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The Semi-Infinite q-Boson System with Boundary Interaction

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Abstract. Upon introducing a one-parameter quadratic deformation of the q-boson algebra and a diagonal perturbation at the end point, we arrive at a semi-infinite q-boson system with a two-parameter boundary interaction. The eigenfunctions are shown to be given by Macdonald's hyperoctahedral Hall–Littlewood functions of type BC. It follows that the n-particle spectrum is bounded and absolutely continuous and that the corresponding scattering matrix factorizes as a product of two-particle bulk and one-particle boundary scattering matrices.

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1. Introduction

The q-boson system [1] is a lattice discretization of the one-dimensional quantum nonlinear Schrödinger equation [10,11,14,17,20] built of particle creation and annihilation operators representing the q-oscillator algebra [13, Ch. 5]. Its n-particle eigenfunctions are given by Hall–Littlewood functions [7,15,22]. This model is a limiting case of a more general quantum particle system arising as a q-deformation of the totally asymmetric simple exclusion process (q-TASEP) [2,19]. In the present letter, we study a system of q-bosons on the semi-infinite lattice with boundary interactions, in the spirit of previous works concerned with the quantum nonlinear Schrödinger equation on the half-line [3,8,9,12,21].

Specifically, by introducing at the end point creation and annihilation operators representing a quadratic deformation of the q-oscillator algebra together with a diagonal perturbation, we arrive at the Hamiltonian of a q-boson system on the

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semi-infinite integer lattice endowed with a two-parameter boundary interaction. By means of an explicit formula for the action of the Hamiltonian in the n-particle subspace, it is deduced that the Bethe Ansatz eigenfunctions are given by Macdonald's three-parameter Hall–Littlewood functions with hyperoctahedral symmetry associated with the BC-type root system [16, §10].

It follows that the q-boson system fits within a large class of discrete quantum models with bounded absolutely continuous spectrum for which the scattering behavior was determined in great detail by means of stationary phase techniques [6,18]. In particular, the n-particle scattering matrix is seen to factorize as a product of explicitly computed two-particle bulk and one-particle boundary scattering matrices.

2. Semi-Infinite q-Boson System

Let

$$\mathcal{F} := \bigoplus_{n\in\mathbb{N}} \mathcal{F}(\Lambda_n) \tag{2.1}$$

denote the algebraic Fock space consisting of finite linear combinations of $f_n \in \mathcal{F}(\Lambda_n)$, $n \in \mathbb{N} := \{0, 1, 2, \ldots\}$, where $\mathcal{F}(\Lambda_n)$ stands for the space of functions $f : \Lambda_n \to \mathbb{C}$ on the set of partitions of length at most n:

$$\Lambda_n := \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}, \tag{2.2}$$

with the additional convention that $\Lambda_0 := \{\emptyset\}$ and $\mathcal{F}(\Lambda_0) := \mathbb{C}$. For $l \in \mathbb{N}$, we introduce the following actions on $f \in \mathcal{F}(\Lambda_n) \subset \mathcal{F}$:

$$(\beta_l f)(\lambda) := f(\beta_l^* \lambda) \quad (\lambda \in \Lambda_{n-1})$$

if $n > 0$ and $\beta_l f := 0$ if $n = 0$,

$$(\beta_l^* f)(\lambda) := \begin{cases} [m_l(\lambda)](1 - c\delta_l q^{m_0(\lambda)-1}) f(\beta_l \lambda) & \text{if } m_l(\lambda) > 0 \\ 0 & \text{otherwise} \end{cases} \quad (\lambda \in \Lambda_{n+1}),$$

$$(q^{N_l+k} f)(\lambda) := q^{m_l(\lambda)+k} f(\lambda) \quad (\lambda \in \Lambda_n),$$

with $q, c \in \mathbb{R}$ such that $|q| \neq 0, 1$ and $k \in \mathbb{Z}$. Here

$$\delta_l := \begin{cases} 1 & \text{for } l = 0, \\ 0 & \text{otherwise} \end{cases}, \quad [m] := \frac{1-q^m}{1-q} = \begin{cases} 0 & \text{for } m = 0 \\ 1 + q + \cdots + q^{m-1} & \text{for } m > 0 \end{cases},$$

