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Bispectral Dual Difference Equations for the Quantum Toda Chain with Boundary Perturbations

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We show that hyperoctahedral Whittaker functions—diagonalizing an open quantum Toda chain with one-sided boundary potentials of Morse type—satisfy a dual system of difference equations in the spectral variable. This extends a well-known bispectral duality between the nonperturbed open quantum Toda chain and a strong-coupling limit of the rational Macdonald-Ruijsenaars difference operators. It is manifest from the difference equations in question that the hyperoctahedral Whittaker function is entire as a function of the spectral variable.

1 Introduction

We consider hyperoctahedral Whittaker functions that arise as eigenfunctions for the Hamiltonian of an open quantum Toda chain with one-sided boundary potentials of Morse type:

$$\mathrm{H} = -\frac{1}{2}\sum_{1\leq j\leq n}\frac{\partial^2}{\partial x_j^2} + \mathrm{e}^{-x_1+x_2} + \mathrm{e}^{-x_2+x_3} + \cdots + \mathrm{e}^{-x_{n-1}+x_n} + a\mathrm{e}^{-x_n} + b\mathrm{e}^{-2x_n}. \tag{1.1}$$

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Here the independent variables $x_1, \ldots, x_n$ represent one-dimensional particle positions and the constants a, b denote two coupling parameters governing the perturbation of the chain at the boundary. It is known that the boundary potentials in question preserve the integrability of the Toda chain [21, 43] and that they deform several integrable boundary perturbations stemming from the simple Lie algebras of classical type [3, 15, 26, 32, 41]. Specifically, for $ab = 0$, the quantum Toda Hamiltonians associated with the Lie algebras of type A_{n-1} ($a = 0$, $b = 0$), type B_n ($a \neq 0$, $b = 0$) and type C_n ($a = 0$, $b \neq 0$) are recovered as degenerations.

Our main objective is to show that the hyperoctahedral Whittaker functions diagonalizing H (1.1) satisfy a dual system of difference equations in the spectral variable, which themselves can be interpreted as eigenvalue equations for a quantum integrable particle system. The commuting quantum integrals for this dual particle system turn out to be given by a strong-coupling limit of rational Macdonald-Ruijsenaars type difference operators with hyperoctahedral symmetry introduced in [8]. This extends a corresponding bispectral duality [11, 16] between the nonperturbed open quantum Toda chain and a strong-coupling limit of the (conventional) rational Macdonald-Ruijsenaars operators [1, 4, 17, 25, 27, 44]. At the level of classical mechanics, the latter bispectral duality for the (standard A_{n-1} type) open Toda chain manifests itself in the fact that the action-angle transforms linearizing the corresponding classical-mechanical particle system and its dual are inverses of each other [13, 40].

For any simple Lie algebra, the quantum Toda chain is diagonalized by an associated (class-one) Whittaker function [12, 15, 26, 41]. These multivariate confluent hypergeometric functions have been studied extensively in the literature in diverse contexts, cf. for example, [2, 4, 5, 7, 14, 18, 22–25, 31, 38, 45, 46] and references therein. In relation to the quantum Toda chain, the Whittaker function arises for example, via a connection formula that both normalizes and symmetrizes the Harish-Chandra series solution of the eigenvalue problem (also known as the fundamental Whittaker function) [2, 18]. It can moreover be seen as a confluent limit [37, 42] of the Heckman–Opdam hypergeometric function (pertaining to the (reduced) root system of the underlying Lie algebra) [19, 35]. With the aid of this kind of hypergeometric confluences, dual difference equations for the Heckman–Opdam hypergeometric function were recently seen to degenerate to corresponding difference equations for the Whittaker function [10]. By extending this scheme to the case of the Heckman–Opdam hypergeometric function associated with the nonreduced root system of type BC_n, we arrive below at the hyperoctahedral Whittaker function diagonalizing H (1.1) together with a corresponding system of dual difference equations in the spectral variable. As an application, the difference equations in question are employed to infer that the dependence of the hyperoctahedral Whittaker function on the spectral variable is holomorphic everywhere.

The presentation is organized as follows. In Sections 2 and 3, the Harish-Chandra series and the hyperoctahedral Whittaker function for the quantum Toda Hamiltonian H (1.1) are constructed for generic values of the spectral variable. The associated bispectral dual difference equations are exhibited in Section 4 and in Section 5, these are subsequently used to deduce that the hyperoctahedral Whittaker function extends to an entire function of the spectral variable. Some of the more technical parts of our discussion are isolated from the main flow of the arguments and postponed towards the end. Specifically, Section 6 confirms the convergence of the Harish-Chandra series for the fundamental Whittaker function, and Section 7 verifies that the function in question arises from the Harish-Chandra series for the hypergeometric equation of type BC_n via a confluent limit of the kind considered by Shimeno and Oshima [37, 42]. In Section 8, this same confluence is employed to retrieve the dual difference equations from their hypergeometric counterparts in [10, Theorem 2].

Remark 1. For $ab \neq 0$, the Morse potential at the boundary depends effectively on a single nontrivial coupling parameter only. Indeed, by translating the center of mass of the particle system ($x_j \to x_j + c$, $j = 1, \ldots, n$), we may normalize *one* of either two coupling parameters a or b ($\neq 0$) to a fixed (nonzero) value of choice. Below we will assume that $b \neq 0$ and use the translational freedom to fix the strength of this coupling at

$$b \equiv \frac{1}{8} \tag{1.2}$$

(while the parameter a is allowed to vanish). Notice in this connection that in H (1.1) the coupling strength of the pair potential $\mathrm{e}^{-x_k+x_{k+1}}$ ($k \in \{1, \ldots, n-1\}$) was also conveniently normalized at unit value in a similar way, by exploiting the center-of-mass translational freedom of the particle cluster corresponding to the positions $x_1, \ldots, x_k$. In all our formulas below, we can in principle undo the above normalization of b (by reverting the translation of the center of mass). With some care and assuming that $a \neq 0$, it is then straightforward to recuperate also the case of a vanishing coupling strength b by performing the limit $b \to 0$. □

2 Harish-Chandra Series: Fundamental Whittaker Function

Following a classical approach going back to Harish-Chandra—cf. for example, Ref. [20, Chapter IV.5] and [18, Section 4]—we first construct a power-series solution of the eigenvalue problem for H (1.1).

Let us denote the unit vectors of the standard basis for $\mathbb{C}^n$ by $\mathrm{e}_1, \ldots, \mathrm{e}_n$ and let $\langle \cdot, \cdot \rangle$ represent the natural bilinear scalar product turning these into an orthonormal basis, that is, for $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$ and $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$:

$$\langle \xi, x \rangle := \xi_1 x_1 + \cdots + \xi_n x_n.$$

The Toda Laplacian with Morse perturbations is defined as

$$L_x := \sum_{1 \le j \le n} \frac{\partial^2}{\partial x_j^2} - \sum_{\alpha \in S} \mathrm{a}_\alpha \mathrm{e}^{-\langle \alpha, x \rangle}, \tag{2.1a}$$

where

$$S := \{\mathrm{e}_1 - \mathrm{e}_2, \mathrm{e}_2 - \mathrm{e}_3, \ldots, \mathrm{e}_{n-1} - \mathrm{e}_n, \mathrm{e}_n, 2\mathrm{e}_n\} \tag{2.1b}$$

and (we have picked the normalization, cf. Remark 1 above)

$$\mathrm{a}_{\mathrm{e}_1 - \mathrm{e}_2} = \cdots = \mathrm{a}_{\mathrm{e}_{n-1} - \mathrm{e}_n} = 2, \quad \mathrm{a}_{\mathrm{e}_n} = g, \quad \mathrm{a}_{2\mathrm{e}_n} = \frac{1}{4}. \tag{2.1c}$$

So $\mathrm{H} = -\frac{1}{2} L_x$ with $a = \frac{g}{2}$ and $b = \frac{1}{8}$.

For $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{Z}^n$, we write that

$$\nu \ge 0 \Leftrightarrow \nu_1 + \cdots + \nu_k \ge 0 \text{ for } k = 1, \ldots, n, \tag{2.2}$$

and that $\nu > 0$ if $\nu \ge 0$ and $\nu \neq 0$. Given a wave vector ξ in the dense domain of $\mathbb{C}^n$ of the form

$$\mathbb{U}^n := \{\xi \in \mathbb{C}^n \mid \langle \nu - 2\xi, \nu \rangle \neq 0, \forall \nu > 0\}, \tag{2.3}$$

the Harish-Chandra series is now defined as the formal power series

$$\phi_\xi(x; g) := \sum_{\nu \ge 0} a_\nu(\xi; g) \mathrm{e}^{\langle \xi - \nu, x \rangle}, \tag{2.4a}$$

with expansion coefficients determined by the recurrence

$$a_\nu(\xi; g) = \frac{1}{\langle \nu - 2\xi, \nu \rangle} \sum_{\alpha \in S} \mathrm{a}_\alpha a_{\nu - \alpha}(\xi; g) \qquad (\nu > 0) \tag{2.4b}$$

and the initial conditions

$$a_\nu(\xi;g) := \begin{cases} 1 & \text{if } \nu = 0, \\ 0 & \text{if } \nu \not\geq 0. \end{cases} \tag{2.4c}$$

Notice that the uniqueness of the expansion coefficients $a_\nu(\xi;g)$ (2.4b), (2.4c) is guaranteed as the cone of integral wave vectors $\nu \geq 0$ is nonnegatively generated by S (2.1b).

The following proposition confirms that the above Harish-Chandra series converges to an eigenfunction of L_x (2.1a)–(2.1c). Adopting standard terminology, we will refer to this eigenfunction as the *fundamental Whittaker function*.

Proposition 1 (Fundamental Whittaker Function)**.** (i) The Harish-Chandra series $\phi_\xi(x;g)$ (2.4a)–(2.4c) converges absolutely and uniformly on compacts to a holomorphic function of $(\xi, x, g) \in \mathbb{U}^n \times \mathbb{C}^n \times \mathbb{C}$.

