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EXACT CUBATURE RULES FOR SYMMETRIC FUNCTIONS

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ABSTRACT. We employ a multivariate extension of the Gauss quadrature formula, originally due to Berens, Schmid, and Xu [Arch. Math. (Basel) 64 (1995), pp. 26–32], so as to derive cubature rules for the integration of symmetric functions over hypercubes (or infinite limiting degenerations thereof) with respect to the densities of unitary random matrix ensembles. Our main application concerns the explicit implementation of a class of cubature rules associated with the Bernstein-Szegö polynomials, which permit the exact integration of symmetric rational functions with prescribed poles at coordinate hyperplanes against unitary circular Jacobi distributions stemming from the Haar measures on the symplectic and the orthogonal groups.

1. Introduction

The study of cubature rules for the numeric integration of functions in several variables has a long fruitful history; see e.g. [S71, DR84, S92, SV97, C97, CMS01, IN06, DX14, CH15] and references therein. Over the past few years significant progress has been reported regarding the construction of explicit cubature rules of Gauss-Chebyshev type, permitting the exact integration of multivariate polynomials [LX10, MP11, NS12, MMP14, HM14, HMP16].

Inspired by these recent developments we invoke the Cauchy-Binet-Andréief formulas to rederive a multivariate lifting of the Gauss quadrature formula due to Berens, Schmid, and Xu [BSX95], in the version designed to integrate symmetric functions over a hypercube, a hyperoctant, or over the entire euclidean space. In the case of the classical Gauss-Hermite, Gauss-Laguerre, and Gauss-Jacobi quadratures [S75, DR84], this readily produces corresponding cubature rules for the exact integration of symmetric polynomials against the densities of ubiquitous unitary random matrix ensembles associated with the Hermite, Laguerre, and Jacobi polynomials [M04, F10], respectively.

At the special parameter values for which the Gauss-Jacobi quadrature simplifies to a Gauss-Chebyshev quadrature, the construction in question leads to cubature rules associated with the classical simple Lie groups that turn out to be closely related to those studied in [MP11, MMP14, HM14, HMP16]. One of our primary concerns is to extend the corresponding Gauss-Chebyshev cubatures to a class of explicit cubature rules arising from the Bernstein-Szegö polynomials [S75, Section 2.6]. It is well known that the Gauss quadrature associated with

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the Bernstein-Szegö polynomials [N90, P93, N00, BCM07, BCGM08] permits the exact integration of rational functions with prescribed poles (outside the integration domain) [DGJ06, BCDG09] (cf. also [G93, VV93, G01, BGHN01] for related approaches). Our aim is to extend this picture to the multivariate setup: we construct an exact cubature rule for a class of symmetric rational functions with prescribed poles at the coordinate hyperplanes, where the integration is against the unitary circular Jacobi distributions stemming from the Haar measures on the symplectic and the orthogonal groups (cf. e.g. [S96, Chapter IX.9], [P07, Chapter 11.10], or [F10, Chapter 2.6]). This way the above-mentioned Gauss-Chebyshev cubature rules originating from parameter specializations of the (more involved) Gauss-Jacobi cubature formulas are generalized so as to allow for pole singularities on the coordinate hyperplanes.

The presentation is structured as follows. After setting up the notation for the Gauss quadrature rule in Section 2, we emphasize in Section 3 the effectiveness of the Cauchy-Binet-Andréief formulas when extending the underlying family of orthogonal polynomials to the multivariate level via associated generalized Schur polynomials [M92, BSX95, NNSY00, SV14]. This readily allows us to recover a Gaussian cubature rule for the integrations of symmetric functions from [BSX95, equation (8)] (with $\rho = 0$) in Section 4. In Section 5 we highlight the explicit cubature rules stemming from the classical Hermite, Laguerre, and Jacobi families, which permit the exact integration of symmetric polynomials with respect to the densities of the corresponding unitary ensembles. In the remainder of the paper the implementation of the construction for the case of Bernstein-Szegö polynomials is carried out. Specifically, after recalling the definition of the Bernstein-Szegö polynomials in Section 6 and providing estimates for the locations of their roots in Section 7, the corresponding Gauss quadrature rule stemming from [DGJ06, BCDG09] is exhibited in Section 8. In Section 9 we then apply the general formalism of Section 4 to lift this quadrature to an explicit cubature rule for an associated class of symmetric rational functions. To enhance the readability, some technical details regarding the explicit computation of the pertinent Christoffel weights associated with the Bernstein-Szegö families are supplemented in Appendix A at the end.

2. Preliminaries and notation regarding the Gauss quadrature

Given a continuous weight function $\mathrm{w}(x) > 0$ on a nonempty interval (a, b) with finite moments, let $p_l(x)$, $l = 0, 1, 2, \ldots$, denote the orthonormal basis obtained from the monomial basis $m_l(x) := x^l$, $l = 0, 1, 2, \ldots$, via Gram-Schmidt orthogonalization with respect to the inner product

$$(f, g)_{\mathrm{w}} := \int_a^b f(x)g(x)\mathrm{w}(x)\mathrm{d}x \tag{2.1}$$

(for $f, g : (a, b) \to \mathbb{R}$ polynomial (say)). It is well known (cf. e.g. [S75, Section 3.3]) that the roots of such orthogonal polynomials $p_l(x)$ are simple and belong to (a, b); i.e., for $m \geq 0$:

$$p_{m+1}(x) = \alpha_{m+1}\big(x - x_0^{(m+1)}\big)\big(x - x_1^{(m+1)}\big)\cdots\big(x - x_m^{(m+1)}\big), \tag{2.2a}$$

with

$$a < x_0^{(m+1)} < x_1^{(m+1)} < \cdots < x_m^{(m+1)} < b \tag{2.2b}$$

and $\alpha_l := 1/(p_l, m_l)_{\mathrm{w}}$ $(l = 0, 1, 2, \ldots)$.

Let $f(x)$ be an arbitrary polynomial of degree at most $2m+1$ in x. The celebrated Gauss quadrature formula states that in this situation (cf. e.g. [S75, Section 3.4], [G81], or for an overview of more recent developments [G04]):

$$\int_a^b f(x)\mathrm{w}(x)\mathrm{d}x = \sum_{0\leq \hat{l}\leq m} f(x_{\hat{l}}^{(m+1)})\mathrm{w}_{\hat{l}}^{(m+1)}, \tag{2.3a}$$

where the corresponding Christoffel weights $\mathrm{w}_0^{(m+1)},\ldots,\mathrm{w}_m^{(m+1)}$ are given by

$$\mathrm{w}_{\hat{l}}^{(m+1)} = \Bigl(\sum_{0\leq l\leq m}\bigl(p_l(x_{\hat{l}}^{(m+1)})\bigr)^2\Bigr)^{-1} \qquad (\hat{l}=0,\ldots,m). \tag{2.3b}$$

This quadrature rule can be reformulated in terms of discrete orthogonality relations for $p_0(x), p_1(x),\ldots,p_m(x)$. Indeed, when applying the Gauss quadrature rule (2.3a), (2.3b) to the product $f(x)=p_l(x)p_k(x)$ with l,k at most m, the defining orthogonality

$$(p_l,p_k)_{\mathrm{w}} = \begin{cases} 1 & \text{if } k=l,\\ 0 & \text{if } k\neq l\end{cases} \tag{2.4}$$

gives rise to the associated finite-dimensional discrete orthogonality:

$$\sum_{0\leq \hat{l}\leq m} p_l(x_{\hat{l}}^{(m+1)})p_k(x_{\hat{l}}^{(m+1)})\mathrm{w}_{\hat{l}}^{(m+1)} = \begin{cases} 1 & \text{if } k=l,\\ 0 & \text{if } k\neq l\end{cases} \tag{2.5a}$$

($l,k\in\{0,\ldots,m\}$). By "column-row duality", one can further reformulate (2.5a) in terms of the equivalent dual orthogonality relations

$$\sum_{0\leq l\leq m} p_l(x_{\hat{l}}^{(m+1)})p_l(x_{\hat{k}}^{(m+1)}) = \begin{cases} 1/\mathrm{w}_{\hat{l}}^{(m+1)} & \text{if } \hat{k}=\hat{l},\\ 0 & \text{if } \hat{k}\neq \hat{l}\end{cases} \tag{2.5b}$$

($\hat{l},\hat{k}\in\{0,\ldots,m\}$).

Remark 2.1. The discrete orthogonality relations in (2.5a) remain in fact valid when either l or k (but not both) become equal to $m+1$ (because $p_{m+1}(x)$ vanishes identically on the nodes $x_0^{(m+1)},\ldots,x_m^{(m+1)}$).

3. Generalized Schur polynomials

Given $\lambda=(\lambda_1,\ldots,\lambda_n)$ in the fundamental cone

$$\Lambda^{(n)} := \{\lambda\in\mathbb{Z}^n \mid \lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n\geq 0\}, \tag{3.1}$$

the generalized Schur polynomial $P_\lambda(\mathbf{x})$ associated with the orthonormal system $p_0(x), p_1(x), p_2(x),\ldots$ is defined via the determinantal formula (cf. [M92, BSX95, NNSY00, SV14])

$$P_\lambda(\mathbf{x}) := \frac{1}{\mathrm{V}(\mathbf{x})}\det[p_{\lambda_j+n-j}(x_k)]_{1\leq j,k\leq n}, \tag{3.2a}$$

where $V(\mathbf{x})$ refers to the Vandermonde determinant

$$\mathrm{V}(\mathbf{x}) := \prod_{1\leq j<k\leq n}(x_j-x_k). \tag{3.2b}$$

Clearly $P_\lambda(\mathbf{x})$ constitutes a (permutation-)symmetric polynomial in the components of $\mathbf{x}:=(x_1,x_2,\ldots,x_n)$ (as the determinant in the numerator produces an antisymmetric polynomial which is therefore divisible by the Vandermonde determinant).

