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Spectrum and Eigenfunctions of the Lattice Hyperbolic Ruijsenaars–Schneider System with Exponential Morse Term

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Abstract. We place the hyperbolic quantum Ruijsenaars–Schneider system with an exponential Morse term on a lattice and diagonalize the resulting n-particle model by means of multivariate continuous dual q-Hahn polynomials that arise as a parameter reduction of the Macdonald–Koornwinder polynomials. This allows to compute the n-particle scattering operator, to identify the bispectral dual system, and to confirm the quantum integrability in a Hilbert space setup.

1. Introduction

It is well known that the hyperbolic Calogero–Moser n-particle system on the line can be placed in an exponential Morse potential without spoiling the integrability [1,15]. An extension of Manin's Painlevé–Calogero correspondence links the particle model in question to a multicomponent Painlevé III equation [26]. Just as for the conventional Calogero–Moser system without Morse potential, the integrability is preserved upon quantization and the corresponding spectral problem gives rise to a rich theory of remarkable novel hypergeometric functions in several variables [9–11,19].

An integrable Ruijsenaars–Schneider type (q-)deformation [20,24] of the hyperbolic Calogero–Moser system with Morse potential was introduced in [25] and in a more general form in [3, Sec. II.B]. Recently, it was pointed out that particle systems of this kind can be recovered from the Heisenberg double of $SU(n,n)$ via Hamiltonian reduction [18]. In the present work, we address the eigenvalue problem for a quantization of the latter hyperbolic Ruijsenaars–Schneider system with Morse term. Specifically, it is shown that the eigenfunctions are given by multivariate continuous dual q-Hahn polynomials that arise as a parameter reduction of the Macdonald–Koornwinder polynomials [14,17].

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As immediate by-products, one reads off the n-particle scattering operator and the commuting quantum integrals of a bispectral dual system [7,8].

The material is organized is as follows. In Sect. 2, we place the hyperbolic Ruijsenaars–Schneider system with Morse term from [3] on a lattice. The diagonalization of the resulting quantum model in terms of multivariate continuous dual q-Hahn polynomials is carried out in Sect. 3. In Sects. 4 and 5, the n-particle scattering operator and the bispectral dual integrable system are exhibited. Finally, the quantum integrability of both the hyperbolic Ruijsenaars–Schneider system with Morse term on the lattice and its bispectral dual system are addressed in Sect. 6.

2. Hyperbolic Ruijsenaars–Schneider System with Morse Term

The hyperbolic quantum Ruijsenaars–Schneider system on the lattice was briefly introduced in [21, Sec. 3C2] and studied in detail from the point of view of its scattering behavior in [23] (see also [5, Sec. 6] for a further generalization in terms of root systems). In this section, we formulate a corresponding lattice version of the hyperbolic quantum Ruijsenaars–Schneider system with Morse term introduced in [3, Sec. II.B].

2.1. Hamiltonian

The Hamiltonian of our n-particle model is given by the formal difference operator [3, Eqs. (2.25), (2.26)]:

$$H := \sum_{j=1}^{n}\Bigg(w_+(x_j)\Bigg(\prod_{\substack{1\le k\le n\\ k\neq j}} \frac{t^{-1}-q^{x_j-x_k}}{1-q^{x_j-x_k}}\Bigg)(T_j-1) + \ w_-(x_j)\Bigg(\prod_{\substack{1\le k\le n\\ k\neq j}} \frac{t-q^{x_j-x_k}}{1-q^{x_j-x_k}}\Bigg)(T_j^{-1}-1)\Bigg), \tag{2.1}$$

where

$$w_+(x) := \sqrt{\frac{qt_0t_3}{t_1t_2}}(1-t_1q^x)(1-t_2q^x),$$
$$w_-(x) := \sqrt{\frac{t_1t_2}{qt_0t_3}}(1-t_0q^x)(1-t_3q^x),$$

and T_j $(j=1,\ldots,n)$ acts on functions $f:\mathbb{R}^n\to\mathbb{C}$ by a unit translation of the jth position variable

$$(T_jf)(x_1,\ldots,x_n)=f(x_1,\ldots,x_{j-1},x_j+1,x_{j+1},\ldots,x_n).$$

Here, q denotes a real-valued scale parameter, t plays the role of the coupling parameter for the Ruijsenaars–Schneider inter-particle interaction, and the parameters t_r $(r=0,\ldots,3)$ are coupling parameters governing the exponential Morse interaction. Upon setting $t_0=\epsilon t^{n-1}q^{-1}$ and $t_r=\epsilon$ for $r=1,2,3$, one has that $w_\pm(x_j)\to t^{\pm(n-1)/2}$ when $\epsilon\to 0$. We thus recover in this limit the Hamiltonian of the hyperbolic quantum Ruijsenaars–Schneider system given in terms of Ruijsenaars–Macdonald difference operators [16,20]. By a translation of the center-of-mass of the form $q^{x_j} \to cq^{x_j}$ $(j = 1, \dots, n)$ for some suitable constant c, it is possible to normalize one of the t_r-parameters to unit value; from now on, it will, therefore, always be assumed that $t_3 \equiv 1$ unless explicitly stated otherwise.

2.2. Restriction to Lattice Functions

Let $\rho + \Lambda := \{\rho + \lambda \mid \lambda \in \Lambda\}$, where Λ denotes the cone of integer partitions $\lambda = (\lambda_1, \dots, \lambda_n)$ with weakly decreasingly ordered parts $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, and $\rho = (\rho_1, \dots, \rho_n)$ with

$$\rho_j = (n - j) \log_q(t) \quad (j = 1, \dots, n). \tag{2.2}$$

The action of H (2.1) (with $t_3 = 1$) preserves the space of lattice functions $f : \rho + \Lambda \to \mathbb{C}$:

$$\begin{aligned}(Hf)(\rho + \lambda) = &\sum_{\substack{1\leq j\leq n \\ \lambda+e_j\in\Lambda}} v_j^+(\lambda)\big(f(\rho + \lambda + e_j) - f(\rho + \lambda)\big) \\ &+ \sum_{\substack{1\leq j\leq n \\ \lambda-e_j\in\Lambda}} v_j^-(\lambda)\big(f(\rho + \lambda - e_j) - f(\rho + \lambda)\big),\end{aligned} \tag{2.3}$$

