_\lambda(\lambda-\nu)}). \end{aligned} \tag{7.16}$$

Since $\langle w_\lambda \nu, \alpha^\vee \rangle \leqslant 1$ for $\alpha \in R(s_j w_{w_\lambda(\lambda-\nu)})$ in view of Lemma 7.8, it follows from Eq. (7.16) that in fact $\langle w_\lambda \nu, \alpha^\vee \rangle = 1$ for $\alpha \in R(s_j w_{w_\lambda(\lambda-\nu)})$. We thus conclude from Eq. (7.15) that

$$e_q(w_\lambda \nu) = q_0^2 \prod_{\alpha \in R(s_j w_{w_\lambda(\lambda-\nu)})} q_\alpha^2 \overset{\text{Lem. 7.8}}{=} \prod_{\alpha \in R(w_{w_\lambda(\lambda-\nu)})} q_\alpha^2 \overset{\text{Eq. (7.14)}}{=} q^2_{w_{w_\lambda(\lambda-\nu)}}.$$

7.2. Proof of Corollary 7.2

It is immediate from Theorem 7.1 that the action of $\widehat{m_\omega(Y)}$ reduces to an action on $C(P)^{W_0} \cong C(P^+)$ of the form in Eq. (7.2a) with

$$\begin{aligned} V_{\lambda,-\nu}(q^2) &= q_{t_\lambda} q_{t_{\lambda-\nu}}^{-1} \sum_{\substack{\nu' \in W_0\omega \\ (\lambda-\nu')_+ = \lambda-\nu}} q^2_{w_{\lambda-\nu'}} \overset{\text{Lem. 7.8}}{=} e_q(\nu) \sum_{\mu \in W_{0,\lambda}(\lambda-\nu)} q^2_{w_\mu} \\ &= e_q(\nu) W_{0,\lambda}^{\lambda-\nu}(q^2) = e_q(\nu) W_{0,\lambda}(q^2)/(W_{0,\lambda} \cap W_{0,\lambda-\nu})(q^2) \end{aligned} \tag{7.17}$$

and

$$\begin{aligned} U_{\lambda,-\omega}(q^2) &= \sum_{\substack{\nu \in W_0\omega \\ (\lambda-\nu)_+ = \lambda}} q^2_{w_{\lambda-\nu}} + (1 - q_0^{-2}) \sum_{\nu \in W_0\omega} \varepsilon_{\lambda,\nu} e_q(\nu) \\ &\overset{\text{Lem. 7.8}}{=} \sum_{\substack{\nu \in W_0\omega \\ (\lambda-\nu)_+ = \lambda}} q^2_{w_{\lambda-\nu}} + (1 - q_0^{-2}) \sum_{\substack{\nu \in W_0\omega \\ w_{\lambda-\nu}\lambda = \lambda}} \theta(\lambda-\nu) e_q(\nu). \end{aligned} \tag{7.18}$$

This proves Corollary 7.2 with $V_{\lambda,-\nu}(q^2)$ and $U_{\lambda,-\omega}(q^2)$ given by Eqs. (7.17) and (7.18), respectively. The coefficient $V_{\lambda,\nu}(q^2)$ can be recasted in the form given by Eq. (7.2b) upon invoking Macdonald's product formula (6.5) and the coefficient $U_{\lambda,\omega}(q^2)$ can be rewritten in the form given by Eq. (7.2c) upon comparing the corresponding Pieri formula of the form in Corollary 7.3 with [DE, Eqs. (2.3a)–(2.3c)].

Acknowledgments

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Appendix A. Braid relation for A_2

In this appendix we verify the braid relation (3.3b) for the root system A_2 via a direct computation. For the root systems B_2 and G_2 the corresponding computation is analogous (though increasingly tedious).

