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Periodic Integrable Systems with Delta-Potentials

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Abstract: In this paper we study root system generalizations of the quantum Bose-gas on the circle with pair-wise delta-function interactions. The underlying symmetry structures are shown to be governed by the associated graded algebra of Cherednik's (suitably filtered) degenerate double affine Hecke algebra, acting by Dunkl-type differential-reflection operators. We use Gutkin's generalization of the equivalence between the impenetrable Bose-gas and the free Fermi-gas to derive the Bethe ansatz equations and the Bethe ansatz eigenfunctions.

1. Introduction

Given any affine root system Σ, Gutkin and Sutherland [10, 31] defined a quantum integrable system whose Hamiltonian $-\Delta + \mathcal{V}$ has a potential $\mathcal{V}$ expressible as a weighted sum of delta-functions at the affine root hyperplanes of Σ. For the affine root system of type A, the quantum integrable system essentially reduces to the quantum Bose-gas on the circle with pair-wise delta-function interactions, which has been the subject of intensive studies over the past 40 years.

The special case of the impenetrable Bose-gas on the circle was exactly solved by relating the model to the free Fermi-gas on the circle (see Girardeau [9]). Soon afterwards fundamental progress was made for arbitrary pair-wise delta function interactions by Lieb & Liniger [22], Yang [32] and Yang & Yang [33], leading to the derivation of the associated Bethe ansatz equations and Bethe ansatz eigenfunctions. Yang & Yang [33] showed that the solutions of the Bethe ansatz equations are controlled by a strictly convex master function. One of the aims of the present paper is to generalize these results to Gutkin's and Sutherland's quantum integrable systems associated to affine root systems.

Quantum Calogero-Moser systems are root system generalizations of quantum Bose-gases on the line or circle with long range pair-wise interactions. In special cases quantum Calogero-Moser systems naturally arose from harmonic analysis on symmetric spaces. A decisive role in the studies of quantum Calogero-Moser systems has been played by certain non-bosonic analogs of these systems, which are defined in terms of Dunkl-type commuting differential-reflection operators. Suitable degenerations of affine Hecke algebras naturally appear here as the fundamental objects governing the algebraic relations between the Dunkl-type operators and the natural Weyl group action.

In this paper we define Dunkl-type commuting differential-reflection operators associated to the root system generalizations of the quantum Bose-gas with delta-function interactions. We furthermore show that the Dunkl-type operators, together with the natural affine Weyl group action, realize a faithful representation of the associated graded algebra of Cherednik's [3] (suitably filtered) degenerate double affine Hecke algebra. These results show that these quantum integrable systems naturally fit into the class of quantum Calogero-Moser integrable systems, a point of view which also has been advertised from the perspective of harmonic analysis in [15, Sect. 5].

The quantum integrable systems under consideration for affine root systems Σ of classical type still have reasonable physical interpretations in terms of interacting one-dimensional quantum bosons. In these cases various results of the present paper can be found in the vast physics literature on this subject. We will give the precise connections to the literature in the main body of the text.

The knowledge on the quantum Bose-gas with pair-wise delta-function interactions still far exceeds the knowledge on its root system generalizations. In fact, an important feature of the quantum Bose-gas with pair-wise delta-function interactions is its realization as the restriction to a fixed particle sector of the quantum integrable field theory in $1 + 1$ dimensions governed by the quantum nonlinear Schrödinger equation. This point of view has led to the study of this model by quantum inverse scattering methods. With these methods a proof of full orthogonality of the Bethe eigenfunctions on a period box (with respect to Lebesgue measure) is derived in [5] and the quadratic norms of the Bethe eigenfunctions are evaluated in terms of the determinant of the Hessian of the master function (conjectured by Gaudin [8, Sect. 4.3.3] and proved by Korepin [20]).

At this point we can only speculate on the generalizations of these results to arbitrary root systems. The quantum inverse scattering techniques are only in reach for classical root systems, in which case we have quantum field theories with (non)periodic integrable boundary conditions at our disposal, see [29]. In general it seems reasonable to expect that the Bethe eigenfunctions are orthogonal on a fundamental domain for the reflection representation of the affine Weyl group (with respect to Lebesgue measure), and that their quadratic norms are expressible in terms of the determinant of the Hessian of the master function at the associated spectral point.

The contents of the paper is as follows. Sections 2 and 3 are meant to introduce the quantum integrable systems and to state and clarify the results on the associated spectral problem. We first introduce in Sect. 2 the relevant notations on affine root systems. Following Gutkin [11] we formulate the spectral problem for the quantum integrable systems under consideration as an explicit boundary value problem. We state the main results on the boundary value problem (Bethe ansatz equations and Bethe ansatz eigenfunctions) and we introduce the associated master function. In Sect. 3 we formulate the analog of Girardeau's equivalence between the impenetrable Bose-gas and the free Fermi-gas on the circle for the quantum integrable systems under consideration.

In Sect. 4 we introduce Dunkl-type commuting differential-reflection operators and show that they realize, together with the natural affine Weyl group action, a faithful realization of the associated graded algebra H of Cherednik's [3] (suitably filtered) degenerate double affine Hecke algebra. In Sect. 5 we show that Gutkin's [11] integral-reflection operators, together with the ordinary directional derivatives, yield an equivalent realization of H. The equivalence is realized by Gutkin's [11] propagation operator. We furthermore show that the Dunkl operators naturally act on a space of functions with higher order normal derivative jumps over the affine root hyperplanes.

In Sect. 6 we return to the boundary value problem of Sect. 2. Using the Hecke-type algebra H we refine and clarify Gutkin's [11] generalization of Girardeau's equivalence between the boundary value problem for the impenetrable Bose gas and the boundary value problem for the free Fermi-gas as formulated in Sect. 3. The results in this section entail that the boundary value problem is equivalent to a boundary value problem with trivial boundary value conditions, at the cost of having to deal with a non-standard affine Weyl group action. In Sect. 7 we study the reformulated boundary value problem, leading in Sect. 8 to the derivation of the Bethe ansatz equations. In Sect. 9 we study the master function and show how it leads to a natural parametrization of the solutions of the Bethe ansatz equations. In Sect. 10 the solutions of the Bethe ansatz equations are further analyzed. In Sect. 11 it is proved that the boundary value problem has solutions if and only if the associated spectral value is a regular solution of the Bethe ansatz equations. In case of root system of type A, this is known as the Pauli principle for the interacting bosons.