where $e_1, \dots, e_n$ denotes the standard basis of $\mathbb{R}^n$ and

$$v_j^+(\lambda) = \sqrt{\frac{qt_0}{t_1t_2}}(1 - t_1t^{n-j}q^{\lambda_j})(1 - t_2t^{n-j}q^{\lambda_j}) \prod_{\substack{1\leq k\leq n \\ k\neq j}} \frac{t^{-1} - t^{k-j}q^{\lambda_j-\lambda_k}}{1 - t^{k-j}q^{\lambda_j-\lambda_k}},$$

$$v_j^-(\lambda) = \sqrt{\frac{t_1t_2}{qt_0}}(1 - t_0t^{n-j}q^{\lambda_j})(1 - t^{n-j}q^{\lambda_j}) \prod_{\substack{1\leq k\leq n \\ k\neq j}} \frac{t - t^{k-j}q^{\lambda_j-\lambda_k}}{1 - t^{k-j}q^{\lambda_j-\lambda_k}}.$$

Indeed, given $\lambda \in \Lambda$, one has that $v_j^+(\lambda) = 0$ if $\lambda + e_j \notin \Lambda$ due to a zero stemming from the factor $t^{-1} - t^{-1}q^{\lambda_{j-1}-\lambda_j}$ when $\lambda_{j-1} = \lambda_j$ and one has that $v_j^-(\lambda) = 0$ if $\lambda - e_j \notin \Lambda$ due to a zero stemming from either the factor $t - tq^{\lambda_j-\lambda_{j+1}}$ when $\lambda_j = \lambda_{j+1}$ or from the factor $(1 - q^{\lambda_n})$ when $\lambda_n = 0$.

3. Spectrum and Eigenfunctions

Ruijsenaars' starting point in [23] is the fact that the hyperbolic quantum Ruijsenaars–Schneider system on the lattice is diagonalized by the celebrated Macdonald polynomials [16, Ch.VI]. In this section, we show that in the presence of the Morse interaction the role of the Macdonald eigenpolynomials is taken over by multivariate continuous dual q-Hahn eigenpolynomials that arise as a parameter reduction of the Macdonald–Koornwinder polynomials [14,17].

3.1. Multivariate Continuous Dual q-Hahn Polynomials

Continuous dual q-Hahn polynomials are a special limiting case of the Askey–Wilson polynomials in which one of the four Askey–Wilson parameters is set to vanish [12, Ch. 14.3]. The corresponding reduction of the Macdonald–Koornwinder multivariate Askey–Wilson polynomials [14,17] is governed by a weight function of the form

$$\hat{\Delta}(\xi) := \frac{1}{(2\pi)^n} \prod_{1\leq j\leq n} \left| \frac{(e^{2i\xi_j})_\infty}{\prod_{0\leq r\leq 2}(\hat{t}_r e^{i\xi_j})_\infty} \right|^2 \prod_{1\leq j<k\leq n} \left| \frac{(e^{i(\xi_j+\xi_k)}, e^{i(\xi_j-\xi_k)})_\infty}{(te^{i(\xi_j+\xi_k)}, te^{i(\xi_j-\xi_k)})_\infty} \right|^2 \tag{3.1}$$

supported on the alcove

$$\mathbb{A} := \{(\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n \mid \pi > \xi_1 > \xi_2 > \cdots > \xi_n > 0\}, \tag{3.2}$$

where $(x)_m := \prod_{l=0}^{m-1}(1 - xq^l)$ and $(x_1, \ldots, x_l)_m := (x_1)_m \cdots (x_l)_m$ refer to the q-Pochhammer symbols, and it is assumed that

$$q, t \in (0, 1) \quad \text{and} \quad \hat{t}_r \in (-1, 1)\setminus\{0\} \quad (r = 0, 1, 2). \tag{3.3}$$

More specifically, the multivariate continuous dual q-Hahn polynomials $P_\lambda(\xi)$, $\lambda \in \Lambda$ are defined as the trigonometric polynomials of the form

$$P_\lambda(\xi) = \sum_{\substack{\mu\in\Lambda \\ \mu\leq\lambda}} c_{\lambda,\mu} m_\mu(\xi) \quad (c_{\lambda,\mu} \in \mathbb{C}) \tag{3.4a}$$

such that

$$c_{\lambda,\lambda} = \prod_{1\leq j\leq n} \frac{\hat{t}_0^{\lambda_j} t^{(n-j)\lambda_j}}{(\hat{t}_0\hat{t}_1 t^{n-j}, \hat{t}_0\hat{t}_2 t^{n-j})_{\lambda_j}} \prod_{1\leq j<k\leq n} \frac{(t^{k-j})_{\lambda_j-\lambda_k}}{(t^{1+k-j})_{\lambda_j-\lambda_k}} \tag{3.4b}$$

and

$$\int_{\mathbb{A}} P_\lambda(\xi)\overline{P_\mu(\xi)}\hat{\Delta}(\xi)\mathrm{d}\xi = 0 \quad \text{if } \mu < \lambda. \tag{3.4c}$$

Here, we have employed the dominance partial order

$$\forall \mu, \lambda \in \Lambda: \quad \mu \leq \lambda \text{ iff } \sum_{1\leq j\leq k} \mu_j \leq \sum_{1\leq j\leq k} \lambda_j \quad \text{for} \quad k = 1, \ldots, n, \tag{3.5}$$

and the symmetric monomials

$$m_\lambda(\xi) := \sum_{\nu\in W\lambda} e^{i(\nu_1\xi_1+\cdots+\nu_n\xi_n)}, \quad \lambda \in \Lambda, \tag{3.6}$$

associated with the hyperoctahedral group $W = S_n \ltimes \{1, -1\}^n$ of signed permutations.

