QUADRATURE RULES FROM FINITE ORTHOGONALITY RELATIONS FOR BERNSTEIN-SZEGÖ POLYNOMIALS

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ABSTRACT. We glue two families of Bernstein-Szegö polynomials to construct the eigenbasis of an associated finite-dimensional Jacobi matrix. This gives rise to finite orthogonality relations for this composite eigenbasis of Bernstein-Szegö polynomials. As an application, a number of Gauss-like quadrature rules are derived for the exact integration of rational functions with prescribed poles against the Chebyshev weight functions.

1. INTRODUCTION

From the three-term recurrence relation for the Chebyshev polynomials of the second kind [OLBC10, Chapter 18], it is immediate that these diagonalize a semi-infinite Jacobi matrix with zeros on the diagonal and units on the sub– and superdiagonals. Upon modifying the orthogonality measure via division by a positive polynomial of degree d, the Chebyshev basis passes over into a basis of orthonormal Bernstein-Szegö polynomials [S75, Section 2.6]. These polynomials turn out to diagonalize a (d-parameter family) of semi-infinite Jacobi matrices that are perturbed at the top left block [S75, DS06, GI17]:

$$\begin{bmatrix} b_0 & a_1 & & & & & \\ a_1 & \ddots & \ddots & & & & \\ & \ddots & \ddots & a_l & & & \\ & & a_l & b_l & 1 & & \\ & & & 1 & 0 & \ddots & \\ & & & & \ddots & \ddots & \end{bmatrix},$$

where $l = \lfloor \frac{d}{2} \rfloor$ (and $b_l \equiv 0$ when d is even). In this note we glue two such families of Bernstein-Szegö polynomials depending on d and $\tilde{d}$ parameters, respectively, so as to diagonalize a corresponding $(d + \tilde{d})$-parameter family of $(m + 1)$-dimensional

Jacobi matrices of the form

$$
\begin{bmatrix}
b_0 & a_1 & & & & & & & & & \\
a_1 & \ddots & \ddots & & & & & & & & \\
& \ddots & \ddots & a_l & & & & & & & \\
& & a_l & b_l & 1 & & & & & & \\
& & & 1 & 0 & \ddots & & & & & \\
& & & & \ddots & \ddots & \ddots & & & & \\
& & & & & \ddots & 0 & 1 & & & \\
& & & & & & 1 & b_{m-\tilde{l}} & a_{m+1-\tilde{l}} & & \\
& & & & & & & a_{m+1-\tilde{l}} & \ddots & \ddots & \\
& & & & & & & & \ddots & \ddots & a_m \\
& & & & & & & & & a_m & b_m
\end{bmatrix},
$$

where $l = \lfloor \frac{d}{2} \rfloor$, $\tilde{l} = \lfloor \frac{\tilde{d}}{2} \rfloor$, and with m positive such that $m+1 \geq \lceil \frac{d}{2} \rceil + \lceil \frac{\tilde{d}}{2} \rceil$. The symmetry of the Jacobi matrix gives rise to a finite-dimensional system of discrete orthogonality relations for the pertinent composite eigenbasis built of Bernstein-Szegö polynomials. By standard arguments (cf., e.g., [S75, G81, DR84]), these orthogonality relations imply in turn Gauss-like quadrature rules for the exact integration of rational functions with prescribed poles against the Chebyshev weight function.

When $\tilde{d} = d = 2$, the present construction recovers orthogonality relations that lie at the basis of a four-parameter family of discrete Fourier transforms unifying all sixteen standard types of discrete (co)sine transforms DST-k and DCT-k ($k = 1, \ldots, 8$) [DE18a], whereas for $\tilde{d} = 0$ with d arbitrary one recuperates discrete orthogonality relations for the Bernstein-Szegö polynomials stemming from the Gauss quadrature rule [DGJ06, BCDG09, DE18b].

The presentation splits into four parts. First our main fundamental result is stated in Section 2. Specifically—after recalling some classical facts concerning the Bernstein-Szegö polynomials—we introduce a finite grid consisting of nodes $\xi_0^{(m)}, \ldots, \xi_m^{(m)}$ on the interval $[0, \pi]$ whose positions are governed by an elementary transcendental equation (cf. equations (2.5a), (2.5b)). To each node $\xi_{\hat{l}}^{(m)}$ ($0 \leq \hat{l} \leq m$), we can associate an $(m+1)$-dimensional vector $\psi^{(m)}\big(\xi_{\hat{l}}^{(m)}\big) = \big[\psi_l^{(m)}\big(\xi_{\hat{l}}^{(m)}\big)\big]_{0 \leq l \leq m}$ with components $\psi_l^{(m)}\big(\xi_{\hat{l}}^{(m)}\big)$ built of two families of Bernstein-Szegö polynomials evaluated at the node (cf. equation (2.7)). Our main theorem states that the vectors $\psi^{(m)}\big(\xi_0^{(m)}\big), \ldots, \psi^{(m)}\big(\xi_m^{(m)}\big)$ thus constructed satisfy an explicit system of orthogonality relations (cf. Theorem 1).

Section 3 is devoted to the proof of these orthogonality relations, which proceeds in two steps. First, the orthogonality is verified by inferring that (after performing a diagonal gauge transformation) the vectors in question provide an eigenbasis of an $(m + 1)$-dimensional Jacobi matrix of the form displayed above, with entries built from the recurrence coefficients of the two underlying families of Bernstein-Szegö polynomials. As a byproduct, this reveals that our nodes parametrize the eigenvalues of the Jacobi matrix in question. Next, the quadratic norms of the

vectors are calculated by evaluating the diagonal of the corresponding Christoffel-Darboux kernel at the nodes.

Section 4 collects some further miscellaneous results of interest. Specifically, we compute the characteristic polynomial of our Jacobi matrix, provide estimates for the locations of the nodes $\xi_0^{(m)}, \ldots, \xi_m^{(m)}$ (and thus for the eigenvalues of the Jacobi matrix), and exhibit—as a special example of our construction—finite orthogonality relations for the Askey-Wilson polynomials at $q = 0$.

As a principal application, we wrap up in Section 5 with a precise description of the quadrature rules stemming from our orthogonality relations (cf. Theorem 5). Indeed, by "column-row duality" the orthogonality established in this note implies discrete orthogonality relations for a finite system of Bernstein-Szegö polynomials supported on the nodes $\xi_0^{(m)}, \ldots, \xi_m^{(m)}$. By comparing these discrete orthogonality relations with the conventional continuous orthogonality relations for the Bernstein-Szegö polynomials, the pertinent quadrature formulas are read off immediately. A key feature of the quadrature rules under consideration is that these allow for the exact integration of rational functions with prescribed poles (outside the integration interval) originating from the denominator of the orthogonality measure for the Bernstein-Szegö polynomials.

Note. In principle the Bernstein-Szegö polynomials associated with the Chebyshev polynomials of the first–, the third–, and the fourth kind may be viewed as parameter degenerations of the ones above stemming from the Chebyshev polynomials of the second kind [S75, G02]. However, to avoid limit transitions we have adopted the standard practice of formulating the statements below uniformly for all four kinds.