and the multiplicity $m_l(\lambda)$ counts the number of parts $\lambda_j, 1 \leq j \leq n$ of size $\lambda_j = l$ (so $m_0(\lambda)$, $\lambda \in \Lambda_n$ is equal to n minus the number of nonzero parts), while $\beta_l^* \lambda \in \Lambda_{n+1}$ and $\beta_l \lambda \in \Lambda_{n-1}$ stand for the partitions obtained from $\lambda \in \Lambda_n$ by inserting/deleting a part of size l, respectively (where it is assumed in the latter situation that $m_l(\lambda) > 0$). It is clear from these definitions that β_l, β_l^* and q^{N_l+k} map $\mathcal{F}(\Lambda_n)$

into $\mathcal{F}(\Lambda_{n-1}), \mathcal{F}(\Lambda_{n+1})$ and $\mathcal{F}(\Lambda_n)$, respectively (with the convention that $\mathcal{F}(\Lambda_{-1})$ is the null space).

The operators in question represent a quadratic deformation of the q-boson field algebra at the boundary site $l=0$ parametrized by the constant c:

$$\beta_l q^{N_l} = q^{N_l+1}\beta_l, \quad \beta_l^* q^{N_l} = q^{N_l-1}\beta_l^*,$$
$$\beta_l\beta_l^* = [N_l+1](1-c\delta_l q^{N_0}), \quad [\beta_l, \beta_l^*]_q = 1 - c\delta_l q^{2N_0} \tag{2.3a}$$

and preserving the ultralocality:

$$[\beta_l, \beta_k] = [\beta_l^*, \beta_k^*] = [N_l, N_k] = [N_l, \beta_k] = [N_l, \beta_k^*] = [\beta_l, \beta_k^*] = 0 \tag{2.3b}$$

for $l \neq k$ (where $[A, B] := AB - BA$, $[A, B]_q := AB - qBA$, and $[N_l + r] := (1 - q^{N_l+r})/(1-q)$).

When interpreting the characteristic function $|\lambda\rangle \in \mathcal{F}(\Lambda_n)$ supported on $\lambda \in \Lambda_n$ as a state representing a configuration of n particles on $\mathbb{N}$ such that $m_l(\lambda)$ particles are occupying the site $l \in \mathbb{N}$ (i.e., each part λ_j encodes a particle at site λ_j), it is clear that the operators β_l and β_l^* act as particle annihilation and creation operators:

$$\beta_l|\lambda\rangle = \begin{cases} |\beta_l\lambda\rangle & \text{if } m_l(\lambda) > 0 \\ 0 & \text{otherwise} \end{cases}, \quad \beta_l^*|\lambda\rangle = [m_l(\lambda)+1](1-c\delta_l q^{m_0(\lambda)})|\beta_l^*\lambda\rangle,$$

while q^{N_l} counts the number of particles at the site l (as a power of q):

$$q^{N_l}|\lambda\rangle = q^{m_l(\lambda)}|\lambda\rangle.$$

The dynamics of our q-boson system is governed by a Hamiltonian built of left and right hopping operators together with a diagonal boundary term:

$$\mathrm{H}_q = a[N_0] + \sum_{l\in\mathbb{N}} (\beta_{l+1}\beta_l^* + \beta_{l+1}^*\beta_l), \tag{2.4}$$

$a \in \mathbb{R}$. The interaction at the lattice end stems from a particle reflection in the boundary governed by the deformation parameter c of the q-boson algebra at $l = 0$, and from the additive potential term at the end point controlled by the coupling parameter a (cf. also Proposition 3.1 below). The Hamiltonian in question constitutes a well-defined operator on $\mathcal{F}$ (2.1) as for any $f \in \mathcal{F}(\Lambda_n)$ and $\lambda \in \Lambda_n$ the infinite sum $(\mathrm{H}_q f)(\lambda)$ contains only a finite number of nonvanishing terms.

3. The n-Particle Hamiltonian and Its Eigenfunctions

By construction H_q (2.4) preserves the n-particle subspace $\mathcal{F}(\Lambda_n) \subset \mathcal{F}$. Let us denote the restriction of H_q to $\mathcal{F}(\Lambda_n)$ by $\mathrm{H}_{q,n}$. The following proposition describes the action of this operator in the n-particle subspace explicitly.