The present choice of the leading coefficient $c_{\lambda,\lambda}$ in Eq. (3.4b) normalizes the polynomials in question such that $P_\lambda(i\hat{\rho}) = 1$, where $\hat{\rho} = (\hat{\rho}_1, \ldots, \hat{\rho}_n)$ is given by $\hat{\rho}_j = (n - j)\log(t) + \log(\hat{t}_0)$, $j = 1, \ldots, n$ (cf. [4, Sec. 6], [17, Ch. 5.3]). With this normalization, the orthogonality relations obtained as the degeneration of those for the Macdonald–Koornwinder polynomials [14, Sec. 5], [4, Sec. 7], [17, Ch. 5.3] read:

$$\int_{\mathbb{A}} P_\lambda(\xi)\overline{P_\mu(\xi)}\hat{\Delta}(\xi)\mathrm{d}\xi = \begin{cases}\Delta_\lambda^{-1} & \text{if } \lambda=\mu,\\ 0 & \text{otherwise,}\end{cases} \tag{3.7a}$$

where

$$\Delta_\lambda := \Delta_0 \prod_{1\le j\le n} \frac{(\hat{t}_0\hat{t}_1 t^{n-j}, \hat{t}_0\hat{t}_2 t^{n-j})_{\lambda_j}}{\hat{t}_0^{2\lambda_j} t^{2(n-j)\lambda_j}(qt^{n-j}, \hat{t}_1\hat{t}_2 t^{n-j})_{\lambda_j}} \times \prod_{1\le j<k\le n} \frac{1-t^{k-j}q^{\lambda_j-\lambda_k}}{1-t^{k-j}} \frac{(t^{1+k-j})_{\lambda_j-\lambda_k}}{(qt^{k-j-1})_{\lambda_j-\lambda_k}} \tag{3.7b}$$

and

$$\Delta_0 := \prod_{1\le j\le n}\left(\frac{(q,t^j)_\infty}{(t)_\infty}\prod_{0\le r<s\le 2}(\hat{t}_r\hat{t}_s t^{n-j})_\infty\right). \tag{3.7c}$$

3.2. Diagonalization

Let $\ell^2(\rho+\Lambda,\Delta)$ denote the Hilbert space of lattice functions $f:\rho+\Lambda\to\mathbb{C}$ determined by the inner product

$$\langle f,g\rangle_\Delta := \sum_{\lambda\in\Lambda} f(\rho+\lambda)\overline{g(\rho+\lambda)}\Delta_\lambda \quad (f,g\in\ell^2(\rho+\lambda,\Delta)), \tag{3.8}$$

with ρ and Δ_λ as in Eqs. (2.2) and (3.7a)–(3.7c), and let $L^2(\mathbb{A},\hat{\Delta}(\xi)\mathrm{d}\xi)$ be the Hilbert space of functions $\hat{f}:\mathbb{A}\to\mathbb{C}$ determined by the inner product

$$\langle\hat{f},\hat{g}\rangle_{\hat{\Delta}} := \int_{\mathbb{A}} \hat{f}(\xi)\overline{\hat{g}(\xi)}\hat{\Delta}(\xi)\mathrm{d}\xi \quad (\hat{f},\hat{g}\in L^2(\mathbb{A},\hat{\Delta}(\xi)\mathrm{d}\xi)), \tag{3.9}$$

with $\hat{\Delta}$ taken from Eq. (3.1). We denote by $\psi_\xi:\rho+\Lambda\to\mathbb{C}$ the lattice wave function given by

$$\psi_\xi(\rho+\lambda) := P_\lambda(\xi) \quad (\xi\in\mathbb{A},\ \lambda\in\Lambda). \tag{3.10}$$

Then, the orthogonality relations in Eqs. (3.7a)–(3.7c) imply that the associated Fourier transform $\boldsymbol{F}:\ell^2(\rho+\Lambda,\Delta)\to L^2(\mathbb{A},\hat{\Delta}\mathrm{d}\xi)$ of the form

$$(\boldsymbol{F}f)(\xi) := \langle f,\psi_\xi\rangle_\Delta = \sum_{\lambda\in\Lambda} f(\rho+\lambda)\overline{\psi_\xi(\rho+\lambda)}\Delta_\lambda \tag{3.11a}$$

$(f\in\ell^2(\rho+\Lambda,\Delta))$ constitutes a Hilbert space isomorphism with an inversion formula given by

$$(\boldsymbol{F}^{-1}\hat{f})(\rho+\lambda) = \langle\hat{f},\overline{\psi(\rho+\lambda)}\rangle_{\hat{\Delta}} = \int_{\mathbb{A}} \hat{f}(\xi)\psi_\xi(\rho+\lambda)\hat{\Delta}(\xi)\mathrm{d}\xi \tag{3.11b}$$

$(\hat{f}\in L^2(\mathbb{A},\hat{\Delta}\mathrm{d}\xi))$.

Theorem 1. *Let $\hat{E}$ denote the bounded real multiplication operator acting on $\hat{f} \in L^2(\mathbb{A}, \hat{\Delta} d\xi)$ by $(\hat{E}\hat{f})(\xi) := \hat{E}(\xi)\hat{f}(\xi)$ with*

$$\hat{E}(\xi) := \sum_{1 \le j \le n} \big(2\cos(\xi_j) - t^{n-j}\hat{t}_0 - t^{j-n}\hat{t}_0^{-1}\big). \tag{3.12a}$$

For

$$t_0 = q^{-1}\hat{t}_1\hat{t}_2, \quad t_1 = \hat{t}_0\hat{t}_2, \quad t_2 = \hat{t}_0\hat{t}_1 \tag{3.12b}$$

with q, t and $\hat{t}_r$ in the parameter domain (3.3) and $\sqrt{\frac{t_1 t_2}{q t_0}} := \hat{t}_0$, the hyperbolic lattice Ruijsenaars–Schneider Hamiltonian with Morse interaction H (2.3) constitutes a bounded self-adjoint operator in the Hilbert space $\ell^2(\rho + \Lambda, \Delta)$ diagonalized by the Fourier transform $\boldsymbol{F}$ (3.11a), (3.11b):

$$H = \boldsymbol{F}^{-1} \circ \hat{E} \circ \boldsymbol{F}. \tag{3.12c}$$

Proof. It suffices to verify that the Fourier kernel ψ_ξ (3.10) satisfies the eigenvalue equation $H\psi_\xi = \hat{E}(\xi)\psi_\xi$, or more explicitly that:

$$\sum_{\substack{1 \le j \le n \\ \lambda + e_j \in \Lambda}} v_j^+(\lambda)\big(\psi_\xi(\rho + \lambda + e_j) - \psi_\xi(\rho + \lambda)\big) + \sum_{\substack{1 \le j \le n \\ \lambda - e_j \in \Lambda}} v_j^-(\lambda)\big(\psi_\xi(\rho + \lambda - e_j) - \psi_\xi(\rho + \lambda)\big) = \hat{E}(\xi)\psi_\xi(\rho + \lambda).$$

This eigenvalue equation amounts to the continuous dual q-Hahn reduction of the Pieri recurrence formula for the Macdonald–Koornwinder polynomials corresponding to Eqs. (6.4), (6.5) and Section 6.1 of [4]. □

It is immediate from Theorem 1 that the hyperbolic lattice Ruijsenaars–Schneider Hamiltonian with Morse interaction H (2.3) has purely absolutely continuous spectrum in $\ell^2(\rho + \Lambda, \Delta)$, with the wave functions ψ_ξ, $\xi \in \mathbb{A}$ in Eq. (3.10) constituting an orthogonal basis of (generalized) eigenfunctions.