It is well known that the symmetric polynomials in question inherit multivariate orthogonality relations from the underlying univariate family (cf. e.g. [BSX95, SV14]).

Proposition 3.1 (Orthogonality relations)**.** *The generalized Schur polynomials $P_\lambda(\mathbf{x})$, $\lambda \in \Lambda^{(n)}$ satisfy the orthogonality relations*

$$\frac{1}{n!}\int_a^b \cdots \int_a^b P_\lambda(\mathbf{x}) P_\mu(\mathbf{x}) \mathrm{W}(\mathbf{x})\, dx_1 \cdots dx_n = \begin{cases} 1 & \text{if } \mu = \lambda, \\ 0 & \text{if } \mu \neq \lambda \end{cases} \tag{3.3a}$$

($\lambda, \mu \in \Lambda^{(n)}$), where

$$\mathrm{W}(\mathbf{x}) := \bigl(\mathrm{V}(\mathbf{x})\bigr)^2 \prod_{1\leq j\leq n} \mathrm{w}(x_j). \tag{3.3b}$$

Proof. This orthogonality is immediate from (i) a classical (Cauchy-Binet type) integration formula for the products of determinants going back to M.C. Andréief [A86] (which is for instance reproduced with proof in [BDS03, Lemma 3.1]), in combination with (ii) the orthogonality of the basis polynomials $p_0(x), p_1(x), p_2(x), \ldots$. More specifically, one has that $\forall \lambda, \mu \in \Lambda^{(n)}$:

$$\begin{aligned} &\frac{1}{n!}\int_a^b \cdots \int_a^b P_\lambda(\mathbf{x}) P_\mu(\mathbf{x}) \mathrm{W}(\mathbf{x})\mathrm{d}x_1 \cdots \mathrm{d}x_n \\ &= \frac{1}{n!}\int_a^b \cdots \int_a^b \det[p_{\lambda_j+n-j}(x_k)]_{1\leq j,k\leq n} \det[p_{\mu_j+n-j}(x_k)]_{1\leq j,k\leq n} \prod_{1\leq j\leq n} \mathrm{w}(x_j)\mathrm{d}x_j \\ &\overset{\text{(i)}}{=} \det\left[\int_a^b p_{\lambda_j+n-j}(x) p_{\mu_k+n-k}(x)\mathrm{w}(x)\mathrm{d}x\right]_{1\leq j,k\leq n} \overset{\text{(ii)}}{=} \begin{cases} 1 & \text{if } \mu = \lambda, \\ 0 & \text{if } \mu \neq \lambda. \end{cases} \end{aligned}$$

□

The following proposition provides a corresponding multivariate generalization of the finite-dimensional discrete orthogonality in (2.5a), which holds for $P_\lambda(\mathbf{x})$ when λ is restricted to the fundamental alcove

$$\Lambda^{(m,n)} := \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid m \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}. \tag{3.4}$$

Proposition 3.2 (Discrete orthogonality relations)**.** *The generalized Schur polynomials $P_\lambda(\mathbf{x})$, $\lambda \in \Lambda^{(m,n)}$ satisfy the discrete orthogonality relations*

$$\sum_{\hat\lambda \in \Lambda^{(m,n)}} P_\lambda\bigl(\mathbf{x}_{\hat\lambda}^{(m,n)}\bigr) P_\mu\bigl(\mathbf{x}_{\hat\lambda}^{(m,n)}\bigr) \mathrm{W}_{\hat\lambda} = \begin{cases} 1 & \text{if } \mu = \lambda, \\ 0 & \text{if } \mu \neq \lambda \end{cases} \tag{3.5a}$$

($\lambda, \mu \in \Lambda^{(m,n)}$), where

$$\mathbf{x}_{\hat\lambda}^{(m,n)} := \Bigl(x^{(m+n)}_{\hat\lambda_1+n-1}, x^{(m+n)}_{\hat\lambda_2+n-2}, \ldots, x^{(m+n)}_{\hat\lambda_{n-1}+1}, x^{(m+n)}_{\hat\lambda_n}\Bigr) \tag{3.5b}$$

and

$$\mathrm{W}_{\hat\lambda} := \Bigl(\mathrm{V}\bigl(\mathbf{x}_{\hat\lambda}^{(m,n)}\bigr)\Bigr)^2 \prod_{1\leq j\leq n} \mathrm{w}^{(m+n)}_{\hat\lambda_j+n-j} \tag{3.5c}$$

(for $\hat{\lambda}\in\Lambda^{(m,n)}$). Here $x^{(m+n)}_{\hat{\lambda}_j+n-j}$ *and* $\mathrm{w}^{(m+n)}_{\hat{\lambda}_j+n-j}$ $(j=1,\ldots,n)$ *are in accordance with the definitions in Equations* (2.2a), (2.2b) *and Equations* (2.3a), (2.3b), *respectively.*

Proof. Similarly to the proof of Proposition 3.1, the asserted orthogonality relations are derived from those in Equation (2.5a) by means of the Cauchy-Binet formula:

$$\sum_{\hat{\lambda}\in\Lambda^{(m,n)}} P_\lambda\bigl(\mathbf{x}^{(m,n)}_{\hat{\lambda}}\bigr)P_\mu\bigl(\mathbf{x}^{(m,n)}_{\hat{\lambda}}\bigr)\mathrm{W}_{\hat{\lambda}}$$

$$= \sum_{m+n>\tilde{\lambda}_1>\tilde{\lambda}_2>\cdots>\tilde{\lambda}_n\geq 0}\Biggl(\det\Bigl[p_{\lambda_j+n-j}\Bigl(x^{(m+n)}_{\tilde{\lambda}_k}\Bigr)\sqrt{\mathrm{w}^{(m+n)}_{\tilde{\lambda}_k}}\Bigr]_{1\leq j,k\leq n} \times\det\Bigl[p_{\mu_j+n-j}\Bigl(x^{(m+n)}_{\tilde{\lambda}_k}\Bigr)\sqrt{\mathrm{w}^{(m+n)}_{\tilde{\lambda}_k}}\Bigr]_{1\leq j,k\leq n}\Biggr)$$

$$\stackrel{\text{(i)}}{=}\det\Bigl[\sum_{0\leq\hat{l}<m+n} p_{\lambda_j+n-j}\Bigl(x^{(m+n)}_{\hat{l}}\Bigr)p_{\mu_k+n-k}\Bigl(x^{(m+n)}_{\hat{l}}\Bigr)\mathrm{w}^{(m+n)}_{\hat{l}}\Bigr]_{1\leq j,k\leq n}$$

$$\stackrel{\text{(ii)}}{=}\begin{cases}1 & \text{if } \mu=\lambda,\\ 0 & \text{if } \mu\neq\lambda\end{cases}$$

(where it was assumed that $\lambda,\mu\in\Lambda^{(m,n)}$). Here the equality (i) hinges on the Cauchy-Binet formula, while the equality (ii) follows using Equation (2.5a). □

Remark 3.1. The alternative dual formulation of the discrete orthogonality in Proposition 3.2 reads (cf. Equation (2.5b))

$$\sum_{\lambda\in\Lambda^{(m,n)}} P_\lambda\bigl(\mathbf{x}^{(m,n)}_{\hat{\lambda}}\bigr)P_\lambda\bigl(\mathbf{x}^{(m,n)}_{\hat{\mu}}\bigr)=\begin{cases}1/\mathrm{W}_{\hat{\lambda}} & \text{if } \hat{\mu}=\hat{\lambda},\\ 0 & \text{if } \hat{\mu}\neq\hat{\lambda}\end{cases} \tag{3.6}$$

$(\hat{\lambda},\hat{\mu}\in\Lambda^{(m,n)})$.

Remark 3.2. The orthogonality in Proposition 3.2 extends in fact to the situation that $\lambda\in\Lambda^{(m+1,n)}$ and $\mu\in\Lambda^{(m,n)}$. Indeed, it is immediate from the definitions that if $\lambda_1=m+1$, then $P_\lambda\bigl(\mathbf{x}^{(m,n)}_{\hat{\lambda}}\bigr)=0$ for all $\hat{\lambda}\in\Lambda^{(m,n)}$ (cf. Remark 2.1).

4. Gaussian cubature for symmetric functions

For $\lambda\in\Lambda^{(n)}$, let us define the symmetric monomial

$$M_\lambda(\mathbf{x}):=\sum_{\mu\in S_n\lambda} x_1^{\mu_1}x_2^{\mu_2}\cdots x_n^{\mu_n}, \tag{4.1}$$

where the summation is meant over the orbit $S_n\lambda$ of λ with respect to the standard action of the permutation group S_n on the components

$$\lambda=(\lambda_1,\ldots,\lambda_n)\stackrel{\sigma}{\to}(\lambda_{\sigma^{-1}(1)},\ldots,\lambda_{\sigma^{-1}(n)})=:\sigma\lambda \tag{4.2}$$

for $\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix} \in S_n$. Clearly $M_\lambda(\mathbf{x})$ is homogeneous of total degree

$$|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_n. \tag{4.3}$$

Upon restricting the (inhomogeneous) dominance order

$$\forall \mu, \lambda \in \mathbb{Z}^n : \qquad \mu \leq \lambda \Leftrightarrow \sum_{1\leq j\leq k} (\lambda_j - \mu_j) \geq 0 \quad \text{for} \quad k = 1, \ldots, n \tag{4.4}$$

from $\mathbb{Z}^n$ to $\Lambda^{(n)}$, this partial ordering is inherited by monomial basis $M_\lambda(\mathbf{x})$, $\lambda \in \Lambda^{(n)}$.