(ii) The (fundamental Whittaker) function in question provides an eigenfunction of the Toda Laplacian L_x (2.1a)–(2.1c):

$$L_x \phi_\xi(x;g) = \langle \xi, \xi \rangle \phi_\xi(x;g) \tag{2.5a}$$

that enjoys a plane-wave asymptotics of the form

$$\lim_{x\to+\infty} |\phi_\xi(x;g) - e^{\langle \xi, x\rangle}| = 0 \quad \text{for} \quad \mathrm{Re}(\xi) = 0, \tag{2.5b}$$

where the notation $x \to +\infty$ means that $x_k - x_{k+1} \to +\infty$ for $k = 1, \ldots, n$ (with the convention that $x_{n+1} \equiv 0$). □

The proof of this proposition hinges on growth estimates for the expansion coefficients $a_\nu(\xi;g)$ stemming from the recurrence relations that ensure the absolute convergence of the power series. The reader is referred to Section 6 below for the particular features of this (classical) argument in the present setting.

In view of Proposition 1, it is immediate that—as a function of the spectral variable ξ—the fundamental Whittaker function extends uniquely to a meromorphic function on $\mathbb{C}^n$, with possible poles at hyperplanes belonging to the locally finite collection $\mathbb{C}^n \setminus \mathbb{U}^n$. Most of these singularities turn out to be removable.

Proposition 2 (Regularity Domain)**.** The fundamental Whittaker function $\phi_\xi(x;g)$ (2.4a)–(2.4c) extends uniquely to a holomorphic function of $(\xi, x, g) \in \mathbb{C}^n_{\mathrm{reg},+} \times \mathbb{C}^n \times \mathbb{C}$, where

$$\mathbb{C}^n_{\mathrm{reg},+} := \{\xi \in \mathbb{C}^n \mid 2\xi_j \notin \mathbb{Z}_{>0},\ \xi_j \pm \xi_k \notin \mathbb{Z}_{>0}\ (j<k)\}. \tag{2.6}$$

Moreover, as a (meromorphic) function of the spectral variable $\xi \in \mathbb{C}^n$ the function under consideration has at most simple poles along the hyperplanes belonging to $\mathbb{C}^n \setminus \mathbb{C}^n_{\mathrm{reg},+}$. □

A corresponding regularity result for the Harish-Chandra series solving the hypergeometric equation associated with root systems was proven by Opdam, cf. [34, Corollary 2.2, Corollary 2.10], [19, Proposition 4.2.5], [35, Lemma 6.5], and [30, Theorem 1.2]. In Section 7, we will deduce Proposition 2 from Opdam's result with the aid of a hypergeometric confluence in the spirit of [37, 42].

3 Connection Formula: Hyperoctahedral Whittaker Function

Upon normalizing with an appropriate c-function, the hyperoctahedral Whittaker function is built from the fundamental Whittaker function through symmetrization with respect to the hyperoctahedral group of signed permutations (acting on the spectral variable). For the quantum Toda systems associated with the simple Lie algebras, an analogous construction of the corresponding Whittaker functions was carried out in Hashizume's seminal paper [18] (cf. also [2, Section 4]). Specifically, our formulas provide a parameter-deformation ($ab = 0 \to ab \neq 0$) that unifies the Hashizume wave functions for the quantum Toda Hamiltonians of type B_n and C_n (cf. Remark 1 above).

We consider the following action of the *hyperoctahedral group* $W := S_n \ltimes \{1,-1\}^n$ of *signed permutations* $w = (\sigma, \epsilon)$ on $\mathbb{C}^n$:

$$\xi = (\xi_1, \ldots, \xi_n) \xrightarrow{w} (\epsilon_1 \xi_{\sigma_1}, \ldots, \epsilon_n \xi_{\sigma_n}) =: w\xi, \tag{3.1}$$

where $\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{pmatrix}$ belongs to the symmetric group S_n and $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ with $\epsilon_j \in \{1,-1\}$ (for $j = 1, \ldots, n$). Let $\mathbb{C}^n_{\mathrm{reg}}$ denote the dense domain of $\mathbb{C}^n$ of the form

$$\mathbb{C}^n_{\mathrm{reg}} := \{\xi \in \mathbb{C}^n \mid 2\xi_j \notin \mathbb{Z},\ \xi_j \pm \xi_k \notin \mathbb{Z}\ (j<k)\}. \tag{3.2}$$

For $(\xi, x, g) \in \mathbb{C}^n_{\mathrm{reg}} \times \mathbb{C}^n \times \mathbb{C}$, the *hyperoctahedral Whittaker function* is now defined via the following *connection formula*:

$$\Phi_\xi(x; g) := \sum_{w \in W} C(w\xi; g)\phi_{w\xi}(x; g), \tag{3.3a}$$

$$C(\xi; g) := \prod_{1 \le j \le n} \frac{\Gamma(2\xi_j)}{\Gamma(\frac{1}{2} + g + \xi_j)} \prod_{1 \le j < k \le n} \Gamma(\xi_j + \xi_k)\Gamma(\xi_j - \xi_k), \tag{3.3b}$$

where $\Gamma(\cdot)$ refers to the gamma function [33, Chapter 5]. Notice that the c-function $C(\xi;g)$ (3.3b) is regular for $(\xi,g) \in \mathbb{C}_{\text{reg},-} \times \mathbb{C}$, where

$$\mathbb{C}^n_{\text{reg},-} := \{\xi \in \mathbb{C}^n \mid 2\xi_j \notin \mathbb{Z}_{\leq 0},\ \xi_j \pm \xi_k \notin \mathbb{Z}_{\leq 0}\ (j<k)\} \tag{3.4}$$

(so $\mathbb{C}^n_{\text{reg}} = \mathbb{C}^n_{\text{reg},+} \cap \mathbb{C}^n_{\text{reg},-}$), whence $\Phi_\xi(x;g)$ is a holomorphic function of $(\xi,x,g) \in \mathbb{C}^n_{\text{reg}} \times \mathbb{C}^n \times \mathbb{C}$. Moreover, since the simple poles along the hyperplanes $\xi_j = 0$ and $\xi_j \pm \xi_k = 0$ $(1 \leq j \neq k \leq n)$ stemming from the c-functions are removable in the final expression due to the hyperoctahedral symmetry, the regularity of $\Phi_\xi(x;g)$ is manifest (in particular) when $\text{Re}(\xi) = 0$ (cf. also Theorem 4 below for a much stronger statement ensuring the regularity of the hyperoctahedral Whittaker function for all values of $\xi \in \mathbb{C}^n$).

Remark 2. It is immediate from Proposition 1 that the hyperoctahedral Whittaker function solves the eigenvalue equation for the Toda Laplacian L_x (2.1a)–(2.1c):

$$L_x\Phi_\xi(x;g) = \langle \xi,\xi\rangle \Phi_\xi(x;g). \tag{3.5a}$$

By construction, this particular solution is W-invariant in the spectral variable $\Phi_{w\xi}(x;g) = \Phi_\xi(x;g)$, $\forall w \in W$, and it enjoys the following plane-wave asymptotics determined by the c-function $C(\xi;g)$ (3.3b):

$$\lim_{x\to+\infty}\left|\Phi_\xi(x;g) - \sum_{w\in W} C(w\xi;g)e^{\langle w\xi,x\rangle}\right| = 0 \quad \text{for} \quad \text{Re}(\xi) = 0. \tag{3.5b}$$

□

4 Bispectral Duality: Difference Equations

The (class-one) Whittaker functions diagonalizing the quantum Toda chains associated with the simple Lie algebras satisfy an explicit system of difference equations in the spectral variable [10, Theorem 3]. The identities in question generalize previously known difference equations for the Whittaker function associated with the Lie algebra of type A_{n-1} [1, 4, 25, 27, 44]. Our main result consists of the following system of difference equations for the hyperoctahedral Whittaker function diagonalizing the Toda Laplacian with Morse perturbation L_x (2.1a)–(2.1c). The proof of these difference equations is relegated to Section 8 below.

Theorem 3 (Dual Difference Equations). For any $(\xi,x,g) \in \mathbb{C}^n_{\text{reg}} \times \mathbb{C}^n \times \mathbb{C}$ and $\ell \in \{1,\ldots,n\}$, the hyperoctahedral Whittaker function $\Phi_\xi(x;g)$ (3.3a), (3.3b) satisfies the difference equation

$$\sum_{\substack{J\subset\{1,\ldots,n\},\,0\leq|J|\leq\ell\\ \epsilon_j\in\{1,-1\},\,j\in J}} U_{J^c,\,\ell-|J|}(\xi;g)V_{\epsilon J}(\xi;g)\Phi_{\xi+e_{\epsilon J}}(x;g)=e^{x_1+\cdots+x_\ell}\,\Phi_\xi(x;g), \tag{4.1a}$$

where

$$V_{\epsilon J}(\xi;g):=\prod_{j\in J}\frac{(\frac12+g+\epsilon_j\xi_j)}{2\xi_j(2\xi_j+\epsilon_j)}\prod_{\substack{j\in J\\ k\notin J}}(\xi_j^2-\xi_k^2)^{-1}$$

$$\times\prod_{\substack{j,j'\in J\\ j<j'}}(\epsilon_j\xi_j+\epsilon_{j'}\xi_{j'})^{-1}(1+\epsilon_j\xi_j+\epsilon_{j'}\xi_{j'})^{-1} \tag{4.1b}$$

and

$$U_{K,p}(\xi;g):=(-1)^{p(p+1)/2}\sum_{\substack{I\subset K,\,|I|=p\\ \epsilon_i\in\{1,-1\},\,i\in I}}\Biggl(\prod_{i\in I}\frac{\frac12+g+\epsilon_i\xi_i}{2\xi_i(2\xi_i+\epsilon_i)}\prod_{\substack{i\in I\\ k\in K\setminus I}}(\xi_i^2-\xi_k^2)^{-1} \tag{4.1c}$$

$$\times\prod_{\substack{i,i'\in I\\ i<i'}}(\epsilon_i\xi_i+\epsilon_{i'}\xi_{i'})^{-1}(1+\epsilon_i\xi_i+\epsilon_{i'}\xi_{i'})^{-1}\Biggr).$$