Remark 2. For $\hat{t}_2 \to 0$, the lattice Hamiltonian H (3.12c) becomes of the form

$$H = \sum_{j=1}^{n} \Bigg(\hat{t}_0^{-1}(1 - \hat{t}_0\hat{t}_1 q^{x_j}) \Bigg(\prod_{\substack{1 \le k \le n \\ k \ne j}} \frac{t^{-1} - q^{x_j - x_k}}{1 - q^{x_j - x_k}} \Bigg) T_j + \hat{t}_0(1 - q^{x_j}) \Bigg(\prod_{\substack{1 \le k \le n \\ k \ne j}} \frac{t - q^{x_j - x_k}}{1 - q^{x_j - x_k}} \Bigg) T_j^{-1} + (\hat{t}_0 + \hat{t}_1) q^{x_j} \Bigg) - \varepsilon_0, \tag{3.13a}$$

with $x = \rho + \lambda$ and

$$\varepsilon_0 := \sum_{j=1}^{n} (\hat{t}_0 t^{n-j} + \hat{t}_0^{-1} t^{j-n}). \tag{3.13b}$$

Indeed, this readily follows from Eqs. (2.1), (3.12b) with the aid of the elementary polynomial identity (cf. Example 2. (a) of [16, Ch. VI.3])

$$\sum_{j=1}^{n}(1+z_j)\prod_{\substack{1\le k\le n\\ k\neq j}}\frac{t-z_j/z_k}{1-z_j/z_k}=\sum_{j=1}^{n}(z_j+t^{n-j}).$$

4. Scattering

In this section, we rely on results from [5], permitting to describe briefly how the n-particle scattering operator for the hyperbolic quantum Ruijsenaars–Schneider system on the lattice computed by Ruijsenaars [23] gets modified due to the presence of the external Morse interactions. Specifically, the scattering process of the present model with Morse terms turns out to be governed by an n-particle scattering matrix $\hat{\mathcal{S}}(\xi)$ that factorizes in two-particle and one-particle matrices:

$$\hat{\mathcal{S}}(\xi):=\prod_{1\le j<k\le n}s(\xi_j-\xi_k)s(\xi_j+\xi_k)\prod_{1\le j\le n}s_0(\xi_j), \tag{4.1a}$$

with

$$s(x):=\frac{(qe^{ix},te^{-ix})_\infty}{(qe^{-ix},te^{ix})_\infty}\quad\text{and}\quad s_0(x):=\frac{(qe^{2ix})_\infty}{(qe^{-2ix})_\infty}\prod_{0\le r\le 2}\frac{(\hat{t}_re^{-ix})_\infty}{(\hat{t}_re^{ix})_\infty}, \tag{4.1b}$$

which compares to Ruijsenaars' scattering matrix $\prod_{1\le j<k\le n}s(\xi_j-\xi_k)$ for the corresponding model without Morse interactions [23].

To substantiate further some additional notation is needed. Let us denote by $\mathcal{H}_0$ the self-adjoint discrete Laplacian in $\ell^2(\Lambda)$ of the form

$$(\mathcal{H}_0f)(\lambda):=\sum_{\substack{1\le j\le n\\ \lambda+e_j\in\Lambda}}f(\lambda+e_j)+\sum_{\substack{1\le j\le n\\ \lambda-e_j\in\Lambda}}f(\lambda-e_j)\quad(f\in\ell^2(\Lambda)),$$

and let

$$\mathcal{H}:=\boldsymbol{\Delta}^{1/2}(H+\varepsilon_0)\boldsymbol{\Delta}^{-1/2}, \tag{4.2}$$

with H and ε_0 taken from (2.3) and (3.13b), respectively. Here, the operator $\boldsymbol{\Delta}^{1/2}:\ell^2(\rho+\Lambda,\Delta)\to\ell^2(\Lambda)$ refers to the Hilbert space isomorphism

$$(\boldsymbol{\Delta}^{1/2}f)(\lambda):=\Delta_\lambda^{1/2}f(\rho+\lambda)\quad(f\in\ell^2(\rho+\Lambda,\Delta)) \tag{4.3}$$

[with $\boldsymbol{\Delta}^{-1/2}:=(\boldsymbol{\Delta}^{1/2})^{-1}$]. Then, (by Theorem 1)

$$\mathcal{H}=\boldsymbol{\mathcal{F}}^{-1}(\hat{E}+\varepsilon_0)\boldsymbol{\mathcal{F}}\quad\text{with}\quad\boldsymbol{\mathcal{F}}:=\hat{\boldsymbol{\Delta}}^{1/2}\boldsymbol{F}\boldsymbol{\Delta}^{-1/2}, \tag{4.4}$$

where $\hat{\boldsymbol{\Delta}}^{1/2}:L^2(\mathbb{A},\hat{\Delta}\mathrm{d}\xi)\to L^2(\mathbb{A})$ denotes the Hilbert space isomorphism

$$(\hat{\boldsymbol{\Delta}}^{1/2}\hat{f})(\xi):=\hat{\Delta}^{1/2}(\xi)\hat{f}(\xi)\quad(\hat{f}\in L^2(\mathbb{A},\hat{\Delta}\mathrm{d}\xi)) \tag{4.5}$$

