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Orthogonality of Macdonald polynomials with unitary parameters

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Abstract For any admissible pair of irreducible reduced crystallographic root systems, we present discrete orthogonality relations for a finite-dimensional system of Macdonald polynomials with parameters on the unit circle subject to a truncation relation.

Keywords Orthogonal polynomials · Macdonald polynomials · Root systems

Mathematics Subject Classification (2000) Primary 33D52; Secondary 11C08 · 20C08

1 Introduction

In [28] a finite-dimensional system of discrete orthogonality relations was found for the Macdonald polynomials [20] with parameters of the form $t = q^{\mathrm{g}}$ and $q = e^{\frac{2\pi i}{(n+1)\mathrm{g}+c}}$, where $n+1$ denotes the number of variables, g is a positive real parameter, and c is a positive integer (so both t and q lie on the unit circle and satisfy the truncation relation $t^{n+1}q^c = 1$). For g integral Macdonald's parameters q and t become roots of unity; the discrete orthogonality relations of [28] specialize in this situation to those considered in [13, Sec. 5]. In particular, when $\mathrm{g} = 1$ elementary orthogonality relations for systems of periodic Schur polynomials are recovered, cf. [12, §13.8], [13, Sec. 6], [24, Sec. 4.2], and [16, Sec. 6.2].

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The purpose of the present paper is to generalize the finite-dimensional discrete orthogonality relations of [28] to Macdonald polynomials with unitary parameters associated with arbitrary admissible pairs of irreducible reduced crystallographic root systems [19] in the spirit of [27], where the case of nonreduced root systems was considered. We thus arrive at a parameter deformation interpolating between discrete orthogonality relations for such Macdonald polynomials with parameter values at roots of unity [5, Sec. 5], containing as a special case elementary orthogonality relations for systems of periodic Weyl characters, cf. [12, §13.8], [13, Sec. 6], [9, Sec. 5.3] and [26, Sec. 8.4].

When the rank of the reduced root system is *one*, the orthogonality considered here reduces to a finite-dimensional discrete orthogonality relation for the q-ultraspherical polynomials [22, Sec. 3C2], [28, Sec. 5.2] that arises as a parameter specialization of the celebrated orthogonality for the q-Racah polynomials found by Askey and Wilson [15]. The full q-Racah orthogonality corresponds from this perspective to the nonreduced rank-one situation [27]. A very different multivariate analog of the q-Racah polynomials was studied recently by Gasper, Rahman and Iliev [7,10].

The paper is organized as follows. Section 2 sets up notation and recalls the definition of the Macdonald polynomials diagonalizing Macdonald's difference operators. In Sect. 3 we formulate our finite-dimensional system of discrete orthogonality relations for the Macdonald polynomials with unitary parameters (subject to a truncation relation). The remainder of the paper is devoted to the proof of these orthogonality relations. Specifically, we infer the orthogonality of the Macdonald polynomials with unitary parameters exploiting explicit formulas for the Macdonald difference operators from Ref. [25] allowing to confirm the self-adjointness and the nondegeneracy of the spectrum in the present setting (Sect. 4). Next, we rely on the duality symmetry [3,4,8,21] to compute the norms of the Macdonald polynomials via Macdonald's Pieri-type recurrence relations associated with the (quasi-)minuscule weights [17,21] (Sect. 5). The total mass of the weight function is expressed compactly in product form by applying a finitely truncated version [23,6] of a basic hypergeometric summation formula associated with root systems due to Aomoto, Ito and Macdonald [1,11,18]. Some technical issues concerning the proof of the nondegeneracy of the eigenvalues of the Macdonald difference operators for exceptional root systems are relegated to an appendix at the end of the paper.

2 Preliminaries

In this section the definitions of the Macdonald polynomials and the Macdonald difference operators are recalled briefly. For more detailed discussions with proofs and further background material the reader is referred to the standard texts [4,8,14,19,21]. Throughout familiarity with the basic properties of root systems and their Weyl groups [2] will be assumed.

2.1 Macdonald polynomials

Let E be a real finite-dimensional Euclidean vector space with inner product $\langle\cdot,\cdot\rangle$ spanned by an irreducible reduced crystallographic root system R. We write Q, P, and W, for the root lattice, the weight lattice, and the Weyl group associated with R. The semigroup of the root lattice generated by a (fixed) choice of positive roots R^+ is denoted by Q^+ whereas P^+ stands for the corresponding cone of dominant weights. The group algebra $\mathbb{C}[P]$ is spanned by formal exponentials e^λ, $\lambda \in P$ characterized by the relations $e^0 = 1$, $e^\lambda e^\mu = e^{\lambda+\mu}$. The

Weyl group acts linearly on $\mathbb{C}[P]$ via $we^{\lambda} := e^{w\lambda}$, $w \in W$, and the W-invariant subalgebra $\mathbb{C}[P]^W$ is spanned by the basis of symmetric monomials $m_\lambda := \sum_{\mu \in W\lambda} e^{\mu}$, $\lambda \in P^+$ (where the sum is meant over the W-orbit of λ). This monomial basis inherits a partial order from the dominance order on the cone of dominant weights:

$$\mu \leq \lambda \quad \text{iff} \quad \lambda - \mu \in Q^+ \quad (\lambda, \mu \in P^+). \tag{2.1}$$

The dual root system $R^\vee := \{\alpha^\vee \mid \alpha \in R\}$ and its positive subsystem $R^{\vee,+}$ are obtained from R and R^+ by applying the involution

$$x \mapsto x^\vee := 2x/\|x\|^2 \quad (x \in E \setminus \{0\}), \tag{2.2}$$

where $\|x\| := \langle x, x\rangle^{1/2}$. Following Macdonald's terminology, we refer to a tuple $(R, \hat{R})$ with $\hat{R}$ being equal either to R or $R^\vee$ as an *admissible pair* of root systems [19]. The assignment $a \mapsto \hat{a}$ for $a \in R \cup \hat{R}$, where $\hat{a} := a$ if $\hat{R} = R$ and $\hat{a} := a^\vee$ if $\hat{R} = R^\vee$, defines an involution on $R \cup \hat{R}$ swapping the roots of R and $\hat{R}$. We write $\hat{Q}$ and $\hat{P}$ for the root lattice and the weight lattice of $\hat{R}$, and more generally: dual objects associated with the admissible pair $(\hat{R}, R)$ (i.e. with the role of R and $\hat{R}$ interchanged) will be endowed with a superscript hat.

For a formal series $f = \sum_{\lambda \in P} f_\lambda e^{\lambda}$, $f_\lambda \in \mathbb{C}$, we define $\int f := f_0$ and $\bar{f} := \sum_{\lambda \in P} \bar{f}_\lambda e^{-\lambda}$ (with $\bar{f}_\lambda$ meaning the complex conjugate of f_λ). The Macdonald inner product on $\mathbb{C}[P]$ is then given by [19,21]

$$\langle f, g\rangle_\delta := |W|^{-1} \int f \bar{g}\, \delta \quad (f, g \in \mathbb{C}[P]), \tag{2.3a}$$

where $|W|$ stands for the order of W and

$$\delta := \delta_+ \bar{\delta}_+, \quad \delta_+ := \prod_{\alpha \in R^+} \frac{(e^{\alpha}; q_\alpha)_\infty}{(t_\alpha e^{\alpha}; q_\alpha)_\infty} \tag{2.3b}$$

(employing standard notation for the q-shifted factorial $(a; q)_m := \prod_{k=0}^{m-1}(1 - aq^k)$ with m nonnegative integral or ∞). Here q is a parameter taking values in $(0, 1)$ and

$$q_a := q^{u_a}, \quad t_a := q_a^{g_a} \quad (a \in R \cup \hat{R}), \tag{2.3c}$$

where $u_a := \|\hat{a}\|/\|a^\vee\|$ (so $u_{\hat{a}} = u_a$) and $g : R \cup \hat{R} \to (0, \infty)$ denotes a root multiplicative function such that $g_{wa} = g_a$ and $g_{\hat{a}} = g_a$ (for all $w \in W$ and $a \in R \cup \hat{R}$). We think of this function as a (pair of) positive parameter(s) attached to the $(W \times \mathbb{Z}_2)$-orbits of $R \cup \hat{R}$ (where the $\mathbb{Z}_2$-action corresponds to the 'hat'-involution).

Definition 2.1 (*Macdonald Polynomials* [19,21]) For $\lambda \in P^+$, the *Macdonald polynomial* is defined as the unique element in $\mathbb{C}[P]^W$ of the form

$$p_\lambda = m_\lambda + \sum_{\mu < \lambda} c_{\lambda\mu}(q, t)\, m_\mu \tag{2.4a}$$

with $c_{\lambda\mu}(q, t) \in \mathbb{C}$ such that

$$\langle p_\lambda, m_\mu\rangle_\delta = 0 \quad \text{for all } \mu < \lambda. \tag{2.4b}$$

The elements of the group algebra will be interpreted as functions on E through the evaluation homomorphism $e^{\lambda}(x) := q^{\langle \lambda, x\rangle}$, $x \in E$. Three remarkable fundamental properties of Macdonald's polynomials are the *orthogonality relations* [21, Eqs. (5.3.4), (5.8.17)]

$$\langle p_\lambda, p_\mu\rangle_\delta = \begin{cases} 0 & \text{if } \lambda \neq \mu \\ \frac{\hat{\delta}_+(\rho_g + \lambda)}{\hat{\delta}_-(\rho_g + \lambda)} & \text{if } \lambda = \mu \end{cases}, \tag{2.5a}$$

the *principal specialization formula* [21, Eq. (5.3.12)]

$$p_\lambda(\hat{\rho}_g) = \frac{\hat{\delta}_+(\rho_g+\lambda)}{\hat{\delta}_+(\rho_g)e^\lambda(\hat{\rho}_g)}$$

$$= \prod_{\alpha\in R^+} t_\alpha^{-\langle\lambda,\alpha^\vee\rangle/2} \frac{\left(t_\alpha q_\alpha^{\langle\rho_g,\alpha^\vee\rangle}; q_\alpha\right)_{\langle\lambda,\alpha^\vee\rangle}}{\left(q_\alpha^{\langle\rho_g,\alpha^\vee\rangle}; q_\alpha\right)_{\langle\lambda,\alpha^\vee\rangle}}, \tag{2.5b}$$

and the *duality symmetry* [21, Eq. (5.3.6)] for the normalized Macdonald polynomials $P_\lambda := p_\lambda/p_\lambda(\hat{\rho}_g)$:

$$P_\lambda(\hat{\rho}_g+\mu) = \hat{P}_\mu\left(\rho_g+\lambda\right) \quad \left(\lambda\in P^+, \mu\in\hat{P}^+\right), \tag{2.5c}$$

where we have employed the additional notation

$$\rho_g := \frac{1}{2}\sum_{\alpha\in R^+} g_\alpha\alpha, \quad \delta_- := \prod_{\alpha\in R^+}\frac{\left(t_\alpha^{-1}q_\alpha e^\alpha; q_\alpha\right)_\infty}{(q_\alpha e^\alpha; q_\alpha)_\infty}.$$

2.2 Macdonald difference operators

Macdonald's polynomials are joint eigenfunctions of an algebra of commuting difference operators that is isomorphic to $\mathbb{C}[\hat{P}]^W$, cf. [21, Eqs. (4.4.12), (5.3.3)]. Explicit formulas for the difference operators associated with the basis elements $\hat{m}_\omega$ with $\omega\in\hat{P}^+\setminus\{0\}$ *minuscule* (i.e. $\langle\omega,\alpha^\vee\rangle\leq 1$ for all $\alpha\in\hat{R}^+$) or *quasi-minuscule* (i.e. $\langle\omega,\alpha^\vee\rangle\leq 1$ for all $\alpha\in\hat{R}^+\setminus\{\omega\}$ and ω is not minuscule) were presented in [19, Secs. 5, 6]. In [25, Sec. 3] more general explicit formulas for the Macdonald difference operators can be found corresponding to the subspace of $\mathbb{C}[\hat{P}]^W$ spanned by the monomials $\hat{m}_\omega$ with $\omega\in\hat{P}^+$ *small* (viz. $\langle\omega,\alpha^\vee\rangle\leq 2$ for all $\alpha\in\hat{R}^+$). To formulate the latter difference operators some further notation is needed. Given $\lambda\in P^+$, let us denote the saturated set in P cut out by the convex hull of $W\lambda$ by

$$P(\lambda) := \bigcup_{\mu\in P^+,\ \mu\leq\lambda} W\mu. \tag{2.6}$$

The stabilizer of $x\in E$ in W and the corresponding parabolic subsystem of R are given by $W_x := \{w\in W \mid wx = x\}$ and $R_x := \{\alpha\in R \mid \langle x,\alpha\rangle = 0\}$ (with $R_x^+ := R_x\cap R^+$). Finally, for $\lambda\in P$ we write w_λ for the unique shortest Weyl group element mapping λ into the dominant cone P^+.

For $\omega\in\hat{P}^+$ small in the sense above, the Macdonald difference operator $\mathcal{D}_\omega$ from [25] acts on meromorphic functions $f: E\to\mathbb{C}$ as

$$(\mathcal{D}_\omega f)(x) = \sum_{\substack{\nu\in\hat{P}(\omega)\\ \eta\in W_\nu(w_\nu^{-1}\omega)}} V_\nu(x)U_{\nu,\eta}(x)(T_\nu f)(x), \tag{2.7a}$$

where $(T_\nu f)(x) := f(x+\nu)$ and the coefficients V_ν and $U_{\nu,\eta}$ are of the form

$$V_\nu(x) = \prod_{\substack{\alpha\in\hat{R}\\ \langle\nu,\alpha^\vee\rangle>0}} \frac{\sin\kappa_\alpha(\langle x,\alpha^\vee\rangle + g_\alpha)}{\sin\kappa_\alpha\langle x,\alpha^\vee\rangle} \prod_{\substack{\alpha\in\hat{R}\\ \langle\nu,\alpha^\vee\rangle=2}} \frac{\sin\kappa_\alpha(\langle x,\alpha^\vee\rangle + 1 + g_\alpha)}{\sin\kappa_\alpha(\langle x,\alpha^\vee\rangle + 1)} \tag{2.7b}$$

and

$$U_{\nu,\eta}(x) = \prod_{\substack{\alpha\in\hat{R}_\nu \\ \langle\eta,\alpha^\vee\rangle>0}} \frac{\sin\kappa_\alpha(\langle x,\alpha^\vee\rangle + \mathrm{g}_\alpha)}{\sin\kappa_\alpha\langle x,\alpha^\vee\rangle} \prod_{\substack{\alpha\in\hat{R}_\nu \\ \langle\eta,\alpha^\vee\rangle=2}} \frac{\sin\kappa_\alpha(\langle x,\alpha^\vee\rangle + 1 - \mathrm{g}_\alpha)}{\sin\kappa_\alpha(\langle x,\alpha^\vee\rangle + 1)}. \tag{2.7c}$$

Here κ is a (for now positive imaginary) parameter that is related to Macdonald's parameter q via

$$q = \exp(2i\kappa) \tag{2.8a}$$

and $\kappa_a := \kappa u_a$, so

$$q_a = \exp(2i\kappa_a) \quad \text{and} \quad t_a = \exp(2i\kappa_a \mathrm{g}_a) \quad (a \in R \cup \hat{R}). \tag{2.8b}$$

One has that [25, Thm. 3.1]

$$\mathcal{D}_\omega p_\lambda = E_\omega(\rho_\mathrm{g} + \lambda) p_\lambda, \tag{2.9a}$$

where

$$\begin{aligned} E_\omega &:= \hat{m}_\omega + \sum_{\mu\in\hat{P}^+,\, \mu<\omega} \epsilon_{\omega,\mu}\hat{m}_\mu \in \mathbb{C}[\hat{P}], \\ \epsilon_{\omega,\mu} &:= \sum_{\eta\in W_\mu\omega} \prod_{\substack{\alpha\in\hat{R}^+_\mu \\ |\langle\eta,\alpha^\vee\rangle|=1}} t_\alpha^{\langle\eta,\alpha^\vee\rangle}. \end{aligned} \tag{2.9b}$$

When ω is minuscule $\hat{P}(\omega) = W\omega$ and

$$\mathcal{D}_\omega = \sum_{\nu\in W\omega} V_\nu(x) T_\nu, \quad E_\omega = \hat{m}_\omega, \tag{2.10a}$$

whereas when ω is quasi-minuscule $\hat{P}(\omega) = W\omega \cup \{0\}$ and

$$\mathcal{D}_\omega = \sum_{\nu\in W\omega} (U_{0,\nu}(x) + V_\nu(x) T_\nu), \quad E_\omega = \hat{m}_\omega + \epsilon_{\omega,0}. \tag{2.10b}$$

The difference equation (2.9a) reduces in these two situations to the explicit eigenvalue equations for the Macdonald polynomials stemming from [19, Sec. 5] and [19, Sec. 6], respectively.

