



Orthogonality of Bethe Ansatz Eigenfunctions for the Laplacian on a Hyperoctahedral Weyl Alcove

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Abstract: We prove the orthogonality of the Bethe Ansatz eigenfunctions for the Laplacian on a hyperoctahedral Weyl alcove with repulsive homogeneous Robin boundary conditions at the walls. To this end these eigenfunctions are retrieved as the continuum limit of an orthogonal basis of algebraic Bethe Ansatz eigenfunctions for a finite q -boson system endowed with diagonal open-end boundary interactions.

1. Introduction

It is well-known that the Lieb–Liniger spectral problem for n bosonic particles on the circle with a pairwise delta potential interaction [LL, G2, KB1, ML] is—in the center-of-mass frame—mathematically equivalent to that of the Laplacian on a Weyl alcove. (This alcove arises as a fundamental domain for the configuration space with respect to the action of the symmetric group S_n permuting the particles.) In this picture, the delta potential interaction gives rise to homogeneous Robin boundary conditions at the walls of the alcove. For $n = 3$, the alcove in question amounts to an equilateral triangle and the corresponding spectral problem for the Laplacian has in such case a rich history going back to Lamé (who considered the special situations of Dirichlet and Neumann boundary conditions at the triangle’s edges) [DF, Mc].

From the point of view of spectral- and harmonic analysis, a crucial issue regarding the eigenfunctions of the Laplacian—which were constructed by Lieb and Liniger with the aid of the coordinate Bethe Ansatz method via a suitable linear combination of plane waves so as to match the boundary conditions at the walls—is the proof of their orthogonality and completeness in an appropriate Hilbert space setting. In the repulsive coupling parameter regime both questions were settled by Dorlas [Do], who combined a variational method of Yang and Yang describing the solutions of the Bethe equations characterizing the spectrum [YY] with the observation that the eigenfunctions under

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consideration can be retrieved via a continuum limit from the algebraic Bethe Ansatz wave functions for a lattice quantum nonlinear Schrödinger equation studied by Bogoliubov, Izergin and Korepin [KB1, Ch. V.III]. The crux is that the diagonalization of the transfer operator for the lattice model allows one to remove possible degeneracies in the spectrum that prevent a direct orthogonality proof based on the self-adjointness of the Laplacian alone. The completeness question is moreover dealt with by exploiting the continuity and monotonicity in the coupling parameter, which allows for a reduction to the elementary case of Neumann boundary conditions (where the completeness of the Bethe Ansatz eigenfunctions follows from the completeness of the standard Fourier basis).

Our main goal is to extend the above picture to the case of the harmonic analysis on a Weyl alcove associated with the *hyperoctahedral group* $W := S_n \times \{1, -1\}^n$ of *signed permutations* (σ, ϵ) :

$$x = (x_1, \dots, x_n) \xrightarrow{(\sigma, \epsilon)} (\epsilon_1 x_{\sigma_1}, \dots, \epsilon_n x_{\sigma_n}),$$

where $\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma_1 & \sigma_2 & \dots & \sigma_n \end{pmatrix} \in S_n$ and $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ with $\epsilon_j \in \{1, -1\}$ ($j = 1, \dots, n$). The underlying particle model is governed by a formal Schrödinger operator of the form [G1, G2, GS, G, HO, D1, EOS1, EOS2, E]:

$$-\Delta + \sum_{1 \leq j < k \leq n} c \left(\delta(x_j - x_k) + \delta(x_j + x_k) \right) + \sum_{1 \leq j \leq n} \left(c_1 \delta(x_j) + c_2 \delta(2x_j) \right), \tag{1.1}$$

where Δ represents the Laplacian $\partial_{x_1}^2 + \dots + \partial_{x_n}^2$ and $\delta(\cdot)$ refers to the *periodic* delta distribution (or Dirac comb) with $\delta(x_j + m) = \delta(x_j)$ ($m \in \mathbb{Z}$), so $\delta(2x_j) = \frac{1}{2}(\delta(x_j) + \delta(x_j + \frac{1}{2}))$. The position variables x_1, \dots, x_n are again assumed to lie on the circle \mathbb{R}/\mathbb{Z} and the constants c, c_1 and c_2 are interpreted as coupling parameters regulating the strengths of the delta potential interactions. Upon restriction to the space of W -invariant functions on the configuration space $(\mathbb{R}/\mathbb{Z})^n$, the eigenvalue problem for the Schrödinger operator (1.1) becomes manifestly equivalent to the following eigenvalue equation [GS, G, EOS1, EOS2]

$$-\Delta \psi = E \psi \tag{1.2a}$$

for the Laplacian on the fundamental hyperoctahedral Weyl alcove

$$A = \{x \in \mathbb{R}^n \mid \frac{1}{2} > x_1 > x_2 > \dots > x_n > 0\}, \tag{1.2b}$$

endowed with homogeneous Robin boundary conditions at the walls of the form

$$(\partial_{x_j} - \partial_{x_{j+1}} - g) \psi \Big|_{x_j - x_{j+1} = 0} = 0 \quad \text{for } j = 1, \dots, n - 1 \tag{1.2c}$$

and

$$(\partial_{x_1} + g_+) \psi \Big|_{x_1 = \frac{1}{2}} = 0, \quad (\partial_{x_n} - g_-) \psi \Big|_{x_n = 0} = 0. \tag{1.2d}$$

Here E represents the energy eigenvalue and the boundary parameters g, g_+, g_- are related to the coupling parameters c, c_1, c_2 via $c = 2g, c_1 = 2(g_- - g_+)$ and $c_2 = 4g_+$.

Below we will demonstrate the orthogonality of the Bethe Ansatz eigenfunctions for the spectral problem in Eqs. (1.2a)–(1.2d), under the assumption that the boundary parameters g, g_+ and g_- are positive (so our walls are repulsive). This is achieved

by showing that the eigenfunctions under consideration arise as the continuum limit of the algebraic Bethe Ansatz eigenfunctions for a q -boson system [Ku, BB, BIK, SW, Ts, D2, Ko, DE2, BCPS] built of a finite number of q -oscillators [Ma, KS] endowed with diagonal open-end boundary interactions [D1, LW, DE1, DE3, WZ]. A key advantage of invoking this q -boson system (rather than a lattice quantum nonlinear Schrödinger equation with boundary interactions) is that the corresponding Bethe Ansatz eigenfunctions can be expressed explicitly in terms of Macdonald's hyperoctahedral Hall–Littlewood polynomials [M2, NR], which greatly facilitates our analysis of the continuum limit. Indeed, analogous explicit expressions for the Bethe Ansatz eigenfunctions of the finite q -boson system with *periodic* boundary conditions in terms of Hall–Littlewood polynomials [BB, BIK, Ts, D2, Ko] were instrumental for a simpler alternative continuum limit reconfirming the orthogonality of the Lieb–Liniger eigenfunctions for the Weyl alcove associated with S_n [D2] (in comparison with the more elaborate continuum limit employed in Dorlas' original approach based on the lattice quantum nonlinear Schrödinger equation). Together, the orthogonality of the Bethe Ansatz eigenfunctions on the Weyl alcove associated with the symmetric group and the hyperoctahedral group imply the orthogonality of a basis of Bethe Ansatz eigenfunctions for the corresponding Laplacian with repulsive Robin type boundary conditions on any Weyl alcove of classical type. Indeed, the cases of the Weyl alcoves of types B , C and D are recovered from the spectral problem in Eqs. (1.2a)–(1.2d) via specializations of the boundary parameters g_+ and g_- .

The material is organized as follows. First our main result, concerning the orthogonality of the Bethe Ansatz eigenfunctions for the eigenvalue problem in Eqs. (1.2a)–(1.2d), is formulated precisely in Sect. 2. Sections 3–12 are devoted to the proof of this orthogonality, which breaks up in several steps. Section 3 describes the transfer operator of a finite q -boson system with diagonal open-end boundary interactions [LW, DE3]. Some key points of Sklyanin's quantum inverse scattering formalism for open systems [S] sustaining our discussion have been briefly summarized in Appendix A (at the end of the paper). Section 4 analyzes the main properties of the underlying q -boson boundary monodromy matrix in further detail. These properties are subsequently exploited in Sect. 5 to identify the q -boson Hamiltonian generated by the boundary transfer operator. In Sect. 6, the Hamiltonian and the boundary transfer operator are represented as (essentially) self-adjoint operators in the q -boson Fock space. The explicit action of the boundary transfer operator in Fock space is computed in Sect. 7. Next, the eigenfunctions are constructed in Sect. 8 by means of Sklyanin's algebraic Bethe Ansatz formalism for open systems (which involves some technical computations supplemented in Appendix B). In Sect. 9 the Bethe Ansatz creation operator is employed to derive a branching rule that permits one to compute the n -particle Bethe Ansatz wave function inductively in the number of particles/variables. After providing the solutions of the Bethe Ansatz equations—via the minima of a family of strictly convex Morse functions following the method of Yang and Yang [YY]—the self-adjointness of the boundary transfer operator is exploited in Sect. 10 to infer that the corresponding Bethe Ansatz eigenfunctions constitute an orthogonal basis for the q -boson Fock space. Next, in Sect. 11, we rely on the branching rule in combination with recent results drawn from Ref. [WZ] to express the n -particle Bethe-Ansatz eigenfunctions in terms of Macdonald's hyperoctahedral Hall–Littlewood polynomials of BC_n -type [M2, §10]. This explicit coordinate representation of the q -boson eigenfunctions is then put to use in Sect. 12, where the orthogonality of the Bethe Ansatz eigenfunctions for the Laplacian on the hyperoctahedral Weyl alcove is retrieved via a continuum limit ($q \rightarrow 1$). Finally, in Sect. 13, some details are provided on how to derive the orthogonality of a basis of Bethe Ansatz eigenfunctions for the

Laplacian with repulsive Robin boundary conditions on any Weyl alcove of classical type.

2. Main Result

For any $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ such that vanishing denominators are avoided, the following linear combination of plane waves

$$\psi(\xi, x) = \sum_{\substack{\sigma \in \mathcal{S}_n \\ \epsilon \in \{1, -1\}^n}} C(\epsilon_1 \xi_{\sigma_1}, \dots, \epsilon_n \xi_{\sigma_n}) \exp(i\epsilon_1 \xi_{\sigma_1} x_1 + \dots + i\epsilon_n \xi_{\sigma_n} x_n), \tag{2.1}$$

with coefficients of the form

$$C(\xi_1, \dots, \xi_n) = \prod_{1 \leq j \leq n} \left(\frac{\xi_j - ig_-}{\xi_j} \right) \prod_{1 \leq j < k \leq n} \left(\frac{\xi_j + \xi_k - ig}{\xi_j + \xi_k} \right) \left(\frac{\xi_j - \xi_k - ig}{\xi_j - \xi_k} \right),$$

provides a solution to the eigenvalue equation (1.2a), (1.2b) that complies with the boundary conditions (1.2c), (1.2d) at those walls that pass through the origin [G1,GS, G,HO,D1]. The corresponding eigenvalue is given by

$$E = E(\xi) := \xi_1^2 + \dots + \xi_n^2. \tag{2.2}$$

In order to satisfy also the boundary condition at the affine wall $x_1 = \frac{1}{2}$, the spectral parameter ξ must additionally obey the Bethe Ansatz equations [EOS1,BDM,EOS2,E]

$$e^{i\xi_j} = \left(\frac{ig_+ + \xi_j}{ig_+ - \xi_j} \right) \left(\frac{ig_- + \xi_j}{ig_- - \xi_j} \right) \prod_{\substack{1 \leq k \leq n \\ k \neq j}} \left(\frac{ig + \xi_j + \xi_k}{ig - \xi_j - \xi_k} \right) \left(\frac{ig + \xi_j - \xi_k}{ig - \xi_j + \xi_k} \right) \tag{2.3}$$

for $j = 1, \dots, n$. Furthermore, following the paradigm instaurated by Yang and Yang [YY], a solution of the Bethe Ansatz equations (2.3) is obtained for positive boundary parameters g, g_+, g_- as the unique global minimum ξ_λ of the following (semibounded) strictly convex Morse function [EOS1,BDM,EOS2,E]

$$\begin{aligned} V_\lambda(\xi) = & \sum_{1 \leq j \leq n} \left(\frac{1}{2} \xi_j^2 - 2\pi(\rho_j + \lambda_j)\xi_j + 2 \int_0^{\xi_j} \arctan\left(\frac{u}{g_+}\right) + \arctan\left(\frac{u}{g_-}\right) du \right) \\ & + \sum_{1 \leq j < k \leq n} 2 \left(\int_0^{\xi_j + \xi_k} \arctan\left(\frac{u}{g}\right) du + \int_0^{\xi_j - \xi_k} \arctan\left(\frac{u}{g}\right) du \right) \end{aligned} \tag{2.4}$$

associated with a partition

$$\lambda \in \Lambda = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0\}, \tag{2.5}$$

where $\rho_j = n + 1 - j$ for $j = 1, \dots, n$. Indeed, the critical equation $\nabla_\xi V_\lambda(\xi) = 0$, i.e.

$$\begin{aligned} & \xi_j + 2 \arctan\left(\frac{\xi_j}{g_+}\right) + 2 \arctan\left(\frac{\xi_j}{g_-}\right) \\ & + \sum_{\substack{1 \leq k \leq n \\ k \neq j}} 2 \left(\arctan\left(\frac{\xi_k + \xi_j}{g}\right) - \arctan\left(\frac{\xi_k - \xi_j}{g}\right) \right) = 2\pi(\rho_j + \lambda_j) \end{aligned} \tag{2.6}$$

($j = 1, \dots, n$), goes over into the Bethe Ansatz equation (2.3) upon exponentiation after multiplication by the imaginary unit, and the Hessian $H_{j,k} = \partial_{\xi_j} \partial_{\xi_k} V_\lambda(\xi)$, i.e.

$$H_{j,k} = \begin{cases} 1 + \frac{2g_+}{g_+^2 + \xi_j^2} + \frac{2g_-}{g_-^2 + \xi_j^2} + \sum_{l \neq j} \left(\frac{2g}{g^2 + (\xi_j + \xi_l)^2} + \frac{2g}{g^2 + (\xi_j - \xi_l)^2} \right) & \text{if } k = j \\ \frac{2g}{g^2 + (\xi_j + \xi_k)^2} - \frac{2g}{g^2 + (\xi_j - \xi_k)^2} & \text{if } k \neq j \end{cases} \quad (2.7)$$

($j, k = 1, \dots, n$), is positive definite. It moreover follows from Eq. (2.6) that the minimum ξ_λ of $V_\lambda(\xi)$ (2.4) is assumed inside the Weyl chamber

$$C = \{ \xi \in \mathbb{R}^n \mid \xi_1 > \xi_2 > \dots > \xi_n > 0 \} \quad (2.8)$$

(cf. Remark 2.3 below).

We are now in the position to formulate the main contribution of the present work.

Theorem 2.1 (Orthogonality). *For $g, g_+, g_- > 0$, the Bethe Ansatz eigenfunctions $\psi(\xi_\lambda, x)$, $\lambda \in \Lambda$ constitute an orthogonal basis for the Hilbert space $L^2(A, dx)$ of quadratically integrable functions with respect to the standard Lebesgue measure dx on $A \subset \mathbb{R}^n$.*

The principal issue to be settled here is the orthogonality of the Bethe Ansatz eigenfunctions, as the completeness of the system $\psi(\xi_\lambda, x)$, $\lambda \in \Lambda$ in $L^2(A, dx)$ is readily deduced with the aid of the ideas in [Do, Sec. 3] and [E]. Indeed, when $g_+ = g_-$ the completeness follows directly from [E, Thm. 1] (upon specialization to the root system $R = C_n$). The extension to the case that $g_+ \neq g_-$ hinges in turn on [Do, Thm. 3.2] and exploits that our Laplacian is continuous and monotonically nondecreasing (as a quadratic form) both in g_+ and in g_- , cf. [Do, Sec. 3] and [E] for the precise details of this argument.

From the diagonalization of the Laplacian by the orthogonal basis in Theorem 2.1, one deduces its spectral properties.

Corollary 2.2 (Spectrum). *For $g, g_+, g_- > 0$, the Laplacian $-\Delta$ in Eqs. (1.2a)–(1.2d) constitutes an unbounded essentially self-adjoint operator in $L^2(A, dx)$ that has purely discrete spectrum given by the positive eigenvalues $E(\xi_\lambda)$, $\lambda \in \Lambda$.*

Remark 2.3. Since the function $\arctan(\cdot)$ is odd and strictly monotonously increasing, it is clear from Eq. (2.6) that $\xi_j > 0$ at the critical point of $V_\lambda(\xi)$. Moreover, subtracting the k th equation from the j th equation in Eq. (2.6) reveals similarly that at this critical point $\xi_j > \xi_k$ if $j < k$. In other words, one has that

$$\xi_1 > \xi_2 > \dots > \xi_n > 0 \quad \text{at } \xi = \xi_\lambda. \quad (2.9)$$

Upon refining this analysis slightly with the aid of the elementary inequality $0 < \arctan(u) - \arctan(v) < u - v$ for $u > v$, one readily finds more precise estimates for ξ_j and the moment gaps $\xi_j - \xi_k$ at the spectral value $\xi = \xi_\lambda$:

$$\frac{2\pi(\rho_j + \lambda_j)}{1 + \kappa} < \xi_j < 2\pi(\rho_j + \lambda_j) \quad \text{at } \xi = \xi_\lambda, \quad (2.10a)$$

for $1 \leq j \leq n$, and

$$\frac{2\pi(\rho_j - \rho_k + \lambda_j - \lambda_k)}{1 + \kappa} < \xi_j - \xi_k < 2\pi(\rho_j - \rho_k + \lambda_j - \lambda_k) \quad \text{at } \xi = \xi_\lambda, \quad (2.10b)$$

for $1 \leq j < k \leq n$, where $\kappa := 2(g_+^{-1} + g_-^{-1} + 2(n - 1)g^{-1})$. The former inequalities confirm in particular that the eigenvalues $E(\xi_\lambda)$ do not remain bounded from above when λ varies over Λ (2.5), and also reveal the behavior of the spectrum in the limit of strong coupling:

$$\lim_{g, g_+, g_- \rightarrow +\infty} \xi_\lambda = 2\pi(\rho + \lambda). \tag{2.11}$$

Remark 2.4. Given a partition $\lambda \in \Lambda$, the dependence of the Bethe Ansatz eigenfunction $\psi(\xi_\lambda, x)$ on the coupling parameters is real-analytic (and thus smooth) in the positivity domain $g, g_+, g_- > 0$. Indeed, for this repulsive regime the critical equation (2.6) is real-analytic in these couplings and so is the critical point ξ_λ (by the implicit function theorem, since the Jacobi matrix of the critical equation equals the Hessian (2.7) of the strictly convex Morse function $V_\lambda(\xi)$ (2.4) and is therefore invertible). The explicit formula for the Bethe Ansatz eigenfunction in Eq. (2.1)—combined with the inequalities in Eq. (2.9)—guarantee that this real-analyticity carries over to $\psi(\xi_\lambda, x)$. When one or more of the three coupling parameters are negative, the corresponding walls of the alcove A (1.2b) are attractive and particle binding becomes feasible. In this situation the Yang–Yang construction that provided us with a complete set of spectral points via the respective minima of $V_\lambda(\xi)$, $\lambda \in \Lambda$ cannot persist in the above straightforward form, because the convexity is no longer guaranteed in the presence of attractive coupling parameters and—more fundamentally—the complex solutions of the Bethe–Ansatz equations relevant for particle binding are excluded from the start.

3. Open-end q -Boson System

In this section we describe the transfer operator of a finite q -boson system with diagonal open-end boundary interactions.

3.1. q -Boson algebra. For a fixed nonnegative integer m and a deformation parameter

$$t := q^2 \quad \text{with} \quad 0 < q < 1, \tag{3.1}$$

let us define the q -boson algebra \mathbb{A}_m on the finite integral lattice

$$\mathbb{N}_m := \{0, 1, 2, \dots, m\} \tag{3.2}$$

as the unital associative algebra over \mathbb{C} with ultralocal generators $\beta_l, \beta_l^*, t^{\pm N_l}$ ($l \in \mathbb{N}_m$) subject to the relations [Ku, Ma, KS]

$$t^{N_l} \beta_l^* = t \beta_l^* t^{N_l}, \quad t^{N_l} \beta_l = t^{-1} \beta_l t^{N_l}, \quad t^{N_l} t^{-N_l} = t^{-N_l} t^{N_l} = 1 \tag{3.3a}$$

and

$$\beta_l \beta_l^* - \beta_l^* \beta_l = t^{N_l}, \quad \beta_l \beta_l^* - t \beta_l^* \beta_l = 1. \tag{3.3b}$$

(By ultralocal it is meant that the generators corresponding to distinct sites $l \in \mathbb{N}_m$ commute.) It follows from the relations (3.3a), (3.3b) that

$$\beta_l \beta_l^* = [N_l + 1], \quad \beta_l^* \beta_l = [N_l], \quad \beta_l [N_l] = [N_l + 1] \beta_l, \quad \beta_l^* [N_l] = [N_l - 1] \beta_l, \tag{3.4a}$$

where

$$[N_l + k] := \frac{1 - t^k t^{N_l}}{1 - t} \quad (k \in \mathbb{Z}). \tag{3.4b}$$

3.2. *Periodic transfer operator.* The periodic q -boson system is a lattice regularization of the quantum nonlinear Schrödinger equation on the circle, which was introduced in Refs. [BB, BIK] within the framework of the quantum inverse scattering formalism originating from Faddeev’s celebrated St. Petersburg school [T, KB1, JM, F].

Let $M_k(\mathbb{A}_m)$ denote the algebra of square matrices of size k with coefficients from \mathbb{A}_m and let $\mathbb{C}[u^{\pm 1}]$ be the algebra of complex Laurent polynomials in the spectral parameter u . For our purposes, a convenient system of generators for $\mathbb{C}[u^{\pm 1}]$ is given by $s(u) := u - u^{-1}$ and $c(u) := u + u^{-1}$.

The q -boson system is characterized by the following Lax matrix [BIK]

$$L_l(u) = \begin{pmatrix} u^{-1} & (1-t)\beta_l^* \\ \beta_l & u \end{pmatrix} \in \mathbb{C}[u^{\pm 1}] \otimes M_2(\mathbb{A}_m) \tag{3.5}$$

satisfying the quantum Yang–Baxter equation (a.k.a. fundamental commutation relation)

$$R(u/v)L_l(u)_1L_l(v)_2 = L_l(v)_1L_l(u)_2R(u/v) \tag{3.6}$$

in $\mathbb{C}[u^{\pm 1}, v^{\pm 1}] \otimes M_4(\mathbb{A}_m)$, where

$$L_l(u)_1 := L_l(u) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad L_l(u)_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes L_l(u),$$

and $R(u) \in \mathbb{C}[u^{\pm 1}] \otimes M_4(\mathbb{C})$ denotes the (trigonometric) R -matrix

$$R(u) = \begin{pmatrix} s(q^{-1}u) & 0 & 0 & 0 \\ 0 & s(q^{-1}) & q^{-1}s(u) & 0 \\ 0 & qs(u) & s(q^{-1}) & 0 \\ 0 & 0 & 0 & s(q^{-1}u) \end{pmatrix}. \tag{3.7}$$

By taking the trace of the associated periodic monodromy matrix

$$\begin{aligned} U_m(u) &:= L_m(u) \cdots L_1(u)L_0(u) \\ &= \begin{pmatrix} A_m(u) & B_m(u) \\ C_m(u) & D_m(u) \end{pmatrix} \in \mathbb{C}[u^{\pm 1}] \otimes M_2(\mathbb{A}_m) \end{aligned} \tag{3.8}$$

solving the quantum Yang–Baxter equation (3.6) (with $U_m(u)$ substituted for $L_l(u)$), one arrives at the transfer operator for the periodic q -boson system

$$T_m(u) = \text{tr } U_m(u) = A_m(u) + D_m(u) \in \mathbb{C}[u^{\pm 1}] \otimes \mathbb{A}_m, \tag{3.9}$$

satisfying the commutativity

$$[T_m(u), T_m(v)] = 0 \tag{3.10}$$

in $\mathbb{C}[u^{\pm 1}, v^{\pm 1}] \otimes \mathbb{A}_m$ (where the bracket $[\cdot, \cdot]$ refers to the commutator product).