[and $\hat{E}$ (3.12a) is now regarded as a self-adjoint bounded multiplication operator in $L^2(\mathbb{A})$]. Furthermore, one has that

$$\mathcal{H}_0=\boldsymbol{\mathcal{F}}_0^{-1}(\hat{E}+\varepsilon_0)\boldsymbol{\mathcal{F}}_0,$$

where $\boldsymbol{\mathcal{F}}_0 : \ell^2(\Lambda) \to L^2(\mathbb{A})$ denotes the Fourier isomorphism recovered from $\mathcal{F}$ in the limit $q, t \to 0$, $\hat{t}_r \to 0$ $(r = 0, 1, 2)$. Specifically, this amounts to the Fourier transform

$$(\boldsymbol{\mathcal{F}}_0 f)(\xi) = \sum_{\lambda \in \Lambda} f(\lambda) \overline{\chi_\xi(\lambda)} \tag{4.6a}$$

$(f \in \ell^2(\Lambda))$ with the inversion formula

$$(\boldsymbol{\mathcal{F}}_0^{-1} \hat{f})(\lambda) = \int_{\mathbb{A}} \hat{f}(\xi) \chi_\xi(\lambda) \mathrm{d}\xi \tag{4.6b}$$

$(\hat{f} \in L^2(\mathbb{A}))$ associated with the anti-invariant Fourier kernel

$$\chi_\xi(\lambda) := \frac{1}{(2\pi)^{n/2}\, i^{n^2}} \sum_{w \in W} \mathrm{sign}(w) e^{i\langle w(\rho_0 + \lambda), \xi \rangle},$$

where $\mathrm{sign}(w) = \epsilon_1 \cdots \epsilon_n \mathrm{sign}(\sigma)$ for $w = (\sigma, \epsilon) \in W = S_n \ltimes \{1, -1\}^n$ and $\rho_0 = (n, n-1, \ldots, 2, 1)$.

Let $C_0(\mathbb{A}_{\mathrm{reg}})$ be the dense subspace of $L^2(\mathbb{A})$ consisting of smooth test functions with compact support in the open dense subset $\mathbb{A}_{\mathrm{reg}} \subset \mathbb{A}$ on which the components of the gradient

$$\nabla \hat{E}(\xi) = (-2\sin(\xi_1), \ldots, -2\sin(\xi_n)), \quad \xi \in \mathbb{A}$$

do not vanish and are all distinct in absolute value. We define the following unitary multiplication operator $\hat{\mathcal{S}} : L^2(\mathbb{A}, \mathrm{d}\xi) \to L^2(\mathbb{A}, \mathrm{d}\xi)$ via its restriction to $C_0(\mathbb{A}_{\mathrm{reg}})$:

$$(\hat{\mathcal{S}} \hat{f})(\xi) := \hat{\mathcal{S}}(w_\xi \xi) \hat{f}(\xi) \quad (\hat{f} \in C_0(\mathbb{A}_{\mathrm{reg}})), \tag{4.7}$$

where $w_\xi \in W$ for $\xi \in \mathbb{A}_{\mathrm{reg}}$ is the signed permutation such that the components of $w_\xi \nabla \hat{E}(\xi)$ are all positive and reordered from large to small.

Theorem 4.2 and Corollary 4.3 of Ref. [5] now provide explicit formulas for the wave operators and scattering operator comparing the large-times asymptotics of the interacting particle dynamics $e^{i\mathcal{H}t}$ relative to the Laplacian's reference dynamics $e^{i\mathcal{H}_0 t}$ as a continuous dual q-Hahn reduction of [5, Thm. 6.7].

Theorem 3 (Wave and scattering operators). *The operator limits*

$$\Omega^\pm := s - \lim_{t \to \pm\infty} e^{it\mathcal{H}} e^{-it\mathcal{H}_0}$$

converge in the strong $\ell^2(\Lambda)$-norm topology and the corresponding wave operators $\Omega^\pm$ intertwining the interacting dynamics $e^{i\mathcal{H}t}$ with the discrete Laplacian's dynamics $e^{i\mathcal{H}_0 t}$ are given by unitary operators in $\ell^2(\Lambda)$ of the form

$$\Omega^\pm = \boldsymbol{\mathcal{F}}^{-1} \circ \hat{\mathcal{S}}^{\mp 1/2} \circ \boldsymbol{\mathcal{F}}_0,$$

where the branches of the square roots are to be chosen such that

$$s(x)^{1/2} = \frac{(qe^{ix})_\infty}{|(qe^{ix})_\infty|} \frac{|(te^{ix})_\infty|}{(te^{ix})_\infty} \quad \text{and} \quad s_0(x)^{1/2} = \frac{(qe^{2ix})_\infty}{|(qe^{2ix})_\infty|} \prod_{0 \le r \le 2} \frac{|(\hat{t}_r e^{ix})_\infty|}{(\hat{t}_r e^{ix})_\infty}.$$

The scattering operator relating the large-times asymptotics of $e^{i\mathcal{H}t}$ *for* $t \to -\infty$ *and* $t \to +\infty$ *is thus given by the unitary operator*

$$\mathcal{S} := (\Omega^+)^{-1}\Omega^- = \boldsymbol{\mathcal{F}}_0^{-1} \circ \hat{\mathcal{S}} \circ \boldsymbol{\mathcal{F}}_0.$$

5. Bispectral Dual System

The bispectral dual in the sense of Duistermaat and Grünbaum [7,8] of the hyperbolic quantum Ruijsenaars–Schneider system on the lattice is given by the trigonometric Ruijsenaars–Macdonald q-difference operators [16,20]. This bispectral duality is a quantum manifestation of the duality between the classical Ruijsenaars–Schneider systems with hyperbolic/trigonometric dependence on the position/momentum variables and vice versa [22], which (at the classical level) states that the respective action-angle transforms linearizing the two systems under consideration are inverses of each other. As a degeneration of the Macdonald–Koornwinder q-difference operator [14, Eq. (5.4)], we immediately arrive at a bispectral dual Hamiltonian for our hyperbolic quantum Ruijsenaars–Schneider system with Morse term.