Here $|J|$ denotes the cardinality of $J\subset\{1,\ldots,n\}$, $J^c:=\{1,\ldots,n\}\setminus J$, and we have employed the compact notation $e_{\epsilon J}:=\sum_{j\in J}\epsilon_j e_j$. □

For $\ell=1$ the difference equation in Theorem 3 boils down to the following identity

$$\sum_{\substack{1\leq j\leq n\\ \epsilon\in\{1,-1\}}} V_j(\epsilon\xi;g)\Bigl(\Phi_{\xi+\epsilon e_j}(x;g)-\Phi_\xi(x;g)\Bigr)=e^{x_1}\,\Phi_\xi(x;g), \tag{4.2a}$$

where

$$V_j(\xi;g)=\frac{\frac12+g+\xi_j}{2\xi_j(2\xi_j+1)}\prod_{\substack{1\leq k\leq n\\ k\neq j}}(\xi_j^2-\xi_k^2)^{-1}. \tag{4.2b}$$

Remark 3. Theorem 3 reveals that $\Phi_\xi(x;g)$ (3.3a), (3.3b) constitutes a joint eigenfunction for a quantum integrable system of discrete difference operators acting on meromorphic functions of the spectral variable ξ:

$$D_{\xi,\ell}:=\sum_{\substack{J\subset\{1,\ldots,n\},\,0\leq|J|\leq\ell\\ \epsilon_j\in\{1,-1\},\,j\in J}} U_{J^c,\,\ell-|J|}(\xi;g)V_{\epsilon J}(\xi;g)T_{\epsilon J,\xi}\quad(\ell=1,\ldots,n), \tag{4.3}$$

where $(T_{\epsilon J,\xi}f)(\xi) := f(\xi + e_{\epsilon J})$ for $f : \mathbb{C}^n \to \mathbb{C}$. Indeed, the difference operators $D_{\xi,1}, \ldots, D_{\xi,n}$ (4.3) amount to a strong-coupling limit of the rational commuting hyperoctahedral Macdonald-Ruijsenaars type operators introduced in [8, Section II] (cf. also [9, Eq. (2.10)]). This identifies the latter integrable quantum system as the bispectral dual [16] of the quantum Toda system with Morse term L_x (2.1a)–(2.1c), via the following (bispectral) extension of the eigenvalue equation in Remark 2:

$$L_x \Phi_\xi(x; g) = \langle \xi, \xi \rangle \Phi_\xi(x; g), \quad D_{\xi,\ell} \Phi_\xi(x; g) = e^{x_1 + \cdots + x_\ell} \Phi_\xi(x; g) \tag{4.4}$$

($\ell = 1, \ldots, n$). □

Remark 4. It is instructive to detail the above formulas somewhat further in the classical situation of a single particle ($n = 1$). It is then well-known—cf. for example, Ref. [28] and references therein—that for the corresponding Schrödinger eigenvalue problem on the line with a Morse potential

$$L_x \phi = \xi^2 \phi \quad \text{with} \quad L_x = \frac{d^2}{dx^2} - g e^{-x} - \frac{1}{4} e^{-2x}, \tag{4.5}$$

the unique solution characterized by a normalized plane-wave asymptotics of the form $\lim_{x \to +\infty} |\phi(x) - e^{\xi x}| = 0$ for $\mathrm{Re}(\xi) = 0$ can be conveniently expressed explicitly in terms of Whittaker's (fundamental) M-function [33, 13.14.2]:

$$\phi_\xi(x; g) = e^{\frac{x}{2}} M_{-g,-\xi}(e^{-x}) = e^{\xi x} e^{-\frac{1}{2} e^{-x}} {}_1F_1(\frac{1}{2} + g - \xi, 1 - 2\xi; e^{-x}) \tag{4.6}$$

($2\xi \notin \mathbb{Z}_{>0}$). From the connection formula (3.3a), (3.3b), it is now manifest that for $n = 1$ our hyperoctahedral Whittaker function reduces essentially to Whittaker's (class-one) W-function [33, 13.14.33]:

$$\Phi_\xi(x; g) = \frac{\Gamma(2\xi)}{\Gamma(\frac{1}{2} + g + \xi)} \phi_\xi(x; g) + \frac{\Gamma(-2\xi)}{\Gamma(\frac{1}{2} + g - \xi)} \phi_{-\xi}(x; g) = e^{\frac{x}{2}} W_{-g,\xi}(e^{-x}) \tag{4.7}$$

($2\xi \notin \mathbb{Z}$). In the present univariate case, Theorem 3 specializes to the following elementary difference equation for this celebrated Whittaker function (cf. Eqs. (4.2a), (4.2b)):

$$\begin{aligned} &\frac{\frac{1}{2} + g + \xi}{2\xi(2\xi + 1)} \big(\Phi_{\xi+1}(x; g) - \Phi_\xi(x; g)\big) + \\ &\frac{(\frac{1}{2} + g - \xi)}{2\xi(2\xi - 1)} \big(\Phi_{\xi-1}(x; g) - \Phi_\xi(x; g)\big) = e^x \Phi_\xi(x; g). \end{aligned} \tag{4.8}$$

At $g = 0$, the latter identity recovers a classical recurrence relation for Macdonald's modified Bessel function $\Phi_\xi(x,0) = \mathrm{e}^{\frac{x}{2}} W_{0,\xi}(\mathrm{e}^{-x}) = \frac{1}{\sqrt{\pi}} K_\xi(\frac{1}{2}\mathrm{e}^{-x})$ [33, 10.29.1, 13.18.9]:

$$\frac{1}{4\xi}\left(\Phi_{\xi+1}(x;0) - \Phi_{\xi-1}(x;0)\right) = \mathrm{e}^x \Phi_\xi(x;0). \tag{4.9}$$

For general g, the difference equation in Eq. (4.8) can be retrieved by combining the recurrence relations in [33, 13.15.10, 13.15.12] (we thank T.H. Koornwinder for pointing this out). □

5 Analytic Continuation in the Spectral Variable

In this section, we employ the difference equations of Theorem 3 to extend the hyperoctahedral Whittaker function analytically in the spectral variable to an entire function of $(\xi, x, g) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}$.

5.1 Regularity

Specifically, by inspecting the singularities of the difference equations we will read-off the following regularity of $\Phi_\xi(x;g)$ in the spectral variable.

Theorem 4 (Analyticity of the Hyperoctahedral Whittaker Function)**.** The singularities of the hyperoctahedral Whittaker function in the spectral variable $\xi \in \mathbb{C}^n$ along the hyperplanes of $\mathbb{C}^n \setminus \mathbb{C}^n_{\text{reg}}$ are removable, that is, by Hartogs' theorem $\Phi_\xi(x;g)$ (3.3a), (3.3b) extends (uniquely) to an entire function of $(\xi, x, g) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}$. □

This analyticity of the hyperoctahedral Whittaker function implies in turn that the difference equations themselves extend as holomorphic identities on $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}$.

Corollary 5 (Analyticity of the Dual Difference Equations)**.** The difference equations of Theorem 3 extend (uniquely) to identities between entire functions of the variables $(\xi, x, g) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}$. □

5.2 Residue calculus

While for any $\ell \in \{1, \ldots, n\}$ the analyticity of both sides of the difference equation in Corollary 5 is an immediate consequence of the asserted analyticity of $\Phi_\xi(x;g)$ (3.3a), (3.3b) in Theorem 4, the verification of the latter regularity requires a detailed study of the residues of the hyperoctahedral Whittaker function in ξ along the hyperplanes of $\mathbb{C}^n \setminus \mathbb{C}^n_{\text{reg}}$. In view of the hyperoctahedral symmetry, it is sufficient to infer that the residues of $\Phi_\xi(x;g)$ along the hyperplanes $2\xi_1 \in \mathbb{Z}_{>0}$ and $\xi_1+\xi_2 \in \mathbb{Z}_{>0}$ vanish.

Specifically, for a fixed integer m and a meromorphic function $f:\mathbb{C}^n \to \mathbb{C}$ with at most simple poles along the hyperplanes

$$H_{1;m} := \{\xi \in \mathbb{C}^n \mid 2\xi_1 = m\} \quad \text{and} \quad H_{12;m} := \{\xi \in \mathbb{C}^n \mid \xi_1+\xi_2 = m\},$$

the residues of f along these hyperplanes are given by the meromorphic functions,

$$\mathrm{Res}_{1;m} f(\xi) := \lim_{2\xi_1 \to m} (2\xi_1 - m) f(\xi),$$

$$\mathrm{Res}_{12;m} f(\xi) := \lim_{\xi_1+\xi_2 \to m} (\xi_1+\xi_2 - m) f(\xi)$$

on $H_{1;m}$ and $H_{12;m}$, respectively. In other words, $\mathrm{Res}_{1;m} f(\xi)$ and $\mathrm{Res}_{12;m} f(\xi)$ are the restrictions $[(2\xi_1-m)f(\xi)]_{2\xi_1=m}$ and $[(\xi_1+\xi_2-m)f(\xi)]_{\xi_1+\xi_2=m}$ of the meromorphic functions $(2\xi_1-m)f(\xi)$ to $H_{1;m}$ and $(\xi_1+\xi_2-m)f(\xi)$ to $H_{12;m}$, respectively. We will now check that for $m \in \mathbb{Z}_{>0}$ both $\mathrm{Res}_{1;m}\Phi_\xi(x;g)$ and $\mathrm{Res}_{12;m}\Phi_\xi(x;g)$ vanish by induction in m, assuming that $\mathrm{Res}_{1;k}\Phi_\xi(x;g) = 0$ and $\mathrm{Res}_{12;k}\Phi_\xi(x;g) = 0$ for $0 \le k < m$. Recall that for $m=1$, this induction hypothesis is fulfilled trivially by virtue of the hyperoctahedral symmetry.