Remark 3.1. Apart from the fundamental commutation relation in Eq. (3.6), the R -matrix (3.7) also satisfies the following (braid-type) quantum Yang–Baxter equation

$$R_{12}(u)R_{23}(uv)R_{12}(v) = R_{23}(v)R_{12}(uv)R_{23}(u) \tag{3.11}$$

in $\mathbb{C}[u^{\pm 1}, v^{\pm 1}] \otimes M_8(\mathbb{C})$, where

$$R_{12}(u) := R(u) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R_{23}(u) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes R(u).$$

(In the literature, such R -matrices satisfying the braid form of the quantum Yang–Baxter equation are often denoted by $\check{R}(u)$, see e.g. Refs. [J, PA].) One obtains the q -boson R -matrix from the standard trigonometric R -matrix for the XXZ chain (cf. e.g. [S, Sec. 4])

$$R_{xxz}(u) := \begin{pmatrix} s(q^{-1}u) & 0 & 0 & 0 \\ 0 & s(u) & s(q^{-1}) & 0 \\ 0 & s(q^{-1}) & s(u) & 0 \\ 0 & 0 & 0 & s(q^{-1}u) \end{pmatrix} \tag{3.12}$$

via the relation $R(u) = D^{-1}PR_{xxz}(u)D$, where $D := \text{diag}(1, q, 1, 1) \in M_4(\mathbb{C})$ and $P \in M_4(\mathbb{C})$ denotes the permutation matrix in Eq. (A.4) below.

3.3. Boundary transfer operator. The quantum inverse scattering formalism was extended through a fundamental contribution of Sklyanin allowing for open-end boundary conditions (as opposed to periodic boundary conditions) [S]. Over the years the technical assumptions on the properties of the permitted R -matrices were relaxed somewhat, putting Sklyanin’s formalism at a similar level of generality as the original quantum inverse scattering formalism for periodic systems [MN, FSHY, V]. The objectives of this paper only require to consider a finite q -boson system with diagonal boundary interactions at the lattice endpoints [LW]. For completeness, the main highlights of Sklyanin’s formalism underlying our presentation have been briefly summarized in Appendix A with pointers to more detailed discussions in the literature.

The diagonal boundary K -matrices in $\mathbb{C}[u^{\pm 1}] \otimes M_2(\mathbb{C})$ of the form [C]

$$K_-(u; a_-) = \begin{pmatrix} e(u; a_-) & 0 \\ 0 & f(u; a_-) \end{pmatrix}, \quad K_+(u; a_+) = \begin{pmatrix} f(u; a_+) & 0 \\ 0 & e(u; a_+) \end{pmatrix} \tag{3.13}$$

with $e(u; a) := au - u^{-1}$ and $f(u; a) := e(q^{-1}u^{-1}; a)$, satisfy respectively the left reflection equation

$$\begin{aligned} R(u/v)K_-(u; a_-)_1 R(quv)K_-(v; a_-)_1 \\ = K_-(v; a_-)_1 R(quv)K_-(u; a_-)_1 R(u/v) \end{aligned} \tag{3.14a}$$

and the right reflection equation

$$\begin{aligned} R(u/v)K_+(u; a_+)_2 R(quv)K_+(v; a_+)_2 \\ = K_+(v; a_+)_2 R(quv)K_+(u; a_+)_2 R(u/v) \end{aligned} \tag{3.14b}$$

in $\mathbb{C}[u^{\pm 1}] \otimes M_4(\mathbb{C}) \subset \mathbb{C}[u^{\pm 1}] \otimes M_4(\mathbb{A}_m)$. Here $a_-, a_+ \in (-1, 1)$ encode two coupling parameters that govern the boundary interactions at the sites $l = 0$ and $l = m$, respectively.

From the boundary monodromy matrix

$$\begin{aligned} \mathcal{U}_m(u; a_-) &:= U_m(u)K_-(u; a_-)U_m^{-1}(q^{-1}u^{-1}) \\ &= \begin{pmatrix} \mathcal{A}_m(u; a_-) & \mathcal{B}_m(u; a_-) \\ \mathcal{C}_m(u; a_-) & \mathcal{D}_m(u; a_-) \end{pmatrix} \in \mathbb{C}[u^{\pm 1}] \otimes M_2(\mathbb{A}_m) \end{aligned} \tag{3.15}$$

solving the left reflection equation (3.14a) (with $\mathcal{U}_m(u; a_-)$ substituted for $K_-(u; a_-)$), one arrives at the boundary transfer operator

$$\begin{aligned} \mathcal{T}_m(u; a_+, a_-) &:= \text{tr} \left(K_+(u^{-1}; a_+) \mathcal{U}_m(u; a_-) \right) \\ &= f(u^{-1}; a_+) \mathcal{A}_m(u; a_-) + e(u^{-1}; a_+) \mathcal{D}_m(u; a_-) \end{aligned} \tag{3.16}$$

in $\mathbb{C}[u^{\pm 1}] \otimes \mathbb{A}_m$, which satisfies the commutativity

$$[\mathcal{T}_m(u; a_+, a_-), \mathcal{T}_m(v; a_+, a_-)] = 0 \tag{3.17}$$

(in $\mathbb{C}[u^{\pm 1}, v^{\pm 1}] \otimes \mathbb{A}_m$).

Remark 3.2. To confirm that the inverse $U_m^{-1}(u)$ of $U_m(u)$ (3.8) and thus $\mathcal{U}_m(u; a_-)$ (3.15) are indeed well-defined matrices in $\mathbb{C}[u^{\pm 1}] \otimes M_2(\mathbb{A}_m)$, it suffices to observe that following matrix in $\mathbb{C}[u^{\pm 1}] \otimes M_2(\mathbb{A}_m)$

$$\begin{pmatrix} u & (1-t^{-1})\beta_l^* \\ -\beta_l & u^{-1}t^{-1} \end{pmatrix} t^{-N_l} = \begin{pmatrix} 1 & 0 \\ 0 & -q \end{pmatrix} L_l(q^{-1}u^{-1}) \begin{pmatrix} 1 & 0 \\ 0 & -q \end{pmatrix}^{-1} q^{-1}t^{-N_l}$$

constitutes an inverse for the Lax matrix $L_l(u)$ (3.5).

4. Monodromy Matrix

This section isolates some salient features of the q -boson monodromy matrices $U_m(u)$ (3.8) and $\mathcal{U}_m(u; a)$ (3.15). (The reader might wish to skip this material at first reading and refer back to it when needed.)

4.1. Periodic monodromy matrix. Let the adjoint $*$: $\mathbb{A}_m \rightarrow \mathbb{A}_m$ be the antilinear anti-involution $a \xrightarrow{*} a^*$ ($a \in \mathbb{A}_m$) induced by the following action on the generators

$$\beta_l \xrightarrow{*} \beta_l^*, \quad \beta_l^* \xrightarrow{*} \beta_l, \quad t^{\pm N_l} \xrightarrow{*} t^{\pm N_l}, \tag{4.1a}$$

and let r : $\mathbb{A}_m \rightarrow \mathbb{A}_m$ denote the linear site-reversal involution $a \xrightarrow{r} a^r$ induced by the action

$$\beta_l \xrightarrow{r} \beta_{m-l}, \quad \beta_l^* \xrightarrow{r} \beta_{m-l}^*, \quad t^{\pm N_l} \xrightarrow{r} t^{\pm N_{m-l}} \tag{4.1b}$$

on the generators (a.k.a. space inversion). Both operations on \mathbb{A}_m are extended naturally to $\mathbb{C}[u^{\pm 1}] \otimes \mathbb{A}_m$ through the prescription $u^* := u =: u^r$ (so in particular, we treat our spectral parameter u here momentarily as a real-valued indeterminate). A further extension to matrices $M(u) \in \mathbb{C}[u^{\pm 1}] \otimes M_k(\mathbb{A}_m)$ is achieved element-wise, applied to the transposed matrix $M^t(u)$ in the case of the adjoint and to the matrix $M(u)$ itself in the case of the site-reversal involution.

Lemma 4.1. *The adjoint $U_m^*(u) = L_0^*(u)L_1^*(u) \cdots L_m^*(u)$ and the reverse $U_m^r(u) = L_0(u)L_1(u) \cdots L_m(u)$ of the periodic monodromy matrix $U_m(u)$ (3.8) are given by*

$$U_m^*(u) = \begin{pmatrix} A_m^*(u) & C_m^*(u) \\ B_m^*(u) & D_m^*(u) \end{pmatrix} = \begin{pmatrix} D_m(u^{-1}) & (1-t)^{-1}B_m(u^{-1}) \\ (1-t)C_m(u^{-1}) & A_m(u^{-1}) \end{pmatrix} \tag{4.2a}$$

and

$$U_m^r(u) = \begin{pmatrix} A_m^r(u) & B_m^r(u) \\ C_m^r(u) & D_m^r(u) \end{pmatrix} = \begin{pmatrix} D_m(u^{-1}) & B_m(u^{-1}) \\ C_m(u^{-1}) & A_m(u^{-1}) \end{pmatrix}. \tag{4.2b}$$

Proof. We employ induction in m starting from $m = 0$, in which case the lemma is immediate from the explicit expression for the Lax matrix in Eq. (3.5). For $m > 0$, the monodromy matrix $U_m(u)$ (3.8) satisfies the recurrence

$$\begin{aligned} \begin{pmatrix} A_m(u) & B_m(u) \\ C_m(u) & D_m(u) \end{pmatrix} &= \begin{pmatrix} u^{-1} & (1-t)\beta_m^* \\ \beta_m & u \end{pmatrix} \begin{pmatrix} A_{m-1}(u) & B_{m-1}(u) \\ C_{m-1}(u) & D_{m-1}(u) \end{pmatrix} \\ &= \begin{pmatrix} u^{-1}A_{m-1}(u) + (1-t)\beta_m^*C_{m-1}(u) & u^{-1}B_{m-1}(u) + (1-t)\beta_m^*D_{m-1}(u) \\ \beta_m A_{m-1}(u) + uC_{m-1}(u) & \beta_m B_{m-1}(u) + uD_{m-1}(u) \end{pmatrix}. \end{aligned} \tag{4.3}$$

To verify the first part of the lemma, we take the adjoint on both sides of Eq. (4.3), invoke the induction hypothesis to rewrite $A_{m-1}^*(u)$, $B_{m-1}^*(u)$, $C_{m-1}^*(u)$ and $D_{m-1}^*(u)$ in terms of $A_{m-1}(u^{-1})$, $B_{m-1}(u^{-1})$, $C_{m-1}(u^{-1})$ and $D_{m-1}(u^{-1})$, and commute on the RHS all generators corresponding site m to the left of the matrix elements (where one uses that generators corresponding to distinct sites commute in $\mathbb{C}[u^\pm] \otimes \mathbb{A}_m$ in view of the ultralocality). Eq. (4.2a) now follows upon comparing with the recurrence originating from Eq. (4.3) with u replaced by u^{-1} . The verification of the second part is similar and hinges on a comparison of the recurrence

$$\begin{aligned} \begin{pmatrix} A_m^r(u) & B_m^r(u) \\ C_m^r(u) & D_m^r(u) \end{pmatrix} &= \begin{pmatrix} A_{m-1}^r(u) & B_{m-1}^r(u) \\ C_{m-1}^r(u) & D_{m-1}^r(u) \end{pmatrix} \begin{pmatrix} u^{-1} & (1-t)\beta_m^* \\ \beta_m & u \end{pmatrix} \\ &= \begin{pmatrix} u^{-1}A_{m-1}^r(u) + \beta_m B_{m-1}^r(u) & (1-t)\beta_m^*A_{m-1}^r(u) + uB_{m-1}^r(u) \\ u^{-1}C_{m-1}^r(u) + \beta_m D_{m-1}^r(u) & (1-t)\beta_m^*C_{m-1}^r(u) + uD_{m-1}^r(u) \end{pmatrix} \end{aligned}$$

with that of Eq. (4.3) with u replaced by u^{-1} . (Here we mean by $A_{m-1}^r(u)$, $B_{m-1}^r(u)$, $C_{m-1}^r(u)$ and $D_{m-1}^r(u)$ the site-reversals of these matrix elements in $\mathbb{C}[u^{\pm 1}] \otimes \mathbb{A}_{m-1}$.) \square

By combining the above formulas for $U_m^*(u)$ and $U_m^r(u)$ with the inverse of the Lax matrix in Remark 3.2, one deduces a compact expression for the inverse of the periodic monodromy matrix.

Lemma 4.2. *The inverse of the periodic monodromy matrix $U_m(\cdot)$ (3.8) is given by*

$$U_m^{-1}(q^{-1}u^{-1}) = \begin{pmatrix} A_m^*(u) & s(q)C_m^*(u) \\ s(q)^{-1}B_m^*(u) & D_m^*(u) \end{pmatrix} \mathcal{N}_m^{-1}, \tag{4.4a}$$

where \mathcal{N}_m encodes a q -deformed number operator defined by

$$\mathcal{N}_m := \prod_{0 \leq l \leq m} qt^{N_l}. \tag{4.4b}$$

Proof. From Remark 3.2 it is immediate that

$$U_m^{-1}(q^{-1}u^{-1}) = \begin{pmatrix} 1 & 0 \\ 0 & -q \end{pmatrix} U_m^r(u) \begin{pmatrix} 1 & 0 \\ 0 & -q \end{pmatrix}^{-1} \mathcal{N}_m^{-1},$$

whence the lemma follows upon making $U_m^r(u)$ explicit by means of Lemma 4.1. \square

Finally, the particle-conservation of the periodic q -boson system reflects itself in the following commutation relations between the q -deformed number operator and the matrix elements of the periodic monodromy matrix $U_m(u)$ (3.8).

Lemma 4.3. *One has that*

$$\mathcal{N}_m \begin{pmatrix} A_m(u) & B_m(u) \\ C_m(u) & D_m(u) \end{pmatrix} = \begin{pmatrix} A_m(u) & t B_m(u) \\ t^{-1} C_m(u) & D_m(u) \end{pmatrix} \mathcal{N}_m \quad (4.5)$$

in $\mathbb{C}[u^{\pm 1}] \otimes M_2(\mathbb{A}_m)$.

Proof. By induction in m using Eq. (4.3) and the ultralocal q -boson commutation relations in Eq. (3.3a). \square

4.2. Boundary monodromy matrix. Upon plugging the inverse of Lemma 4.2 into the definition of the boundary monodromy matrix $\mathcal{U}_m(u; a)$ (3.15):

$$\mathcal{U}_m(u; a) = \begin{pmatrix} A_m(u) & B_m(u) \\ C_m(u) & D_m(u) \end{pmatrix} \begin{pmatrix} e(u; a) & 0 \\ 0 & f(u; a) \end{pmatrix} \begin{pmatrix} A_m^*(u) & s(q)C_m^*(u) \\ s(q)^{-1}B_m^*(u) & D_m^*(u) \end{pmatrix} \mathcal{N}_m^{-1}, \quad (4.6)$$

the following expressions for its matrix elements become evident (cf. also Lemma 4.1).

Lemma 4.4. *The matrix elements of the boundary monodromy matrix $\mathcal{U}_m(u; a)$ (3.15) are given by*

$$\begin{aligned} \mathcal{A}_m(u; a) &= \left(e(u; a)A_m(u)A_m^*(u) + s(q)^{-1}f(u; a)B_m(u)B_m^*(u) \right) \mathcal{N}_m^{-1} \\ &= \left(e(u; a)A_m(u)D_m(u^{-1}) - qf(u; a)B_m(u)C_m(u^{-1}) \right) \mathcal{N}_m^{-1}, \\ \mathcal{B}_m(u; a) &= \left(s(q)e(u; a)A_m(u)C_m^*(u) + f(u; a)B_m(u)D_m^*(u) \right) \mathcal{N}_m^{-1} \\ &= \left(-q^{-1}e(u; a)A_m(u)B_m(u^{-1}) + f(u; a)B_m(u)A_m(u^{-1}) \right) \mathcal{N}_m^{-1}, \\ \mathcal{C}_m(u; a) &= \left(e(u; a)C_m(u)A_m^*(u) + s(q)^{-1}f(u; a)D_m(u)B_m^*(u) \right) \mathcal{N}_m^{-1} \\ &= \left(e(u; a)C_m(u)D_m(u^{-1}) - qf(u; a)D_m(u)C_m(u^{-1}) \right) \mathcal{N}_m^{-1}, \\ \mathcal{D}_m(u; a) &= \left(s(q)e(u; a)C_m(u)C_m^*(u) + f(u; a)D_m(u)D_m^*(u) \right) \mathcal{N}_m^{-1} \\ &= \left(-q^{-1}e(u; a)C_m(u)B_m(u^{-1}) + f(u; a)D_m(u)A_m(u^{-1}) \right) \mathcal{N}_m^{-1}. \end{aligned}$$

With the aid of these expressions, we confirm the commutation relations between the q -deformed number operator \mathcal{N}_m (4.4b) and the matrix elements of the boundary monodromy matrix $\mathcal{U}_m(u)$ (3.15) (cf. Lemma 4.3).

Lemma 4.5. *One has that*

$$\mathcal{N}_m \begin{pmatrix} \mathcal{A}_m(u; a) & \mathcal{B}_m(u; a) \\ \mathcal{C}_m(u; a) & \mathcal{D}_m(u; a) \end{pmatrix} = \begin{pmatrix} \mathcal{A}_m(u; a) & t\mathcal{B}_m(u; a) \\ t^{-1}\mathcal{C}_m(u; a) & \mathcal{D}_m(u; a) \end{pmatrix} \mathcal{N}_m \quad (4.7)$$

in $\mathbb{C}[u^{\pm 1}] \otimes M_2(\mathbb{A}_m)$.

Proof. Immediate from Lemmas 4.3 and 4.4. \square

Finally, we compute the adjoint of the boundary monodromy matrix in terms of previously known matrix elements (cf. Lemma 4.1).

Lemma 4.6. *The adjoint of the boundary monodromy matrix $\mathcal{U}_m(u; a)$ (3.15) reads*

$$\mathcal{U}_m^*(u; a) = \begin{pmatrix} \mathcal{A}_m^*(u; a) & \mathcal{C}_m^*(u; a) \\ \mathcal{B}_m^*(u; a) & \mathcal{D}_m^*(u; a) \end{pmatrix} = \begin{pmatrix} \mathcal{A}_m(u; a) & s(q)^{-1}t^{-1}\mathcal{B}_m(u; a) \\ s(q)t\mathcal{C}_m(u; a) & \mathcal{D}_m(u; a) \end{pmatrix}. \tag{4.8}$$

Proof. Immediate from Lemmas 4.4 and 4.5. \square

5. Hamiltonian

In this section, we identify the q -boson Hamiltonian generated by our boundary transfer operator $\mathcal{T}_m(u; a_+, a_-)$ (3.16). It corresponds to an open q -boson system on \mathbb{N}_m (3.2) with integrable boundary interactions of a type considered in Ref. [LW] (on the finite lattice) and more generally in Ref. [DE3] (on the semi-infinite lattice).

5.1. Particle conservation, integrability and hermiticity. The commuting quantum integrals generated by our boundary transfer operator $\mathcal{T}_m(u; a_+, a_-)$ (3.16) are retrieved from its power expansion in the spectral parameter u . The simplest quantum integral corresponds to the q -deformed number operator \mathcal{N}_m (4.4b).

Proposition 5.1 (Particle Conservation). *The q -deformed number operator $\mathcal{N}_m = \prod_{l=0}^m qt^{N_l}$ commutes with the transfer operators $\mathcal{T}_m(u)$ (3.9) and $\mathcal{T}_m(u; a_+, a_-)$ (3.16):*

$$[\mathcal{N}_m, \mathcal{T}_m(u)] = 0 \quad \text{and} \quad [\mathcal{N}_m, \mathcal{T}_m(u; a_+, a_-)] = 0 \tag{5.1}$$

in $\mathbb{C}[u^{\pm 1}] \otimes \mathbb{A}_m$.

Proof. Immediate from Lemmas 4.3 and 4.5. \square

The next term in the expansion gives rise to the following q -boson Hamiltonian

$$\mathcal{H}_m := a_- \beta_0^* \beta_0 + a_+ \beta_m^* \beta_m + \sum_{0 \leq l < m} (\beta_{l+1}^* \beta_l + \beta_l^* \beta_{l+1}), \tag{5.2}$$

which is endowed with quadratic boundary interactions that can alternatively be rewritten in terms of local q -deformed number operators at the end-points through the relation $\beta_l^* \beta_l = [N_l]$ (cf. Eqs. (3.4a)–(3.4b)). Notice in this connection also that it is already manifest from the ultralocal q -boson commutation relations (3.3a) that \mathcal{H}_m (5.2) commutes in \mathbb{A}_m with the total q -deformed number operator \mathcal{N}_m (4.4b).

Proposition 5.2 (Quantum Integrability). *The q -boson Hamiltonian \mathcal{H}_m (5.2) commutes with the boundary transfer operator $\mathcal{T}_m(u; a_+, a_-)$ (3.16) in $\mathbb{C}[u^{\pm 1}] \otimes \mathbb{A}_m$:*

$$[\mathcal{H}_m, \mathcal{T}_m(u; a_+, a_-)] = 0. \tag{5.3}$$

Before entering the proof of this proposition (below), let us observe that it is clear from the explicit representations in terms of the generators that the q -deformed number operator \mathcal{N}_m (4.4b) and the Hamiltonian \mathcal{H}_m (5.2) are Hermitian elements in \mathbb{A}_m , i.e. $\mathcal{N}_m^* = \mathcal{N}_m$ and $\mathcal{H}_m^* = \mathcal{H}_m$. This hermiticity turns out to extend to all commuting quantum integrals generated by the boundary transfer operator.

Proposition 5.3 (Hermiticity). *The boundary transfer operator $\mathcal{T}_m(u; a_+, a_-)$ (3.16) constitutes a Hermitian Laurent polynomial in $\mathbb{C}[u^{\pm 1}] \otimes \mathbb{A}_m$:*

$$\mathcal{T}_m^*(u; a_+, a_-) = \mathcal{T}_m(u; a_+, a_-). \tag{5.4}$$

Proof. Immediate from Eq. (3.16) and Lemma 4.6. \square

5.2. *Proof of Proposition 5.2.* Our proof consists of showing that the boundary transfer operator is a Laurent polynomial in u of the form

$$\mathcal{T}_m(u; a_+, a_-) = \sum_{k=-m-2}^{m+2} \tau_{m,k}(a_+, a_-) u^{-2k} \quad (\tau_{m,k}(a_+, a_-) \in \mathbb{A}_m), \tag{5.5a}$$

with

$$\tau_{m,k}(a_+, a_-) = \begin{cases} q\mathcal{N}_m^{-1} & \text{if } k = m + 2, \\ ((1 - t)\mathcal{H}_m - a_+ - a_-)q\mathcal{N}_m^{-1} & \text{if } k = m + 1. \end{cases} \tag{5.5b}$$

Because the expansion coefficients $\tau_{m,k}(a_+, a_-)$ ($k = -m - 2, \dots, m + 2$) commute with $\mathcal{T}_m(u; a_+, a_-)$ in view of Eq. (3.17), this confirms that both \mathcal{N}_m and \mathcal{H}_m commute with $\mathcal{T}_m(u; a_+, a_-)$ (and among themselves).

To verify Eqs. (5.5a), (5.5b), we first compute the leading terms of the matrix elements of the periodic monodromy matrix $U_m(u)$ (3.8) (cf. also the proof of [Ko, Prp. 3.10] for the full power expansions of $u^{m+1}A_m(u)$ and $u^{m+1}D_m(u)$).

Lemma 5.4. *The matrix elements of $u^{m+1}U_m(u)$ are polynomials in the spectral parameter u of the form:*

$$\begin{aligned} u^{m+1}A_m(u) &= 1 + u^2(1 - t) \sum_{0 \leq l < m} \beta_l \beta_{l+1}^* + r_{m,A}(u), \\ u^{m+1}B_m(u) &= u(1 - t)\beta_0^* + r_{m,B}(u), \\ u^{m+1}C_m(u) &= u\beta_m + r_{m,C}(u), \\ u^{m+1}D_m(u) &= u^2(1 - t)\beta_m \beta_0^* + r_{m,D}(u), \end{aligned}$$

with $r_{m,A}(u), r_{m,D}(u) \in u^4\mathbb{C}[u^2] \otimes \mathbb{A}_m$ of degree $2m$ and $2m + 2$, respectively, and $r_{m,B}(u), r_{m,C}(u) \in u^3\mathbb{C}[u^2] \otimes \mathbb{A}_m$ of degree $2m + 1$.