Indeed, the continuous dual q-Hahn reduction of the q-difference equation satisfied by the Macdonald–Koornwinder polynomials [14, Thm. 5.4] reads

$$\hat{H}P_\lambda = E_\lambda P_\lambda \quad \text{with} \quad E_\lambda = \sum_{j=1}^{n} t^{j-1}(q^{-\lambda_j} - 1) \quad (\lambda \in \Lambda), \tag{5.1a}$$

where

$$\hat{H} = \sum_{j=1}^{n} \Big(\hat{v}_j(\xi)(\hat{T}_{j,q} - 1) + \hat{v}_j(-\xi)(\hat{T}_{j,q}^{-1} - 1) \Big), \tag{5.1b}$$

and

$$\hat{v}_j(\xi) = \frac{\prod_{0\leq r\leq 2}(1 - \hat{t}_r e^{i\xi_j})}{(1 - e^{2i\xi_j})(1 - qe^{2i\xi_j})} \prod_{\substack{1\leq k\leq n \\ k\neq j}} \frac{1 - te^{i(\xi_j+\xi_k)}}{1 - e^{i(\xi_j+\xi_k)}} \frac{1 - te^{i(\xi_j-\xi_k)}}{1 - e^{i(\xi_j-\xi_k)}}. \tag{5.1c}$$

Here, $\hat{T}_{j,q}$ acts on trigonometric (Laurent) polynomials $\hat{p}(e^{i\xi_1}, \ldots, e^{i\xi_n})$ by a q-shift of the jth variable:

$$(\hat{T}_{j,q}\hat{p})(e^{i\xi_1}, \ldots, e^{i\xi_n}) := \hat{p}(e^{i\xi_1}, \ldots, e^{i\xi_{j-1}}, qe^{i\xi_j}, e^{i\xi_{j+1}}, \ldots, e^{i\xi_n}).$$

In other words, the bispectral dual Hamiltonian $\hat{H}$ (5.1b),(5.1c) constitutes a nonnegative unbounded self-adjoint operator with purely discrete spectrum in $L^2(\mathbb{A}, \hat{\Delta}\mathrm{d}\xi)$ that is diagonalized by the (inverse) Fourier transform $\boldsymbol{F}$ (3.11a), (3.11b):

$$\hat{H} = \boldsymbol{F} \circ E \circ \boldsymbol{F}^{-1}, \tag{5.2}$$

where E denotes the self-adjoint multiplication operator in $\ell^2(\rho + \Lambda, \Delta)$ of the form $(Ef)(\rho + \lambda) := E_\lambda f(\rho + \lambda)$ (for $\lambda \in \Lambda$ and $f \in \ell^2(\rho + \Lambda, \Delta)$ with $\langle Ef, Ef\rangle_\Delta < \infty$).

6. Quantum Integrability

In this final section, we provide explicit formulas for a complete system of commuting quantum integrals for the hyperbolic quantum Ruijsenaars–Schneider Hamiltonian with Morse term on the lattice H (2.3) and for its bispectral dual Hamiltonian $\hat{H}$ (5.1b), (5.1c). This confirms the quantum integrability of both Hamiltonians in the present Hilbert space setup.

6.1. Hamiltonian

The quantum integrals for the hyperbolic Ruijsenaars–Schneider Hamiltonian with Morse term are given by commuting difference operators $H_1, \ldots, H_n$ that are defined via their action on $f \in \ell^2(\rho + \Lambda, \Delta)$ (cf. [3, Eqs. (2.20)–(2.23)]):

$$(H_l f)(\rho+\lambda) := \sum_{\substack{J_+, J_- \subset \{1,\ldots,n\} \\ J_+ \cap J_- = \emptyset,\, |J_+|+|J_-| \leq l \\ \lambda + e_{J_+} - e_{J_-} \in \Lambda}} U_{J_+^c \cap J_-^c, l-|J_+|-|J_-|}(\lambda) V_{J_+,J_-}(\lambda) f(\rho + \lambda + e_{J_+} - e_{J_-}) \tag{6.1}$$

($\lambda \in \Lambda$, $l = 1, \ldots, n$), where $e_J := \sum_{j \in J} e_j$ for $J \subset \{1, \ldots, n\}$, $J^c := \{1, \ldots, n\} \backslash J$ and

$$\begin{aligned}
V_{J_+,J_-}(\lambda) &= t^{-\frac{1}{2}|J_+|(|J_+|-1)+\frac{1}{2}|J_-|(|J_-|-1)} \\
&\quad \times \prod_{j \in J_+} \sqrt{\frac{qt_0}{t_1 t_2}} (1 - t_1 t^{n-j} q^{\lambda_j})(1 - t_2 t^{n-j} q^{\lambda_j}) \\
&\quad \times \prod_{j \in J_-} \sqrt{\frac{t_1 t_2}{qt_0}} (1 - t_0 t^{n-j} q^{\lambda_j})(1 - t^{n-j} q^{\lambda_j}) \\
&\quad \times \prod_{\substack{j \in J_+ \\ k \in J_-}} \left(\frac{1 - t^{1+k-j} q^{\lambda_j - \lambda_k}}{1 - t^{k-j} q^{\lambda_j - \lambda_k}} \right) \left(\frac{t^{-1} - t^{k-j} q^{\lambda_j - \lambda_k + 1}}{1 - t^{k-j} q^{\lambda_j - \lambda_k + 1}} \right) \\
&\quad \times \prod_{\substack{j \in J_+ \\ k \notin J_+ \cup J_-}} \frac{t^{-1} - t^{k-j} q^{\lambda_j - \lambda_k}}{1 - t^{k-j} q^{\lambda_j - \lambda_k}} \prod_{\substack{j \in J_- \\ k \notin J_+ \cup J_-}} \frac{t - t^{k-j} q^{\lambda_j - \lambda_k}}{1 - t^{k-j} q^{\lambda_j - \lambda_k}}, \\
U_{K,p}(\lambda) &= (-1)^p \\
&\quad \times \sum_{\substack{I_+, I_- \subset K \\ I_+ \cap I_- = \emptyset,\, |I_+|+|I_-| = p}} \Bigg(\prod_{j \in I_+} \sqrt{\frac{qt_0}{t_1 t_2}} (1 - t_1 t^{n-j} q^{\lambda_j})(1 - t_2 t^{n-j} q^{\lambda_j}) \\
&\quad \times \prod_{j \in I_-} \sqrt{\frac{t_1 t_2}{qt_0}} (1 - t_0 t^{n-j} q^{\lambda_j})(1 - t^{n-j} q^{\lambda_j})
\end{aligned}$$

$$\times \prod_{\substack{j\in I_+\\ k\in I_-}} \left(\frac{1-t^{1+k-j}q^{\lambda_j-\lambda_k}}{1-t^{k-j}q^{\lambda_j-\lambda_k}}\right)\left(\frac{1-t^{-1+k-j}q^{\lambda_j-\lambda_k+1}}{1-t^{k-j}q^{\lambda_j-\lambda_k+1}}\right)$$

$$\times \prod_{\substack{j\in I_+\\ k\in K\setminus(I_+\cup I_-)}} \frac{t^{-1}-t^{k-j}q^{\lambda_j-\lambda_k}}{1-t^{k-j}q^{\lambda_j-\lambda_k}} \prod_{\substack{j\in I_-\\ k\in K\setminus(I_+\cup I_-)}} \frac{t-t^{k-j}q^{\lambda_j-\lambda_k}}{1-t^{k-j}q^{\lambda_j-\lambda_k}}\Bigg).$$

For $l=1$, the action of H_l (6.1) is seen to reduce to that of H (2.3). The diagonalization in Theorem 1 generalizes to these higher commuting quantum integrals as follows.

