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Branching formula for Macdonald–Koornwinder polynomials ☆

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ABSTRACT

We present an explicit branching formula for the six-parameter Macdonald–Koornwinder polynomials with hyperoctahedral symmetry.

1. Introduction

Branching formulas constitute a powerful tool in algebraic combinatorics providing a recursive scheme to build symmetric polynomials via induction in the number of variables [11,10]. The combinatorial aspects of the hyperoctahedral-symmetric

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Macdonald–Koornwinder polynomials [12,9] were explored in seminal works of Okounkov and Rains [14,15]. In particular, the structure of a branching formula for the Macdonald–Koornwinder polynomials has been outlined at the end of [15, Sec. 5]. The aim of the present note is to make the branching polynomials under consideration explicit. Following the ideas underlying the proof of the branching formula for the Macdonald polynomials [11, Ch. VI.7], our main tools to achieve this goal consist of: Mimachi's Cauchy formula for the Macdonald–Koornwinder polynomials [13], (a special 'column–row' case of) the Cauchy formula for Okounkov's hyperoctahedral interpolation polynomials [14], and explicitly known Pieri coefficients for the Macdonald–Koornwinder polynomials [3,5].

The material is structured as follows. After recalling some necessary preliminaries regarding the Macdonald–Koornwinder polynomials and their Pieri formulas in Section 2, our branching formula is first stated in Section 3 and then proven in Section 4.

2. Preliminaries

2.1. *Macdonald–Koornwinder polynomials [9]*

For a partition

$$\lambda \in \Lambda_n := \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}, \tag{2.1}$$

the monic Macdonald–Koornwinder polynomial $P_\lambda(z_1, \ldots, z_n; q, t, \mathbf{t})$ is a Laurent polynomial in the complex variables $z_1, \ldots, z_n$ that depends rationally on the parameters q, t and $\mathbf{t} := (t_0, t_1, t_2, t_3)$. It is determined by a leading monomial of the form $z_1^{\lambda_1} z_2^{\lambda_2} \cdots z_n^{\lambda_n}$, while being symmetric with respect to the action of the hyperoctahedral group $W = S_n \ltimes \mathbb{Z}_2^n$ by permutations and inversions of the variables. For real parameter values in the domain $0 < q, |t|, |t_l| < 1$ ($l = 0, 1, 2, 3$), the Macdonald–Koornwinder polynomials form an orthogonal system on the n-dimensional torus $|z_j| = 1$, $j = 1, \ldots, n$. The orthogonality measure is given by Gustafson's q-Selberg type density [6]

$$\Delta = \prod_{1 \leq j \leq n} \frac{(z_j^2, z_j^{-2}; q)_\infty}{\prod_{0 \leq l \leq 3} (t_l z_j, t_l z_j^{-1}; q)_\infty} \prod_{1 \leq j < k \leq n} \frac{(z_j z_k, z_j z_k^{-1}, z_j^{-1} z_k, z_j^{-1} z_k^{-1}; q)_\infty}{(t z_j z_k, t z_j z_k^{-1}, t z_j^{-1} z_k, t z_j^{-1} z_k^{-1}; q)_\infty}$$

with respect to the Haar measure on this torus. Here and below we employ standard conventions for the q-Pochhammer symbols: $(a; q)_k := (1 - a)(1 - aq) \cdots (1 - aq^{k-1})$ (with $(a; q)_0 := 1$) and $(a_1, \ldots, a_l; q)_k := (a_1; q)_k \cdots (a_l; q)_k$.

2.2. *Pieri coefficients [3,5,16]*

The W-invariant Laurent polynomials

$$E_r(z_1, \ldots, z_n; t, t_0) = \sum_{1 \leq j_1 < \cdots < j_r \leq n} \langle z_{j_1}; t^{j_1 - 1} t_0 \rangle \cdots \langle z_{j_r}; t^{j_r - r} t_0 \rangle \tag{2.2}$$

($r = 1, \ldots, n$), with $\langle z; x\rangle := z + z^{-1} - x - x^{-1}$, are special instances of Okounkov's hyperoctahedral interpolation polynomials [14]—with shifted variables as considered by Rains [15]—that correspond to the partitions with only a single column [8]. They describe the eigenvalues of commuting difference operators diagonalized by the Macdonald–Koornwinder polynomials [2] and are also instrumental in Ito's Aomoto-style proof [7] of the hyperoctahedral ${}_6\Psi_6$ sum evaluated in Ref. [4].