PROPOSITION 3.1 (n-Particle Hamiltonian). *For any $f \in \mathcal{F}(\Lambda_n)$ and $\lambda \in \Lambda_n$, one has that*

$$
\begin{aligned}
(H_{q,n} f)(\lambda) &= a[m_0(\lambda)] f(\lambda) \\
&\quad + \sum_{\substack{1 \le j \le n \\ \lambda + e_j \in \Lambda_n}} (1 - c\delta_{\lambda_j} q^{m_0(\lambda)-1})[m_{\lambda_j}(\lambda)] f(\lambda + e_j) + \sum_{\substack{1 \le j \le n \\ \lambda - e_j \in \Lambda_n}} [m_{\lambda_j}(\lambda)] f(\lambda - e_j),
\end{aligned}
$$

where $e_1, \ldots, e_n$ refer to the unit vectors comprising the standard basis of $\mathbb{Z}^n$.

Proof. It is clear from the definitions that $([N_0]f)(\lambda) = [m_0(\lambda)] f(\lambda)$, and that for any $l \in \mathbb{N}$:

$$
(\beta_{l+1}\beta_l^* f)(\lambda) = \begin{cases} [m_l(\lambda)](1 - c\delta_l q^{m_0(\lambda)-1}) f(\beta_{l+1}^* \beta_l \lambda) & \text{if } m_l(\lambda) > 0, \\ 0 & \text{otherwise,} \end{cases}
$$

where $\beta_{l+1}^* \beta_l \lambda = \lambda + e_j$ with $j = \min\{k \mid \lambda_k = l\}$ (so $l = \lambda_j$), and

$$
(\beta_{l+1}^* \beta_l f)(\lambda) = \begin{cases} [m_{l+1}(\lambda)] f(\beta_{l+1} \beta_l^* \lambda) & \text{if } m_{l+1}(\lambda) > 0, \\ 0 & \text{otherwise,} \end{cases}
$$

where $\beta_{l+1}\beta_l^* \lambda = \lambda - e_j$ with $j = \max\{k \mid \lambda_k = l+1\}$ (so $l = \lambda_j - 1$). □

The n-particle Hamiltonian $\mathrm{H}_{q,n}$ has Bethe Ansatz eigenfunctions given by the following plane wave expansion

$$
\phi_\xi(\lambda) := \sum_{\substack{\sigma \in S_n \\ \epsilon \in \{\pm 1\}^n}} C(\epsilon \xi_\sigma) \mathrm{e}^{i\langle \lambda, \epsilon \xi_\sigma \rangle}, \tag{3.1a}
$$

with expansion coefficients of the form

$$
\begin{aligned}
C(\xi) &:= \prod_{1 \le j \le n} \frac{1 - a\mathrm{e}^{-i\xi_j} + c\mathrm{e}^{-2i\xi_j}}{1 - \mathrm{e}^{-2i\xi_j}} \\
&\quad \times \prod_{1 \le j < k \le n} \left(\frac{1 - q\mathrm{e}^{-i(\xi_j - \xi_k)}}{1 - \mathrm{e}^{-i(\xi_j - \xi_k)}} \right) \left(\frac{1 - q\mathrm{e}^{-i(\xi_j + \xi_k)}}{1 - \mathrm{e}^{-i(\xi_j + \xi_k)}} \right).
\end{aligned} \tag{3.1b}
$$

Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$, $\epsilon \xi_\sigma := (\epsilon_1 \xi_{\sigma_1}, \epsilon_2 \xi_{\sigma_2}, \ldots, \epsilon_n \xi_{\sigma_n})$, and the summation is meant over all permutations σ in the symmetric group S_n and all sign configurations $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{1, -1\}^n$. Viewed as a function of the spectral parameter $\xi = (\xi_1, \ldots, \xi_n)$ in the fundamental alcove

$$
A := \{(\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n \mid \pi > \xi_1 > \xi_2 > \cdots > \xi_n > 0\}, \tag{3.2}
$$

the expression $\phi_\xi(\lambda), \lambda \in \Lambda_n$ amounts to Macdonald's three-parameter Hall–Littlewood polynomial with hyperoctahedral symmetry associated with the root system BC_n [16, §10].