Theorem 4. *For parameters of the form as in Theorem 1, the difference operators $H_1,\ldots,H_n$ (6.1) constitute bounded commuting self-adjoint operators in the Hilbert space $\ell^2(\rho+\Lambda,\Delta)$ that are simultaneously diagonalized by the Fourier transform $\boldsymbol{F}$ (3.11a), (3.11b):*

$$H_l=\boldsymbol{F}^{-1}\circ\hat{E}_l\circ\boldsymbol{F}\quad (l=1,\ldots,n), \tag{6.2a}$$

where $\hat{E}_l$ denotes the bounded real multiplication operator acting on $\hat{f}\in L^2(\mathbb{A},\hat{\Delta}d\xi)$ by $(\hat{E}_l\hat{f})(\xi):=\hat{E}_l(\xi)\hat{f}(\xi)$ with

$$\hat{E}_l(\xi):=\sum_{1\le j_1<\cdots<j_l\le n} \tag{6.2b}$$
$$(2\cos(\xi_{j_1})-t^{j_1-1}\hat{t}_0-t^{-(j_1-1)}\hat{t}_0^{-1})\cdots(2\cos(\xi_{j_l})-t^{j_l-l}\hat{t}_0-t^{-(j_l-l)}\hat{t}_0^{-1}).$$

Proof. The eigenvalue equation $H_l\psi_\xi=\hat{E}_l(\xi)\psi_\xi$ reads explicitly

$$\sum_{\substack{J_+,J_-\subset\{1,\ldots,n\}\\ J_+\cap J_-=\emptyset,\,|J_+|+|J_-|\le l\\ \lambda+e_{J_+}-e_{J_-}\in\Lambda}} U_{J_+^c\cap J_-^c,\,l-|J_+|-|J_-|}(\lambda)V_{J_+,J_-}(\lambda)\psi_\xi(\rho+\lambda+e_{J_+}-e_{J_-})$$
$$=\hat{E}_l(\xi)\psi_\xi(\rho+\lambda).$$

This eigenvalue identity corresponds to the continuous dual q-Hahn reduction of the Pieri recurrence formula for the Macdonald–Koornwinder polynomials in [4, Thm. 6.1], where we have expressed the eigenvalues $\hat{E}_l(\xi)$ in a compact form stemming from [13, Eq. (5.1)] (cf. also [6, Sec. 2.2]). □

6.2. Bispectral Dual Hamiltonian

The continuous dual q-Hahn reduction of the system of higher q-difference equations for the Macdonald–Koornwinder polynomials in [4, Sec. 5.1] reads

$$\hat{H}_lP_\lambda=E_{\lambda,l}P_\lambda\quad (\lambda\in\Lambda,\ l=1,\ldots,n), \tag{6.3a}$$

where

$$E_{\lambda,l}:=t^{-l(l-1)/2}\sum_{1\le j_1<\cdots<j_l\le n}(t^{j_1-1}q^{-\lambda_{j_1}}-t^{n-j_1})\cdots(t^{j_l-1}q^{-\lambda_{j_l}}-t^{n+l-j_l-1}) \tag{6.3b}$$

(cf. [13, Eq. (5.1)]), and

$$\hat{H}_l := \sum_{\substack{J\subset\{1,\ldots,n\},\, 0\le|J|\le l\\ \epsilon_j\in\{1,-1\}, j\in J}} \hat{U}_{J^c,l-|J|}\hat{V}_{\epsilon J}\hat{T}_{\epsilon J,q}, \tag{6.3c}$$

with $\hat{T}_{\epsilon J,q} := \prod_{j\in J}\hat{T}_{j,q}^{\epsilon_j}$ and

$$\hat{V}_{\epsilon J} = \prod_{j\in J}\frac{\prod_{0\le r\le 2}(1-\hat{t}_r e^{i\epsilon_j\xi_j})}{(1-e^{2i\epsilon_j\xi_j})(1-qe^{2i\epsilon_j\xi_j})}\prod_{\substack{j\in J\\ k\notin J}}\frac{1-te^{i(\epsilon_j\xi_j+\xi_k)}}{1-e^{i(\epsilon_j\xi_j+\xi_k)}}\frac{1-te^{i(\epsilon_j\xi_j-\xi_k)}}{1-e^{i(\epsilon_j\xi_j-\xi_k)}}$$
$$\times\prod_{\substack{j,k\in J\\ j<k}}\frac{1-te^{i(\epsilon_j\xi_j+\epsilon_k\xi_k)}}{1-e^{i(\epsilon_j\xi_j+\epsilon_k\xi_k)}}\frac{1-tqe^{i(\epsilon_j\xi_j+\epsilon_j\xi_k)}}{1-qe^{i(\epsilon_j\xi_j+\epsilon_k\xi_k)}},$$

$$\hat{U}_{K,p} = (-1)^p\sum_{\substack{I\subset K,\,|I|=p\\ \epsilon_j\in\{1,-1\}, j\in I}}\Bigg(\prod_{j\in I}\frac{\prod_{0\le r\le 2}(1-\hat{t}_r e^{i\epsilon_j\xi_j})}{(1-e^{2i\epsilon_j\xi_j})(1-qe^{2i\epsilon_j\xi_j})}$$
$$\times\prod_{\substack{j\in I\\ k\in K\setminus I}}\frac{1-te^{i(\epsilon_j\xi_j+\xi_k)}}{1-e^{i(\epsilon_j\xi_j+\xi_k)}}\frac{1-te^{i(\epsilon_j\xi_j-\xi_k)}}{1-e^{i(\epsilon_j\xi_j-\xi_k)}}$$
$$\times\prod_{\substack{j,k\in I\\ j<k}}\frac{1-te^{i(\epsilon_j\xi_j+\epsilon_k\xi_k)}}{1-e^{i(\epsilon_j\xi_j+\epsilon_k\xi_k)}}\frac{t-qe^{i(\epsilon_j\xi_j+\epsilon_j\xi_k)}}{1-qe^{i(\epsilon_j\xi_j+\epsilon_k\xi_k)}}\Bigg).$$

For $l = 1$, this reproduces the continuous dual q-Hahn reduction of the Macdonald–Koornwinder q-difference equation in Eqs. (5.1a)–(5.1c).