2. STATEMENT OF THE MAIN RESULT

2.1. **Preliminaries on Bernstein-Szegö polynomials.** The Bernstein-Szegö polynomials constitute an orthogonal basis for the Hilbert space $L^2\big((0, \pi), w(\xi)\mathrm{d}\xi\big)$, where the weight function is of the form

$$(2.1a) \qquad w(\xi) := \frac{1}{2\pi |c(\xi)|^2} \qquad (0 < \xi < \pi),$$

with

$$(2.1b) \qquad c(\xi) := (1 + \epsilon_+ e^{-i\xi})^{-1}(1 - \epsilon_- e^{-i\xi})^{-1} \prod_{1 \le r \le d} (1 + \alpha_r e^{-i\xi})$$

(and $i := \sqrt{-1}$). Here (and below) it is always assumed that

$$\epsilon_\pm \in \{0, 1\} \quad \text{and} \quad 0 < |\alpha_r| < 1 \quad (r = 1, \ldots, d),$$

with possible complex parameters α_r occurring in complex conjugate pairs. More specifically, the Bernstein-Szegö basis $p_l(\xi)$, $l = 0, 1, 2, \ldots$ enjoys a unitriangular expansion on the Fourier-cosine monomials $1, (e^{i\xi} + e^{-i\xi}), (e^{2i\xi} + e^{-2i\xi}), \ldots$ and is obtained from them via Gram-Schmidt orthogonalization with respect to the weight function $w(\xi)$. Rather than working with a monic– or an orthonormal basis, it will be more convenient for our purposes to employ the following normalized version of the Bernstein-Szegö polynomials:

$$(2.2) \qquad \mathfrak{p}_l(\xi) := \Delta_l^{-1} p_l(\xi) \quad \text{with} \quad \Delta_l := \int_0^\pi p_l^2(\xi) \, w(\xi)\mathrm{d}\xi.$$

A crucial observation going back to Bernstein and Szegö—cf. [S75, Section 2.6]—now states that for $\boxed{l \geq d_\epsilon := \frac{1}{2}(d - \epsilon_+ - \epsilon_-)}$ one has explicitly:

$$(2.3a) \qquad \mathrm{p}_l(\xi) = c(\xi)e^{il\xi} + c(-\xi)e^{-il\xi}$$

and (consequently)

$$(2.3b) \qquad \Delta_l = \begin{cases} \left(1 + (-1)^{\epsilon_-} \prod_{1 \leq r \leq d} \alpha_r\right)^{-1} & \text{if } l = d_\epsilon, \\ 1 & \text{if } l > d_\epsilon. \end{cases}$$

2.2. **Composite Bernstein-Szegö basis.** Let us fix two families of Bernstein-Szegö polynomials $\mathrm{p}_l(\xi)$ and $\tilde{\mathrm{p}}_l(\xi)$ associated with the parameters $\epsilon_\pm$, α_r ($r = 1, \ldots, d$) and $\tilde{\epsilon}_\pm$, $\tilde{\alpha}_r$ ($r = 1, \ldots, \tilde{d}$), respectively (subject to the domain restrictions specified above). For positive m such that

$$(2.4) \qquad \boxed{m > \lceil d_\epsilon \rceil + \lceil \tilde{d}_{\tilde{\epsilon}} \rceil}$$

and $\hat{l} \in \{0, \ldots, m\}$, let $\xi_{\hat{l}}^{(m)}$ be defined as the unique real solution of the transcendental equation

$$(2.5a) \quad 2\big(m - d_\epsilon - \tilde{d}_{\tilde{\epsilon}}\big)\xi + \sum_{1 \leq r \leq d} \int_0^\xi u_{\alpha_r}(x)\mathrm{d}x + \sum_{1 \leq r \leq \tilde{d}} \int_0^\xi u_{\tilde{\alpha}_r}(x)\mathrm{d}x = \pi(2\hat{l} + \epsilon_- + \tilde{\epsilon}_-),$$

where for $x \in \mathbb{R}$:

$$(2.5b) \qquad u_\alpha(x) := \frac{1 - \alpha^2}{1 + 2\alpha\cos(x) + \alpha^2} \qquad (|\alpha| < 1)$$
$$= 1 + 2\sum_{l > 0}(-\alpha)^l \cos(lx).$$

Indeed—since $\int_0^\pi u_\alpha(x)\mathrm{d}x = \pi$ and the LHS of equation (2.5a) constitutes a (smooth) strictly increasing function of ξ—it is clear from this transcendental equation (via the mean value theorem) that

$$(2.6) \qquad 0 \leq \xi_0^{(m)} < \xi_1^{(m)} < \cdots < \xi_m^{(m)} \leq \pi.$$

Notice that the equality $\xi_0^{(m)} = 0$ is reached iff $\epsilon_- = \tilde{\epsilon}_- = 0$, and that the equality $\xi_m^{(m)} = \pi$ is reached iff $\epsilon_+ = \tilde{\epsilon}_+ = 0$. The nodes therefore never hit a pole of $c(\pm\xi)$ or $\tilde{c}(\pm\xi)$, even in the situations that the extremal values $\xi_0^{(m)} = 0$ or $\xi_m^{(m)} = \pi$ are reached.

We are now in the position to introduce the following *composite Bernstein-Szegö basis* $\psi_0^{(m)}, \ldots, \psi_m^{(m)}$ on the nodes (2.6):

$$(2.7) \qquad \psi_l^{(m)}(\xi) := \begin{cases} e^{\frac{i}{2}m\xi} \frac{\mathrm{p}_l(\xi)}{c(-\xi)} & \text{if } 0 \leq l < d_\epsilon, \\ e^{\frac{i}{2}m\xi} \frac{\mathrm{p}_l(\xi)}{c(-\xi)} = e^{-\frac{i}{2}m\xi} \frac{\tilde{\mathrm{p}}_{m-l}(\xi)}{\tilde{c}(\xi)} & \text{if } d_\epsilon \leq l \leq m - \tilde{d}_{\tilde{\epsilon}}, \\ e^{-\frac{i}{2}m\xi} \frac{\tilde{\mathrm{p}}_{m-l}(\xi)}{\tilde{c}(\xi)} & \text{if } m - \tilde{d}_{\tilde{\epsilon}} < l \leq m, \end{cases}$$

where $\xi \in \{\xi_0^{(m)}, \ldots, \xi_m^{(m)}\}$ and $l \in \{0, \ldots, m\}$. To avoid possible confusion with customary notation for the higher order derivatives of functions, let us emphasize at this point that throughout the presentation the superscript (m) merely reflects the dependence of our construction on the number of nodes.

2.3. **Finite orthogonality relations.** For $l, \hat{l} \in \{0, \ldots, m\}$, we define the (positive) weights

$$(2.8a) \qquad \Delta_l^{(m)} := \begin{cases} \Delta_l & \text{if } 0 \le l \le d_\epsilon, \\ 1 & \text{if } d_\epsilon < l < m - \tilde{d}_{\tilde{\epsilon}}, \\ \tilde{\Delta}_{m-l} & \text{if } m - \tilde{d}_{\tilde{\epsilon}} \le l \le m, \end{cases}$$

and the dual (positive) weights

$$(2.8b) \quad \hat{\Delta}_{\hat{l}}^{(m)} := \gamma_{\hat{l}}^{(m)} \left(2\big(m - d_\epsilon - \tilde{d}_{\tilde{\epsilon}}\big) + \sum_{1 \le r \le d} u_{\alpha_r}\big(\xi_{\hat{l}}^{(m)}\big) + \sum_{1 \le r \le \tilde{d}} u_{\tilde{\alpha}_r}\big(\xi_{\hat{l}}^{(m)}\big) \right)^{-1}.$$

Here $\gamma_{\hat{l}}^{(m)} := \left(\frac{1}{2}\right)^{(1-\epsilon_-)(1-\tilde{\epsilon}_-)\delta_{\hat{l}} + (1-\epsilon_+)(1-\tilde{\epsilon}_+)\delta_{m-\hat{l}}}$ with $\delta_x := 1$ if $x = 0$ and $\delta_x := 0$ otherwise.