2. The Boundary Value Problem

In this section we recall Gutkin's [11] reformulation of the spectral problem for periodic integrable systems with delta-potentials in terms of a concrete boundary value problem. We furthermore state the main results on the solutions of the boundary value problem and we detail the physical background.

In order to fix notations we start by recalling some well known facts on affine root systems, see e.g. [17] for a detailed exposition. Let V be an Euclidean space of dimension n. Let Σ_0 be a finite, irreducible crystallographic root system in the dual Euclidean space V^*. We denote $\langle\cdot,\cdot\rangle$ for the inner product on V^* and $\|\cdot\|$ for the corresponding norm. The co-root of $\alpha \in \Sigma_0$ is the unique vector $\alpha^\vee \in V$ satisfying

$$\xi(\alpha^\vee) = \frac{2\langle \xi, \alpha\rangle}{\|\alpha\|^2}, \qquad \forall \xi \in V^*.$$

We write $\Sigma_0^\vee = \{\alpha^\vee\}_{\alpha\in\Sigma_0}$ for the resulting co-root system in V. We fix a basis $I_0 = \{a_1, \ldots, a_n\}$ for the root system Σ_0. Let $\Sigma_0 = \Sigma_0^+ \cup \Sigma_0^-$ be the corresponding decomposition in positive and negative roots. We denote $\rho \in V^*$ for the half sum of positive roots and $\varphi \in \Sigma_0^+$ for the highest root with respect to the basis I_0. The highest root φ is a long root in Σ_0. We define the fundamental Weyl chamber in V^* by

$$V_+^* = \{\xi \in V^* \mid \xi(\alpha^\vee) > 0 \quad \forall\, \alpha \in \Sigma_0^+\}. \tag{2.1}$$

Let $\widehat{V}$ be the vector space of affine linear functionals on V. Then $\widehat{V} \simeq V^* \oplus \mathbb{R}$ as vector spaces, where the second component is identified with the constant functions on V. The gradient map $D : \widehat{V} \to V^*$ is the projection onto V^* along this decomposition.

The subset $\Sigma = \Sigma_0 + \mathbb{Z} \subset \widehat{V}$ is the affine root system associated to Σ_0. We extend the basis I_0 of Σ_0 to a basis $I = \{a_0 = -\varphi + 1, a_1, \ldots, a_n\}$ of the affine root system Σ. Observe that D maps Σ onto Σ_0.

For a root $a \in \Sigma$,

$$s_a(v) = v - a(v) Da^\vee, \qquad v \in V$$

defines the orthogonal reflection in the root hyperplane $V_a := a^{-1}(0)$. The affine Weyl group W associated to Σ is the sub-group of the affine linear isomorphisms of V generated by the orthogonal reflections s_a ($a \in \Sigma$). The sub-group $W_0 \subset W$ generated by the orthogonal reflections s_α ($\alpha \in \Sigma_0$) is the Weyl group associated to Σ_0. We denote w_0 for the longest Weyl group element in W_0. It is well known that W (respectively W_0) is a Coxeter group with Coxeter generators the simple reflections $s_j = s_{a_j}$ for $j = 0, \ldots, n$ (respectively s_j for $j = 1, \ldots, n$).

A second important presentation of W is given by

$$W \simeq W_0 \ltimes Q^\vee, \tag{2.2}$$

with $Q^\vee = \mathbb{Z}\Sigma_0^\vee \subset V$ the co-root lattice of Σ_0, acting by translations on V. The gradient map D induces a surjective group homomorphism $D : W \to W_0$ by $D(s_a) = s_{Da}$ for $a \in \Sigma$. Alternatively, $Dw = v$ if $v \in W_0$ is the W_0-component of w in the semi-direct product decomposition (2.2).

The space $\widehat{V}$ of affine linear functionals on V is a W-module by $(wf)(v) = f(w^{-1}v)$ ($w \in W$, $f \in \widehat{V}$, $v \in V$). Observe that V^* is W_0-stable, and

$$s_\alpha(\xi) = \xi - \xi(\alpha^\vee)\alpha, \qquad \xi \in V^*$$

for roots $\alpha \in \Sigma_0$. Furthermore,

$$s_\alpha(\Sigma_0) = \Sigma_0, \qquad s_a(\Sigma) = \Sigma$$

for $\alpha \in \Sigma_0$ and $a \in \Sigma$. The length of $w \in W$ is defined by $l(w) = \#\big(\Sigma^+ \cap w^{-1}\Sigma^-\big)$. Alternatively, $l(w)$ is the minimal positive integer r such that $w \in W$ can be written as a product of r simple reflections. Such an expression $w = s_{j_1}s_{j_2}\cdots s_{j_{l(w)}}$ ($j_k \in \{0, \ldots, n\}$) is called reduced.

The weight lattice of Σ_0 is defined by

$$P = \{\lambda \in V^* \mid \lambda(\alpha^\vee) \in \mathbb{Z} \quad \forall \alpha \in \Sigma_0\}.$$

Another convenient description is

$$P = \{\lambda \in V^* \mid w\lambda(\varphi^\vee) \in \mathbb{Z} \quad \forall w \in W_0\}, \tag{2.3}$$

which follows from the fact that $Q^\vee$ is already spanned over $\mathbb{Z}$ by the short co-roots in $\Sigma_0^\vee$. We denote P^+ (respectively P^{++}) for the cone of dominant (respectively strictly dominant) weights with respect to the choice Σ_0^+ of positive roots in Σ_0. Recall that $P^{++} = \rho + P^+$.

We write $V_{irreg} = \bigcup_{a\in\Sigma^+} V_a$ for the irregular vectors in V with respect to the affine root hyperplane arrangement $\{V_a \mid a \in \Sigma^+\}$. Its open, dense complement $V_{reg} := V \setminus V_{irreg}$ is called the set of regular vectors in V.

We denote $\mathcal{C}$ for the collection of connected components of V_{reg}. An element $C \in \mathcal{C}$ is called an alcove. The affine Weyl group W acts simply transitively on $\mathcal{C}$. Explicitly, $V_{reg} = \bigcup_{w\in W} w(C_+)$ (disjoint union) with the fundamental alcove C_+ defined by

$$C_+ = \{\, v \in V \mid a_j(v) > 0 \ \ (j = 0, \ldots, n)\,\}.$$

We call a vector $v \in V_a$ ($a \in \Sigma^+$) sub-regular if it does not lie on any other root hyperplane V_b ($a \neq b \in \Sigma^+$).