PROPOSITION 3.2 (Bethe Ansatz eigenfunctions). *The n-particle Bethe Ansatz wave function $\phi_\xi, \xi \in A$ solves the eigenvalue equation*

$$H_{q,n}\phi_\xi = E_n(\xi)\phi_\xi, \quad E_n(\xi) := 2\sum_{j=1}^{n}\cos(\xi_j). \tag{3.3}$$

Proof. It follows from Proposition 3.1 that the stated eigenvalue equation boils down to the following identity

$$a[m_0(\lambda)]\phi_\xi(\lambda) + \sum_{\substack{1\le j\le n\\ \lambda+e_j\in\Lambda_n}} (1 - c\delta_{\lambda_j} q^{m_0(\lambda)-1})[m_{\lambda_j}(\lambda)]\phi_\xi(\lambda+e_j)$$

$$+ \sum_{\substack{1\le j\le n\\ \lambda-e_j\in\Lambda_n}} [m_{\lambda_j}(\lambda)]\phi_\xi(\lambda-e_j) = 2\phi_\xi(\lambda)\sum_{j=1}^{n}\cos(\xi_j),$$

which is in turn equivalent to the Pieri formula for the hyperoctahedral Hall–Littlewood function in Equation (A.3) of Appendix A. □

4. Diagonalization

From now on it will be assumed unless stated otherwise that $0 < |q| < 1$ and that the boundary parameters a and c are chosen such that the roots r_1, r_2 of the quadratic polynomial $r^2 - ar + c$ belong to the interval $(-1, 1)$:

$$a = r_1 + r_2 \quad \text{and} \quad c = r_1 r_2 \quad \text{with } r_1, r_2 \in (-1, 1). \tag{4.1}$$

Let $L^2(A, \Delta d\xi)$ be the Hilbert space of functions $\hat{f} : A \to \mathbb{C}$ characterized by the inner product

$$\langle \hat{f}, \hat{g}\rangle_\Delta = \int_A \hat{f}(\xi)\overline{\hat{g}(\xi)}\Delta(\xi)\, d\xi, \quad \text{where } \Delta(\xi) := \frac{1}{(2\pi)^n |C(\xi)|^2} \tag{4.2}$$

with $C(\xi)$ given by Equation (3.1b). It is well known that for the parameter regime in question Macdonald's hyperoctahedral Hall–Littlewood functions form an orthogonal basis of $L^2(A, \Delta d\xi)$ [16, §10]:

$$\langle \phi(\lambda), \phi(\mu)\rangle_\Delta = \begin{cases} \mathcal{N}(\lambda) & \text{if } \lambda = \mu, \\ 0 & \text{otherwise,} \end{cases} \tag{4.3a}$$

where

$$\mathcal{N}(\lambda) := (c;q)_{m_0(\lambda)} \prod_{\ell\in\mathbb{N}} [m_\ell(\lambda)]! \tag{4.3b}$$

with $(c;q)_m := (1-c)(1-cq)\cdots(1-cq^{m-1})$ (and the convention that $(c;q)_0 := 1$) and $[m]! := (q;q)_m/(q;q)_1^m = [m][m-1]\cdots[2][1]$. By combining the orthogonality in Equations (4.3a), (4.3b) with Proposition 3.2, the spectral decomposition of H_q in the n-particle Hilbert space $\ell^2(\Lambda_n, \mathcal{N}^{-1}) \subset \mathcal{F}(\Lambda_n)$ characterized by the inner product

$$\langle f, g\rangle_n := \sum_{\lambda\in\Lambda_n} f(\lambda)\overline{g(\lambda)}\mathcal{N}^{-1}(\lambda) \tag{4.4}$$

becomes immediate.