The following two (equivalent) orthogonality relations satisfied by the composite Bernstein-Szegö basis constitute our principal result.

Theorem 1 (Orthogonality relations). *For positive $m > \lceil d_\epsilon \rceil + \lceil \tilde{d}_{\tilde{\epsilon}} \rceil$, the composite Bernstein-Szegö basis* (2.7) *satisfies the orthogonality relations*

$$(2.9a) \qquad \sum_{0 \le \hat{l} \le m} \psi_l^{(m)}\big(\xi_{\hat{l}}^{(m)}\big) \overline{\psi_k^{(m)}\big(\xi_{\hat{l}}^{(m)}\big)} \hat{\Delta}_{\hat{l}}^{(m)} = \begin{cases} 1/\Delta_l^{(m)} & \text{if } k = l, \\ 0 & \text{if } k \ne l, \end{cases}$$

$(0 \le l, k \le m)$, *or equivalently* (*by column-row duality*)

$$(2.9b) \qquad \sum_{0 \le l \le m} \psi_l^{(m)}\big(\xi_{\hat{l}}^{(m)}\big) \overline{\psi_l^{(m)}\big(\xi_{\hat{k}}^{(m)}\big)} \Delta_l^{(m)} = \begin{cases} 1/\hat{\Delta}_{\hat{l}}^{(m)} & \text{if } \hat{k} = \hat{l}, \\ 0 & \text{if } \hat{k} \ne \hat{l}, \end{cases}$$

$(0 \le \hat{l}, \hat{k} \le m)$.

For $d = \tilde{d} = 2$ and $\epsilon_\pm = \tilde{\epsilon}_\pm = 1$, this orthogonality can be found in [DE18a, Section 2.3], while the degenerate case $\tilde{d} = 0$ and $\tilde{\epsilon}_\pm = 1$ is immediate from the Gauss quadrature rule associated with the Bernstein-Szegö polynomials [DE18b, Section 8].

3. PROOF OF THEOREM 1

3.1. **Preparatives.** Before proving the main theorem, let us first corroborate that (the gluing in) the definition of $\psi_l^{(m)}(\xi)$ (2.7) is legitimate, i.e., that at $\xi = \xi_{\hat{l}}^{(m)}$, $\hat{l} \in \{0, \ldots, m\}$:

$$(3.1a) \qquad e^{\frac{1}{2}im\xi} \frac{\mathsf{p}_l(\xi)}{c(-\xi)} = e^{-\frac{1}{2}im\xi} \frac{\tilde{\mathsf{p}}_{m-l}(\xi)}{\tilde{c}(\xi)} \quad \text{for} \quad d_\epsilon \le l \le m - \tilde{d}_{\tilde{\epsilon}}.$$

Recalling equation (2.3a), one readily infers that equation (3.1a) is satisfied when

$$(3.1b) \qquad e^{2im\xi} = \frac{c(-\xi)\tilde{c}(-\xi)}{c(\xi)\tilde{c}(\xi)},$$

or more explicitly (by equation (2.1b)):

$$(3.1c) \qquad e^{2im\xi} = (-1)^{\epsilon_- + \tilde{\epsilon}_-} e^{2i(d_\epsilon + \tilde{d}_{\tilde{\epsilon}})\xi} \prod_{1 \le r \le d} \frac{1 + \alpha_r e^{i\xi}}{e^{i\xi} + \alpha_r} \prod_{1 \le r \le \tilde{d}} \frac{1 + \tilde{\alpha}_r e^{i\xi}}{e^{i\xi} + \tilde{\alpha}_r}.$$

That equation (3.1c) holds at $\xi = \xi_{\hat{l}}^{(m)}$ is immediate from equations (2.5a), (2.5b), upon multiplying this defining transcendental equation for $\xi_{\hat{l}}^{(m)}$ by the imaginary unit and exponentiating both sides with the aid of the identity

$$(3.2) \qquad \exp\left(-i\int_0^\xi u_\alpha(x)\mathrm{d}x\right) = \frac{1 + \alpha e^{i\xi}}{e^{i\xi} + \alpha} \qquad (|\alpha| < 1).$$

Now turning to the proof of the theorem, we observe that both types of orthogonality relations formulated in Theorem 1 amount to the claim that the $(m+1)$-dimensional matrix

$$\left[\sqrt{\Delta_l^{(m)}\hat{\Delta}_{\hat{l}}^{(m)}}\,\psi_l^{(m)}\big(\xi_{\hat{l}}^{(m)}\big)\right]_{0\leq l,\hat{l}\leq m}$$

is unitary. It is therefore sufficient to verify either one of them, and here we choose to infer equation (2.9b). Rewritten explicitly in terms of Bernstein-Szegö polynomials, this orthogonality relation states that for any $\ell \in \{0,\ldots,m\}$ such that $\lceil \tilde{d}_{\tilde{\epsilon}}\rceil < \ell \leq m - \lceil d_\epsilon\rceil$ (which exists because $m > \lceil d_\epsilon\rceil + \lceil \tilde{d}_{\tilde{\epsilon}}\rceil$):

$$(3.3) \qquad \frac{1}{c\big(-\xi_{\hat{l}}^{(m)}\big)c\big(\xi_{\hat{k}}^{(m)}\big)} \sum_{l=0}^{m-\ell} \mathsf{p}_l\big(\xi_{\hat{l}}^{(m)}\big)\mathsf{p}_l\big(\xi_{\hat{k}}^{(m)}\big)\Delta_l$$
$$+ \frac{1}{\tilde{c}\big(\xi_{\hat{l}}^{(m)}\big)\tilde{c}\big(-\xi_{\hat{k}}^{(m)}\big)} \sum_{l=0}^{\ell-1} \tilde{\mathsf{p}}_l\big(\xi_{\hat{l}}^{(m)}\big)\tilde{\mathsf{p}}_l\big(\xi_{\hat{k}}^{(m)}\big)\tilde{\Delta}_l = \begin{cases} 1/\hat{\Delta}_{\hat{l}}^{(m)} & \text{if } \hat{k} = \hat{l}, \\ 0 & \text{if } \hat{k} \neq \hat{l}, \end{cases}$$

$(0 \leq \hat{l}, \hat{k} \leq m)$.

3.2. **Orthogonality.** In the normalization (2.2), the three-term recurrence relation for the Bernstein-Szegö polynomials takes the form

$$(3.4\mathrm{a}) \quad 2\cos(\xi)\mathsf{p}_l(\xi) = \mathsf{p}_{l-1}(\xi) + b_l\mathsf{p}_l(\xi) + a_{l+1}^2\mathsf{p}_{l+1}(\xi) \quad \text{with } a_{l+1} := \left(\frac{\Delta_{l+1}}{\Delta_l}\right)^{1/2}$$

for certain (real) coefficients b_l, $l = 0,1,2,\ldots$ (and $\mathsf{p}_{-1}(\xi) := 0$). From the explicit formulas (2.3a), (2.3b) it is moreover seen that

$$(3.4\mathrm{b}) \qquad a_{l+1} = 1 \text{ for } l > d_\epsilon \quad \text{and} \quad b_l = 0 \text{ for } l > \lceil d_\epsilon\rceil.$$