Let $C^{\mu,n}_{\lambda,r}(q,t,\mathbf{t})$ denote the coefficients in the Macdonald–Koornwinder Pieri-expansions associated with these one-column interpolation polynomials:

$$E_r(z_1,\ldots,z_n;t,t_0)P_\lambda(z_1,\ldots,z_n;q,t,\mathbf{t}) = \sum_{\substack{\mu\in\Lambda_n\\ \mu\sim_r\lambda}} C^{\mu,n}_{\lambda,r}(q,t,\mathbf{t})P_\mu(z_1,\ldots,z_n;q,t,\mathbf{t}), \tag{2.3}$$

$r = 1, \ldots, n$. To filter only the nonvanishing coefficients, we have employed the following proximity relation within Λ_n restricting the sum on the RHS: $\mu \sim_r \lambda$ iff there exists a partition $\nu \in \Lambda_n$ with $\nu \subset \lambda$ and $\nu \subset \mu$ such that the skew diagrams λ/ν and μ/ν are vertical strips with $|\lambda/\nu| + |\mu/\nu| \leq r$. Here $|\cdot|$ refers to the number of boxes of the diagram, $\nu \subset \lambda$ means that $\nu \in \Lambda_n$ is contained in λ: $\nu_j \leq \lambda_j$ ($j = 1, \ldots, n$), and (recall) the skew diagram λ/ν is a vertical strip iff $\nu_j \leq \lambda_j \leq \nu_j + 1$ ($j = 1, \ldots, n$).

Upon writing $J = \{1 \leq j \leq n \mid \lambda_j \neq \mu_j\}$, $J^c = \{1, \ldots, n\} \setminus J$, and $\epsilon_j = \mu_j - \lambda_j$ for $j \in J$ (so, if $\mu \sim_r \lambda$ the cardinality $|J|$ of J is at most r and $\epsilon_j \in \{1, -1\}$), one can express the Pieri coefficients in question explicitly as follows:

$$C^{\mu,n}_{\lambda,r}(q,t,\mathbf{t}) = \frac{P_\lambda(\tau_1,\ldots,\tau_n;q,t,\mathbf{t})}{P_\mu(\tau_1,\ldots,\tau_n;q,t,\mathbf{t})} V^n_{\epsilon J}(\lambda;q,t,\mathbf{t}) U^n_{J^c,r-|J|}(\lambda;q,t,\mathbf{t}), \tag{2.4}$$

where

$$P_\lambda(\tau_1,\ldots,\tau_n;q,t,\mathbf{t}) =$$
$$\prod_{1\leq j\leq n} \frac{\prod_{0\leq l\leq 3}(\hat{t}_l\hat{\tau}_j;q)_{\lambda_j}}{\tau_j^{\lambda_j}(\hat{\tau}_j^2;q)_{2\lambda_j}} \prod_{1\leq j<k\leq n} \frac{(t\hat{\tau}_j\hat{\tau}_k;q)_{\lambda_j+\lambda_k}(t\hat{\tau}_j\hat{\tau}_k^{-1};q)_{\lambda_j-\lambda_k}}{(\hat{\tau}_j\hat{\tau}_k;q)_{\lambda_j+\lambda_k}(\hat{\tau}_j\hat{\tau}_k^{-1};q)_{\lambda_j-\lambda_k}},$$