THEOREM 4.1 (Diagonalization). *For $0 < |q| < 1$ and values of the boundary parameters a and c in the orthogonality domain (4.1), the q-boson Hamiltonian H_q (2.4) restricts to a bounded self-adjoint operator in $\ell^2(\Lambda_n, \mathcal{N}^{-1})$ with purely absolutely continuous spectrum. More specifically, the spectral decomposition of $H_{q,n}$ in $\ell^2(\Lambda_n, \mathcal{N}^{-1})$ reads explicitly*

$$H_{q,n} = \boldsymbol{F}_{\boldsymbol{q}}^{-1} \circ \hat{E}_n \circ \boldsymbol{F}_{\boldsymbol{q}}, \tag{4.5}$$

where $\boldsymbol{F}_{\boldsymbol{q}} : \ell^2(\Lambda_n, \mathcal{N}^{-1}) \to L^2(A, \Delta\mathrm{d}\xi)$ denotes the unitary Fourier transform associated with the hyperoctahedral Macdonald–Hall–Littlewood basis:

$$(\boldsymbol{F}_{\boldsymbol{q}} f)(\xi) := \langle f, \phi_\xi\rangle_n = \sum_{\lambda\in\Lambda_n} f(\lambda)\overline{\phi_\xi(\lambda)}\mathcal{N}^{-1}(\lambda) \tag{4.6a}$$

($f \in \ell^2(\Lambda_n, \mathcal{N}^{-1})$) with the inversion formula given by

$$(\boldsymbol{F}_{\boldsymbol{q}}^{-1} \hat{f})(\lambda) = \langle \hat{f}, \overline{\phi(\lambda)}\rangle_\Delta = \int_A \hat{f}(\xi)\phi_\xi(\lambda)\Delta(\xi)\ \mathrm{d}\xi \tag{4.6b}$$

($\hat{f} \in L^2(A, \Delta\mathrm{d}\xi)$), and $(\hat{E}_n \hat{f})(\xi) := E_n(\xi)\hat{f}(\xi)$ stands for the bounded real multiplication operator in $L^2(A, \Delta\mathrm{d}\xi)$ associated with the n-particle eigenvalue $E_n(\xi)$ (3.3).

In the Fock space $\mathcal{H} := \bigoplus_{n\geq 0} \ell^2(\Lambda_n, \mathcal{N}^{-1})$, built of all linear combinations $\sum_{n\geq 0} c_n f_n$ with $c_n \in \mathbb{C}$ and $f_n \in \ell^2(\Lambda_n, \mathcal{N}^{-1})$ such that $\sum_{n\geq 0} |c_n|^2 \langle f_n, f_n\rangle_n < \infty$, the q-boson Hamiltonian H_q (2.4) constitutes an unbounded operator that is essentially self-adjoint on the dense domain $\mathcal{D} := \mathcal{F} \cap \mathcal{H}$ (because for $z \in \mathbb{C}\setminus\mathbb{R}$ the range $(\mathrm{H}_q - z)\mathcal{D}$ is dense in $\mathcal{H}$ and $\lim_{n\to\infty} \sup_{\xi\in A} |E_n(\xi)| = \infty$). The representation of the deformed q-boson field algebra in Section 2 on the other hand gives rise to a bounded representation on $\mathcal{H}$:

$$\langle \beta_l f, \beta_l f\rangle_{n-1} \leq \frac{1+|c|\delta_l}{1-q}\langle f, f\rangle_n,$$
$$\langle \beta_l^* f, \beta_l^* f\rangle_{n+1} \leq \frac{1+|c|\delta_l}{1-q}\langle f, f\rangle_n,$$
$$\langle q^{N_l} f, q^{N_l} f\rangle_n \leq \langle f, f\rangle_n,$$

preserving the $*$-structure:

$$\langle \beta_l^* f, g\rangle_{n+1} = \langle f, \beta_l g\rangle_n \quad \text{and} \quad \langle q^{N_l} f, g\rangle_n = \langle f, q^{N_l} g\rangle_n.$$

Remark 4.2. Upon rescaling the lattice Λ_n (2.2) and performing an appropriate continuum limit [5, Sec. 5], Macdonald's hyperoctahedral Hall–Littlewood functions tend to the eigenfunctions of the quantum nonlinear Schrödinger equation on the half-line with a boundary interaction [3,8,9,12,21]. In particular, it follows from [5, Sec. 5.3] that for $a=0$ (which corresponds to a reduction from type BC to type C root systems), a renormalized version of the n-particle q-boson Hamiltonian $\mathrm{H}_{q,n}$ then converges in the strong resolvent sense to a Schrödinger operator that can be written formally as:

$$-\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}+g\sum_{1\le j<k\le n}(\delta(x_j-x_k)+\delta(x_j+x_k))+g_0\sum_{1\le j\le n}\delta(x_j)$$

with $g, g_0>0$ (where $\delta(\cdot)$ stands for the 'delta potential').