With the aid of the corresponding three-term recurrences for $\mathsf{p}_l(\xi)$ and $\tilde{\mathsf{p}}_l(\xi)$, we now construct a real tridiagonal matrix that is diagonalized by the composite Bernstein-Szegö basis (2.7):

$$(3.5\mathrm{a}) \qquad\qquad L^{(m)} := \left[L_{l,k}^{(m)}\right]_{0\leq l,k\leq m}$$

with

$$(3.5\mathrm{b}) \quad L_{l,k}^{(m)} := \begin{cases} \delta_{l-k-1} + b_l\delta_{l-k} + a_{l+1}^2\delta_{l-k+1} & \text{if } 0 \leq l \leq \lceil d_\epsilon\rceil, \\ \delta_{l-k-1} + \delta_{l-k+1} & \text{if } \lceil d_\epsilon\rceil < l < m - \lceil \tilde{d}_{\tilde{\epsilon}}\rceil, \\ \tilde{a}_{m-l+1}^2\delta_{l-k-1} + \tilde{b}_{m-l}\delta_{l-k} + \delta_{l-k+1} & \text{if } m - \lceil \tilde{d}_{\tilde{\epsilon}}\rceil \leq l \leq m. \end{cases}$$

Upon interpreting functions $f : \{0, \ldots, m\} \to \mathbb{C}$ as column vectors $[f_l]_{0 \leq l \leq m}$, the action of $L^{(m)}$ (3.5a), (3.5b) on f becomes:

$$(3.6) \qquad (L^{(m)}f)_l := \begin{cases} f_{l-1} + b_l f_l + a_{l+1}^2 f_{l+1} & \text{if } 0 \leq l \leq \lceil d_\epsilon \rceil, \\ f_{l-1} + f_{l+1} & \text{if } \lceil d_\epsilon \rceil < l < m - \lceil \tilde{d}_{\tilde{\epsilon}} \rceil, \\ \tilde{a}_{m-l+1}^2 f_{l-1} + \tilde{b}_{m-l} f_l + f_{l+1} & \text{if } m - \lceil \tilde{d}_{\tilde{\epsilon}} \rceil \leq l \leq m, \end{cases}$$

(where $f_{-1} = f_{m+1} := 0$).

Proposition 2 (Diagonalization of $L^{(m)}$). *Viewed as a function of $l \in \{0, \ldots, m\}$, the composite Bernstein-Szegö vector $\psi_l^{(m)}(\xi)$ (2.7) solves the eigenvalue equation*

$$(3.7) \qquad\qquad L^{(m)} \psi^{(m)}(\xi) = 2\cos(\xi)\psi^{(m)}(\xi)$$

for any $\xi \in \{\xi_0^{(m)}, \ldots, \xi_m^{(m)}\}$.

Proof. If $0 \leq l < m - \lceil \tilde{d}_{\tilde{\epsilon}} \rceil$, then

$$\left(L^{(m)} \psi^{(m)}(\xi)\right)_l \overset{i)}{=} \psi_{l-1}^{(m)}(\xi) + b_l \psi_l^{(m)}(\xi) + a_{l+1}^2 \psi_{l-1}^{(m)}(\xi) \overset{ii)}{=} 2\cos(\xi)\psi_l^{(m)}(\xi),$$

and if $\lceil d_\epsilon \rceil < l \leq m$, then

$$\left(L^{(m)} \psi^{(m)}(\xi)\right)_l \overset{i)}{=} \tilde{a}_{m-l+1}^2 \psi_{l-1}^{(m)}(\xi) + \tilde{b}_{m-l} \psi_l^{(m)}(\xi) + \psi_{l+1}^{(m)}(\xi) \overset{ii)}{=} 2\cos(\xi)\psi_l^{(m)}(\xi).$$

Here we used i) equations (3.6), (3.4b) and ii) equations (2.7), (3.4a). $\square$

Since the (eigen)vectors $\psi^{(m)}(\xi_0^{(m)}), \ldots, \psi^{(m)}(\xi_m^{(m)})$ are nonzero (because for any $0 \leq \hat{l} \leq m$: $\psi_0^{(m)}(\xi_{\hat{l}}^{(m)}) \neq 0$, as $\mathfrak{p}_0(\xi) = \Delta_0^{-1} \neq 0$), and the eigenvalues $2\cos(\xi_0^{(m)}), \ldots, 2\cos(\xi_m^{(m)})$ are simple in view of equation (2.6), it is evident that Proposition 2 indeed provides an eigenbasis for $L^{(m)}$ (3.5a), (3.5b).

Let $\Delta^{(m)}$ denote the positive $(m+1)$-dimensional diagonal matrix

$$(3.8) \qquad\qquad \Delta^{(m)} := \mathrm{diag}\left(\Delta_0^{(m)}, \Delta_1^{(m)}, \ldots, \Delta_m^{(m)}\right).$$

The orthogonality in equation (2.9b) asserts that the vectors

$$\left(\Delta^{(m)}\right)^{1/2} \psi^{(m)}(\xi_0^{(m)}), \ldots, \left(\Delta^{(m)}\right)^{1/2} \psi^{(m)}(\xi_m^{(m)})$$

are orthogonal. In view of the diagonalization stemming from Proposition 2, this orthogonality is clear after corroborating that the similarity transformation

$$(3.9) \qquad\qquad J^{(m)} := \left(\Delta^{(m)}\right)^{1/2} L^{(m)} \left(\Delta^{(m)}\right)^{-1/2}$$

yields a Jacobi matrix.

Proposition 3 (Jacobi matrix $J^{(m)}$). *The (real) tridiagonal matrix $J^{(m)}$ (3.9) is symmetric, with elements $a_1^{(m)}, \ldots, a_m^{(m)}$ on the sub- and superdiagonals given by*

$$(3.10a) \qquad a_{l+1}^{(m)} := \begin{cases} a_{l+1} & \text{if } 0 \leq l < \lceil d_\epsilon \rceil, \\ \left(\Delta_l \tilde{\Delta}_{m-l-1}\right)^{-1/2} & \text{if } \lceil d_\epsilon \rceil \leq l < m - \lceil \tilde{d}_{\tilde{\epsilon}} \rceil, \\ \tilde{a}_{m-l} & \text{if } m - \lceil \tilde{d}_{\tilde{\epsilon}} \rceil \leq l < m, \end{cases}$$

and elements $b_0^{(m)}, \ldots, b_m^{(m)}$ on the principal diagonal of the form

$$(3.10b) \qquad\qquad b_l^{(m)} := \begin{cases} b_l & \text{if } 0 \leq l \leq \lceil d_\epsilon \rceil, \\ 0 & \text{if } \lceil d_\epsilon \rceil < l < m - \lceil \tilde{d}_{\tilde{\epsilon}} \rceil, \\ \tilde{b}_{m-l} & \text{if } m - \lceil \tilde{d}_{\tilde{\epsilon}} \rceil \leq l \leq m. \end{cases}$$

Proof. Starting from the formulas for $L^{(m)}$ and $\Delta^{(m)}$ in equations (3.5a), (3.5b) and equation (3.8), respectively, one readily verifies (upon recalling the definitions in equations (2.8a) and (3.4a)) that $J_{l,l+1}^{(m)} = \left(\Delta_l^{(m)}\right)^{1/2} L_{l,l+1}^{(m)} \left(\Delta_{l+1}^{(m)}\right)^{-1/2} = a_{l+1}^{(m)}$ and that $J_{l+1,l}^{(m)} = \left(\Delta_{l+1}^{(m)}\right)^{1/2} L_{l+1,l}^{(m)} \left(\Delta_l^{(m)}\right)^{-1/2} = a_{l+1}^{(m)}$ $(l = 0, \ldots, m-1)$, whereas on the diagonal it is obvious that $J_{l,l}^{(m)} = L_{l,l}^{(m)} = b_l^{(m)}$ $(l = 0, \ldots, m)$.   □

This completes the proof of equation (2.9b) when $\hat{k} \neq \hat{l}$.