$$V^n_{\epsilon J}(\lambda;q,t,\mathbf{t}) = \prod_{j\in J} \frac{\prod_{0\leq l\leq 3}(1-\hat{t}_l\hat{\tau}_j^{\epsilon_j}q^{\epsilon_j\lambda_j})}{t_0(1-\hat{\tau}_j^{2\epsilon_j}q^{2\epsilon_j\lambda_j})(1-\hat{\tau}_j^{2\epsilon_j}q^{2\epsilon_j\lambda_j+1})}$$
$$\times \prod_{\substack{j,j'\in J\\ j<j'}} \frac{(1-t\hat{\tau}_j^{\epsilon_j}\hat{\tau}_{j'}^{\epsilon_{j'}}q^{\epsilon_j\lambda_j+\epsilon_{j'}\lambda_{j'}})(1-t\hat{\tau}_j^{\epsilon_j}\hat{\tau}_{j'}^{\epsilon_{j'}}q^{\epsilon_j\lambda_j+\epsilon_{j'}\lambda_{j'}+1})}{t(1-\hat{\tau}_j^{\epsilon_j}\hat{\tau}_{j'}^{\epsilon_{j'}}q^{\epsilon_j\lambda_j+\epsilon_{j'}\lambda_{j'}})(1-\hat{\tau}_j^{\epsilon_j}\hat{\tau}_{j'}^{\epsilon_{j'}}q^{\epsilon_j\lambda_j+\epsilon_{j'}\lambda_{j'}+1})}$$
$$\times \prod_{j\in J,k\in J^c} \frac{(1-t\hat{\tau}_j^{\epsilon_j}\hat{\tau}_k q^{\epsilon_j\lambda_j+\lambda_k})(1-t\hat{\tau}_j^{\epsilon_j}\hat{\tau}_k^{-1}q^{\epsilon_j\lambda_j-\lambda_k})}{t(1-\hat{\tau}_j^{\epsilon_j}\hat{\tau}_k q^{\epsilon_j\lambda_j+\lambda_k})(1-\hat{\tau}_j^{\epsilon_j}\hat{\tau}_k^{-1}q^{\epsilon_j\lambda_j-\lambda_k})},$$

and

$$U^n_{K,p}(\lambda; q, t, \mathbf{t}) = (-1)^p \sum_{\substack{I\subset K,\, |I|=p \\ \epsilon_i\in\{1,-1\}, i\in I}} \Bigg(\prod_{i\in I} \frac{\prod_{0\le l\le 3}(1-\hat{t}_l \hat{\tau}_i^{\epsilon_i} q^{\epsilon_i\lambda_i})}{t_0(1-\hat{\tau}_i^{2\epsilon_i} q^{2\epsilon_i\lambda_i})(1-\hat{\tau}_i^{2\epsilon_i} q^{2\epsilon_i\lambda_i+1})}$$

$$\times \prod_{\substack{i,i'\in I \\ i<i'}} \frac{(1-t\hat{\tau}_i^{\epsilon_i}\hat{\tau}_{i'}^{\epsilon_{i'}} q^{\epsilon_i\lambda_i+\epsilon_{i'}\lambda_{i'}})(1-t^{-1}\hat{\tau}_i^{\epsilon_i}\hat{\tau}_{i'}^{\epsilon_{i'}} q^{\epsilon_i\lambda_i+\epsilon_{i'}\lambda_{i'}+1})}{(1-\hat{\tau}_i^{\epsilon_i}\hat{\tau}_{i'}^{\epsilon_{i'}} q^{\epsilon_i\lambda_i+\epsilon_{i'}\lambda_{i'}})(1-\hat{\tau}_i^{\epsilon_i}\hat{\tau}_{i'}^{\epsilon_{i'}} q^{\epsilon_i\lambda_i+\epsilon_{i'}\lambda_{i'}+1})}$$

$$\times \prod_{i\in I, k\in K\setminus I} \frac{(1-t\hat{\tau}_i^{\epsilon_i}\hat{\tau}_k q^{\epsilon_i\lambda_i+\lambda_k})(1-t\hat{\tau}_i^{\epsilon_i}\hat{\tau}_k^{-1} q^{\epsilon_i\lambda_i-\lambda_k})}{t(1-\hat{\tau}_i^{\epsilon_i}\hat{\tau}_k q^{\epsilon_i\lambda_i+\lambda_k})(1-\hat{\tau}_i^{\epsilon_i}\hat{\tau}_k^{-1} q^{\epsilon_i\lambda_i-\lambda_k})} \Bigg)$$

for $p = 0, \ldots, |K|$ (with the convention that $V^n_{\epsilon J}(\lambda; q, t, \mathbf{t}) = 1$ if J is empty and $U^n_{K,p}(\lambda; q, t, \mathbf{t}) = 1$ if $p = 0$). Here

$$\tau_j = t^{n-j} t_0, \quad \hat{\tau}_j = t^{n-j}\hat{t}_0 \quad (j = 1, \ldots, n),$$

and

$$\hat{t}_0^2 = q^{-1} t_0 t_1 t_2 t_3, \qquad \hat{t}_0 \hat{t}_l = t_0 t_l \quad (l = 1, 2, 3).$$