For $R = A_2$ the braid relation reads:

$$\widehat{T}_1\widehat{T}_2\widehat{T}_1 = \widehat{T}_2\widehat{T}_1\widehat{T}_2, \tag{A.1a}$$

with

$$\widehat{T}_1 = q + \chi_1(s_1 - 1), \qquad \widehat{T}_2 = q + \chi_2(s_2 - 1). \tag{A.1b}$$

Multiplication of the product

$$\widehat{T}_1\widehat{T}_2 = q^2 + q\chi_1(s_1 - 1) + q\chi_2(s_2 - 1) + \chi_1(s_1 - 1)\chi_2(s_2 - 1)$$

from the right by $\widehat{T}_1$ produces

$$\begin{aligned}\widehat{T}_1\widehat{T}_2\widehat{T}_1 = {} & q^3 + 2q^2\chi_1(s_1 - 1) + q^2\chi_2(s_2 - 1) + \chi_1(s_1 - 1)\chi_2(s_2 - 1)\chi_1(s_1 - 1)\\ & + q\chi_1(s_1 - 1)\chi_1(s_1 - 1) + q\chi_1(s_1 - 1)\chi_2(s_2 - 1) + q\chi_2(s_2 - 1)\chi_1(s_1 - 1).\end{aligned}$$

Swapping the indices 1 and 2 yields a corresponding formula for the product $\widehat{T}_2\widehat{T}_1\widehat{T}_2$. By comparing both formulas it is seen that the braid relation (A.1a) amounts to the following identity:

$$\begin{aligned}& q^2\chi_1(s_1 - 1) + q\chi_1(s_1 - 1)\chi_1(s_1 - 1) + \chi_1(s_1 - 1)\chi_2(s_2 - 1)\chi_1(s_1 - 1)\\ &\quad = q^2\chi_2(s_2 - 1) + q\chi_2(s_2 - 1)\chi_2(s_2 - 1) + \chi_2(s_2 - 1)\chi_1(s_1 - 1)\chi_2(s_2 - 1).\end{aligned} \tag{A.2}$$

Upon acting with both sides of Eq. (A.2) on an arbitrary function $f : P \to \mathbb{C}$, it is sufficient to verify the resulting equality evaluated at the points of a finite W_0-invariant set of weights representing the facets of the Coxeter complex for W_0 (in view of Lemma 3.2). A convenient choice for such a set of facet representatives is displayed in Fig. 1 and the corresponding values confirming the equality of both sides of the identity at these points are collected in Fig. 2, where

$$\begin{aligned}f_0 := {} & f(\omega_1 + \omega_2) + f(\omega_1 - 2\omega_2) + f(-2\omega_1 + \omega_2)\\ & - f(-\omega_1 - \omega_2) - f(-\omega_1 + 2\omega_2) - f(2\omega_1 - \omega_2).\end{aligned} \tag{A.3}$$

Appendix B. Affine intertwining relations

In this appendix we prove the affine intertwining relations in Lemma 5.3 (therewith completing the proof of the intertwining property in Theorem 5.1).

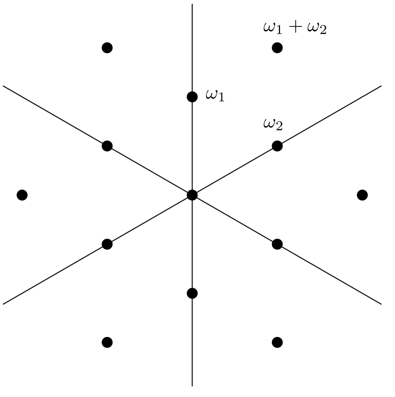

Fig. 1. The set $W_0\{0,\omega_1,\omega_2,\omega_1+\omega_2\}$ consisting of 13 weights representing the facets of the Coxeter complex of W_0 for $R=A_2$.

λ	LHS = RHS
$0, \pm\omega_1, \pm\omega_2, \pm(\omega_1-\omega_2)$	0
$\omega_1+\omega_2$	$-q^{-3}f_0$
$-\omega_1-\omega_2$	$q^3 f_0$
$\omega_1-2\omega_2,\ -2\omega_1+\omega_2$	$-qf_0$
$-\omega_1+2\omega_2,\ 2\omega_1-\omega_2$	$q^{-1}f_0$

Fig. 2. Values of both sides of Eq. (A.2) upon acting on an arbitrary function $f: P\to\mathbb{C}$ and evaluation at the points λ of $W_0\{0,\omega_1,\omega_2,\omega_1+\omega_2\}$. (Here f_0 is given by Eq. (A.3).)