5.2.1 Verification that $\mathrm{Res}_{1;m}\Phi_\xi(x;g) = 0$ for $m \in \mathbb{Z}_{>0}$

We start from the simplest difference equation in Theorem 3 corresponding to $\ell = 1$ (cf. Eqs. (4.2a), (4.2b)):

$$\begin{aligned} V_1(\xi)(\Phi_{\xi+e_1} - \Phi_\xi) + V_{-1}(\xi)(\Phi_{\xi-e_1} - \Phi_\xi) \\ + \sum_{2\le j\le n, \epsilon\in\{1,-1\}} V_{\epsilon j}(\xi)(\Phi_{\xi+\epsilon e_j} - \Phi_\xi) = e^{x_1}\Phi_\xi \end{aligned} \tag{5.1}$$

(where the dependence on x and g is suppressed and we have written $V_{\epsilon j}(\xi)$ for $V_j(\epsilon\xi)$).

If $\underline{m=1}$, then multiplication of Eq. (5.1) by $(2\xi_1+1)^2$ and performing the limit $2\xi_1 + 1 \to 0$ yields the identity

$$\mathrm{Res}_{1;-1} V_1(\xi)\, \mathrm{Res}_{1;-1}\Phi_{\xi+e_1} = 0.$$

Since

$$\mathrm{Res}_{1;-1} V_1(\xi) = -g \prod_{1<k\le n} (\tfrac{1}{4} - \xi_k^2)^{-1} \not\equiv 0,$$

it follows that $\mathrm{Res}_{1;-1}\Phi_{\xi+\mathrm{e}_1}(x,g)\equiv 0$ for $(\xi,x,g)\in H_{1;-1}\times\mathbb{C}^n\times\mathbb{C}$, and thus $\mathrm{Res}_{1;1}\Phi_\xi(x;g)\equiv 0$ on $H_{1;1}\times\mathbb{C}^n\times\mathbb{C}$.

If $\underline{m=2}$, then multiplication of Eq. (5.1) by $4\xi_1^2$ and performing the limit $2\xi_1\to 0$ yields that

$$\mathrm{Res}_{1;0}V_1(\xi)\mathrm{Res}_{1;0}\Phi_{\xi+\mathrm{e}_1}+\mathrm{Res}_{1;0}V_{-1}(\xi)\mathrm{Res}_{1;0}\Phi_{\xi-\mathrm{e}_1}=0.$$

Since

$$\mathrm{Res}_{1;0}V_1(\xi)=-\mathrm{Res}_{1;0}V_{-1}(\xi)=(-1)^{n-1}(\tfrac{1}{2}+g)\prod_{1<k\leq n}\xi_k^{-2}\not\equiv 0,$$

and

$$\mathrm{Res}_{1;0}\Phi_{\xi+\mathrm{e}_1}=-\mathrm{Res}_{1;0}\Phi_{\xi-\mathrm{e}_1}$$

by the hyperoctahedral symmetry, it follows that $\mathrm{Res}_{1;0}\Phi_{\xi+\mathrm{e}_1}(x,g)\equiv 0$ for $(\xi,x,g)\in H_{1;0}\times\mathbb{C}^n\times\mathbb{C}$, and thus $\mathrm{Res}_{1;2}\Phi_\xi(x;g)\equiv 0$ on $H_{1;2}\times\mathbb{C}^n\times\mathbb{C}$.

If $\underline{m=3}$, then multiplication of Eq. (5.1) by $2\xi_1-1$ and performing the limit $2\xi_1-1\to 0$ yields that

$$\big[V_1(\xi)\big]_{2\xi_1=1}\mathrm{Res}_{1;1}\Phi_{\xi+\mathrm{e}_1}+\mathrm{Res}_{1;1}V_{-1}(\xi)\big[\Phi_{\xi-\mathrm{e}_1}-\Phi_\xi\big]_{2\xi_1=1}=0.$$

Since $\big[V_1(\xi)\big]_{2\xi_1=1}\not\equiv 0$, and

$$\big[\Phi_{\xi-\mathrm{e}_1}\big]_{2\xi_1=1}=\Phi_{(-\frac{1}{2},\xi_2,\ldots,\xi_n)}=\Phi_{(\frac{1}{2},\xi_2,\ldots,\xi_n)}=\big[\Phi_\xi\big]_{2\xi_1=1}$$

by the hyperoctahedral symmetry, it follows that $\mathrm{Res}_{1;1}\Phi_{\xi+e_1}(x,g)\equiv 0$ for $(\xi,x,g)\in H_{1;1}\times\mathbb{C}^n\times\mathbb{C}$, and thus $\mathrm{Res}_{1;3}\Phi_\xi(x;g)\equiv 0$ on $H_{1;3}\times\mathbb{C}^n\times\mathbb{C}$.

If $\underline{m\geq 4}$, then multiplication of Eq. (5.1) by $2\xi_1-m+2$ and performing the limit $2\xi_1-m+2\to 0$ yields that

$$\big[V_1(\xi)\big]_{2\xi_1=m-2}\mathrm{Res}_{1;m-2}\Phi_{\xi+e_1}=0.$$

Since $\big[V_1(\xi)\big]_{2\xi_1=m-2}\not\equiv 0$, it follows that $\mathrm{Res}_{1;m-2}\Phi_{\xi+e_1}(x,g)\equiv 0$ for $(\xi,x,g)\in H_{1;m-2}\times\mathbb{C}^n\times\mathbb{C}$, and thus $\mathrm{Res}_{1;m}\Phi_\xi(x;g)\equiv 0$ on $H_{1;m}\times\mathbb{C}^n\times\mathbb{C}$.

5.2.2 Verification that $\mathrm{Res}_{12;m}\Phi_\xi(x;g)=0$ for $m\in\mathbb{Z}_{>0}$

The required computations are similar to the previous case, but based instead on the second difference equation of Theorem 3 corresponding to $\ell=2$:

$$\sum_{\substack{1\le j<j'\le n\\ \epsilon,\epsilon'\in\{1,-1\}}} V_{\{\epsilon j,\epsilon' j'\}}(\xi)\left(\Phi_{\xi+\epsilon e_j+\epsilon' e_{j'}}-\Phi_\xi\right)+ \tag{5.2}$$
$$\sum_{1\le j\le n,\epsilon\in\{1,-1\}} U_{\{1,\ldots,n\}\setminus\{j\},1}(\xi)V_{\epsilon j}(\xi)\Phi_{\xi+\epsilon e_j}=e^{x_1+x_2}\Phi_\xi,$$

where

$$V_{\{\epsilon j,\epsilon' j'\}}(\xi)=\frac{(\frac12+g+\epsilon\xi_j)}{2\xi_j(2\xi_j+\epsilon)}\frac{(\frac12+g+\epsilon'\xi_{j'})}{2\xi_{j'}(2\xi_{j'}+\epsilon')}\prod_{\substack{1\le k\le n\\ k\ne j,j'}}(\xi_j^2-\xi_k^2)^{-1}(\xi_{j'}^2-\xi_k^2)^{-1}$$
$$\times(\epsilon\xi_j+\epsilon'\xi_{j'})^{-1}(1+\epsilon\xi_j+\epsilon'\xi_{j'})^{-1},$$

$$U_{\{1,\ldots,n\}\setminus\{j\},1}(\xi)=-\sum_{\substack{1\le i\le n,\,i\ne j\\ \epsilon\in\{1,-1\}}}\frac{\frac12+g+\epsilon\xi_i}{2\xi_i(2\xi_i+\epsilon)}\prod_{\substack{1\le k\le n\\ k\ne j,i}}(\xi_i^2-\xi_k^2)^{-1},$$

and $V_{\epsilon j}(\xi)$ is in accordance with Eq. (5.1).

If $m=1$, then multiplication of Eq. (5.2) by $(\xi_1+\xi_2+1)^2$ and performing the limit $\xi_1+\xi_2+1\to 0$ yields that

$$\mathrm{Res}_{12;-1}V_{\{+1,+2\}}(\xi)\left(\mathrm{Res}_{12;-1}\Phi_{\xi+e_1+e_2}-\mathrm{Res}_{12;-1}\Phi_\xi\right)=0.$$

Since

$$\mathrm{Res}_{12;-1}V_{\{+1,+2\}}(\xi)=-\left[\prod_{j\le 2}\frac{\frac12+g+\xi_j}{2\xi_j(2\xi_j+1)}\prod_{\substack{j\le 2\\ k\ge 3}}(\xi_j^2-\xi_k^2)\right]_{\xi_1+\xi_2=1}\not\equiv 0,$$

and

$$\mathrm{Res}_{12;-1}\Phi_{\xi+e_1+e_2}=-\mathrm{Res}_{12;-1}\Phi_\xi$$

by the hyperoctahedral symmetry, it follows that $\mathrm{Res}_{12;-1}\Phi_{\xi+e_1+e_2}(x;g)\equiv 0$ for $(\xi,x,g)\in H_{12;-1}\times\mathbb{C}^n\times\mathbb{C}$, and thus $\mathrm{Res}_{12;1}\Phi_\xi(x;g)\equiv 0$ on $H_{12;1}\times\mathbb{C}^n\times\mathbb{C}$.

If $\underline{m=2}$, then multiplication of Eq. (5.2) by $(\xi_1+\xi_2)^2$ and performing the limit $\xi_1+\xi_2\to 0$ yields that

$$\mathrm{Res}_{12;0}V_{\{+1,+2\}}(\xi)\mathrm{Res}_{12;0}\Phi_{\xi+e_1+e_2}+\mathrm{Res}_{12;0}V_{\{-1,-2\}}(\xi)\mathrm{Res}_{12;0}\Phi_{\xi-e_1-e_2}=0.$$

Since

$$\begin{aligned}\mathrm{Res}_{12;0}V_{\{+1,+2\}}(\xi)&=-\mathrm{Res}_{12;0}V_{\{-1,-2\}}(\xi)\\&=\left[\prod_{j\le 2}\frac{\frac{1}{2}+g+\xi_j}{2\xi_j(2\xi_j+1)}\prod_{\substack{j\le 2\\k\ge 3}}(\xi_j^2-\xi_k^2)\right]_{\xi_1+\xi_2=0}\not\equiv 0,\end{aligned}$$

and

$$\mathrm{Res}_{12;0}\Phi_{\xi+e_1+e_2}=-\mathrm{Res}_{12;0}\Phi_{\xi-e_1-e_2}$$

by the hyperoctahedral symmetry, it follows that $\mathrm{Res}_{12;0}\Phi_{\xi+e_1+e_2}(x;g)\equiv 0$ for $(\xi,x,g)\in H_{12;0}\times\mathbb{C}^n\times\mathbb{C}$, and thus $\mathrm{Res}_{12;2}\Phi_\xi(x;g)\equiv 0$ on $H_{12;2}\times\mathbb{C}^n\times\mathbb{C}$.