5. Factorized Scattering

The similarity transformation

$$H_n:=\mathcal{N}^{-1/2}\,\mathrm{H}_{q,n}\,\mathcal{N}^{1/2} \tag{5.1}$$

turns the n-particle q-boson Hamiltonian in Proposition 3.1 into a self-adjoint operator in $\ell^2(\Lambda_n)$ diagonalized by the normalized wave function

$$\begin{aligned}\Psi_\xi(\lambda)&:=(2\pi)^{-n/2}\,\mathrm{e}^{\frac{\pi i}{2}n^2}|C(\xi)|^{-1}\mathcal{N}(\lambda)^{-1/2}\phi_\xi(\lambda)\\&=(2\pi)^{-n/2}\mathcal{N}(\lambda)^{-1/2}\sum_{\substack{\sigma\in S_n\\ \epsilon\in\{\pm1\}^n}}\mathrm{sign}(\epsilon\sigma)\hat{\mathcal{S}}(\epsilon\xi_\sigma)^{1/2}\mathrm{e}^{i\langle\rho+\lambda,\epsilon\xi_\sigma\rangle},\end{aligned} \tag{5.2a}$$

with $\xi\in A$ (3.2), $\mathrm{sign}(\epsilon\sigma):=\epsilon_1\cdots\epsilon_n\mathrm{sign}(\sigma)$, $\rho:=(n,n-1,\ldots,2,1)$, and

$$\hat{\mathcal{S}}(\xi):=\prod_{1\le j<k\le n}s(\xi_j-\xi_k)s(\xi_j+\xi_k)\prod_{1\le j\le n}s_0(\xi_j), \tag{5.2b}$$

where

$$s(x):=\frac{1-q\mathrm{e}^{-ix}}{1-q\mathrm{e}^{ix}}\quad\text{with}\quad s(x)^{1/2}=\frac{1-q\mathrm{e}^{-ix}}{|1-q\mathrm{e}^{ix}|} \tag{5.2c}$$

and

$$s_0(x):=\frac{1-a\mathrm{e}^{-ix}+c\mathrm{e}^{-2ix}}{1-a\mathrm{e}^{ix}+c\mathrm{e}^{2ix}}\quad\text{with}\quad s_0(x)^{1/2}=\frac{1-a\mathrm{e}^{-ix}+c\mathrm{e}^{-2ix}}{|1-a\mathrm{e}^{ix}+c\mathrm{e}^{2ix}|}. \tag{5.2d}$$

Specifically, one has that $H_n=\boldsymbol{F}^{-1}\circ\hat{E}_n\circ\boldsymbol{F}$ where $\boldsymbol{F}:\ell^2(\Lambda_n)\to L^2(A,\mathrm{d}\xi)$ denotes the unitary Fourier transformation determined by the kernel $\Psi_\xi(\lambda)$ (and $\hat{E}_n$ is now interpreted as a bounded multiplication operator in $L^2(A,\mathrm{d}\xi)$). For $q,a,c\to 0$

the n-particle q-boson Hamiltonian H_n (5.1) simplifies to a Hamiltonian modeling impenetrable bosons on $\mathbb{N}$:

$$(H_{n,0}f)(\lambda)=\sum_{\substack{1\leq j\leq n\\ \lambda+e_j\in\Lambda_n}} f(\lambda+e_j)+\sum_{\substack{1\leq j\leq n\\ \lambda-e_j\in\Lambda_n}} f(\lambda-e_j) \tag{5.3}$$

($f\in\ell^2(\Lambda_n)$), which is diagonalized by the conventional Fourier transform $\boldsymbol{F_0}:\ell^2(\Lambda_n)\to L^2(A,\mathrm{d}\xi)$ obtained from $\boldsymbol{F}$ by setting $\hat{\mathcal{S}}(\xi)\equiv 1, \mathcal{N}(\lambda)\equiv 1$.