2.3 Generic complex parameters

From a Taylor expansion of $\hat{m}_\omega(\rho_\mathrm{g} + \lambda)$ in κ:

$$\begin{aligned} \hat{m}_\omega(\rho_\mathrm{g} + \lambda) &= \sum_{\nu\in W\omega} \exp\left(2i\kappa\langle\nu, \rho_\mathrm{g} + \lambda\rangle\right) \\ &= |W\omega| - 2\kappa^2 \sum_{\nu\in W\omega} \langle\nu, \rho_\mathrm{g} + \lambda\rangle^2 + O\left(\kappa^3\right) \\ &= |W\omega| \left(1 - \frac{2\kappa^2}{n} \|\omega\|^2 \|\rho_\mathrm{g} + \lambda\|^2\right) + O\left(\kappa^3\right), \end{aligned}$$

one reads-off that $\hat{m}_\omega(\rho_\mathrm{g} + \mu) \neq \hat{m}_\omega(\rho_\mathrm{g} + \lambda)$ as an analytic expression in the parameters κ and g when $\mu < \lambda$ (because the latter inequality implies that $\|\rho_\mathrm{g} + \mu\|^2 < \|\rho_\mathrm{g} + \lambda\|^2$

for g > 0). Here n refers to the rank of the root system ($:=\dim E$) and $|W\omega|$ stands for the order of the orbit $W\omega$. The upshot is that—for generic values in our parameter domain—the Macdonald polynomial p_λ can be alternatively characterized as the unique polynomial of the form in Eq. (2.4a) satisfying the eigenvalue equation (2.9a) for the Macdonald operator $\mathcal{D}_\omega$ with ω (quasi-)minuscule, i.e. one then has explicitly (cf. [19, Sec. 4]):

$$p_\lambda = \left(\prod_{\substack{\mu\in P^+ \\ \mu<\lambda}} \frac{\mathcal{D}_\omega - E_\omega(\rho_{\mathrm{g}}+\mu)}{E_\omega(\rho_{\mathrm{g}}+\lambda) - E_\omega(\rho_{\mathrm{g}}+\mu)} \right) m_\lambda \tag{2.11}$$

(where ω is assumed to be (quasi-)minuscule).

It is immediate from (2.11) and the explicit formulas for $\mathcal{D}_\omega$ and E_ω in (2.10a) and (2.10b) that the Macdonald polynomial p_λ is meromorphic in the parameters κ and g. Hence, the Macdonald polynomial p_λ extends meromorphically in these parameters to an eigenpolynomial of the form in (2.4a) solving the eigenvalue equation (2.9a), (2.9b) for generic complex parameter values (and $\omega \in \hat{P}^+$ small).

3 Main result: orthogonality relations for unitary parameters

We exploit the above meromorphy of the Macdonald polynomials in the parameters to continue the parameter κ analytically from the imaginary to the real axis while leaving g positive (so q, q_a and t_a become unitary) and consider the normalized Macdonald polynomials satisfying the duality symmetry (2.5c):

$$P_\lambda = c_\lambda p_\lambda \tag{3.1a}$$

with

$$c_\lambda := 1/p_\lambda(\hat{\rho}_{\mathrm{g}}) \overset{\text{Eq. (2.5b)}}{=} \prod_{\alpha\in R^+} \frac{(\langle \rho_{\mathrm{g}}, \alpha^\vee\rangle : \kappa_\alpha)_{\langle \lambda,\alpha^\vee\rangle}}{\left(\langle \rho_{\mathrm{g}}, \alpha^\vee\rangle + \mathrm{g}_\alpha : \kappa_\alpha\right)_{\langle \lambda,\alpha^\vee\rangle}}. \tag{3.1b}$$

Here we have employed trigonometric Pochhammer symbols of the form

$$(a:\kappa)_l := 2^l \sin(\kappa a) \sin\kappa(a+1)\cdots \sin\kappa(a+l-1) \tag{3.2}$$

when l is positive integral, subject to the convention that $(a:\kappa)_0 := 1$. Following the standard conventions for Pochhammer symbols, we will occasionally abbreviate products of the form $(a_1:\kappa)_l \cdots (a_k:\kappa)_l$ by $(a_1,\ldots,a_k:\kappa)_l$.

For c nonnegative integral, let

$$P_c := \{\lambda \in P^+ \mid \langle \lambda, \hat{\psi}^\vee\rangle \le c\} \quad \text{and so} \quad \hat{P}_c = \{\mu \in \hat{P}^+ \mid \langle \mu, \hat{\varphi}^\vee\rangle \le c\}, \tag{3.3}$$

where φ and ψ refer to the maximal roots of R and $\hat{R}$, respectively. Let us furthermore denote the maximal *short* root of R by ϑ (with the convention that all roots of R are short if the root system is simply laced). (So ϑ is equal to the unique quasi-minuscule weight of R and $\vartheta^\vee$ is the maximal coroot in $R^\vee$.) From now it will be always assumed—unless explicitly stated otherwise—that $c > 1$ and that *for R of type E_7 the value of c is not a proper multiple of* 6 (cf. Remark 5.3 below).

Theorem 3.1 (Finite-Dimensional Discrete Orthogonality Relations) *For*

$$\kappa = \frac{\pi}{u_\varphi(h_g + c)} \quad \textit{with} \quad h_g := \langle \rho_g, \hat{\psi}^\vee\rangle + g_\psi, \tag{3.4a}$$

the Macdonald polynomials P_λ, $\lambda \in P_c$ are analytic in $g > 0$ and satisfy the orthogonality relations

$$\sum_{\lambda \in P_c} P_\lambda(\hat{\rho}_g + \mu)\overline{P_\lambda(\hat{\rho}_g + \tilde{\mu})}\Delta(\lambda) = \begin{cases} 0 & \text{if } \mu \neq \tilde{\mu} \\ \frac{\mathcal{N}_0}{\hat{\Delta}(\mu)} & \text{if } \mu = \tilde{\mu} \end{cases} \tag{3.4b}$$

($\mu, \tilde{\mu} \in \hat{P}_c$), or equivalently

$$\sum_{\mu \in \hat{P}_c} P_\lambda(\hat{\rho}_g + \mu)\overline{P_{\tilde{\lambda}}(\hat{\rho}_g + \mu)}\hat{\Delta}(\mu) = \begin{cases} 0 & \text{if } \lambda \neq \tilde{\lambda} \\ \frac{\mathcal{N}_0}{\Delta(\lambda)} & \text{if } \lambda = \tilde{\lambda} \end{cases} \tag{3.4c}$$

($\lambda, \tilde{\lambda} \in P_c$), where

$$\Delta(\lambda) := \prod_{\alpha \in R^+} \frac{\sin \kappa_\alpha \langle \rho_g + \lambda, \alpha^\vee \rangle}{\sin \kappa_\alpha \langle \rho_g, \alpha^\vee \rangle} \frac{(\langle \rho_g, \alpha^\vee \rangle + g_\alpha ; \kappa_\alpha)_{\langle \lambda, \alpha^\vee \rangle}}{(\langle \rho_g, \alpha^\vee \rangle + 1 - g_\alpha ; \kappa_\alpha)_{\langle \lambda, \alpha^\vee \rangle}}, \tag{3.4d}$$

$$\mathcal{N}_0 := \sum_{\lambda \in P_c} \Delta(\lambda) = \sum_{\mu \in \hat{P}_c} \hat{\Delta}(\mu) = \hat{\mathcal{N}}_0. \tag{3.4e}$$

Furthermore, the total mass $\mathcal{N}_0$ of the positive *discrete orthogonality measure Δ admits a compact product representation of the form $\mathcal{N}_0 = Ind(R)\mathcal{N}_c$ with $Ind(R) := |P/Q|$ and $\mathcal{N}_c$ given by* Tables 1 *(for R simply laced),* 2 *(for R multiply laced with $\hat{R} = R$) and* 3 *(for R multiply laced with $\hat{R} = R^\vee$).*

Table 1 Value of $\mathcal{N}_c = \frac{\mathcal{N}_0}{\text{Ind}(R)}$ when R is simply laced

R	$\mathcal{N}_c$	h_{g}
A_n	$\prod_{k=1}^{n}(1+k\text{g};\kappa_\varphi)_{c-1}$	$(n+1)\text{g}$
D_n	$(1+(n-1)\text{g};\kappa_\varphi)_{c-1}\prod_{k=1}^{n-1}(1+(2k-1)\text{g};\kappa_\varphi)_{c-1}$	$2(n-1)\text{g}$
E_6	$(1+\text{g},1+4\text{g},1+5\text{g},1+7\text{g},1+8\text{g},1+11\text{g};\kappa_\varphi)_{c-1}$	12g
E_7	$(1+\text{g},1+5\text{g},1+7\text{g},1+9\text{g},1+11\text{g},1+13\text{g},1+17\text{g};\kappa_\varphi)_{c-1}$	18g
E_8	$(1+\text{g},1+7\text{g},1+11\text{g},1+13\text{g},1+17\text{g},1+19\text{g},1+23\text{g},1+29\text{g};\kappa_\varphi)_{c-1}$	30g

Table 2 Value of $\mathcal{N}_c = \frac{\mathcal{N}_0}{\text{Ind}(R)}$ when R is multiply laced with $\hat{R} = R$

R	$\mathcal{N}_c$	h_{g}
B_n	$(1+\text{g}_\vartheta+2(n-1)\text{g}_\varphi;\kappa_\vartheta)_{2c-1}\prod_{k=1}^{n-1}\frac{(1+k\text{g}_\varphi,1+\text{g}_\vartheta+(n+k-2)\text{g}_\varphi;\kappa_\varphi)_{c-1}}{(1+2k\text{g}_\varphi;\kappa_\vartheta)_{c-1}^2}$	$2(n-1)\text{g}_\varphi+\text{g}_\vartheta$
C_n	$(1+(n-1)\text{g}_\vartheta+2\text{g}_\varphi;\kappa_\vartheta)_{2c-1}\prod_{k=1}^{n-2}(1+(n+k)\text{g}_\vartheta+2\text{g}_\varphi;\kappa_\vartheta)_c^2$ $\times\prod_{k=0}^{n-1}\frac{(1+k\text{g}_\vartheta+\text{g}_\varphi;\kappa_\varphi)_{c-1}}{(1+2k\text{g}_\vartheta+2\text{g}_\varphi;\kappa_\vartheta)_{2c-1}}$	$2\text{g}_\varphi+(n-1)\text{g}_\vartheta$
F_4	$(1+\text{g}_\varphi,1+\text{g}_\vartheta+3\text{g}_\varphi,1+2\text{g}_\vartheta+3\text{g}_\varphi,1+3\text{g}_\vartheta+5\text{g}_\varphi;\kappa_\varphi)_{c-1}$ $\times\frac{(1+3\text{g}_\vartheta+4\text{g}_\varphi;\kappa_\vartheta)_{c-1}^2(1+5\text{g}_\vartheta+6\text{g}_\varphi;\kappa_\vartheta)_c^2}{(1+4\text{g}_\varphi,1+2\text{g}_\vartheta+6\text{g}_\varphi;\kappa_\vartheta)_{c-1}^2}$	$6\text{g}_\varphi+3\text{g}_\vartheta$
G_2	$\frac{(1+\text{g}_\varphi,1+\text{g}_\vartheta+2\text{g}_\varphi;\kappa_\varphi)_{c-1}(1+2\text{g}_\vartheta+3\text{g}_\varphi;\kappa_\vartheta)_c^2}{(1+3\text{g}_\varphi;\kappa_\vartheta)_{c-1}^2}$	$3\text{g}_\varphi+\text{g}_\vartheta$

Table 3 Value of $\mathcal{N}_c = \frac{\mathcal{N}_0}{\mathrm{Ind}(R)}$ when R is multiply laced with $\hat{R} = R^\vee$

R	$\mathcal{N}_c$	h_{g}
B_n	$\frac{\prod_{k=1}^{n}(1+\mathrm{g}_\vartheta+(n+k-2)\mathrm{g}_\varphi;\kappa)_{c-1}\prod_{k=1}^{[n/2]}(1+(2k-1)\mathrm{g}_\varphi;\kappa)_{c-1}}{\prod_{k=1}^{[n/2]}(1+2(n-k)\mathrm{g}_\varphi;\kappa)_{c-1}}$	$2(n-1)\mathrm{g}_\varphi+2\mathrm{g}_\vartheta$
C_n	$\frac{\prod_{k=1}^{n}(1+(k-1)\mathrm{g}_\vartheta+\mathrm{g}_\varphi;\kappa)_{c-1}\prod_{k=1}^{[n/2]}(1+(2n-2k-1)\mathrm{g}_\vartheta+2\mathrm{g}_\varphi;\kappa)_{c-1}}{\prod_{k=1}^{[n/2]}(1+2(k-1)\mathrm{g}_\vartheta+2\mathrm{g}_\varphi;\kappa)_{c-1}}$	$2\mathrm{g}_\varphi+2(n-1)\mathrm{g}_\vartheta$
F_4	$\frac{(1+\mathrm{g}_\varphi,1+\mathrm{g}_\vartheta+3\mathrm{g}_\varphi,1+2\mathrm{g}_\vartheta+3\mathrm{g}_\varphi,1+3\mathrm{g}_\vartheta+4\mathrm{g}_\varphi,1+3\mathrm{g}_\vartheta+5\mathrm{g}_\varphi,1+5\mathrm{g}_\vartheta+6\mathrm{g}_\varphi;\kappa)_{c-1}}{(1+4\mathrm{g}_\varphi,1+2\mathrm{g}_\vartheta+6\mathrm{g}_\varphi;\kappa)_{c-1}}$	$6\mathrm{g}_\varphi+6\mathrm{g}_\vartheta$
G_2	$\frac{(1+\mathrm{g}_\varphi,1+\mathrm{g}_\vartheta+2\mathrm{g}_\varphi,1+2\mathrm{g}_\vartheta+3\mathrm{g}_\varphi;\kappa)_{c-1}}{(1+3\mathrm{g}_\varphi;\kappa)_{c-1}}$	$3\mathrm{g}_\varphi+3\mathrm{g}_\vartheta$

Remark 3.2 For R of type A Theorem 3.1 amounts to [28, Eq. (4.15b)] whereas for $R = \hat{R}$ of type C one recovers a special case of the orthogonality in [27, Sec. 6] (with $\mathrm{g} = \mathrm{g}_\vartheta$, $\mathrm{g}_a = \mathrm{g}_b = \mathrm{g}_\varphi$ and $\mathrm{g}_c = \mathrm{g}_d = 0$).

Remark 3.3 For κ as in Theorem 3.1, the Macdonald parameters q and t satisfy the truncation relation

$$t_\vartheta^{m\langle\rho_\vartheta,\hat{\psi}^\vee\rangle} t_\varphi^{\langle\rho_{\varphi\setminus\vartheta},\hat{\psi}^\vee\rangle} t_\psi\, q_\varphi^c = 1, \tag{3.5a}$$

where $m := u_\varphi/u_\vartheta (\in \{1, 2, 3\})$ and

$$\rho := \frac{1}{2}\sum_{\alpha\in R^+}\alpha, \quad \rho_\vartheta := \frac{1}{2}\sum_{\substack{\alpha\in R^+\\ \|\alpha\|=\|\vartheta\|}}\alpha, \quad \rho_{\varphi\setminus\vartheta} := \frac{1}{2}\sum_{\substack{\alpha\in R^+\\ \|\alpha\|\neq\|\vartheta\|}}\alpha = \rho - \rho_\vartheta \tag{3.5b}$$

(so $\rho_{\mathrm{g}} = \mathrm{g}_\vartheta\rho_\vartheta + \mathrm{g}_\varphi\rho_{\varphi\setminus\vartheta}$). If the dual root system $R^\vee$ is isomorphic to R, the truncation relation in Eq. (3.5a) becomes of the form $t_\vartheta^{h/2}t_\varphi^{h/2}q_\varphi^c = 1$, where $h = h(R) := \langle\rho, \vartheta^\vee\rangle + 1$ (the Coxeter number of R). In particular, for R simply laced the truncation relation reads $t^h q_\varphi^c = 1$.

Remark 3.4 Since for κ as in Theorem 3.1 one has that $q_a = \exp(\frac{2\pi i}{m_a(h_{\mathrm{g}}+c)})$ and $t_a = \exp(\frac{2\pi i \mathrm{g}_a}{m_a(h_{\mathrm{g}}+c)})$ with $m_a := u_\varphi/u_a (\in \{1, m\}$, cf. Remark 3.3) for $a \in R \cup \hat{R}$, it is clear that when is g integral-valued q_a and t_a are roots of unity, cf. [5, Sec. 5] and [13, Sec. 5].

Remark 3.5 Both orthogonality relations in Eqs. (3.4b) and (3.4c) are equivalent to the unitarity of the matrix $[S_{\lambda,\mu}]_{\lambda\in P_c,\mu\in\hat{P}_c}$ with

$$S_{\lambda,\mu} := \left(\frac{\Delta(\lambda)\hat{\Delta}(\mu)}{\mathcal{N}_0}\right)^{1/2} P_\lambda(\hat{\rho}_{\mathrm{g}} + \mu). \tag{3.6}$$

Since the parameter specialization in Eq. (3.4a) preserves the duality symmetry in the sense that $\hat{\kappa} = \frac{\pi}{u_\psi(\hat{h}_{\mathrm{g}}+c)} = \kappa$ (because $\hat{h}_{\mathrm{g}} = \langle\hat{\rho}_{\mathrm{g}}, \hat{\varphi}^\vee\rangle + \mathrm{g}_\varphi = h_{\mathrm{g}}$ and $u_\psi = u_\varphi$), the matrix in question inherits from Eq. (2.5c) in addition the duality symmetry $\hat{S}_{\mu,\lambda} = S_{\lambda,\mu}$.