Remark 1. Below we will employ a trivially extended notion of $E_r(z_1, \ldots, z_n; t, t_0)$ and $C^{\mu,n}_{\lambda,r}(q, t, \mathbf{t})$ that allows r and n to become equal to zero. By convention $E_0(z_1, \ldots, z_n; t, t_0) := 1$, whence for the corresponding coefficients $C^{\mu,n}_{\lambda,0}(q, t, \mathbf{t}) = 1$ if $\mu = \lambda$ and vanishes otherwise.

3. Branching formula

Let us recall that for $\mu \subset \lambda \in \Lambda_n$, the skew diagram λ/μ is a horizontal strip provided the parts of λ and μ interlace as follows:

$$\lambda_1 \ge \mu_1 \ge \lambda_2 \ge \mu_2 \ge \cdots \ge \lambda_n \ge \mu_n.$$

We will need the following relation in Λ_n expressing that the partition λ can be obtained from $\mu \subset \lambda$ by adding at most two horizontal strips: $\mu \preceq \lambda$ iff there exists a $\nu \in \Lambda_n$ with $\mu \subset \nu \subset \lambda$ such that the skew diagrams λ/ν and ν/μ are horizontal strips. From now on we will think of Λ_n as being embedded in Λ_{n+1} in the natural way (i.e. 'by adding a part of size zero'). The main result of this note is given by the following branching formula for the Macdonald–Koornwinder polynomials, the proof of which is delayed until Section 4 below.

We denote by $m^n \in \Lambda_n$ the rectangular partition such that $(m^n)_j = m$ $(j = 1, \ldots, n)$ and—more generally—by $m^n - \mu$ with $\mu \subset m^n$ the partition such that $(m^n - \mu)_j = m - \mu_{n+1-j}$ $(j = 1, \ldots, n)$. Finally, we write $\lambda' \in \Lambda_m$ $(m \ge \lambda_1)$ for the conjugate partition of $\lambda \in \Lambda_n$, i.e. with λ'_i counting the number of parts of λ that are greater or equal than i $(i = 1, \ldots, m)$.

Theorem 1 *(Branching formula). For $\lambda \in \Lambda_{n+1}$, the Macdonald–Koornwinder polynomial in $(n+1)$ variables expands in terms of the n-variable polynomials as*

$$P_\lambda(z_1,\ldots,z_n,x;q,t,\mathbf{t}) = \sum_{\substack{\mu\in\Lambda_n\\ \mu\preceq\lambda}} P_\mu(z_1,\ldots,z_n;q,t,\mathbf{t})P_{\lambda/\mu}(x;q,t,\mathbf{t}), \tag{3.1a}$$

with one-variable branching polynomials of degree $d = |\{1 \leq j \leq m \mid \lambda'_j = \mu'_j + 1\}|$ whose expansion

$$P_{\lambda/\mu}(x;q,t,\mathbf{t}) = \sum_{0\leq k\leq d} B^k_{\lambda/\mu}(q,t,\mathbf{t})\langle x;t_0\rangle_{q,k} \tag{3.1b}$$

in the basis of the interpolation polynomials in one variable

$$\langle x;t_0\rangle_{q,k} := \langle x;t_0\rangle\langle x;qt_0\rangle\cdots\langle x;q^{k-1}t_0\rangle \qquad (\text{with } \langle x;t_0\rangle_{q,0} := 1) \tag{3.1c}$$

has coefficients that are given explicitly by the Macdonald–Koornwinder Pieri coefficients in $m = \lambda_1$ variables:

$$B^k_{\lambda/\mu}(q,t,\mathbf{t}) = (-1)^{k+|\lambda|-|\mu|} C^{(n+1)^m-\lambda',m}_{n^m-\mu',m-k}(t,q,\mathbf{t}) \quad (k=0,\ldots,d). \tag{3.1d}$$

It is well-known [9] that in the case of only a single variable the Macdonald–Koornwinder polynomials reduce to the five-parameter monic Askey–Wilson polynomials $P_m(z;q,\mathbf{t})$, $m = 0,1,2,\ldots$ [1]. On the other hand, if we formally set $n = 0$ and $\mu = 0$ then (the proof of) Theorem 1 remains valid. This gives rise to the following expansion of the Askey–Wilson polynomials in terms of the one-variable interpolation polynomials $\langle x;t_0\rangle_{q,k}$.