The symmetric algebra $S(V)$ is canonically a W_0-module algebra. Using the standard identification $S(V) \simeq P(V^*)$, where $P(V^*)$ is the algebra of real-valued polynomial functions on V^*, the W_0-module structure takes the form

$$(wp)(\xi) = p(w^{-1}\xi), \qquad w \in W_0, \ \xi \in V^*.$$

We denote $S(V)^{W_0}$ and $P(V^*)^{W_0}$ for the subalgebra of W_0-invariants in $S(V)$ and $P(V^*)$, respectively.

Let ∂_v ($v \in V$) be the derivative in direction v,

$$(\partial_v f)(u) = \frac{d}{dt}\bigg|_{t=0} f(u+tv)$$

for f continuously differentiable at $u \in V$. The assignment $v \mapsto \partial_v$ uniquely extends to an algebra isomorphism of $S(V)$ onto the algebra of constant coefficient differential operators on V (say acting on $C^\infty(V)$). We denote $p(\partial)$ for the constant coefficient differential operator corresponding to $p \in S(V) \simeq P(V^*)$. For example, the W_0-invariant constant coefficient differential operator $p_2(\partial)$ associated to the polynomial $p_2(\cdot) = \|\cdot\|^2 \in P(V^*)^{W_0}$ is the Laplacian Δ on V.

The quantum integrable system which we will define now in a moment depends on certain coupling constants called multiplicity functions.

Definition 2.1. *A multiplicity function k is a W-invariant function $k : \Sigma \to \mathbb{R}$ satisfying $k(a) = k(Da)$ for all $a \in \Sigma$.*

Unless stated explicitly otherwise, we fix a strictly positive multiplicity function $k : \Sigma \to \mathbb{R}_{>0}$. To simplify notations we write k_a for the value of k at the root $a \in \Sigma$.

We define the quantum Hamiltonian $\mathcal{H}_k$ by

$$\mathcal{H}_k = -\Delta + \sum_{a \in \Sigma} k_a \delta(a(\cdot)), \tag{2.4}$$

where δ is the Kronecker delta-function. Here we interpret $\mathcal{H}_k$ as a linear map $\mathcal{H}_k : C(V) \to D'(V)$, with $C(V)$ the complex-valued continuous functions on V and $D'(V)$ the space of distributions on V, as

$$\big(\mathcal{H}_k f\big)(\phi) := -\int_V f(v)\big(\Delta\phi\big)(v)dv + \sum_{a \in \Sigma} \frac{k_a}{\|Da^\vee\|} \int_{V_a} f(v)\phi(v)d_a v \tag{2.5}$$

for a test function ϕ, with dv the Euclidean volume measure on V and $d_a v$ ($a \in \Sigma^+$) the corresponding volume measure on the root hyperplane V_a.

The quantum Hamiltonian $\mathcal{H}_k$ and the associated quantum physical system has been studied in e.g. [31, 10, 11]. A key step in these investigations is the reformulation of the spectral problem for $\mathcal{H}_k$ in terms of an explicit boundary value problem for the Laplacian Δ on V, which we now proceed to recall.

Let $CB^1(V)$ be the space of complex valued continuous functions f on V whose restriction $f|_C$ to an alcove $C \in \mathcal{C}$ has a continuously differentiable extension to some open neighborhood $\widetilde{C} \supset \overline{C}$. Let $C^{1,(k)}(V)$ be the space of functions $f \in CB^1(V)$ which satisfy the derivative jump conditions

$$\big(\partial_{Da^\vee} f\big)(v + 0Da^\vee) - \big(\partial_{Da^\vee} f\big)(v - 0Da^\vee) = 2k_a f(v) \tag{2.6}$$

for sub-regular vectors $v \in V_a$ ($a \in \Sigma^+$).

Proposition 2.2. *For $f \in CB^1(V)$ and $E \in \mathbb{C}$ the following two statements are equivalent.*
(i) $\mathcal{H}_k f = Ef$ *as distributions on* V.
(ii) $f \in C^{1,(k)}(V)$ *and* $\Delta f|_{V_{reg}} = -Ef|_{V_{reg}}$ *as distributions on* V_{reg}.
A function $f \in CB^1(V)$ *satisfying these equivalent conditions is smooth on* V_{reg}.

Proof. The first part of the proposition follows from a straightforward application of Green's identity (cf. the proof of [11, Thm. 2.7]). The last statement follows from the fact that the constant coefficient differential operator $\Delta + E$ on V is (hypo)elliptic. □

The quantum physical system with quantum Hamiltonian $\mathcal{H}_k$ is known to be integrable. The common spectral problem for the associated quantum conserved integrals has been translated by Sutherland and Gutkin [31, 10] into the following boundary value problem.

Definition 2.3. *Fix a spectral parameter* $\lambda \in V_{\mathbb{C}}^* := \mathbb{C} \otimes_{\mathbb{R}} V^*$. *We denote* $\mathrm{BVP}_k(\lambda)$ *for the space of functions* $f \in C^{1,(k)}(V)$ *solving (in the distributional sense) the system*

$$p(\partial) f\big|_{V_{reg}} = p(\lambda) f\big|_{V_{reg}} \qquad \forall\, p \in S(V)^{W_0} \tag{2.7}$$

of differential equations away from the root hyperplane configuration $\bigcup_{a\in\Sigma^+} V_a$.

Remark 2.4. Since $\Delta = p_2(\partial)$ is the Laplacian on V, Proposition 2.2 implies that a function $f \in \mathrm{BVP}_k(\lambda)$ is smooth on V_{reg} and satisfies the differential equations (2.7) in the strong sense. The fact that f is an eigenfunction of all W_0-invariant constant coefficient differential operators on V_{reg} in fact implies that $f|_C$ is the restriction of a (necessarily unique) analytic function on V for all alcoves $C \in \mathcal{C}$, see [30].

The central theme of this paper is the study of the subspace $\mathrm{BVP}_k(\lambda)^W \subset \mathrm{BVP}(\lambda)$ of $W = W_0 \ltimes Q^\vee$-invariant solutions, where W acts on $\mathrm{BVP}_k(\lambda) \subset C^{1,(k)}(V)$ by

$$(wf)(v) = f(w^{-1}v) \tag{2.8}$$

for $w \in W$ and $v \in V$. Our focus on W-invariant solutions thus amounts to studying the bosonic (=W_0-invariant) theory of the quantum system under $Q^\vee$-periodicity constraints (or equivalently, we view the quantum system on the torus $V/Q^\vee$).