Let $\mathbb{P}^{(m,n)}$ denote the $\binom{m+n}{n}$-dimensional subspace of the algebra of symmetric polynomials spanned by $M_\lambda(\mathbf{x})$, $\lambda \in \Lambda^{(m,n)}$. Notice that these are precisely the monomials $M_\lambda(\mathbf{x})$ with $\lambda \in \Lambda^{(n)}$ such that $\lambda \subseteq (m)_n := (m, \ldots, m) \in \Lambda^{(n)}$ (where for $\lambda, \mu \in \Lambda^{(n)}$ one writes $\lambda \subseteq \mu$ iff $\lambda_j \leq \mu_j$ for $j = 1, \ldots, n$). In other words, $\mathbb{P}^{(m,n)}$ consists of all symmetric polynomials of degree at most m in each of the variables x_j ($j \in \{1, \ldots, n\}$).

Lemma 4.1 (Generalized Schur basis)**.** *The generalized Schur polynomials $P_\lambda(\mathbf{x})$, $\lambda \in \Lambda^{(m,n)}$ constitute a basis for $\mathbb{P}^{(m,n)}$.*

Proof. Since the numerator is a polynomial in x_j of degree at most $\lambda_1 + n - 1$ and the Vandermonde determinant is of degree $n - 1$ in x_j, it is clear that the generalized Schur polynomial $P_\lambda(\mathbf{x})$ ((3.2a), (3.2b)) belongs to $\mathbb{P}^{(m,n)}$ when $\lambda \in \Lambda^{(m,n)}$. Moreover, if we replace $p_l(x)$ on the RHS of Equation (3.2a) by x^l, then we recover a classic determinantal formula for the conventional Schur polynomial $S_\lambda(\mathbf{x})$ (cf. e.g. [M92, equation (0.1)]). Hence, up to normalization, the top-degree terms of $P_\lambda(\mathbf{x})$ are given by $S_\lambda(\mathbf{x})$. It is therefore enough to infer that the Schur polynomials $S_\lambda(\mathbf{x})$, $\lambda \in \Lambda^{(m,n)}$ provide a basis for $\mathbb{P}^{(m,n)}$. This, however, is immediate from the well-known fact that the expansion of the Schur polynomials on the monomial basis is unitriangular with respect to the dominance partial order (cf. e.g. [M95, Chapter I.6]):

$$S_\lambda(\mathbf{x}) = M_\lambda(\mathbf{x}) + \sum_{\substack{\mu\in\Lambda^{(n)},\, \mu<\lambda \\ |\mu|=|\lambda|}} C_\lambda^\mu M_\mu(\mathbf{x})$$

for certain (nonnegative integral) coefficients C_λ^μ. □

After these preparations we are now in a position to reformulate the orthogonality relations of Propositions 3.1 and 3.2 as a cubature rule for the integration of symmetric functions in n variables over $(a, b)^n \subseteq \mathbb{R}^n$. The resulting cubature formula, which provides a multivariate extension of the celebrated Gauss quadrature rule (2.3a), (2.3b), was originally found by Berens, Schmid, and Xu; cf. [BSX95, equation (8)] (with $\rho = 0$) and [DX14, Chapter 5.4].

Proposition 4.2 (Exact Gaussian cubature rule in $\mathbb{P}^{(2m+1,n)}$)**.** *For $f(\mathbf{x}) \in \mathbb{P}^{(2m+1,n)}$, one has that*

$$\frac{1}{n!}\int_a^b \cdots \int_a^b f(\mathbf{x})\mathrm{W}(\mathbf{x})\mathrm{d}x_1 \cdots \mathrm{d}x_n = \sum_{\hat{\lambda}\in\Lambda^{(m,n)}} f\big(\mathbf{x}_{\hat{\lambda}}^{(m,n)}\big)\mathrm{W}_{\hat{\lambda}}, \tag{4.5}$$

where $\mathrm{W}(\mathbf{x})$, $\mathbf{x}_{\hat\lambda}^{(m,n)}$, *and* $\mathrm{W}_{\hat\lambda}$ *are drawn from* (3.3b), (3.5b), *and* (3.5c), *respectively.*

Proof. By comparing the orthogonality relations in Propositions 3.1 and 3.2, while also recalling Remark 3.2, it is plain that the cubature rule in (4.5) is valid for $f(\mathbf{x}) = P_\lambda(\mathbf{x})P_\mu(\mathbf{x})$ with $\lambda \in \Lambda^{(m+1,n)}$ and $\mu \in \Lambda^{(m,n)}$. By Lemma 4.1 and the bilinearity, the same is thus true for $f(\mathbf{x}) = M_\lambda(\mathbf{x})M_\mu(\mathbf{x})$ with $\lambda \in \Lambda^{(m+1,n)}$ and $\mu \in \Lambda^{(m,n)}$. Since $\sigma\lambda \leq \lambda$ for all $\lambda \in \Lambda^{(n)}$ and $\sigma \in S_n$ (cf. e.g. [H78, Chapter III.13.2]), it is clear from (4.1) that

$$M_\lambda(\mathbf{x})M_\mu(\mathbf{x}) = M_{\lambda+\mu}(\mathbf{x}) + \sum_{\substack{\nu\in\Lambda^{(n)},\, \nu<\lambda+\mu \\ |\nu|=|\lambda+\mu|}} C^{\nu}_{\lambda,\mu} M_\nu(\mathbf{x})$$

for certain (nonnegative integral) coefficients $C^{\nu}_{\lambda,\mu}$. Thus, the products $M_\lambda(\mathbf{x})M_\mu(\mathbf{x})$ span the space $\mathbb{P}^{(2m+1,n)}$ as λ and μ vary over $\Lambda^{(m+1,n)}$ and $\Lambda^{(m,n)}$, respectively. The asserted cubature rule now follows for general $f(\mathbf{x}) \in \mathbb{P}^{(2m+1,n)}$ by linearity. □

Remark 4.1. Since any symmetric polynomial can be uniquely written as a polynomial expression in the elementary symmetric monomials, the change of variables $\mathbf{x} \to \mathbf{y} = (y_1, \ldots, y_n)$ given by

$$y_k = E_k(x_1, \ldots, x_n) := \sum_{1\leq j_1<j_2<\cdots<j_k\leq n} x_{j_1}x_{j_2}\cdots x_{j_k} \qquad (k = 1, \ldots, n), \tag{4.6}$$

induces a linear isomorphism between $\mathbb{P}^{(m,n)}$ and the space $\Pi^{(m,n)}$ of all (not necessarily symmetric) polynomials in the variables $y_1, \ldots, y_n$ of total degree $\leq m$. In particular, $\dim(\Pi^{(m,n)}) = \dim(\mathbb{P}^{(m,n)}) = \binom{m+n}{m}$. Under this change of variables, the cubature formula in (4.5) transforms into an exact cubature formula in $\Pi^{(2m+1,n)}$ supported on $\dim(\Pi^{(m,n)})$ nodes that was detailed explicitly in [BSX95, equation (2)]. Since it is well known that any exact cubature rule in $\Pi^{(2m+1,n)}$ involves function evaluations on at least $\dim(\Pi^{(m,n)})$ nodes (cf. e.g. [DX14, Chapter 3.8] and references therein), it follows via the change of variables in (4.6) that similarly any exact cubature rule in $\mathbb{P}^{(2m+1,n)}$ involves function evaluations on at least $\dim(\mathbb{P}^{(m,n)})$ nodes. Following standard terminology [DX14, Chapter 3.8], here we refer to exact cubature rules in $\mathbb{P}^{(2m+1,n)}$ supported on precisely (the minimal possible number of) $\dim(\mathbb{P}^{(m,n)})$ nodes as being *Gaussian*. From this perspective, Proposition 4.2 is to be viewed as a concrete example of a Gaussian cubature rule in $\mathbb{P}^{(2m+1,n)}$. Notice in this connection also that, in view of Remark 3.2, the nodes $\mathbf{x}_{\hat\lambda}^{(m,n)}$, $\hat\lambda \in \Lambda^{(m,n)}$ are common zeros of all $\binom{m+n}{m+1}$ basis polynomials $P_\lambda(\mathbf{x})$, $\lambda \in \Lambda^{(n)}$ that are precisely of degree $m+1$ in each of the variables x_j ($j \in \{1, \ldots, n\}$); cf. [DX14, Theorem 3.8.4].

5. The classical orthogonal families: cubature rules for unitary random matrix ensembles

By specializing $p_l(x)$, $l = 0, 1, 2, \ldots$, to the classical orthogonal families of Hermite, Laguerre, and Jacobi type, Proposition 4.2 provides cubature rules for the exact integration of $f(\mathbf{x}) \in \mathbb{P}^{(2m+1,n)}$ with respect to the densities of the Gaussian unitary ensemble, the Laguerre unitary ensemble, and the Jacobi unitary ensemble, respectively.