Proof. For $m = 0$ this is clear from the Lax matrix $L_l(u)$ (3.5). The general case then follows inductively in m via Eq. (4.3). \square

Armed with these formulas, it is not difficult to infer that

$$\begin{aligned} f(u^{-1}; a_+)e(u; a_-)A_m(u)A_m^*(u) &= qu^{-2m-4} + qu^{-2m-2} \left((1 - t) \sum_{0 \leq l < m} (\beta_l \beta_{l+1}^* + \beta_l^* \beta_{l+1}) - a_- - t^{-1}a_+ \right) + O(u^{-2m}), \\ s(q)^{-1} f(u^{-1}; a_+)f(u; a_-)B_m(u)B_m^*(u) &= qu^{-2m-2}a_-(1 - t)[N_0] + O(u^{-2m}), \\ s(q)e(u^{-1}; a_+)e(u; a_-)C_m(u)C_m^*(u) &= qu^{-2m-2}a_+ \left((1 - t)[N_m] - 1 + t^{-1} \right) + O(u^{-2m}) \end{aligned}$$

and

$$e(u^{-1}; a_+) f(u; a_-) D_m(u) D_m^*(u) = O(u^{-2m}),$$

where $O(u^{-2m})$ stands for certain higher-degree terms in the expansions corresponding to the powers u^{2k} for $k = -m, -m + 1, \dots, m + 2$ (at most). When substituting these expansions in the boundary transfer operator

$$\begin{aligned} & \mathcal{T}_m(u; a_+, a_-) \\ & \stackrel{\text{Eq. (3.16)}}{=} f(u^{-1}; a_+) \mathcal{A}_m(u; a_-) + e(u^{-1}; a_+) \mathcal{D}_m(u; a_-) \\ & \stackrel{\text{Lem. 4.4}}{=} f(u^{-1}; a_+) \left(e(u; a_-) A_m(u) A_m^*(u) + s(q)^{-1} f(u; a_-) B_m(u) B_m^*(u) \right) \mathcal{N}_m^{-1} \\ & \quad + e(u^{-1}; a_+) \left(s(q) e(u; a_-) C_m(u) C_m^*(u) + f(u; a_-) D_m(u) D_m^*(u) \right) \mathcal{N}_m^{-1}, \end{aligned}$$

we end up with Eqs. (5.5a), (5.5b).

6. Fock Space Representation

The algebra \mathbb{A}_m will now be represented unitarily on the q -boson Fock space. In this representation, the actions of the boundary transfer operator $\mathcal{T}_m(u; a_+, a_-)$ (3.16) and the Hamiltonian \mathcal{H}_m (5.2) give rise to (essentially) self-adjoint operators in Fock space. Let us recall/emphasize in this connection that throughout

$$t = q^2, \quad 0 < q < 1 \quad \text{and} \quad -1 < a_+, a_- < 1. \tag{6.1}$$

6.1. q -Boson Fock space. For a positive integer n , we think of

$$\Lambda_{n,m} := \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid m \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0\} \tag{6.2}$$

as a q -boson configuration space with the parts of $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda_{n,m}$ encoding the positions of a configuration of n particles (q -bosons) placed on the finite lattice \mathbb{N}_m (3.2). Let us define

$$\mathcal{F}_{n,m} := \ell^2(\Lambda_{n,m}, \delta_{n,m}) \tag{6.3a}$$

as the (n -particle) Hilbert space of complex functions $f : \Lambda_{n,m} \rightarrow \mathbb{C}$, endowed with the inner product

$$(f, g)_{n,m} := \sum_{\lambda \in \Lambda_{n,m}} f(\lambda) \overline{g(\lambda)} \delta_{n,m}(\lambda) \tag{6.3b}$$

determined by the weight function

$$\delta_{n,m}(\lambda) := \frac{1}{\prod_{l \in \mathbb{N}_m} [m_l(\lambda)]!}. \tag{6.3c}$$

Here $m_l(\lambda)$ is given by the number of parts λ_j ($1 \leq j \leq n$) such that $\lambda_j = l$, i.e. $m_l(\lambda)$ counts the multiplicity of (the number of particles at site) l in (the configuration) λ , and we have also employed q -deformed integers and q -deformed factorials of the form

$$[k] := \frac{1 - t^k}{1 - t} \quad \text{and} \quad [k]! := [k][k - 1] \dots [2][1] = \prod_{1 \leq j \leq k} \frac{1 - t^j}{1 - t}$$

for k nonnegative integral (with the convention that $[0]! := 1$). Notice that our Hilbert space $\mathcal{F}_{n,m}$ (6.3a)–(6.3c) is actually finite-dimensional:

$$\dim(\mathcal{F}_{n,m}) = \frac{(n+m)!}{n!m!} \tag{6.4}$$

(= the cardinality of $\Lambda_{n,m}$ (6.2)).

By collecting the n -particle Hilbert spaces $\mathcal{F}_{n,m}$ for $n = 0, 1, 2, \dots$ (with the convention that $\Lambda_{0,m} := \{\emptyset\}$ and $\delta_{0,m}(\emptyset) := 1$, so $\mathcal{F}_{0,m} \cong \mathbb{C}$) via a direct orthogonal sum, one ends up with the following (infinite-dimensional) q -boson Fock space:

$$\mathcal{F}_m := \bigoplus_{n \geq 0} \mathcal{F}_{n,m}, \tag{6.5a}$$

consisting of all series $F = \sum_{n \geq 0} f_n$ of $f_n \in \mathcal{F}_{n,m}$ such that

$$(F, F)_m := \sum_{n \geq 0} (f_n, f_n)_{n,m} < \infty. \tag{6.5b}$$

6.2. Unitary representation of the q -boson algebra. For $\lambda \in \Lambda_{n,m}$ and $l \in \mathbb{N}_m$, let $\beta_l^* \lambda \in \Lambda_{n+1,m}$ be obtained from λ by inserting an additional part of size l and—assuming $m_l(\lambda) > 0$ —let $\beta_l \lambda \in \Lambda_{n-1,m}$ be the result of the inverse operation removing a part of size l from λ .

We are now in the position to represent the q -boson algebra \mathbb{A}_m on \mathcal{F}_m (6.5a), (6.5b) by specifying the action of the generators. For any $f \in \mathcal{F}_{n,m}$ and $l \in \mathbb{N}_m$, the functions $\beta_l f \in \mathcal{F}_{n-1,m}$, $\beta_l^* f \in \mathcal{F}_{n+1,m}$ and $t^{\pm N_l} f \in \mathcal{F}_{n,m}$ are defined as:

$$(\beta_l f)(\lambda) := f(\beta_l^* \lambda) \quad \text{if } n > 0 \tag{6.6a}$$

($\lambda \in \Lambda_{n-1,m}$) and $\beta_l f = 0$ if $n = 0$ (where we have used the convention that $\mathcal{F}_{-1,m} := \{0\}$),

$$(\beta_l^* f)(\lambda) := \begin{cases} [m_l(\lambda)] f(\beta_l \lambda) & \text{if } m_l(\lambda) > 0 \\ 0 & \text{otherwise} \end{cases} \tag{6.6b}$$

($\lambda \in \Lambda_{n+1,m}$), and

$$(t^{\pm N_l} f)(\lambda) := t^{\pm m_l(\lambda)} f(\lambda) \tag{6.6c}$$

($\lambda \in \Lambda_{n,m}$). Indeed, it is readily verified that the operations in question satisfy the ultralocal q -boson algebra relations in Eqs. (3.3a), (3.3b). Moreover, the representation at issue is unitary (i.e. it preserves the $*$ -structure):

$$(\beta_l f, g)_{n,m} = (f, \beta_l^* g)_{n+1,m} \quad (f \in \mathcal{F}_{n+1,m}, g \in \mathcal{F}_{n,m}) \tag{6.7a}$$

(where one exploits that $\delta_{n+1,m}(\beta_l^* \lambda) = \delta_{n,m}(\lambda) / [m_l(\lambda) + 1]$) and

$$(t^{\pm N_l} f, g)_{n,m} = (f, t^{\pm N_l} g)_{n,m} \quad (f, g \in \mathcal{F}_{n,m}). \tag{6.7b}$$

It is clear from these definitions that the q -boson annihilation-, creation-, and q -deformed number operators at site $l \in \mathbb{N}_m$ are bounded on \mathcal{F}_m :

$$\begin{aligned} (\beta_l f, \beta_l f)_{n-1,m} &\leq (1-t)^{-1} (f, f)_{n,m}, \\ (\beta_l^* f, \beta_l^* f)_{n+1,m} &\leq (1-t)^{-1} (f, f)_{n,m}, \\ (t^{N_l} f, t^{N_l} f)_{n,m} &\leq (f, f)_{n,m} \end{aligned} \tag{6.8}$$

for any $f \in \mathcal{F}_{n,m}$.

Remark 6.1. If we denote the characteristic function in $\mathcal{F}_{n,m}$ supported on $\lambda \in \Lambda_{n,m}$ by $|\lambda\rangle$, then the functions $|\lambda\rangle, \lambda \in \Lambda_{n,m}$ constitute a standard basis for $\mathcal{F}_{n,m}$ satisfying the orthogonality:

$$\langle \mu | \lambda \rangle := (|\lambda\rangle, |\mu\rangle)_{n,m} = \begin{cases} \delta_{n,m}(\lambda) & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu \end{cases}$$

($\lambda, \mu \in \Lambda_{n,m}$). The state $|\lambda\rangle$ represents a configuration of n q -bosons on \mathbb{N}_m (with $m_l(\lambda)$ particles occupying the site $l \in \mathbb{N}_m$). The action of the q -boson generators on this standard basis reads:

$$\beta_l |\lambda\rangle = \begin{cases} |\beta_l \lambda\rangle & \text{if } m_l(\lambda) > 0 \\ 0 & \text{otherwise} \end{cases}, \quad \beta_l^* |\lambda\rangle = [m_l(\lambda) + 1] |\beta_l^* \lambda\rangle, \quad t^{\pm N_l} |\lambda\rangle = t^{\pm m_l(\lambda)} |\lambda\rangle.$$

In this picture, the vacuum sector $\mathcal{F}_{0,m}$ amounts to $\mathbb{C}|\emptyset\rangle$ with $|\emptyset\rangle$ playing the role of the vacuum state.

6.3. n -Particle hamiltonian. Let $\mathcal{D}_m \subset \mathcal{F}_m$ denote the dense subspace of *terminating series* $F = \sum_{n \geq 0} f_n$ of $f_n \in \mathcal{F}_{n,m}$ ($n = 0, 1, 2, \dots$). The boundary transfer operator $\mathcal{T}_m(u; a_+, a_-)$ (3.16) (with $u \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$), the number operator \mathcal{N}_m (4.4b), and the Hamiltonian \mathcal{H}_m (5.2) are all mapped to essentially self-adjoint operators on $\mathcal{D}_m \subset \mathcal{F}_m$ by the representation of the q -boson algebra \mathbb{A}_m in Eqs. (6.6a)–(6.6c). The finite-dimensional n -particle sector $\mathcal{F}_{n,m}$ (6.3a)–(6.3c) constitutes a stable subspace for these operators. In particular, one has that

$$\mathcal{N}_m f = q^{m+1} t^n f \quad \text{for all } f \in \mathcal{F}_{n,m}. \tag{6.9}$$

Proposition 6.2 (Self-adjointness). *On the dense domain $\mathcal{D}_m \subset \mathcal{F}_m$, the action of the boundary transfer operator $\mathcal{T}_m(u; a_+, a_-)$ (3.16), with $u \in \mathbb{R}^*$, is essentially self-adjoint in the q -boson Fock space \mathcal{F}_m (6.5a), (6.5b). Moreover, the finite-dimensional n -particle sector $\mathcal{F}_{n,m} \subset \mathcal{D}_m$ constitutes an invariant subspace for this action.*

Proof. Since the n -particle sector $\mathcal{F}_{n,m}$ constitutes an eigenspace with eigenvalue $q^{m+1} t^n$ for the action of \mathcal{N}_m (4.4b), it follows from Propositions 5.1 and 5.3 that the action of $\mathcal{T}_m(u; a_+, a_-)$ restricts to a self-adjoint operator in the invariant subspace $\mathcal{F}_{n,m}$. This means that the action of the boundary transfer operator gives rise to a symmetric operator on \mathcal{D}_m . Moreover, as for $z \in \mathbb{C} \setminus \mathbb{R}$ the range $(\mathcal{T}_m(u; a_+, a_-) - z)\mathcal{D}_m$ is equal to \mathcal{D}_m (because $(\mathcal{T}_m(u; a_+, a_-) - z)\mathcal{F}_{n,m} = \mathcal{F}_{n,m}$), the deficiency indices for the action of the boundary transfer operator on \mathcal{D}_m vanish in \mathcal{F}_m , i.e. the symmetric operator under consideration is essentially self-adjoint. \square

As the value of the spectral parameter $u \in \mathbb{R}^*$ was arbitrary, it follows from Proposition 6.2 that the actions on \mathcal{D}_m of all commuting quantum integrals $\tau_{m,k}(a_+, a_-)$ (5.5a), (5.5b) generated by the boundary transfer operator are essentially self-adjoint in \mathcal{F}_m and leave the n -particle sector $\mathcal{F}_{n,m}$ invariant. This holds thus in particular for the Hamiltonian \mathcal{H}_m (5.2) (which is actually seen to extend to a bounded self-adjoint operator on \mathcal{F}_m in view of Eq. (6.8)). The following proposition makes the action of \mathcal{H}_m in $\mathcal{F}_{n,m}$ explicit.

Proposition 6.3 (n -Particle Hamiltonian). *For any $f \in \mathcal{F}_{n,m}$ (6.3a)–(6.3c) and $\lambda \in \Lambda_{n,m}$ (6.2), one has that*

$$\begin{aligned}
 (\mathcal{H}_m f)(\lambda) &= \left(a_- [m_0(\lambda)] + a_+ [m_m(\lambda)] \right) f(\lambda) \\
 &+ \sum_{\substack{1 \leq j \leq n \\ \lambda + e_j \in \Lambda_{n,m}}} [m_{\lambda_j}(\lambda)] f(\lambda + e_j) + \sum_{\substack{1 \leq j \leq n \\ \lambda - e_j \in \Lambda_{n,m}}} [m_{\lambda_j}(\lambda)] f(\lambda - e_j),
 \end{aligned}
 \tag{6.10}$$

where e_1, \dots, e_n stand for the unit vectors of the standard basis in \mathbb{Z}^n .

Proof. It is clear from the definitions that $([N_0]f)(\lambda) = [m_0(\lambda)]f(\lambda)$ and $([N_m]f)(\lambda) = [m_m(\lambda)]f(\lambda)$. Moreover, for any $0 \leq l < m$:

$$(\beta_{l+1} \beta_l^* f)(\lambda) = \begin{cases} [m_l(\lambda)] f(\beta_{l+1}^* \beta_l \lambda) & \text{if } m_l(\lambda) > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\beta_{l+1}^* \beta_l \lambda = \lambda + e_j$ with $j = \min\{k \mid \lambda_k = l\}$ (so $l = \lambda_j$), and

$$(\beta_{l+1}^* \beta_l f)(\lambda) = \begin{cases} [m_{l+1}(\lambda)] f(\beta_{l+1} \beta_l^* \lambda) & \text{if } m_{l+1}(\lambda) > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\beta_{l+1} \beta_l^* \lambda = \lambda - e_j$ with $j = \max\{k \mid \lambda_k = l + 1\}$ (so $l = \lambda_j - 1$). \square

7. Action of the Boundary Transfer Operator in Fock Space

In this section, the action of the boundary transfer operator in the q -boson Fock space is calculated explicitly. To this end, we first determine the corresponding actions of the matrix elements of the q -boson monodromy matrices.

7.1. Action of the periodic monodromy matrix elements. The explicit actions in Fock space of the elements of the periodic q -boson monodromy matrix were established in [Ko, Prp. 4.1], by viewing the matrix elements in question as partition functions for an associated statistical vertex model.

To describe the result, some further notation is needed. For $\lambda \in \Lambda_{n,m}$ and $\mu \in \Lambda_{n,m} \cup \Lambda_{n-1,m}$ ($n \geq 1$), let us write $\mu \leq \lambda$ if $\mu_j \leq \lambda_j$ for all j and the skew diagram λ/μ is a horizontal strip:

$$m \geq \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \dots \geq \mu_{n-1} \geq \lambda_n \geq \begin{cases} \mu_n \geq 0 & \text{if } \mu \in \Lambda_{n,m}, \\ 0 & \text{if } \mu \in \Lambda_{n-1,m}. \end{cases}
 \tag{7.1}$$

In this situation, we define

$$\varphi_{\lambda/\mu}(t) := \prod_{\substack{0 \leq l \leq m \\ m_l(\lambda) = m_l(\mu) + 1}} (1 - t^{m_l(\lambda)})
 \tag{7.2a}$$

and

$$\psi_{\lambda/\mu}(t) := \prod_{\substack{0 \leq l \leq m \\ m_l(\lambda) = m_l(\mu) - 1}} (1 - t^{m_l(\mu)}).
 \tag{7.2b}$$

The latter two combinatorial quantities go back to [MI, Ch. III.5], with the caveat that here we consider (shifted) partitions with possibly ‘parts of size zero’ whose multiplicity

is taken into account (cf. also Refs. [B] and [WZ] for a similar state of affairs). Finally, we also employ the standard abbreviation for the total number of boxes

$$|\lambda| := \lambda_1 + \lambda_2 + \dots + \lambda_n \quad (\lambda \in \Lambda_{n,m}).$$

Proposition 7.1 (Periodic Monodromy Matrix in Fock Space [Ko, Prp. 4.1]). *For $u \in \mathbb{C}^*$ and $n \geq 1$, the action of the periodic monodromy matrix elements on $f \in \mathcal{F}_{n,m}$ is given by*

$$(A_m(u)f)(\lambda) = u^{-m-1} \sum_{\substack{\mu \in \Lambda_{n,m} \\ \mu \leq \lambda}} u^{2(|\lambda|-|\mu|)} \varphi_{\lambda/\mu}(t) f(\mu) \quad (\lambda \in \Lambda_{n,m}), \quad (7.3a)$$

$$(B_m(u)f)(\lambda) = u^{-m} \sum_{\substack{\mu \in \Lambda_{n,m} \\ \mu \leq \lambda}} u^{2(|\lambda|-|\mu|)} \varphi_{\lambda/\mu}(t) f(\mu) \quad (\lambda \in \Lambda_{n+1,m}), \quad (7.3b)$$

$$(C_m(u)f)(\lambda) = u^m \sum_{\substack{\mu \in \Lambda_{n,m} \\ \lambda \leq \mu}} u^{2(|\lambda|-|\mu|)} \psi_{\mu/\lambda}(t) f(\mu) \quad (\lambda \in \Lambda_{n-1,m}), \quad (7.3c)$$

$$(D_m(u)f)(\lambda) = u^{m+1} \sum_{\substack{\mu \in \Lambda_{n,m} \\ \lambda \leq \mu}} u^{2(|\lambda|-|\mu|)} \psi_{\mu/\lambda}(t) f(\mu) \quad (\lambda \in \Lambda_{n,m}). \quad (7.3d)$$

For the reader’s convenience, the end of this section (below) includes an alternative direct proof of these formulas that is inductive in the number of sites and avoids the digression into vertex models altogether.

7.2. Action of the boundary monodromy matrix elements. By combining Lemma 4.4 and Proposition 7.1, one arrives at the action of the elements of the boundary monodromy matrix in the q -boson Fock space.

To describe these actions, it is convenient to introduce the following three relations on our partitions. Firstly, for $\lambda \in \Lambda_{n,m}$ and $\mu \in \Lambda_{n-1,m}$, we write that $\mu \leq \lambda$ if there exists a $\nu \in \Lambda_{n,m} \cup \Lambda_{n-1,m}$ such that $\mu \leq \nu \leq \lambda$. Secondly, for $\lambda, \mu \in \Lambda_{n,m}$, we write that $\mu \sim_- \lambda$ if there exists a $\nu \in \Lambda_{n,m} \cup \Lambda_{n-1,m}$ such that $\nu \leq \lambda$ and $\nu \leq \mu$, and we write $\mu \sim_+ \lambda$ if there exists a $\nu \in \Lambda_{n,m} \cup \Lambda_{n+1,m}$ such that $\lambda \leq \nu$ and $\mu \leq \nu$.

Proposition 7.2 (Boundary Monodromy Matrix in Fock Space). *For $u \in \mathbb{C}^*$ and $n \geq 1$, the action of the boundary monodromy matrix elements on $f \in \mathcal{F}_{n,m}$ is given by*

$$(\mathcal{A}_m(u; a)f)(\lambda) = q^{-m-1} t^{-n} u^{-1} \sum_{\substack{\mu \in \Lambda_{n,m} \\ \mu \sim_- \lambda}} A_{\lambda,\mu}^{(n)}(u^2; t, a) f(\mu) \quad (\lambda \in \Lambda_{n,m}), \quad (7.4a)$$

$$(\mathcal{B}_m(u; a)f)(\lambda) = q^{-m} t^{-n-1} \sum_{\substack{\mu \in \Lambda_{n,m} \\ \mu \leq \lambda}} B_{\lambda,\mu}^{(n)}(u^2; t, a) f(\mu) \quad (\lambda \in \Lambda_{n+1,m}), \quad (7.4b)$$

$$(\mathcal{C}_m(u; a)f)(\lambda) = q^{-m-1} t^{-n} \sum_{\substack{\mu \in \Lambda_{n,m} \\ \lambda \leq \mu}} C_{\lambda,\mu}^{(n)}(u^2; t, a) f(\mu) \quad (\lambda \in \Lambda_{n-1,m}), \quad (7.4c)$$

$$(\mathcal{D}_m(u; a)f)(\lambda) = q^{-m} t^{-n-1} u \sum_{\substack{\mu \in \Lambda_{n,m} \\ \mu \sim_+ \lambda}} D_{\lambda,\mu}^{(n)}(u^2; t, a) f(\mu) \quad (\lambda \in \Lambda_{n,m}), \quad (7.4d)$$

where

$$A_{\lambda,\mu}^{(n)}(z; t, a) := z^{-m}(a - z^{-1}) \sum_{\substack{v \in \Lambda_{n,m} \\ v \leq \lambda, v \leq \mu}} \varphi_{\lambda/v}(t) \psi_{\mu/v}(t) z^{|\lambda|+|\mu|-2|v|} \tag{7.5a}$$

$$+ z^{-m}(tz - a) \sum_{\substack{v \in \Lambda_{n-1,m} \\ v \leq \lambda, v \leq \mu}} \varphi_{\lambda/v}(t) \psi_{\mu/v}(t) z^{|\lambda|+|\mu|-2|v|},$$

$$B_{\lambda,\mu}^{(n)}(z; t, a) := (z^{-1} - a) \sum_{\substack{v \in \Lambda_{n+1,m} \\ \mu \leq v \leq \lambda}} \varphi_{\lambda/v}(t) \varphi_{v/\mu}(t) z^{|\lambda|+|\mu|-2|v|} \tag{7.5b}$$

$$+ (a - tz) \sum_{\substack{v \in \Lambda_{n,m} \\ \mu \leq v \leq \lambda}} \varphi_{\lambda/v}(t) \varphi_{v/\mu}(t) z^{|\lambda|+|\mu|-2|v|},$$

$$C_{\lambda,\mu}^{(n)}(z; t, a) := (a - z^{-1}) \sum_{\substack{v \in \Lambda_{n,m} \\ \lambda \leq v \leq \mu}} \psi_{v/\lambda}(t) \psi_{\mu/v}(t) z^{|\lambda|+|\mu|-2|v|} \tag{7.5c}$$

$$+ (tz - a) \sum_{\substack{v \in \Lambda_{n-1,m} \\ \lambda \leq v \leq \mu}} \psi_{v/\lambda}(t) \psi_{\mu/v}(t) z^{|\lambda|+|\mu|-2|v|},$$

$$D_{\lambda,\mu}^{(n)}(z; t, a) := z^m(z^{-1} - a) \sum_{\substack{v \in \Lambda_{n+1,m} \\ \lambda \leq v, \mu \leq v}} \psi_{v/\lambda}(t) \varphi_{v/\mu}(t) z^{|\lambda|+|\mu|-2|v|} \tag{7.5d}$$

$$+ z^m(a - tz) \sum_{\substack{v \in \Lambda_{n,m} \\ \lambda \leq v, \mu \leq v}} \psi_{v/\lambda}(t) \varphi_{v/\mu}(t) z^{|\lambda|+|\mu|-2|v|}.$$

Proof. We will compute the action of $B_m(u; a)$. The actions of the other three matrix elements follow analogously by combining Lemma 4.4 and Proposition 7.1. Specifically, upon composing the actions in Eqs. (7.3a) and (7.3b), it is seen that for any $f \in \mathcal{F}_{n,m}$ and $\lambda \in \Lambda_{n+1,m}$:

$$(A_m(u)B_m(u^{-1})f)(\lambda) = u^{-1} \sum_{\substack{v \in \Lambda_{n+1,m} \\ \mu \in \Lambda_{n,m} \\ \mu \leq v \leq \lambda}} u^{2(|\lambda|+|\mu|-2|v|)} \varphi_{\lambda/v}(t) \varphi_{v/\mu}(t) f(\mu)$$

and

$$(B_m(u)A_m(u^{-1})f)(\lambda) = u \sum_{\substack{v \in \Lambda_{n,m} \\ \mu \in \Lambda_{n,m} \\ \mu \leq v \leq \lambda}} u^{2(|\lambda|+|\mu|-2|v|)} \varphi_{\lambda/v}(t) \varphi_{v/\mu}(t) f(\mu).$$

Armed with these two formulas, the asserted action follows upon acting on f with the expression for $B_m(u; a)$ in Lemma 4.4. \square

7.3. Action of the boundary transfer operator. From Proposition 7.2, the following explicit formula for the action of the boundary transfer operator $\mathcal{T}_m(u; a_+, a_-)$ (3.16) in the q -boson Fock space is immediate.