Example 2.5 (Free case $k \equiv 0$). *A function* $f \in \mathrm{BVP}_0(\lambda)$ *is a distribution solution of the (hypo)elliptic constant coefficient differential operator* $\Delta - p_2(\lambda)$ *on* V, *hence* f *is smooth on* V *(cf. Proposition 2.2). Combined with Remark 2.4 we conclude that a function* $f \in \mathrm{BVP}_0(\lambda)$ *is analytic on* V. *Then* $\mathrm{BVP}_0(\lambda)^W$ $(\lambda \in V_{\mathbb{C}}^*)$ *are the common eigenspaces of the quantum conserved integrals for the free bosonic quantum integrable system on* $V/Q^\vee$ *associated to the Laplacian* Δ *on* V. *It is easy to show that* $\mathrm{BVP}_0(\lambda)^W$ *is zero-dimensional unless* $\lambda \in 2\pi i P$, *in which case it is spanned by the plane wave*

$$\phi_\lambda^0 = \frac{1}{\# W_0} \sum_{w\in W_0} e^{w\lambda}$$

(cf. the analysis in the impenetrable case $k \equiv \infty$ *in Sect. 3).*

The quantum Hamiltonian (2.4) for Σ_0 of type A_n takes the explicit form

$$-\Delta + k \sum_{m\in\mathbb{Z}} \sum_{1\le i\ne j\le n+1} \delta(x_i - x_j + m).$$

Here we have embedded V into $\mathbb{R}^{n+1}$ as the hyperplane defined by $x_1 + \cdots + x_{n+1} = 0$. The study of W-invariant solutions to the boundary value problem then essentially amounts to analyzing the spectral problem for the system describing $n+1$ quantum bosons on the circle with pair-wise repulsive delta-function interactions. In this special case the quantum system has been extensively studied in the physics literature, see e.g. [9, 22, 32, 33, 8, 18]. The upgrade to other classical root systems amounts to adding particular reflection terms to the physical model, see e.g. [29, 2, 8, 16, 19, 24].

We are now in a position to formulate the main results on the solution space of the boundary value problem. We call the spectral value $\lambda \in V_{\mathbb{C}}^* = V^* \oplus iV^*$ regular if its isotropy sub-group in W_0 is trivial (equivalently, $\lambda(\alpha^\vee) \ne 0$ for all $\alpha \in \Sigma_0$). We call λ singular otherwise. Furthermore, λ is called real (respectively purely imaginary) if $\lambda \in V^*$ (respectively $\lambda \in iV^*$). Define the c-function by

$$\widetilde{c}_k(\lambda) = \prod_{\alpha\in\Sigma_0^+} \frac{\lambda(\alpha^\vee) + k_\alpha}{\lambda(\alpha^\vee)} \tag{2.9}$$

as rational function of $\lambda \in V_{\mathbb{C}}^*$, cf. [8, 15].

Theorem 2.6. *Let $\lambda \in V_{\mathbb{C}}^*$. The space $\mathrm{BVP}_k(\lambda)^W$ of W-invariant solutions to the boundary value problem is one-dimensional or zero-dimensional. It is one-dimensional if and only if the spectral value λ is a purely imaginary, regular solution of the Bethe ansatz equations*

$$e^{w\lambda(\varphi^\vee)} = \prod_{\alpha\in\Sigma_0^+} \left(\frac{w\lambda(\alpha^\vee) - k_\alpha}{w\lambda(\alpha^\vee) + k_\alpha} \right)^{\alpha(\varphi^\vee)} \qquad \forall\, w \in W_0. \tag{2.10}$$

If $\mathrm{BVP}_k(\lambda)^W$ is one-dimensional, then there exists a unique $\phi_\lambda^k \in \mathrm{BVP}_k(\lambda)^W$ normalized by $\phi_\lambda^k(0) = 1$. The solution ϕ_λ^k is the unique W-invariant function satisfying

$$\phi_\lambda^k(v) = \frac{1}{\#W_0} \sum_{w\in W_0} \widetilde{c}_k(w\lambda) e^{w\lambda(v)}, \qquad v \in \overline{C_+}. \tag{2.11}$$

We give a reformulation of Theorem 2.6 in Sect. 3. The Bethe ansatz equations are derived in Sect. 8. The regularity constraint on λ is proved in Sect. 11.

Remark 2.7. The Bethe ansatz equations (2.10) can be rewritten as

$$e^{w\lambda(\varphi^\vee)} = \frac{w\lambda(\varphi^\vee) - k_\varphi}{w\lambda(\varphi^\vee) + k_\varphi} \prod_{\alpha\in\Sigma_0^+\cap s_\varphi\Sigma_0^-} \frac{w\lambda(\alpha^\vee) - k_\alpha}{w\lambda(\alpha^\vee) + k_\alpha}, \qquad \forall\, w \in W_0, \tag{2.12}$$

due to the fact that for $\alpha \in \Sigma_0^+$,

$$\alpha(\varphi^\vee) = \begin{cases} 2 & \text{if } \alpha = \varphi, \\ 1 & \text{if } \alpha \in \left(\Sigma_0^+ \cap s_\varphi \Sigma_0^-\right) \setminus \{\varphi\}, \\ 0 & \text{if } \alpha \in \Sigma_0^+ \cap s_\varphi \Sigma_0^+. \end{cases} \tag{2.13}$$

A key role in the analysis of the Bethe ansatz equations (2.10) is played by the following *master function*.

Definition 2.8. *The master function* $S_k : P \times V^* \to \mathbb{R}$ *is defined by*

$$S_k(\mu, \xi) = \frac{1}{2}\|\xi\|^2 - 2\pi\langle\mu, \xi\rangle + \frac{1}{2}\sum_{\alpha\in\Sigma_0}\|\alpha\|^2\int_0^{\xi(\alpha^\vee)} \arctan\left(\frac{t}{k_\alpha}\right)dt. \quad (2.14)$$

The master function S_k enters into the description of the set BAE_k of solutions $\lambda \in iV^*$ of the Bethe ansatz equations (2.10) in the following way.

Proposition 2.9. *For* $\mu \in P$ *there exists a unique extremum* $\widehat{\mu}_k \in V^*$ *of the master function* $S_k(\mu, \cdot)$. *The assignment* $\mu \mapsto i\widehat{\mu}_k$ *defines a* W_0*-equivariant bijection* $P \xrightarrow{\sim} \mathrm{BAE}_k$.