Corollary 2 *(Askey–Wilson polynomials). The monic Askey–Wilson polynomial of degree m is given by*

$$P_m(z;q,\mathbf{t}) = \sum_{0\leq k\leq m} B^k_{m/0}(q,\mathbf{t})\langle z;t_0\rangle_{q,k} \tag{3.2a}$$

with

$$B^k_{m/0}(q,\mathbf{t}) = (-1)^{m+k} C^{0^m,m}_{0^m,m-k}(t,q,\mathbf{t}). \tag{3.2b}$$

This formula for the Askey–Wilson polynomials amounts to the $n = 1$ case of Okounkov's binomial formula for the Macdonald–Koornwinder polynomials [14, Thm. 7.1] with the binomial coefficients written explicitly in terms of the m-variable Macdonald–Koornwinder Pieri coefficients. Notice that since the Askey–Wilson polynomials on the LHS are independent of t, it follows that the t-dependence of the corresponding branching coefficients drops out as well. In this special situation, alternative expressions for the relevant binomial coefficients are available in a much more compact form [15, Prp. 4.1] and the binomial formula is in fact seen to reduce to the usual ${}_4\phi_3$ representation of the Askey–Wilson polynomial [8, p. 25].

By iterating the branching formula in Theorem 1, one finds the general Macdonald–Koornwinder branching polynomial as a sum of factorized contributions over ascending chains of partitions.

Corollary 3 *(Branching polynomials). For $\lambda \in \Lambda_{n+l}$, one has that*

$$P_\lambda(z_1,\ldots,z_n,x_1,\ldots,x_l;q,t,\mathbf{t}) = \sum_{\substack{\mu^{(i)}\in\Lambda_{n+i}, i=0,\ldots,l \\ \mu=\mu^{(0)}\preceq\mu^{(1)}\preceq\cdots\preceq\mu^{(l)}=\lambda}} P_\mu(z_1,\ldots,z_n;q,t,\mathbf{t}) \prod_{1\leq i\leq l} P_{\mu^{(i)}/\mu^{(i-1)}}(x_i;q,t,\mathbf{t}). \tag{3.3}$$

Setting $n = 1$ in the latter formula, leads us to an explicit formula for the Macdonald–Koornwinder polynomials generalizing the formula for the Askey–Wilson polynomials in Corollary 2.

Corollary 4 *(Macdonald–Koornwinder polynomials). For $\lambda \in \Lambda_n$, the monic Macdonald–Koornwinder polynomial is given by*

$$P_\lambda(z_1,\ldots,z_n;q,t,\mathbf{t}) = \sum_{\substack{\mu^{(i)}\in\Lambda_i, i=1,\ldots,n \\ \mu^{(1)}\preceq\mu^{(2)}\preceq\cdots\preceq\mu^{(n)}=\lambda}} \prod_{1\leq i\leq n} P_{\mu^{(i)}/\mu^{(i-1)}}(z_i;q,t,\mathbf{t}), \tag{3.4}$$

where $P_{\mu^{(1)}/\mu^{(0)}}(z;q,t,\mathbf{t}) := P_{\mu^{(1)}}(z;q,\mathbf{t})$ (3.2a), (3.2b) by convention.

This is the analog of a classic formula for the usual permutation-symmetric Macdonald polynomials in terms of semistandard tableaux, cf. e.g. [11, Ch. VI.7] and [10, Sec. 1].

Remark 2. It follows from the proof in Section 4 below that the branching formula in Theorem 1 holds in fact *for any $m \geq \lambda_1$*, i.e. the expressions for the branching coefficients $B^k_{\lambda/\mu}$ in Eq. (3.1d) *do not depend on $m \geq \lambda_1$*.