If $\underline{m=3}$, then multiplication of Eq. (5.2) by $\xi_1+\xi_2-1$ and performing the limit $\xi_1+\xi_2\to 1$ yields that

$$\begin{aligned}&\big[V_{\{+1,+2\}}(\xi)\big]_{\xi_1+\xi_2=1}\mathrm{Res}_{12;1}\Phi_{\xi+e_1+e_2}\\&\qquad+\mathrm{Res}_{12;1}V_{\{-1,-2\}}(\xi)\big[\Phi_{\xi-e_1-e_2}-\Phi_\xi\big]_{\xi_1+\xi_2=1}=0.\end{aligned}$$

Since $\big[V_{\{+1,+2\}}(\xi)\big]_{\xi_1+\xi_2=1}\not\equiv 0$, and

$$\big[\Phi_{\xi-e_1-e_2}\big]_{\xi_1+\xi_2=1}=\big[\Phi_\xi\big]_{\xi_1+\xi_2=1}$$

by the hyperoctahedral symmetry, it follows that $\mathrm{Res}_{12;1}\Phi_{\xi+e_1+e_2}(x;g)\equiv 0$ for $(\xi,x,g)\in H_{12;1}\times\mathbb{C}^n\times\mathbb{C}$, and thus $\mathrm{Res}_{12;3}\Phi_\xi(x;g)\equiv 0$ on $H_{12;3}\times\mathbb{C}^n\times\mathbb{C}$.

If $\underline{m\ge 4}$, then multiplication of Eq. (5.2) by $\xi_1+\xi_2-m+2$ and performing the limit $\xi_1+\xi_2\to m-2$ yields that

$$\big[V_{\{+1,+2\}}(\xi)\big]_{\xi_1+\xi_2=m-2}\mathrm{Res}_{12;m-2}\Phi_{\xi+e_1+e_2}=0.$$

Since $\big[V_{\{+1,+2\}}(\xi)\big]_{\xi_1+\xi_2=m-2}\not\equiv 0$, it follows that $\mathrm{Res}_{12;m-2}\Phi_{\xi+e_1+e_2}(x;g)\equiv 0$ for $(\xi,x,g)\in H_{12;m-2}\times\mathbb{C}^n\times\mathbb{C}$, and thus $\mathrm{Res}_{12;m}\Phi_\xi(x;g)\equiv 0$ on $H_{12;m}\times\mathbb{C}^n\times\mathbb{C}$.

5.2.3 Conclusion

The above residue computations confirm that all singularities of $\Phi_\xi(x;g)$ (3.3a), (3.3b) in (ξ, x, g) along the hyperplanes of $\mathbb{C}^n \setminus \mathbb{C}^n_{\text{reg}} \times \mathbb{C}^n \times \mathbb{C}$ are removable, which completes the proof of Theorem 4.

6 Proof of Proposition 1

The statements of the proposition are verified via a Harish-Chandra type analysis—cf. for example, [20, Chapter IV.5] and [18, Section 4]—which is tailored towards the current setup of the Toda Laplacian with Morse perturbations.

6.1 Growth estimates for $a_\nu(\xi;g)$

We will first infer that—given a (any) compact subset $K \subset \mathbb{U}^n \times \mathbb{C}^n \times \mathbb{C}$—there exists a constant $C > 0$ (depending only on K) such that $\forall(\xi, x, g) \in K$ and $\forall \nu \geq 0$:

$$|a_\nu(\xi;g)e^{-\langle x,\nu\rangle}| \leq \frac{C^{\langle \nu,\rho\rangle}}{\langle \nu, \rho\rangle!} \quad \text{where} \quad \rho := (n, n-1, \ldots, 2, 1). \tag{6.1}$$

This bound hinges on the following (well-known) elementary estimates.

Lemma 6. There exist constants $a, b, c > 0$ such that $\forall(\xi, x, g) \in K$:

(i) $|\langle \nu - 2\xi, \nu\rangle| \geq a\langle \nu, \rho\rangle^2$, $\forall \nu \geq 0$;

(ii) $|e^{-\langle \nu,x\rangle}| \leq b^{\langle \nu,\rho\rangle}$, $\forall \nu \geq 0$;

(iii) $|a_\alpha| \leq c$, $\forall \alpha \in S$. □

Proof. Since the integral cone $\{\nu \in \mathbb{Z}^n \mid \nu \geq 0\}$ is nonnegatively generated by S (2.1b), the angle between ρ and the vectors in this cone stays strictly sharp and bounded away from $\frac{\pi}{2}$, that is, there exist positive constants c_1, c_2 such that

$$\forall \nu \geq 0: \quad c_1\langle \nu, \nu\rangle \leq \langle \nu, \rho\rangle^2 \leq c_2\langle \nu, \nu\rangle. \tag{6.2}$$

Let $k := \max_{(\xi,x,g)\in K}\langle 2\xi, 2\xi\rangle$ and pick $c_0 > 1$. Then for $\langle \nu, \nu\rangle > kc_0^2$ the estimate in part (i) holds because of Eq. (6.2) and the observation that $|\langle \nu - 2\xi, \nu\rangle| \geq (1 - \frac{1}{c_0})\langle \nu, \nu\rangle$, whereas for $\langle \nu, \nu\rangle \leq kc_0^2$ the estimate in question follows from Eq. (6.2) and the fact that for $\nu > 0$ the quantity $|\langle \nu - 2\xi, \nu\rangle|$ stays bounded from below on K away from zero. The estimate in part (ii) follows from the Cauchy-Schwarz inequality, the compactness of K, and Eq. (6.2), while the estimate in part (iii) is plain from the compactness of K. ■

To see how the asserted bound in Eq. (6.1) arises from the estimates in Lemma 6, we pick a point $(\xi, x, g) \in K$ and define

$$A_m(\xi; g) := \max_{\substack{\nu \geq 0 \\ \langle \nu, \rho \rangle = m}} |a_\nu(\xi; g)| \tag{6.3a}$$

for $m \in \mathbb{Z}_{\geq 0}$. The estimates in parts (i) and (iii) of Lemma 6 then imply that

$$A_m(\xi; g) \leq \frac{A^m}{m!} \quad \text{with} \quad A := 1 + \frac{\mathrm{c}n}{\mathrm{a}}. \tag{6.3b}$$

Indeed—while for $m = 0$ this bound holds trivially—it is readily seen inductively in m via the recurrence relations (2.4b), (2.4c) that for $m > 0$:

$$\begin{aligned} A_m(\xi; g) &\leq \frac{\mathrm{c}}{\mathrm{a}\, m^2} \left(n A_{m-1}(\xi; g) + A_{m-2}(\xi; g) \right) \\ &\leq \frac{\mathrm{c}}{\mathrm{a}\, m^2} \left(\frac{n A^{m-1}}{(m-1)!} + \frac{A^{m-2}}{(m-2)!} \right) \\ &\leq \frac{A^m}{m!} \frac{\mathrm{c}}{\mathrm{a}} \left(\frac{n}{A} + \frac{1}{A^2} \right) \leq \frac{A^m}{m!} \end{aligned}$$

(with the implicit understanding that the terms involving $A_{m-2}(\xi; g)$ and $\frac{A^{m-2}}{(m-2)!}$ in the intermediate expressions are absent when $m = 1$). Notice at this point that the last of these successive estimates, is immediate from the manifest inequality $A^2 > \frac{\mathrm{c}}{\mathrm{a}}(1 + nA)$. By combining the bound in Eq. (6.3b) and the estimate in part (ii) of Lemma 6, the desired upper bound on the growth of the expansion coefficients in Eq. (6.1) follows with $C = \mathrm{b}A$ (which is independent of the choice of $(\xi, x, g) \in K$ and thus holds uniformly).

6.2 Convergence of the Harish-Chandra series

From the estimate in Eq. (6.1) it is clear that the Harish-Chandra series $\phi_\xi(x; g)$ (2.4a)–(2.4c) converges uniformly on the compact set $K \subset \mathbb{U}^n \times \mathbb{C}^n \times \mathbb{C}$ in absolute value:

$$\begin{aligned} |\mathrm{e}^{-\langle \xi, x \rangle} \phi_\xi(x; g)| &\leq \sum_{\nu \geq 0} |a_\nu(\xi; g) \mathrm{e}^{-\langle \nu, x \rangle}| \leq \sum_{\nu \geq 0} \frac{C^{\langle \nu, \rho \rangle}}{\langle \nu, \rho \rangle !} \\ &= \sum_{m \geq 0} \left(\sum_{\substack{\nu \geq 0 \\ \langle \nu, \rho \rangle = m}} \frac{C^{\langle \nu, \rho \rangle}}{\langle \nu, \rho \rangle !} \right) = \sum_{m \geq 0} \binom{n+m-1}{m} \frac{C^m}{m!} < +\infty. \end{aligned}$$

Here we exploited that the number of $\nu \geq 0$, for which $\langle \nu, \rho \rangle = m$ is given explicitly by the binomial $\binom{n+m-1}{m}$ (and thus grows at most polynomially in m).