B.1. Preparations: some properties related to (quasi-)minuscule weights

The proof of the affine intertwining relations is based on properties of certain special elements in W and $\mathcal{H}$ associated with the minuscule and quasi-minuscule weights. Let us recall in this connection that the minuscule weights ω are characterized by the property that $0\leqslant\langle\omega,\alpha^\vee\rangle\leqslant 1$ for all $\alpha\in R^+$, whereas the quasi-minuscule weight $\omega=\alpha_0$ is characterized by the property that $0\leqslant\langle\omega,\alpha^\vee\rangle\leqslant 2$ for all $\alpha\in R^+$ with the upper bound 2 being reached only *once* (viz. for $\alpha=\alpha_0$).

Lemma B.1. *Let $\mu\in P$.*

(i) *If $\omega\in P^+$ is minuscule, then $\mu_++w_\mu\omega\in P^+$.*
(iia) *If $\mu_++w_\mu\alpha_0\notin P^+$, then $w_\mu\alpha_0=-\alpha_j$ for some $1\leqslant j\leqslant n$ and moreover $\langle\mu_+,\alpha_j^\vee\rangle=1$.*
(iib) *If $\mu_++w_\mu\alpha_0\in P^+$ with $w_\mu\alpha_0\in R^-$, then $\langle\mu_+,w_\mu\alpha_0^\vee\rangle\leqslant-2$.*

Proof. (i) For any $1\leqslant j\leqslant n$, one has that $\langle\mu_++w_\mu\omega,\alpha_j^\vee\rangle\geqslant\langle\mu_+,\alpha_j^\vee\rangle-1\geqslant-1$. The statement now amounts to the observation that lower bound -1 cannot be reached. Indeed, if $\langle\mu_+,\alpha_j^\vee\rangle=0$ then $s_j\in W_{\mu_+}$, whence $\ell(s_jw_\mu)=\ell(w_\mu)+1$, i.e. $w_\mu^{-1}\alpha_j\in R^+$, and thus $\langle\mu_++w_\mu\omega,\alpha_j^\vee\rangle\geqslant\langle w_\mu\omega,\alpha_j^\vee\rangle\geqslant 0$.

(iia) By definition the assumption implies that $\langle\mu_++w_\mu\alpha_0,\alpha_j^\vee\rangle<0$ for some $1\leqslant j\leqslant n$. Hence $\langle w_\mu\alpha_0,\alpha_j^\vee\rangle=-2$ with $0\leqslant\langle\mu_+,\alpha_j^\vee\rangle\leqslant 1$ or $\langle w_\mu\alpha_0,\alpha_j^\vee\rangle=-1$ with $\langle\mu_+,\alpha_j^\vee\rangle=0$. By repeating the argument of part (i), it is seen that $\langle\mu_+,\alpha_j^\vee\rangle$ cannot be zero. It thus follows that $\langle\mu_+,\alpha_j^\vee\rangle=1$ and that $\langle w_\mu\alpha_0,\alpha_j^\vee\rangle=-2$, i.e. $w_\mu\alpha_0=-\alpha_j$.

(iib) Immediate from the estimate $\langle\mu_+,w_\mu\alpha_0^\vee\rangle=\langle\mu_++w_\mu\alpha_0,w_\mu\alpha_0^\vee\rangle-2\leqslant-2$. $\square$

Lemma B.2. *(See [M4].) Let $w \in W_0$.*

(i) *If $\omega \in P^+$ is minuscule, then*

$$T_w^{-1} Y^{\omega} T_{v_\omega}^{-1} = Y^{w^{-1}\omega} T_{v_\omega w}^{-1}.$$

(ii) *If $s = s_{\alpha_0,0}$, then*

$$T_w^{-1} T_0^{\mathrm{sign}(w^{-1}\alpha_0)} = Y^{w^{-1}\alpha_0} T_{sw}^{-1}.$$

Proof. Both relations are a consequence of [M4, (3.3.2)]. More specifically, (i) and (ii) amount to [M4] (3.3.3) and (3.3.6), respectively. □

B.2. Proof of $u\mathcal{J} = \mathcal{J} I_u$

It is sufficient to verify the intertwining relation for $u = u_\omega$ with $\omega \in P^+$ minuscule (cf. Eq. (2.3)). Let $f \in C(P)$ and let $\mu, \omega \in P$ with ω minuscule. By definition, we have that

$$(u_\omega \mathcal{J} f)(w_o\mu) = q_{t_\nu} q_{w_\nu} \bigl(I_{w_\nu^{-1}}^{-1} f\bigr)(\nu_+) \quad \text{with } \nu := u_\omega^{-1} w_o \mu.$$