The q-difference operators $\hat{H}_1,\ldots,\hat{H}_n$ extend the bispectral dual Hamiltonian $\hat{H}$ (5.1b)–(5.1c) into a complete system of commuting quantum integrals that are simultaneously diagonalized by the multivariate continuous dual q-Hahn polynomials.

Theorem 5. *For parameter values in the domain* (3.3)*, the q-difference operators* $\hat{H}_1,\ldots,\hat{H}_n$ *constitute* nonnegative *unbounded self-adjoint operators with purely discrete spectra in* $L^2(\mathbb{A},\hat{\Delta}d\xi)$ *that are simultaneously diagonalized by the* (*inverse*) *Fourier transform* $\boldsymbol{F}$ (3.11a), (3.11b)*:*

$$\hat{H}_l = \boldsymbol{F}\circ E_l\circ\boldsymbol{F}^{-1},\quad l=1,\ldots,n, \tag{6.4}$$

where E_l *denotes the self-adjoint multiplication operator in* $\ell^2(\rho+\Lambda,\Delta)$ *given by* $(E_lf)(\rho+\lambda) := E_{\lambda,l}f(\rho+\lambda)$ *(on the domain of* $f\in\ell^2(\rho+\Lambda,\Delta)$ *such that* $\langle E_lf, E_lf\rangle_\Delta<\infty$*).*

Notice in this connection that although the domain of the unbounded operator $\hat{H}_l$ in $L^2(\mathbb{A},\hat{\Delta}\mathrm{d}\xi)$ depends on l, the resolvent operators $(\hat{H}_1-z_1)^{-1},\ldots,$ $(\hat{H}_n-z_n)^{-1}$ (with $z_1,\ldots,z_n\in\mathbb{C}\setminus[0,+\infty)$) commute as bounded operators on $L^2(\mathbb{A},\hat{\Delta}\mathrm{d}\xi)$, and the q-difference operators $\hat{H}_1,\ldots,\hat{H}_n$ moreover commute themselves on the joint polynomial eigenbasis P_λ, $\lambda\in\Lambda$.

Remark 6. To infer that the eigenvalues $E_{\lambda,l}$ (6.3b) are nonnegative—thus indeed giving rise to a *nonnegative* operator $\hat{H}_l$ in Theorem 5—it is helpful to note that these can be rewritten as (cf. [4, Sec. 5.1]):

$$E_{\lambda,l} = t^{-l(l-1)/2} E_{l,n}(q^{-\lambda_1}, tq^{-\lambda_2}, \ldots, t^{n-1}q^{-\lambda_n}; t^{l-1}, t^l, \ldots, t^{n-1})$$

with

$$E_{l,n}(z_1,\ldots,z_n; y_l,\ldots,y_n) := \sum_{0\leq k\leq l} (-1)^{l+k} \boldsymbol{e}_k(z_1,\ldots,z_n)\boldsymbol{h}_{l-k}(y_l,\ldots,y_n).$$

Here, $\boldsymbol{e}_k(z_1,\ldots,z_n)$ and $\boldsymbol{h}_k(y_l,\ldots,y_n)$ refer to the elementary and the complete symmetric functions of degree k (cf. [16, Ch. I.2]), with the convention that $\boldsymbol{e}_0 = \boldsymbol{h}_0 \equiv 1$. The nonnegativity of the eigenvalues now readily follows inductively in the particle number n by means of the recurrence (cf. [2, Lem. B.2])

$$\begin{aligned} &E_{l,n}(q^{-\lambda_1}, tq^{-\lambda_2}, \ldots, t^{n-1}q^{-\lambda_n}; t^{l-1}, t^l, \ldots, t^{n-1}) \\ &\quad = (q^{-\lambda_1} - t^{l-1}) E_{l-1,n-1}(tq^{-\lambda_2}, \ldots, t^{n-1}q^{-\lambda_n}; t^{l-1}, \ldots, t^{n-1}) \\ &\qquad + E_{l,n-1}(tq^{-\lambda_2}, \ldots, t^{n-1}q^{-\lambda_n}; t^l, \ldots, t^{n-1}) \end{aligned}$$

and the homogeneity

$$E_{l,n}(tz_1,\ldots,tz_n; ty_l,\ldots,ty_n) = t^l E_{l,n}(z_1,\ldots,z_n; y_l,\ldots,y_n).$$

Remark 7. The hyperbolic Ruijsenaars–Schneider Hamiltonian with Morse term (2.1) can be retrieved as a limit of the Macdonald–Koornwinder q-difference operator [3]. In this limit, the center-of-mass is sent to infinity, which causes the hyperoctahedral symmetry of the Macdonald–Koornwinder operator to be broken: while the permutation-symmetry still persists, the parity-symmetry is no longer present. Indeed, the limit in question restores the translational invariance of the interparticle pair interactions enjoyed by the original Ruijsenaars–Schneider model and gives moreover rise to additional Morse terms that are not parity-invariant. It turns out that most of our results above can in fact be lifted to the Macdonald–Koornwinder level, even though such a generalization is presumably somewhat less relevant from a physical point of view. Specifically, the scattering of the corresponding quantum lattice model associated with the full six-parameter family of Macdonald–Koornwinder polynomials was briefly discussed in [5, Sec. 6.4], its commuting quantum integrals can be read off from the Pieri formulas for the Macdonald–Koornwinder polynomials in [4, Thm. 6.1], and the pertinent bispectral dual Hamiltonian and its commuting quantum integrals are given by the Macdonald–Koornwinder q-difference operator [14] and its higher-order commuting q-difference operators [4, Thm. 5.1].

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