Let $C_0(A_{\text{reg}})$ denote the dense subspace of $L^2(A,\mathrm{d}\xi)$ consisting of smooth test functions with compact support in an open dense subset A_{reg} of A (3.2) determined by the condition that the components of $\nabla E_n(\xi)=-2(\sin(\xi_1),\ldots,\sin(\xi_n))$ do not vanish and are all distinct in absolute value on it (so A_{reg} is the part of A on which the gradient $\nabla E_n(\xi)$ is regular with respect to the action of the hyperoctahedral group of permutations and sign flips of the components). We will now apply the results in [6, Sec. 4] to conclude that the wave- and scattering operators that relate the q-boson dynamics

$$(\mathrm{e}^{itH_n}f)(\lambda)=\int_A \mathrm{e}^{itE_n(\xi)}\hat{f}(\xi)\Psi_\xi(\lambda)\ \mathrm{d}\xi \quad (\hat{f}=\boldsymbol{F}f) \tag{5.4}$$

to the corresponding impenetrable boson dynamics generated by $H_{n,0}$ (5.3) are governed by an unitary S-matrix $\hat{\mathcal{S}}:L^2(A,\mathrm{d}\xi)\to L^2(A,\mathrm{d}\xi)$ obtained as the closure of the densely defined multiplication operator

$$(\hat{\mathcal{S}}\hat{f})(\xi):=\hat{\mathcal{S}}(\epsilon_\xi\xi_{\sigma_\xi})\hat{f}(\xi) \quad (\hat{f}\in C_0(A_{\text{reg}})), \tag{5.5}$$

where the sign-configuration ϵ_ξ and the permutation σ_ξ are such that the components of $\nabla E_n(\epsilon_\xi\xi_{\sigma_\xi})$ are all positive and ordered from large to small. Specifically, by comparing the large-times asymptotics of oscillatory integrals of the form in Equation (5.4) for the dynamics generated by H_n and $H_{n,0}$ one concludes that [6, Thm. 4.2 and Cor. 4.3]:

THEOREM 5.1 (Wave and scattering operators). *The operator limits*

$$\Omega^{\pm}:=s-\lim_{t\to\pm\infty}\mathrm{e}^{itH_n}\mathrm{e}^{-itH_{n,0}} \tag{5.6a}$$

converge in the strong $\ell^2(\Lambda_n)$-norm topology and the corresponding wave operators $\Omega^{\pm}$ intertwining the dynamics of H_n and $H_{n,0}$ are given by unitary operators in $\ell^2(\Lambda_n)$ of the form

$$\Omega^{\pm}=\boldsymbol{F}^{-1}\circ\hat{\mathcal{S}}^{\mp 1/2}\circ\boldsymbol{F_0}. \tag{5.6b}$$

Hence, the scattering operator relating the large-times asymptotics of the q-boson dynamics for $t\to-\infty$ and $t\to+\infty$ is given by the unitary operator

$$\mathcal{S}:=(\Omega^{+})^{-1}\Omega^{-}=\boldsymbol{F_0}^{-1}\circ\hat{\mathcal{S}}\circ\boldsymbol{F_0}. \tag{5.6c}$$

Remark 5.2. Recently, an interesting two-parameter extension of the hyperoctahedral Hall–Littlewood functions of Macdonald was studied [23]. It is natural to expect that these generalized hyperoctahedral Hall–Littlewood functions arise as the eigenfunctions of a corresponding q-boson system on the semi-infinite lattice involving a more general four-parameter interaction at the boundary.

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Appendix A: Pieri Formula for Macdonald's Hyperoctahedral Hall–Littlewood Function

Let $x := (x_1, \ldots, x_n) = (e^{i\xi_1}, \ldots, e^{i\xi_n})$ and $\tau := (\tau_1, \ldots, \tau_n)$, where $\tau_j = rq^{n-j}$ $(j = 1, \ldots, n)$ with $r = \frac{a}{2} + \sqrt{(\frac{a}{2})^2 - c}$ [cf. Equation (4.1)]. Upon setting

$$P_\lambda(x) := \frac{\tau_1^{\lambda_1} \cdots \tau_n^{\lambda_n}}{\mathcal{N}(0)} \phi_\xi(\lambda) \quad (\lambda \in \Lambda_n), \tag{A.1}$$