Upon setting $\tilde{w} := w_\mu w_o v_\omega^{-1}$, it is readily seen that $\tilde{w}\nu = \mu_+ + w_\mu \omega^* = \nu_+$ in view of Lemma B.1 part (i). Hence, invoking of Eq. (5.5) infers that

$$(u_\omega \mathcal{J} f)(w_o\mu) = q_{t_\nu} q_{\tilde{w}} \bigl(I_{\tilde{w}^{-1}}^{-1} f\bigr)(\nu_+).$$

Similarly, we have that

$$\begin{aligned}(\mathcal{J} I_{u_\omega} f)(w_o\mu) &= \bigl(\mathcal{J} t_\omega I_{v_\omega}^{-1} f\bigr)(w_o\mu)\\ &= q_{t_{w_o\mu}} q_{w_{w_o\mu}} \bigl(I_{w_{w_o\mu}^{-1}}^{-1} t_\omega I_{v_\omega}^{-1} f\bigr)(\mu_+)\\ &= q_{t_\mu} q_{w_\mu w_o} \bigl(I_{w_o w_\mu^{-1}}^{-1} t_\omega I_{v_\omega}^{-1} f\bigr)(\mu_+)\end{aligned}$$

(where in the last step we have again applied Eq. (5.5)). The stated equality now follows because

$$q_{t_\nu} q_{\tilde{w}} = q_{t_{\mu_+ + w_\mu\omega^*}} q_{v_\omega w_o w_\mu^{-1}} \overset{\text{(i)}}{=} q_{t_{\mu_+}} q_{w_o w_\mu^{-1}} = q_{t_\mu} q_{w_\mu w_o}$$

and

$$\bigl(I_{\tilde{w}^{-1}}^{-1} f\bigr)(\nu_+) = \bigl(t_{w_\mu w_o \omega} I_{v_\omega w_o w_\mu^{-1}}^{-1} f\bigr)(\mu_+) \overset{\text{(ii)}}{=} \bigl(I_{w_o w_\mu^{-1}}^{-1} t_\omega I_{v_\omega}^{-1} f\bigr)(\mu_+),$$

where in steps (i) and (ii) we relied on the relation

$$v_\omega w_o w_\mu^{-1} t_{\mu_+ + w_\mu\omega^*} = u_\omega^{-1} w_o w_\mu^{-1} t_{\mu_+}$$

(with $u_\omega \in \Omega$) and Lemma B.2 part (i) with $w = w_o w_\mu^{-1}$, respectively.

B.3. Proof of $\widehat{T}_0\mathcal{J} = \mathcal{J} I_0$

Let $f \in C(P)$ and let $\mu \in P$. By definition (and application of Eq. (5.5)) it is immediate that

$$(\widehat{T}_0\mathcal{J} f)(w_o\mu) = q_0 q_{t_\mu} q_{w_\mu w_o}\big(I^{-1}_{w_o w_\mu^{-1}} f\big)(\mu_+) \\ + \chi_0(w_o\mu)\big(q_{t_\nu} q_{w_\nu}\big(I^{-1}_{w_\nu^{-1}} f\big)(\nu_+) - q_{t_\mu} q_{w_\mu w_o}\big(I^{-1}_{w_o w_\mu^{-1}} f\big)(\mu_+)\big) \tag{B.1}$$

and

$$(\mathcal{J} I_0 f)(w_o\mu) = q_{t_\mu} q_{w_\mu w_o}\big(I^{-1}_{w_o w_\mu^{-1}} I_0 f\big)(\mu_+)$$

($= q_{t_\mu} q_{w_\mu w_o}(I^{-1}_{w_o w_\mu^{-1}} t_{\alpha_0} I_s^{-1} f)(\mu_+)$), with $\nu := s_0 w_o\mu = s w_o(\mu + \alpha_0)$ and $s := s_{\alpha_0,0}$, respectively. We will distinguish three disjoint situations.