6.3 Eigenvalue equation

The uniform convergence of the Harish-Chandra series on compacts guarantees that $\phi_\xi(x;g)$ (2.4a)–(2.4c), is holomorphic for $(\xi,x,g) \in \mathbb{U}^n \times \mathbb{C}^n \times \mathbb{C}$. Substitution of the series in the eigenvalue equation (2.5a) entails for the LHS:

$$L_x\phi_\xi(x;g) = \sum_{\nu\geq 0} a_\nu(\xi;g)\mathrm{e}^{\langle \xi-\nu,x\rangle}\left(\langle \xi-\nu,\xi-\nu\rangle - \sum_{\alpha\in S} \mathrm{a}_\alpha \mathrm{e}^{-\langle\alpha,x\rangle}\right),$$

where we were allowed to differentiate termwise by virtue of the uniform convergence of the power series. Upon recollecting the terms and subsequently employing the recurrence relations (2.4b), (2.4c), the expression under consideration is rewritten as:

$$\sum_{\nu\geq 0}\mathrm{e}^{\langle\xi-\nu,x\rangle}\left(\big(\langle\xi,\xi\rangle + \langle\nu,\nu-2\xi\rangle\big)a_\nu(\xi;g) - \sum_{\substack{\alpha\in S\\ \nu-\alpha\geq 0}} \mathrm{a}_\alpha a_{\nu-\alpha}(\xi;g)\right)$$
$$= \sum_{\nu\geq 0}\langle\xi,\xi\rangle a_\nu(\xi;g)e^{\langle\xi-\nu,x\rangle} = \langle\xi,\xi\rangle\phi_\xi(x;g),$$

which confirms that our fundamental Whittaker function $\phi_\xi(x;g)$ (2.4a)–(2.4c) solves the asserted eigenvalue equation in Eq. (2.5a) for the Toda Laplacian with Morse perturbations L_x (2.1a)–(2.1c).

6.4 Plane-wave asymptotics

Given a (any) wave vector $\xi \in \mathbb{U}^n$ and coupling value $g\in\mathbb{C}$, let us pick for $K \subset \mathbb{U}^n\times\mathbb{C}^n\times\mathbb{C}$ the singleton containing only the point $(\xi,0,g)$. In this situation, the estimate in Eq. (6.1) reveals that there exists a $C>0$ (depending only on ξ and g) such that $|a_\nu(\xi;g)| \leq \frac{C^{\langle\nu,\rho\rangle}}{\langle\nu,\rho\rangle!}$ for all $\nu\geq 0$. The plane-wave asymptotics in Eq. (2.5b) now readily follows by dominated convergence (assuming the additional requirement that $\mathrm{Re}(\xi)=0$, so $|e^{\langle\xi,x\rangle}|=1$ for all $x\in\mathbb{R}^n$):

$$\lim_{x\to+\infty}|\phi_\xi(x;g)-\mathrm{e}^{\langle\xi,x\rangle}| \leq \lim_{x\to+\infty}\sum_{\nu>0}|a_\nu(\xi;g)\mathrm{e}^{-\langle\nu,x\rangle}|$$
$$\leq \lim_{x\to+\infty}\sum_{\nu>0}\frac{C^{\langle\nu,\rho\rangle}\mathrm{e}^{-\langle\nu,x\rangle}}{\langle\nu,\rho\rangle!} = \sum_{\nu>0}\left(\lim_{x\to+\infty}\frac{C^{\langle\nu,\rho\rangle}\mathrm{e}^{-\langle\nu,x\rangle}}{\langle\nu,\rho\rangle!}\right) = 0,$$

where we have used that for $\nu>0$: $\mathrm{e}^{-\langle\nu,x\rangle}\leq 1$ if $x_1\geq x_2\geq\cdots\geq x_n\geq 0$ and $\lim_{x\to+\infty}\mathrm{e}^{-\langle\nu,x\rangle}=0$.

7 Proof of Proposition 2

To prove the proposition, we will show that our fundamental Whittaker function arises via a confluent limit—in the sense of Shimeno and Oshima [37, 42]—from the Harish-Chandra series for the hypergeometric equation with hyperoctahedral symmetry. The analytic continuation of the fundamental Whittaker function in the spectral parameter then follows from Opdam's corresponding result for the Harish-Chandra solution of the hypergeometric equation.

7.1 Hyperoctahedral Calogero–Sutherland Laplacian

The Hamiltonian of the hyperoctahedral Calogero–Sutherland system is given by [19, 32, 35]

$$L_x^{\mathrm{cs}} = \sum_{1\leq j\leq n}\left(\frac{\partial^2}{\partial x_j^2} - \frac{\frac{1}{4}k_1(k_1+2k_2-1)}{\sinh^2\frac{1}{2}(x_j)} - \frac{k_2(k_2-1)}{\sinh^2(x_j)}\right) - \sum_{1\leq j<k\leq n}\left(\frac{\frac{1}{2}k_0(k_0-1)}{\sinh^2\frac{1}{2}(x_j+x_k)} + \frac{\frac{1}{2}k_0(k_0-1)}{\sinh^2\frac{1}{2}(x_j-x_k)}\right). \tag{7.1}$$

It is well-known that Toda potentials can be interpreted as limit degenerations of Calogero–Sutherland potentials [21, 36, 40, 47]. Specifically, upon setting $k_r = k_r^{(c)}$ with

$$k_0^{(c)}(k_0^{(c)}-1) = \mathrm{e}^c, \quad k_1^{(c)} = 2g, \quad k_2^{(c)}(k_2^{(c)}-1) = \frac{\mathrm{e}^{2c}}{16} \quad (\text{and } k_0^{(c)}, k_2^{(c)} > 0), \tag{7.2a}$$

and performing a coordinate translation of the form

$$x \to x + c\rho \tag{7.2b}$$

(with ρ as in Eq. (6.1)), the potential of the hyperoctahedral Calogero–Sutherland Laplacian L_x^{cs} (7.1) passes in the limit $c\to+\infty$ over into the potential of the Toda Laplacian with Morse term L_x (2.1a)–(2.1c), that is, formally [21, 36]:

$$L_x = \lim_{c\to+\infty} L_x^{\mathrm{cs}}\left({\scriptstyle k_r\to k_r^{(c)} \atop x\to x+c\rho}\right). \tag{7.3}$$

7.2 Harish-Chandra series

Upon expanding the Calogero–Sutherland potentials in the right half-plane by means of the power series $(\mathrm{e}^z-\mathrm{e}^{-z})^{-2} = \sum_{l\geq 1} l\mathrm{e}^{-2lz}$ ($\mathrm{Re}(z)>0$), it is readily seen that substitution of a formal power series of the form

$$\phi_\xi^{\mathrm{cs}}(x;k_r)=\sum_{\nu\geq 0}a_\nu^{\mathrm{cs}}(\xi;k_r)\mathrm{e}^{\langle\xi-\nu,x\rangle} \tag{7.4a}$$

in the eigenvalue equation $L_x^{\mathrm{cs}}\phi_\xi=\langle\xi,\xi\rangle\phi_\xi$, gives rise to the following recurrence relation for the expansion coefficients:

$$\langle\nu-2\xi,\nu\rangle a_\nu^{\mathrm{cs}}(\xi;k_r)=\sum_{\substack{\alpha\in R_+\\ l\geq 1}}l\mathrm{a}_\alpha^{\mathrm{cs}}(k_r)a_{\nu-l\alpha}^{\mathrm{cs}}(\xi;k_r)\qquad(\nu>0), \tag{7.4b}$$

where $R_+:=\{\mathrm{e}_j,2\mathrm{e}_j\mid 1\leq j\leq n\}\cup\{\mathrm{e}_j\pm\mathrm{e}_k\mid 1\leq j<k\leq n\}$,

$$\mathrm{a}_\alpha^{\mathrm{cs}}(k_r):=\begin{cases}k_1(k_1+2k_2-1) & \text{if } \langle\alpha,\alpha\rangle=1,\\ 2k_0(k_0-1) & \text{if } \langle\alpha,\alpha\rangle=2,\\ 4k_2(k_2-1) & \text{if } \langle\alpha,\alpha\rangle=4,\end{cases} \tag{7.4c}$$

and we have assumed the initial condition

$$a_\nu^{\mathrm{cs}}(\xi;k_r):=\begin{cases}1 & \text{if } \nu=0,\\ 0 & \text{if } \nu\not\geq 0.\end{cases} \tag{7.4d}$$

Notice that the initial condition (7.4d) guarantees in particular that the series on the RHS of the recurrence relation (7.4b) amounts to a *finite* sum (because the requirement that $\nu-l\alpha\geq 0$ implies that l remains bounded from above by $\langle\nu,\rho\rangle$).

The following result goes back to Opdam, cf. [34, Corollary 2.2, Corollary 2.10], [19, Proposition 4.2.5], [35, Lemma 6.5], and [30, Theorem 1.2].

Proposition 7. The Harish-Chandra series $\phi_\xi^{\mathrm{cs}}(x;k_r)$ (7.4a)–(7.4d) constitutes an analytic function of $(\xi,x,k_r)\in\mathbb{C}^n_{\mathrm{reg},+}\times\mathbb{A}^n\times\mathbb{C}^3$, where

$$\mathbb{A}^n:=\{x\in\mathbb{R}^n\mid x_1>x_2>\cdots>x_n>0\}. \tag{7.5}$$

Moreover, as a (meromorphic) function of the spectral variable $\xi\in\mathbb{C}^n$ the function under consideration has at most simple poles along the hyperplanes belonging to $\mathbb{C}^n\setminus\mathbb{C}^n_{\mathrm{reg},+}$. □

7.3 Calogero-Sutherland → Toda confluence

Following [37, 42], we will now lift the limit transition (7.3) between the Calogero-Sutherland Laplacian L_x^{cs} (7.1) and the Toda Laplacian with Morse term L_x (2.1a)–(2.1c) to the level of the Harish-Chandra series.

To this end let us define for a (any) bounded domain $U \subset \mathbb{C}^n$, the normalization factor

$$\Delta_U(\xi) := \prod_{\substack{\mu>0 \\ H_\mu \cap \overline{U} \neq \emptyset}} \langle \mu - 2\xi, \mu \rangle, \tag{7.6}$$

where $H_\mu := \{\xi \in \mathbb{C}^n \mid \langle \mu - 2\xi, \mu\rangle = 0\}$ and $\overline{U}$ refers to the (compact) closure of U. It is clear from the recurrence relations in Eqs. (7.4b)–(7.4d) that for any $\nu \geq 0$ the normalized Harish-Chandra coefficient $\Delta_U(\xi) a_\nu^{\mathrm{cs}}(\xi; k_r)$ is holomorphic and bounded in ξ on the bounded open connected set U.