Remark 3.6 When R is either simply laced or multiply laced with $\hat{R} = R^\vee$, the product formulas in Tables 1 and 3 follow from the terminating Aomoto-Ito-Macdonald-type basic hypergeometric summation formula in [6, Thm. 3]. In this situation the value of $\mathcal{N}_c$ can be rewritten as (cf. [6, Eq. (4.6)])

$$\mathcal{N}_c = \frac{\prod_{\alpha\in R^+}(1+\langle\rho_{\mathrm{g}},\alpha^\vee\rangle : \kappa_\alpha)_{c-1}}{\prod_{\alpha\in R^+\setminus I}(1+\langle\rho_{\mathrm{g}},\alpha^\vee\rangle - \mathrm{g}_\alpha : \kappa_\alpha)_{c-1}}, \tag{3.7}$$

where $I \subseteq R^+$ consists of the simple roots of R. In the equal label case, i.e. with the root multiplicity function g being constant (so in particular when R is simply laced), the product formula in (3.7) simplifies to (cf. [6, Eq. (4.4)])

$$\mathcal{N}_c = \prod_{k=1}^{n}(1+\mathrm{g}e_k : \kappa_\varphi)_{c-1}, \tag{3.8}$$

where $e_1, \ldots, e_n$ refer to the exponents of the Weyl group (thus explaining the structure of the formulas in Table 1 and those in Table 3 when $\mathrm{g}_\vartheta = \mathrm{g}_\varphi = \mathrm{g}$).

Remark 3.7 For $\mathrm{g} = 1$, Macdonald's polynomials become Weyl characters [19]. Theorem 3.1 then boils down to the following elementary orthogonality relations for the antisymmetric monomials

$$\chi_\lambda := \sum_{w\in W}\det(w)e^{w\lambda} \quad (\lambda \in P^+) \tag{3.9}$$

at $\kappa = \frac{\pi}{u_\varphi(\hbar+c)}$ with $\hbar = \hbar(R,\hat{R}) := \langle\rho,\hat{\psi}^\vee\rangle + 1 (\in \{h, h^\vee\})$,[1] cf. [12, §13.8], [13, Sec. 6], [9, Sec. 5.3], [26, Sec. 8.4] (and also [24, Sec. 4.2] and [16, Sec. 6.2] for the special case when R is of type A):

$$\sum_{\lambda\in P_c}\chi_{\rho+\lambda}(\hat{\rho}+\mu)\overline{\chi_{\rho+\lambda}(\hat{\rho}+\tilde{\mu})} = \begin{cases}0 & \text{if } \mu\neq\tilde{\mu}\\ \mathrm{Ind}(R,\hat{R})(\hbar+c)^n & \text{if } \mu=\tilde{\mu}\end{cases} \tag{3.10}$$

($\mu, \tilde{\mu} \in \hat{P}_c$). Here the index governing the value of the quadratic norms is defined as $\mathrm{Ind}(R,\hat{R}) := |P/(u_\varphi\hat{Q}^\vee)| = |P/Q|\,|Q/(u_\varphi\hat{Q}^\vee)|$ (so $\mathrm{Ind}(R,\hat{R}) = \mathrm{Ind}(R)$ if R is simply-laced or $\hat{R} = R^\vee$, and $\mathrm{Ind}(R,\hat{R}) = m^{n_\vartheta}\,\mathrm{Ind}(R)$ with n_ϑ denoting the number of short simple roots of R otherwise). The orthogonality in (3.10) readily follows from the orthogonality of the discrete Fourier basis on $P/(\hbar+c)u_\varphi\hat{Q}^\vee$:

$$\sum_{\lambda\in P/(\hbar+c)u_\varphi\hat{Q}^\vee} e^{\frac{2\pi i}{u_\varphi(\hbar+c)}\langle\lambda,\mu\rangle} = \begin{cases}0 & \text{if } \mu\in\hat{P}\setminus(\hbar+c)u_\varphi Q^\vee\\ |P/(\hbar+c)u_\varphi\hat{Q}^\vee| & \text{if } \mu\in(\hbar+c)u_\varphi Q^\vee\end{cases}, \tag{3.11}$$

upon antisymmetrization and using that $|P/(\hbar+c)u_\varphi\hat{Q}^\vee| = \mathrm{Ind}(R,\hat{R})(\hbar+c)^n$. When comparing the values of the quadratic norms in (3.10) with the ones obtained from Theorem 3.1 through specialization, one deduces the following remarkable trigonometric identity for root systems at $\mathrm{g} = 1$ (and thus $\kappa_\alpha = \frac{\pi}{m_\alpha(\hbar+c)}$):

$$\mathcal{N}_c\prod_{\alpha\in R^+}\sin\kappa_\alpha\langle\rho,\alpha^\vee\rangle = \frac{\mathrm{Ind}(R,\hat{R})}{\mathrm{Ind}(R)}(\hbar+c)^n \tag{3.12}$$

(cf. also Remark 3.6).

[1] Here $h^\vee = h^\vee(R) := \langle\rho,\varphi^\vee\rangle + 1$ (the dual Coxeter number of R).

4 Analyticity and orthogonality

From now on it is always assumed—unless explicitly stated otherwise—that κ takes the value in Eq. (3.4a) (with $g > 0$).

4.1 Meromorphy

The regularity of (the expansion coefficients of) the Macdonald polynomials in Sect. 3 at the above value of κ hinges for generic $g > 0$ on the following lemma.

Lemma 4.1 (Nondegeneracy Eigenvalues) *For any $\lambda, \mu \in P_c$ (3.3) with $\lambda \neq \mu$, there exists a small weight $\omega \in \hat{P}^+$ such that the eigenvalues E_ω (2.9b) are distinct:*

$$E_\omega(\rho_g + \lambda) \neq E_\omega(\rho_g + \mu) \tag{4.1}$$

as analytic functions in $g > 0$.

Proof For the classical root systems all fundamental weights are small. The stated nondegeneracy of the eigenvalues follows in this situation from the fact that the trigonometric polynomials $\hat{m}_\omega$ (and thus E_ω) corresponding to the fundamental weights $\omega \in \hat{P}^+$ separate the points of $\rho_g + P_c \subset \frac{\pi}{\kappa}\hat{A}$, where $\hat{A}$ refers to the open fundamental alcove $\{x \in V \mid 0 < \langle x, \alpha\rangle < 1,\ \forall \alpha \in \hat{R}^+\}$ of the affine Weyl group $W \ltimes \hat{Q}^\vee$. For the exceptional root systems the nondegeneracy in question follows in turn from a case by case examination of the relevant eigenvalues that is performed in the appendix at the end of the paper. □

To infer the regularity of p_λ for $\lambda \in P_c$ and generic values of the multiplicity parameter, we combine Lemma 4.1 with the expressions for the Macdonald difference operators in Eqs. (2.7a)–(2.7c) to conclude that the Macdonald polynomials at issue are meromorphic in $g > 0$ as a consequence of an explicit representation similar to that in Eq. (2.11).

Proposition 4.2 (Meromorphy) *The Macdonald polynomials p_λ, $\lambda \in P_c$ are meromorphic in $g > 0$.*

Proof For $\lambda \in P_c$ and $\mu < \lambda$, one has that $\mu \in P_c$ (because $\hat{\psi} \in P^+$). In this situation, let $\omega_{\lambda\mu}$ denote a small weight in $\hat{P}^+$ from Lemma 4.1 such that $E_{\omega_{\lambda\mu}}(\rho_g + \lambda) \neq E_{\omega_{\lambda\mu}}(\rho_g + \mu)$ as analytic expressions in g. The meromorphy of p_λ in $g > 0$ is now immediate from the formula (cf. Eq. 2.11)

$$p_\lambda = \left(\prod_{\substack{\mu \in P_c \\ \mu < \lambda}} \frac{\mathcal{D}_{\omega_{\lambda\mu}} - E_{\omega_{\lambda\mu}}(\rho_g + \mu)}{E_{\omega_{\lambda\mu}}(\rho_g + \lambda) - E_{\omega_{\lambda\mu}}(\rho_g + \mu)} \right) m_\lambda,$$

combined with the explicit expression for $\mathcal{D}_{\omega_{\lambda\mu}}$ stemming from Eqs. (2.7a)–(2.7c). □

4.2 Finite Macdonald difference operators

For the parameter regime in Theorem 3.1 (all factors of) $\Delta(\lambda)(\lambda \in P_c)$ and $\hat{\Delta}(\mu)(\mu \in \hat{P}_c)$ are positive because the arguments of the sine functions take values in the interval $(0, \pi)$, as is readily seen from the following estimates.

Lemma 4.3 (Moment Bounds) *For any $\lambda \in P_c$ and $\alpha \in R^+$, the following inequalities hold:*

$$(i)\ \langle \lambda, \alpha^\vee\rangle \leq m_\alpha \langle \lambda, \hat{\psi}^\vee\rangle \leq m_\alpha c \quad and \quad (ii)\ g_\alpha \leq \langle \rho_g, \alpha^\vee\rangle \leq m_\alpha h_g - g_\alpha, \tag{4.2}$$

i.e. $0 < g_\alpha \leq \langle \rho_g + \lambda, \alpha^\vee \rangle \leq m_\alpha(h_g + c) - g_\alpha < m_\alpha(h_g + c)$ *(where* $m_\alpha = u_\varphi/u_\alpha$*, cf.* Remark 3.4*).*

Proof Part (i) is straightforward: $\langle \lambda, \alpha^\vee \rangle = u_\alpha^{-1}\langle \lambda, \hat{\alpha} \rangle \leq u_\alpha^{-1}\langle \lambda, \psi \rangle = m_\alpha \langle \lambda, \hat{\psi}^\vee \rangle \leq m_\alpha c$. For the proof of part (ii) we write $\rho_{\mathrm{g}} = \mathrm{g}_\vartheta \rho_\vartheta + \mathrm{g}_\varphi \rho_{\varphi\setminus\vartheta}$ (cf. Remark 3.3), i.e. $\langle \rho_{\mathrm{g}}, \alpha^\vee \rangle = \mathrm{g}_\vartheta \langle \rho_\vartheta, \alpha^\vee \rangle + \mathrm{g}_\varphi \langle \rho_{\varphi\setminus\vartheta}, \alpha^\vee \rangle$. The lower bound g_α is now immediate from the fact that $\langle \rho_\vartheta, \alpha^\vee \rangle > 0$ if $\|\alpha\| = \|\vartheta\|$ and $\langle \rho_{\varphi\setminus\vartheta}, \alpha^\vee \rangle > 0$ if $\|\alpha\| \neq \|\vartheta\|$, because the (possibly empty) parabolic subsystems of R corresponding to the stabilizers of ρ_ϑ and $\rho_{\varphi\setminus\vartheta}$ are generated by the simple roots β with $\|\beta\| \neq \|\vartheta\|$ and by the simple roots β with $\|\beta\| = \|\vartheta\|$, respectively. To infer the upper bound, we compute:

$$m_\alpha h_{\mathrm{g}} - \langle \rho_{\mathrm{g}}, \alpha^\vee \rangle - \mathrm{g}_\alpha = \mathrm{g}_\vartheta \langle \rho_\vartheta, m_\alpha \hat{\psi}^\vee - \alpha^\vee \rangle + \mathrm{g}_\varphi \langle \rho_{\varphi\setminus\vartheta}, m_\alpha \hat{\psi}^\vee - \alpha^\vee \rangle + m_\alpha \mathrm{g}_\psi - \mathrm{g}_\alpha$$
$$= \begin{cases} \mathrm{g}_\vartheta(\langle \rho_\vartheta, \vartheta^\vee - \alpha^\vee \rangle + 1) + \mathrm{g}_\varphi \langle \rho_{\varphi\setminus\vartheta}, \vartheta^\vee - \alpha^\vee \rangle - \mathrm{g}_\alpha & \text{if } \hat{R} = R^\vee, \\ \mathrm{g}_\vartheta \langle \rho_\vartheta, m_\alpha \varphi^\vee - \alpha^\vee \rangle + \mathrm{g}_\varphi(\langle \rho_{\varphi\setminus\vartheta}, m_\alpha \varphi^\vee - \alpha^\vee \rangle + m_\alpha) - \mathrm{g}_\alpha & \text{if } \hat{R} = R. \end{cases}$$

If $\hat{R} = R^\vee$ the nonnegativity of the expression on the RHS is manifest when $\|\alpha\| = \|\vartheta\|$, while for $\|\alpha\| \neq \|\vartheta\|$ the nonnegativity follows from the fact that $\langle \rho_{\varphi\setminus\vartheta}, \vartheta^\vee - \alpha^\vee \rangle \geq \langle \rho_{\varphi\setminus\vartheta}, \vartheta^\vee - \varphi^\vee \rangle > 0$ (as for R multiply laced the decomposition $\varphi - \vartheta$ in the simple basis contains a simple root β with $\|\beta\| = \|\vartheta\|$). Similarly, if $\hat{R} = R$ the nonnegativity of the RHS is manifest when $\|\alpha\| = \|\varphi\|$ (so $m_\alpha = 1$), while for $\|\alpha\| \neq \|\varphi\|$ the nonnegativity in question follows from the estimates $\langle \rho_\vartheta, m_\alpha \varphi^\vee - \alpha^\vee \rangle = u_\alpha^{-1}\langle \rho_\vartheta, \varphi - \alpha \rangle \geq u_\alpha^{-1}\langle \rho_\vartheta, \varphi - \vartheta \rangle > 0$ and $\langle \rho_{\varphi\setminus\vartheta}, m_\alpha \varphi^\vee - \alpha^\vee \rangle = u_\alpha^{-1}\langle \rho_{\varphi\setminus\vartheta}, \varphi - \alpha \rangle \geq 0$. □

Let $\ell^2(\hat{\rho}_{\mathrm{g}} + \hat{P}_c, \hat{\Delta})$ denote the finite-dimensional Hilbert space of functions $f : \hat{\rho}_{\mathrm{g}} + \hat{P}_c \to \mathbb{C}$ endowed with the inner product

$$\langle f, g \rangle_{\hat{\Delta}} := \sum_{\mu \in \hat{P}_c} f(\hat{\rho}_{\mathrm{g}} + \mu)\overline{g(\hat{\rho}_{\mathrm{g}} + \mu)}\hat{\Delta}(\mu) \quad (f, g \in \ell^2(\hat{\rho}_{\mathrm{g}} + \hat{P}_c, \hat{\Delta})). \tag{4.3}$$

For $\omega \in \hat{P}^+$ small, we consider the following finite analog of the Macdonald difference operator $\mathcal{D}_\omega$ (2.7a)–(2.7c) in the Hilbert space $\ell^2(\hat{\rho}_{\mathrm{g}} + \hat{P}_c, \hat{\Delta})$:

$$(D_\omega f)(\hat{\rho}_{\mathrm{g}} + \mu) = \sum_{\substack{\nu \in \hat{P}(\omega) \\ \mu + \nu \in \hat{P}_c}} \sum_{\eta \in W_\nu(w_\nu^{-1}\omega)}{}' V_\nu(\hat{\rho}_{\mathrm{g}} + \mu) U_{\nu,\eta}(\hat{\rho}_{\mathrm{g}} + \mu) f(\hat{\rho}_{\mathrm{g}} + \mu + \nu) \tag{4.4}$$

($f \in \ell^2(\hat{\rho}_{\mathrm{g}} + \hat{P}_c, \hat{\Delta})$), where the prime indicates that the second sum is restricted to those $\eta \in W_\nu(w_\nu^{-1}\omega)$ for which the denominator of $U_{\nu,\eta}(\hat{\rho}_{\mathrm{g}} + \mu)$ does not vanish. The operator D_ω is well-defined because of the following lemma.

Lemma 4.4 (Regularity of V) *For any* $\nu \in \hat{P}(\omega)$ *with* $\omega \in \hat{P}^+$ *small, the denominator of the coefficient* $V_\nu(x)$ (2.7b) *is nonzero at* $x = \hat{\rho}_g + \mu$ *for all* $\mu \in \hat{P}_c$ *such that* $\mu + \nu \in \hat{P}_c$.