Theorem 7.3 (Boundary Transfer Operator in Fock Space). *For $u \in \mathbb{C}^*$ and $n \geq 1$, the action of the boundary transfer operator on $f \in \mathcal{F}_{n,m}$ reads:*

$$\begin{aligned}
 (\mathcal{T}_m(u; a_+, a_-)f)(\lambda) &= q^{-m}t^{-n-1}(a_+ - tu^{-2}) \sum_{\substack{\mu \in \Lambda_{n,m} \\ \mu \sim -\lambda}} A_{\lambda,\mu}^{(n)}(u^2; t, a_-)f(\mu) \\
 &\quad + q^{-m}t^{-n-1}(a_+ - u^2) \sum_{\substack{\mu \in \Lambda_{n,m} \\ \mu \sim +\lambda}} D_{\lambda,\mu}^{(n)}(u^2; t, a_-)f(\mu),
 \end{aligned}$$

($\lambda \in \Lambda_{n,m}$), where the coefficients $A_{\lambda,\mu}^{(n)}(z; t, a)$ and $D_{\lambda,\mu}^{(n)}(z; t, a)$ are given by Eqs. (7.5a) and (7.5d), respectively.

7.4. Proof of Proposition 7.1. We will first verify the formulas for the actions of $A_m(u)$ and $C_m(u)$ by induction in m with the aid of the recursion in Eq. (4.3). Next, by computing the adjoints of these actions in \mathcal{F}_m , one recovers the actions of $B_m(u)$ and $D_m(u)$ upon recalling that $B_m(u) = (1 - t)C_m(u^{-1})^*$ and $D_m(u) = A_m(u^{-1})^*$ for $u \in \mathbb{R}^*$ (by Lemma 4.1 and the unitarity of the representation of \mathbb{A}_m in \mathcal{F}_m).

For $m = 0$, one has that $\mathbb{N}_m = \mathbb{N}_0 = \{0\}$, $\Lambda_{n,m} = \Lambda_{n,0} = \{0^n\}$, $A_m(u) = A_0(u) = u^{-1}$ and $C_m(u) = C_0(u) = \beta_0$ (cf. Eq. (3.5)). Indeed, in this trivial situation the asserted formulas correctly reproduce that

$$(A_0(u)f)(0^n) = u^{-1}\varphi_{0^n/0^n}(t)u^{2(|0^n|-|0^n|)}f(0^n) = u^{-1}f(0^n)$$

and that

$$(C_0(u)f)(0^{n-1}) = \psi_{0^n/0^{n-1}}(t)u^{2(|0^n|-|0^{n-1}|)}f(0^n) = f(0^n) = (\beta_0f)(0^{n-1}).$$

For $m > 0$, we read-off from Eq. (4.3) that

$$A_m(u) = u^{-1}A_{m-1}(u) + (1 - t)\beta_m^*C_{m-1}(u), \quad C_m(u) = uC_{m-1}(u) + \beta_m A_{m-1}(u),$$

which enables to compute the action of $A_m(u)$ and $C_m(u)$ by means of the induction hypothesis (IH). Indeed, given $f \in \mathcal{F}_{n,m}$ and $\lambda \in \Lambda_{n,m}$ with $r = m_m(\lambda)$, we have that

$$\begin{aligned}
 (A_{m-1}(u)f)(\lambda) &= (A_{m-1}(u)\beta_m^r f)(\beta_m^r \lambda) \\
 &\stackrel{(IH)}{=} u^{-m} \sum_{\substack{\mu \in \Lambda_{n-r,m-1} \\ \mu \leq \beta_m^r \lambda}} \varphi_{\beta_m^r \lambda/\mu}(t)u^{2(|\beta_m^r \lambda|-|\mu|)}(\beta_m^r f)(\mu) \\
 &= u^{-m} \sum_{\substack{\mu \in \Lambda_{n,m}, \mu \leq \lambda \\ m_m(\mu) = m_m(\lambda)}} \varphi_{\lambda/\mu}(t)u^{2(|\lambda|-|\mu|)}f(\mu)
 \end{aligned}$$

and (if $r > 0$)

$$\begin{aligned}
 (\beta_m^* C_{m-1}(u)f)(\lambda) &= [\mathfrak{m}_m(\lambda)](C_{m-1}(u)f)(\beta_m \lambda) = [\mathfrak{m}_m(\lambda)](C_{m-1}(u)\beta_m^{r-1}f)(\beta_m^r \lambda) \\
 &\stackrel{(IH)}{=} u^{m-1}[\mathfrak{m}_m(\lambda)] \sum_{\substack{\mu \in \Lambda_{n+1-r, m-1} \\ \beta_m^r \lambda \leq \mu}} \psi_{\mu/\beta_m^r \lambda}(t) u^{2(|\beta_m^r \lambda| - |\mu|)} (\beta_m^{r-1}f)(\mu) \\
 &= u^{m-1}[\mathfrak{m}_m(\lambda)] \sum_{\substack{\mu \in \Lambda_{n, m}, \mu \leq \lambda \\ \mathfrak{m}_m(\mu) = \mathfrak{m}_m(\lambda) - 1}} \psi_{\beta_m^{r-1} \mu/\beta_m^r \lambda}(t) u^{2(|\lambda| - |\mu|)} f(\mu) \\
 &= u^{m-1}(1-t)^{-1} \sum_{\substack{\mu \in \Lambda_{n, m}, \mu \leq \lambda \\ \mathfrak{m}_m(\mu) = \mathfrak{m}_m(\lambda) - 1}} \varphi_{\lambda/\mu}(t) u^{2(|\lambda| - |\mu|)} f(\mu)
 \end{aligned}$$

(while $(\beta_m^* C_{m-1}(u)f)(\lambda) = 0$ if $r = 0$), whence

$$(A_m(u)f)(\lambda) = u^{-m-1} \sum_{\mu \in \Lambda_{n, m}, \mu \leq \lambda} \varphi_{\lambda/\mu}(t) u^{2(|\lambda| - |\mu|)} f(\mu)$$

as claimed. Similarly, we see that for $f \in \mathcal{F}_{n, m}$ and $\lambda \in \Lambda_{n-1, m}$ with $r = \mathfrak{m}_m(\lambda)$:

$$\begin{aligned}
 (C_{m-1}(u)f)(\lambda) &= (C_{m-1}(u)\beta_m^r f)(\beta_m^r \lambda) \\
 &\stackrel{(IH)}{=} u^{m-1} \sum_{\substack{\mu \in \Lambda_{n-r, m-1} \\ \beta_m^r \lambda \leq \mu}} \psi_{\mu/\beta_m^r \lambda}(t) u^{2(|\beta_m^r \lambda| - |\mu|)} (\beta_m^r f)(\mu) \\
 &= u^{m-1} \sum_{\substack{\mu \in \Lambda_{n, m}, \lambda \leq \mu \\ \mathfrak{m}_m(\mu) = \mathfrak{m}_m(\lambda)}} \psi_{\mu/\lambda}(t) u^{2(|\lambda| - |\mu|)} f(\mu)
 \end{aligned}$$

and

$$\begin{aligned}
 (\beta_m A_{m-1}(u)f)(\lambda) &= (A_{m-1}(u)f)(\beta_m^* \lambda) = (A_{m-1}(u)\beta_m^{r+1}f)(\beta_m^r \lambda) \\
 &\stackrel{(IH)}{=} u^{-m} \sum_{\substack{\mu \in \Lambda_{n-r-1, m-1} \\ \mu \leq \beta_m^r \lambda}} \varphi_{\beta_m^r \lambda/\mu}(t) u^{2(|\beta_m^r \lambda| - |\mu|)} (\beta_m^{r+1}f)(\mu) \\
 &= u^m \sum_{\substack{\mu \in \Lambda_{n, m}, \lambda \leq \mu \\ \mathfrak{m}_m(\mu) = \mathfrak{m}_m(\lambda) + 1}} \psi_{\mu/\lambda}(t) u^{2(|\lambda| - |\mu|)} f(\mu),
 \end{aligned}$$

whence

$$(C_m(u)f)(\lambda) = u^m \sum_{\mu \in \Lambda_{n, m}, \lambda \leq \mu} u^{2(|\lambda| - |\mu|)} \psi_{\mu/\lambda}(t) f(\mu)$$

as claimed.

The computation of the adjoints hinges on the following two elementary identities:

$$\varphi_{\lambda/\mu}(t) \delta_{n+1, m}(\lambda) = (1-t) \psi_{\lambda/\mu}(t) \delta_{n, m}(\mu) \quad (\lambda \in \Lambda_{n+1, m}, \mu \in \Lambda_{n, m}), \quad (7.6a)$$

$$\varphi_{\lambda/\mu}(t) \delta_{n, m}(\lambda) = \psi_{\lambda/\mu}(t) \delta_{n, m}(\mu) \quad (\lambda, \mu \in \Lambda_{n, m}). \quad (7.6b)$$

Indeed, for any $f \in \mathcal{F}_{n,m}$, $g \in \mathcal{F}_{n+1,m}$ and $u \in \mathbb{R}^*$ one has that:

$$\begin{aligned} (B_m(u)f, g)_{n+1,m} &= (f, (1-t)C_m(u^{-1})g)_{n,m} \\ &= (1-t)u^{-m} \sum_{\mu \in \Lambda_{n,m}} \left(\delta_{n,m}(\mu) f(\mu) \sum_{\substack{\lambda \in \Lambda_{n+1,m} \\ \mu \leq \lambda}} \psi_{\lambda/\mu}(t) u^{2(|\lambda|-|\mu|)} \overline{g(\lambda)} \right) \\ &\stackrel{\text{Eq.(7.6a)}}{=} u^{-m} \sum_{\lambda \in \Lambda_{n+1,m}} \left(\delta_{n+1,m}(\lambda) \overline{g(\lambda)} \sum_{\substack{\mu \in \Lambda_{n,m} \\ \mu \leq \lambda}} \varphi_{\lambda/\mu}(t) u^{2(|\lambda|-|\mu|)} f(\mu) \right), \end{aligned}$$

whence

$$(B_m(u)f)(\lambda) = u^{-m} \sum_{\substack{\mu \in \Lambda_{n,m} \\ \mu \leq \lambda}} \varphi_{\lambda/\mu}(t) u^{2(|\lambda|-|\mu|)} f(\mu)$$

as claimed. Similarly, for any $f, g \in \mathcal{F}_{n,m}$ and $u \in \mathbb{R}^*$ one has that:

$$\begin{aligned} (D_m(u)f, g)_{n,m} &= (f, A_m(u^{-1})g)_{n,m} \\ &= u^{m+1} \sum_{\mu \in \Lambda_{n,m}} \left(\delta_{n,m}(\mu) f(\mu) \sum_{\substack{\lambda \in \Lambda_{n,m} \\ \lambda \leq \mu}} \varphi_{\mu/\lambda}(t) u^{2(|\lambda|-|\mu|)} \overline{g(\lambda)} \right) \\ &\stackrel{\text{Eq.(7.6b)}}{=} u^{m+1} \sum_{\lambda \in \Lambda_{n,m}} \left(\delta_{n,m}(\lambda) \overline{g(\lambda)} \sum_{\substack{\mu \in \Lambda_{n,m} \\ \lambda \leq \mu}} \psi_{\mu/\lambda}(t) u^{2(|\lambda|-|\mu|)} f(\mu) \right), \end{aligned}$$

whence

$$(D_m(u)f)(\lambda) = u^{m+1} \sum_{\substack{\mu \in \Lambda_{n,m} \\ \lambda \leq \mu}} \psi_{\mu/\lambda}(t) u^{2(|\lambda|-|\mu|)} f(\mu)$$

as claimed.

8. Algebraic Bethe Ansatz

In this section we construct eigenfunctions for the boundary transfer operator in the q -boson Fock space by means of Sklyanin’s extension [S] of the algebraic Bethe Ansatz formalism [T,KB1,JM,F] allowing for open-end boundary interactions.

8.1. Bethe Ansatz eigenfunction. It is clear from Lemma 4.3 and confirmed explicitly by Proposition 7.2 that the upper diagonal matrix element $\mathcal{B}_m(u; a)$ of the boundary monodromy matrix acts as a particle-creation operator in \mathcal{F}_m . For our purposes it turns out convenient to modify the normalization of this Bethe Ansatz creation operator somewhat:

$$\hat{\mathcal{B}}_m(u; a) := b(u)^{-1} \mathcal{B}_m(u; a) \mathcal{N}_m \quad \text{with} \quad b(u) := qs(q)s(qu^2). \tag{8.1}$$

The following proposition summarizes the main features of the normalized Bethe Ansatz creation operator.

Proposition 8.1 (Bethe Ansatz Creation Operator). *The operator $\hat{\mathcal{B}}_m(u; a)$ (8.1) enjoys the following properties:*

- (i) Particle creation operator: $\mathcal{N}_m \hat{\mathcal{B}}_m(u; a) = t \hat{\mathcal{B}}_m(u; a) \mathcal{N}_m$,
- (ii) Commuting family: $[\hat{\mathcal{B}}_m(u; a), \hat{\mathcal{B}}_m(v; a)] = 0$ in $\mathbb{C}[u^{\pm 1}, v^{\pm 1}] \otimes \mathbb{A}_m$,
- (iii) Reflection symmetry: $\hat{\mathcal{B}}_m(u^{-1}; a) = \hat{\mathcal{B}}_m(u; a)$,
- (iv) Even Laurent polynomial: $\hat{\mathcal{B}}_m(u; a) \in \mathbb{C}[u^{\pm 2}] \otimes \mathbb{A}_m$.

Proof. Parts (i) and (ii) are immediate from the commutation relations in Lemmas 4.3 and B.2. For part (iii), it suffices to verify that $b(u^{-1})\mathcal{B}_m(u; a)\mathcal{N}_m = b(u)\mathcal{B}_m(u^{-1}; a)\mathcal{N}_m$. With the aid of Lemma 4.4, the LHS is rewritten as

$$b(u^{-1})\left(-q^{-1}e(u; a)A_m(u)B_m(u^{-1}) + f(u; a)B_m(u)A_m(u^{-1})\right)$$

and the RHS is rewritten as

$$b(u)\left(-q^{-1}e(u^{-1}; a)A_m(u^{-1})B_m(u) + f(u^{-1}; a)B_m(u^{-1})A_m(u)\right).$$

After further rewriting of the RHS by means of the relations (cf. Lemma B.1)

$$A_m(u^{-1})B_m(u) = \frac{s(q^{-1})}{s(q^{-1}u^{-2})}A_m(u)B_m(u^{-1}) + \frac{qs(u^{-2})}{s(q^{-1}u^{-2})}B_m(u)A_m(u^{-1})$$

and

$$B_m(u^{-1})A_m(u) = \frac{q^{-1}s(u^{-2})}{s(q^{-1}u^{-2})}A_m(u)B_m(u^{-1}) + \frac{s(q^{-1})}{s(q^{-1}u^{-2})}B_m(u)A_m(u^{-1}),$$

one readily infers the desired equality upon comparing the coefficients of the terms involving $A_m(u)B_m(u^{-1})$ and $B_m(u)A_m(u^{-1})$ on both sides. Finally, for part (iv) let us recall that it is clear from the definitions that $\mathcal{B}_m(u; a) \in \mathbb{C}[u^{\pm 1}] \otimes \mathbb{A}_m$. To see that $\hat{\mathcal{B}}_m(u; a)$ (8.1) is also a Laurent polynomial in u , one invokes the reflection-symmetry of part (iii) to conclude that the apparent poles at $qu^2 = \pm 1$ (arising from the normalization) are cancelled. The fact that the operator in question is actually even in u is seen from Lemmas 4.4 and 5.4. \square

After these preparations, we are now in the position to define the following n -particle Bethe Ansatz wave function:

$$\Psi_{(v_1, \dots, v_n)} := \hat{\mathcal{B}}_m(v_1; a_-) \cdots \hat{\mathcal{B}}_m(v_n; a_-)|\emptyset\rangle \in \mathcal{F}_{n,m}, \tag{8.2}$$

obtained by acting with Bethe Ansatz creation operators on the vacuum state specified in Remark 6.1. Here (and below), it is assumed that the spectral variables v_1, \dots, v_n take values in $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. It is immediate from Proposition 8.1 that $\Psi_{(v_1, \dots, v_n)}$ (8.2) constitutes an even Laurent polynomial in v_1, \dots, v_n that is invariant with respect to the action of the hyperoctahedral group by permutations and reflections $v_j \rightarrow v_j^{-1}$ of these spectral variables. The main result of this section states that the n -particle Bethe Ansatz wave function (8.2) satisfies the eigenvalue equation for the boundary transfer operator $\mathcal{T}_m(u; a_+, a_-)$ (3.16) in $\mathcal{F}_{n,m}$, provided the spectral variables satisfy a corresponding algebraic system of Bethe Ansatz equations.

Theorem 8.2 (Bethe Ansatz Eigenfunction). *Let $u \in \mathbb{R}^* \setminus \{1, -1\}$ and $(v_1, \dots, v_n) \in (\mathbb{C}^*)^n$ be generic in the sense that $u^2 v_j^{\pm 2} \neq 1$, $v_j^2 \neq a_{\pm}$, and $v_j^2 v_k^{\pm 2} \neq t$ ($1 \leq j \neq k \leq n$). The n -particle Bethe Ansatz wave function $\Psi_{(v_1, \dots, v_n)}$ (8.2) solves the eigenvalue equation*

$$\mathcal{T}_m(u; a_+, a_-)\Psi_{(v_1, \dots, v_n)} = E_{n,m}(u; v_1, \dots, v_n)\Psi_{(v_1, \dots, v_n)} \tag{8.3a}$$

with eigenvalue

$$\begin{aligned} E_{n,m}(u; v_1, \dots, v_n) & \tag{8.3b} \\ := q^{-m-1} & \left(u^{-2m-2} \frac{s(q^{-1}u^2)}{s(u^2)} e(u; a_+)e(u; a_-) \prod_{1 \leq j \leq n} \frac{s(quv_j)s(quv_j^{-1})}{s(uv_j)s(uv_j^{-1})} \right. \\ & \left. + u^{2m+2} \frac{s(q^{-1}u^{-2})}{s(u^{-2})} e(u^{-1}; a_+)e(u^{-1}; a_-) \prod_{1 \leq j \leq n} \frac{s(qu^{-1}v_j)s(qu^{-1}v_j^{-1})}{s(u^{-1}v_j)s(u^{-1}v_j^{-1})} \right), \end{aligned}$$

provided the spectral variables v_1, \dots, v_n satisfy the following algebraic system of Bethe Ansatz equations:

$$v_j^{4m+4} = \frac{e(v_j; a_+)e(v_j; a_-)}{e(v_j^{-1}; a_+)e(v_j^{-1}; a_-)} \prod_{\substack{1 \leq k \leq n \\ k \neq j}} \frac{s(qv_j v_k)s(qv_j v_k^{-1})}{s(q^{-1}v_j v_k)s(q^{-1}v_j v_k^{-1})}, \tag{8.3c}$$

for $j = 1, 2, \dots, n$.

The eigenvalue $E_{n,m}(u; v_1, \dots, v_n)$ (8.3b) is given by a rational expression in u that becomes a Laurent polynomial in u when the Bethe Ansatz equations (8.3c) are satisfied. Its expansion around $u = 0$ is of the form

$$\begin{aligned} E_{n,m}(u; v_1, \dots, v_n) & = q^{-m} t^{-n} u^{-2m-4} \tag{8.4a} \\ & + q^{-m} t^{-n} \left((1-t)E_n(v_1, \dots, v_n) - a_+ - a_- \right) u^{-2m-2} + O(u^{-2m}), \end{aligned}$$

where

$$E_n(v_1, \dots, v_n) := \sum_{1 \leq j \leq n} v_j^2 + v_j^{-2}. \tag{8.4b}$$

By comparing with the expansion of the boundary transfer operator in Eqs. (5.5a), (5.5b), we read-off that the eigenvalue of the q -boson Hamiltonian \mathcal{H}_m (5.2) on our n -particle Bethe Ansatz eigenfunction is given by $E_n(v_1, \dots, v_n)$ (8.4b).

Corollary 8.3 (Eigenvalue of the Hamiltonian). *For $(v_1, \dots, v_n) \in (\mathbb{C}^*)^n$ generic in the sense that $v_j^2 \neq a_{\pm}$ and $v_j^2 v_k^{\pm 2} \neq t$ ($1 \leq j \neq k \leq n$), the n -particle Bethe Ansatz wave function $\Psi_{(v_1, \dots, v_n)}$ (8.2) solves the eigenvalue equation*

$$\mathcal{H}_m \Psi_{(v_1, \dots, v_n)} = E_n(v_1, \dots, v_n)\Psi_{(v_1, \dots, v_n)} \tag{8.5}$$

for the q -boson Hamiltonian \mathcal{H}_m (5.2), provided the spectral variables v_1, \dots, v_n satisfy the algebraic system of Bethe Ansatz equations in Eq. (8.3c).

8.2. *Proof of Theorem 8.2.* We first compute the action of the lower triangular matrix elements of the monodromy matrices on the vacuum state.

Lemma 8.4. *For $u \in \mathbb{C}^*$, one has that*

$$A_m(u)|\emptyset\rangle = u^{-m-1}|\emptyset\rangle, \quad C_m(u)|\emptyset\rangle = 0, \quad D_m(u)|\emptyset\rangle = u^{m+1}|\emptyset\rangle. \quad (8.6)$$

Proof. For $m = 0$ these actions are immediate from the Lax matrix (3.5) and the fact that $|\emptyset\rangle$ is a vacuum state annihilated by the q -boson operators β_l , $l = 0, \dots, m$ (cf. Remark 6.1). The case of general $m > 0$ follows inductively via the recurrence in Eq. (4.3) (using also the ultralocality). \square

In the case of the boundary monodromy matrix, it turns out to be convenient to employ the modified matrix element [S] (cf. also Appendix B below):

$$\hat{\mathcal{D}}_m(u; a) := \mathcal{D}_m(u; a) + \frac{s(q)}{s(u^2)}\mathcal{A}_m(u; a). \quad (8.7)$$

Lemma 8.5. *For $u \in \mathbb{C}^*$, one has that*

$$\mathcal{A}_m(u; a)|\emptyset\rangle = \alpha_m(u)|\emptyset\rangle, \quad C_m(u; a)|\emptyset\rangle = 0, \quad \hat{\mathcal{D}}_m(u; a)|\emptyset\rangle = \delta_m(u)|\emptyset\rangle, \quad (8.8)$$

with

$$\alpha_m(u) := q^{-m-1}u^{-2m-2}e(u; a) \quad \text{and} \quad \delta_m(u) := q^{-m-1}u^{2m+2}\frac{s(qu^2)}{s(u^2)}e(u^{-1}; a).$$

Proof. The stated actions are readily computed with the aid of Lemmas 4.4 and 8.4. For $\mathcal{A}_m(u; a)$ and $C_m(u; a)$ the computations in question are straightforward, whereas for $\hat{\mathcal{D}}_m(u; a)$ one relies on Lemma B.1 in the form of the identity

$$C_m(u)B_m(u^{-1}) = tB_m(u^{-1})C_m(u) + \frac{1-t}{s(u^2)}\left(A_m(u^{-1})D_m(u) - A_m(u)D_m(u^{-1})\right)$$

to deduce that

$$C_m(u)B_m(u^{-1})|\emptyset\rangle = (1-t)\frac{s(u^{2m+2})}{s(u^2)}|\emptyset\rangle.$$

The latter formula renders the computation of the action of $\mathcal{D}_m(u; a)$ and thus that of $\hat{\mathcal{D}}_m(u; a)$ straightforward, similar to the two previous cases. \square

One learns from these lemmas in particular that $|\emptyset\rangle$ constitutes a vacuum state for the Bethe Ansatz annihilation operators $C_m(u)$ and $C_m(u; a)$ (cf. Lemmas 4.3, 4.5 and Propositions 7.1, 7.2). We now extend the above actions of $\mathcal{A}_m(u; a)$ and $\hat{\mathcal{D}}_m(u; a)$ to arbitrary (unnormalized) n -particle Bethe Ansatz wave functions of the form

$$\psi_{(v_1, \dots, v_n)} := \mathcal{B}_m(v_1; a) \cdots \mathcal{B}_m(v_n; a)|\emptyset\rangle. \quad (8.9)$$

Proposition 8.6. For $u \in \mathbb{C}^*$ and $(v_1, \dots, v_n) \in (\mathbb{C}^*)^n$ generic, in the sense that $u^2 \neq \pm 1$, $v_j^2 \neq \pm 1$, $u^2 v_j^{\pm 2} \neq 1$ and $v_j^2 v_k^{\pm 2} \neq 1$ ($1 \leq j \neq k \leq n$), one has that

$$\begin{aligned} \mathcal{A}_m(u; a)\psi_{(v_1, \dots, v_n)} &= a_n(u; v_1, \dots, v_n)\psi_{(v_1, \dots, v_n)} \\ &\quad + \sum_{j=1}^n a_{n,j}(u; v_1, \dots, v_n)\psi_{(v_1, \dots, v_{j-1}, u, v_{j+1}, \dots, v_n)} \end{aligned} \tag{8.10a}$$

and

$$\begin{aligned} \hat{\mathcal{D}}_m(u; a)\psi_{(v_1, \dots, v_n)} &= d_n(u; v_1, \dots, v_n)\psi_{(v_1, \dots, v_n)} \\ &\quad + \sum_{j=1}^n d_{n,j}(u; v_1, \dots, v_n)\psi_{(v_1, \dots, v_{j-1}, u, v_{j+1}, \dots, v_n)}, \end{aligned} \tag{8.10b}$$

with

$$a_n(u; v_1, \dots, v_n) = \alpha_m(u) \prod_{k=1}^n f_1(u, v_k), \tag{8.11a}$$

$$d_n(u; v_1, \dots, v_n) = \delta_m(u) \prod_{k=1}^n g_1(u, v_k), \tag{8.11b}$$

and

$$a_{n,j}(u; v_1, \dots, v_n) \tag{8.12a}$$

$$= \alpha_m(v_j) f_2(u, v_j) \prod_{\substack{1 \leq k \leq n \\ k \neq j}} f_1(v_j, v_k) + \delta_m(v_j) f_3(u, v_j) \prod_{\substack{1 \leq k \leq n \\ k \neq j}} g_1(v_j, v_k),$$

$$d_{n,j}(u; v_1, \dots, v_n) \tag{8.12b}$$

$$= \delta_m(v_j) g_2(u, v_j) \prod_{\substack{1 \leq k \leq n \\ k \neq j}} g_1(v_j, v_k) + \alpha_m(v_j) g_3(u, v_j) \prod_{\substack{1 \leq k \leq n \\ k \neq j}} f_1(v_j, v_k).$$

Here α_m and δ_m denote the Laurent polynomials defined in Lemma 8.4, while f_r and g_r ($r = 1, 2, 3$) refer to the rational functions defined in Lemma B.3.