Remark 3. For $x = t_0 q^h$, $h = 0, 1, 2, \ldots$, the degree of the branching polynomials $P_{\lambda/\mu}(x;q,t,\mathbf{t})$ (3.1b) remains bounded by h as the sum in question truncates beyond $k = h$. In particular, for $x = t_0$ only the first (constant) term survives and the complexity of the branching coefficients reduces considerably (cf. [15, p. 100]):

$$P_{\lambda/\mu}(t_0;q,t,\mathbf{t}) = B^0_{\lambda/\mu}(q,t,\mathbf{t}) = (-1)^{|\lambda|-|\mu|} C^{(n+1)^m-\lambda',m}_{n^m-\mu',m}(t,q,\mathbf{t}). \tag{3.5}$$

4. Proof of the branching formula

Let

$$\prod(x_1,\ldots,x_m;z_1,\ldots,z_n) := \prod_{\substack{1\leq i\leq m \\ 1\leq j\leq n}} \langle x_i;z_j\rangle. \tag{4.1}$$

Mimachi's Cauchy formula [13, Thm. 2.1] states that this kernel expands in terms of Macdonald–Koornwinder polynomials as:

$$\prod(x_1,\ldots,x_m;z_1,\ldots,z_n) = \sum_{\lambda\subset n^m}(-1)^{mn-|\lambda|}P_\lambda(x_1,\ldots,x_m;q,t,\mathbf{t})P_{m^n-\lambda'}(z_1,\ldots,z_n;t,q,\mathbf{t}). \tag{4.2}$$

A similar expansion of the kernel at issue in terms of Okounkov's hyperoctahedral interpolation polynomials is given by the Cauchy formula in [14, Thm. 6.2] (with shifted variables as in [15, Thm. 3.16]). For $n=1$, the latter Cauchy formula becomes of the form [8, Lem. 5.1]:

$$\prod(x_1,\ldots,x_m;z) = \sum_{0\leq r\leq m}(-1)^{m-r}E_r(x_1,\ldots,x_m;t,t_0)\,\langle z;t_0\rangle_{t,m-r}. \tag{4.3}$$

By expanding the first two factors of the trivial identity

$$\prod(x_1,\ldots,x_m;z_1,\ldots,z_n,z) = \prod(x_1,\ldots,x_m;z_1,\ldots,z_n)\prod(x_1,\ldots,x_m;z)$$

by means of Mimachi's Cauchy formula (4.2) and the last factor by means of the 'column–row' case of Okounkov's Cauchy formula in Eq. (4.3), one arrives at the equality

$$\sum_{\lambda\subset(n+1)^m}(-1)^{m(n+1)-|\lambda|}P_\lambda(x_1,\ldots,x_m;q,t,\mathbf{t})P_{m^{n+1}-\lambda'}(z_1,\ldots,z_n,z;t,q,\mathbf{t})$$
$$= \sum_{\substack{\mu\subset n^m\\ 0\leq r\leq m}}(-1)^{m(n+1)-|\mu|-r}\Big(P_{m^n-\mu'}(z_1,\ldots,z_n;t,q,\mathbf{t})\langle z;t_0\rangle_{t,m-r}$$
$$\times E_r(x_1,\ldots,x_m;t,t_0)P_\mu(x_1,\ldots,x_m;q,t,\mathbf{t})\Big).$$

Upon rewriting the RHS with the aid of the Pieri formula (2.3)

$$= \sum_{\substack{\mu\subset n^m\\ 0\leq r\leq m}}(-1)^{m(n+1)-|\mu|-r}\Big(P_{m^n-\mu'}(z_1,\ldots,z_n;t,q,\mathbf{t})\langle z;t_0\rangle_{t,m-r}$$
$$\times \sum_{\substack{\lambda\subset(n+1)^m\\ \lambda\sim_r\mu}}C^{\lambda,m}_{\mu,r}(q,t,\mathbf{t})P_\lambda(x_1,\ldots,x_m;q,t,\mathbf{t})\Big),$$

and reordering the sums

$$= \sum_{\lambda\subset(n+1)^m}(-1)^{m(n+1)-|\lambda|}\Big(P_\lambda(x_1,\ldots,x_m;q,t,\mathbf{t})$$
$$\times \sum_{\substack{\mu\subset n^m,\,0\leq r\leq m\\ \mu\sim_r\lambda}}(-1)^{r+|\lambda|-|\mu|}C^{\lambda,m}_{\mu,r}(q,t,\mathbf{t})P_{m^n-\mu'}(z_1,\ldots,z_n;t,q,\mathbf{t})\langle z;t_0\rangle_{t,m-r}\Big),$$