The proof of Proposition 2.9, which hinges on the strict convexity of $S_k(\mu, \cdot)$ ($\mu \in P$), is given in Sect. 9. The regularity condition on the spectrum in Theorem 2.6 also turns out to be a consequence of the strict convexity of the master function $S_k(\mu, \cdot)$ ($\mu \in P$), see Sect. 11.

The following proposition yields precise information on the location of the deformed weight $i\widehat{\mu}_k \in \mathrm{BAE}_k$.

Proposition 2.10. *For* $\mu \in P^+$ *and* $\beta \in \Sigma_0^+$ *we have*

$$\frac{2\pi\mu(\beta^\vee)}{\left(1 + \frac{h_k}{n}\right)} \leq \widehat{\mu}_k(\beta^\vee) \leq 2\pi\mu(\beta^\vee), \quad (2.15)$$

where $h_k = 2\sum_{\alpha\in\Sigma_0} k_\alpha^{-1}$. *Furthermore,* $\mu \in P^+$ *if and only if* $\widehat{\mu}_k \in \overline{V_+^*}$.

Proposition 2.10 is proved in Sect. 10. The lower bound in (2.15) shows how far away the spectral values $\widehat{\mu}_k \in V_+^*$ ($\mu \in P^{++}$) are from being singular.

The Bethe ansatz functions ϕ_λ^k and the necessity of the Bethe ansatz equations (2.10) on the allowed spectrum were obtained by Lieb and Liniger [22] for a root system Σ_0 of type A_n, and soon after generalized to a root system Σ_0 of type D_n by Gaudin [7, 8] (see also [19]). For Σ_0 of type A_n, Yang and Yang [33] introduced the master function S (also known as the Yang-Yang action) and derived the special case of Proposition 2.9 using its strict convexity.

In physics literature the regularity of the spectral parameter λ (see Theorem 2.6) is usually imposed as an additional requirement, since it automatically ensures that eigenstates admit a plane wave expansion within any alcove $C \in \mathcal{C}$. The regularity condition for a root system Σ_0 of type A_n can be viewed as a Pauli type principle for the interacting quantum *bosons*, since it implies that the momenta of the quantum bosons are pair-wise different. An actual proof of the regularity of the spectrum was obtained by Izergin and Korepin [18] using quantum inverse scattering methods. In this derivation the regularity condition again follows from the strict convexity of the master function. Estimates for the momenta gaps of the quantum particles play a role in the study of the thermodynamical limit, see [22, 33]. See e.g. [8, Sect 4.3.2] for the exact analog of the estimates (2.15) for Σ_0 of type A_n.

It is believed [18] that quantum integrable systems governed by a strictly convex master function always have a regularity constraint on the spectrum, although a conceptual understanding is not known as far as we know. We remark though that our derivation of the regularity constraint on the spectrum is in accordance with this point of view. A conceptual understanding of the partly fermionic nature of the quantum integrable system at hand is given in the next section.

3. Generalization of Girardeau's Isomorphism

Let $C^\omega(V)$ be the space of complex valued, real analytic functions on V, which we consider as a W-module with respect to the usual action (2.8). Consider for $\lambda \in V_{\mathbb{C}}^*$ the space

$$E(\lambda) = \{f \in C^\omega(V) \mid p(\partial)f = p(\lambda)f \qquad \forall\, p \in S(V)_{\mathbb{C}}^{W_0}\}, \tag{3.1}$$

which is a W-submodule of $C^\omega(V)$. We observed in Example 2.5 that

$$E(\lambda) = \mathrm{BVP}_0(\lambda), \qquad \lambda \in V_{\mathbb{C}}^*. \tag{3.2}$$

In this section we give a convenient description of the solution space $\mathrm{BVP}_k(\lambda)^W$ of the boundary value problem (Definition 2.3) in terms of the space of invariants in $E(\lambda)$ with respect to a k-dependent W-action by integral-reflection operators. We will view this result as a natural generalization of Girardeau's [9] equivalence between the impenetrable quantum Bose-gas and the free quantum Fermi-gas on the circle to arbitrary root systems and to arbitrary multiplicity functions k.

We start by generalizing Girardeau's [9] results on the impenetrable quantum Bose-gas on the circle to arbitrary affine root systems. Denote $E(\lambda)^{Q^\vee}$ for the subspace of $Q^\vee$-translation invariant functions in $E(\lambda)$.

Lemma 3.1. *For $\lambda \in V_{\mathbb{C}}^*$, we have $E(\lambda)^{Q^\vee} = \{0\}$ unless $\lambda \in 2\pi i P$. For $\lambda \in 2\pi i P$ the space $E(\lambda)^{Q^\vee}$ is spanned by e^μ ($\mu \in W_0\lambda$).*

Proof. By [30], a function $f \in E(\lambda)$ can be uniquely expressed as

$$f(v) = \sum_{\mu \in W_0\lambda} p_\mu(v) e^{\mu(v)},$$

where $p_\mu \in P(V)_{\mathbb{C}}$, see also Sect. 7. Such a nonzero function f is $Q^\vee$-translation invariant iff

$$p_\mu(v+\gamma)e^{\mu(\gamma)} = p_\mu(v) \tag{3.3}$$

for all $\mu \in W_0\lambda$, $v \in V$ and $\gamma \in Q^\vee$. This implies that $\lambda \in iV^*$ and that p_μ is bounded on V for all $\mu \in W_0\lambda$. The latter condition implies that p_μ is constant for all $\mu \in W_0\lambda$. Returning to (3.3) with $p_\mu \in \mathbb{C}$, the $Q^\vee$-translation invariance of f is equivalent to $\mu(Q^\vee) \subset 2\pi i\mathbb{Z}$ if $p_\mu \neq 0$. Hence $E(\lambda)^{Q^\vee} = \{0\}$ unless $\lambda \in 2\pi i P$, in which case $E(\lambda)^{Q^\vee}$ is spanned by e^μ ($\mu \in W_0\lambda$). $\square$

We denote $E(\lambda)^{-W}$ for the space of functions $f \in E(\lambda)$ satisfying $f(w^{-1}v) = (-1)^{l(w)}f(v)$ for all $w \in W$ and $v \in V$. Since translations $\mu \in Q^\vee \subset W$ have even length, $E(\lambda)^{-W}$ consists of $Q^\vee$-translation invariant functions. In particular, $E(\lambda)^{-W}$ is the solution space to the spectral problem for a free fermionic quantum integrable system on $V/Q^\vee$ associated to the Laplacian Δ on V.