3.3. Quadratic norms.
It remains to infer the quadratic norm formula in equation (2.9b) when $\hat{k} = \hat{l}$. This amounts to the following summation formula.

Proposition 4 (Summation formula). *For any $\xi \in \{\xi_0^{(m)}, \ldots, \xi_m^{(m)}\}$, one has that*

$$\sum_{0 \leq l \leq m} |\psi_l^{(m)}(\xi)|^2 \Delta_l^{(m)} = 2^{\delta_\xi + \delta_{\pi-\xi}} \left(2\big(m - d_\epsilon - \tilde{d}_{\tilde{\epsilon}}\big) + \sum_{1 \leq r \leq d} u_{\alpha_r}(\xi) + \sum_{1 \leq r \leq \tilde{d}} u_{\tilde{\alpha}_r}(\xi) \right).$$

Proof. For $\xi \in (0, \pi)$ the proof hinges on the Christoffel-Darboux kernel for the Bernstein-Szegö polynomials [S75, G07] evaluated on the diagonal:

$$\sum_{0 \leq l \leq N} \mathrm{p}_l^2(\xi) \Delta_l = \frac{\Delta_{N+1}}{2\sin(\xi)} \Big(\mathrm{p}_{N+1}(\xi) \mathrm{p}_N'(\xi) - \mathrm{p}_{N+1}'(\xi) \mathrm{p}_N(\xi) \Big).$$

For $N \geq d_\epsilon$, the RHS can be computed explicitly by means of equations (2.3a), (2.3b):

$$\sum_{0 \leq l \leq N} \mathrm{p}_l^2(\xi) \Delta_l = c(\xi) c(-\xi) \left[\frac{1}{i} \left(\frac{c'(\xi)}{c(\xi)} + \frac{c'(-\xi)}{c(-\xi)} \right) + 2N + 1 \right]$$
$$+ (2i \sin \xi)^{-1} \left(e^{i(2N+1)\xi} c^2(\xi) - e^{-i(2N+1)\xi} c^2(-\xi) \right).$$

By using the definition of the composite Bernstein-Szegö basis in equation (2.7) and splitting the sum as in equation (3.3), we get (assuming $\xi \in \{\xi_0^{(m)}, \ldots, \xi_m^{(m)}\}$):

$$\sum_{0 \leq l \leq m} |\psi_l^{(m)}(\xi)|^2 \Delta_l^{(m)} = \frac{1}{c(\xi) c(-\xi)} \sum_{l=0}^{m-\ell} \mathrm{p}_l^2(\xi) \Delta_l + \frac{1}{\tilde{c}(\xi) \tilde{c}(-\xi)} \sum_{l=0}^{\ell-1} \tilde{\mathrm{p}}_l^2(\xi) \tilde{\Delta}_l$$

for $\ell \in \{0, \ldots, m\}$ such that $\lceil \tilde{d}_{\tilde{\epsilon}} \rceil < \ell \leq m - \lceil d_\epsilon \rceil$. Summation of both parts with the aid of the explicit formula for the diagonal of the Christoffel-Darboux kernel yields:

$$\frac{1}{\tilde{c}(\xi) \tilde{c}(-\xi)} \sum_{l=0}^{\ell-1} \tilde{\mathrm{p}}_l^2(\xi) \tilde{\Delta}_l = \left[\frac{1}{i} \left(\frac{\tilde{c}'(\xi)}{\tilde{c}(\xi)} + \frac{\tilde{c}'(-\xi)}{\tilde{c}(-\xi)} \right) + 2\ell - 1 \right]$$
$$+ (2i \sin \xi)^{-1} \left((e^{i(2\ell-1)\xi} \frac{\tilde{c}(\xi)}{\tilde{c}(-\xi)} - e^{-i(2\ell-1)\xi} \frac{\tilde{c}(-\xi)}{\tilde{c}(\xi)} \right)$$

and

$$\frac{1}{c(\xi)c(-\xi)} \sum_{l=0}^{m-\ell} \mathsf{p}_l^2(\xi)\Delta_l$$

$$= \left[\frac{1}{i} \left(\frac{c'(\xi)}{c(\xi)} + \frac{c'(-\xi)}{c(-\xi)} \right) + 2(m-\ell) + 1 \right]$$

$$+ (2i\sin\xi)^{-1} \left(e^{i(2(m-\ell)+1)\xi} \frac{c(\xi)}{c(-\xi)} - e^{-i(2(m-\ell)+1)\xi} \frac{c(-\xi)}{c(\xi)} \right)$$

$$= \left[\frac{1}{i} \left(\frac{c'(\xi)}{c(\xi)} + \frac{c'(-\xi)}{c(-\xi)} \right) + 2(m-\ell) + 1 \right]$$

$$+ (2i\sin\xi)^{-1} \left(e^{-i(2\ell-1)\xi} \frac{\tilde{c}(-\xi)}{\tilde{c}(\xi)} - e^{i(2\ell-1)\xi} \frac{\tilde{c}(\xi)}{\tilde{c}(-\xi)} \right),$$

where in the last step equation (3.1b) was used. Pasting everything together finally entails the desired result:

$$\sum_{0 \le l \le m} |\psi_l^{(m)}(\xi)|^2 \Delta_l^{(m)} = 2m + \frac{1}{i} \left(\frac{c'(\xi)}{c(\xi)} + \frac{c'(-\xi)}{c(-\xi)} + \frac{\tilde{c}'(\xi)}{\tilde{c}(\xi)} + \frac{\tilde{c}'(-\xi)}{\tilde{c}(-\xi)} \right)$$

(which simplifies to the stated norm formula upon inserting the explicit expressions for $c(\xi)$ and $\tilde{c}(\xi)$ stemming from equation (2.1b) into the logarithmic derivatives). When $\xi = 0$ or $\xi = \pi$, the summation formula follows similarly upon applying L'Hôpital's rule to the expression for the diagonal of the Christoffel-Darboux kernel at the start (which entails an extra factor 2). □

This completes the proof of equation (2.9b) when $\hat{k} = \hat{l}$.

4. Miscellanea

4.1. **Characteristic polynomial.** It is instructive to observe that the spectral values providing our nodes $0 \le \xi_0^{(m)} < \cdots < \xi_m^{(m)} \le \pi$ can be retrieved as the roots of the following polynomial of degree $m + 1$ in $\cos(\xi)$:

$$(4.1a) \qquad Q_{m+1}(\xi) := \sum_{0 \le k \le 2(\tilde{d}_{\tilde{\epsilon}}+1)} \mathsf{e}_k(\tilde{\alpha}; 1 - \tilde{\epsilon}_+; 1 - \tilde{\epsilon}_-) \, \mathsf{q}_{m+1-k}(\xi),$$

where $\mathsf{q}_l(\xi) := c(\xi)e^{il\xi} + c(-\xi)e^{-il\xi}$ for $l \in \mathbb{Z}$ (so $\mathsf{q}_l(\xi) = \mathsf{p}_l(\xi)$ if $l \ge d_\epsilon$) and

$$\mathsf{e}_k(\tilde{\alpha}; \tilde{\epsilon}_+; \tilde{\epsilon}_-) := \mathsf{e}_k(\tilde{\alpha}) + (\tilde{\epsilon}_+ - \tilde{\epsilon}_-)\mathsf{e}_{k-1}(\tilde{\alpha}) - \tilde{\epsilon}_+\tilde{\epsilon}_-\mathsf{e}_{k-2}(\tilde{\alpha}),$$

with

$$\mathsf{e}_k(\tilde{\alpha}) := \sum_{1 \le r_1 < \cdots < r_k \le \tilde{d}} \tilde{\alpha}_{r_1} \cdots \tilde{\alpha}_{r_k}$$

subject to the conventions that $\mathsf{e}_0(\tilde{\alpha}) := 1$ and $\mathsf{e}_k(\tilde{\alpha}) := 0$ if $k \notin \{0, 1, \ldots, \tilde{d}\}$.