5.1. **Gaussian unitary ensemble.** The normalized Hermite polynomials

$$h_l(x) = \tfrac{1}{\sqrt{2^l l!}} H_l(x), \qquad l = 0, 1, 2, \ldots,$$

constitute an orthonormal basis on the interval $(a, b) = (-\infty, \infty)$ with respect to the weight function $w(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$ [OLBC10, Chapter 18]. At the $(\hat{l}+1)$th root $x_{\hat{l}}^{(m+1)}$ of $h_{m+1}(x)$ the corresponding Christoffel weight is given by (cf. e.g. [S75, Chapter 15.3] or [DR84, Chapter 3.6])

$$\mathrm{w}_{\hat{l}}^{(m+1)} = \Bigl((m+1) h_m^2 \bigl(x_{\hat{l}}^{(m+1)} \bigr) \Bigr)^{-1} \qquad (0 \leq \hat{l} \leq m).$$

In this situation Proposition 4.2 gives rise to the following Gauss-Hermite cubature rule for the integration of $f(\mathbf{x}) \in \mathbb{P}^{(2m+1,n)}$ with respect to the density of the Gaussian unitary ensemble (cf. e.g. [M04, Chapter 3.3] or [F10, Chapter 1.3]):

$$\text{(5.1a)} \qquad \begin{aligned} &\frac{1}{\pi^{\frac{n}{2}} n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(\mathbf{x}) \prod_{1 \leq j \leq n} e^{-x_j^2} \prod_{1 \leq j < k \leq n} (x_j - x_k)^2 \mathrm{d}x_1 \cdots \mathrm{d}x_n \\ &\quad = \sum_{\hat{\lambda} \in \Lambda^{(m,n)}} f\bigl(\mathbf{x}_{\hat{\lambda}}^{(m,n)} \bigr) \mathrm{W}_{\hat{\lambda}}, \end{aligned}$$

where

$$\text{(5.1b)} \qquad \begin{aligned} \mathrm{W}_{\hat{\lambda}} = &\frac{1}{(m+n)^n} \prod_{1 \leq j \leq n} \Bigl(h_{m+n-1} \bigl(x_{\hat{\lambda}_j + n - j}^{(m+n)} \bigr) \Bigr)^{-2} \\ &\times \prod_{1 \leq j < k \leq n} \Bigl(x_{\hat{\lambda}_j + n - j}^{(m+n)} - x_{\hat{\lambda}_k + n - k}^{(m+n)} \Bigr)^2 . \end{aligned}$$

5.2. **Laguerre unitary ensemble.** For $\alpha > -1$ the normalized Laguerre polynomials

$$\ell_l^{(\alpha)}(x) = \sqrt{\tfrac{l!}{\Gamma(l+1+\alpha)}} L_l^{(\alpha)}(x), \qquad l = 0, 1, 2, \ldots,$$

are orthonormal on the interval $(a, b) = (0, \infty)$ with respect to the weight function $w(x) = x^\alpha e^{-x}$ [OLBC10, Chapter 18]. The Christoffel weight at the $(\hat{l}+1)$th root $x_{\hat{l}}^{(m+1)}$ of $\ell_{m+1}^{(\alpha)}(x)$ reads (cf. e.g. [S75, Chapter 15.3] or [DR84, Chapter 3.6])

$$\mathrm{w}_{\hat{l}}^{(m+1)} = \left((m+1)\, x_{\hat{l}}^{(m+1)} \left(\ell_m^{(\alpha+1)} \bigl(x_{\hat{l}}^{(m+1)} \bigr) \right)^2 \right)^{-1} \qquad (0 \leq \hat{l} \leq m).$$

The corresponding Gauss-Laguerre cubature rule from Proposition 4.2 permits the exact integration of $f(\mathbf{x}) \in \mathbb{P}^{(2m+1,n)}$ with respect to the density of the Laguerre unitary ensemble (cf. e.g. [M04, Chapter 19] or [F10, Chapter 3]):

$$\text{(5.2a)} \qquad \begin{aligned} &\frac{1}{n!} \int_0^{\infty} \cdots \int_0^{\infty} f(\mathbf{x}) \prod_{1 \leq j \leq n} x_j^\alpha e^{-x_j} \prod_{1 \leq j < k \leq n} (x_j - x_k)^2 \mathrm{d}x_1 \cdots \mathrm{d}x_n \\ &\quad = \sum_{\hat{\lambda} \in \Lambda^{(m,n)}} f\bigl(\mathbf{x}_{\hat{\lambda}}^{(m,n)} \bigr) \mathrm{W}_{\hat{\lambda}}, \end{aligned}$$

where

$$\mathrm{W}_{\hat\lambda} = \frac{1}{(m+n)^n} \prod_{1\le j\le n} \Bigl(x^{(m+n)}_{\hat\lambda_j+n-j}\Bigr)^{-1} \Bigl(\ell^{(\alpha+1)}_{m+n-1}\bigl(x^{(m+n)}_{\hat\lambda_j+n-j}\bigr)\Bigr)^{-2} \times \prod_{1\le j<k\le n} \Bigl(x^{(m+n)}_{\hat\lambda_j+n-j} - x^{(m+n)}_{\hat\lambda_k+n-k}\Bigr)^2. \tag{5.2b}$$

5.3. **Jacobi unitary ensemble.** For $\alpha,\beta > -1$ the normalized Jacobi polynomials

$$p_l^{(\alpha,\beta)}(x) = \sqrt{\tfrac{(2l+1+\alpha+\beta)\Gamma(l+1+\alpha+\beta)l!}{2^{\alpha+\beta+1}\Gamma(l+1+\alpha)\Gamma(l+1+\beta)}}\,P_l^{(\alpha,\beta)}(x), \qquad l = 0,1,2,\ldots,$$

are orthonormal on the interval $(a,b) = (-1,1)$ with respect to the weight function $w(x) = (1-x)^\alpha(1+x)^\beta$ [OLBC10, Chapter 18]. The Christoffel weight at the $(\hat l+1)$th root $x_{\hat l}^{(m+1)}$ of $p_{m+1}^{(\alpha,\beta)}(x)$ is given by (cf. e.g. [S75, Chapter 15.3])

$$\mathrm{w}_{\hat l}^{(m+1)} = \frac{(2m+3+\alpha+\beta)}{(m+1)(m+2+\alpha+\beta)\Bigl(1-\bigl(x_{\hat l}^{(m+1)}\bigr)^2\Bigr)\Bigl(p_m^{(\alpha+1,\beta+1)}\bigl(x_{\hat l}^{(m+1)}\bigr)\Bigr)^2}$$

($0 \le \hat l \le m$). The corresponding Gauss-Jacobi cubature rule from Proposition 4.2 permits the exact integration of $f(\mathbf{x}) \in \mathbb{P}^{(2m+1,n)}$ with respect to the density of the Jacobi unitary ensemble (cf. e.g. [M04, Chapter 19] or [F10, Chapter 3]):

$$\frac{1}{n!}\int_{-1}^{1}\cdots\int_{-1}^{1} f(\mathbf{x}) \prod_{1\le j\le n}(1-x_j)^\alpha(1+x_j)^\beta \prod_{1\le j<k\le n}(x_j-x_k)^2 \mathrm{d}x_1\cdots\mathrm{d}x_n = \sum_{\hat\lambda\in\Lambda^{(m,n)}} f\bigl(\mathbf{x}_{\hat\lambda}^{(m,n)}\bigr)\mathrm{W}_{\hat\lambda}, \tag{5.3a}$$

where

$$\mathrm{W}_{\hat\lambda} = \tfrac{(2m+2n+1+\alpha+\beta)^n}{(m+n)^n(m+n+1+\alpha+\beta)^n} \prod_{1\le j\le n}\Bigl(1-\bigl(x^{(m+n)}_{\hat\lambda_j+n-j}\bigr)^2\Bigr)^{-1}\Bigl(p^{(\alpha+1,\beta+1)}_{m+n-1}\bigl(x^{(m+n)}_{\hat\lambda_j+n-j}\bigr)\Bigr)^{-2} \times \prod_{1\le j<k\le n}\Bigl(x^{(m+n)}_{\hat\lambda_j+n-j} - x^{(m+n)}_{\hat\lambda_k+n-k}\Bigr)^2. \tag{5.3b}$$

Remark 5.1. For the Hermite, Laguerre, and Jacobi families the orthogonality relations of the associated symmetric polynomials $P_\lambda(\mathbf{x})$, $\lambda \in \Lambda^{(n)}$ ((3.2a), (3.2b)) originating from Proposition 3.1 were pointed out in [L91a, L91b, L91c]. In these special cases, Proposition 3.2 now provides the complementary discrete orthogonality relations underpinning the cubature rules in equations (5.1a), (5.1b), equations (5.2a), (5.2b), and equations (5.3a), (5.3b).

Remark 5.2. For $n = 2$ the bivariate Gaussian cubature rule stemming from Proposition 4.2 was first formulated in [SX94, Equation (1.4)] (in the symmetrized coordinates $y_1 = x_1 + x_2$ and $y_2 = x_1x_2$ of Remark 4.1). A more detailed study of the corresponding bivariate Gauss-Jacobi cubature ((5.3a), (5.3b)) can be found in [X12, X17].