Proof The denominator of the coefficient $V_\nu(\hat{\rho}_{\mathrm{g}} + \mu)$ is built of factors of the form $\sin \kappa_\alpha(\langle \hat{\rho}_{\mathrm{g}} + \mu, \alpha^\vee \rangle)$ and $\sin \kappa_\alpha(\langle \hat{\rho}_{\mathrm{g}} + \mu, \alpha^\vee \rangle + 1)$ with $\alpha \in \hat{R}$ and $\langle \nu, \alpha^\vee \rangle > 0$. These factors can only become zero when (i) $\kappa_\alpha \langle \hat{\rho}_{\mathrm{g}} + \mu, \alpha^\vee \rangle \in \pi\mathbb{Z}$, i.e. $\langle \hat{\rho}_{\mathrm{g}} + \mu, \alpha^\vee \rangle \in m_\alpha(h_{\mathrm{g}} + c)\mathbb{Z}$, or when (ii) $\kappa_\alpha(\langle \hat{\rho}_{\mathrm{g}} + \mu, \alpha^\vee \rangle + 1) \in \pi\mathbb{Z}$, i.e. $\langle \hat{\rho}_{\mathrm{g}} + \mu, \alpha^\vee \rangle + 1 \in m_\alpha(h_{\mathrm{g}} + c)\mathbb{Z}$, respectively. Since for any $\alpha \in \hat{R}$ and $\mu \in \hat{P}_c$: $0 < |\langle \hat{\rho}_{\mathrm{g}}, \alpha^\vee \rangle| < m_\alpha h_{\mathrm{g}}$ and $0 \leq |\langle \mu, \alpha^\vee \rangle| \leq m_\alpha c$ (cf. Lemma 4.3), the zeros of type (i) do not occur as $0 < |\langle \hat{\rho}_{\mathrm{g}} + \mu, \alpha^\vee \rangle| < m_\alpha(h_{\mathrm{g}} + c)$ (so the absolute value of the argument of the sine function in the factor $\sin \kappa_\alpha(\langle \hat{\rho}_{\mathrm{g}} + \mu, \alpha^\vee \rangle)$ stays between

0 and π). When $\alpha \in \hat{R}^+$, the estimates in Lemma 4.3 reveal that a zero of type (ii) can only occur if $\langle \hat{\rho}_{\mathrm{g}} + \mu, \alpha^\vee \rangle + 1 = m_\alpha(h_{\mathrm{g}} + c)$, i.e. $\langle \mu, \alpha^\vee \rangle - m_\alpha c + 1 = m_\alpha h_{\mathrm{g}} - \langle \hat{\rho}_{\mathrm{g}}, \alpha^\vee \rangle (> 0)$, which implies that $\langle \mu, \alpha^\vee \rangle = m_\alpha c$ and $\langle \hat{\rho}_{\mathrm{g}}, \alpha^\vee \rangle = m_\alpha h_{\mathrm{g}} - 1$. But then $\langle \mu + \nu, \alpha^\vee \rangle > m_\alpha c$, i.e. $\mu + \nu \notin \hat{P}_c$ (cf. part (i) of Lemma 4.3). Similarly, when $-\alpha \in \hat{R}^+$ a zero of type (ii) can only occur if $\langle \hat{\rho}_{\mathrm{g}} + \mu, \alpha^\vee \rangle + 1 = 0$, i.e. $\langle \mu, \alpha^\vee \rangle + 1 = -\langle \hat{\rho}_{\mathrm{g}}, \alpha^\vee \rangle (> 0)$, which implies that $\langle \mu, \alpha^\vee \rangle = 0$ and $\langle \hat{\rho}_{\mathrm{g}}, \alpha^\vee \rangle = -1$. But then $\langle \mu + \nu, -\alpha^\vee \rangle < 0$, i.e. $\mu + \nu \notin \hat{P}_c$. □

The next lemma provides an explicit criterion characterizing which weights $\eta \in W_\nu(w_\nu^{-1}\omega)$ are omitted in the second summation of Eq. (4.4). It shows in particular that for generic g, or when $\mu \in \hat{P}_c$ is regular with respect to the action of the affine Weyl group $W \ltimes (cu_\varphi Q^\vee)$, the sum in question is simply over the full orbit $W_\nu(w_\nu^{-1}\omega)$.

Lemma 4.5 (Singularities of U) *For any $\nu \in \hat{P}(\omega)$ and $\eta \in W_\nu(w_\nu^{-1}\omega)$ with $\omega \in \hat{P}^+$ small, the denominator of $U_{\nu,\eta}(x)$ (2.7c) is zero at $x = \hat{\rho}_g + \mu,\ \mu \in \hat{P}_c$ iff there exists an $\alpha \in \hat{R}_\nu$ with $\langle \eta, \alpha^\vee \rangle = 2$ such that $(i) \langle \mu, \alpha^\vee \rangle = 0$ and $\langle \hat{\rho}_g, \alpha^\vee \rangle = -1$,* or *$(ii) \langle \mu, \alpha^\vee \rangle = m_\alpha c$ and $\langle \hat{\rho}_g, \alpha^\vee \rangle = m_\alpha h_g - 1$ (so in both cases $\mu + \eta \notin P_c$).*

Proof The proof goes along the same lines as that of Lemma 4.4, upon replacing ν by η and $\hat{R}$ by $\hat{R}_\nu$. □

By Lemma 4.4, the denominator of $V_\nu(x)$, $\nu \in \hat{P}(\omega)$ can only become zero at $x = \hat{\rho}_{\mathrm{g}} + \mu,\ \mu \in \hat{P}_c$ if $\mu + \nu \notin \hat{P}_c$. For such μ and ν, however, the numerator of the coefficient at issue turns out to vanish identically.

Lemma 4.6 (Vanishing boundary terms) *Let $\nu \in \hat{P}(\omega)$ with $\omega \in \hat{P}^+$ small, and let $\mu \in P_c$. Then the numerator of the coefficient $V_\nu(x)$ (2.7b) vanishes at $x = \hat{\rho}_{\mathrm{g}} + \mu$ iff $\mu + \nu \notin \hat{P}_c$.*

Proof The proof is similar to that of Lemma 4.4. The numerator of the coefficient $V_\nu(\hat{\rho}_{\mathrm{g}} + \mu)$ consists of factors of the form $\sin \kappa_\alpha(\langle \hat{\rho}_{\mathrm{g}} + \mu, \alpha^\vee \rangle + \mathrm{g}_\alpha)$ with $\alpha \in \hat{R}$ and $\langle \nu, \alpha^\vee \rangle > 0$ and $\sin \kappa_\alpha(\langle \hat{\rho}_{\mathrm{g}} + \mu, \alpha^\vee \rangle + 1 + \mathrm{g}_\alpha)$ with $\alpha \in \hat{R}$ and $\langle \nu, \alpha^\vee \rangle = 2$. These factors only become zero when $(i)\, \kappa_\alpha(\langle \hat{\rho}_{\mathrm{g}} + \mu, \alpha^\vee \rangle + \mathrm{g}_\alpha) \in \pi\mathbb{Z}$, i.e. $\langle \hat{\rho}_{\mathrm{g}} + \mu, \alpha^\vee \rangle + \mathrm{g}_\alpha \in m_\alpha(h_{\mathrm{g}} + c)\mathbb{Z}$, or when $(ii)\, \kappa_\alpha(\langle \hat{\rho}_{\mathrm{g}} + \mu, \alpha^\vee \rangle + 1 + \mathrm{g}_\alpha) \in \pi\mathbb{Z}$, i.e. $\langle \hat{\rho}_{\mathrm{g}} + \mu, \alpha^\vee \rangle + 1 + \mathrm{g}_\alpha \in m_\alpha(h_{\mathrm{g}} + c)\mathbb{Z}$. Upon invoking of Lemma 4.3 it is seen that when $\alpha \in \hat{R}^+$, the zeros in question can only occur (i) if $\langle \hat{\rho}_{\mathrm{g}} + \mu, \alpha^\vee \rangle + \mathrm{g}_\alpha = m_\alpha(h_{\mathrm{g}} + c)$, so $\langle \mu, \alpha^\vee \rangle \geq m_\alpha c$, or (ii) if $\langle \hat{\rho}_{\mathrm{g}} + \mu, \alpha^\vee \rangle + 1 + \mathrm{g}_\alpha = m_\alpha(h_{\mathrm{g}} + c)$, so $\langle \mu, \alpha^\vee \rangle \geq m_\alpha c - 1$. In both cases this implies that $\langle \mu + \nu, \alpha^\vee \rangle > m_\alpha c$, whence $\mu + \nu \notin \hat{P}_c$. Similarly, when $-\alpha \in \hat{R}^+$ the zeros in question can only occur (i) if $\langle \hat{\rho}_{\mathrm{g}} + \mu, \alpha^\vee \rangle + \mathrm{g}_\alpha = 0$, so $\langle \mu, \alpha^\vee \rangle \geq 0$, or (ii) if $\langle \hat{\rho}_{\mathrm{g}} + \mu, \alpha^\vee \rangle + 1 + \mathrm{g}_\alpha = 0$, so $\langle \mu, \alpha^\vee \rangle \geq -1$. In both cases this implies that $\langle \mu + \nu, -\alpha^\vee \rangle < 0$, whence $\mu + \nu \notin \hat{P}_c$. Reversely, if $\mu + \nu \notin \hat{P}_c$ then either $(i) \langle \mu + \nu, \beta^\vee \rangle < 0$ for some *simple* root $\beta \in \hat{R}^+$ or $(ii) \langle \mu + \nu, \hat{\varphi}^\vee \rangle > c$. Since $\mu \in \hat{P}_c$ and $|\langle \nu, \alpha^\vee \rangle| \leq 2$ for all $\alpha \in \hat{R}$, in either case we are in one of two situations: $(ia) \langle \mu, \beta^\vee \rangle = 0$ with $\langle \nu, \beta^\vee \rangle < 0$ or $(ib) \langle \mu, \beta^\vee \rangle = 1$ with $\langle \nu, \beta^\vee \rangle = -2$, and $(iia) \langle \mu, \hat{\varphi}^\vee \rangle = c$ with $\langle \nu, \hat{\varphi}^\vee \rangle > 0$ or $(iib) \langle \mu, \hat{\varphi}^\vee \rangle = c - 1$ with $\langle \nu, \hat{\varphi}^\vee \rangle = 2$. In each situation the numerator of $V_\nu(x)$ picks up a zero at $x = \hat{\rho}_{\mathrm{g}} + \mu$ from the factor $(ia) \sin \kappa_\alpha(\langle \hat{\rho}_{\mathrm{g}} + \mu, \alpha^\vee \rangle + \mathrm{g}_\alpha)$ or $(ib) \sin \kappa_\alpha(\langle \hat{\rho}_{\mathrm{g}} + \mu, \alpha^\vee \rangle + 1 + \mathrm{g}_\alpha)$, where $\alpha = -\beta$, and $(iia) \sin \kappa_{\hat{\varphi}}(\langle \hat{\rho}_{\mathrm{g}} + \mu, \hat{\varphi}^\vee \rangle + \mathrm{g}_{\hat{\varphi}})$ or $(iib) \sin \kappa_{\hat{\varphi}}(\langle \hat{\rho}_{\mathrm{g}} + \mu, \hat{\varphi}^\vee \rangle + 1 + \mathrm{g}_{\hat{\varphi}})$. Indeed, the arguments of these sine functions either (i) vanish (since $\langle \hat{\rho}_{\mathrm{g}}, \beta^\vee \rangle = \mathrm{g}_\beta$ for $\beta \in \hat{R}^+$ simple) or (ii) they are equal to π (since $\langle \hat{\rho}_{\mathrm{g}}, \hat{\varphi}^\vee \rangle + \mathrm{g}_{\hat{\varphi}} = \hat{h}_{\mathrm{g}} = h_{\mathrm{g}}$). □

The numerator of $U_{\nu,\eta}(x)$ enjoys an analogous vanishing property at the points $x = \hat{\rho}_g + \mu,\ \mu \in \hat{P}_c$ for which the denominator might become zero (where it is assumed that $\mu + \nu \in \hat{P}_c$ in view of Lemma 4.6).

Lemma 4.7 (Vanishing of U) *Let $\nu \in \hat{P}(\omega)$ and $\eta \in W_\nu(w_\nu^{-1}\omega)$ with $\omega \in \hat{P}^+$ small, and let $\mu \in \hat{P}_c$ with $\mu + \nu \in \hat{P}_c$. Then the numerator of $U_{\nu,\eta}(x)$ (2.7c) vanishes at $x = \hat{\rho}_g + \mu$ if there exists an $\alpha \in \hat{R}_\nu$ with $\langle \eta, \alpha^\vee \rangle = 2$ such that $(i) \langle \mu, \alpha^\vee \rangle = 0$ and $\alpha \notin \hat{R}^+$*, or *$(ii) \langle \mu, \alpha^\vee \rangle = m_\alpha c$.*

Proof In the first case (i), let $-\beta \in \hat{R}^+$ be any simple root in the decomposition of α with respect to the simple basis of $\hat{R}$ for which $\langle \eta, \beta^\vee \rangle > 0$. Since α belongs to the parabolic subsystem $\hat{R}_\mu \cap \hat{R}_\nu = \hat{R}_\mu \cap \hat{R}_{\mu+\nu}$ with μ and $\mu + \nu$ dominant, the same is true for β. Hence, the numerator of $U_{\nu,\eta}(x)$ picks up a zero at $x = \hat{\rho}_g + \mu$ from the factor $\sin \kappa_\beta(\langle \hat{\rho}_g + \mu, \beta^\vee \rangle + g_\beta)$ (using that $\langle \hat{\rho}_g, \beta^\vee \rangle = -g_\beta$ when $-\beta$ is simple). In the second case (ii), it follows from the first inequality in Lemma 4.3 that $\langle \mu, \hat{\varphi}^\vee \rangle = c$ and $\langle \mu + \nu, \hat{\varphi}^\vee \rangle = c$, so $\langle \nu, \hat{\varphi}^\vee \rangle = 0$. Since $\langle \eta, \hat{\varphi}^\vee - \alpha^\vee \rangle = \langle \eta, \hat{\varphi}^\vee \rangle - 2 \leq 0$, we either have that $(iia) \langle \eta, \hat{\varphi}^\vee \rangle = 2$ or (iib) there exists a simple root $-\beta \in \hat{R}^+$ such that $-\hat{\beta}$ is contained in the decomposition of $\varphi - \hat{\alpha} \in Q^+$ with respect to the simple basis of R and $\langle \eta, \beta^\vee \rangle > 0$. Clearly $\varphi - \hat{\alpha}$ belongs to the root lattice of the parabolic subsystem $R_\mu \cap R_{\mu+\nu}$, whence $\hat{\beta} \in R_\mu \cap R_{\mu+\nu}$, i.e. $\beta \in \hat{R}_\nu \cap \hat{R}_\mu$. The upshot is that in the former case (iia) the numerator of $U_{\nu,\eta}(x)$ picks up a zero at $x = \hat{\rho}_g + \mu$ from the factor $\sin \kappa_{\hat{\varphi}}(\langle \hat{\rho}_g + \mu, \hat{\varphi}^\vee \rangle + g_{\hat{\varphi}})$, whereas in the latter case (iib) we pick up a zero from the factor $\sin \kappa_\beta(\langle \hat{\rho}_g + \mu, \beta^\vee \rangle + g_\beta)$ (using again that $\langle \hat{\rho}_g, \beta^\vee \rangle = -g_\beta$). □

We will refer to g being *regular* if

$$\langle \hat{\rho}_g, \alpha^\vee \rangle \notin \{1, m_\alpha h_g - 1\} \quad \text{for all} \quad \alpha \in \hat{R}^+ \tag{4.5}$$

(which is the case generically). The above analysis reveals that when g is regular the denominators of $V_\nu(x)$ (2.7b) and $U_{\nu,\eta}(x)$ (2.7c) do not have zeros at $x = \hat{\rho}_g + \mu$ for all $\mu \in \hat{P}_c$, while the vanishing of the numerators—at places where the denominators might get annihilated if the root multiplier fails to be regular—does persist independent of whether g is regular or not. In other words, for regular root multipliers there are no poles coalescing with the zeros in the numerators and we arrive in this situation at the finite operator D_ω (4.4) by restricting the action of the Macdonald difference operator $\mathcal{D}_\omega$ in Eqs. (2.7a)–(2.7c) to functions f supported on $\hat{\rho}_g + \hat{P}_c$. In general, the finite operator D_ω (4.4) is to be viewed as the continuation of this restriction of the Macdonald operator $\mathcal{D}_\omega$ from regular g to arbitrary $g > 0$.

We will next determine the adjoint of D_ω in the Hilbert space $\ell^2(\hat{\rho}_g + \hat{P}_c, \hat{\Delta})$. The computation hinges on the following elementary recurrence relation for the orthogonality measure $\hat{\Delta}$ in terms of the coefficients V_ν.