where $\mathcal{N}(0)$ is given by Equation (4.3b) with $\lambda = 0$, the hyperoctahedral Hall–Littlewood function is renormalized to have unital principal specialization values: $P_\lambda(\tau) = 1$ $(\forall \lambda \in \Lambda_n)$ [16, §12]. With this normalization, the following Pieri formula holds:

$$\begin{aligned} &P_\lambda(x) \sum_{j=1}^{n} (x_j + x_j^{-1} - \tau_j - \tau_j^{-1}) \\ &= \sum_{\substack{1 \le j \le n \\ \lambda + e_j \in \Lambda_n}} V_j^+(\lambda) \left(P_{\lambda + e_j}(x) - P_\lambda(x)\right) + \sum_{\substack{1 \le j \le n \\ \lambda - e_j \in \Lambda_n}} V_j^-(\lambda) \left(P_{\lambda - e_j}(x) - P_\lambda(x)\right), \end{aligned} \tag{A.2}$$

where

$$V_j^+(\lambda) = \tau_j^{-1} \left(\frac{1 - c^2 \delta_{\lambda_j} q^{2(n-j)}}{1 + c\delta_{\lambda_j} q^{2(n-j)}} \right) \prod_{\substack{j < k \le n \\ \lambda_k = \lambda_j}} \left(\frac{1 - q^{1+k-j}}{1 - q^{k-j}} \right) \left(\frac{1 + c\delta_{\lambda_j} q^{1+2n-k-j}}{1 + c\delta_{\lambda_j} q^{2n-k-j}} \right),$$

$$V_j^-(\lambda) = \tau_j \prod_{\substack{1 \le k < j \\ \lambda_k = \lambda_j}} \left(\frac{1 - q^{1+j-k}}{1 - q^{j-k}} \right).$$

The formula in question is readily obtained through degeneration from an analogous Pieri formula for a BC_n-type Macdonald function that arises as a special case of the Pieri formulas in [4, Sec. 6.1]. Specifically, by substituting $t_2 = q^{1/2}, t_3 = -q^{1/2}$ (which amounts to a reduction from the Macdonald–Koornwinder function to the BC_n-type Macdonald function) in the Pieri formula of [4, Eqs. (6.4), (6.5)] with coefficients taken from [4, Eqs. (6.12), (6.13)], the relation in Equation (A.2) is retrieved for $q \to 0$ (which corresponds to a transition from Macdonald type functions to Hall–Littlewood type functions). Notice in this connection that the parameters q, a, c (and r) of the present paper are related to the parameters t, t_0, t_1 of Ref. [4] via $q = t, a = t_0 + t_1, c = t_0 t_1$ (and $r = t_0$).

Since

$$V_j^+(\lambda) = \tau_j^{-1}(1 - c\delta_{\lambda_j} q^{m_0(\lambda)-1})[m_{\lambda_j}(\lambda)], \quad V_j^-(\lambda) = \tau_j[m_{\lambda_j}(\lambda)],$$

and

$$\sum_{j=1}^{n}(\tau_j + \tau_j^{-1}) - \sum_{\substack{1\le j\le n \\ \lambda - e_j \in \Lambda_n}} \tau_j[m_{\lambda_j}(\lambda)] - \sum_{\substack{1\le j\le n \\ \lambda + e_j \in \Lambda_n}} \tau_j^{-1}[m_{\lambda_j}(\lambda)] = r[m_0(\lambda)],$$

the Pieri formula (A.2) can be condensed into the more compact form

$$P_\lambda(x)\sum_{j=1}^{n}(x_j + x_j^{-1}) = a[m_0(\lambda)]P_\lambda(x) + \sum_{\substack{1\le j\le n \\ \lambda - e_j \in \Lambda_n}} \tau_j[m_{\lambda_j}(\lambda)]P_{\lambda - e_j}(x)$$
$$+ \sum_{\substack{1\le j\le n \\ \lambda + e_j \in \Lambda_n}} \tau_j^{-1}(1 - c\delta_{\lambda_j} q^{m_0(\lambda)-1})[m_{\lambda_j}(\lambda)]P_{\lambda + e_j}(x). \qquad \text{(A.3)}$$

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