Case (A): $\mu_+ + w_\mu\alpha_0 \notin P^+$. By Lemma B.1 part (iia) we have in this case that $w_\mu\alpha_0 = -\alpha_j$ with $\langle \mu_+, \alpha_j^\vee\rangle = 1$ for some $1 \leqslant j \leqslant n$. But then $q_0 = q_j$ and $w_o\mu \in V_0$, i.e. $s_0(w_o\mu) = w_o\mu$, $\chi_0(w_o\mu) = 1$, $\nu_+ = \mu_+$. The stated equality thus reduces to

$$\big(I^{-1}_{w_o w_\mu^{-1}} I_0 f\big)(\mu_+) = q_0\big(I^{-1}_{w_o w_\mu^{-1}} f\big)(\mu_+)$$

(because the terms within the bracket on the second line of Eq. (B.1) now cancel each other by Eq. (5.5)). In view of Lemma B.2 part (ii) (with $w = w_o w_\mu^{-1}$) this amounts to the equation $(t_{\alpha_j} I^{-1}_{s w_o w_\mu^{-1}} f)(\mu_+) = q_0(I^{-1}_{w_o w_\mu^{-1}} f)(\mu_+)$. Since $s w_o w_\mu^{-1} = w_o w_\mu^{-1} s_j$, $\ell(w_o w_\mu^{-1} s_j) = \ell(w_o w_\mu^{-1}) + 1$, and $q_0 = q_j$, the latter equation can be rewritten as

$$\big(t_{\alpha_j} I_j^{-1} g\big)(\mu_+) = q_j g(\mu_+) \quad \text{with } g := I^{-1}_{w_o w_\mu^{-1}} f.$$

This last equality is immediate (for any $g \in C(P)$) from the quadratic relation $I_j^{-1} = I_j - (q_j - q_j^{-1})$ together with the definition of I_j (taking into account that $s_j\mu_+ = t_{\alpha_j}^{-1}\mu_+$).

From now on we will assume that $\mu_+ + w_\mu\alpha_0 \in P^+$ (i.e. we are not in case (A)). Then—upon setting $\tilde{w} := w_\mu w_o s$—it is clear that $\tilde{w}\nu = \mu_+ + w_\mu\alpha_0 = \nu_+$. Hence, application of Eq. (5.5) allows us to rewrite $q_{w_\nu}(I^{-1}_{w_\nu^{-1}} f)(\nu_+)$ as

$$q_{\tilde{w}}\big(I^{-1}_{\tilde{w}^{-1}} f\big)(\nu_+) = q_{\tilde{w}}\big(t_{w_\mu w_o\alpha_0} I^{-1}_{s w_o w_\mu^{-1}} f\big)(\mu_+).$$

Combining this with the relation

$$q_{\tilde{w}} q_{t_\nu} = q_{w_\mu w_o} q_{t_\mu} q_0^{\operatorname{sign}(w_\mu\alpha_0)} \tag{B.2}$$

(proven below) and division by common factors turns Eq. (B.1) into

$$q_{t_\mu}^{-1} q_{w_\mu w_o}^{-1}(\widehat{T}_0\mathcal{J} f)(w_o\mu) \\ = q_0\big(I^{-1}_{w_o w_\mu^{-1}} f\big)(\mu_+) + \chi_0(w_o\mu)\big(q_0^{\operatorname{sign}(w_\mu\alpha_0)}\big(t_{w_\mu w_o\alpha_0} I^{-1}_{s w_o w_\mu^{-1}} f\big)(\mu_+) - \big(I^{-1}_{w_o w_\mu^{-1}} f\big)(\mu_+)\big), \tag{B.3}$$

which must now be shown to coincide with $(I^{-1}_{s w_o w_\mu^{-1}} I_0 f)(\mu_+)$.

Case (B): $w_\mu\alpha_0 \in R^-$ (and $\mu_+ + w_\mu\alpha_0 \in P^+$). By Lemma B.1 part (iib) we have in this case that $\langle \mu_+, w_\mu\alpha_0^\vee\rangle \leqslant -2$, whence $\chi_0(w_o\mu) = q_0$. The stated equality therefore reduces to

$$\bigl(t_{w_\mu w_o\alpha_0} I^{-1}_{sw_o w_\mu^{-1}} f\bigr)(\mu_+) = \bigl(I^{-1}_{w_o w_\mu^{-1}} I_0 f\bigr)(\mu_+),$$

which follows from Lemma B.2 part (ii) (with $w = w_o w_\mu^{-1}$).