Corollary 3.2. *Let $\lambda \in V_{\mathbb{C}}^*$. The space $E(\lambda)^{-W}$ is zero-dimensional or one-dimensional. It is one-dimensional iff λ is a regular element from $2\pi i P$, in which case $E(\lambda)^{-W}$ is spanned by*

$$\psi_\lambda^\infty = \frac{1}{\#W_0}\prod_{\alpha\in\Sigma_0}\lambda(\alpha^\vee)^{-1}\sum_{w\in W_0}(-1)^{l(w)}e^{w\lambda}. \tag{3.4}$$

Proof. Let $\lambda \in 2\pi i P$ and $f = \sum_{\mu\in W_0\lambda} c_\mu e^\mu \in E(\lambda)^{Q^\vee}$ with $c_\mu \in \mathbb{C}$, cf. Lemma 3.1. Then we have $f \in E(\lambda)^{-W}$ iff $c_{w\lambda} = (-1)^{l(w)}c_\lambda$ for all $w \in W_0$. For singular λ this implies $c_\mu = 0$ for all $\mu \in W_0\lambda$. For regular λ we conclude that f is a constant multiple of $\psi_\lambda^\infty \in E(\lambda)^{-W}$. □

Following the analogy with Girardeau's [9] analysis of the impenetrable quantum Bose-gas on the circle, we define now a linear map $G : C^\omega(V) \to C(V)^W$ by

$$\big(Gf\big)(w^{-1}v) := f(v), \qquad w \in W,\ v \in \overline{C_+}. \tag{3.5}$$

The map G is injective: for $g \in C(V)^W$ in the image of G, the function $G^{-1}g$ is the unique analytic continuation of $g|_{C_+}$ to V.

For $k \equiv \infty$ we interpret the boundary conditions (2.6) as $f|_{V_a} \equiv 0$ for all $a \in \Sigma^+$. The solution spaces $\mathrm{BVP}_\infty(\lambda)^W$ of the associated boundary value problem (see Definition 2.3) can now be analyzed as follows.

Proposition 3.3. *For $\lambda \in V_{\mathbb{C}}^*$ we have*
(i) The map G restricts to a linear isomorphism $G : E(\lambda)^{-W} \xrightarrow{\sim} \mathrm{BVP}_\infty(\lambda)^W$.
(ii) The space $\mathrm{BVP}_\infty(\lambda)^W$ is zero-dimensional or one-dimensional. It is one-dimensional iff λ is a regular element from $2\pi i P$. In that case $\mathrm{BVP}_\infty(\lambda)^W$ is spanned by $\phi_\lambda^\infty := G(\psi_\lambda^\infty)$, which is the unique W-invariant function satisfying

$$\phi_\lambda^\infty(v) = \frac{1}{\#W_0}\prod_{\alpha\in\Sigma_0^+}\lambda(\alpha^\vee)^{-1}\sum_{w\in W_0}(-1)^{l(w)}e^{w\lambda(v)}, \qquad v \in \overline{C_+}.$$

Proof. (i) A function $f \in E(\lambda)^{-W}$ vanishes on the root hyperplanes V_a ($a \in \Sigma^+$), hence so does $g := G(f) \in C(V)^W$. The function g furthermore satisfies the differential equations (2.7), hence $g \in \mathrm{BVP}_\infty(\lambda)^W$.

For $g \in \mathrm{BVP}_\infty(\lambda)^W$ we define $f = \widetilde{G}(g) \in C(V)^{-W}$ by $f(w^{-1}v) := (-1)^{l(w)}g(v)$ for $w \in W$ and $v \in \overline{C_+}$. This is well defined since g vanishes on the root hyperplanes V_a ($a \in \Sigma^+$). Since f is W-alternating we have $f \in C^{1,(0)}(V)$. The function f satisfies the differential equations (2.7), hence $f \in \mathrm{BVP}_0(\lambda)^{-W} = E(\lambda)^{-W}$, where the last equality follows from (3.2). The proof is now completed by observing that $\widetilde{G} : \mathrm{BVP}_\infty(\lambda)^W \to E(\lambda)^{-W}$ is the inverse of the map $G : E(\lambda)^{-W} \to \mathrm{BVP}_\infty(\lambda)^W$.
(ii) This follows from (i) and Corollary 3.2. □

For a root system Σ_0 of type A, Proposition 3.3 is due to Girardeau [9].

For the generalization of Proposition 3.3 to arbitrary multiplicity function k it is convenient to reinterpret the space $E(\lambda)^{-W}$ as follows. Consider the integral operator

$$\big(\mathcal{I}(a)f\big)(v) = \int_0^{a(v)} f(v - tDa^\vee)dt \qquad (a \in \Sigma) \tag{3.6}$$

as a linear operator on $C(V)$. The integral operators $\mathcal{I}(a)$ ($a \in I$) satisfy the braid relations of Σ as well as the quadratic relations $\mathcal{I}(a)^2 = 0$, cf. e.g. [13]. In particular, given a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_{l(w)}}$ for $w \in W$, the operator $Q_\infty(w) := \mathcal{I}(a_{i_1})\mathcal{I}(a_{i_2})\cdots\mathcal{I}(a_{i_{l(w)}})$ is well defined. Denote

$$E(\lambda)^W_{Q_\infty} := \{f \in E(\lambda) \mid Q_\infty(w)f = 0, \quad \forall\, w \in W \setminus \{e\}\},$$

where $e \in W$ is the unit element of W. We now have the following simple observation.

Lemma 3.4. *For $f \in C(V)$ and $b \in \Sigma$ we have $s_b f = -f$ if and only if $\mathcal{I}(b)f = 0$. In particular, $E(\lambda)^{-W} = E(\lambda)^W_{Q_\infty}$ for all $\lambda \in V^*_{\mathbb{C}}$.*

Proof. It is immediate that $\mathcal{I}(b)f = 0$ if $s_b f = -f$. The converse follows from the fact that

$$\partial_{Db^\vee}\big(\mathcal{I}(b)f\big) = f + s_b f, \tag{3.7}$$

cf. [11, Lem. 2.1(iii)]. □

By Lemma 3.4, Proposition 3.3 (i) can be reformulated as the statement that the map G restricts to an isomorphism

$$G : E(\lambda)^W_{Q_\infty} \xrightarrow{\sim} \mathrm{BVP}_\infty(\lambda)^W. \tag{3.8}$$