Proof. For $n = 0$ the proposition reduces to Lemma 8.5. To deal with the case $n > 0$, we proceed by induction. From the relations in Lemma B.3 it is seen that

$$\begin{aligned} \mathcal{A}_m(u; a)\psi_{(v_1, \dots, v_n)} &\equiv \mathcal{A}_m(u; a)\mathcal{B}(v_n; a)\psi_{(v_1, \dots, v_{n-1})} \\ &= f_1(u, v_n)\mathcal{B}_m(v_n; a)\mathcal{A}_m(u; a)\psi_{(v_1, \dots, v_{n-1})} \\ &\quad + f_2(u, v_n)\mathcal{B}_m(u; a)\mathcal{A}_m(v_n; a)\psi_{(v_1, \dots, v_{n-1})} \\ &\quad + f_3(u, v_n)\mathcal{B}_m(u; a)\hat{\mathcal{D}}_m(v_n; a)\psi_{(v_1, \dots, v_{n-1})} \end{aligned}$$

and that

$$\begin{aligned} \hat{\mathcal{D}}_m(u; a)\psi_{(v_1, \dots, v_n)} &\equiv \hat{\mathcal{D}}_m(u; a)\mathcal{B}(v_n; a)\psi_{(v_1, \dots, v_{n-1})} \\ &= g_1(u, v_n)\mathcal{B}_m(v_n; a)\hat{\mathcal{D}}_m(u; a)\psi_{(v_1, \dots, v_{n-1})} \\ &\quad + g_2(u, v_n)\mathcal{B}_m(u; a)\hat{\mathcal{D}}_m(v_n; a)\psi_{(v_1, \dots, v_{n-1})} \\ &\quad + g_3(u, v_n)\mathcal{B}_m(u; a)\mathcal{A}_m(v_n; a)\psi_{(v_1, \dots, v_{n-1})}. \end{aligned}$$

When working out the actions of the operators on the RHS with the aid of the induction hypothesis, we end up with an expansion for the LHS of the form in Eqs. (8.10a), (8.10b), respectively, with

$$\begin{aligned} a_n(u; v_1, \dots, v_n) &= f_1(u, v_n)a_{n-1}(u; v_1, \dots, v_{n-1}), \\ d_n(u; v_1, \dots, v_n) &= g_1(u, v_n)d_{n-1}(u; v_1, \dots, v_{n-1}), \end{aligned}$$

and

$$\begin{aligned} a_{n,j}(u; v_1, \dots, v_n) &= f_1(u, v_n) \left(\alpha_m(v_j) f_2(u, v_j) \prod_{\substack{1 \leq k < n \\ k \neq j}} f_1(v_j, v_k) + \delta_m(v_j) f_3(u, v_j) \prod_{\substack{1 \leq k < n \\ k \neq j}} g_1(v_j, v_k) \right) \\ &+ f_2(u, v_n) \left(\alpha_m(v_j) f_2(v_n, v_j) \prod_{\substack{1 \leq k < n \\ k \neq j}} f_1(v_j, v_k) + \delta_m(v_j) f_3(v_n, v_j) \prod_{\substack{1 \leq k < n \\ k \neq j}} g_1(v_j, v_k) \right) \\ &+ f_3(u, v_n) \left(\alpha_m(v_j) g_3(v_n, v_j) \prod_{\substack{1 \leq k < n \\ k \neq j}} f_1(v_j, v_k) + \delta_m(v_j) g_2(v_n, v_j) \prod_{\substack{1 \leq k < n \\ k \neq j}} g_1(v_j, v_k) \right), \\ d_{n,j}(u; v_1, \dots, v_n) &= g_1(u, v_n) \left(\delta_m(v_j) g_2(u, v_j) \prod_{\substack{1 \leq k < n \\ k \neq j}} g_1(v_j, v_k) + \alpha_m(v_j) g_3(u, v_j) \prod_{\substack{1 \leq k < n \\ k \neq j}} f_1(v_j, v_k) \right) \\ &+ g_2(u, v_n) \left(\delta_m(v_j) g_2(v_n, v_j) \prod_{\substack{1 \leq k < n \\ k \neq j}} g_1(v_j, v_k) + \alpha_m(v_j) g_3(v_n, v_j) \prod_{\substack{1 \leq k < n \\ k \neq j}} f_1(v_j, v_k) \right) \\ &+ g_3(u, v_n) \left(\delta_m(v_j) f_3(v_n, v_j) \prod_{\substack{1 \leq k < n \\ k \neq j}} g_1(v_j, v_k) + \alpha_m(v_j) f_2(v_n, v_j) \prod_{\substack{1 \leq k < n \\ k \neq j}} f_1(v_j, v_k) \right), \end{aligned}$$

for $j = 1, \dots, n$. The proposition now follows by virtue of the following elementary rational identities:

$$\begin{aligned} f_1(u, v) f_2(u, w) + f_2(u, v) f_2(v, w) + f_3(u, v) g_3(v, w) &= f_2(u, w) f_1(w, v), \\ f_1(u, v) f_3(u, w) + f_2(u, v) f_3(v, w) + f_3(u, v) g_2(v, w) &= f_3(u, w) g_1(w, v) \end{aligned}$$

and

$$\begin{aligned} g_1(u, v) g_2(u, w) + g_2(u, v) g_2(v, w) + g_3(u, v) f_3(v, w) &= g_2(u, w) g_1(w, v), \\ g_1(u, v) g_3(u, w) + g_2(u, v) g_3(v, w) + g_3(u, v) f_2(v, w) &= g_3(u, w) f_1(w, v). \end{aligned}$$

□

Let us next rewrite the boundary transfer operator $\mathcal{T}_m(u; a_+, a_-)$ (3.16) in terms of the modified matrix element $\hat{\mathcal{D}}_m(u; a)$ (8.7):

$$\mathcal{T}_m(u; a_+, a_-) = \frac{s(q^{-1}u^2)}{s(u^2)} e(u; a_+) \mathcal{A}_m(u; a_-) + e(u^{-1}; a_+) \hat{\mathcal{D}}_m(u; a_-). \quad (8.13)$$

We temporarily assume that the domain restrictions and the genericity assumptions of Theorem 8.2 and Proposition 8.6 are simultaneously satisfied. From the proposition

it is then immediate that $\psi_{(v_1, \dots, v_n)}$ (with $a = a_-$)—and thus also $\Psi_{(v_1, \dots, v_n)}$ (8.2)—constitutes an eigenfunction of $\mathcal{T}_m(u; a_+, a_-)$ with eigenvalue

$$E_{n,m}(u; v_1, \dots, v_n) = \frac{s(q^{-1}u^2)}{s(u^2)} e(u; a_+) a_n(u; v_1, \dots, v_n) + e(u^{-1}; a_+) d_n(u; v_1, \dots, v_n) \tag{8.14a}$$

provided

$$\frac{s(q^{-1}u^2)}{s(u^2)} e(u; a_+) a_{n,j}(u; v_1, \dots, v_n) + e(u^{-1}; a_+) d_{n,j}(u; v_1, \dots, v_n) = 0, \tag{8.14b}$$

for $j = 1, \dots, n$. The expressions for the eigenvalue and the Bethe Ansatz equations formulated in the theorem are now readily recovered upon making $a_n, a_{n,j}$ and $d_n, d_{n,j}$ explicit (by means of the definitions detailed in Proposition 8.6). Finally, we relax our genericity assumptions to those of Theorem 8.2 by recalling that $\mathcal{T}_m(u; a_+, a_-)$ (3.16) and $\Psi_{(v_1, \dots, v_n)}$ (8.2) are Laurent polynomials in u and v_1, \dots, v_n , respectively.

9. Branching Rule

In this section, we employ the Bethe Ansatz creation operator $\hat{\mathcal{B}}_m(u; a)$ (8.1) to derive a branching rule that permits to compute the n -particle Bethe Ansatz wave function $\Psi_{(v_1, \dots, v_n)}$ (8.2) inductively in the number of particles/variables. This explicit construction confirms that our Bethe Ansatz wave function is not identically zero on $\Lambda_{n,m}$ (6.2), and as such constitutes a genuine nontrivial eigenfunction of the q -boson boundary transfer operator in $\mathcal{F}_{n,m}$ (if the spectral variables satisfy the Bethe Ansatz equations).

9.1. Bethe Ansatz wave function. By evaluating the action of the Bethe-Ansatz creation operator on the vacuum state, one arrives at a closed expression for the one-particle Bethe Ansatz wave function $\Psi_v = \hat{\mathcal{B}}_m(v; a_-)|\emptyset\rangle$ in terms of Chebyshev polynomials.

Proposition 9.1 (One-Particle Bethe Ansatz Wave Function). *For $v \in \mathbb{C}^*$, the value of the one-particle Bethe Ansatz wave function $\Psi_v = \hat{\mathcal{B}}_m(v; a_-)|\emptyset\rangle$ at $l \in \mathbb{N}_m$ (4.4b) is given by*

$$\Psi_v(l) = \mathbf{s}_l(v^2) - a_- \mathbf{s}_{l-1}(v^2) \quad \text{with} \quad \mathbf{s}_l(z) := \frac{z^{l+1} - z^{-l-1}}{z - z^{-1}} \tag{9.1}$$

(where $\mathbf{s}_{-1}(z) = 0$ by convention).

Before verifying this explicit formula for the one-particle Bethe Ansatz wave function below, let us first elucidate how one can iteratively augment the number of particles by means of a branching rule originating from the Bethe Ansatz creation operator.

Theorem 9.2 (Branching Rule). *For $n > 1$ and $(v_1, \dots, v_n) \in (\mathbb{C}^*)^n$ generic such that $tv_n^4 \neq 1$, the value of the n -particle Bethe Ansatz wave function $\Psi_{(v_1, \dots, v_n)}$ (8.2) at $\lambda \in \Lambda_{n,m}$ is determined by the following branching rule:*

$$\Psi_{(v_1, \dots, v_n)}(\lambda) = \sum_{\substack{\mu \in \Lambda_{n-1,m} \\ \mu \leq \lambda}} \hat{\mathbf{B}}_{\lambda/\mu}^{(n)}(v_n^2; t, a_-) \Psi_{(v_1, \dots, v_{n-1})}(\mu), \tag{9.2a}$$

with

$$\hat{B}_{\lambda/\mu}^{(n)}(z; t, a) := \frac{(z^{-1} - a)}{(1 - t)(z^{-1} - tz)} \sum_{\substack{v \in \Lambda_{n,m} \\ \mu \leq v \leq \lambda}} \varphi_{\lambda/v}(t) \varphi_{v/\mu}(t) z^{|\lambda|+|\mu|-2|v|} \tag{9.2b}$$

$$+ \frac{(a - tz)}{(1 - t)(z^{-1} - tz)} \sum_{\substack{v \in \Lambda_{n-1,m} \\ \mu \leq v \leq \lambda}} \varphi_{\lambda/v}(t) \varphi_{v/\mu}(t) z^{|\lambda|+|\mu|-2|v|}.$$

Proof. The asserted branching rule follows by computing the explicit action of the Bethe Ansatz creation operator $\hat{B}_m(u; a)$ (8.1) on the Bethe Ansatz wave function in $\mathcal{F}_{n-1,m}$ with the aid of Proposition 7.2:

$$\begin{aligned} \Psi_{(v_1, \dots, v_n)}(\lambda) &= \left(\hat{B}_m(v_n; a_-) \Psi_{(v_1, \dots, v_{n-1})} \right)(\lambda) \\ &= \sum_{\substack{\mu \in \Lambda_{n-1,m} \\ \mu \leq \lambda}} \hat{B}_{\lambda/\mu}^{(n)}(v_n^2; t; a_-) \Psi_{(v_1, \dots, v_{n-1})}(\mu) \end{aligned}$$

($\lambda \in \Lambda_{n,m}$, $n > 1$), with $\hat{B}_{\lambda/\mu}^{(n)}(u^2; t, a) = B_{\lambda,\mu}^{(n-1)}(u^2; t, a) b(u)^{-1} q^{-1}$ (cf. Eqs. (7.4b), (7.5b) and (8.1)). \square

By iterating the above branching rule, one ends up with a closed expression for the n -particle Bethe Ansatz wave function in terms one-particle wave functions taken from Proposition 9.1.

Corollary 9.3 (*n*-Particle Bethe Ansatz Wave Function). *For $n \geq 1$ and $(v_1, \dots, v_n) \in (\mathbb{C}^*)^n$ generic such that $tv_j^4 \neq 1$ ($1 < j \leq n$), one has that at $\lambda \in \Lambda_{n,m}$*

$$\Psi_{(v_1, \dots, v_n)}(\lambda) = \sum_{\substack{\mu^{(j)} \in \Lambda_{j,m}, j=1, \dots, n \\ \mu^{(1)} \leq \mu^{(2)} \leq \dots \leq \mu^{(n)} = \lambda}} \Psi_{v_1}(\mu^{(1)}) \prod_{1 < j \leq n} \hat{B}_{\mu^{(j)}/\mu^{(j-1)}}^{(j)}(v_j^2; t, a_-).$$

The value for the n -particle Bethe Ansatz wave function becomes particularly simple at the origin.

Corollary 9.4 (Evaluation at the Origin). *For $n \geq 1$ and $(v_1, \dots, v_n) \in (\mathbb{C}^*)^n$, the value of the n -particle Bethe Ansatz wave function at $\lambda = 0^n := \underbrace{(0, \dots, 0)}_{n \text{ times}} \in \Lambda_{n,m}$ is given*

by

$$\Psi_{(v_1, \dots, v_n)}(0^n) = [n]!. \tag{9.3}$$

Proof. For $n = 1$, Eq. (9.3) is immediate from Proposition 9.1. For $n > 1$, we deduce via the branching rule in Eqs. (9.2a), (9.2b) that

$$\Psi_{(v_1, \dots, v_n)}(0^n) = \hat{B}_{0^n/0^{n-1}}^{(n)}(v_n^2; t, a_-) \Psi_{(v_1, \dots, v_{n-1})}(0^{n-1})$$

with

$$\hat{B}_{0^n/0^{n-1}}^{(n)}(v_n^2; t, a) = \frac{(v_n^{-2} - a)(1 - t^n)}{(1 - t)(v_n^{-2} - tv_n^2)} + \frac{(a - tv_n^2)(1 - t^n)}{(1 - t)(v_n^{-2} - tv_n^2)} = \frac{1 - t^n}{1 - t} = [n]$$

(since $\varphi_{0^n/0^{n-1}}(t) = 1 - t^n$), whence $\Psi_{(v_1, \dots, v_n)}(0^n) = [n][n - 1] \cdots [2][1] = [n]!$. \square

The latter corollary reveals in particular that $\Psi_{(v_1, \dots, v_n)}$ (8.2) is a nontrivial n -particle wave function in $\mathcal{F}_{n,m}$, which confirms that the algebraic Bethe Ansatz formalism provides us in Theorem 8.2 with a genuine (nonzero) eigenfunction of the q -boson boundary transfer operator $\mathcal{T}_m(u; a_+, a_-)$ (upon solving the Bethe Ansatz equations). In the remainder of this section, the formula for the one-particle wave function in Proposition 9.1 is verified.

9.2. *Proof of Proposition 9.1.* First we compute the action of the Bethe Ansatz creation operator on the vacuum state in the periodic setting.

Lemma 9.5. *For $u \in \mathbb{C}^*$ and $l \in \mathbb{N}_m$, one has that*

$$(B_m(u)|\emptyset\rangle)(l) = (1 - t)u^{2l-m}. \tag{9.4}$$

Proof. The case $m = 0$ is immediate from the Lax matrix (3.5) and the action of the q -boson creation operators β_l^* , $l = 0, \dots, m$ on the vacuum state $|\emptyset\rangle$ (cf. Remark 6.1). The case of general $m > 0$ then follows inductively via the recurrence in Eq. (4.3) and Lemma 8.4. \square

The computation of the action of the Bethe Ansatz creation operator $\hat{\mathcal{B}}_m(u; a)$ (8.1) on the vacuum state now hinges on Lemma 4.4. Indeed, it is seen from Lemmas 8.4 and 9.5 that for $u \in \mathbb{C}^*$ and $l \in \mathbb{N}_m$:

$$(B_m(u)A_m(u^{-1})|\emptyset\rangle)(l) = (1 - t)u^{2l+1} \tag{9.5a}$$

and

$$(A_m(u)B_m(u^{-1})|\emptyset\rangle)(l) = (1 - t)u^{-2l-1} + (1 - t)^2 u s_{l-1}(u^2), \tag{9.5b}$$

where in the latter case we relied on Eq (B.2c) of Lemma B.1 in the form of the identity

$$A_m(u)B_m(u^{-1}) = \frac{qs(q)}{s(u^{-2})} B_m(u)A_m(u^{-1}) + \frac{qs(qu^2)}{s(u^2)} B_m(u^{-1})A_m(u)$$

to reduce it to the former case. From Eqs. (9.5a), (9.5b) and Lemma 4.4, it then follows that $(\mathcal{B}_m(u; a)|\emptyset\rangle)(l) = q^{-m-1}b(u) (s_l(u^2) - a s_{l-1}(u^2))$, whence

$$(\hat{\mathcal{B}}_m(u; a)|\emptyset\rangle)(l) = s_l(u^2) - a s_{l-1}(u^2). \tag{9.6}$$

10. Orthogonality and Completeness

In this section we first implement the method of Yang and Yang [YY] providing the solutions of the Bethe Ansatz equations via the minima of a family of strictly convex Morse functions. Next, the self-adjointness of the boundary transfer operator is exploited to deduce that the corresponding Bethe Ansatz eigenfunctions at these spectral values constitute an orthogonal basis for the q -boson Fock space.

10.1. *Solutions of the Bethe Ansatz equations.* We seek for solutions of the Bethe Ansatz equations (8.3c) corresponding to spectral variables v_1, \dots, v_n placed on the unit circle. This ensures in particular that all genericity conditions in Theorem 8.2 and Corollary 8.3 are automatically satisfied for our parameter domain (cf. Eq. (6.1)). Specifically, for $v_j = e^{i\xi_j/2}$ ($j = 1, \dots, n$) with $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ the Bethe Ansatz equations in Theorem 8.2 are rewritten as:

$$e^{2i(m+1)\xi_j} = \left(\frac{1 - a_+ e^{i\xi_j}}{e^{i\xi_j} - a_+} \right) \left(\frac{1 - a_- e^{i\xi_j}}{e^{i\xi_j} - a_-} \right) \prod_{\substack{1 \leq k \leq n \\ k \neq j}} \left(\frac{1 - t e^{i(\xi_j + \xi_k)}}{e^{i(\xi_j + \xi_k)} - t} \right) \left(\frac{1 - t e^{i(\xi_j - \xi_k)}}{e^{i(\xi_j - \xi_k)} - t} \right), \tag{10.1}$$

$j = 1, \dots, n$. Analogous Bethe Ansatz equations for the periodic q -boson system were considered in Refs. [BIK, Ts, D2, Ko], and more generally in Ref. [DE1] where generalizations of the q -boson system arising from double affine Hecke algebras at critical level were studied.

For any $\lambda \in \Lambda_{n,m}$, the (unique) global minimum $\xi_\lambda^{(n,m)} \in \mathbb{R}^n$ of the strictly convex Morse function $V_\lambda^{(n,m)} : \mathbb{R}^n \rightarrow \mathbb{R}$ given by

$$V_\lambda^{(n,m)}(\xi) := \sum_{1 \leq j \leq n} \left((m+1)\xi_j^2 - 2\pi(\rho_j + \lambda_j)\xi_j + \int_0^{\xi_j} (v_{a_+}(u) + v_{a_-}(u)) du \right) + \sum_{1 \leq j < k \leq n} \left(\int_0^{\xi_j + \xi_k} v_t(u) du + \int_0^{\xi_j - \xi_k} v_t(u) du \right), \tag{10.2a}$$

with $\rho_j = n + 1 - j$ ($j = 1, \dots, n$) and

$$v_a(\theta) := \int_0^\theta \frac{(1 - a^2) du}{1 - 2a \cos(u) + a^2} = i \log \left(\frac{1 - a e^{i\theta}}{e^{i\theta} - a} \right) \quad (-1 < a < 1), \tag{10.2b}$$

provides a solution of the Bethe Ansatz equations in Eq. (10.1). Here the branches of the logarithmic function are assumed to be chosen such that $v_a(\theta)$ is quasi-periodic ($v_a(\theta + 2\pi) = v_a(\theta) + 2\pi$) and varies from $-\pi$ to π as θ varies from $-\pi$ to π (this corresponds to the principal branch). Moreover, the parameter constraint $a \in (-1, 1)$ guarantees that the odd function $v_a(\theta)$ is smooth and strictly monotonously increasing on \mathbb{R} . Notice that the equation $\nabla_\xi V_\lambda^{(n,m)}(\xi) = 0$ characterizing the critical point $\xi_\lambda^{(n,m)}$ is given by the system

$$2(m+1)\xi_j + v_{a_+}(\xi_j) + v_{a_-}(\xi_j) + \sum_{\substack{1 \leq k \leq n \\ k \neq j}} \left(v_t(\xi_k + \xi_j) - v_t(\xi_k - \xi_j) \right) = 2\pi(\rho_j + \lambda_j), \tag{10.3}$$

$j = 1, \dots, n$.