Case (C): $w_\mu\alpha_0 \in R^+$ (and $\mu_+ + w_\mu\alpha_0 \in P^+$). Now $\chi_0(w_o\mu) = q_0^{-1}$, whence the stated equality becomes

$$\bigl(q_0 - q_0^{-1}\bigr)\bigl(I^{-1}_{w_o w_\mu^{-1}} f\bigr)(\mu_+) + \bigl(t_{w_\mu w_o\alpha_0} I^{-1}_{sw_o w_\mu^{-1}} f\bigr)(\mu_+) = \bigl(I^{-1}_{w_o w_\mu^{-1}} I_0 f\bigr)(\mu_+).$$

Applying the quadratic relation $I_0 = I_0^{-1} + (q_0 - q_0^{-1})$ rewrites this as

$$\bigl(t_{w_\mu w_o\alpha_0} I^{-1}_{sw_o w_\mu^{-1}} f\bigr)(\mu_+) = \bigl(I^{-1}_{w_o w_\mu^{-1}} I_0^{-1} f\bigr)(\mu_+),$$

which again follows by Lemma B.2 part (ii) (with $w = w_o w_\mu^{-1}$).

It remains to verify the above relation in Eq. (B.2) for the length multiplicative function. Indeed, straightforward manipulations reveal that

$$\begin{aligned} q_{\tilde{w}} q_{t_\nu} &= q_{sw_o w_\mu^{-1}} q_{t_{\mu_+ + w_\mu\alpha_0}} \overset{\text{(i)}}{=} q_{s_0 w_o w_\mu^{-1} t_{\mu_+}} \\ &\overset{\text{(ii)}}{=} q_{w_o w_\mu^{-1}} q_{t_{\mu_+}} q_0^{\operatorname{sign}(w_\mu\alpha_0)} = q_{w_\mu w_o} q_{t_\mu} q_0^{\operatorname{sign}(w_\mu\alpha_0)}, \end{aligned}$$

where in steps (i) and (ii) we relied on the elementary relations

$$s_0 w_o w_\mu^{-1} t_{\mu_+} = s w_o w_\mu^{-1} t_{\mu_+ + w_\mu\alpha_0}$$

and

$$\ell\bigl(s_0 w_o w_\mu^{-1} t_{\mu_+}\bigr) = \ell\bigl(w_o w_\mu^{-1} t_{\mu_+}\bigr) + \operatorname{sign}(w_\mu\alpha_0),$$

respectively.

Appendix C. Explicit formulas for $R = A_{N-1}$

In this appendix we exhibit explicit formulas describing the differential-reflection representation and the integral-reflection representation for the root system A_{N-1}, as well as the corresponding discrete difference operators diagonalized by the Hall-Littlewood polynomials. To facilitate their direct use in the theory of symmetric functions it will be convenient to employ a central extension of the A_{N-1}-type extended affine Weyl group and its Hecke algebra (associated with GL_N rather than SL_N).

C.1. Affine permutation group

For $R = A_{N-1}$ the finite Weyl group amounts to the permutation group S_N and the corresponding extended affine Weyl group is given by the affine permutation group $W = S_N \ltimes \mathbb{Z}^N$, which acts on $\mathbb{R}^N$ by permuting the elements of the standard basis $e_1, \ldots, e_N$ and translating over vectors in the integral lattice, i.e. for $w \in S_N$, $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N$ and $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$:

$$wx = (x_{w^{-1}(1)}, \ldots, x_{w^{-1}(N)}), \tag{C.1a}$$

$$t_\lambda x = (x_1 + \lambda_1, \ldots, x_N + \lambda_N). \tag{C.1b}$$

The group W is generated by the finite transpositions

$$s_j x = (x_1, \ldots, x_{j-1}, x_{j+1}, x_j, x_{j+2}, \ldots, x_N) \quad (1 \leqslant j < N) \tag{C.2a}$$

and the affine generator

$$ux = (x_N + 1, x_1, \ldots, x_{N-1}) \tag{C.2b}$$

(so $u^N = t_{e_1+\cdots+e_n}$ lies in the center of W). The translations over the vectors of the standard basis can be expressed in terms of these generators as:

$$t_{e_j} = s_{j-1} \cdots s_2 s_1 u s_{N-1} s_{N-2} \cdots s_j \quad (1 \leqslant j \leqslant N). \tag{C.3}$$