Lemma 4.8 (Recurrence relation) *Let $\omega \in \hat{P}^+$ be small and let $\mu \in \hat{P}_c$. Then for any $\nu \in \hat{P}(\omega)$ such that $\mu + \nu \in \hat{P}_c$, one has that*

$$\hat{\Delta}(\mu + \nu) V_{-\nu}(\hat{\rho}_g + \mu + \nu) = \hat{\Delta}(\mu) V_\nu(\hat{\rho}_g + \mu). \tag{4.6}$$

Proof Multiplication of

$$\begin{aligned}\hat{\Delta}(\mu + \nu) &= \prod_{\alpha \in \hat{R}^+} \frac{\sin \kappa_\alpha \langle \hat{\rho}_g + \mu + \nu, \alpha^\vee \rangle}{\sin \kappa_\alpha \langle \hat{\rho}_g, \alpha^\vee \rangle} \frac{(\langle \hat{\rho}_g, \alpha^\vee \rangle + g_\alpha ; \kappa_\alpha)_{\langle \mu+\nu, \alpha^\vee \rangle}}{(\langle \hat{\rho}_g, \alpha^\vee \rangle + 1 - g_\alpha ; \kappa_\alpha)_{\langle \mu+\nu, \alpha^\vee \rangle}} \\ &= \prod_{\alpha \in \hat{R}^+} \frac{\sin \kappa_\alpha \langle \hat{\rho}_g + \mu, \alpha^\vee \rangle}{\sin \kappa_\alpha \langle \hat{\rho}_g, \alpha^\vee \rangle} \frac{(\langle \hat{\rho}_g, \alpha^\vee \rangle + g_\alpha ; \kappa_\alpha)_{\langle \mu, \alpha^\vee \rangle}}{(\langle \hat{\rho}_g, \alpha^\vee \rangle + 1 - g_\alpha ; \kappa_\alpha)_{\langle \mu, \alpha^\vee \rangle}}\end{aligned}$$

$$\times \prod_{\substack{\alpha\in\hat{R}^+\\ \langle\nu,\alpha^\vee\rangle>0}} \frac{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle+1)}{\sin\kappa_\alpha\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle} \frac{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle+{\rm g}_\alpha)}{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle+1-{\rm g}_\alpha)}$$
$$\times \prod_{\substack{\alpha\in\hat{R}^+\\ \langle\nu,\alpha^\vee\rangle=2}} \frac{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle+2)}{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle+1)} \frac{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle+1+{\rm g}_\alpha)}{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle+2-{\rm g}_\alpha)}$$
$$\times \prod_{\substack{\alpha\in\hat{R}^+\\ \langle\nu,\alpha^\vee\rangle<0}} \frac{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle-1)}{\sin\kappa_\alpha\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle} \frac{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle-{\rm g}_\alpha)}{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle-1+{\rm g}_\alpha)}$$
$$\times \prod_{\substack{\alpha\in\hat{R}^+\\ \langle\nu,\alpha^\vee\rangle=-2}} \frac{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle-2)}{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle-1)} \frac{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle-1-{\rm g}_\alpha)}{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle-2+{\rm g}_\alpha)}$$

by

$$\begin{aligned}
&V_{-\nu}(\hat{\rho}_{\rm g}+\mu+\nu) = V_\nu(-\hat{\rho}_{\rm g}-\mu-\nu)\\
&= \prod_{\substack{\alpha\in\hat{R}^+\\ \langle\nu,\alpha^\vee\rangle>0}} \frac{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle+1-{\rm g}_\alpha)}{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle+1)} \prod_{\substack{\alpha\in\hat{R}^+\\ \langle\nu,\alpha^\vee\rangle=2}} \frac{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle+2-{\rm g}_\alpha)}{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle+2)}\\
&\quad\times \prod_{\substack{\alpha\in\hat{R}^+\\ \langle\nu,\alpha^\vee\rangle<0}} \frac{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle-1+{\rm g}_\alpha)}{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle-1)} \prod_{\substack{\alpha\in\hat{R}^+\\ \langle\nu,\alpha^\vee\rangle=-2}} \frac{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle-2+{\rm g}_\alpha)}{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle-2)}
\end{aligned}$$

entails

$$\begin{aligned}
&\prod_{\alpha\in\hat{R}^+} \frac{\sin\kappa_\alpha\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle}{\sin\kappa_\alpha\langle\hat{\rho}_{\rm g},\alpha^\vee\rangle} \frac{(\langle\hat{\rho}_{\rm g},\alpha^\vee\rangle+{\rm g}_\alpha:\kappa_\alpha)_{\langle\mu,\alpha^\vee\rangle}}{(\langle\hat{\rho}_{\rm g},\alpha^\vee\rangle+1-{\rm g}_\alpha:\kappa_\alpha)_{\langle\mu,\alpha^\vee\rangle}}\\
&\quad\times \prod_{\substack{\alpha\in\hat{R}^+\\ \langle\nu,\alpha^\vee\rangle>0}} \frac{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle+{\rm g}_\alpha)}{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle)} \prod_{\substack{\alpha\in\hat{R}^+\\ \langle\nu,\alpha^\vee\rangle=2}} \frac{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle+1+{\rm g}_\alpha)}{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle+1)}\\
&\quad\times \prod_{\substack{\alpha\in\hat{R}^+\\ \langle\nu,\alpha^\vee\rangle<0}} \frac{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle-{\rm g}_\alpha)}{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle)} \prod_{\substack{\alpha\in\hat{R}^+\\ \langle\nu,\alpha^\vee\rangle=-2}} \frac{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle-1-{\rm g}_\alpha)}{\sin\kappa_\alpha(\langle\hat{\rho}_{\rm g}+\mu,\alpha^\vee\rangle-1)}\\
&\quad= \hat{\Delta}(\mu)V_\nu(\hat{\rho}_{\rm g}+\mu).
\end{aligned}$$

□

Proposition 4.9 (Adjoint) *For any $\omega\in\hat{P}^+$ small, the finite Macdonald difference operator D_ω (4.4) satisfies the adjointness relation*

$$\langle D_\omega f, g\rangle_{\hat{\Delta}} = \langle f, D_{\omega^*} g\rangle_{\hat{\Delta}} \quad (f,g\in\ell^2(\hat{\rho}_g+\hat{P}_c,\hat{\Delta})). \tag{4.7}$$

Here $\omega^* := -w_0\omega(\in\hat{P}^+)$, where w_0 refers to the longest element *of W.*

Proof The stated equality is readily inferred via the following sequence of elementary manipulations:

$$
\begin{aligned}
\langle f, D_{\omega^*} g\rangle_{\hat{\Delta}} &= \sum_{\mu\in\hat{P}_c} f(\hat{\rho}_{\mathrm{g}}+\mu)\overline{(D_{\omega^*}g)(\hat{\rho}_{\mathrm{g}}+\mu)}\hat{\Delta}(\mu)\\
&= \sideset{}{'}\sum_{\substack{\nu\in\hat{P}(\omega^*)\\ \eta\in W_\nu(w_\nu^{-1}\omega^*)}} \sum_{\substack{\mu\in\hat{P}_c\\ \mu+\nu\in\hat{P}_c}} f(\hat{\rho}_{\mathrm{g}}+\mu)V_\nu(\hat{\rho}_{\mathrm{g}}+\mu)U_{\nu,\eta}(\hat{\rho}_{\mathrm{g}}+\mu)\overline{g(\hat{\rho}_{\mathrm{g}}+\mu+\nu)}\hat{\Delta}(\mu)\\
&\overset{(i)}{=} \sideset{}{'}\sum_{\substack{\nu\in\hat{P}(\omega)\\ \eta\in W_\nu(w_\nu^{-1}\omega)}} \sum_{\substack{\mu\in\hat{P}_c\\ \mu-\nu\in\hat{P}_c}} f(\hat{\rho}_{\mathrm{g}}+\mu)V_{-\nu}(\hat{\rho}_{\mathrm{g}}+\mu)U_{\nu,-\eta}(\hat{\rho}_{\mathrm{g}}+\mu)\overline{g(\hat{\rho}_{\mathrm{g}}+\mu-\nu)}\hat{\Delta}(\mu)\\
&\overset{(ii)}{=} \sideset{}{'}\sum_{\substack{\nu\in\hat{P}(\omega)\\ \eta\in W_\nu(w_\nu^{-1}\omega)}} \sum_{\substack{\mu\in\hat{P}_c\\ \mu+\nu\in\hat{P}_c}} f(\hat{\rho}_{\mathrm{g}}+\mu+\nu)V_{-\nu}(\hat{\rho}_{\mathrm{g}}+\mu+\nu)U_{\nu,-\eta}(\hat{\rho}_{\mathrm{g}}+\mu)\overline{g(\hat{\rho}_{\mathrm{g}}+\mu)}\hat{\Delta}(\mu+\nu)\\
&\overset{(iii)}{=} \sideset{}{'}\sum_{\substack{\nu\in\hat{P}(\omega)\\ \eta\in W_\nu(w_\nu^{-1}\omega)}} \sum_{\substack{\mu\in\hat{P}_c\\ \mu+\nu\in\hat{P}_c}} f(\hat{\rho}_{\mathrm{g}}+\mu+\nu)V_\nu(\hat{\rho}_{\mathrm{g}}+\mu)U_{\nu,\eta}(\hat{\rho}_{\mathrm{g}}+\mu)\overline{g(\hat{\rho}_{\mathrm{g}}+\mu)}\hat{\Delta}(\mu)\\
&= \sum_{\mu\in\hat{P}_c}(D_\omega f)(\hat{\rho}_{\mathrm{g}}+\mu)\overline{g(\hat{\rho}_{\mathrm{g}}+\mu)}\hat{\Delta}(\mu) = \langle D_\omega f, g\rangle_{\hat{\Delta}}.
\end{aligned}
$$

Here we have used: (i) that $\hat{P}(\omega^*) = -\hat{P}(\omega)$, $W_{-\nu}(w_{-\nu}^{-1}\omega^*) = -W_\nu(w_\nu^{-1}\omega)$ (as $w_{-\nu} = w_0 w_\nu$ and $W_{-\nu} = W_\nu$) and $U_{-\nu,-\eta} = U_{\nu,-\eta}$, (ii) a translation of μ by ν and the equality $U_{\nu,-\eta}(\hat{\rho}_{\mathrm{g}}+\mu+\nu) = U_{\nu,-\eta}(\hat{\rho}_{\mathrm{g}}+\mu)$, (iii) Lemma 4.8 and the identity

$$\sum_{\eta\in W_\nu(w_\nu^{-1}\omega)} U_{\nu,-\eta} = \sum_{\eta\in W_\nu(w_\nu^{-1}\omega)} U_{\nu,w_0\eta} = \sum_{\eta\in W_\nu(w_\nu^{-1}\omega)} U_{\nu,\eta}.$$

□

Remark 4.10 The recurrence in Lemma 4.8 determines the value of $\hat{\Delta}(\mu+\nu)$ in terms of $\hat{\Delta}(\mu)$ (and vice versa), as the coefficients $V_{-\nu}(\hat{\rho}_{\mathrm{g}}+\mu+\nu)$ and $V_\nu(\hat{\rho}_{\mathrm{g}}+\mu)$ on both sides do not vanish (cf. Lemma 4.6). A careful examination of the proofs of Lemmas 4.4 and 4.6 confirms that these coefficients are in fact always positive.

4.3 Orthogonality

We now interpret the polynomials in the finite-dimensional subspace of $\mathbb{C}[P]^W$ spanned by m_λ, $\lambda\in P_c$ as elements of $\ell^2(\hat{\rho}_{\mathrm{g}}+\hat{P}_c,\hat{\Delta})$ by restricting the polynomial variable x to the finite lattice $\hat{\rho}_{\mathrm{g}}+\hat{P}_c\subset E$. Our main concern is to show that the Macdonald polynomials p_λ, $\lambda\in P_c$ then constitute an orthogonal eigenbasis of D_ω (4.4) in this Hilbert space. Recall in this connection that for $\lambda\in P_c$ the inequality $\mu<\lambda$ implies that $\mu\in P_c$ (cf. the proof of Proposition 4.2), i.e. the polynomials m_λ, $\lambda\in P_c$ and p_λ, $\lambda\in P_c$ span the same subspace of $\mathbb{C}[P]^W$. At this point it has only been demonstrated that the Macdonald polynomials are meromorphic in $\mathrm{g}>0$ (cf. Proposition 4.2). For the moment all proofs in this subsection therefore assume that the positive root multiplicity parameters are *generic* (and thus in particular regular) avoiding possible poles of the expansion coefficients and degeneracies of the eigenvalues (cf. Lemma 4.1), while the statements of the propositions are formulated more generally for all $\mathrm{g}>0$. A simple continuity argument in the next subsection will remove this discrepancy.

Proposition 4.11 (Diagonalization) *Let $\omega \in \hat{P}^+$ be small. For any $g > 0$, the Macdonald polynomials p_λ, $\lambda \in P_c$ form a basis of eigenfunctions for D_ω (4.4) in $\ell^2(\hat{\rho}_g + \hat{P}_c, \hat{\Delta})$:*

$$D_\omega p_\lambda = E_\omega(\rho_g + \lambda) p_\lambda, \quad \lambda \in P_c, \tag{4.8}$$

where the eigenvalues are given by E_ω (2.9b).

Proof By virtue of Proposition 4.2—the Macdonald polynomial p_λ satisfies the eigenvalue equation in Eqs. (2.9a), (2.9b) as a meromorphic identity in the positive root multiplicity parameter(s). Let us pick $\mathrm{g} > 0$ generic (see above). The eigenvalue equation in question then reduces to Eq. (4.8) upon restriction of the polynomial variable x to $\hat{\rho}_\mathrm{g} + \hat{P}_c$ (by the argument following Eq. 4.5). From the principal specialization formula in Eq. (3.1b) and the estimates in Lemma 4.3 it is moreover seen that $p_\lambda(\hat{\rho}_\mathrm{g}) > 0$ for $\lambda \in P_c$ (as the arguments of the sine functions in the product formula again take values between 0 and π), so p_λ constitutes for such λ a true (i.e. nonvanishing) eigenfunction of D_ω in $\ell^2(\hat{\rho}_\mathrm{g} + \hat{P}_c, \hat{\Delta})$. The nondegeneracy of the eigenvalues in Lemma 4.1 furthermore implies that the eigenfunctions p_λ, $\lambda \in P_c$ are linearly independent in $\ell^2(\hat{\rho}_\mathrm{g} + \hat{P}_c, \hat{\Delta})$. Hence, they form a basis of this Hilbert space as $\dim \ell^2(\hat{\rho}_\mathrm{g} + \hat{P}_c, \hat{\Delta}) = |\hat{P}_c| = |P_c|$ (cf. Remark 4.13 below). □

Proposition 4.12 (Orthogonality) *For any $g > 0$, the Macdonald polynomials p_λ, $\lambda \in P_c$ form an orthogonal basis of $\ell^2(\hat{\rho}_g + \hat{P}_c, \hat{\Delta})$:*

$$\langle p_\lambda, p_{\tilde{\lambda}} \rangle_{\hat{\Delta}} = 0 \quad \textit{iff}\ \lambda \neq \tilde{\lambda} \tag{4.9}$$

($\lambda, \tilde{\lambda} \in P_c$).

Proof Let us assume that $\mathrm{g} > 0$ is generic (see above). The adjointness relation in Proposition 4.9 and the eigenvalue equation in Proposition 4.11 then lead to the stated orthogonality via a standard argument involving the nondegeneracy of the eigenvalues in Lemma 4.1:

$$0 = \langle D_\omega p_\lambda, p_{\tilde{\lambda}} \rangle_{\hat{\Delta}} - \langle p_\lambda, D_{\omega^*} p_{\tilde{\lambda}} \rangle_{\hat{\Delta}} = (E_\omega(\rho_\mathrm{g} + \lambda) - E_\omega(\rho_\mathrm{g} + \tilde{\lambda})) \langle p_\lambda, p_{\tilde{\lambda}} \rangle_{\hat{\Delta}}$$

(using that $E_{\omega^*} = \overline{E_\omega}$), i.e. $\langle p_\lambda, p_{\tilde{\lambda}} \rangle_{\hat{\Delta}} = 0$ if $\lambda \neq \tilde{\lambda}$ because in this situation $E_\omega(\rho_\mathrm{g} + \lambda) \neq E_\omega(\rho_\mathrm{g} + \tilde{\lambda})$ for some $\omega \in \hat{P}^+$ small. (Notice also that $\langle p_\lambda, p_\lambda \rangle_{\hat{\Delta}} \geq |p_\lambda(\hat{\rho}_\mathrm{g})|^2 > 0$ by the principal specialization formula 3.1b.) □

Remark 4.13 Let $\varphi^\vee = k_1\alpha_1^\vee + \cdots + k_n\alpha_n^\vee$ and $\vartheta^\vee = m_1\alpha_1^\vee + \cdots + m_n\alpha_n^\vee$ be the decompositions of $\varphi^\vee$ and $\vartheta^\vee$ with respect to the simple coroots of R. Then the generating function for the cardinalities of P_c, $c = 0, 1, 2, \ldots$ reads

$$\sum_{c=0}^{\infty} |P_c|\, z^c = (1-z)^{-1} \times \begin{cases} \prod_{j=1}^n (1 - z^{k_j})^{-1} & \text{if } \hat{R} = R \\ \prod_{j=1}^n (1 - z^{m_j})^{-1} & \text{if } \hat{R} = R^\vee \end{cases}$$

($|z| < 1$). In particular, one always has that $|P_c| = |\hat{P}_c|$.

4.4 Analyticity

The triangularity of the monomial expansion of p_λ in Eq. (2.4a) and the orthogonality in Proposition 4.12 implies that (for generic $\mathrm{g} > 0$):

$$p_\lambda = m_\lambda - \sum_{\substack{\mu \in P_c \\ \mu < \lambda}} \frac{\langle m_\lambda, p_\mu \rangle_{\hat{\Delta}}}{\langle p_\mu, p_\mu \rangle_{\hat{\Delta}}} p_\mu \quad (\lambda \in P_c). \tag{4.10}$$

From this Gram-Schmidt type formula it is manifest—by induction on the dominant weight λ with respect to the dominance ordering—that p_λ is in fact *analytic* in $g > 0$ (since the positive weight function $\hat{\Delta}$ is analytic in $g > 0$ and the denominators $\langle p_\mu, p_\mu\rangle_{\hat{\Delta}}$ remain positively bounded from below). As a consequence, the statements in Propositions 4.11 and 4.12 extend from generic g to the full parameter domain $g > 0$ by continuity.