Proposition 10.1 (Spectral Points). *For $\lambda \in \Lambda_{n,m}$ (6.2), the unique global minimum $\xi_\lambda^{(n,m)} \in \mathbb{R}^n$ of the strictly convex Morse function $V_\lambda^{(n,m)}(\xi)$ (10.2a), (10.2b) enjoys the following properties.*

- (i) *At the spectral point $\xi = \xi_\lambda^{(n,m)}$, the Bethe Ansatz equations (10.1) are satisfied.*

- (ii) The mapping $\lambda \mapsto \xi_\lambda^{(n,m)}$ is injective.
- (iii) The minimum $\xi_\lambda^{(n,m)}$ is assumed inside the alcove $2\pi A$ (where A refers to the hyperoctahedral Weyl alcove in Eq. (1.2b)).
- (iv) At the spectral point $\xi = \xi_\lambda^{(n,m)}$, the following estimates hold for ξ_j :

$$\frac{\pi(\rho_j + \lambda_j)}{m + 1 + \kappa_-} < \xi_j < \frac{\pi(\rho_j + \lambda_j)}{m + 1 + \kappa_+} \tag{10.4a}$$

(for $1 \leq j \leq n$), and for the moment gaps $\xi_j - \xi_k$:

$$\frac{\pi(\rho_j - \rho_k + \lambda_j - \lambda_k)}{m + 1 + \kappa_-} < \xi_j - \xi_k < \frac{\pi(\rho_j - \rho_k + \lambda_j - \lambda_k)}{m + 1 + \kappa_+} \tag{10.4b}$$

(for $1 \leq j < k \leq n$), where

$$\begin{aligned} \kappa_\pm &= \kappa_\pm(t, a_+, a_-) \\ &:= \frac{1}{2} \left(\frac{1 - a_+^2}{(1 \pm |a_+|)^2} + \frac{1 - a_-^2}{(1 \pm |a_-|)^2} \right) + \frac{(n - 1)(1 - t^2)}{(1 \pm t)^2}. \end{aligned} \tag{10.4c}$$

- (v) The position of $\xi_\lambda^{(n,m)}$ depends analytically on the parameters t, a_+, a_- (taken from the domain in Eq. (6.1)), and one has that

$$\lim_{t, a_+, a_- \rightarrow 0} \xi_\lambda^{(n,m)} = \frac{\pi(\rho + \lambda)}{m + n + 1} \tag{10.4d}$$

(where $\rho = (\rho_1, \dots, \rho_n)$).

Analogs of this proposition for the periodic q -boson system and its generalization associated with the double affine Hecke algebra at critical level were previously formulated in [D2, Sec. 4] and [DE1, Prp. 7.3], respectively. To keep the presentation self-contained, the main ingredients of the proof are briefly adapted to the present setup in Sect. 10.3 below.

10.2. Diagonalization of the open-end q -boson system. Let us abbreviate the notation for n -particle Bethe Ansatz wave function with spectral parameters on the unit circle:

$$\Psi^{(n,m)}(\xi, \mu) := \Psi_{(\exp(\frac{i\xi_1}{2}), \dots, \exp(\frac{i\xi_n}{2}))}(\mu) \quad (\xi \in \mathbb{R}^n, \mu \in \Lambda_{n,m}). \tag{10.5}$$

Our spectral analysis of the open-end q -boson system—by means of Sklyanin’s version of the algebraic Bethe Ansatz formalism—now culminates in the following principal conclusion, which confirms that the Bethe Ansatz eigenfunctions $\Psi^{(n,m)}(\xi_\lambda^{(n,m)}) := \Psi^{(n,m)}(\xi_\lambda^{(n,m)}, \cdot), \lambda \in \Lambda_{n,m}$ constitute an orthogonal basis for the n -particle sector $\mathcal{F}_{n,m}$ (6.3a)–(6.3c) of the q -boson Fock space \mathcal{F}_m (6.5a), (6.5b).

Theorem 10.2 (Diagonalization and Orthogonality). *Let $u \in \mathbb{R}^* \setminus \{1, -1\}$. The n -particle Bethe Ansatz wave functions $\Psi^{(n,m)}(\xi_\lambda^{(n,m)})$, $\lambda \in \Lambda_{n,m}$, which satisfy the eigenvalue equations*

$$T_m(u; a_+, a_-)\Psi^{(n,m)}(\xi_\lambda^{(n,m)}) = E^{(n,m)}(u; \xi_\lambda^{(n,m)})\Psi^{(n,m)}(\xi_\lambda^{(n,m)}) \tag{10.6a}$$

with

$$\begin{aligned}
 E^{(n,m)}(u; \xi) &:= q^{-m} t^{-n} & (10.6b) \\
 &\times \left(u^{-2(m+2)} \frac{(1-t^{-1}u^4)}{(1-u^4)} (1-a_+u^2)(1-a_-u^2) \right. \\
 &\quad \times \prod_{1 \leq j \leq n} \frac{(1-tu^2e^{i\xi_j})(1-tu^2e^{-i\xi_j})}{(1-u^2e^{i\xi_j})(1-u^2e^{-i\xi_j})} \\
 &\quad + u^{2(m+2)} \frac{(1-t^{-1}u^{-4})}{(1-u^{-4})} (1-a_+u^{-2})(1-a_-u^{-2}) \\
 &\quad \left. \times \prod_{1 \leq j \leq n} \frac{(1-tu^{-2}e^{i\xi_j})(1-tu^{-2}e^{-i\xi_j})}{(1-u^{-2}e^{i\xi_j})(1-u^{-2}e^{-i\xi_j})} \right),
 \end{aligned}$$

and

$$\mathcal{H}_m \Psi^{(n,m)}(\xi_\lambda^{(n,m)}) = E^{(n)}(\xi_\lambda^{(n,m)}) \Psi^{(n,m)}(\xi_\lambda^{(n,m)}) \tag{10.7a}$$

with

$$E^{(n)}(\xi) := \sum_{1 \leq j \leq n} 2 \cos(\xi_j), \tag{10.7b}$$

constitute an orthogonal basis for $\mathcal{F}_{n,m}$:

$$\left(\Psi^{(n,m)}(\xi_\lambda^{(n,m)}), \Psi^{(n,m)}(\xi_\mu^{(n,m)}) \right)_{n,m} = 0 \text{ iff } \lambda \neq \mu \tag{10.8}$$

($\lambda, \mu \in \Lambda_{n,m}$).

It is clear from Theorem 8.2, Corollary 8.3, Corollary 9.4 and Proposition 10.1, that the n -particle Bethe-Ansatz wave function $\Psi^{(n,m)}(\xi, \mu)$ (10.5) provides a nonvanishing eigenfunction of the q -boson boundary transfer operator and of the q -boson Hamiltonian in the n -particle sector $\mathcal{F}_{n,m}$, at any spectral point $\xi = \xi_\lambda^{(n,m)}$, $\lambda \in \Lambda_{n,m}$. In view of Proposition 6.2, for confirming that the eigenfunctions in question are orthogonal it is enough to verify that the corresponding eigenvalues of the boundary transfer operator are simple (as Laurent polynomials in u), i.e. that for any $\lambda, \mu \in \Lambda_{n,m}$:

$$E^{(n,m)}(u; \xi_\lambda^{(n,m)}) \neq E^{(n,m)}(u; \xi_\mu^{(n,m)}) \text{ if } \lambda \neq \mu. \tag{10.9}$$

The verification of this nondegeneracy of the n -particle Bethe Ansatz eigenvalues is relegated to Sect. 10.4 below.

From Theorem 10.2, we read-off the n -particle spectrum of the boundary transfer operator and the Hamiltonian for our open-end q -boson system.

Corollary 10.3 (Open-End q -Boson Spectrum). *Let $u \in \mathbb{R}^* \setminus \{1, -1\}$. The n -particle spectrum of the self-adjoint boundary transfer operator $\mathcal{T}_m(u; a_+, a_-)$ (3.16) and the Hamiltonian (5.2) in $\mathcal{F}_{n,m}$ is given, respectively, by the eigenvalues $E^{(n,m)}(u; \xi_\lambda^{(n,m)})$ and $E^{(n)}(\xi_\lambda^{(n,m)})$ (where λ runs through $\Lambda_{n,m}$ (6.2)).*

Remark 10.4. Since $E^{(n,m)}(u^{-1}; \xi) = E^{(n,m)}(u; \xi)$, it is clear that the commuting quantum integrals $\tau_{m,k}(a_+, a_-)$, $k = -m - 2, -m - 1, \dots, m + 2$ generated by the bound-

ary transfer operator $\mathcal{T}_m(u; a_+, a_-)$ (3.16) in Eqs. (5.5a), (5.5b), satisfy the symmetry $\tau_{m,-k}(a_+, a_-) = \tau_{m,k}(a_+, a_-)$ (as operators in the q -boson Fock space \mathcal{F}_m (6.5a), (6.5b)). In other words, apart from the number operator \mathcal{N}_m (4.4b) and the Hamiltonian \mathcal{H}_m (5.2) the boundary transfer operator generates at most $m + 1$ independent quantum integrals $\tau_{m,k}(a_+, a_-)$, $k = 0, \dots, m$, in \mathcal{F}_m .

10.3. Proof of Proposition 10.1. It is immediate from its explicit formula that the value of the smooth function $V_\lambda^{(n,m)}(\xi)$ (10.2a), (10.2b) grows unboundedly as the norm of ξ becomes large, whence the function under consideration is bounded from below on \mathbb{R}^n and possesses a minimum. To confirm that this minimum is unique, we check that the function $V_\lambda^{(n,m)}(\xi)$ is strictly convex. Indeed, an explicit computation of

$$H_{j,k}^{(n,m)} := \partial_{\xi_j} \partial_{\xi_k} V_\lambda^{(n,m)}(\xi) \tag{10.10}$$

$$= \begin{cases} 2(m+1) + v'_{a_+}(\xi_j) + v'_{a_-}(\xi_j) + \sum_{l \neq j} (v'_t(\xi_j + \xi_l) + v'_t(\xi_j - \xi_l)) & \text{if } k = j \\ v'_t(\xi_j + \xi_k) - v'_t(\xi_j - \xi_k) & \text{if } k \neq j \end{cases},$$

reveals that the Hessian under consideration is positive definite:

$$\begin{aligned} \sum_{1 \leq j, k \leq n} H_{j,k}^{(n,m)} x_j x_k &= \sum_{1 \leq j \leq n} \left(2(m+1) + v'_{a_+}(\xi_j) + v'_{a_-}(\xi_j) \right) x_j^2 \\ &\quad + \sum_{1 \leq j < k \leq n} \left(v'_t(\xi_j + \xi_k)(x_j + x_k)^2 + v'_t(\xi_j - \xi_k)(x_j - x_k)^2 \right) \\ &\geq 2(m+1) \sum_{1 \leq j \leq n} x_j^2 > 0 \end{aligned}$$

(if $(x_1, x_2, \dots, x_n) \neq (0, 0, \dots, 0)$), because

$$v'_a(\theta) = \frac{1 - a^2}{1 - 2a \cos(\theta) + a^2} > 0 \quad \text{for } a \in (-1, 1).$$

The properties (i)–(v) for the unique global minimum $\xi_\lambda^{(n,m)}$ are now verified by exploiting that Eq. (10.3) is satisfied at $\xi = \xi_\lambda^{(n,m)}$.

(i) That the Bethe–Ansatz equations (10.1) hold at $\xi = \xi_\lambda^{(n,m)}$ follows upon multiplication of Eq. (10.3) by the imaginary unit and exponentiation of both sides (while using that $e^{-i v_a(\theta)} = (1 - a e^{i\theta}) / (e^{i\theta} - a)$).

(ii) The injectivity of $\lambda \rightarrow \xi_\lambda^{(n,m)}$ is immediate from Eq. (10.3) (if the LHS’s are the same then RHS’s must also be the same).

(iii) Since the LHS of Eq. (10.3) is odd and monotonously increasing in ξ_j , it follows that $\xi_j > 0$ at $\xi = \xi_\lambda^{(n,m)}$ because $2\pi(\rho_j + \lambda_j) > 0$. Furthermore, by subtracting the k th equation from the j th equation:

$$\begin{aligned} &2(m+1)(\xi_j - \xi_k) + (v_{a_+}(\xi_j) - v_{a_+}(\xi_k)) + (v_{a_-}(\xi_j) - v_{a_-}(\xi_k)) + 2v_t(\xi_j - \xi_k) \\ &\quad + \sum_{\substack{1 \leq l \leq n \\ l \neq j, k}} \left((v_t(\xi_l + \xi_j) - v_t(\xi_l + \xi_k)) + (v_t(\xi_l - \xi_k) - v_t(\xi_l - \xi_j)) \right) \\ &= 2\pi(\rho_j - \rho_k + \lambda_j - \lambda_k), \end{aligned}$$

one sees in the same way that for $j < k$: $\xi_j > \xi_k$ at $\xi = \xi_\lambda^{(n,m)}$. Finally, if $\xi_j \geq \pi$ then the LHS of Eq. (10.3) is larger than $2(m+1)\pi + 2(n-1)\pi = 2(n+m)\pi$ by the quasi-periodicity of $v_a(\theta)$. This implies that on the RHS $2\pi(\rho_j + \lambda_j) > 2\pi(m+n)$, i.e. $\lambda_j > m+n - \rho_j \geq m$ so $\lambda \notin \Lambda_{n,m}$. The upshot is that $\xi_\lambda^{(n,m)} \in 2\pi\mathbb{A}$ for $\lambda \in \Lambda_{n,m}$.

(iv) Since the integrand of $v_a(\theta)$ oscillates between $\frac{1-a^2}{(1+|a|)^2}$ and $\frac{1-a^2}{(1-|a|)^2}$, it follows from Eq. (10.3) that

$$(m+1+\kappa_+)\xi_j < \pi(\rho_j + \lambda_j) < (m+1+\kappa_-)\xi_j$$

at $\xi = \xi_\lambda^{(n,m)}$. In the same way, one deduces from the equation in the proof of part iii) that for $j < k$:

$$(m+1+\kappa_+)(\xi_j - \xi_k) < \pi(\rho_j - \rho_k + \lambda_j - \lambda_k) < (m+1+\kappa_-)(\xi_j - \xi_k)$$

at $\xi = \xi_\lambda^{(n,m)}$.

(v) Since $V_\lambda^{(n,m)}(\xi)$ is strictly convex, the Jacobian with respect to ξ of the equation $\nabla_\xi V_\lambda^{(n,m)}(\xi) = 0$ in Eq. (10.3) is nonzero (viz. positive). Invoking of the implicit function theorem then entails that the critical point $\xi_\lambda^{(n,m)}$ solving Eq. (10.3) inherits the analytic dependence on the parameters $t, a_+, a_- \in (-1, 1)$ from this equation. The limit (10.4d) is now immediate from the estimate in Eq. (10.4a).

10.4. Proof of Theorem 10.2. Let us assume that λ, μ are two *not necessarily distinct* partitions belonging to $\Lambda_{n,m}$ (6.2). Following Dorlas' approach in [Do, Lem. 4.2], we will check the nondegeneracy of the Bethe Ansatz spectrum with the aid of the Casoratian (or q -difference Wronskian)

$$W(u) := F(qu)G(u) - F(u)G(qu) \tag{10.11a}$$

of the associated spectral polynomials

$$F(u) := \prod_{1 \leq j \leq n} (u^2 - v_j^2)(u^2 - v_j^{-2}), \tag{10.11b}$$

$$G(u) := \prod_{1 \leq j \leq n} (u^2 - w_j^2)(u^2 - w_j^{-2}). \tag{10.11c}$$

Here (and throughout the rest of the proof below) the vectors (v_1, \dots, v_n) and (w_1, \dots, w_n) in $(\mathbb{C}^*)^n$ are equal to $(e^{i\xi_1/2}, \dots, e^{i\xi_n/2})$ with $\xi \in 2\pi\mathbb{A}$ taken at $\xi = \xi_\lambda^{(n,m)}$ and $\xi = \xi_\mu^{(n,m)}$, respectively (cf. Part iii) of Proposition 10.1).

Lemma 10.5. *If $E^{(n,m)}(u; \xi_\lambda^{(n,m)}) = E^{(n,m)}(u; \xi_\mu^{(n,m)})$ as an identity between Laurent polynomials in u , then the associated Casorati polynomial $W(u)$ (10.11a)–(10.11c) is quasi-periodic:*

$$W(q^{-1}u) = t^{-2n} \frac{\hat{\alpha}_m(u)}{\hat{\alpha}_m(u^{-1})} W(u) \tag{10.12a}$$

with

$$\hat{\alpha}_m(u) := u^{-2(m+2)} \left(\frac{1-t^{-1}u^4}{1-u^4} \right) (1-a_+u^2)(1-a_-u^2). \tag{10.12b}$$

Proof. Because

$$q^m t^n F(u) E^{(n,m)}(u; \xi_\lambda^{(n,m)}) = \hat{\alpha}_m(u) F(qu) + t^{2n} \hat{\alpha}_m(u^{-1}) F(q^{-1}u)$$

and

$$q^m t^n G(u) E^{(n,m)}(u; \xi_\mu^{(n,m)}) = \hat{\alpha}_m(u) G(qu) + t^{2n} \hat{\alpha}_m(u^{-1}) G(q^{-1}u),$$

we have that

$$\begin{aligned} W(q^{-1}u) &= F(u)G(q^{-1}u) - F(q^{-1}u)G(u) \\ &= F(u) \left(\frac{q^m G(u) E^{(n,m)}(u; \xi_\mu^{(n,m)})}{t^n \hat{\alpha}_m(u^{-1})} - \frac{\hat{\alpha}_m(u)}{t^{2n} \hat{\alpha}_m(u^{-1})} G(qu) \right) \\ &\quad - G(u) \left(\frac{q^m F(u) E^{(n,m)}(u; \xi_\lambda^{(n,m)})}{t^n \hat{\alpha}_m(u^{-1})} - \frac{\hat{\alpha}_m(u)}{t^{2n} \hat{\alpha}_m(u^{-1})} F(qu) \right) \\ &= t^{-2n} \frac{\hat{\alpha}_m(u)}{\hat{\alpha}_m(u^{-1})} W(u), \end{aligned}$$

where the cancellations in the last step hinge on our assumption that $E^{(n,m)}(u; \xi_\lambda^{(n,m)})$ be equal to $E^{(n,m)}(u; \xi_\mu^{(n,m)})$. \square

The quasi-periodicity of the Casorati polynomial $W(u)$ in Lemma 10.5 implies that in fact $W(u) \equiv 0$. This becomes manifest after rewriting the quasi-periodicity in Eqs. (10.12a), (10.12b) as a polynomial identity by clearing denominators and negative powers of u :

$$\begin{aligned} u^{4(m+1)} t^{2n+1} (u^4 - t^{-1})(u^2 - a_+)(u^2 - a_-) W(q^{-1}u) \\ = (u^4 - t)(a_+ u^2 - 1)(a_- u^2 - 1) W(u). \end{aligned}$$

When comparing the degrees in u of the polynomial expressions on both sides, it is seen that the degree of the LHS exceeds the degree of the RHS unless $W(u) \equiv 0$. The vanishing of the Casoratian implies in particular that $W(v_j) = F(qv_j)G(v_j) = 0$ for $j = 1, \dots, n$. Because qv_j is not on the unit circle, it is clear that $F(qv_j) \neq 0$ and thus $G(v_j) = 0$, whence

$$v_j \in \{w_k, w_k^{-1}, -w_k, -w_k^{-1} \mid k = 1, \dots, n\} \quad \text{for } j = 1, \dots, n. \tag{10.13}$$

Since our spectral points belong to $2\pi A$ [by Part (iii) of Proposition 10.1], the principal arguments of v_1, \dots, v_n and those of w_1, \dots, w_n correspond to angles between 0 and $\pi/2$ in strictly decreasing order. Hence there is no possible ambiguity in the identification of the roots in Eq. (10.13): $v_j = w_j$ (for $j = 1, \dots, n$), i.e. $\xi_\lambda^{(n,m)} = \xi_\mu^{(n,m)}$, and thus $\lambda = \mu$ [by the injectivity in Part (ii) of Proposition 10.1]. The upshot is that the equality of the Laurent polynomials $E^{(n,m)}(u; \xi_\lambda^{(n,m)})$ and $E^{(n,m)}(u; \xi_\mu^{(n,m)})$ implies that $\lambda = \mu$, which confirms the desired nondegeneracy of the Bethe Ansatz spectrum in Eq. (10.9), and therewith completes the proof of Theorem 10.2.

11. Hyperoctahedral Hall–Littlewood Polynomials

In this section we combine the branching rule in Theorem 9.2 with recent results from Ref. [WZ], to express our Bethe Ansatz wave functions in terms of Macdonald’s hyperoctahedral Hall–Littlewood polynomials associated with the root system BC_n .

11.1. *Macdonald’s formula* [M2, § 10]. For $\lambda \in \Lambda_{n,m}$ (6.2), Macdonald’s Hall–Littlewood polynomial associated with the root system BC_n is a three-parameter hyperoctahedral-symmetric Laurent polynomial in the variables z_1, \dots, z_n of the form

$$P_\lambda(z_1, \dots, z_n; t, a, \hat{a}) = \sum_{\substack{\sigma \in S_n \\ \epsilon \in \{1, -1\}^n}} C(z_{\sigma_1}^{\epsilon_1}, \dots, z_{\sigma_n}^{\epsilon_n}; t, a, \hat{a}) z_{\sigma_1}^{\epsilon_1 \lambda_1} \cdots z_{\sigma_n}^{\epsilon_n \lambda_n}, \quad (11.1a)$$

with

$$C(z_1, \dots, z_n; t, a, \hat{a}) := \prod_{1 \leq j \leq n} \frac{(z_j - a)(z_j - \hat{a})}{z_j^2 - 1} \prod_{1 \leq j < k \leq n} \left(\frac{z_j z_k - t}{z_j z_k - 1} \right) \left(\frac{z_j z_k^{-1} - t}{z_j z_k^{-1} - 1} \right). \quad (11.1b)$$

In the present normalization, the coefficient of the leading monomial $z_1^{\lambda_1} \cdots z_n^{\lambda_n}$ in $P_\lambda(z_1, \dots, z_n; t, a, \hat{a})$ is given by

$$l.c. = (1 - a\hat{a})(1 - a\hat{a}t) \cdots (1 - a\hat{a}t^{m_0(\lambda)-1}) \prod_{l \in \mathbb{N}_m} [m_l(\lambda)]!. \quad (11.2)$$

In particular, for $\lambda = 0^n$ we have that

$$P_{0^n}(z_1, \dots, z_n; t, a, \hat{a}) = (1 - a\hat{a})(1 - a\hat{a}t) \cdots (1 - a\hat{a}t^{n-1})[n]!. \quad (11.3)$$

11.2. *Branching rule* [WZ, Sec. 3.3]. In [WZ, Sec. 3.3] a realization of Macdonald’s hyperoctahedral Hall–Littlewood polynomials was found in terms of lattice paths. By expressing the polynomials in question as matrix elements of a q -boson system with integrable boundary interactions, this approach gives rise to the following branching rule for $\hat{a} = 0$:

$$P_\lambda(z_1, \dots, z_n; t, a, 0) = \sum_{\substack{\mu \in \Lambda_{n-1,m} \\ \mu \leq \lambda}} \hat{B}_{\lambda/\mu}^{(n)}(z_n; t, a) P_\mu(z_1, \dots, z_{n-1}; t, a, 0), \quad (11.4)$$

where $\hat{B}_{\lambda/\mu}^{(n)}(z; t, a)$ is given by Eq. (9.2b). When also $a = 0$ the contributions from the boundary site are trivial, and in this case the branching rule was derived in Remark 4 of [WZ, Sec. 3.3]. When $a \neq 0$, there is a single nonzero contribution from the boundary site that can be absorbed in the (diagonal) K -matrix, after which one recovers the branching rule in Eq. (11.4) by repeating the steps in Remark 4 of [WZ, Sec. 3.3] with the modified K -matrix.

A branching rule of the form in Eq. (11.4), but with much more intricate branching coefficients $\hat{B}_{\lambda/\mu}^{(n)}$, was recently found for the five-parameter Macdonald–Koornwinder q -deformation of the hyperoctahedral Hall–Littlewood polynomials [DE4].

11.3. *Relation with the n -particle Bethe Ansatz wave function.* For $\hat{a} = 0$, Macdonald’s hyperoctahedral Hall–Littlewood polynomials in Eqs. (11.1a), (11.1b) reduce in the case of a single variable ($n = 1$) to

$$P_l(z; a, 0) = \left(\frac{z-a}{z^2-1}\right)z^{l+1} + \left(\frac{z^{-1}-a}{z^{-2}-1}\right)z^{-l-1} \stackrel{\text{Eq. (9.1)}}{=} \mathbf{s}_l(z) - a \mathbf{s}_{l-1}(z) \quad (11.5)$$

($l \in \mathbb{N}_m$). By comparing Eq. (11.5) with Proposition 9.1 and Eq. (11.4) with Theorem 9.2, it is now immediate that our n -particle Bethe Ansatz wave function can be expressed explicitly in terms of Macdonald’s hyperoctahedral Hall–Littlewood polynomials as follows:

$$\begin{aligned} \Psi_{(v_1, \dots, v_n)}(\lambda) &= P_\lambda(v_1^2, \dots, v_n^2; t, a_-, 0) \\ &= \sum_{\substack{\sigma \in S_n \\ \epsilon \in \{1, -1\}^n}} C(v_{\sigma_1}^{2\epsilon_1}, \dots, v_{\sigma_n}^{2\epsilon_n}; t, a_-, 0) v_{\sigma_1}^{2\epsilon_1 \lambda_1} \dots v_{\sigma_n}^{2\epsilon_n \lambda_n} \end{aligned} \quad (11.6)$$

($\lambda \in \Lambda_{n,m}$).