6. The Bernstein-Szegö polynomials

For parameters $\alpha, \beta \in \{\frac{1}{2}, -\frac{1}{2}\}$, the Gauss-Jacobi cubature ((5.3a), (5.3b)) specializes to more elementary Gauss-Chebyshev cubature rules. For $n = 2$ such bivariate Gauss-Chebyshev cubatures were highlighted in [X12, X17] (cf. Remark 5.2 above). For general n, a systematic study of closely related Gauss-Chebyshev cubature formulas was carried out in [MP11, MMP14, HM14, HMP16] within the framework of compact simple Lie groups. From this perspective, the Gauss-Chebyshev cubatures arising here turn out to be associated with the classical Lie groups of types B_n, C_n, and D_n. In the remainder of the paper, we employ the Bernstein-Szegö polynomials [S75, Section 2.6] to construct (rational) generalizations of the Gauss-Chebyshev cubatures stemming from (5.3a), (5.3b) when $\alpha, \beta \in \{\frac{1}{2}, -\frac{1}{2}\}$. To this end it will be convenient to pass to trigonometric variables from now on:

$$\boxed{x = \cos(\xi), \qquad 0 \leq \xi \leq \pi.}$$

By definition (cf. [S75, Section 2.6]), the Bernstein-Szegö polynomial $p_l\big(\cos(\xi)\big)$ serving our purposes is a polynomial of degree l in $x = \cos(\xi)$ such that the sequence $p_0\big(\cos(\xi)\big), p_1\big(\cos(\xi)\big), p_2\big(\cos(\xi)\big), \ldots$ provides an orthonormal basis of the Hilbert space $L^2((0,\pi), w(\xi)\mathrm{d}\xi)$, where the weight function is of the form

$$w(\xi) := \frac{2^{\epsilon_+ + \epsilon_-}\big(1 + \epsilon_+ \cos(\xi)\big)\big(1 - \epsilon_- \cos(\xi)\big)}{2\pi \prod_{1\leq r\leq d}(1 + 2a_r\cos(\xi) + a_r^2)} \qquad (0 < |a_r| < 1) \tag{6.1}$$

$(r = 1, \ldots, d)$. Here $\epsilon_\pm \in \{0, 1\}$ and, moreover, it is assumed (throughout) that any complex parameters a_r occur in complex conjugate pairs (so $w(\xi)$ remains positive and bounded on the interval $(0, \pi)$).

It is well known (cf. [S75, Section 2.6]) that for $\boxed{l \geq d_\epsilon := \frac{d-\epsilon_+-\epsilon_-}{2}}$, the Bernstein-Szegö polynomial is given by an explicit formula of the form

$$p_l(\cos(\xi)) = \Delta_l^{1/2}\Big(c(\xi)e^{il\xi} + c(-\xi)e^{-il\xi}\Big), \tag{6.2a}$$

where

$$c(\xi) := (1 + \epsilon_+ e^{-i\xi})^{-1}(1 - \epsilon_- e^{-i\xi})^{-1} \prod_{1\leq r\leq d}(1 + a_r e^{-i\xi}) \tag{6.2b}$$

(so $w(\xi) = 1/(2\pi c(\xi)c(-\xi)) = 1/(2\pi|c(\xi)|^2)$) and

$$\Delta_l := \begin{cases} \big(1 + (-1)^{\epsilon_-}\prod_{1\leq r\leq d} a_r\big)^{-1} & \text{if } l = d_\epsilon, \\ 1 & \text{if } l > d_\epsilon. \end{cases} \tag{6.2c}$$

Remark 6.1. For $d = 0$ the Bernstein-Szegö polynomials degenerate to

$$p_l\big(\cos(\xi)\big) = \begin{cases} 2^{1-\delta_l/2}\cos(l\xi) & \text{if } (\epsilon_+, \epsilon_-) = (0,0), \\ \frac{\epsilon_+ \cos((l+\frac{1}{2})\xi)\sin(\frac{\xi}{2}) + \epsilon_- \sin((l+\frac{1}{2})\xi)\cos(\frac{\xi}{2})}{(\epsilon_+ + \epsilon_-)\sin(\frac{\xi}{2})\cos(\frac{\xi}{2})} & \text{if } (\epsilon_+, \epsilon_-) \neq (0,0) \end{cases}$$

(where $\delta_l := 1$ if $l = 0$ and $\delta_l := 0$ otherwise). These are, respectively, the Chebyshev polynomials of the first kind $(\epsilon_+, \epsilon_-) = (0, 0)$, of the second kind $(\epsilon_+, \epsilon_-) = (1, 1)$ (so $p_l\big(\cos(\xi)\big) = \frac{\sin((l+1)\xi)}{\sin(\xi)}$), of the third kind $(\epsilon_+, \epsilon_-) = (0, 1)$ (so $p_l\big(\cos(\xi)\big) = \frac{\sin((l+\frac{1}{2})\xi)}{\sin(\frac{1}{2}\xi)}$), and of the fourth kind $(\epsilon_+, \epsilon_-) = (1, 0)$ (so $p_l\big(\cos(\xi)\big) = \frac{\cos((l+\frac{1}{2})\xi)}{\cos(\frac{1}{2}\xi)}$) (cf. e.g. [OLBC10, Chapter 18]).

Remark 6.2. The formula in (6.2a)–(6.2c) is immediate from the following elementary asymptotics in the complex plane for $l \geq d_\epsilon$:

$$c(\xi)e^{il\xi} + c(-\xi)e^{-il\xi} = \Delta_l^{-1} e^{il\xi} + o(e^{il\xi}) \quad \text{as } |e^{i\xi}| \to +\infty, \tag{6.3a}$$

in combination with the relatively straightforward integration formula for $0 \leq k \leq l$ (cf. the end of this remark below for some additional indications concerning the evaluation of this integral):

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{il\xi}}{c(-\xi)} c_k(\xi)\mathrm{d}\xi = \begin{cases} 1 & \text{if } k = l, \\ 0 & \text{if } k < l, \end{cases} \tag{6.3b}$$

where $c_k(\xi) := 2^{1-\delta_k}\cos(k\xi)$. Indeed, since the (possible) singularities at $e^{i\xi} = \pm 1$ (stemming from $c(\xi)$, if $\epsilon_+ + \epsilon_- > 0$) are removable in the even expression on the LHS of (6.3a), it is clear from the asymptotics that we are dealing with a polynomial of degree $l(\geq d_\epsilon)$ in $\cos(\xi)$. Moreover, it follows from (6.3b) that

$$\int_0^{\pi} \big(c(\xi)e^{il\xi} + \mathrm{c}(-\xi)e^{-il\xi}\big) c_k(\xi) w(\xi)\mathrm{d}\xi = \begin{cases} 1 & \text{if } k = l, \\ 0 & \text{if } k < l \end{cases} \tag{6.4}$$

(where the overall numerical factor is absorbed in the weight function $w(\xi)$). The upshot is that for $l \geq d_\epsilon$ the RHS of (6.2a) satisfies the defining orthogonality relations for $p_l\big(\cos(\xi)\big)$. Notice that this also reveals that the (leading) coefficient α_l of $\big(\cos(\xi)\big)^l$ in $p_l\big(\cos(\xi)\big)$ is given by

$$\alpha_l = 2^l \Delta_l^{-1/2} \tag{6.5}$$

in this situation. Finally, to infer the identity in (6.3b) it suffices to observe that the integral under consideration picks up the constant term of the (Fourier) expansion in $e^{i\xi}$ of the integrand. Indeed, after expanding the d factors stemming from the denominator of $1/c(-\xi)$ in terms of geometric series, it readily follows that the constant term in question is equal to 0 if $l > k \geq 0$ and equal to 1 if $l = k \geq 0$.

7. On the roots of Bernstein-Szegö polynomials

For $m+1 \geq d_\epsilon$, the explicit representation in (6.2a)–(6.2c) permits us to compute the $(\hat{l}+1)$th root $\xi_{\hat{l}}^{(m+1)}$ of the Bernstein-Szegö polynomial

$$p_{m+1}\big(\cos(\xi)\big) = \alpha_{m+1} \prod_{0 \leq \hat{l} \leq m} \left(\cos(\xi) - \cos(\xi_{\hat{l}}^{(m+1)})\right), \tag{7.1a}$$

with the convention

$$0 < \xi_0^{(m+1)} < \xi_1^{(m+1)} < \cdots < \xi_m^{(m+1)} < \pi, \tag{7.1b}$$

as the unique real solution of an elementary transcendental equation.