Remark 4.14 It is an immediate consequence of Proposition 4.12 that the matrix $[m_\lambda(\hat{\rho}_g + \mu)]_{\lambda\in P_c,\mu\in\hat{P}_c}$ is invertible, i.e. the evaluation homomorphism mapping the subspace of $\mathbb{C}[P]^W$ spanned by m_λ, $\lambda \in P_c$ into $\ell^2(\hat{\rho}_g + \hat{P}_c, \hat{\Delta})$ is a linear isomorphism.

5 Normalization

5.1 Finite Pieri identity

By combining Proposition 4.11 with the duality symmetry in Eq. (2.5c), one arrives at the following finite Pieri identity in $\ell(\hat{\rho}_g + \hat{P}_c, \hat{\Delta})$ associated with $\omega \in P^+$ small:

$$\hat{E}_\omega P_\lambda = \sum_{\substack{\nu\in P(\omega)\\ \lambda+\nu\in P_c}} \sum_{\eta\in W_\nu(w_\nu^{-1}\omega)}{}' \hat{V}_\nu(\rho_g + \lambda)\hat{U}_{\nu,\eta}(\rho_g + \lambda)P_{\lambda+\nu}. \tag{5.1}$$

Indeed—upon evaluating the difference equation $\hat{D}_\omega \hat{P}_\mu = \hat{E}_\omega(\hat{\rho}_g + \mu)\hat{P}_\mu$ for $\mu \in \hat{P}_c$ at $\rho_g + \lambda$, $\lambda \in P_c$ and invoking of the duality symmetry—Eq. (5.1) follows immediately. With the aid of the above Pieri identity and the recurrence in Lemma 4.8, it is not difficult to express the quadratic norms $\langle P_\lambda, P_\lambda\rangle_{\hat{\Delta}}$ in terms of the norms of the unit polynomial $\langle 1, 1\rangle_{\hat{\Delta}}$. For this purposes it suffices to restrict attention to the Pieri identities associated with the (quasi-)minuscule weights only.

Lemma 5.1 ((Quasi-)Minuscule Path Connectedness) *For any $\lambda \in P_c$, there exists a path $0 = \lambda^{(0)} \to \lambda^{(1)} \to \cdots \to \lambda^{(\ell)} = \lambda$ of weights in P_c such that the increments $\lambda^{(k)} - \lambda^{(k-1)}$, $k = 1, \ldots, \ell$ are given either by positive roots in the orbit $W\vartheta$ or by minuscule weights.*

Proof From the tables in Bourbaki [2], it is readily inferred that the fundamental weights $\omega_1, \ldots, \omega_n$ of R can be grouped—by means of the dominance order on P^+—in Ind(R) linearly ordered chains with minimal elements given by the (quasi-)minuscule fundamental weights. Subsequent fundamental weights in a chain differ moreover by a root in $W\vartheta$. The existence of the path claimed by the lemma is clear if the decomposition of $\lambda \in P_c$ in the basis of the fundamental weights

$$\lambda = \lambda_1\omega_1 + \cdots + \lambda_n\omega_n$$

contains at most nonzero coefficients corresponding to fundamental weights that are either minuscule or quasi-minuscule. Otherwise, if $\lambda_j > 0$ with ω_j neither minuscule nor quasi-minuscule, then the weight $\tilde{\lambda}$ obtained by subtracting the positive root $\omega_j - \omega_{j'} \in W\vartheta$—where $\omega_{j'}$ refers to the fundamental weight preceding ω_j in the respective chain—belongs to P_c as $\tilde{\lambda} < \lambda$ (cf. the proof of Proposition 4.2). The lemma now follows by induction with respect to the dominance order on P^+. □

After these preparations the computation of the quadratic norms is standard.

Proposition 5.2 (Normalization) *For any $\lambda \in P_c$, the quadratic norm of the normalized Macdonald polynomial P_λ (3.1a), (3.1b) is given by*

$$\langle P_\lambda, P_\lambda\rangle_{\hat{\Delta}} = \frac{\mathcal{N}_0}{\Delta(\lambda)} \quad with \quad \mathcal{N}_0 = \langle 1, 1\rangle_{\hat{\Delta}}. \tag{5.2}$$

Proof For $\omega \in P^+$ (quasi)-minuscule, $\lambda \in P_c$ and $\nu \in W\omega$ such that $\lambda + \nu \in P_c$, an expansion of the products on both sides of the identity

$$\langle \hat{E}_\omega P_\lambda, P_{\lambda+\nu}\rangle_{\hat{\Delta}} = \langle P_\lambda, \hat{E}_{\omega^*} P_{\lambda+\nu}\rangle_{\hat{\Delta}}$$

by means of the corresponding RHS of the Pieri formula (5.1) entails (using the orthogonality of Proposition 4.12) that

$$\hat{V}_\nu(\lambda+\rho_{\mathrm{g}})\langle P_{\lambda+\nu}, P_{\lambda+\nu}\rangle_{\hat{\Delta}} = \hat{V}_{-\nu}(\lambda+\nu+\rho_{\mathrm{g}})\langle P_\lambda, P_\lambda\rangle_{\hat{\Delta}}$$

(because $\eta = \nu$ if $\nu \in W\omega$, and $\hat{U}_{\nu,\nu} = 1$). This relation can be recasted—with the aid of the dual version of the recurrence in Lemma 4.8—in terms of the following translational symmetry:

$$\Delta(\lambda+\nu)\langle P_{\lambda+\nu}, P_{\lambda+\nu}\rangle_{\hat{\Delta}} = \Delta(\lambda)\langle P_\lambda, P_\lambda\rangle_{\hat{\Delta}}.$$

By applying the translational symmetry along the increments of a path in Lemma 5.1, we conclude that $\Delta(\lambda)\langle P_\lambda, P_\lambda\rangle_{\hat{\Delta}}$ is equal to $\langle 1, 1\rangle_{\hat{\Delta}}$ (and thus independent of λ), i.e. $\langle P_\lambda, P_\lambda\rangle_{\hat{\Delta}} = \langle 1, 1\rangle_{\hat{\Delta}}/\Delta(\lambda)$. □

5.2 Total mass of the weight function

In this subsection we momentarily allow the multiplicity parameter g to be negative and even complex valued. When $\hat{R} = R^\vee$ the product formula for the total mass of the weight function in Sect. 3 follows from the trigonometric identity in [6, Rem. 4.6]. This trigonometric identity was obtained by truncating a basic hypergeometric summation formula due to Aomoto [1], Ito [11] and Macdonald [18]. It is straightforward to adapt the techniques of Ref. [6] to incorporate the case that $\hat{R} = R$. Indeed, the appropriate Aomoto-Ito-Macdonald sum for our purposes reads (cf. [1, Sec. 1], [11, Sec. 4], and [18, Sec. 9]):

$$\sum_{\lambda\in P} q^{-2\langle\hat{\rho}_{\mathrm{g}},\lambda\rangle} \prod_{\alpha\in R^+}\left(\frac{1-q_\alpha^{\langle x+\lambda,\alpha^\vee\rangle}}{1-q_\alpha^{\langle x,\alpha^\vee\rangle}}\right)\frac{\left(q_\alpha^{\langle x,\alpha^\vee\rangle+\mathrm{g}_\alpha}; q_\alpha\right)_{\langle\lambda,\alpha^\vee\rangle}}{\left(q_\alpha^{\langle x,\alpha^\vee\rangle+1-\mathrm{g}_\alpha}; q_\alpha\right)_{\langle\lambda,\alpha^\vee\rangle}}$$
$$= \mathcal{N}\prod_{\alpha\in R}\frac{\left(q_\alpha^{\langle x,\alpha^\vee\rangle+1}; q_\alpha\right)_\infty}{\left(q_\alpha^{\langle x,\alpha^\vee\rangle+1-\mathrm{g}_\alpha}; q_\alpha\right)_\infty}, \tag{5.3a}$$

with $x \in E$, $0 < q < 1$ and

$$\mathcal{N} := \mathrm{Ind}(R)\prod_{\alpha\in\hat{R}^+}\frac{\left(q_\alpha^{-\langle\hat{\rho}_{\mathrm{g}},\alpha^\vee\rangle+1-\mathrm{g}_\alpha}, q_\alpha^{-\langle\hat{\rho}_{\mathrm{g}},\alpha^\vee\rangle+\mathrm{g}_\alpha+\delta_\alpha}; q_\alpha\right)_\infty}{\left(q_\alpha^{-\langle\hat{\rho}_{\mathrm{g}},\alpha^\vee\rangle+1}, q_\alpha^{-\langle\hat{\rho}_{\mathrm{g}},\alpha^\vee\rangle}; q_\alpha\right)_\infty}, \tag{5.3b}$$

where the value of δ_α is 1 if α is simple and 0 otherwise. Here we have employed the convention that $(a; q)_m := (a; q)_\infty/(aq^m; q)_\infty$ for $m < 0$. The basic hypergeometric sum in Eqs. (5.3a), (5.3b) is normalized such that the term on the LHS is equal to 1 when $\lambda = 0$.

To ensure convergence and avoid poles it is assumed that $\mathrm{g} < 0$ and that for all $\alpha \in R$: $\langle x, \alpha^\vee \rangle \neq 0$ and $\mathrm{g}_\alpha - \langle x, \alpha^\vee \rangle \notin \mathbb{N}$.

At $x = \rho_\mathrm{g}$ with $\mathrm{g} < 0$ such that $\mathrm{g}_\alpha - \langle \rho_\mathrm{g}, \alpha^\vee \rangle \notin \mathbb{N}$ for all $\alpha \in R^+$, this Aomoto-Ito-Macdonald sum reduces to a sum over the dominant weights:

$$\sum_{\lambda \in P^+} q^{-2\langle \hat{\rho}_\mathrm{g}, \lambda \rangle} \prod_{\alpha \in R^+} \left(\frac{1 - q_\alpha^{\langle \rho_\mathrm{g} + \lambda, \alpha^\vee \rangle}}{1 - q_\alpha^{\langle \rho_\mathrm{g}, \alpha^\vee \rangle}} \right) \frac{\left(q_\alpha^{\langle \rho_\mathrm{g}, \alpha^\vee \rangle + \mathrm{g}_\alpha}; q_\alpha \right)_{\langle \lambda, \alpha^\vee \rangle}}{\left(q_\alpha^{\langle \rho_\mathrm{g}, \alpha^\vee \rangle + 1 - \mathrm{g}_\alpha}; q_\alpha \right)_{\langle \lambda, \alpha^\vee \rangle}}$$
$$= \mathrm{Ind}(R) \prod_{\alpha \in R^+} \frac{\left(q_\alpha^{\langle \rho_\mathrm{g}, \alpha^\vee \rangle + 1}; q_\alpha \right)_\infty}{\left(q_\alpha^{-\langle \hat{\rho}_\mathrm{g}, \hat{\alpha}^\vee \rangle}; q_\alpha \right)_\infty} \prod_{\alpha \in R^+ \setminus I} \frac{\left(q_\alpha^{-\langle \hat{\rho}_\mathrm{g}, \hat{\alpha}^\vee \rangle + \mathrm{g}_\alpha}; q_\alpha \right)_\infty}{\left(q_\alpha^{\langle \rho_\mathrm{g}, \alpha^\vee \rangle + 1 - \mathrm{g}_\alpha}; q_\alpha \right)_\infty}, \tag{5.4}$$

where $I \subseteq R^+$ refers to the basis of the simple roots. Indeed, for $\lambda \notin P^+$ there exists a simple root β such that $\langle \lambda, \beta^\vee \rangle < 0$. The term on the LHS then picks up a zero from the factor

$$\frac{1}{\left(q_\beta^{\langle \rho_\mathrm{g}, \beta^\vee \rangle + 1 - g_\beta}; q_\beta \right)_{\langle \lambda, \beta^\vee \rangle}} = \frac{1}{(q_\beta; q_\beta)_{\langle \lambda, \beta^\vee \rangle}} = 0.$$

By exploiting the analyticity in g, the summation formula in Eq. (5.4) can be extended to complex g with $\mathrm{Re}(\mathrm{g}) < 0$ such that $\mathrm{g}_\alpha - \langle \rho_\mathrm{g}, \alpha^\vee \rangle \notin (\mathbb{N} + 2\pi i \mathbb{Z} / \log q_\alpha)$ for all $\alpha \in R^+$. Upon choosing such g subject to the additional constraint that $h_\mathrm{g} + c = 2\pi i / \log q_\varphi$, so $q_\varphi^{h_\mathrm{g} + c} = 1$, the LHS of Eq. (5.4) truncates to a finite sum of the form:

$$\sum_{\lambda \in P_c} q^{-2\langle \hat{\rho}_\mathrm{g}, \lambda \rangle} \prod_{\alpha \in R^+} \left(\frac{1 - q_\alpha^{\langle \rho_\mathrm{g} + \lambda, \alpha^\vee \rangle}}{1 - q_\alpha^{\langle \rho_\mathrm{g}, \alpha^\vee \rangle}} \right) \frac{\left(q_\alpha^{\langle \rho_\mathrm{g}, \alpha^\vee \rangle + \mathrm{g}_\alpha}; q_\alpha \right)_{\langle \lambda, \alpha^\vee \rangle}}{\left(q_\alpha^{\langle \rho_\mathrm{g}, \alpha^\vee \rangle + 1 - \mathrm{g}_\alpha}; q_\alpha \right)_{\langle \lambda, \alpha^\vee \rangle}},$$

since for $\lambda \in P^+ \setminus P_c$ one has that $\langle \lambda, \hat{\psi}^\vee \rangle > c$, i.e. the corresponding term on the LHS of Eq. (5.4) picks up a zero from the factor

$$(q_{\hat{\psi}}^{\langle \rho_\mathrm{g}, \hat{\psi}^\vee \rangle + g_\psi}; q_{\hat{\psi}})_{\langle \lambda, \hat{\psi}^\vee \rangle} = (q_\varphi^{h_\mathrm{g}}; q_\varphi)_{\langle \lambda, \hat{\psi}^\vee \rangle} = (q_\varphi^{-c}; q_\varphi)_{\langle \lambda, \hat{\psi}^\vee \rangle} = 0.$$

In this situation the RHS of Eq. (5.4) can be reduced accordingly to a quotient of finite q-factorials, upon canceling common factors in the numerator and the denominator with the aid of the relation $q_\varphi^{h_\mathrm{g} + c} = 1$. By passing from q-factorials to trigonometric factorials via the substitution $q = e^{2i\kappa}$ with $\kappa = \pi / (u_\varphi (h_\mathrm{g} + c))$, one ends up with the summation formula

$$\sum_{\lambda \in P_c} \Delta(\lambda) = \mathrm{Ind}(R) \mathcal{N}_c, \tag{5.5}$$

with $\mathcal{N}_c$ given by the tables in Sect. 3. The product formula for the total mass of the weight function then follows by continuing analytically to the parameter domain $\mathrm{g} > 0$.

Remark 5.3 Throughout the paper it was assumed that c is not a proper multiple of 6 when $R = E_7$. The reason being that in this particular situation simultaneous degenerations in the spectrum of D_ω (4.4) with $\omega \in \hat{P}^+$ small do occur (cf. the Appendix below), causing Lemma 4.1 (and thus the proof of Theorem 3.1) to break down at this point (only). In fact, our proof of the orthogonality relations in Eq. (3.4c) applies verbatim in this situation for weights belonging to $P_{\tilde{c}} \subseteq P_c$ with $\tilde{c} = \lceil \frac{11}{12} c \rceil$, in view of Remark 6.1. To extend the proof under consideration to the complete basis of Macdonald polynomials for $\ell^2(\hat{P}_c, \hat{\Delta})$, one more independent commuting difference operator is needed to separate the spectrum. In principle the complete algebra of commuting difference operators containing the explicit Macdonald difference operators $\mathcal{D}_\omega$ (2.7a)–(2.7c) can be obtained from Cherednik's representation of the double affine Hecke algebra [21, Eqs. (4.4.12), (5.3.3)]. In this approach it therefore suffices to verify that these difference operators restrict to normal operators in the finite-dimensional Hilbert space $\ell^2(\hat{P}_c, \hat{\Delta})$, cf. [21, Secs. 4.5, 5.3].

Acknowledgments The computations sustaining our case-by-case analysis to verify the nondegeneracy of the spectrum of the Macdonald operators with unitary parameters for the exceptional root systems benefitted much from Stembridge's Maple packages COXETER and WEYL.

6 Appendix: Nondegeneracy of the Eigenvalues for exceptional root systems

In this appendix the nondegeneracy of the eigenvalues in Lemma 4.1 is verified for the exceptional root systems. Specifically, we will check that if for certain $\lambda, \mu \in P_c$ the equality

$$\hat{m}_\omega(\rho_{\mathrm{g}} + \lambda) = \hat{m}_\omega(\rho_{\mathrm{g}} + \mu) \tag{6.1}$$

holds for all $\omega \in \hat{P}^+$ small (as an identity in g), then necessarily $\lambda = \mu$. This implies that the same holds true for the equality $E_\omega(\rho_{\mathrm{g}} + \lambda) = E_\omega(\rho_{\mathrm{g}} + \mu)$ in view of the triangularity of E_ω (2.9b) with respect to the monomial basis.