For the q -boson system with periodic boundary conditions an analogous formula for the algebraic Bethe Ansatz wave functions in terms of Hall–Littlewood polynomials can be found in [Ts, Prp 4] and [Ko, Rem. 4.2]. The latter formula amounts to a q -deformation of a previously known expression for the Bethe Ansatz wave functions of the ($q = 1$) phase model in terms of Schur polynomials [B, Sec. 5] (cf. also [KoS, Part I]). For the q -boson systems on the infinite and semi-infinite integral lattices, corresponding formulas for the q -boson eigenfunctions in terms of the Hall–Littlewood polynomials (infinite lattice) and the hyperoctahedral Hall–Littlewood polynomials (semi-infinite lattice) can be found in [DE2, Sec. 5] and [DE3, Sec. 3], respectively. Recently it has moreover been shown in Ref. [BCPS] that, on the infinite lattice, the explicit formula in terms of Hall–Littlewood polynomials arises as a degeneration of a more general formula for the Bethe Ansatz eigenfunction of a stochastic q -boson system related to the asymmetric exclusion process [SW].

11.4. Affine Pieri formula and orthogonality. With the aid of Eq. (11.6), we can reformulate the results of Theorem 10.2 completely in terms of Macdonald’s hyperoctahedral Hall–Littlewood polynomials.

For $\lambda, \mu \in \Lambda_{n,m}$, let (z_1, \dots, z_n) and (y_1, \dots, y_n) be equal to $(e^{i\xi_1}, \dots, e^{i\xi_n})$ at $\xi = \xi_\lambda^{(n,m)}$ and $\xi = \xi_\mu^{(n,m)}$, respectively (cf. Proposition 10.1). Then it follows from Proposition 6.3 and Theorem 10.2 that at these spectral points Macdonald’s hyperoctahedral Hall–Littlewood polynomials satisfy the following affine Pieri formula

$$\begin{aligned} P_\nu(z_1, \dots, z_n; t, a_-, 0) &\sum_{1 \leq j \leq n} (z_j + z_j^{-1}) \\ &= (a_- [m_0(\nu)] + a_+ [m_m(\nu)]) P_\nu(z_1, \dots, z_n; t, a_-, 0) \\ &\quad + \sum_{\substack{1 \leq j \leq n \\ \nu \pm e_j \in \Lambda_{n,m}}} [m_{\nu_j}(\nu)] P_{\nu \pm e_j}(z_1, \dots, z_n; t, a_-, 0), \end{aligned} \quad (11.7)$$

($\nu \in \Lambda_{n,m}$), and the following finite-dimensional discrete orthogonality relations

$$\sum_{\nu \in \Lambda_{n,m}} P_\nu(z_1, \dots, z_n; t, a_-, 0) \overline{P_\nu(y_1, \dots, y_n; t, a_-, 0)} \delta_{n,m}(\nu) = 0 \quad \text{iff } \lambda \neq \mu. \quad (11.8)$$

For the Hall–Littlewood polynomials diagonalizing the q -boson system with periodic boundary conditions, corresponding affine Pieri formulas and finite-dimensional discrete orthogonality relations were presented in Theorems 5.1 and 5.2 of Ref. [D2], respectively. A further generalization of those formulas, related to a generalized q -boson system associated with the double affine Hecke algebra at critical level, was recently studied in Ref. [DE1].

Remark 11.1. Theorem 7.3 and Theorem 10.2 imply in fact a system of affine Pieri formulas for the hyperoctahedral Hall–Littlewood polynomials generalizing the one in Eq. (11.7). Specifically, for $\lambda, \nu \in \Lambda_{n,m}$ one has that at $(z_1, \dots, z_n) = (e^{i\xi_1}, \dots, e^{i\xi_n})$ with $\xi = \xi_\lambda^{(n,m)}$:

$$\begin{aligned}
 P_\nu(z_1, \dots, z_n; t, a_-, 0) E^{(n,m)}(u; z_1, \dots, z_n) & \tag{11.9} \\
 = q^{-m} t^{-n-1} (a_+ - tu^{-2}) \sum_{\substack{\mu \in \Lambda_{n,m} \\ \mu \sim_- \nu}} A_{\nu,\mu}^{(n)}(u^2; t, a_-) P_\mu(z_1, \dots, z_n; t, a_-, 0) \\
 + q^{-m} t^{-n-1} (a_+ - u^2) \sum_{\substack{\mu \in \Lambda_{n,m} \\ \mu \sim_+ \nu}} D_{\nu,\mu}^{(n)}(u^2; t, a_-) P_\mu(z_1, \dots, z_n; t, a_-, 0)
 \end{aligned}$$

(as an identity between Laurent polynomials in u). Here $E^{(n,m)}(u; z_1, \dots, z_n)$ is given by Eq. (10.6b) with $e^{i\xi_j} = z_j$ ($j = 1, \dots, n$) and the coefficients $A_{\nu,\mu}^{(n)}(z; t, a)$ and $D_{\nu,\mu}^{(n)}(z; t, a)$ are of the form in Eqs. (7.5a) and (7.5d), respectively.

12. Continuum Limit

In this section we obtain the orthogonality in Theorem 2.1 as a continuum limit of the orthogonality in Eq. (11.8), by adapting the analysis in [D2, Sec. 6] to the present setup. Similar techniques were employed previously by Ruijsenaars when studying the continuum limit of the infinite isotropic Heisenberg chain [R].

Throughout the section we put

$$t = e^{-\frac{g}{2m}}, \quad a_+ = e^{-\frac{g_+}{2m}}, \quad a_- = e^{-\frac{g_-}{2m}} \quad \text{with } g, g_+, g_- > 0. \tag{12.1}$$

12.1. Staircase functions. For any $x = (x_1, \dots, x_n)$ in the closure \bar{C} of the chamber C (2.8), we consider its lattice approximation $\lfloor x \rfloor \in \Lambda$ (2.5) defined by

$$\lfloor x \rfloor := \sum_{1 \leq j \leq n} \lfloor x_j - x_{j+1} \rfloor (e_1 + \dots + e_j) \quad (\text{with } x_{n+1} := 0), \tag{12.2}$$

where $\lfloor \cdot \rfloor$ refers to the floor function extracting the integral part of a nonnegative real number by truncation. Armed with this lattice approximation, one can embed $\mathcal{F}_{n,m}$ (6.3a)–(6.3c) into $L^2(C, dx)$ by means of a linear injection $J^{(n,m)} : \mathcal{F}_{n,m} \rightarrow L^2(C, dx)$, which associates to $f : \Lambda_{n,m} \rightarrow \mathbb{C}$ a staircase function $J^{(n,m)}(f) : \bar{C} \rightarrow \mathbb{C}$ of the form

$$(J^{(n,m)} f)(x) := \begin{cases} \sqrt{\delta_{n,m}(\lfloor 2mx \rfloor)} f(\lfloor 2mx \rfloor) & \text{for } \lfloor 2mx \rfloor \in \Lambda_{n,m}, \\ 0 & \text{for } \lfloor 2mx \rfloor \notin \Lambda_{n,m}. \end{cases} \tag{12.3}$$

Clearly the staircase function $J^{(n,m)}(f)$ is compactly supported inside $(1 + \frac{n}{m})\overline{A} \subset \overline{C}$, where \overline{A} denotes the closure of the alcove A (1.2b), and moreover

$$\int_C (J^{(n,m)} f)(x) \overline{(J^{(n,m)} g)(x)} dx = (2m)^{-n} (f, g)_{n,m} \tag{12.4}$$

$(f, g \in \mathcal{F}_{n,m})$.

12.2. *Convergence of the Bethe Ansatz wave function.* Let $\psi^{(n,m)}(\xi, x)$ denote the embedding in $L^2(C, dx)$ of the hyperoctahedral Hall–Littlewood polynomial

$$2^n P_\lambda(e^{i\xi_1}, \dots, e^{i\xi_n}; e^{-\frac{g}{2m}}, e^{-\frac{g-m}{2m}}, 0)$$

(viewed as a function of $\lambda \in \Lambda_{n,m}$):

$$\begin{aligned} \psi^{(n,m)}(\xi, x) &:= 2^n (J^{(n,m)} P(e^{i\xi_1}, \dots, e^{i\xi_n}; e^{-\frac{g}{2m}}, e^{-\frac{g-m}{2m}}, 0))(x) \\ &= 2^n \sqrt{\delta_{n,m}(\lfloor 2mx \rfloor)} P_{\lfloor 2mx \rfloor}(e^{i\xi_1}, \dots, e^{i\xi_n}; e^{-\frac{g}{2m}}, e^{-\frac{g-m}{2m}}, 0), \end{aligned} \tag{12.5}$$

for $\lfloor 2mx \rfloor \in \Lambda_{n,m}$ (and zero otherwise).

We now retrieve the Bethe Ansatz wave function $\psi(\xi, x)$ (2.1) as a pointwise continuum limit of the staircase q -boson wave function $\psi^{(n,m)}(\xi, x)$ (12.5).

Lemma 12.1. *For any $\xi \in C$, one has that*

$$\lim_{m \rightarrow \infty} \psi^{(n,m)}(\frac{1}{2m}\xi, x) = \begin{cases} \psi(\xi, x) & \text{if } x \in A, \\ 0 & \text{if } x \in C \setminus \overline{A}, \end{cases} \tag{12.6}$$

with $\psi(\xi, x)$ given by Eq. (2.1).

Proof. Since $\psi^{(n,m)}(\xi, x)$ is supported inside $(1 + \frac{n}{m})\overline{A} \subset \overline{C}$, it is clear that for m sufficiently large $\psi^{(n,m)}(\xi, x) = 0$ if $x \in C \setminus \overline{A}$, i.e. $\lim_{m \rightarrow \infty} \psi^{(n,m)}(\frac{1}{2m}\xi, x) = 0$ in this situation. If on the other hand $x \in A$, then $\lfloor 2mx \rfloor \in \Lambda_{n,m}$ with—for m sufficiently large—all parts being distinct, whence

$$\lim_{m \rightarrow \infty} \delta_{n,m}(\lfloor 2mx \rfloor) = 1 \quad \text{for } x \in A.$$

In this situation, the pointwise limit of the staircase wave function is then immediate from the explicit expression for the hyperoctahedral Hall–Littlewood polynomials in Eqs. (11.1a), (11.1b), upon observing that

$$\lim_{m \rightarrow \infty} \frac{1}{2m} \lfloor 2mx \rfloor = x$$

and

$$\lim_{m \rightarrow \infty} 2^n C(e^{\frac{i\xi_1}{2m}}, \dots, e^{\frac{i\xi_n}{2m}}; e^{-\frac{g}{2m}}, e^{-\frac{g-m}{2m}}, 0) = C(\xi_1, \dots, \xi_n),$$

with $C(\xi_1, \dots, \xi_n)$ taken from Eq. (2.1). \square

12.3. *Convergence of the spectral points.* Next, we retrieve the Bethe Ansatz spectral points ξ_λ , $\lambda \in \Lambda$ for the Laplacian with repulsive Robin boundary conditions on the hyperoctahedral Weyl alcove, as scaled limits of the q -boson spectral points when the number of sites grows to infinity.

Lemma 12.2. *For any $\lambda \in \Lambda$, one has that*

$$\lim_{m \rightarrow \infty} 2m\xi_\lambda^{(n,m)} = \xi_\lambda, \tag{12.7}$$

where ξ_λ denotes the unique global minimum of $V_\lambda(\xi)$ (2.4).

Proof. Given $\lambda \in \Lambda$, let us pick m sufficiently large such that $\lambda \in \Lambda_{n,m}$. The rescaled spectral point $2m\xi_\lambda^{(n,m)}$ then corresponds to the unique global minimum of the rescaled Morse function $2mV_\lambda^{(n,m)}(\frac{1}{2m}\xi)$, which for $m \rightarrow \infty$ converges uniformly on compacts to the Morse function $V_\lambda(\xi)$ (2.4) (cf. Remark 12.3 below). Hence the unique global minimum $2m\xi_\lambda^{(n,m)}$ of $2mV_\lambda^{(n,m)}(\frac{1}{2m}\xi)$ converges to the unique global minimum ξ_λ of $V_\lambda(\xi)$ as $m \rightarrow \infty$. \square

Remark 12.3. The proof of Lemma 12.2 asserts that $2mV_\lambda^{(n,m)}(\frac{1}{2m}\xi)$ converges uniformly on compacts to $V_\lambda(\xi)$ (2.4) as m grows to infinity. Indeed, for $-\pi < \theta < \pi$ the function $v_a(\theta)$ (10.2b) can be rewritten as

$$v_a(\theta) = 2 \arctan \left(\frac{1+a}{1-a} \tan \left(\frac{\theta}{2} \right) \right). \tag{12.8}$$

Hence, for $a = e^{-\varepsilon g}$ (with $\varepsilon, g > 0$) the rescaled function $v_a(\varepsilon\theta)$ converges uniformly on compacts to $2 \arctan(\frac{\theta}{g})$ as $\varepsilon \rightarrow 0$ (because $\varepsilon^{-1} \tan(\varepsilon\theta)$ converges uniformly on compacts to θ). It thus follows that $\varepsilon^{-1} \int_0^{\varepsilon\theta} v_a(u)du = \int_0^\theta v_a(\varepsilon u)du$ converges in this situation uniformly on compacts to $2 \int_0^\theta \arctan(\frac{u}{g})du$.

12.4. *Proof of the orthogonality.* After these preparations, we are now in the position to retrieve the orthogonality in Theorem 2.1 as the continuum limit of the orthogonality in Eq. (11.8).

Proposition 12.4 (Continuum Limit). *For all $\lambda, \mu \in \Lambda$, one has that*

$$\lim_{m \rightarrow \infty} \int_C \psi^{(n,m)}(\xi_\lambda^{(n,m)}, x) \overline{\psi^{(n,m)}(\xi_\mu^{(n,m)}, x)} dx = \int_A \psi(\xi_\lambda, x) \overline{\psi(\xi_\mu, x)} dx. \tag{12.9}$$

Proof. Given $\lambda, \mu \in \Lambda$, let us pick m sufficiently large such that $\lambda, \mu \in \Lambda_{n,m}$. To verify the asserted continuum limit, we make the expressions under consideration more explicit:

$$\begin{aligned} & \int_C \psi^{(n,m)}(\xi_\lambda^{(n,m)}, x) \overline{\psi^{(n,m)}(\xi_\mu^{(n,m)}, x)} dx \\ &= \sum_{\substack{\sigma, \hat{\sigma} \in \mathcal{S}_n \\ \varepsilon, \hat{\varepsilon} \in \{1, -1\}^n}} \left(C_{\sigma, \varepsilon}^{(n,m)}(\xi_\lambda^{(n,m)}) C_{\hat{\sigma}, \hat{\varepsilon}}^{(n,m)}(-\xi_\mu^{(n,m)}) \right. \\ & \quad \left. \times \int_{(1+\frac{n}{m})A} \delta_{n,m}(\lfloor 2mx \rfloor) \prod_{1 \leq j \leq n} e^{i \lfloor 2mx \rfloor_j (\varepsilon_j (\xi_\lambda^{(n,m)})_{\sigma_j} - \hat{\varepsilon}_j (\xi_\mu^{(n,m)})_{\hat{\sigma}_j})} dx \right), \end{aligned}$$

where $C_{\sigma,\epsilon}^{(n,m)}(\xi) := 2^n C(e^{i\epsilon_1 \xi_{\sigma_1}}, \dots, e^{i\epsilon_n \xi_{\sigma_n}}; e^{-\frac{g}{2m}}, e^{-\frac{g}{2m}}, 0)$ and we have used the convention that $\delta_{n,m}(\lfloor 2mx \rfloor) := 0$ if $\lfloor 2mx \rfloor \notin \Lambda_{n,m}$. It is clear from our previous analysis above that for $m \rightarrow \infty$

$$C_{\sigma,\epsilon}^{(n,m)}(\xi_\lambda^{(n,m)}) \longrightarrow C(\epsilon_1(\xi_\lambda)_{\sigma_1}, \dots, \epsilon_n(\xi_\lambda)_{\sigma_n})$$

and

$$e^{i\epsilon_j \lfloor 2mx \rfloor_j (\xi_\lambda^{(n,m)})_{\sigma_j}} \longrightarrow e^{i\epsilon_j x_j (\xi_\lambda)_{\sigma_j}},$$

$$\delta_{n,m}(\lfloor 2mx \rfloor) \longrightarrow \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in C \setminus \overline{A} \end{cases}$$

(pointwise). Moreover, since $|e^{i\epsilon_j \lfloor 2mx \rfloor_j (\xi_\lambda^{(n,m)})_{\sigma_j}}| = 1$ and $0 \leq \delta_{n,m}(\lfloor 2mx \rfloor) \leq 1$, the dominated convergence theorem of Lebesgue then guarantees that for $m \rightarrow \infty$ our integral converges to

$$\sum_{\substack{\sigma, \hat{\sigma} \in \mathcal{S}_n \\ \epsilon, \hat{\epsilon} \in \{1, -1\}^n}} \left(C(\epsilon_1(\xi_\lambda)_{\sigma_1}, \dots, \epsilon_n(\xi_\lambda)_{\sigma_n}) C(-\hat{\epsilon}_1(\xi_\mu)_{\hat{\sigma}_1}, \dots, -\hat{\epsilon}_n(\xi_\mu)_{\hat{\sigma}_n}) \right. \\ \left. \times \int_A \prod_{1 \leq j \leq n} e^{ix_j(\epsilon_j(\xi_\lambda)_{\sigma_j} - \hat{\epsilon}_j(\xi_\mu)_{\hat{\sigma}_j})} dx \right),$$

which is precisely $\int_A \psi(\xi_\lambda, x) \overline{\psi(\xi_\mu, x)} dx$. \square

From Proposition 12.4, we see that

$$\int_A \psi(\xi_\lambda, x) \overline{\psi(\xi_\mu, x)} dx \tag{12.10} \\ = \lim_{m \rightarrow \infty} 4^n \int_C (J^{(n,m)} P(z_1, \dots, z_n; t, a_-, 0))(x) \overline{(J^{(n,m)} P(y_1, \dots, y_n; t, a_-, 0))(x)} dx,$$

where (z_1, \dots, z_n) and (y_1, \dots, y_n) are given by $(e^{i\xi_1}, \dots, e^{i\xi_n})$ at $\xi = \xi_\lambda^{(n,m)}$ and $\xi = \xi_\mu^{(n,m)}$, respectively. Since it is immediate from Eqs. (11.8) and (12.4) that the RHS of Eq. (12.10) vanishes when $\lambda \neq \mu$, the orthogonality in Theorem 2.1 thus follows.

13. Orthogonality of the Bethe Ansatz on Classical Weyl Alcoves

The eigenvalue problem for the Laplacian on the hyperoctahedral Weyl alcove in Eqs. (1.2a)–(1.2d) admits a generalization in terms of the Weyl alcove associated with an arbitrary Weyl group [G1, G2, GS, G, HO, D1, EOS1, BDM]. The completeness of the corresponding Bethe Ansatz eigenfunctions was shown in Ref. [E]. For all classical Weyl alcoves, i.e. those of type *A*, *B*, *C* and *D*, an orthogonal basis of these eigenfunctions now follows from [Do, D2] (type *A*) and Theorem 2.1 (types *B*, *C* and *D*).

To formulate this orthogonality result more precisely, some additional definitions and notations are needed. Let *R* be a reduced crystallographic root system spanning a real Euclidean vector space *V* with inner product $\langle \cdot, \cdot \rangle$, and let *W* be the Weyl group

generated by the orthogonal reflections $r_\alpha : V \rightarrow V$ in the root hyperplanes of the form $r_\alpha(x) := x - \langle x, \alpha^\vee \rangle \alpha$ ($\alpha \in R$), where $\alpha^\vee := 2\alpha / \langle \alpha, \alpha \rangle$ (and $x \in V$). For a fixed choice of positive roots $R_+ \subset R$ and a root multiplicity parameter $g_\alpha > 0$ on R such that $g_\alpha = g_\beta$ if $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle$, let $\xi_\lambda \in V$ denote the unique global minimum of the strictly convex function

$$V_\lambda(\xi) := \frac{1}{2} \langle \xi, \xi \rangle - 2\pi \langle \rho + \lambda, \xi \rangle + \sum_{\alpha \in R_+} \langle \alpha, \alpha \rangle \int_0^{\langle \xi, \alpha^\vee \rangle} \arctan\left(\frac{u}{g_\alpha}\right) du, \tag{13.1}$$

where $\rho := \frac{1}{2} \sum_{\alpha \in R_+} \alpha$ and λ is taken from the cone of dominant weights

$$\Lambda := \{ \lambda \in V \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}, \forall \alpha \in R_+ \}. \tag{13.2}$$

This minimum is determined by the critical equation $\nabla_\xi V_\lambda(\xi) = 0$:

$$\xi + 2 \sum_{\alpha \in R_+} \arctan\left(\frac{\langle \xi, \alpha^\vee \rangle}{g_\alpha}\right) \alpha = 2\pi(\rho + \lambda), \tag{13.3}$$

and (thus) satisfies the following Bethe Ansatz equations

$$e^{i \langle \xi, \alpha^\vee \rangle} = \prod_{\substack{\beta \in R \\ \langle \beta, \alpha^\vee \rangle > 0}} \left(\frac{i g_\beta + \langle \xi, \beta^\vee \rangle}{i g_\beta - \langle \xi, \beta^\vee \rangle} \right)^{\langle \beta, \alpha^\vee \rangle} \quad (\alpha \in R_+). \tag{13.4}$$

It was shown in Ref. [E] that the linear span of the Bethe Ansatz wave functions $\psi(\xi_\lambda, x)$, $\lambda \in \Lambda$, where

$$\psi(\xi, x) := \sum_{w \in W} e^{i \langle w\xi, x \rangle} \prod_{\alpha \in R^+} \frac{\langle w\xi, \alpha^\vee \rangle - i g_\alpha}{\langle w\xi, \alpha^\vee \rangle}, \tag{13.5}$$

is dense in the Hilbert space $L^2(A, dx)$ of quadratically integrable functions on the Weyl alcove

$$A = \{ x \in V \mid 0 < \langle \alpha, x \rangle < 1, \forall \alpha \in R^+ \}. \tag{13.6}$$

(Here the integration is meant with respect to the standard Lebesgue measure dx inherited from the Euclidean space V .)

By specialization of the boundary parameters in Theorem 2.1, we extend the orthogonality for the root system of type A from Refs. [Do, D2] so as to arrive at an orthogonal basis of Bethe Ansatz wave functions on the Weyl alcove for any classical root system.

Theorem 13.1 (Types A,B,C). *When R is of type A, B or C , the repulsive Bethe Ansatz wave functions $\psi(\xi_\lambda, x)$, $\lambda \in \Lambda$ constitute an orthogonal basis for the Hilbert space $L^2(A, dx)$.*

Proof. The proof hinges on a case-by-case analysis based on Bourbaki’s tables for the irreducible root systems [Bo].

For $R = A_n$ the stated orthogonality amounts to [Do, Thm. 5.1], upon projection onto the center-of-mass plane as in [D2, Thm. 6.3] (where attention was restricted to the Laplacian in the center-of-mass plane).

For $R = B_n$ the stated orthogonality is retrieved from Theorem 2.1 through the parameter identification $g_\alpha = g$ when α is long and $g_\alpha = 2g_-$ when α is short (cf. e.g.

[BDM, Sec. 4.2]). Indeed, for $g_+ \rightarrow 0$ and $g_+ \rightarrow +\infty$ one recovers the Bethe Ansatz wave functions on the B_n -type Weyl alcove

$$\{x \in \mathbb{R}^n \mid 1 - x_2 > x_1 > x_2 > \dots > x_{n-1} > x_n > 0\} \tag{13.7}$$

that are respectively even and odd with respect to the extended affine reflection of length zero. (This reflection acts on the B_n -type Weyl alcove (13.7) as the reflection-symmetry $x_1 \rightarrow 1 - x_1$ in the affine hyperplane $x_1 = \frac{1}{2}$ for which A (1.2b) determines a fundamental domain).

For $R = C_n$ the stated orthogonality is retrieved from that of Theorem 2.1 through the parameter identification $g_\alpha = g$ when α is short and $g_\alpha = g_+ = g_-$ when α is long (cf. e.g. [BDM, Sec. 4.3]). \square

For $R = D_n$, let $\lambda \rightarrow \lambda^*$ be the orthogonal involution on the cone of dominant weights Λ (13.2) induced by the Dynkin diagram automorphism interchanging the two fundamental spin weights. We define

$$\psi_\pm(\xi_\lambda, x) := \frac{1}{1 + \delta_{\lambda, \lambda^*}} (\psi(\xi_\lambda, x) \pm \psi(\xi_{\lambda^*}, x)), \tag{13.8}$$

where $\delta_{\lambda, \mu} := 1$ if $\lambda = \mu$ and $\delta_{\lambda, \mu} := 0$ otherwise (so $\psi_+(\xi_\lambda, x) = \psi(\xi_\lambda, x)$ and $\psi_-(\xi_\lambda, x) = 0$ if $\lambda = \lambda^*$).