More precisely, one has that

$$(4.1b) \qquad Q_{m+1}(\xi) = 2^{m+1} \left(\cos(\xi) - \cos\left(\xi_0^{(m)}\right) \right) \cdots \left(\cos(\xi) - \cos\left(\xi_m^{(m)}\right) \right).$$

Indeed, by means of the explicit formula for $\mathsf{q}_l(\xi)$ it is readily seen that the equation $Q_{m+1}(\xi) = 0$ boils down to equations (3.1b) and (3.1c). To this end $Q_{m+1}(\xi)$ (4.1a) is first split in two parts, with common factors $c(\xi)e^{i(m+1)\xi}$ and $c(-\xi)e^{-i(m+1)\xi}$,

respectively, and then the terms of each factor are collected with the aid of the generating function

$$(1 + \tilde{\epsilon}_+ z)(1 - \tilde{\epsilon}_- z) \prod_{1 \le r \le \tilde{d}} (1 + \tilde{\alpha}_r z) = \sum_{l=0}^{\tilde{d}+\tilde{\epsilon}_+ + \tilde{\epsilon}_-} e_k(\tilde{\alpha}; \tilde{\epsilon}_+; \tilde{\epsilon}_-) z^k$$

(at $z = e^{\mp i\xi}$). To confirm that the degree of $Q_{m+1}(\xi)$ in $\cos(\xi)$ is $m+1$, one uses that

$$\deg(q_l) = \begin{cases} l & \text{if } l \ge d_\epsilon, \\ 2d_\epsilon - l & \text{if } l < d_\epsilon, \end{cases}$$

in combination with our condition on m in equation (2.4), whereas to compute the coefficient of the corresponding leading term (stemming from $q_{m+1}(\xi) = p_{m+1}(\xi)$) one employs equations (2.2) and (2.3a), (2.3b) (thus verifying the factorization of $Q_{m+1}(\xi)$ in equation (4.1b)).

If we compare equation (4.1b) with equation (3.7), then it is clear that $Q_{m+1}(\xi)$ (4.1a) coincides with the characteristic polynomials of the $(m+1)$-dimensional tridiagonal matrices $L^{(m)}$ and $J^{(m)}$ from Propositions 2 and 3, respectively:

$$(4.2) \qquad \det\left(2\cos(\xi)I^{(m)} - L^{(m)}\right) = \det\left(2\cos(\xi)I^{(m)} - J^{(m)}\right) = Q_{m+1}(\xi)$$

(where $I^{(m)}$ refers to the $(m+1)$-dimensional identity matrix).

4.2. **Estimates for the locations of the nodes.** With the aid of the mean value theorem it is immediate from the transcendental equation for $\xi_{\hat{l}}^{(m)}$ in equations (2.5a), (2.5b), that the nodes/spectral values $\xi_0^{(m)}, \ldots, \xi_m^{(m)}$ (2.6) satisfy the following inequalities:

$$(4.3a) \qquad \frac{\pi\left(\hat{l} + \frac{\epsilon_- + \tilde{\epsilon}_-}{2}\right)}{m - d_\epsilon - \tilde{d}_{\tilde{\epsilon}} + \kappa_-} \le \xi_{\hat{l}}^{(m)} \le \frac{\pi\left(\hat{l} + \frac{\epsilon_- + \tilde{\epsilon}_-}{2}\right)}{m - d_\epsilon - \tilde{d}_{\tilde{\epsilon}} + \kappa_+}$$

(for $0 \le \hat{l} \le m$), and

$$(4.3b) \qquad \frac{\pi(\hat{k} - \hat{l})}{m - d_\epsilon - \tilde{d}_{\tilde{\epsilon}} + \kappa_-} \le \xi_{\hat{k}}^{(m)} - \xi_{\hat{l}}^{(m)} \le \frac{\pi(\hat{k} - \hat{l})}{m - d_\epsilon - \tilde{d}_{\tilde{\epsilon}} + \kappa_+}$$

(for $0 \le \hat{l} < \hat{k} \le m$), where

$$(4.3c) \qquad \kappa_\pm := \frac{1}{2} \sum_{1 \le r \le d} \left(\frac{1 - |\alpha_r|}{1 + |\alpha_r|}\right)^{\pm 1} + \frac{1}{2} \sum_{1 \le r \le \tilde{d}} \left(\frac{1 - |\tilde{\alpha}_r|}{1 + |\tilde{\alpha}_r|}\right)^{\pm 1}.$$

Here one uses that for ξ real

$$\text{Re}\left(u_\alpha(\xi)\right) = \frac{1}{2}\left(u_{|\alpha|}\left(\xi + \text{Arg}(\alpha)\right) + u_{|\alpha|}\left(\xi - \text{Arg}(\alpha)\right)\right),$$

and thus

$$\frac{1 - |\alpha|}{1 + |\alpha|} \le \text{Re}\left(u_\alpha(\xi)\right) \le \frac{1 + |\alpha|}{1 - |\alpha|} \qquad (|\alpha| < 1).$$

(To recover the second estimate, for the distances between the nodes, one first subtracts the $\hat{l}$th equation in equation (2.5a) from the $\hat{k}$th equation before invoking the mean value theorem.)

Note. When all Bernstein-Szegö parameters α_r and $\tilde{\alpha}_r$ tend to 0, the above estimates become exact and we reduce to the elementary situation corresponding to $d = \tilde{d} = 0$:

$$(4.4) \qquad \xi_{\hat{l}}^{(m)} = \frac{\pi(2\hat{l} + \epsilon_- + \tilde{\epsilon}_-)}{2m + \epsilon_+ + \epsilon_- + \tilde{\epsilon}_+ + \tilde{\epsilon}_-} \qquad \text{(for } d, \tilde{d} = 0).$$

More generally the positions of the nodes $\xi_{\hat{l}}^{(m)}$ can be effectively retrieved numerically from equations (2.5a), (2.5b), e.g., by means of a standard fixed-point iteration scheme like Newton's method. Notice in this connection that the numerical integration of $u_\alpha(x)$ in equation (2.5a) can be avoided at this point, since (cf. equation (3.2))

$$\int_0^\xi u_\alpha(x)\mathrm{d}x = i\mathrm{Log}\left(\frac{1 + \alpha e^{i\xi}}{e^{i\xi} + \alpha}\right) = 2\mathrm{Arctan}\left(\frac{1 - \alpha}{1 + \alpha}\tan\left(\frac{\xi}{2}\right)\right)$$

(for $|\alpha| < 1$ and $-\pi < \xi < \pi$). Convenient initial values for starting up such numerical calculations are provided by the locations of the elementary nodes at vanishing parameter values from equation (4.4). Indeed, this initial configuration automatically complies with all inequalities in equations (4.3a)–(4.3c).