C.2. Affine Hecke algebra

The extended affine Hecke algebra $\mathcal{H}$ associated with W is the complex associative algebra generated by the invertible elements $T_1, \ldots, T_{N-1}$ and T_u subject to the relations:

$$(T_j - q)\big(T_j + q^{-1}\big) = 0 \quad (1 \leqslant j < N), \tag{C.4a}$$

$$T_j T_k = T_k T_j \quad (1 \leqslant j < k - 1 < N - 1), \tag{C.4b}$$

$$T_j T_{j+1} T_{j+1} = T_{j+1} T_j T_{j+1} \quad (1 \leqslant j < N - 1), \tag{C.4c}$$

$$T_u T_j = T_{j+1} T_u \quad (1 \leqslant j < N - 1), \tag{C.4d}$$

$$T_u^N T_j = T_j T_u^N \quad (1 \leqslant j < N). \tag{C.4e}$$

The Bernstein–Lusztig–Zelevinsky basis for $\mathcal{H}$ is in this situation of the form $T_w Y^\lambda$ ($w \in S_N$, $\lambda \in \mathbb{Z}^N$), where $T_w := T_{s_{j_1}} \cdots T_{s_{j_\ell}}$ for $w = s_{j_1} \cdots s_{j_\ell}$ a reduced expression ($\ell = \ell(w)$) and $Y^\lambda := Y_1^{\lambda_1} \ldots Y_N^{\lambda_N}$ with (cf. Eq. (C.3))

$$Y_j := T_{j-1}^{-1} \cdots T_2^{-1} T_1^{-1} T_u T_{N-1} T_{N-2} \cdots T_{j+1} T_j \quad (1 \leqslant j \leqslant N) \tag{C.5}$$

pairwise commutative.

C.3. Difference-reflection representation

For $1 \leqslant j < N$, let $\widehat{T}_j : C(\mathbb{Z}^N) \to C(\mathbb{Z}^N)$ be defined as

$$(\widehat{T}_j f)(\lambda) = \begin{cases} (q - q^{-1}) f(\lambda) + q^{-1} f(s_j \lambda) & \text{if } \lambda_j > \lambda_{j+1}, \\ q f(\lambda) & \text{if } \lambda_j = \lambda_{j+1}, \\ q f(s_j \lambda) & \text{if } \lambda_j < \lambda_{j+1} \end{cases} \tag{C.6}$$

($f \in C(\mathbb{Z}^N)$, $\lambda \in \mathbb{Z}^N$). The difference-reflection representation $h \mapsto \widehat{T}(h)$ ($h \in \mathcal{H}$) of the extended affine Hecke algebra on $\mathcal{C}(\mathbb{Z}^N)$ is determined by the assignment $T_j \mapsto \widehat{T}_j$ ($1 \leqslant j < N$) and $T_u \mapsto u$.

C.4. Integral-reflection representation

For $1 \leqslant j < N$, let $I_j : C(\mathbb{Z}^N) \to C(\mathbb{Z}^N)$ be defined as

$$(I_j f)(\lambda) = qf(s_j\lambda) + \big(q - q^{-1}\big) \times \begin{cases} -\sum_{l=1}^{\lambda_j-\lambda_{j+1}} f(\lambda_1, \ldots, \lambda_j - l, \lambda_{j+1} + l, \ldots, \lambda_N) & \text{if } \lambda_j > \lambda_{j+1}, \\ 0 & \text{if } \lambda_j = \lambda_{j+1}, \\ \sum_{l=0}^{\lambda_{j+1}-\lambda_j-1} f(\lambda_1, \ldots, \lambda_j + l, \lambda_{j+1} - l, \ldots, \lambda_N) & \text{if } \lambda_j < \lambda_{j+1} \end{cases} \tag{C.7}$$

($f \in C(\mathbb{Z}^N)$, $\lambda \in \mathbb{Z}^N$). The integral-reflection representation $h \mapsto I(h)$ ($h \in \mathcal{H}$) of the extended affine Hecke algebra on $\mathcal{C}(\mathbb{Z}^N)$ is determined by the assignment $T_j \mapsto I_j$ ($1 \leqslant j < N$) and $Y^\lambda \mapsto t_\lambda$ ($\lambda \in \mathbb{Z}^N$).