Proposition 8 (Confluent limit of the Harish-Chandra Series)**.** For any $(\xi, x, g) \in U \times \mathbb{A}^n \times \mathbb{C}$, one has that

$$\lim_{c \to +\infty} \mathrm{e}^{-c\langle \xi, \rho\rangle} \Delta_U(\xi) \phi_\xi^{cs}(x + c\rho; k_r^{(c)}) = \Delta_U(\xi) \phi_\xi(x; g). \tag{7.7}$$

□

Proof. It is immediate from the Harish-Chandra series (7.4a)–(7.4d) that

$$\mathrm{e}^{-c\langle \xi, \rho\rangle} \Delta_U(\xi) \phi_\xi^{\mathrm{cs}}(x + c\rho; k_r^{(c)}) = \sum_{\nu \geq 0} \hat{a}_\nu^{\mathrm{cs}}(\xi; k_r^{(c)}) \mathrm{e}^{\langle \xi - \nu, x\rangle}, \tag{7.8}$$

where $\hat{a}_\nu^{\mathrm{cs}}(\xi; k_r^{(c)})$, $\nu \geq 0$ is determined by the rescaled recurrence

$$\langle \nu - 2\xi, \nu\rangle \hat{a}_\nu^{\mathrm{cs}}(\xi; k_r^{(c)}) = \sum_{\substack{\alpha \in R_+ \\ l \geq 1}} l e^{-c l \langle \alpha, \rho\rangle} \mathrm{a}_\alpha^{\mathrm{cs}}(k_r^{(c)}) \hat{a}_{\nu - l\alpha}^{\mathrm{cs}}(\xi; k_r^{(c)}) \qquad (\nu > 0), \tag{7.9a}$$

with the modified initial condition

$$\hat{a}_\nu^{\mathrm{cs}}(\xi; k_r^{(c)}) := \begin{cases} \Delta_U(\xi) & \text{if } \nu = 0, \\ 0 & \text{if } \nu \not\geq 0. \end{cases} \tag{7.9b}$$

Since in the finite sum on the RHS of the recurrence (7.9a)

$$\lim_{c \to +\infty} e^{-c l \langle \alpha, \rho\rangle} \mathrm{a}_\alpha^{\mathrm{cs}}(k_r^{(c)}) = \begin{cases} \mathrm{a}_\alpha & \text{if } \alpha \in S \text{ and } l = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{7.10}$$

we recover for $c \to +\infty$ the recurrence in Eq. (2.4b) with the initial condition (2.4c) multiplied by $\Delta_U(\xi)$, that is,

$$\lim_{c \to +\infty} \hat{a}_\nu^{\mathrm{cs}}(\xi; k_r^{(c)}) = \Delta_U(\xi) a_\nu(\xi; g) \quad (\forall \xi \in U, \nu \geq 0). \tag{7.11}$$

Moreover, by a standard argument (cf. Subsection 7.4 below for the precise details in the present setting) the above recurrence relations imply that for any $\varepsilon > 0$ there exists a constant $A > 0$ (depending only on $\xi \in U$, $g \in \mathbb{C}$ and ε) such that

$$\forall c \geq 0, \nu \geq 0: \quad |\hat{a}_\nu^{\mathrm{cs}}(\xi; k_r^{(c)})| \leq A\, e^{\varepsilon\langle \nu,\rho\rangle}. \tag{7.12}$$

Upon picking $\varepsilon > 0$ sufficiently small such that $x \in \varepsilon\rho + \mathbb{A}^n$, it is seen that the series on the RHS of Eq. (7.8) can be bounded term wise and uniformly in c by a absolutely convergent series:

$$\sum_{\nu \geq 0} |\hat{a}_\nu^{\mathrm{cs}}(\xi; k_r^{(c)}) \mathrm{e}^{\langle \xi-\nu, x\rangle}| \leq A|\mathrm{e}^{\langle \xi, x\rangle}| \sum_{\nu\geq 0} \mathrm{e}^{-\langle \nu, x-\varepsilon\rho\rangle} < +\infty. \tag{7.13}$$

The upshot is that the asserted limit transition follows from Eqs. (7.8) and (7.11) by dominated convergence. ■

Since the multiplication by $\Delta_U(\xi)$ regularizes the coefficients of the Harish-Chandra series (2.4a)–(2.4c) for $\xi \in U$, it is clear from its local absolute uniform convergence (cf. Section 6) that the normalized Harish-Chandra series $\Delta_U(\xi)\phi_\xi(x; g)$ constitutes an analytic function of $(\xi, x, g) \in U \times \mathbb{C}^n \times \mathbb{C}$. One concludes from Propositions 7 and 8 that for $(\xi, x, g) \in (U \cap H_\mu) \times \mathbb{A}^n \times \mathbb{C}$ $(\mu > 0)$ the function in question vanishes—and is thus divisible by $\langle \mu - 2\xi, \mu\rangle$—except (possibly) when the hyperplane H_μ belongs to $\mathbb{C}^n \setminus \mathbb{C}^n_{\mathrm{reg},+}$. But then, by analytic continuation, it is clear that in fact $\Delta_U(\xi)\phi_\xi(x; g) = 0$ for $(\xi, x, g) \in (U \cap H_\mu) \times \mathbb{C}^n \times \mathbb{C}$ (if H_μ, $\mu > 0$ does not belong to $\mathbb{C}^n \setminus \mathbb{C}^n_{\mathrm{reg},+}$).

Since the bounded domain $U \subset \mathbb{C}^n$ was chosen arbitrarily, this means that the fundamental Whittaker function $\phi_\xi(x; g)$ is regular in ξ along the hyperplane H_ν for all $\nu > 0$ such that $H_\nu \not\subset \mathbb{C}^n \setminus \mathbb{C}^n_{\mathrm{reg},+}$, and that it possesses at most a simple pole along the hyperplane H_ν for all $\nu > 0$ such that $H_\nu \subset \mathbb{C}^n \setminus \mathbb{C}^n_{\mathrm{reg},+}$, which completes the proof of Proposition 2.

7.4 Proof of the bound in Eq. (7.12)

After fixing $\xi \in U$ and $g \in \mathbb{C}$, we follow the proof of a similar bound in [20, Chapter IV, Lemma 5.3]. To this end let us pick $\mathrm{a} > 0$ and $N > 0$ such that:

$$\langle \nu, \nu - 2\xi\rangle \geq \mathrm{a}\langle \nu, \rho\rangle^2 \qquad \forall \nu \geq 0 \text{ with } \langle \nu, \rho\rangle \geq N \tag{7.14}$$

(cf. Lemma 6). In view of Eqs. (7.2a) and (7.4c), it is clear that for a given $\varepsilon > 0$ there exists a $C > 0$ such that $\forall c \geq 0$:

$$\frac{1}{\mathrm{a}}\sum_{\substack{\alpha\in R_+\\ l\geq 1}} \mathrm{e}^{-(\varepsilon+c)l\langle\alpha,\rho\rangle}|\mathrm{a}_\alpha^{\mathrm{cs}}(k_r^{(c)})|\leq C. \tag{7.15}$$

Upon fixing a positive integer $M\geq \max(N,C)$, we now pick $A>0$ sufficiently large such that $\forall c\geq 0$: $|\hat{a}_\nu^{\mathrm{cs}}(\xi;k_r^{(c)})|\leq Ae^{\varepsilon\langle\nu,\rho\rangle}$ for all $\nu\geq 0$ with $\langle\nu,\rho\rangle<M$. (The existence of A is clear from the limit in Eq. (7.11) and the continuity in c.)

One then sees inductively from the recurrence in Eq. (7.9a) that also for all $\nu\geq 0$ such that $\langle\nu,\rho\rangle\geq M$:

$$\begin{aligned}
|\hat{a}_\nu^{\mathrm{cs}}(\xi;k_r^{(c)})| &\leq \frac{1}{\mathrm{a}\langle\nu,\rho\rangle^2}\sum_{\substack{\alpha\in R_+\\ l\geq 1}} l\mathrm{e}^{-c\,l\langle\alpha,\rho\rangle}|\mathrm{a}_\alpha^{\mathrm{cs}}(k_r^{(c)})\hat{a}_{\nu-l\alpha}^{\mathrm{cs}}(\xi;k_r^{(c)})|\\
&\leq \frac{1}{\mathrm{a}\langle\nu,\rho\rangle}\sum_{\substack{\alpha\in R_+\\ l\geq 1}} \mathrm{e}^{-c\,l\langle\alpha,\rho\rangle}|\mathrm{a}_\alpha^{\mathrm{cs}}(k_r^{(c)})|A\mathrm{e}^{\varepsilon\langle\nu-l\alpha,\rho\rangle}\\
&\leq \tfrac{M}{\langle\nu,\rho\rangle}A\mathrm{e}^{\varepsilon\langle\nu,\rho\rangle}\leq A\mathrm{e}^{\varepsilon\langle\nu,\rho\rangle}
\end{aligned}$$

(where we exploited that $l\leq\langle\nu,\rho\rangle$ if $\nu-l\alpha\geq 0$).

8 Proof of Theorem 3

In this section we retrieve our difference equations for the hyperoctahedral Whittaker function from the hypergeometric difference equations in [10, Theorem 2], with the aid of the Calogero–Sutherland → Toda confluence from Section 7.

8.1 Hyperoctahedral hypergeometric function

For $(\xi,x,k_r)\in\mathbb{C}^n_{\mathrm{reg}}\times\mathbb{A}^n\times\mathbb{C}^3$, let us consider the following eigenfunction of the Calogero–Sutherland Laplacian L_x^{cs} (7.1):

$$\Phi_\xi^{\mathrm{cs}}(x;k_r):=\sum_{w\in W} C^{\mathrm{cs}}(w\xi;k_r)\phi_{w\xi}^{\mathrm{cs}}(x;k_r), \tag{8.1a}$$

where

$$\begin{aligned}
C^{\mathrm{cs}}(\xi;k_r):=&\prod_{1\leq j\leq n}\frac{\Gamma(2\xi_j)\Gamma(\frac{1}{2}k_1+\xi_j)}{\Gamma(k_1+2\xi_j)\Gamma(\frac{1}{2}k_1+k_2+\xi_j)}\\
&\times\prod_{1\leq j<k\leq n}\frac{\Gamma(\xi_j+\xi_k)\Gamma(\xi_j-\xi_k)}{\Gamma(k_0+\xi_j+\xi_k)\Gamma(k_0+\xi_j-\xi_k)}.
\end{aligned} \tag{8.1b}$$

Up to an overall normalization factor—depending on x and k_r but *not* on ξ—this is the hyperoctahedral hypergeometric function of Heckman and Opdam associated with the root system BC_n [19, 34, 35] (cf. Remark 5 below).