Proposition 7.1 (Bernstein-Szegö roots). *Given $m+1 \geq d_\epsilon$ and $\hat{l} \in \{0, \ldots, m\}$, the root $\xi_{\hat{l}}^{(m+1)}$ (7.1b) of the Bernstein-Szegö polynomial $p_{m+1}\big(\cos(\xi)\big)$ can be retrieved as the unique real solution of the transcendental equation*

$$2(m+1-d_\epsilon)\xi + \sum_{1 \leq r \leq d} v_{a_r}(\xi) = \pi\left(2\hat{l} + 1 + \epsilon_-\right), \tag{7.2a}$$

where

$$v_a(\xi) := \int_0^\xi \frac{1-a^2}{1+2a\cos(\theta)+a^2}\,d\theta \qquad (|a|<1). \tag{7.2b}$$

Proof. Since for $|a|<1$ and ξ real $v_a'(\xi)+v_{\bar a}'(\xi)>0$, it is clear that the (odd) real function of ξ on the LHS of (7.2a) is monotonously increasing and unbounded (as $v_a(\xi+2\pi)=v_a(\xi)+2\pi$). The transcendental equation in question has therefore a unique real solution $\hat\xi_{\hat l}^{(m+1)}$ (say). Moreover, from the RHS (and the monotonicity of the LHS) we see that $\hat\xi_{\hat k}^{(m+1)} > \hat\xi_{\hat l}^{(m+1)}$ if $\hat k > \hat l$. At $\xi=0$ and $\xi=\pi$ the LHS of (7.2a) takes the values 0 and $(2m+2+\epsilon_+ +\epsilon_-)\pi$, respectively (because $v_a(\pi)=\pi$), so it is clear (by comparing with the values on the RHS) that $0 < \hat\xi_0^{(m+1)} < \hat\xi_1^{(m+1)} < \cdots < \hat\xi_m^{(m+1)} < \pi$. It remains to infer that at $\xi=\hat\xi_{\hat l}^{(m+1)}$ $(0\le \hat l\le m)$ our Bernstein-Szegö polynomial $p_{m+1}\bigl(\cos(\xi)\bigr)$ vanishes or, equivalently (when $m+1\ge d_\epsilon$), that $e^{2i(m+1)\xi}=-\frac{c(-\xi)}{c(\xi)}$ or, more explicitly:

$$e^{2i(m+1-d_\epsilon)\xi} = (-1)^{\epsilon_- +1}\prod_{1\le r\le d}\frac{1+a_r e^{i\xi}}{e^{i\xi}+a_r}. \tag{7.3}$$

Multiplication of (7.2a) by i and exponentiation of both sides with the aid of the identity (cf. (7.5) below)

$$e^{-iv_a(\xi)} = \frac{1+ae^{i\xi}}{e^{i\xi}+a} \qquad (|a|<1)$$

reveals that (7.3) is automatically satisfied at solutions of (7.2a); i.e., $p_{m+1}(\hat\xi_{\hat l}^{(m+1)}) = 0$ and thus $\hat\xi_{\hat l}^{(m+1)} = \xi_{\hat l}^{(m+1)}$ (for $\hat l = 0,\ldots,m$). □

Proposition 7.1 entails the following estimates for the Bernstein-Szegö roots and their distances.

Proposition 7.2 (Estimates for the Bernstein-Szegö roots)**.** *For* $m+1\ge d_\epsilon$, *the Bernstein-Szegö roots* (7.1b) *obey the following inequalities:*

$$\frac{\pi\bigl(\hat l+\frac12+\frac{\epsilon_-}{2}\bigr)}{m+1-d_\epsilon+\kappa_-} \le \xi_{\hat l}^{(m+1)} \le \frac{\pi\bigl(\hat l+\frac12+\frac{\epsilon_-}{2}\bigr)}{m+1-d_\epsilon+\kappa_+} \tag{7.4a}$$

(for $0\le \hat l\le m$*) and*

$$\frac{\pi(\hat k-\hat l)}{m+1-d_\epsilon+\kappa_-} \le \xi_{\hat k}^{(m+1)} - \xi_{\hat l}^{(m+1)} \le \frac{\pi(\hat k-\hat l)}{m+1-d_\epsilon+\kappa_+} \tag{7.4b}$$

(for $0\le \hat l<\hat k\le m$*), where*

$$\kappa_\pm := \frac12\sum_{1\le r\le d}\left(\frac{1-|a_r|}{1+|a_r|}\right)^{\pm1}. \tag{7.4c}$$

Proof. The estimate in (7.4a) readily follows from the transcendental equation for $\xi_{\hat l}^{(m+1)}$ in (7.2a) through the mean value theorem. Here one uses that for ξ real

$$\mathrm{Re}\bigl(v_a'(\xi)\bigr) = \frac12\Bigl(v_{|a|}'\bigl(\xi+\mathrm{Arg}(a)\bigr)+v_{|a|}'\bigl(\xi-\mathrm{Arg}(a)\bigr)\Bigr),$$

whence

$$\frac{1-|a|}{1+|a|} \leq \mathrm{Re}\big(v_a'(\xi)\big) \leq \frac{1+|a|}{1-|a|} \qquad (|a|<1).$$

The estimate in (7.4b) for the distance between the zeros follows in an analogous way after subtracting the $\hat{l}$th equation in (7.2a) from the $\hat{k}$th equation. □

Remark 7.1. The transcendental equation in Proposition 7.1 is well-suited for computing $\xi_{\hat{l}}^{(m+1)}$ ($m+1 \geq d_\epsilon$) numerically (e.g. by means of a standard fixed-point iteration scheme like Newton's method). Notice in this connection that for $-\pi < \xi < \pi$ (and $|a|<1$):

$$v_a(\xi) = i\mathrm{Log}\left(\frac{1+ae^{i\xi}}{e^{i\xi}+a}\right) = 2\mathrm{Arctan}\left(\frac{1-a}{1+a}\tan\left(\frac{\xi}{2}\right)\right), \tag{7.5}$$

so numerical integration can be readily avoided when evaluating $v_a(\xi)$ (7.2b). A natural initial estimate for starting up the numerical computation of $\xi_{\hat{l}}^{(m+1)}$ is provided by the exact $(\hat{l}+1)$th Chebyshev root $\frac{\pi\left(\hat{l}+\frac{1}{2}+\frac{\epsilon_-}{2}\right)}{m+1+\frac{\epsilon_+}{2}+\frac{\epsilon_-}{2}}$ (which corresponds to the case $d=0$; cf. Remark 6.1). Indeed, these Chebyshev roots automatically comply with all inequalities in Proposition 7.2.

8. Gauss-Chebyshev quadrature for rational functions with prescribed poles

For $\epsilon_\pm = 0$ compact expressions for the Christoffel weights associated with the Bernstein-Szegö polynomials were computed in [DGJ06, Theorem 4.4], while for general $\epsilon_\pm \in \{0,1\}$ the corresponding formulas can be gleaned from [BCDG09, Theorems 5.3–5.5]:

$$\begin{aligned} w_{\hat{l}}^{(m+1)} :=& \Big(\sum_{0\leq l\leq m} p_l^2\big(\cos(\xi_{\hat{l}}^{(m+1)})\big)\Big)^{-1} \\ =& \Big(|c(\xi_{\hat{l}}^{(m+1)})|^2 h^{(m+1)}(\xi_{\hat{l}}^{(m+1)})\Big)^{-1} \end{aligned} \tag{8.1}$$

with

$$h^{(m+1)}(\xi) := 2(m+1-d_\epsilon) + \sum_{1\leq r\leq d} v_{a_r}'(\xi)$$

($\hat{l} = 0,\ldots,m$), where it was assumed that $m+1 \geq d_\epsilon$. To keep our presentation self-contained, a short verification of (8.1) is provided in Appendix A.

The Gauss quadrature ((2.3a), (2.3b)) now gives rise to the following exact quadrature rule for the integration of rational functions with prescribed poles against the Chebyshev weight function (cf. [DGJ06, Section 4] and [BCDG09, Section 5]):

$$\frac{1}{2\pi}\int_0^\pi R(\xi)\rho(\xi)\mathrm{d}\xi = \sum_{0\leq \hat{l}\leq m} R\big(\xi_{\hat{l}}^{(m+1)}\big)\rho\big(\xi_{\hat{l}}^{(m+1)}\big)\Big(h^{(m+1)}\big(\xi_{\hat{l}}^{(m+1)}\big)\Big)^{-1}. \tag{8.2a}$$

Here $\rho(\cdot)$ refers to the Chebyshev weight function

$$\rho(\xi) := 2^{\epsilon_+ + \epsilon_-}(1+\epsilon_+\cos(\xi))(1-\epsilon_-\cos(\xi)), \tag{8.2b}$$

and $R(\cdot)$ is of the form

$$R(\xi) = \frac{f\bigl(\cos(\xi)\bigr)}{\prod_{1\leq r\leq d}\bigl(1+2a_r\cos(\xi)+a_r^2\bigr)} \tag{8.2c}$$

with $d \leq 2(m+1)+\epsilon_+ + \epsilon_-$, where $f(\cos(\xi))$ denotes an arbitrary polynomial of degree at most $2m+1$ in $\cos(\xi)$. For $d=0$, the quadrature rule in (8.2a)–(8.2c) reproduces the standard Gauss-Chebyshev quadratures (cf. Remark 6.1).

Remark 8.1. Assuming $m+1 \geq d_\epsilon$, the underlying discrete orthogonality relations for the Bernstein-Szegö polynomials (cf. (2.5a) and (2.5b)) become explicitly

$$\sum_{0\leq \hat{l}\leq m} p_l\bigl(\cos(\xi_{\hat{l}}^{(m+1)})\bigr)p_k\bigl(\cos(\xi_{\hat{l}}^{(m+1)})\bigr)\Bigl(|c(\xi_{\hat{l}}^{(m+1)})|^2 h^{(m+1)}(\xi_{\hat{l}}^{(m+1)})\Bigr)^{-1} = \begin{cases} 1 & \text{if } k=l, \\ 0 & \text{if } k\neq l \end{cases} \tag{8.3a}$$

($l,k \in \{0,\ldots,m\}$) and

$$\sum_{0\leq l\leq m} p_l\bigl(\cos(\xi_{\hat{l}}^{(m+1)})\bigr)p_l\bigl(\cos(\xi_{\hat{k}}^{(m+1)})\bigr) = \begin{cases} |c(\xi_{\hat{l}}^{(m+1)})|^2 h^{(m+1)}(\xi_{\hat{l}}^{(m+1)}) & \text{if } \hat{k}=\hat{l}, \\ 0 & \text{if } \hat{k}\neq \hat{l} \end{cases} \tag{8.3b}$$

($\hat{l},\hat{k} \in \{0,\ldots,m\}$), respectively.