one deduces by comparing with the LHS that for any $\lambda \subset (n+1)^m$:

$$P_{m^{n+1}-\lambda'}(z_1,\ldots,z_n,z;t,q,\mathbf{t}) = \sum_{\substack{\mu\subset n^m,\, 0\leq r\leq m \\ \mu\sim_r\lambda}} (-1)^{r+|\lambda|-|\mu|} C^{\lambda,m}_{\mu,r}(q,t,\mathbf{t}) P_{m^n-\mu'}(z_1,\ldots,z_n;t,q,\mathbf{t})\langle z;t_0\rangle_{t,m-r},$$

i.e. for any $\lambda \subset m^{n+1}$:

$$P_\lambda(z_1,\ldots,z_n,z;q,t,\mathbf{t}) = \sum_{\substack{\mu\subset m^n,\, 0\leq r\leq m \\ n^m-\mu'\sim_r(n+1)^m-\lambda'}} (-1)^{m-r+|\lambda|-|\mu|} C^{(n+1)^m-\lambda',m}_{n^m-\mu',r}(t,q,\mathbf{t}) P_\mu(z_1,\ldots,z_n;q,t,\mathbf{t})\langle z;t_0\rangle_{q,m-r}. \tag{4.4}$$

To finish the proof we invoke the following lemma.

Lemma 5. *Let $\lambda \subset m^{n+1}$ and $\mu \subset m^n$. Then $n^m - \mu' \sim_r (n+1)^m - \lambda'$ iff $\mu \preceq \lambda$ and $m - d \leq r \leq m$ with $d = \{1 \leq j \leq m \mid \lambda'_j = \mu'_j + 1\}$.*

Proof. The statement of the lemma is immediate upon combining the following two properties: (i) $n^m - \mu' \sim_m (n+1)^m - \lambda'$ iff $\mu \preceq \lambda$ and (ii) $n^m - \mu' \sim_r (n+1)^m - \lambda'$ iff $n^m - \mu' \sim_m (n+1)^m - \lambda'$ and $r \geq m - d$.

Firstly, $n^m - \mu' \sim_m (n+1)^m - \lambda' \Leftrightarrow \exists \nu \subset n^m$ such that $(n^m - \mu')/\nu$ and $((n+1)^m - \lambda')/\nu$ are vertical strips $\Leftrightarrow \exists \kappa \subset m^n$ such that $(m^n - \mu)/\kappa$ and $(m^{n+1} - \lambda)/\kappa$ are horizontal strips $\Leftrightarrow \exists \kappa \subset m^n$ such that $(m^n - \kappa)/\mu$ and $\lambda/(m^n - \kappa)$ are horizontal strips $\Leftrightarrow \mu \preceq \lambda$, which proves part (i).

Secondly—assuming that $n^m - \mu' \sim_m (n+1)^m - \lambda'$ and picking $\nu \subset n^m$ such that $(n^m - \mu')/\nu$ and $((n+1)^m - \lambda')/\nu$ are vertical strips with $|(n^m - \mu')/\nu| + |((n+1)^m - \lambda')/\nu|$ minimal—one has that $|(n^m - \mu')/\nu| + |((n+1)^m - \lambda')/\nu| \leq r \Leftrightarrow |\{1 \leq j \leq m \mid \lambda'_j \neq \mu'_j + 1\}| \leq r \Leftrightarrow m - d \leq r$, which completes the proof of part (ii). □

The upshot is that Eq. (4.4) can be rewritten as

$$P_\lambda(z_1,\ldots,z_n,z;q,t,\mathbf{t}) = \sum_{\substack{\mu\subset m^n,\, \mu\preceq\lambda \\ m-d\leq r\leq m}} (-1)^{m-r+|\lambda|-|\mu|} C^{(n+1)^m-\lambda',m}_{n^m-\mu',r}(t,q,\mathbf{t}) P_\mu(z_1,\ldots,z_n;q,t,\mathbf{t})\langle z;t_0\rangle_{q,m-r}$$

and Theorem 1 follows (where our choice of picking m equal to λ_1 corresponds to the minimal value of m such that $\lambda \subset m^{n+1}$, cf. Remark 2 at the end of the previous section).

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