4.3. Finite orthogonality for Askey-Wilson polynomials at $q = 0$.

If $d = \tilde{d} = 4$ and $\epsilon_\pm = \tilde{\epsilon}_\pm = 1$ (so $d_\epsilon = \tilde{d}_{\tilde{\epsilon}} = 1$), then the Bernstein-Szegö weight function coincides with that of the Askey-Wilson polynomials [AW85, KLS10] at $q = 0$:

$$(4.5) \qquad w(\xi) = \frac{1}{2\pi|c(\xi)|^2}, \quad c(\xi) = \frac{\prod_{r=1}^4(1 + \alpha_r e^{-i\xi})}{1 - e^{-2i\xi}} \qquad (0 < \xi < \pi).$$

We then have explicitly

$$(4.6\text{a}) \qquad \mathrm{p}_l(\xi) = \begin{cases} \Delta_0^{-1} & \text{if } l = 0, \\ c(\xi)e^{il\xi} + c(-\xi)e^{-il\xi} & \text{if } l > 0, \end{cases}$$

and

$$(4.6\text{b}) \qquad \Delta_l = \begin{cases} \frac{1 - \alpha_1\alpha_2\alpha_3\alpha_4}{\prod_{1 \leq r < s \leq 4}(1 - \alpha_r\alpha_s)} & \text{if } l = 0, \\ \frac{1}{1 - \alpha_1\alpha_2\alpha_3\alpha_4} & \text{if } l = 1, \\ 1 & \text{if } l > 1. \end{cases}$$

For any $m > 2$, the associated composite $q = 0$ Askey-Wilson vectors

(4.7a)

$$\psi_l^{(m)}(\xi) = \begin{cases} e^{\frac{i}{2}m\xi}\frac{\Delta_0^{-1}}{c(-\xi)} & \text{if } l = 0, \\ e^{\frac{i}{2}m\xi}\left(\frac{c(\xi)}{c(-\xi)}e^{il\xi} + e^{-il\xi}\right) = e^{-\frac{i}{2}m\xi}\left(e^{i(m-l)\xi} + \frac{\tilde{c}(-\xi)}{\tilde{c}(\xi)}e^{-i(m-l)\xi}\right) & \text{if } 0 < l < m, \\ e^{-\frac{i}{2}m\xi}\frac{\Delta_0^{-1}}{\tilde{c}(\xi)} & \text{if } l = m, \end{cases}$$

at $\xi = \xi_{\hat{l}}^{(m)}$ solving

$$(4.7\text{b}) \qquad 2(m - 2)\xi + \sum_{1 \leq r \leq 4}\int_0^\xi\left(u_{\alpha_r}(x) + u_{\tilde{\alpha}_r}(x)\right)\mathrm{d}x = 2\pi(\hat{l} + 1)$$

$(0 \leq \hat{l} \leq m)$, diagonalize the $(m+1)$-dimensional tridiagonal matrix

$$(4.8) \qquad L^{(m)} = \begin{bmatrix} b_0 & a_1^2 & & & & & \\ 1 & b_1 & a_2^2 & & & & \\ & 1 & 0 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & 0 & 1 & \\ & & & & \tilde{a}_2^2 & \tilde{b}_1 & 1 \\ & & & & & \tilde{a}_1^2 & \tilde{b}_0 \end{bmatrix},$$

where

$$a_1 = \left(\Delta_1/\Delta_0\right)^{1/2}, \qquad a_2 = \Delta_1^{-1/2},$$

$$b_0 = (\Delta_1 - 1) \sum_{1 \leq r \leq 4} \alpha_r^{-1} - \Delta_1 \sum_{1 \leq r \leq 4} \alpha_r,$$

$$b_1 = (\Delta_1 - 1) \sum_{1 \leq r \leq 4} (\alpha_r - \alpha_r^{-1})$$

(and with analog formulas for $\tilde{a}_{l+1}$, $\tilde{b}_l$ and $\tilde{\Delta}_l$, $l = 0, 1$).

Given (any) $1 < \ell \leq m - 1$, the $q = 0$ Askey-Wilson basis satisfies the composite finite orthogonality relations (cf. equation (3.3)):

$$(4.9) \qquad \frac{1}{c\left(-\xi_{\hat{l}}^{(m)}\right) c\left(\xi_{\hat{k}}^{(m)}\right)} \sum_{l=0}^{m-\ell} \mathrm{p}_l\left(\xi_{\hat{l}}^{(m)}\right) \mathrm{p}_l\left(\xi_{\hat{k}}^{(m)}\right) \Delta_l$$

$$+ \frac{1}{\tilde{c}\left(\xi_{\hat{l}}^{(m)}\right) \tilde{c}\left(-\xi_{\hat{k}}^{(m)}\right)} \sum_{l=0}^{\ell-1} \tilde{\mathrm{p}}_l\left(\xi_{\hat{l}}^{(m)}\right) \tilde{\mathrm{p}}_l\left(\xi_{\hat{k}}^{(m)}\right) \tilde{\Delta}_l$$

$$= \begin{cases} 2(m-2) + \sum_{1 \leq r \leq 4} \left(u_{\alpha_r}\left(\xi_{\hat{l}}^{(m)}\right) + u_{\tilde{\alpha}_r}\left(\xi_{\hat{l}}^{(m)}\right) \right) & \text{if } \hat{k} = \hat{l}, \\ 0 & \text{if } \hat{k} \neq \hat{l}, \end{cases}$$

$(0 \leq \hat{l}, \hat{k} \leq m)$.

5. Exact quadratures for rational functions with prescribed poles

From the orthogonality in equation (2.9a), the following quadrature formulae for the exact integration of rational functions with prescribed poles against the Chebyshev weight functions are immediate via standard arguments (cf., e.g., [S75, G81, DR84]).

Theorem 5 (Exact quadrature rules for rational functions). *Let m be positive such that $m > \lceil d_\epsilon \rceil + \lceil \tilde{d}_{\tilde{\epsilon}} \rceil$ (and let us assume that the parameters meet the restrictions in Section 2). Then the following (positive) quadrature rule holds true:*

$$(5.1a) \qquad \frac{1}{2\pi} \int_0^\pi R(\xi) \rho(\xi) \, d\xi = \sum_{0 \leq \hat{l} \leq m} R\left(\xi_{\hat{l}}^{(m)}\right) \rho\left(\xi_{\hat{l}}^{(m)}\right) \hat{\Delta}_{\hat{l}}^{(m)},$$

where the nodes $0 \leq \xi_0^{(m)} < \xi_2^{(m)} < \cdots < \xi_m^{(m)} \leq \pi$ are determined by equations (2.5a), (2.5b) and the Christoffel weights $\hat{\Delta}_0, \ldots, \hat{\Delta}_m$ are given by equation (2.8b). In this identity $\rho(\cdot)$ refers to the Chebyshev weight function

$$(5.1b) \qquad \rho(\xi) := 2^{\epsilon_+ + \epsilon_-} (1 + \epsilon_+ \cos(\xi))(1 - \epsilon_- \cos(\xi)),$$

and $R(\cdot)$ denotes a rational function of the form

$$(5.1c) \qquad R(\xi) = \frac{f\big(\cos(\xi)\big)}{\prod_{1 \leq r \leq d}\big(1 + 2\alpha_r \cos(\xi) + \alpha_r^2\big)},$$

where $f\big(\cos(\xi)\big)$ represents an arbitrary polynomial in $\cos(\xi)$ of degree at most

$$(5.1d) \qquad D := 2\big(m - \tilde{d}_{\tilde{\epsilon}}\big) - 1.$$

Proof. In the normalization from equation (2.2), the orthogonality relations for the Bernstein-Szegö polynomials read:

$$(5.2a) \qquad \int_0^\pi \mathtt{p}_l(\xi)\mathtt{p}_k(\xi)w(\xi)\mathrm{d}x = \begin{cases} \Delta_l^{-1} & \text{if } k = l, \\ 0 & \text{if } k \neq l, \end{cases}$$

for $l, k = 0, 1, 2, \ldots$, where

$$(5.2b) \qquad w(\xi) = \frac{1}{2\pi|c(\xi)|^2} = \frac{2^{\epsilon_+ + \epsilon_-}(1 + \epsilon_+ \cos(\xi))(1 - \epsilon_- \cos(\xi))}{2\pi \prod_{1 \leq r \leq d}\big(1 + 2\alpha_r \cos(\xi) + \alpha_r^2\big)}.$$