Theorem 13.2 (Type D). *When R is of type D, the (nonvanishing) repulsive Bethe Ansatz wave functions $\psi_\pm(\xi_\lambda, x)$, $\lambda \in \Lambda \pmod{\star}$ constitute an orthogonal basis for the Hilbert space $L^2(A, dx)$.*

Proof. The case $R = D_n$ is retrieved from the case $R = B_n$ using the standard realizations of these root systems in accordance with the tables in Bourbaki [Bo]. Indeed, the limits $g_\alpha \rightarrow 0$ and $g_\alpha \rightarrow +\infty$ for α short (in the second case upon renormalizing through an overall multiplication of the wave function by $(i/g_\alpha)^n \prod_{\alpha \in R_+} \langle \xi, \alpha^\vee \rangle$) recover the Bethe Ansatz wave functions $\psi_+(\xi_\lambda, x)$ and $\psi_-(\xi_\lambda, x)$, respectively (cf. e.g. [BDM, Secs. 4.2, 4.4]). Since the Bethe Ansatz wave functions $\psi(\xi_\lambda, x)$ and $\psi(\xi_{\lambda^*}, x)$ on the D_n -type Weyl alcove

$$\{x \in \mathbb{R}^n \mid 1 - x_2 > x_1 > x_2 > \dots > x_{n-1} > |x_n|\} \tag{13.9}$$

are related by the reflection-symmetry $x_n \rightarrow -x_n$ in the hyperplane $x_n = 0$, it is clear that $\psi_+(\xi_\lambda, x)$ and $\psi_-(\xi_\lambda, x)$ are respectively even and odd with respect to this symmetry (for which the B_n -type Weyl alcove (13.7) determines a fundamental domain). \square

Remark 13.3. The analog of the eigenvalue problem (1.2a)–(1.2d) satisfied by the Bethe Ansatz wave function $\psi(\xi_\lambda, x)$ (13.5) reads [G1, G2, GS, G, HO, D1, EOS1, BDM]:

$$-\Delta\psi = \langle \xi_\lambda, \xi_\lambda \rangle \psi, \tag{13.10a}$$

where Δ denotes the Laplacian on the alcove A (13.6) with repulsive Robin boundary conditions at the walls of the form

$$\langle \nabla\psi, \alpha_j^\vee \rangle - g_{\alpha_j} \psi = 0 \quad \text{when} \quad \langle \alpha_j, x \rangle = 0 \tag{13.10b}$$

($j = 1, \dots, n$) and

$$\langle \nabla\psi, \alpha_0^\vee \rangle + g_{\alpha_0} \psi = 0 \quad \text{when} \quad \langle \alpha_0, x \rangle = 1. \tag{13.10c}$$

Here $\alpha_1, \dots, \alpha_n$ and α_0 refer to the basis of simple roots and to the maximal root of R_+ , respectively, and ∇ denotes the gradient with respect to x . When R is of type D_n , the Bethe Ansatz wave functions $\psi_+(\xi_\lambda, x)$ and $\psi_-(\xi_\lambda, x)$ (13.8) also satisfy the corresponding eigenvalue problem because $\langle \xi_\lambda, \xi_\lambda \rangle = \langle \xi_{\lambda^*}, \xi_{\lambda^*} \rangle$ (as the orthogonal reflection mapping λ to λ^* also maps ξ_λ to ξ_{λ^*}). As a consequence of Theorems 13.1 and 13.2, we may thus conclude that for any classical root system R the repulsive Laplacian $-\Delta$ in Eqs. (13.10a)–(13.10c) constitutes an unbounded essentially self-adjoint operator in $L^2(A, dx)$ that has purely discrete spectrum given by the positive eigenvalues $\langle \xi_\lambda, \xi_\lambda \rangle$, $\lambda \in \Lambda$. Notice in this connection that it is immediate from (the gradient of) the Morse function $V_\lambda(\xi)$ (13.1) that the eigenvalues $\langle \xi_\lambda, \xi_\lambda \rangle$ do not remain bounded as λ varies over the dominant cone Λ (13.2) (cf. also [EOS1, Secs. 9, 10]).

Remark 13.4. From the point of view of root systems, our eigenvalue problem on the hyperoctahedral Weyl alcove in Eqs. (1.2a)–(1.2d) corresponds to the case of a nonreduced root system of type BC_n (cf. also Ref. [EOS2], where the present eigenvalue problem is interpreted in terms of an affine root system associated with $R = C_n$).

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Appendix A: Sklyanin’s Boundary Transfer Operator

This appendix summarizes a special instance of Sklyanin’s extension of the quantum inverse scattering formalism enabling the construction of transfer operators for open systems with boundary interactions [S]. Section 3 relies on this formalism to provide the boundary transfer operator for the open finite q -boson system with diagonal boundary interactions at the lattice endpoints.

Let us assume that $R(u) \in \mathbb{C}[u^{\pm 1}] \otimes M_4(\mathbb{C})$ enjoys the following three symmetries:

(i) *PT-symmetry*

$$R^t(u) = PR(u)P, \tag{A.1}$$

(ii) *Unitarity*

$$R(u)R(u^{-1}) = \rho(u)I, \tag{A.2}$$

(iii) *Crossing unitarity*

$$(R(qu)P)^{t_1}(PR(qu^{-1}))^{t_2} = \hat{\rho}(u)I, \tag{A.3}$$

for certain nonvanishing $\rho(u), \hat{\rho}(u) \in \mathbb{C}[u^{\pm 1}]$ and a suitable shift $q \in \mathbb{C} \setminus \{0\}$. Here $I \in M_4(\mathbb{C})$ stands for the identity matrix, $P \in M_4(\mathbb{C})$ denotes the permutation matrix

$$P := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{A.4}$$

R^t represents the transposed of R , and R^{t_1}, R^{t_2} refer to the partially transposed matrices characterized by the property that $(A \otimes B)^{t_1} = A^t \otimes B$ and $(A \otimes B)^{t_2} = A \otimes B^t$ for $A, B \in M_2(\mathbb{C})$ (and extended to $\mathbb{C}[u^{\pm 1}] \otimes M_4(\mathbb{C})$ by linearity). For instance, our

particular R -matrix in Eq. (3.7) turns out to satisfy these symmetries with $\rho(u) = \hat{\rho}(u) = s(qu)s(qu^{-1})$.

Let \mathbb{A}_+ and \mathbb{A}_- be unital associative algebras over \mathbb{C} , and let $K_{\pm}(u) \in \mathbb{C}[u^{\pm 1}] \otimes M_2(\mathbb{A}_{\pm})$ be solutions of the left reflection equation

$$R(u/v)K_-(u)_1R(quv)K_-(v)_1 = K_-(v)_1R(quv)K_-(u)_1R(u/v) \tag{A.5a}$$

in $\mathbb{C}[u^{\pm 1}, v^{\pm 1}] \otimes M_4(\mathbb{A}_-)$, and the right reflection equation

$$R(u/v)K_+(u)_2R(quv)K_+(v)_2 = K_+(v)_2R(quv)K_+(u)_2R(u/v) \tag{A.5b}$$

in $\mathbb{C}[u^{\pm 1}, v^{\pm 1}] \otimes M_4(\mathbb{A}_+)$, respectively. Following standard conventions, both \mathbb{A}_+ and \mathbb{A}_- are thought of as subalgebras of $\mathbb{A}_+ \otimes \mathbb{A}_-$ via the embeddings $a_+ \rightarrow a_+ \otimes 1$ ($a_+ \in \mathbb{A}_+$) and $a_- \rightarrow 1 \otimes a_-$ ($a_- \in \mathbb{A}_-$), which gives rise to the commutativity

$$a_+a_- := (a_+ \otimes 1)(1 \otimes a_-) = a_+ \otimes a_- = (1 \otimes a_-)(a_+ \otimes 1) =: a_-a_+.$$

Theorem A.1 ([S]). *The boundary transfer operator*

$$\mathcal{T}(u) := \text{tr} \left(K_+^t(u^{-1})K_-(u) \right) \in \mathbb{C}[u^{\pm 1}] \otimes \mathbb{A}_+ \otimes \mathbb{A}_-$$

satisfies the commutativity

$$[\mathcal{T}(u), \mathcal{T}(v)] = 0$$

in $\mathbb{C}[u^{\pm 1}, v^{\pm 1}] \otimes \mathbb{A}_+ \otimes \mathbb{A}_-$.

For a complex unital associative algebra \mathbb{A} , let $U(u)$ be an invertible element of $\mathbb{C}[u^{\pm 1}] \otimes M_2(\mathbb{A})$ satisfying the quantum Yang–Baxter equation

$$R(u/v)U(u)_1U(v)_2 = U(v)_1U(u)_2R(u/v) \tag{A.6}$$

in $\mathbb{C}[u^{\pm 1}, v^{\pm 1}] \otimes M_4(\mathbb{A})$.

Theorem A.2 ([S]). *The boundary monodromy matrix*

$$\mathcal{U}(u) := U(u)K_-(u)U^{-1}(q^{-1}u^{-1}) \in \mathbb{C}[u^{\pm 1}] \otimes M_2(\mathbb{A}_- \otimes \mathbb{A})$$

solves the left reflection equation (A.5a) in $\mathbb{C}[u^{\pm 1}, v^{\pm 1}] \otimes M_4(\mathbb{A}_- \otimes \mathbb{A})$ (upon substituting $\mathcal{U}(u)$ for $K_-(u)$).

By combining these two theorems, it is immediate that the boundary transfer operator

$$\mathcal{T}(u) = \text{tr} \left(K_+^t(u^{-1})\mathcal{U}(u) \right) \in \mathbb{C}[u^{\pm 1}] \otimes \mathbb{A}_+ \otimes \mathbb{A}_- \otimes \mathbb{A} \tag{A.7}$$

satisfies the desired commutativity $[\mathcal{T}(u), \mathcal{T}(v)] = 0$ in $\mathbb{C}[u^{\pm 1}, v^{\pm 1}] \otimes \mathbb{A}_+ \otimes \mathbb{A}_- \otimes \mathbb{A}$, which is the form in which Sklyanin’s formalism was applied in Sect. 3 (with $\mathbb{A}_+ = \mathbb{A}_- = \mathbb{C}$ and $\mathbb{A} = \mathbb{A}_m$).

Both theorems were proven (within a broader setup) by Sklyanin [S], who assumed additionally that $R^t(u) = R(u)$. While minor variations on Sklyanin’s arguments allow to suppress the latter assumption that $R(u)$ be symmetric, in the mean time revised versions of the theorems at issue have been formulated requiring milder conditions on $R(u)$ [MN, FSHY, V]. In particular, the formulation in Ref. [V] only imposes that a partial transposition of the R -matrix be invertible. We close this appendix by a) indicating how Theorem A.1 can be retrieved upon specialization from a more general statement taken from [V, Thm. 2.4] and b) recalling how Theorem A.2 is verified via an elementary computation going back to [S, Prp. 2].

Proof of Theorem A.1. We apply [V, Thm. 2.4] with $\mathcal{U}^+(u) = K_+^t(q^{1/2}u^{-1})$, $\mathcal{U}^-(u) = K_-(q^{-1/2}u)$ and $R_V(u) = R(u)P$, where R_V refers to the R -matrix employed in Ref. [V]. To conclude the commutativity for the boundary transfer operator

$$\mathcal{T}(u) = \text{tr} \left(K_+^t(u^{-1})K_-(u) \right) = \text{tr} \left(\mathcal{U}^+(q^{1/2}u)\mathcal{U}^-(q^{1/2}u) \right)$$

from [V, Thm. 2.4], it is sufficient to check that these \mathcal{U}^+ , \mathcal{U}^- and R_V comply with the assumptions stated in the theorem. Indeed, the crossing unitarity (A.3) of $R(u)$ implies that $R_V^t(u)$ is invertible in $\mathbb{C}[u^{\pm 1}] \otimes M_4(\mathbb{C})$, and the left reflection equation (A.5a) implies that

$$PR_V(u/v)P\mathcal{U}^-(u)_1R_V(uv)\mathcal{U}^-(v)_2 = \mathcal{U}^-(v)_2PR_V(uv)P\mathcal{U}^-(u)_1R_V(u/v),$$

which confirms that $\mathcal{U}^-(u)$ solves the reflection equation [V, Eq. (2.5)]. Finally, the right reflection equation (A.5b) implies that

$$\begin{aligned} R_V(v/u)P(\mathcal{U}^+(u)_2)^{t_2}R_V(q^2u^{-1}v^{-1})P(\mathcal{U}^+(v)_2)^{t_2} \\ = (\mathcal{U}^+(v)_2)^{t_2}R_V(q^2u^{-1}v^{-1})P(\mathcal{U}^+(u)_2)^{t_2}R_V(v/u)P. \end{aligned}$$

Upon exploiting the PT-symmetry (A.1), the unitarity (A.2), and the crossing unitarity (A.3) of $R(u)$, this identity can be recasted in the form

$$\begin{aligned} PR_V^{-t}(u/v)P(\mathcal{U}^+(u)_1)^{t_1}\tilde{R}_V^t(uv)(\mathcal{U}^+(v)_2)^{t_2} \\ = (\mathcal{U}^+(v)_2)^{t_2}P\tilde{R}_V^t(uv)P(\mathcal{U}^+(u)_1)^{t_1}R_V^{-t}(u/v), \end{aligned}$$

where $\tilde{R}_V(u) := (((R_V(u))^{t_1})^{-1})^{t_1}$. This confirms that $\mathcal{U}^+(u)$ solves the dual reflection equation [V, Eq. (2.7)].

Proof of Theorem A.2. Elementary manipulations involving the left reflection equation (A.5a) and the quantum Yang–Baxter equation (A.6) readily entail the desired equality [S, Prp. 2]:

$$\begin{aligned} R(u/v)\mathcal{U}(u)_1R(quv)\mathcal{U}(v)_1 \\ &= R(u/v)U(u)_1K_-(u)_1U^{-1}(q^{-1}u^{-1})_1R(quv)U(v)_1K_-(v)_1U^{-1}(q^{-1}v^{-1})_1 \\ &\stackrel{YB}{=} R(u/v)U(u)_1K_-(u)_1U(v)_2R(quv)U^{-1}(q^{-1}u^{-1})_2K_-(v)_1U^{-1}(q^{-1}v^{-1})_1 \\ &= R(u/v)U(u)_1U(v)_2K_-(u)_1R(quv)K_-(v)_1U^{-1}(q^{-1}u^{-1})_2U^{-1}(q^{-1}v^{-1})_1 \\ &\stackrel{YB}{=} U(v)_1U(u)_2R(u/v)K_-(u)_1R(quv)K_-(v)_1U^{-1}(q^{-1}u^{-1})_2U^{-1}(q^{-1}v^{-1})_1 \\ &\stackrel{RE}{=} U(v)_1U(u)_2K_-(v)_1R(quv)K_-(u)_1R(u/v)U^{-1}(q^{-1}u^{-1})_2U^{-1}(q^{-1}v^{-1})_1 \\ &\stackrel{YB}{=} U(v)_1U(u)_2K_-(v)_1R(quv)K_-(u)_1U^{-1}(q^{-1}v^{-1})_2U^{-1}(q^{-1}u^{-1})_1R(u/v) \\ &= U(v)_1K_-(v)_1U(u)_2R(quv)U^{-1}(q^{-1}v^{-1})_2K_-(u)_1U^{-1}(q^{-1}u^{-1})_1R(u/v) \\ &\stackrel{YB}{=} U(v)_1K_-(v)_1U^{-1}(q^{-1}v^{-1})_1R(quv)U(u)_1K_-(u)_1U^{-1}(q^{-1}u^{-1})_1R(u/v) \\ &= \mathcal{U}(v)_1R(quv)\mathcal{U}(u)_1R(u/v). \end{aligned}$$

Appendix B: Relations Between the Entries of the Monodromy Matrix

The quantum Yang–Baxter equation (A.6) and the left reflection equation (A.5a) each encode up to sixteen relations between the entries of the monodromy matrices

$$U(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \quad \text{and} \quad \mathcal{U}(u) = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix}, \tag{B.1}$$

respectively (cf. Theorem A.2). In this appendix we exhibit the (for our purposes) most relevant ones explicitly, assuming that $R(u)$ is of the concrete form in Eq. (3.7).

(i) *Relations for the periodic monodromy matrix [T, KBI, JM, F].*

Lemma B.1. *The entries $A(u)$, $B(u)$, $C(u)$ and $D(u)$ in $\mathbb{C}[u^{\pm 1}] \otimes \mathbb{A}$ of a (periodic monodromy) matrix $U(u)$ (B.1) solving the quantum Yang–Baxter equation (A.6) associated with $R(u)$ (3.7), satisfy the following relations in $\mathbb{C}[u^{\pm 1}, v^{\pm 1}] \otimes \mathbb{A}$:*

$$[X(u), X(v)] = 0 \text{ for } X = A, B, C, D, \tag{B.2a}$$

$$s(q^{-1}u/v)A(u)B(v) = s(q^{-1})A(v)B(u) + qs(u/v)B(v)A(u), \tag{B.2b}$$

$$s(q^{-1}u/v)B(u)A(v) = q^{-1}s(u/v)A(v)B(u) + s(q^{-1})B(v)A(u), \tag{B.2c}$$

$$C(u)B(v) - tB(v)C(u) = \frac{1-t}{s(u/v)} \left(A(v)D(u) - A(u)D(v) \right), \tag{B.2d}$$

$$tB(u)C(v) - C(v)B(u) = \frac{1-t}{s(u/v)} \left(D(v)A(u) - D(u)A(v) \right), \tag{B.2e}$$

$$A(u)D(u) + D(v)A(u) = A(u)D(v) + D(u)A(v). \tag{B.2f}$$

Proof. The stated relations follow by comparing matrix entries on both sides of the quantum Yang–Baxter equation (A.6). For (B.2a) one considers the matrix entries at the four corners (1, 1), (1, 4), (4, 1) and (4, 4). Eqs. (B.2b), (B.2c), (B.2d) and (B.2e) correspond to the matrix entries at the positions (1, 2), (1, 3), (2, 2) and (3, 3), respectively. Finally, Eq. (B.2f) follows from Eq. (B.2a) and the commutativity $[T(u), T(v)] = 0$ of the transfer operator $T(u) = A(u) + D(u)$. \square

(ii) *Relations for the boundary monodromy matrix [S].*

Lemma B.2. *The entries $\mathcal{A}(u)$, $\mathcal{B}(u)$, $\mathcal{C}(u)$ and $\mathcal{D}(u)$ in $\mathbb{C}[u^{\pm 1}] \otimes \mathbb{A}_- \otimes \mathbb{A}$ of a (boundary monodromy) matrix $\mathcal{U}(u)$ (B.1) solving the left reflection equation (A.5a) (upon substituting $\mathcal{U}(u)$ for $K_-(u)$) associated with $R(u)$ (3.7), satisfy the following relations in $\mathbb{C}[u^{\pm 1}, v^{\pm 1}] \otimes \mathbb{A}_- \otimes \mathbb{A}$:*

$$[\mathcal{B}(u), \mathcal{B}(v)] = [\mathcal{C}(u), \mathcal{C}(v)] = 0, \tag{B.3a}$$

$$\mathcal{A}(u)\mathcal{B}(v) \tag{B.3b}$$

$$= \frac{s(qu/v)s(quv)}{s(u/v)s(uv)} \mathcal{B}(v)\mathcal{A}(u) + \frac{s(q^{-1})s(quv)}{s(u/v)s(uv)} \mathcal{B}(u)\mathcal{A}(v) + \frac{s(q)}{s(uv)} \mathcal{B}(u)\mathcal{D}(v),$$

and

$$\begin{aligned} \mathcal{D}(u)\mathcal{B}(v) &= \frac{s(q^{-1}u/v)s(q^{-1}uv)}{s(u/v)s(uv)}\mathcal{B}(v)\mathcal{D}(u) + \frac{s(q)s(q^{-1}uv)}{s(u/v)s(uv)}\mathcal{B}(u)\mathcal{D}(v) \quad (\text{B.3c}) \\ &+ \frac{s(q)s(q^{-1})c(q)}{s(u/v)s(uv)}\mathcal{B}(v)\mathcal{A}(u) + \frac{s(q^{-1})s(q^{-2}u/v)}{s(u/v)s(uv)}\mathcal{B}(u)\mathcal{A}(v). \end{aligned}$$

Proof. The stated relations follow by comparing matrix entries on both sides of the left reflection equation (A.5a). For the relations in Eq. (B.3a) one considers the two corner matrix entries (1,4) and (4,1), respectively. The relation in Eq. (B.3b) is read-off in turn from the matrix entries at the position (1, 2). For Eq. (B.3c) one considers the matrix entries at position (2,4) to deduce that

$$\begin{aligned} \mathcal{D}(u)\mathcal{B}(v) &= \frac{s(q^{-1}u/v)s(uv)}{s(u/v)s(quv)}\mathcal{B}(v)\mathcal{D}(u) + \frac{s(q)s(uv)}{s(u/v)s(quv)}\mathcal{B}(u)\mathcal{D}(v) \\ &+ \frac{s(q)s(q^{-1})}{s(u/v)s(quv)}\mathcal{A}(u)\mathcal{B}(v) + \frac{s(q^{-1})s(q^{-1}u/v)}{s(u/v)s(quv)}\mathcal{A}(v)\mathcal{B}(u). \end{aligned}$$

Invoking of Eq. (B.3b) allows to rewrite $\mathcal{A}(u)\mathcal{B}(v)$ in terms of $\mathcal{B}(v)\mathcal{A}(u)$, $\mathcal{B}(u)\mathcal{A}(v)$ and $\mathcal{B}(u)\mathcal{D}(v)$, and to rewrite $\mathcal{A}(v)\mathcal{B}(u)$ in terms of $\mathcal{B}(u)\mathcal{A}(v)$, $\mathcal{B}(v)\mathcal{A}(u)$ and $\mathcal{B}(v)\mathcal{D}(u)$, respectively. Upon simplification of the coefficients, Eq. (B.3c) now follows. \square

Following Sklyanin [S] we put

$$\hat{\mathcal{D}}(u) := \mathcal{D}(u) + \frac{s(q)}{s(u^2)}\mathcal{A}(u), \quad (\text{B.4})$$

which permits to rewrite the last two relations of Lemma B.2 in a somewhat more symmetric form.

Lemma B.3. *One has that*

$$\mathcal{A}(u)\mathcal{B}(v) = f_1(u, v)\mathcal{B}(v)\mathcal{A}(u) + f_2(u, v)\mathcal{B}(u)\mathcal{A}(v) + f_3(u, v)\mathcal{B}(u)\hat{\mathcal{D}}(v) \quad (\text{B.5a})$$

and that

$$\hat{\mathcal{D}}(u)\mathcal{B}(v) = g_1(u, v)\mathcal{B}(v)\hat{\mathcal{D}}(u) + g_2(u, v)\mathcal{B}(u)\hat{\mathcal{D}}(v) + g_3(u, v)\mathcal{B}(u)\mathcal{A}(v), \quad (\text{B.5b})$$

where

$$\begin{aligned} f_1(u, v) &= \frac{s(quv)s(qu/v)}{s(u/v)s(uv)}, & g_1(u, v) &= \frac{s(q^{-1}u/v)s(q^{-1}uv)}{s(u/v)s(uv)}, \\ f_2(u, v) &= \frac{s(q^{-1})s(qv^2)}{s(u/v)s(v^2)}, & g_2(u, v) &= \frac{s(q)s(q^{-1}u^2)}{s(u/v)s(u^2)}, \\ f_3(u, v) &= \frac{s(q)}{s(uv)}, & g_3(u, v) &= \frac{s(q^{-1})s(q^{-1}u^2)s(qv^2)}{s(uv)s(u^2)s(v^2)}. \end{aligned}$$

Proof. Substitution of $\hat{\mathcal{D}}(u)$ (B.4) into Eq. (B.5a) readily reproduces Eq. (B.3b) upon simplification of the coefficients. Similarly, substitution of $\hat{\mathcal{D}}(u)$ (B.4) into Eq. (B.5b) results in an expression for $\mathcal{D}(u)\mathcal{B}(v)$ in terms of $\mathcal{B}(v)\mathcal{D}(u)$, $\mathcal{B}(u)\mathcal{D}(v)$, $\mathcal{B}(v)\mathcal{A}(u)$, $\mathcal{B}(u)\mathcal{A}(v)$ and $\mathcal{A}(u)\mathcal{B}(v)$. After rewriting this last term with the aid of Eq. (B.3b), one reproduces Eq. (B.3c) upon simplification of the coefficients. \square

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