The isomorphism (3.8) can now be generalized to arbitrary multiplicity function k as follows. In the terminology of Gutkin [13], the system of integral operators $\{k_b\mathcal{I}(b)\}_{b\in\Sigma^+}$ is an operator calculus with respect to the affine Weyl group W for arbitrary multiplicity function k. This implies that the assignment $s_a \mapsto Q_{k,a}$ $(a \in I)$, with $Q_{k,a}$ the integral-reflection operators

$$\big(Q_{k,a}f\big)(v) = f(s_a v) + k_a\big(\mathcal{I}(a)f\big)(v), \qquad a \in \Sigma,\ \ f \in C(V), \tag{3.9}$$

uniquely defines a W-action on $C(V)$, cf. [11, 13] or Sect. 5. Accordingly, we write

$$Q_k(w) := Q_{k,a_{i_1}} Q_{k,a_{i_2}} \cdots Q_{k,a_{i_r}}$$

for $w = s_{i_1}s_{i_2}\cdots s_{i_r} \in W$. Note that

$$Q_\infty(w) = \lim_{k\to\infty} k_w^{-1} Q_k(w), \qquad \forall\, w \in W,$$

where $k_w := k_{a_{i_1}}k_{a_{i_2}}\cdots k_{a_{i_r}}$ for a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r} \in W$. The generalization of (3.8) for arbitrary multiplicity function k is now the statement that the map G restricts to a linear isomorphism

$$G : E(\lambda)^W_{Q_k} \xrightarrow{\sim} \mathrm{BVP}_k(\lambda)^W \tag{3.10}$$

for arbitrary positive multiplicity function k, where $E(\lambda)^W_{Q_k}$ is the subspace of $Q_k(W)$-invariant functions in $E(\lambda)$. The proof of (3.10) will be given in Sect. 6.

With the isomorphism (3.10) at hand, Theorem 2.6 is equivalent to the following theorem.

Theorem 3.5. *Let $\lambda \in V^*_{\mathbb{C}}$. The space $E(\lambda)^W_{Q_k}$ is one-dimensional or zero-dimensional. It is one-dimensional if and only if λ is a purely imaginary, regular solution of the Bethe ansatz equations* (2.10). *If $E(\lambda)^W_{Q_k}$ is one-dimensional then*

$$\psi^k_\lambda(v) = \frac{1}{\#W_0}\sum_{w\in W_0} \widetilde{c}_k(w\lambda)e^{w\lambda(v)}, \qquad \forall v \in V \tag{3.11}$$

is the unique function in $E(\lambda)^W_{Q_k}$ normalized by $\psi^k_\lambda(0) = 1$.

Theorem 3.5 is proved in Sect. 8 under the assumption that λ is regular. The assertion that λ is necessarily regular is proved in Sect. 11.

In order to reveal the full symmetry structures underlying the isomorphism (3.10), we will consider the upgrade of the map G to a k-dependent linear isomorphism T_k of $C(V)$ which intertwines the $Q_k(W)$-action with the usual W-action (2.8), and which acts as G when applied to $Q_k(W)$-invariant functions. The map which does the job is Gutkin's [11] propagation operator, defined by $\big(T_k f\big)(w^{-1}v) = \big(Q_k(w)f\big)(v)$ for $w \in W$ and $v \in \overline{C_+}$ (see Sect. 5 for details). The propagation operator T_k now restricts to an isomorphism

$$T_k : E(\lambda) \xrightarrow{\sim} \mathrm{BVP}_k(\lambda) \tag{3.12}$$

for all $\lambda \in V_{\mathbb{C}}^*$ (cf. [11] and Theorem 6.3), which implies (3.10) by restricting to the subspaces of W-invariant functions.

We conclude this section by considering the limit to the impenetrable case $k \equiv \infty$. The Bethe ansatz equations (2.10) then reduce to

$$e^{w\lambda(\varphi^\vee)} = 1 \qquad \forall\, w \in W_0,$$

which has $2\pi i P$ as purely imaginary solutions λ (see (2.3)). Furthermore we have

$$\lim_{k\to\infty} \widehat{\mu}_k = 2\pi\mu \tag{3.13}$$

for $\mu \in P^+$, which follows by taking the limit $k \to \infty$ in (2.15). For $\lambda = i\widehat{\mu}_k \in iV^*$ ($\mu \in P^{++}$) a regular solution to the Bethe ansatz equation, $\psi_\lambda^k \in E(\lambda)_{Q_k}^W$ (see (3.11)) can alternatively be written as

$$\psi_\lambda^k = \frac{1}{\#W_0} \sum_{w\in W_0} Q_k(w)(e^\lambda),$$

see [15] or Sect. 7. It follows that

$$\lim_{k\to\infty} k_{w_0}^{-1}\psi_{i\widehat{\mu}_k}^k = \frac{1}{\#W_0} Q_\infty(w_0)(e^{2\pi i\mu}) = \psi_{2\pi i\mu}^\infty$$

for $\mu \in P^{++}$, uniformly on compacta. Pulling the limits through the map G, we obtain

$$\lim_{k\to\infty} k_{w_0}^{-1}\phi_{i\widehat{\mu}_k}^k = \phi_{2\pi i\mu}^\infty$$

for $\mu \in P^{++}$, uniformly on compacta.

4. Dunkl Operators and Hecke Algebras

It is well known that conserved integrals for quantum integrable systems of Calogero-Moser type can be conveniently expressed in terms of Dunkl-type operators, which are explicit commuting first-order differential-reflection operators, see e.g. [14, 6, 1]. The Dunkl operators, together with the usual Weyl group action (2.8), form a faithful representation of suitable degenerations of affine Hecke algebras, see [26, Cor. 2.9]. The exploration of these structures has been instrumental in solving the corresponding quantum integrable systems.

In this section we derive the Dunkl-type operators and the underlying Hecke algebra structures for the periodic quantum integrable systems with delta-potentials as introduced in Sect. 2. We initially define the Dunkl operators as explicit differential-reflection operators on the space $C^\infty(V_{reg})$ of smooth functions on V_{reg}. In Sect. 6 we obtain the key result that these Dunkl operators act on the solution space $\mathrm{BVP}_k(\lambda)$ to the boundary value problem. Together with the usual W-action (2.8), the space $\mathrm{BVP}_k(\lambda)$ then becomes a module over the associated graded algebra H_k of Cherednik's [3] (suitably filtered) degenerate double affine Hecke algebra.