Lemma 9. For any $(\xi, g) \in \mathbb{C}^n_{\mathrm{reg},-} \times \mathbb{C}$ (cf. Eq. (3.4)), one has that

$$\lim_{c\to+\infty} \gamma(k_r^{(c)})\, \mathrm{e}^{c\langle\xi,\rho\rangle} C^{cs}(\xi; k_r^{(c)}) = C(\xi; g), \tag{8.2a}$$

where

$$\gamma(k_r) := (\Gamma(k_0))^{n(n-1)} \left(\frac{\Gamma(k_1)\Gamma(\frac{1}{2}k_1 + k_2)}{\Gamma(\frac{1}{2}k_1)\Gamma(\frac{1}{2} + \frac{1}{2}k_1)} \right)^n . \tag{8.2b}$$

□

Proof. This lemma is immediate from the duplication formula for the gamma function [33, 5.5.5]

$$\Gamma(2z) = \pi^{-\frac{1}{2}} 2^{2z-1} \Gamma(z)\Gamma(\tfrac{1}{2} + z)$$

($2z \notin \mathbb{Z}_{\leq 0}$) and the asymptotics [33, 5.11.12]

$$\lim_{c\to+\infty} \frac{c^z \Gamma(c)}{\Gamma(z+c)} = 1.$$

■

The upshot is that the confluence limit for the Harish-Chandra series in Proposition 8 persists at the level of the connection formulas.

Proposition 10 (Confluent limit of the Connection Formula). For any $(\xi, x, g) \in \mathbb{C}^n_{\mathrm{reg}} \times \mathbb{A}^n \times \mathbb{C}$, one has that

$$\lim_{c\to+\infty} \gamma(k_r^{(c)}) \Phi^{cs}_\xi(x + c\rho; k_r^{(c)}) = \Phi_\xi(x; g). \tag{8.3}$$

□

Proof. It is clear from Proposition 8, Lemma 9 and the connection formula in Eqs (8.1a), (8.1b), that the limit on the LHS of Eq. (8.3) reproduces the connection formula for the hyperoctahedral Whittaker function in Eqs (3.3a), (3.3b). ■

For reduced root systems such confluences from the hypergeometric function of Heckman and Opdam to the class-one Whittaker function studied by Hashizume were established by Shimeno [42]. In the case of a single variable ($n = 1$), the limit in Proposition 10 was detailed in [37, Section 2].

Remark 5. The precise relation between $\Phi_\xi^{\mathrm{cs}}(x;k_r)$ (8.1a), (8.1b) and the hyperoctahedral hypergeometric function $F_{BC_n}(\xi,x;k_r)$ of Heckman and Opdam [19, 34, 35] is given by:

$$\Phi_\xi^{\mathrm{cs}}(x;k_r)=\delta(x;k_r)C^{\mathrm{cs}}(\rho(k_r);k_r)F_{BC_n}(\xi,x;k_r) \tag{8.4a}$$

($(\xi,x)\in\mathbb{C}^n_{\mathrm{reg}}\times\mathbb{A}^n$), where

$$\begin{aligned}\delta(x;k_r):=&\prod_{1\le j\le n}(\mathrm{e}^{\frac12 x_j}-\mathrm{e}^{-\frac12 x_j})^{k_1}(\mathrm{e}^{x_j}-\mathrm{e}^{-x_j})^{k_2}\\&\times\prod_{1\le j<k\le n}(\mathrm{e}^{\frac12(x_j+x_k)}-\mathrm{e}^{-\frac12(x_j+x_k)})^{k_0}(\mathrm{e}^{\frac12(x_j-x_k)}-\mathrm{e}^{-\frac12(x_j-x_k)})^{k_0}\end{aligned} \tag{8.4b}$$

and

$$\rho(k_r):=((n-1)k_0+\tfrac12k_1+k_2,(n-2)k_0+\tfrac12k_1+k_2,\ldots,\tfrac12k_1+k_2). \tag{8.4c}$$

□

8.2 Hypergeometric difference equations

In view of Remark 5, it follows from [10, Theorem 2] that for any $x\in\mathbb{A}^n$ (7.5), $k_r\in\mathbb{C}$ ($r=0,1,2$) and $\ell\in\{1,\ldots,n\}$, the normalized hypergeometric function $\Phi_\xi^{\mathrm{cs}}(x;k_r)$ (8.1a), (8.1b) satisfies the difference equation

$$\sum_{\substack{J\subset\{1,\ldots,n\},\,0\le|J|\le\ell\\ \varepsilon_j=\pm1,\,j\in J}}U^{\mathrm{cs}}_{J^c,\ell-|J|}(\xi;k_r)V^{\mathrm{cs}}_{\varepsilon J}(\xi;k_r)\Phi^{\mathrm{cs}}_{\xi+e_{\varepsilon J}}(x;k_r)=E_\ell(x)\Phi_\xi^{\mathrm{cs}}(x;k_r)$$

(as an identity between meromorphic functions of $\xi\in\mathbb{C}^n$), where

$$\begin{aligned}V^{\mathrm{cs}}_{\varepsilon J}(\xi;k_r):=&\prod_{j\in J}\frac{(\varepsilon_j\xi_j+\frac12k_1+k_2)(1+2\varepsilon_j\xi_j+k_1)}{\varepsilon_j\xi_j(1+2\varepsilon_j\xi_j)}\prod_{\substack{j\in J\\k\notin J}}\left(\frac{\varepsilon_j\xi_j+\xi_k+k_0}{\varepsilon_j\xi_j+\xi_k}\right)\left(\frac{\varepsilon_j\xi_j-\xi_k+k_0}{\varepsilon_j\xi_j-\xi_k}\right)\\&\times\prod_{\substack{j,j'\in J\\j<j'}}\left(\frac{\varepsilon_j\xi_j+\varepsilon_{j'}\xi_{j'}+k_0}{\varepsilon_j\xi_j+\varepsilon_{j'}\xi_{j'}}\right)\left(\frac{1+\varepsilon_j\xi_j+\varepsilon_{j'}\xi_{j'}+k_0}{1+\varepsilon_j\xi_j+\varepsilon_{j'}\xi_{j'}}\right),\end{aligned}$$

$$\begin{aligned}U^{\mathrm{cs}}_{K,p}(\xi;k_r):=(-1)^p\sum_{\substack{I\subset K,\,|I|=p\\ \varepsilon_i=\pm1,\,i\in I}}\Biggl(&\prod_{i\in I}\frac{(\varepsilon_i\xi_i+\frac12k_1+k_2)(1+2\varepsilon_i\xi_i+k_1)}{\varepsilon_i\xi_i(1+2\varepsilon_i\xi_i)}\\&\times\prod_{\substack{i\in I\\k\in K\setminus I}}\left(\frac{\varepsilon_i\xi_i+\xi_k+k_0}{\varepsilon_i\xi_i+\xi_k}\right)\left(\frac{\varepsilon_i\xi_i-\xi_k+k_0}{\varepsilon_i\xi_i-\xi_k}\right)\end{aligned}$$

$$\times \prod_{\substack{i,i' \in I \\ i<i'}} \left(\frac{\varepsilon_i\xi_i + \varepsilon_{i'}\xi_{i'} + k_0}{\varepsilon_i\xi_i + \varepsilon_{i'}\xi_{i'}}\right)\left(\frac{1 + \varepsilon_i\xi_i + \varepsilon_{i'}\xi_{i'} - k_0}{1 + \varepsilon_i\xi_i + \varepsilon_{i'}\xi_{i'}}\right)\Bigg),$$

and

$$E_\ell(x) := 4^\ell \sum_{\substack{J\subset\{1,\ldots,n\} \\ |J|=\ell}} \prod_{j\in J} \sinh^2\left(\frac{x_j}{2}\right).$$

(For $\ell = 1$, this hypergeometric difference equation had been found before by Chalykh, cf. [6, Theorem 6.12]). If we now pick $(\xi, x, g) \in \mathbb{C}^n_{\text{reg}} \times \mathbb{A}^n \times \mathbb{C}$ and perform the substitution (7.2a), (7.2b) into the above hypergeometric identities, then—after multiplication of both sides by an overall scaling factor of the form $\mathrm{e}^{-\frac{c}{2}\ell(2n+1-\ell)}\gamma(k_r^{(c)})$—the difference equations in Theorem 3 are recovered in the limit $c \to +\infty$. Indeed, this is immediate from Proposition 10 and the elementary limits

$$\lim_{c\to+\infty} \mathrm{e}^{-\frac{c}{2}|J|(2n+1-|J|)} V^{\text{cs}}_{\varepsilon J}(\xi; k_r^{(c)}) = V_{\varepsilon J}(\xi),$$

$$\lim_{c\to+\infty} \mathrm{e}^{-\frac{c}{2}p(2|K|+1-p)} U^{\text{cs}}_{K,p}(\xi; k_r^{(c)}) = U_{K,p}(\xi),$$

$$\lim_{c\to+\infty} \mathrm{e}^{-\frac{c}{2}\ell(2n+1-\ell)} E_\ell(x + c\rho) = \mathrm{e}^{x_1+\cdots+x_\ell},$$

upon noting that $|J|(2n+1-|J|) + p(2|K|+1-p) = \ell(2n+1-\ell)$ when $|K| = n - |J|$ and $p = \ell - |J|$. This completes the proof of Theorem 3 for $x \in \mathbb{A}^n$ (7.5). The extension to arbitrary $x \in \mathbb{C}^n$ is plain by analytic continuation, as $\Phi_\xi(x; g)$ (3.3a), (3.3b) constitutes an entire function of x (cf. Propositions 1 and 2).

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