Since for R exceptional the dual root system $R^\vee$ is isomorphic to R, the truncation relation in Remark 3.3 reads $t_\vartheta^{h/2} t_\varphi^{h/2} q_\varphi^c = 1$ (with h being the Coxeter number of R). We write $\tilde{h}$ for the Coxeter number of the simply laced subsystem $W\varphi \subseteq R$ (so $\tilde{h} = h$ if R is simply laced and $\tilde{h} = h/2$—in our situation—if R is multiply laced). Let us furthermore denote the primitive root of unity $e^{2\pi i/\tilde{h}}$ by ε. Upon writing $\hat{m}_\omega(\rho_{\mathrm{g}} + \lambda) = \sum_{\nu \in W\omega} q^{\langle \nu, \lambda \rangle} \prod_{\alpha \in R^+} t_\alpha^{\langle \nu, \hat{\alpha}^\vee \rangle / 2}$ and elimination of t_ϑ by means of the relations $t_\vartheta = \varepsilon q_\varphi^{-c/\tilde{h}}$ if R is simply laced or $t_\vartheta t_\varphi = \varepsilon q_\varphi^{-c/\tilde{h}}$ if R is multiply laced, both sides of the equality in Eq. (6.1) become Laurent polynomials in t_φ with coefficients built of terms that are products of powers of ε and q (so the Laurent polynomials in question are of degree zero if R is simply laced). For R multiply laced both sides of Eq. (6.1) are equal as analytic functions in g iff all coefficients of the corresponding Laurent polynomials in t_φ match. (Indeed, the polar angles of $q = \exp\left(\frac{2\pi i}{u_\varphi(h_{\mathrm{g}}+c)}\right)$ and $t_\varphi = q^{u_\varphi \mathrm{g}_\varphi} = \exp\left(\frac{2\pi i \mathrm{g}_\varphi}{h_{\mathrm{g}}+c}\right)$ are controlled by two independent parameters g_ϑ and g_φ, so by varying these parameters over the positive reals the tuple of the respective angles covers an open subset of $\left(0, \frac{2\pi}{u_\varphi c}\right) \times \left(0, \frac{2\pi}{\tilde{h}}\right)$.)

The expressions (for the coefficients of the Laurent polynomials in t_φ) on both sides of Eq. (6.1) are themselves polynomials in the primitive root of unity ε of degree $\leq \tilde{h} - 1$ (possibly up to an overall factor $\varepsilon^{1/2}$ when $\mathrm{Ind}(R) > 1$), with coefficients that are sums of powers of q. To eliminate linear dependencies between these roots of unity, the powers $\varepsilon^{\phi(\tilde{h})}, \ldots, \varepsilon^{\tilde{h}-1}$—where ϕ refers to Euler's totient function counting the number of coprimes not exceeding its argument—are expressed in terms of the basis $1, \varepsilon, \ldots, \varepsilon^{\phi(\tilde{h})-1}$ via their residues modulo the cyclotomic polynomial $\Phi_{\tilde{h}}(\varepsilon)$ of degree $\phi(\tilde{h})$. Upon differentiating the coefficients with respect to q and subsequently evaluating at $q = 1$, a pairwise comparison of terms from both sides provides linear relations of the form $\langle \lambda - \mu, v \rangle = 0$ with $v \in Q^\vee$ (where we exploit the fact that the roots of unity $1, \varepsilon, \ldots, \varepsilon^{\phi(\tilde{h})-1}$ are linearly independent over the rationals).

By varying over the different coefficients and small weights $\omega \in \hat{P}$, we deduce this way that the equality in Eq. (6.1) implies that $\lambda - \mu$ must be orthogonal to $n(=\text{rank}(R))$ linearly independent vectors $v \in Q^\vee$ unless R is of type E_7, whence μ must be equal to λ in these cases.

When R is of type E_7, the relevant vectors $v \in Q^\vee$ turn out to span a hyperplane, viz. the equality in Eq. (6.1) now permits to conclude only that $\lambda - \mu$ must belong to the line perpendicular to this hyperplane. A comparison of the quadratic terms—obtained by first applying the differential operator $(q\frac{\mathrm{d}}{\mathrm{d}q})^2$ to the coefficients of the expression on both sides of Eq. (6.1) and then evaluating at $q = 1$—under the additional assumption that μ differs from λ by a *nonzero* vector belonging to this perpendicular line, now entails a nonhomogeneous linear system for λ. When c is not a multiple of 6, its (two-dimensional) solution space does not intersect P, whence the equality in Eq. (6.1) still implies that $\lambda = \mu$ in this situation.

Below we identify for each exceptional root system (ordered by increasing rank), a minimal choice of small weights ω and the corresponding coefficients of $\hat{m}_\omega(\rho_{\mathrm{g}} + \lambda)$ giving rise to a maximal system of linearly independent vectors $v \in Q^\vee$ that are orthogonal to $\lambda - \mu$ when Eq. (6.1) holds. Here the weights λ (and μ) will be expressed in the basis of fundamental weights $\lambda = \lambda_1\omega_1 + \cdots + \lambda_n\omega_n$, and the relevant vectors $v \in Q^\vee$ will be represented by the components $(v_1, v_2, \ldots, v_n)$ with respect to the dual basis of simple coroots (i.e. $v = v_1\alpha_1^\vee + \cdots + v_n\alpha_n^\vee$). In each case, the normalization of the root system, the choice of the positive subsystem, and the numbering of the elements of the simple and fundamental bases will follow the conventions of the tables in Bourbaki [2]. We end the appendix by providing some details regarding the additional analysis of the quadratic terms required to rule out the degeneracies when R is of type E_7.

6.1 Type G

The quasi-minuscule weight ω of $\hat{R}$ is equal to $\varphi^\vee$ if $\hat{R} = R^\vee$ and equal to ϑ is $\hat{R} = R$. For R of type G_2, the corresponding monomials $\hat{m}_\omega(\rho_{\mathrm{g}} + \lambda)$ are of the form $\hat{m}_\omega(\rho_{\mathrm{g}} + \lambda) = \hat{m}^+_\omega(\rho_{\mathrm{g}} + \lambda) + \overline{\hat{m}^+_\omega(\rho_{\mathrm{g}} + \lambda)}$ with

$$\hat{m}^+_{\varphi^\vee}(\rho_{\mathrm{g}} + \lambda) = t_\vartheta t_\varphi^2 q^{\lambda_1+2\lambda_2} + t_\vartheta t_\varphi q^{\lambda_1+\lambda_2} + t_\varphi q^{\lambda_2} \quad (\hat{R} = R^\vee),$$
$$\hat{m}^+_{\vartheta}(\rho_{\mathrm{g}} + \lambda) = t_\vartheta^2 t_\varphi q^{2\lambda_1+3\lambda_2} + t_\vartheta t_\varphi q^{\lambda_1+3\lambda_2} + t_\vartheta q^{\lambda_1} \quad (\hat{R} = R).$$

We have that $\tilde{h} = 3$ and $\varepsilon = e^{2\pi i/3}$. Elimination of t_ϑ via the truncation relation $t_\vartheta t_\varphi = \varepsilon q_\varphi^{-c/3}$ and calculation of the residues modulo the cyclotomic polynomial $\Phi_3(\varepsilon) = \varepsilon^2+\varepsilon+1$ gives

$$\begin{aligned}\hat{m}_{\varphi^\vee}(\rho_{\mathrm{g}} + \lambda) = &\left(q^{\lambda_2} + \varepsilon q^{\lambda_1+2\lambda_2-\frac{c}{3}}\right) t_\varphi + \left(q^{-\lambda_2} - q^{-\lambda_1-2\lambda_2+\frac{c}{3}} - \varepsilon q^{-\lambda_1-2\lambda_2+\frac{c}{3}}\right) t_\varphi^{-1} \\ &+ \left(-q^{-\lambda_1-\lambda_2+\frac{c}{3}} + \varepsilon\left(q^{\lambda_1+\lambda_2-\frac{c}{3}} - q^{-\lambda_1-\lambda_2+\frac{c}{3}}\right)\right) \quad (\hat{R} = R^\vee)\end{aligned}$$

and

$$\begin{aligned}\hat{m}_{\vartheta}(\rho_{\mathrm{g}} + \lambda) = &\left(-q^{-\lambda_1+\frac{c}{3}u_\varphi} + \varepsilon\left(q^{-2\lambda_1-3\lambda_2+\frac{2c}{3}u_\varphi} - q^{-\lambda_1+\frac{c}{3}u_\varphi}\right)\right) t_\varphi \\ &+ \left(-q^{2\lambda_1+3\lambda_2-\frac{2c}{3}u_\varphi} + \varepsilon\left(q^{\lambda_1-\frac{c}{3}u_\varphi} - q^{2\lambda_1+3\lambda_2-\frac{2c}{3}u_\varphi}\right)\right) t_\varphi^{-1} \\ &+ \left(-q^{-\lambda_1-3\lambda_2+\frac{c}{3}u_\varphi} + \varepsilon\left(q^{\lambda_1+3\lambda_2-\frac{c}{3}u_\varphi} - q^{-\lambda_1-3\lambda_2+\frac{c}{3}u_\varphi}\right)\right) \quad (\hat{R} = R).\end{aligned}$$

Differentiation with respect to q of the coefficients of the Laurent polynomials in t_φ on both sides of Eq. (6.1) and subsequent evaluation at $q = 1$ leads—upon comparing the coefficients of t_φ and εt_φ from both sides—to the relations $\lambda_2 = \mu_2,\ \lambda_1 + 2\lambda_2 = \mu_1 + 2\mu_2$ if $\hat{R} = R^\vee$ and $\lambda_1 = \mu_1,\ \lambda_1 + 3\lambda_2 = \mu_1 + 3\mu_2$ if $\hat{R} = R$. In other words, the equality in Eq. (6.1) implies that $\lambda - \mu$ must be orthogonal to $\alpha_2^\vee$ and $\alpha_1^\vee + 2\alpha_2^\vee$ if $\hat{R} = R^\vee$ and to $\alpha_1^\vee$ and $\alpha_1^\vee + 3\alpha_2^\vee$ if $\hat{R} = R$. In both cases, the equality in Eq. (6.1) therefore holds only when $\lambda = \mu$.

6.2 Type F

Proceeding as for G_2, we compute for $\omega \in \hat{P}^+$ quasi-minuscule $\hat{m}_\omega(\rho_{\mathrm{g}} + \lambda) = \hat{m}_\omega^+(\rho_{\mathrm{g}} + \lambda) + \overline{\hat{m}_\omega^+(\rho_{\mathrm{g}} + \lambda)}$, with $\omega = \varphi^\vee$ and

$$
\begin{aligned}
\hat{m}_{\varphi^\vee}^+(\rho_{\mathrm{g}} + \lambda) = {} & t_\vartheta^3 t_\varphi^5 q^{2\lambda_1+3\lambda_2+2\lambda_3+\lambda_4} + t_\vartheta^3 t_\varphi^4 q^{\lambda_1+3\lambda_2+2\lambda_3+\lambda_4} + t_\vartheta^3 t_\varphi^3 q^{\lambda_1+2\lambda_2+2\lambda_3+\lambda_4} \\
& + t_\vartheta^2 t_\varphi^3 q^{\lambda_1+2\lambda_2+\lambda_3+\lambda_4} + t_\vartheta^2 t_\varphi^2 q^{\lambda_1+\lambda_2+\lambda_3+\lambda_4} \\
& + t_\vartheta t_\varphi^3 q^{\lambda_1+2\lambda_2+\lambda_3} + t_\vartheta^2 t_\varphi q^{\lambda_2+\lambda_3+\lambda_4} \\
& + t_\vartheta t_\varphi^2 q^{\lambda_1+\lambda_2+\lambda_3} + t_\vartheta t_\varphi q^{\lambda_2+\lambda_3} + t_\varphi^2 q^{\lambda_1+\lambda_2} + t_\varphi \left(q^{\lambda_1} + q^{\lambda_2}\right)
\end{aligned}
$$

if $\hat{R} = R^\vee$, and with $\omega = \vartheta$ and

$$
\begin{aligned}
\hat{m}_\vartheta^+(\rho_{\mathrm{g}} + \lambda) = {} & t_\vartheta^5 t_\varphi^3 q^{2\lambda_1+4\lambda_2+3\lambda_3+2\lambda_4} + t_\vartheta^4 t_\varphi^3 q^{2\lambda_1+4\lambda_2+3\lambda_3+\lambda_4} + t_\vartheta^3 t_\varphi^3 q^{2\lambda_1+4\lambda_2+2\lambda_3+\lambda_4} \\
& + t_\vartheta^3 t_\varphi^2 q^{2\lambda_1+2\lambda_2+2\lambda_3+\lambda_4} + t_\vartheta^3 t_\varphi q^{2\lambda_2+2\lambda_3+\lambda_4} + t_\vartheta^2 t_\varphi^2 q^{2\lambda_1+2\lambda_2+\lambda_3+\lambda_4} \\
& + t_\vartheta^2 t_\varphi q^{2\lambda_2+\lambda_3+\lambda_4} + t_\vartheta t_\varphi^2 q^{2\lambda_1+2\lambda_2+\lambda_3} + t_\vartheta^2 q^{\lambda_3+\lambda_4} + t_\vartheta t_\varphi q^{2\lambda_2+\lambda_3} + t_\vartheta \left(q^{\lambda_3} + q^{\lambda_4}\right)
\end{aligned}
$$

if $\hat{R} = R$. In the present case $\tilde{h} = 6,\ \varepsilon = e^{2\pi i/6}$, and elimination of t_ϑ via $t_\vartheta t_\varphi = \varepsilon q_\varphi^{-c/6}$ yields modulo the cyclotomic polynomial $\Phi_6(\varepsilon) = \varepsilon^2 - \varepsilon + 1$:

$$
\begin{aligned}
\hat{m}_{\varphi^\vee}(\rho_{\mathrm{g}} + \lambda) = {} & \Big(q^{\lambda_1+\lambda_2} - q^{2\lambda_1+3\lambda_2+2\lambda_3+\lambda_4-\frac{c}{2}} + \varepsilon q^{\lambda_1+2\lambda_2+\lambda_3-\frac{c}{6}}\Big) t_\varphi^2 \\
& + \Big(q^{\lambda_1} + q^{\lambda_2} - q^{\lambda_1+2\lambda_2+\lambda_3+\lambda_4-\frac{c}{3}} - q^{\lambda_1+3\lambda_2+2\lambda_3+\lambda_4-\frac{c}{2}} \\
& + \varepsilon \left(q^{\lambda_1+2\lambda_2+\lambda_3+\lambda_4-\frac{c}{3}} + q^{\lambda_1+\lambda_2+\lambda_3-\frac{c}{6}} - q^{-\lambda_2-\lambda_3-\lambda_4+\frac{c}{3}}\right)\Big) t_\varphi \\
& \Big(-q^{-\lambda_1-2\lambda_2-2\lambda_3-\lambda_4+\frac{c}{2}} + q^{-\lambda_2-\lambda_3+\frac{c}{6}} - q^{\lambda_1+\lambda_2+\lambda_3+\lambda_4-\frac{c}{3}} - q^{\lambda_1+2\lambda_2+2\lambda_3+\lambda_4-\frac{c}{2}} \\
& + \varepsilon \left(q^{\lambda_1+\lambda_2+\lambda_3+\lambda_4-\frac{c}{3}} + q^{\lambda_2+\lambda_3-\frac{c}{6}} - q^{-\lambda_2-\lambda_3+\frac{c}{6}} - q^{-\lambda_1-\lambda_2-\lambda_3-\lambda_4+\frac{c}{3}}\right)\Big) \\
& + \Big(q^{-\lambda_1-\lambda_2-\lambda_3+\frac{c}{6}} - q^{-\lambda_1-3\lambda_2-2\lambda_3-\lambda_4+\frac{c}{2}} + q^{-\lambda_2} + q^{-\lambda_1} - q^{\lambda_2+\lambda_3+\lambda_4-\frac{c}{3}} \\
& + \varepsilon \left(-q^{-\lambda_1-2\lambda_2-\lambda_3-\lambda_4+\frac{c}{3}} + q^{\lambda_2+\lambda_3+\lambda_4-\frac{c}{3}} - q^{-\lambda_1-\lambda_2-\lambda_3+\frac{c}{6}}\right)\Big) t_\varphi^{-1} \\
& + \Big(-q^{-2\lambda_1-3\lambda_2-2\lambda_3-\lambda_4+\frac{c}{2}} + q^{-\lambda_1-2\lambda_2-\lambda_3+\frac{c}{6}} + q^{-\lambda_1-\lambda_2} - \varepsilon q^{-\lambda_1-2\lambda_2-\lambda_3+\frac{c}{6}}\Big) t_\varphi^{-2}
\end{aligned}
$$