On the other hand, the discrete orthogonality in equation (2.9a) guarantees that

$$(5.3a) \qquad \sum_{0 \leq \hat{l} \leq m} \mathtt{p}_l\big(\xi_{\hat{l}}^{(m)}\big)\mathtt{p}_k\big(\xi_{\hat{l}}^{(m)}\big)\left|c\big(\xi_{\hat{l}}^{(m)}\big)\right|^{-2}\hat{\Delta}_{\hat{l}}^{(m)} = \begin{cases} \Delta_l^{-1} & \text{if } k = l, \\ 0 & \text{if } k \neq l, \end{cases}$$

for $0 \leq l, k \leq m$ such that

$$(5.3b) \qquad \begin{cases} 0 \leq l, k \leq m - \lceil \tilde{d}_{\tilde{\epsilon}} \rceil & \text{if } \tilde{d}_{\tilde{\epsilon}} \notin \mathbb{Z}, \\ 0 \leq l \leq m - \lceil \tilde{d}_{\tilde{\epsilon}} \rceil \text{ and } 0 \leq k < m - \lceil \tilde{d}_{\tilde{\epsilon}} \rceil & \text{if } \tilde{d}_{\tilde{\epsilon}} \in \mathbb{Z}. \end{cases}$$

Upon comparing the orthogonality relations in equations (5.2a), (5.2b) and equations (5.3a), (5.3b), it is clear that the exact quadrature rule in equations (5.1a)–(5.1c) is valid if we pick $f\big(\cos(\xi)\big) = \mathtt{p}_l(\xi)\mathtt{p}_k(\xi)$ with $0 \leq l, k \leq m$ as in equation (5.3b). Since the products in question span the space of polynomials in $\cos(\xi)$ of degree at most D (5.1d) (unless $\tilde{d}_{\tilde{\epsilon}} = -1$), the asserted quadrature formula follows in this situation by linearity. If $\tilde{d}_{\tilde{\epsilon}} = -1$, then the present arguments only recover the stated quadrature rule for polynomials $f\big(\cos(\xi)\big)$ up to degree $2m$. However, this case corresponds to that of the classical Gauss quadrature and is thus known to extend up to degree $D = 2m + 1$ (cf. the note below). $\qquad\square$

Note. i). If $\tilde{d} = 0$ and $\tilde{\epsilon}_{\pm} = 1$ (so $\tilde{d}_{\tilde{\epsilon}} = -1$ and $D = 2m + 1$), then Theorem 5 amounts to the conventional Gauss quadrature formula associated with the Bernstein-Szegö polynomials supported on the roots of $\mathtt{p}_{m+1}(\xi)$ [DGJ06, BCDG09, DE18b]. The case $\tilde{d} = 1$ and $\tilde{\epsilon}_{\pm} = 1$ (so $\tilde{d}_{\tilde{\epsilon}} = -\frac{1}{2}$ and $D = 2m$) provides, on the other hand, a corresponding Gauss-type quadrature supported on the roots of the quasi-orthogonal polynomial $\mathtt{p}_{m+1}(\xi) + \tilde{\alpha}_1 \mathtt{p}_m(\xi)$; cf. [MR73, G81, A86].

ii). If $\epsilon_- = \tilde{\epsilon}_- = 0$ and $\epsilon_+ + \tilde{\epsilon}_+ \neq 0$ (so $\tilde{d}_{\tilde{\epsilon}} = \frac{\tilde{d} - \tilde{\epsilon}_+}{2}$ and $D = 2m + \tilde{\epsilon}_+ - \tilde{d} - 1$) or if $\epsilon_- + \tilde{\epsilon}_- \neq 0$ and $\epsilon_+ = \tilde{\epsilon}_+ = 0$ (so $\tilde{d}_{\tilde{\epsilon}} = \frac{\tilde{d} - \tilde{\epsilon}_-}{2}$ and $D = 2m + \tilde{\epsilon}_- - \tilde{d} - 1$), then we arrive at Radau-type quadrature rules with a left-boundary node $x_0^{(m)} = 0$ or a right-boundary node $x_m^{(m)} = \pi$, respectively; cf. [G81, DR84, G04, JB09, P17]. If $\epsilon_{\pm} = \tilde{\epsilon}_{\pm} = 0$ (so $\tilde{d}_{\tilde{\epsilon}} = \frac{\tilde{d}}{2}$ and $D = 2m - \tilde{d} - 1$), then both boundary nodes $x_0^{(m)} = 0$ and $x_m^{(m)} = \pi$ are present, whence our quadrature is of Lobatto-type

in this situation; cf. [G81, DR84, G04, JB09, P17]. It is important to emphasize at this point that the particular kinds of generalized Radau– and Lobatto-type quadratures appearing here do not involve function evaluations of the derivatives at the end-nodes (in contrast to those studied in [G04, JB09, P17]).

iii). Clearly the degree of exactness D (5.1d) becomes unacceptably low if $\tilde{d}$ is too large. By requiring our quadrature to be interpolatory, i.e., $D \geq m$ (thus giving rise to the exact integration of interpolation functions $R(\xi)$ (5.1c) with $\deg(f) \leq m$ and (arbitrarily) prescribed values on the nodes), one has to pick $2\tilde{d}_{\tilde{\epsilon}} \leq m - 1$. In this situation, the expansion of the characteristic polynomial $Q_{m+1}(\xi)$ (4.1a), (4.1b) in the orthogonal Bernstein-Szegö basis is of the form [MR73, G81, A86, P90, X94, EG99]

$$(5.4a) \qquad Q_{m+1}(\xi) = \mathrm{p}_{m+1}(\xi) + \sum_{1 \leq k \leq 2(\tilde{d}_{\tilde{\epsilon}}+1)} c_{2m+1-k}\, \mathrm{p}_{m+1-k}(\xi)$$

for certain coefficients $c_{2m+1-k} \in \mathbb{R}$ $(k = 1, 2, \ldots, 2(\tilde{d}_{\tilde{\epsilon}}+1))$. In other words, $Q_{m+1}(\xi)$ is orthogonal to $\mathrm{p}_{m+1-k}(\xi)$ if $2(\tilde{d}_{\tilde{\epsilon}}+1) < k \leq m+1$, which is readily seen upon integrating the product $f\big(\cos(\xi)\big) = Q_{m+1}(\xi)\mathrm{p}_{m+1-k}(\xi)$ against the weight function $w(\xi)$ (2.1a), (2.1b) by means of the quadrature rule of Theorem 5. If in addition $2\tilde{d}_{\tilde{\epsilon}} \leq m - 1 - d_{\epsilon}$ (so $m + 1 - k \geq d_{\epsilon}$ for $1 \leq k \leq 2(\tilde{d}_{\tilde{\epsilon}}+1)$), then a comparison with equation (4.1a) reveals that in this case one has explicitly:

$$(5.4b) \qquad c_{2m+1-k} = \mathrm{e}_k(\tilde{\alpha}; 1 - \tilde{\epsilon}_+; 1 - \tilde{\epsilon}_-).$$

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