9. Gauss-Chebyshev cubature for symmetric rational functions with prescribed poles at coordinate hyperplanes

The specialization of Proposition 4.2 to the case of the Bernstein-Szegö polynomials now immediately culminates in the principal result of this paper: an explicit cubature rule for the integration of symmetric functions—with prescribed poles at coordinate hyperplanes—against the distributions of the unitary circular Jacobi ensembles. The cubature in question generalizes the quadrature in (8.2a)–(8.2c) to the situation of an arbitrary number of variables $n \geq 1$.

Theorem 9.1 (Gauss-Chebyshev cubature rule for symmetric functions)**.** *Let $\epsilon_\pm \in \{0,1\}$ and $|a_r|<1$ $(r=1,\ldots,d)$ with (possible) complex parameters a_r arising in complex conjugate pairs. Then assuming*

$$d \leq 2(m+n)+\epsilon_+ + \epsilon_-,$$

one has that

$$\frac{1}{(2\pi)^n\, n!}\int_0^\pi\cdots\int_0^\pi R(\boldsymbol{\xi})\rho(\boldsymbol{\xi})\,d\xi_1\cdots d\xi_n = \sum_{\hat{\lambda}\in\Lambda^{(m,n)}} R\bigl(\boldsymbol{\xi}_{\hat{\lambda}}^{(m,n)}\bigr)\rho\bigl(\boldsymbol{\xi}_{\hat{\lambda}}^{(m,n)}\bigr)\Bigl(H^{(m,n)}\bigl(\boldsymbol{\xi}_{\hat{\lambda}}^{(m,n)}\bigr)\Bigr)^{-1}. \tag{9.1a}$$

Here the nodes $\boldsymbol{\xi}_{\hat{\lambda}}^{(m,n)}$ are of the form in (3.5b) *with $\xi_{\hat{l}}^{(m+1)}$ as in* (7.1a), (7.1b) *(cf. also Propositions* 7.1, 7.2*), the weight function $\rho(\cdot)$ refers to the unitary circular*

Jacobi distribution

$$\rho(\boldsymbol{\xi}) := \prod_{1\le j\le n} 2^{\epsilon_+ + \epsilon_-}\bigl(1+\epsilon_+\cos(\xi_j)\bigr)\bigl(1-\epsilon_-\cos(\xi_j)\bigr) \times \prod_{1\le j<k\le n}\bigl(\cos(\xi_j)-\cos(\xi_k)\bigr)^2, \tag{9.1b}$$

the Christoffel weights are governed by

$$H^{(m,n)}(\boldsymbol{\xi}) := \prod_{1\le j\le n} h^{(m+n)}(\xi_j) \tag{9.1c}$$

with $h^{(m+1)}(\cdot)$ *taken from* (8.1)*, and* $R(\cdot)$ *is of the form*

$$R(\boldsymbol{\xi}) = \frac{f\bigl(\cos(\xi_1),\ldots,\cos(\xi_n)\bigr)}{\prod_{\substack{1\le r\le d\\ 1\le j\le n}}\bigl(1+2a_r\cos(\xi_j)+a_r^2\bigr)}, \tag{9.1d}$$

where $f(x_1,\ldots,x_n)=f(\mathbf{x})$ *denotes an arbitrary symmetric polynomial in* $\mathbb{P}^{(2m+1,n)}$.

When $d=0$, Theorem 9.1 reduces to a Gauss-Chebyshev cubature of the form (cf. Remark 6.1)

$$\frac{1}{(2\pi)^n\, n!}\int_0^\pi\cdots\int_0^\pi f\bigl(\cos(\boldsymbol{\xi})\bigr)\rho(\boldsymbol{\xi})\mathrm{d}\xi_1\cdots\mathrm{d}\xi_n = \frac{1}{\bigl(2(m+n)+\epsilon_+ + \epsilon_-\bigr)^n}\sum_{\hat\lambda\in\Lambda^{(m,n)}} f\left(\cos\bigl(\boldsymbol{\xi}_{\hat\lambda}^{(m,n)}\bigr)\right)\rho\bigl(\boldsymbol{\xi}_{\hat\lambda}^{(m,n)}\bigr), \tag{9.2a}$$

where $f\bigl(\cos(\boldsymbol{\xi})\bigr) := f\bigl(\cos(\xi_1),\ldots,\cos(\xi_n)\bigr)$ with $f(x_1,\ldots,x_n)=f(\mathbf{x})\in\mathbb{P}^{(2m+1,n)}$, and with explicit nodes $\boldsymbol{\xi}_{\hat\lambda}^{(m,n)}$ (3.5b) governed by the Chebyshev roots:

$$\xi_{\hat l}^{(m+n)} = \frac{\pi\bigl(\hat l+\frac12+\frac{\epsilon_-}{2}\bigr)}{m+n+\frac{\epsilon_+}{2}+\frac{\epsilon_-}{2}} \qquad (0\le \hat l< m+n). \tag{9.2b}$$

The latter cubature formula turns out to be closely related to a class of integration rules of Lie-theoretic nature studied in [MP11, MMP14, HM14, HMP16], upon restricting to the classical simple Lie groups of types B_n, C_n, and D_n (cf. Remarks 9.2 and 9.3 below for some further details).

Remark 9.1. By exploiting the symmetry of the integrand in the coordinates, the LHS of (9.1a) can be readily rewritten as a multivariate integral over the fundamental domain

$$\mathbb{A}^{(n)} := \{\boldsymbol{\xi}\in\mathbb{R}^n \mid \pi>\xi_1>\xi_2>\cdots>\xi_n>0\}. \tag{9.3}$$

Convention (7.1b) then ensures that the cubature nodes $\xi_{\hat\lambda}^{(m,n)}$, $\hat\lambda\in\Lambda^{(m,n)}$ lie inside this ordered domain of integration.

Remark 9.2. The unitary circular Jacobi distributions $\rho(\cdot)$ (9.1b) correspond to the Haar measures on the compact simple Lie groups $\mathrm{SO}(2n+1;\mathbb{R})$ (type B_n: $\epsilon_+\neq\epsilon_-$), $\mathrm{Sp}(n;\mathbb{H})$ (type C_n: $\epsilon_\pm=1$), and $\mathrm{SO}(2n;\mathbb{R})$ (type D_n: $\epsilon_\pm=0$); cf. e.g. [S96, Chapter IX.9], [P07, Chapter 11.10], or [F10, Chapter 2.6].

Remark 9.3. The Gauss-Chebyshev cubature formula in (9.2a), (9.2b) should be viewed as a counterpart pertaining to the *nonreduced* root system $R=BC_n$ of the cubature rules in [MP11, Theorem 7.2] (where the situation of *reduced* crystallographic root systems was considered). The choice for the kind of underlying Chebyshev polynomials originates in this perspective from a freedom in the weight function $\rho(\boldsymbol{\xi})$ (9.1b), which is given (up to normalization) by the squared modulus of the Weyl denominator of one of the following three reduced subsystems of $R = BC_n$: type B_n ($\epsilon_+ \neq \epsilon_-$), type C_n ($\epsilon_\pm = 1$), or type D_n ($\epsilon_\pm = 0$), respectively (cf. Remarks 6.1 and 9.2 above). For $\epsilon_- = 0$, the cubature in (9.2a), (9.2b) can actually already be retrieved from [HM14, Theorem 5.2] (second part). Notice in this connection that both in [MP11] and in [HM14] the corresponding cubature rules are formulated in symmetrized coordinates involving transformations analogous to the one in Remark 4.1. To further facilitate the comparison of (9.2a), (9.2b) with the formulas in [MP11, Theorem 7.2], let us briefly recall that the Weyl group of the root system BC_n is given by the hyperoctahedral group of signed permutations (acting on the coordinates $\xi_1, \ldots, \xi_n$ through permutations and sign flips). The closure of the ordered integration domain $\mathbb{A}^{(n)}$ (9.3), which constitutes a fundamental domain for this Weyl group action on $\mathbb{T}^{(n)} := \mathbb{R}^n/(2\pi\mathbb{Z})^n$, coincides (up to rescaling) with the positive Weyl alcove of the root system. Similarly, the set $\Lambda^{(m,n)}$ (3.4), which labels both the Schur (character) basis of $\mathbb{P}^{(m,n)}$ and the cubature nodes $\xi^{(m,n)}_{\hat{\lambda}}$, arises as a fundamental domain for the Weyl group action on $\mathbb{Z}^n/(2m\mathbb{Z})^n$. This fundamental domain is built of the BC_n root system's dominant weights of the form

$$\lambda = l_1\varpi_1 + \cdots + l_n\varpi_n \quad \text{with} \quad l_1, \ldots, l_n \geq 0 \quad \text{and} \quad l_1 + \cdots + l_n \leq m, \tag{9.4}$$

where $\varpi_k := e_1 + \cdots + e_k$ ($k = 1, \ldots, n$) refers to the basis of the fundamental weights (and $e_1, \ldots, e_n$ denotes the standard unit basis of (the weight lattice) $\mathbb{Z}^n$). The fact that in the present situation both the (Schur) character basis and the cubature nodes are labeled by the same dominant weights reflects the self-duality of the root system BC_n, whereas in general one has to resort to both weights (for labeling the character basis) and coweights (for labeling the nodes) [MP11].