if $\hat{R} = R^\vee$, and

$$
\begin{aligned}
\hat{m}_\vartheta(\rho_{\mathrm{g}} + \lambda) = {} & \left(-q^{-2\lambda_2-2\lambda_3-\lambda_4+\frac{c}{2}u_\varphi} + \varepsilon \left(q^{-2\lambda_1-4\lambda_2-3\lambda_3-2\lambda_4+\frac{5c}{6}u_\varphi} - q^{-\lambda_3-\lambda_4+\frac{c}{3}u_\varphi}\right)\right) t_\varphi^2 \\
& + \Big(q^{-\lambda_4+\frac{c}{6}u_\varphi} + q^{-\lambda_3+\frac{c}{6}u_\varphi} - q^{-2\lambda_1-2\lambda_2-2\lambda_3-\lambda_4+\frac{c}{2}u_\varphi} - q^{-2\lambda_1-4\lambda_2-3\lambda_3-\lambda_4+\frac{2c}{3}u_\varphi} \\
& + \varepsilon \Big(q^{2\lambda_1+2\lambda_2+\lambda_3-\frac{c}{6}u_\varphi} - q^{-\lambda_4+\frac{c}{6}u_\varphi} - q^{-\lambda_3+\frac{c}{6}u_\varphi} - q^{-2\lambda_2-\lambda_3-\lambda_4+\frac{c}{3}u_\varphi} \\
& + q^{-2\lambda_1-4\lambda_2-3\lambda_3-\lambda_4+\frac{2c}{3}u_\varphi}\Big)\Big) t_\varphi
\end{aligned}
$$

$$+\Big(-q^{2\lambda_1+4\lambda_2+2\lambda_3+\lambda_4-\frac{c}{2}u_\varphi} - q^{2\lambda_1+2\lambda_2+\lambda_3+\lambda_4-\frac{c}{3}u_\varphi} + q^{-2\lambda_2-\lambda_3+\frac{c}{6}u_\varphi}$$
$$-q^{-2\lambda_1-4\lambda_2-2\lambda_3-\lambda_4+\frac{c}{2}u_\varphi}$$
$$\varepsilon\left(q^{2\lambda_2+\lambda_3-\frac{c}{6}u_\varphi} + q^{2\lambda_1+2\lambda_2+\lambda_3+\lambda_4-\frac{c}{3}u_\varphi} - q^{-2\lambda_1-2\lambda_2-\lambda_3-\lambda_4+\frac{c}{3}u_\varphi}\right.$$
$$\left.\left.-q^{-2\lambda_2-\lambda_3+\frac{c}{6}u_\varphi}\right)\right)$$
$$+\Big(q^{-2\lambda_1-2\lambda_2-\lambda_3+\frac{c}{6}u_\varphi} - q^{2\lambda_1+2\lambda_2+2\lambda_3+\lambda_4-\frac{c}{2}u_\varphi} - q^{2\lambda_2+\lambda_3+\lambda_4-\frac{c}{3}u_\varphi}$$
$$+\varepsilon\left(q^{\lambda_3-\frac{c}{6}u_\varphi} + q^{\lambda_4-\frac{c}{6}u_\varphi} + q^{2\lambda_2+\lambda_3+\lambda_4-\frac{c}{3}u_\varphi} - q^{-2\lambda_1-2\lambda_2-\lambda_3+\frac{c}{6}u_\varphi}\right.$$
$$\left.\left.-q^{2\lambda_1+4\lambda_2+3\lambda_3+\lambda_4-\frac{2c}{3}u_\varphi}\right)\right)t_\varphi^{-1}$$
$$+\Big(q^{2\lambda_1+4\lambda_2+3\lambda_3+2\lambda_4-\frac{5c}{6}u_\varphi} - q^{2\lambda_2+2\lambda_3+\lambda_4-\frac{c}{2}u_\varphi} - q^{\lambda_3+\lambda_4-\frac{c}{3}u_\varphi}$$
$$\left.+\varepsilon\left(q^{\lambda_3+\lambda_4-\frac{c}{3}u_\varphi} - q^{2\lambda_1+4\lambda_2+3\lambda_3+2\lambda_4-\frac{5c}{6}u_\varphi}\right)\right)t_\varphi^{-2}$$

if $\hat{R} = R$. Comparison of the coefficients of t_φ, t_φ^2, εt_φ and εt_φ^2 on both sides of Eq. (6.1) now leads (upon differentiation at $q = 1$) to the following linearly independent vectors $v \in Q^\vee$ that are orthogonal to $\lambda - \mu$ if the equality holds: $(1, 4, 3, 2)$, $(1, 2, 2, 1)$, $(2, 4, 3, 2)$ and $(1, 2, 1, 0)$ if $\hat{R} = R^\vee$, and $(4, 6, 4, 1)$, $(0, 2, 2, 1)$, $(0, 0, 0, 1)$ and $(2, 4, 2, 1)$ if $\hat{R} = R$ (where—recall—the components are with respect to the basis of simple coroots of R).

6.3 Type E

For R of type E_6, one has that $\tilde{h} = h = 12$, so $t_\vartheta = \varepsilon q^{-c/12}$ with $\varepsilon = e^{2\pi i/12}$, and the relevant cyclotomic polynomial is $\Phi_{12}(\varepsilon) = \varepsilon^4 - \varepsilon^2 + 1$. We consider $\hat{m}_\omega(\rho_{\rm g} + \lambda)$ with ω being equal either to the minuscule weight ω_6 or to the quasi-minuscule weight $\omega_2 = \varphi$. In the minuscule case the LHS of Eq. (6.1) becomes explicitly:

$$\hat{m}_{\omega_6}(\rho_{\rm g}+\lambda) = \varepsilon^{11}\left(q^{\frac{1}{3}(-\lambda_1-2\lambda_3-\lambda_5+\lambda_6)+\frac{c}{12}} + q^{\frac{1}{3}(-\lambda_1+\lambda_3-\lambda_5-2\lambda_6)+\frac{c}{12}}\right)$$
$$+\varepsilon^{10}\left(q^{\frac{1}{3}(-\lambda_1-2\lambda_3-3\lambda_4-\lambda_5+\lambda_6)+\frac{c}{6}} + q^{\frac{1}{3}(-\lambda_1-2\lambda_3-\lambda_5-2\lambda_6)+\frac{c}{6}}\right)$$
$$+\varepsilon^{9}\left(q^{\frac{1}{3}(-\lambda_1-3\lambda_2-2\lambda_3-3\lambda_4-\lambda_5+\lambda_6)+\frac{c}{4}} + q^{\frac{1}{3}(-\lambda_1-2\lambda_3-3\lambda_4-\lambda_5-2\lambda_6)+\frac{c}{4}}\right)$$
$$+\varepsilon^{8}\Big(q^{\frac{1}{3}(2\lambda_1+3\lambda_2+4\lambda_3+6\lambda_4+5\lambda_5+4\lambda_6)-\frac{2c}{3}}$$
$$\left.+q^{\frac{1}{3}(-\lambda_1-3\lambda_2-2\lambda_3-3\lambda_4-\lambda_5-2\lambda_6)+\frac{c}{3}} + q^{\frac{1}{3}(-\lambda_1-2\lambda_3-3\lambda_4-4\lambda_5-2\lambda_6)+\frac{c}{3}}\right)$$
$$+\varepsilon^{7}\left(q^{\frac{1}{3}(2\lambda_1+3\lambda_2+4\lambda_3+6\lambda_4+5\lambda_5+\lambda_6)-\frac{7c}{12}} + q^{\frac{1}{3}(-\lambda_1-3\lambda_2-2\lambda_3-3\lambda_4-4\lambda_5-2\lambda_6)+\frac{5c}{12}}\right)$$
$$+\varepsilon^{6}\left(q^{\frac{1}{3}(2\lambda_1+3\lambda_2+4\lambda_3+6\lambda_4+2\lambda_5+\lambda_6)-\frac{c}{2}} + q^{\frac{1}{3}(-\lambda_1-3\lambda_2-2\lambda_3-6\lambda_4-4\lambda_5-2\lambda_6)+\frac{c}{2}}\right)$$
$$+\varepsilon^{5}\left(q^{\frac{1}{3}(-\lambda_1-3\lambda_2-5\lambda_3-6\lambda_4-4\lambda_5-2\lambda_6)+\frac{7c}{12}} + q^{\frac{1}{3}(2\lambda_1+3\lambda_2+4\lambda_3+3\lambda_4+2\lambda_5+\lambda_6)-\frac{5c}{12}}\right)$$
$$+\varepsilon^{4}\Big(q^{\frac{1}{3}(-4\lambda_1-3\lambda_2-5\lambda_3-6\lambda_4-4\lambda_5-2\lambda_6)+\frac{2c}{3}}$$
$$\left.+q^{\frac{1}{3}(2\lambda_1+4\lambda_3+3\lambda_4+2\lambda_5+\lambda_6)-\frac{c}{3}} + q^{\frac{1}{3}(2\lambda_1+3\lambda_2+\lambda_3+3\lambda_4+2\lambda_5+\lambda_6)-\frac{c}{3}}\right)$$
$$+\varepsilon^{3}\left(q^{\frac{1}{3}(2\lambda_1+\lambda_3+3\lambda_4+2\lambda_5+\lambda_6)-\frac{c}{4}} + q^{\frac{1}{3}(-\lambda_1+3\lambda_2+\lambda_3+3\lambda_4+2\lambda_5+\lambda_6)-\frac{c}{4}}\right)$$

$$+\varepsilon^2\left(q^{\frac{1}{3}(-\lambda_1+\lambda_3+3\lambda_4+2\lambda_5+\lambda_6)-\frac{c}{6}}+q^{\frac{1}{3}(2\lambda_1+\lambda_3+2\lambda_5+\lambda_6)-\frac{c}{6}}\right)$$
$$+\varepsilon\left(q^{\frac{1}{3}(2\lambda_1+\lambda_3-\lambda_5+\lambda_6)-\frac{c}{12}}+q^{\frac{1}{3}(-\lambda_1+\lambda_3+2\lambda_5+\lambda_6)-\frac{c}{12}}\right)$$
$$+q^{\frac{1}{3}(-\lambda_1+\lambda_3-\lambda_5+\lambda_6)}+q^{\frac{1}{3}(-\lambda_1-2\lambda_3+2\lambda_5+\lambda_6)}+q^{\frac{1}{3}(2\lambda_1+\lambda_3-\lambda_5-2\lambda_6)},$$

with $\varepsilon^4=\varepsilon^2-1$, $\varepsilon^5=\varepsilon^3-\varepsilon$, $\varepsilon^6=-1$, $\varepsilon^7=-\varepsilon$, $\varepsilon^8=-\varepsilon^2$, $\varepsilon^9=-\varepsilon^3$, $\varepsilon^{10}=1-\varepsilon^2$, and $\varepsilon^{11}=\varepsilon-\varepsilon^3$. Differentiation at $q=1$ of the coefficients of ε^0, ε^1, ε^2 and ε^3 on both sides of Eq. (6.1) produces the following four linearly independent vectors $v\in Q^\vee$: $(1,0,2,1,0,0)$, $(1,0,0,0,0,-1)$, $(1,0,2,2,2,1)$ and $(2,2,2,3,2,1)$, respectively. A similar computation for $\omega=\omega_2=\varphi$ complements these with two more linearly independent vectors v: $(0,1,1,1,1,0)$ and $(0,1,1,3,1,0)$, stemming from the coefficients of ε^0 and ε^3.

For R of type E_7, one has that $\tilde{h}=h=18$, so $t_\vartheta=\varepsilon q^{-c/18}$ with $\varepsilon=e^{2\pi i/18}$, and the corresponding cyclotomic polynomial is $\Phi_{18}(\varepsilon)=\varepsilon^6-\varepsilon^3+1$. We consider $\hat{m}_\omega(\rho_{\rm g}+\lambda)$ with ω being equal either to the minuscule weight ω_7 or to the quasi-minuscule weight $\omega_1=\varphi$. In the minuscule case we divide out an overall factor $\varepsilon^{1/2}q^{-c/(2h)}$ from Eq. (6.1) before proceeding. The relevant linearly independent vectors $v\in Q^\vee$ are: $(2,2,3,4,3,2,2)$(ε^0-term), $(1,0,0,0,0,-1,0)$(ε^1-term) and $(0,1,0,2,3,2,1)$(ε^5-term) for $\omega=\omega_7$, and $(1,1,2,2,2,1,0)$(ε^0-term), $(1,0,1,2,1,1,0)$(ε^1-term) and $(1,2,2,4,2,1,0)$(ε^4-term) for $\omega=\omega_1$.

For R of type E_8, one has that $\tilde{h}=h=30$, so $t_\vartheta=\varepsilon q^{-c/30}$ with $\varepsilon=e^{2\pi i/30}$, and the corresponding cyclotomic polynomial is $\Phi_{30}(\varepsilon)=\varepsilon^8+\varepsilon^7-\varepsilon^5-\varepsilon^4-\varepsilon^3+\varepsilon+1$. We consider $\hat{m}_\omega(\rho_{\rm g}+\lambda)$ with ω being equal either to the quasi-minuscule weight $\omega_8=\varphi$ or to the only other small weight ω_1. The relevant linearly independent vectors $v\in Q^\vee$ are for $\omega=\omega_8$: $(1,1,4,5,4,2,1,1)$(ε^0-term), $(2,3,6,7,5,4,2,1)$(ε^1-term), $(2,3,2,4,3,2,2,0)$(ε^2-term) and $(0,0,2,2,1,0,1,0)$(ε^3-term), and for $\omega=\omega_1$: $(7,7,30,39,29,14,6,7)$(ε^0-term), $(14,21,44,51,35,28,12,5)$(ε^1-term), $(16,24,16,30,23,14,15,0)$(ε^2-term) and $(-2,0,14,15,6,2,6,-1)$(ε^3-term).

6.4 Type E_7 revisited

In the case that R is of type E_7, it follows from the previous considerations that the equality in Eq. (6.1) can hold only if $\lambda-\mu$ is an integral multiple of the weight

$$\nu=2\omega_1+2\omega_2-\omega_3-\omega_4-\omega_5+2\omega_6-\omega_7=\alpha_1+\alpha_2+\alpha_6 \tag{6.2}$$

(which spans the orthogonal complement of the hyperplane spanned by the above vectors $v\in Q^\vee$ for this case). Substituting $\mu=\lambda+k\nu\,(k\in\mathbb{Z})$ and application of the operator $(q\frac{\mathrm{d}}{\mathrm{d}q})^2$ to the coefficients on both sides of the equality entails a system of quadratic relations in λ and k (upon evaluation at $q=1$). In each of these relations the LHS cancels against the quadratic terms in λ on the RHS (viz. the k^0-terms) and —more surprisingly—the quadratic terms in k on the RHS also turn out to cancel against each other. From the remaining linear terms in k we then deduce that the equality in Eq. (6.1) implies that either $k=0$ or that λ must satisfy a nonhomogenous system of five linearly independent equations: $2\lambda_1+2\lambda_2+3\lambda_3+4\lambda_4+3\lambda_5+2\lambda_6+2\lambda_7=c$($\varepsilon^0$-term), $\lambda_1-\lambda_6=0$(ε^1-term) for $\omega=\omega_7$, and $\lambda_1+\lambda_2+2\lambda_3+2\lambda_4+2\lambda_5+\lambda_6=c/2$($\varepsilon^0$-term), $\lambda_1+\lambda_3+2\lambda_4+\lambda_5+\lambda_6=c/3$($\varepsilon^1$-term) and $2\lambda_2+\lambda_3+2\lambda_4+\lambda_5=c/3$($\varepsilon^5$-term) for $\omega=\omega_1$. The intersection of its two-dimensional plane of solutions with the convex hull of P_c is given by the triangle

$$\lambda^{(0)}-s\nu-r\eta,\quad |r|\le s\le\frac{c}{12}, \tag{6.3}$$

where $\lambda^{(0)} := \frac{c}{6}(\omega_1 + \omega_2 + \omega_6)$, ν is given by Eq. (5.7), and $\eta := \omega_3 - \omega_5$. Our condition that c not be an integral multiple of 6 when R is of type E_7 guarantees that the intersection of the triangle with P_c is empty, i.e. the equality in Eq. (5.6) can only hold if $k = 0$ (so $\lambda = \mu$).

Remark 6.1 When c is a multiple of 6 the intersection of the triangle (5.8) with P_c is given by weights of the form $\lambda^{(0)} - k\nu - l\eta$ with $k, l \in \mathbb{Z}$ such that $|l| \leq k \leq [\frac{c}{12}]$. For instance, for $c = 6$ the intersection consists only of $\lambda^{(0)}$ (so degenerations are not possible in this case) whereas for proper multiples of 6 a pair of weights λ and μ in the triangle corresponding to the same value for l and different values for k may lead to equal expressions on both sides of Eq. (5.6) for all $\omega \in \hat{P}$ small. Explicit computations for a few multiples of 6 suggest that for fixed l and any $\omega \in \hat{P}$ small, the expression for $\hat{m}_\omega(\rho_g + \lambda^{(0)} - k\nu - l\eta)$ is in fact independent of $k = |l|, \ldots, [\frac{c}{12}]$. Such degenerations only occur for weights near the affine wall of P_c. Indeed, since $\langle \lambda^{(0)} - s\nu - r\eta, \varphi^\vee \rangle \geq \langle \lambda^{(0)} - \frac{c}{12}\nu, \varphi^\vee \rangle = \frac{11}{12}c$ for $|r| \leq s \leq \frac{c}{12}$, the degenerations in question are restricted to weights outside $P_{\tilde{c}} \subseteq P_c$ with $\tilde{c} = \lceil \frac{11}{12}c \rceil$.

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