C.5. Central difference operators

The elementary symmetric polynomials

$$m_r(Y) := \sum_{\substack{J \subset \{1,\ldots,N\} \\ |J|=r}} \prod_{j \in J} Y_j \quad (r = 1, \ldots, N) \tag{C.8}$$

lie in the center of $\mathcal{H}$. The explicit action in $C(\mathbb{Z}^N)$ of the corresponding operators $\widehat{m_1(Y)}, \ldots, \widehat{m_N(Y)}$ under the difference-reflection representation is of the form:

$$\widehat{m_r(Y)} = \epsilon M_r \epsilon^{-1} \quad (r = 1, \ldots, N), \tag{C.9a}$$

with $\epsilon : \mathcal{C}(\mathbb{Z}^N) \to \mathcal{C}(\mathbb{Z}^N)$ and $M_r : \mathcal{C}(\mathbb{Z}^N) \to \mathcal{C}(\mathbb{Z}^N)$ given by

$$(\epsilon f)(\lambda_1, \ldots, \lambda_N) = q^{2\langle \rho, \lambda_+ \rangle} f(\lambda_N, \lambda_{N-1}, \ldots, \lambda_1), \tag{C.9b}$$

$$(M_r f)(\lambda) = \sum_{\substack{J \subset \{1,2,\ldots,N\} \\ |J|=r}} q^{2\ell(w_{w_\lambda(\lambda - e_J)})} f(\lambda - e_J) \tag{C.9c}$$

($f \in C(\mathbb{Z}^N)$, $\lambda \in \mathbb{Z}^N$). Here $\rho := \frac{1}{2}(N-1, N-3, \ldots, 3-N, 1-N)$, λ_+ is obtained from λ by reordering the components of λ in (weakly) decreasing order, w_λ denotes the shortest permutation in S_N taking λ to λ_+, and $e_J := \sum_{j \in J} e_j$.

The restriction of the action of $\widehat{m_r(Y)}$ to $C(\mathbb{Z}^N)^{S_N} \simeq C(\mathbb{Z}^N_{\geqslant})$ with

$$\mathbb{Z}^N_{\geqslant} := \big\{ \lambda \in \mathbb{Z}^N \bigm| \lambda_1 \geqslant \cdots \geqslant \lambda_N \big\}$$

is given by

$$\big(\widehat{m_r(Y)} f\big)(\lambda) = \sum_{\substack{J \subset \{1,2,\ldots,N\},\, |J|=r \\ \lambda - e_J \in \mathbb{Z}^N_{\geqslant}}} V_{\lambda, J^c}\big(q^2\big) f(\lambda - e_J) \quad \big(f \in \mathcal{C}\big(\mathbb{Z}^N_{\geqslant}\big),\ \lambda \in \mathbb{Z}^N_{\geqslant}\big), \tag{C.10a}$$

where $J^c := \{1, \ldots, N\} \setminus J$ and

$$V_{\lambda,J}(q^2) := q^{-2\langle \rho, e_J \rangle} \prod_{\substack{1 \leqslant k < l \leqslant N \\ k \in J, l \in J^c \\ \lambda_k = \lambda_l}} \frac{1 - q^{2(l-k+1)}}{1 - q^{2(l-k)}}. \tag{C.10b}$$

The diagonal action of $\widehat{m_r(Y)}$ on the Hall–Littlewood basis entails the following Pieri formula:

$$m_r p_\lambda = \sum_{\substack{J \subset \{1,2,\ldots,N\},\, |J| = r \\ \lambda + e_J \in \mathbb{Z}^N_{\geqslant}}} V_{\lambda,J}(q^2) p_{\lambda + e_J} \quad (r = 1, \ldots, N) \tag{C.11}$$

for the Hall–Littlewood polynomials

$$p_\lambda = q^{2\langle \rho, \lambda \rangle} \sum_{w \in S_N} x^{w\lambda} \prod_{1 \leqslant k < l \leqslant N} \frac{x_{wk} - q^2 x_{wl}}{x_{wk} - x_{wl}} \quad \left(\lambda \in \mathbb{Z}^N_{\geqslant}\right), \tag{C.12}$$

where $x^\mu := x_1^{\mu_1} \cdots x_N^{\mu_N}$ $(\mu \in \mathbb{Z}^N)$. This Pieri formula amounts to a classic Pieri formula for the Hall-Littlewood polynomials due to Morris [Mo].

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