Remark 9.4. For $\lambda \in \Lambda^{(n)}$, let $P_\lambda\big(\cos(\boldsymbol{\xi})\big) := P_\lambda\big(\cos(\xi_1), \ldots, \cos(\xi_n)\big)$ denote the generalized Schur polynomial from (3.2a), (3.2b) associated with the orthonormal Bernstein-Szegö family $p_l\big(\cos(\xi)\big)$ from Section 6. The orthogonality relations from Proposition 3.1 now become:

$$\frac{1}{(2\pi)^n\, n!}\int_0^\pi \cdots \int_0^\pi P_\lambda\big(\cos(\boldsymbol{\xi})\big)P_\mu\big(\cos(\boldsymbol{\xi})\big)|C(\boldsymbol{\xi})|^{-2}\mathrm{d}\xi_1\cdots\mathrm{d}\xi_n = \begin{cases} 1 & \text{if } \mu = \lambda, \\ 0 & \text{if } \mu \neq \lambda \end{cases} \tag{9.5a}$$

($\lambda, \mu \in \Lambda^{(n)}$), where

$$C(\boldsymbol{\xi}) := 2^{n(n-1)/2} \prod_{1\leq j\leq n} c(\xi_j) \prod_{1\leq j<k\leq n} (1 - e^{-i(\xi_j+\xi_k)})^{-1}(1 - e^{-i(\xi_j-\xi_k)})^{-1} \tag{9.5b}$$

(so $|C(\boldsymbol{\xi})|^{-2} = \prod_{1\leq j\leq n} |c(\xi_j)|^{-2} \prod_{1\leq j<k\leq n}\big(\cos(\xi_j) - \cos(\xi_k)\big)^2$) with $c(\xi)$ taken from (6.2b). Upon expanding the pertinent determinant from P_λ (3.2a), (3.2b), one arrives at a multivariate generalization of the explicit formula in (6.2a)–(6.2c)

that is valid for $\lambda \in \Lambda^{(n)}$ with $\lambda_n \geq d_\epsilon$:

(9.6a)

$$P_\lambda\bigl(\cos(\boldsymbol{\xi})\bigr) = \Delta_\lambda^{1/2} \sum_{\substack{\varepsilon_1,\ldots,\varepsilon_n\in\{-1,1\}\\ \sigma\in S_n}} C(\varepsilon_1\xi_{\sigma_1},\ldots,\varepsilon_n\xi_{\sigma_n}) \exp\bigl(i\varepsilon_1\lambda_1\xi_{\sigma_1}+\cdots+i\varepsilon_n\lambda_n\xi_{\sigma_n}\bigr),$$

where

$$\Delta_\lambda := \prod_{1\leq j\leq n} \Delta_{\lambda_j+n-j} = \Delta_{\lambda_n} \tag{9.6b}$$

and $C(\xi_1,\ldots,\xi_n) = C(\boldsymbol{\xi})$. This explicit formula reveals that the polynomials in question are special instances of the multivariate Bernstein-Szegö polynomials associated with root systems appearing in [D06] (for nonreduced root systems) and in [DMR07] (for reduced root systems).

Remark 9.5. In the situation of the previous remark, the orthogonality relations of Proposition 3.2 and Remark 3.1 give rise to the following multivariate generalization of the discrete orthogonality relations for the Bernstein-Szegö polynomials in Remark 8.1 when $m + n \geq d_\epsilon$:

$$\sum_{\hat\lambda\in\Lambda^{(m,n)}} P_\lambda\bigl(\cos(\boldsymbol{\xi}^{(m,n)}_{\hat\lambda})\bigr) P_\mu\bigl(\cos(\boldsymbol{\xi}^{(m,n)}_{\hat\lambda})\bigr) \Bigl(|C(\boldsymbol{\xi}^{(m,n)}_{\hat\lambda})|^2 H^{(m,n)}(\boldsymbol{\xi}^{(m,n)}_{\hat\lambda})\Bigr)^{-1} = \begin{cases} 1 & \text{if } \mu=\lambda,\\ 0 & \text{if } \mu\neq\lambda \end{cases} \tag{9.7a}$$

$(\lambda,\mu\in\Lambda^{(m,n)})$ and

$$\sum_{\lambda\in\Lambda^{(m,n)}} P_\lambda\bigl(\cos(\boldsymbol{\xi}^{(m,n)}_{\hat\lambda})\bigr) P_\lambda\bigl(\cos(\boldsymbol{\xi}^{(m,n)}_{\hat\mu})\bigr) = \begin{cases} |C(\boldsymbol{\xi}^{(m,n)}_{\hat\lambda})|^2 H^{(m,n)}(\boldsymbol{\xi}^{(m,n)}_{\hat\lambda}) & \text{if } \hat\mu=\hat\lambda,\\ 0 & \text{if } \hat\mu\neq\hat\lambda \end{cases} \tag{9.7b}$$

$(\hat\lambda,\hat\mu\in\Lambda^{(m,n)})$, respectively. (Here the parameter restrictions and the notation are in accordance with Theorem 9.1 and Remark 9.4.)

Appendix A. Explicit Christoffel weights for the Gauss quadrature associated with Bernstein-Szegö polynomials

In this appendix we provide a short verification of equation (8.1) based on the Christoffel-Darboux kernel

$$\sum_{0\leq l\leq m} p_l(x)p_l(y) = \frac{\alpha_m}{\alpha_{m+1}} \frac{p_{m+1}(x)p_m(y)-p_m(x)p_{m+1}(y)}{x-y} \tag{A.1a}$$

on the diagonal $y\to x$

$$\sum_{0\leq l\leq m} \bigl(p_l(x)\bigr)^2 = \frac{\alpha_m}{\alpha_{m+1}} \Bigl(p'_{m+1}(x)p_m(x)-p_{m+1}(x)p'_m(x)\Bigr). \tag{A.1b}$$

At the root $x=\cos(\xi_{\hat{l}}^{(m+1)})$ of $p_{m+1}(x)$, the kernel in (A.1b) produces (cf. e.g. [S75, equation (3.4.7)]):

$$w_{\hat{l}}^{(m+1)} = \frac{-\alpha_{m+2}/\alpha_{m+1}}{p_{m+2}\big(\cos(\xi_{\hat{l}}^{(m+1)})\big)p'_{m+1}\big(\cos(\xi_{\hat{l}}^{(m+1)})\big)} \qquad (0\le \hat{l}\le m), \tag{A.2}$$

where we have used that $\alpha_m\alpha_{m+2}p_m\big(\cos(\xi_{\hat{l}}^{(m+1)})\big) = -\alpha_{m+1}^2 p_{m+2}\big(\cos(\xi_{\hat{l}}^{(m+1)})\big)$ (by the three-term recurrence relation). Combined with the explicit expressions for the Bernstein-Szegö polynomials in (6.2a)–(6.2c) and (6.5) for $l\ge m+1\ge d_\epsilon$, the formula in (A.2) readily produces (8.1) upon invoking that $p_{m+1}\big(\cos(\xi_{\hat{l}}^{(m+1)})\big)=0$, i.e.,

$$e^{2i(m+1)\xi} = -\frac{c(-\xi)}{c(\xi)} \quad \text{at} \quad \xi=\xi_{\hat{l}}^{(m+1)} \tag{A.3}$$

(cf. the proof of Proposition 7.2). Indeed, we see from (6.2a)–(6.2c) that

$$\begin{aligned} p_{m+2}\big(\cos(\xi)\big) &= c(-\xi)e^{-i(m+1)\xi}\left(\frac{c(\xi)}{c(-\xi)}e^{i(2m+3)\xi}+e^{-i\xi}\right)\\ &\overset{(A.3)}{=} -2ic(-\xi)e^{-i(m+1)\xi}\sin(\xi) \end{aligned} \tag{A.4}$$

and that

$$\begin{aligned} p'_{m+1}\big(\cos(\xi)\big) &= -\big(\sin(\xi)\big)^{-1}\Delta_{m+1}^{1/2}c(\xi)e^{i(m+1)\xi}\\ &\times\left(\frac{c'(\xi)}{c(\xi)}-\frac{c'(-\xi)}{c(\xi)}e^{-2i(m+1)\xi}+i(m+1)-i(m+1)\frac{c(-\xi)}{c(\xi)}e^{-2i(m+1)\xi}\right)\\ &\overset{(A.3)}{=} -i\big(\sin(\xi)\big)^{-1}\Delta_{m+1}^{1/2}c(\xi)e^{i(m+1)\xi}\left(2(m+1)+\frac{1}{i}\left(\frac{c'(\xi)}{c(\xi)}+\frac{c'(-\xi)}{c(-\xi)}\right)\right). \end{aligned} \tag{A.5}$$

Substitution of (A.4), (A.5), and (6.5) into (A.2) entails that

$$w_{\hat{l}}^{(m+1)} = \frac{1}{c(\xi_{\hat{l}}^{(m+1)})c(-\xi_{\hat{l}}^{(m+1)})}\left(2(m+1)+\frac{1}{i}\left(\frac{c'(\xi_{\hat{l}}^{(m+1)})}{c(\xi_{\hat{l}}^{(m+1)})}+\frac{c'(-\xi_{\hat{l}}^{(m+1)})}{c(-\xi_{\hat{l}}^{(m+1)})}\right)\right)^{-1}.$$

Equation (8.1) now follows upon making (the imaginary part of) the logarithmic derivative of $c(\xi)$ (6.2b) explicit:

$$\frac{1}{i}\left(\frac{c'(\xi)}{c(\xi)}+\frac{c'(-\xi)}{c(-\xi)}\right) = \epsilon_+ + \epsilon_- + \sum_{1\le r\le d}\left(\frac{1-a_r^2}{1+a_r^2+2a_r\cos(\xi)}-1\right).$$

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