On the other hand, we will show in Sect. 5 that the W-action Q_k on $E(\lambda)$ together with the directional derivatives ∂_v $(v \in V)$ makes $E(\lambda)$ into a H_k-module. With these upgraded symmetry structures, Gutkin's propagation operator T_k turns out to yield an isomorphism

$$T_k : E(\lambda) \xrightarrow{\sim} \mathrm{BVP}_k(\lambda)$$

of H_k-modules for all $\lambda \in V_{\mathbb{C}}^*$. It is this particular isomorphism which is explored in Sect. 6 to (re-)prove and clarify crucial results on the boundary value problem (see Definition 2.3), as well as on the associated bosonic theory.

We denote $\chi : \mathbb{R}\backslash\{0\} \to \{0, 1\}$ for the characteristic function of the interval $(-\infty, 0)$, so $\chi(x) = 1$ if $x < 0$ and $\chi(x) = 0$ if $x > 0$. For $a \in \Sigma$ the function $\chi_a(v) := \chi(a(v))$ $(v \in V_{reg})$ defines a smooth function on V_{reg}, which is constant on the alcoves C of V_{reg}. In fact, for $w \in W$ and $a \in \Sigma^+$ we have

$$\chi_a|_{w^{-1}C_+} \equiv \begin{cases} 1 & \text{if } wa \in \Sigma^- \\ 0 & \text{if } wa \in \Sigma^+, \end{cases} \tag{4.1}$$

hence χ_a is nonzero on a given alcove $w^{-1}C_+$ $(w \in W)$ for only finitely many positive roots $a \in \Sigma^+$. The Dunkl-type operators

$$\mathcal{D}_v^k = \partial_v + \sum_{a\in\Sigma^+} k_a Da(v)\chi_a(\cdot)s_a \qquad (v \in V), \tag{4.2}$$

thus define linear operators on $C^\infty(V_{reg})$, which depend linearly on $v \in V$. For $f \in C^\infty(V_{reg})$ and $w \in W$ we have by (4.1),

$$\mathcal{D}_v^k f|_{w^{-1}C_+} = \Big(\partial_v f + \sum_{a\in\Sigma^+\cap w^{-1}\Sigma^-} k_a Da(v)s_a f\Big)\Big|_{w^{-1}C_+}. \tag{4.3}$$

In particular, for the fundamental alcove C_+ we simply have

$$\mathcal{D}_v^k f|_{C_+} = \partial_v f|_{C_+}. \tag{4.4}$$

The Dunkl operators $\mathcal{D}_v^k$ $(v \in V)$ and the W-action (2.8) on $C^\infty(V_{reg})$ satisfy the following fundamental commutation relations.

Theorem 4.1. *(i) We have the cross relation*

$$s_a\mathcal{D}_v^k = \mathcal{D}_{s_{Da}v}^k s_a + k_a Da(v), \qquad v \in V,\ a \in I.$$

(ii) The Dunkl operators $\mathcal{D}_v^k$ $(v \in V)$ pair-wise commute.

Proof. (i) Fix $v \in V$ and $a \in I$. By a direct computation we have

$$s_a \mathcal{D}_v^k s_a = \partial_{s_{Da}v} + \sum_{b \in s_a \Sigma^+} k_b Db(s_{Da}v)\chi_b(\cdot)s_b.$$

Since $s_a\Sigma^+ = \big(\Sigma^+ \setminus \{a\}\big) \cup \{-a\}$ we obtain

$$\begin{aligned} s_a \mathcal{D}_v^k &= \mathcal{D}_{s_{Da}v}^k s_a - k_a Da(s_{Da}v)\big(\chi_a(\cdot) + \chi_{-a}(\cdot)\big) \\ &= \mathcal{D}_{s_{Da}v}^k s_a + k_a Da(v), \end{aligned}$$

which is the desired cross relation.

(ii) We derive the commutativity of the Dunkl operators $\mathcal{D}_v^k$ $(v \in V)$ as a direct consequence of (4.4) and the cross relation. Let $f \in C^\infty(V_{reg})$ and $v, v' \in V$. We show by induction on the length $l(w)$ of $w \in W$ that

$$[\mathcal{D}_v^k, \mathcal{D}_{v'}^k] f|_{w^{-1}C_+} = 0. \tag{4.5}$$

By (4.4), Eq. (4.5) is obviously valid for $w = e$ the unit element of W. To prove the induction step, it suffices to show that

$$s_a[\mathcal{D}_v^k, \mathcal{D}_{v'}^k] = [\mathcal{D}_{s_{Da}v}^k, \mathcal{D}_{s_{Da}v'}^k]s_a \tag{4.6}$$

for all $a \in I$. For the proof of (4.6), first observe that

$$s_a \mathcal{D}_v^k \mathcal{D}_{v'}^k - \mathcal{D}_{s_{Da}v}^k \mathcal{D}_{s_{Da}v'}^k s_a = k_a\big(Da(v')\mathcal{D}_v^k + Da(v)\mathcal{D}_{v'}^k - Da(v)Da(v')\mathcal{D}_{Da^\vee}^k\big) \tag{4.7}$$

for all $a \in I$, which follows from applying the cross relation twice. Now (4.6) follows from the fact that the right hand side of (4.7) is symmetric in v and v'. □

By Theorem 4.1 (ii), the assignment $v \mapsto \mathcal{D}_v^k$ uniquely extends to an algebra morphism $S(V)_{\mathbb{C}} \to \mathrm{End}(C^\infty(V_{reg}))$. We denote $p(\mathcal{D}^k)$ for the differential-reflection operator on $C^\infty(V_{reg})$ associated to $p \in S(V)_{\mathbb{C}}$.

Theorem 4.2. *(i) There exists a unique complex unital associative algebra $H_k = H_k(\Sigma)$ satisfying*

(a) $H_k = S(V)_{\mathbb{C}} \otimes \mathbb{C}[W]$ as vector spaces, with $\mathbb{C}[W]$ the group algebra of W.
(b) The maps $p \mapsto p \otimes e$ and $w \mapsto 1 \otimes w$, with $e \in W$ the unit element of W, are algebra embeddings of $S(V)_{\mathbb{C}}$ and $\mathbb{C}[W]$ into H_k.
(c) The cross relations

$$s_a \cdot v - \big(s_{Da}v\big) \cdot s_a = k_a Da(v)$$

holds in H_k for $a \in I$ and $v \in V \subset S(V)_{\mathbb{C}}$. Here we have identified $S(V)_{\mathbb{C}}$ and $\mathbb{C}[W]$ with their images in H_k through the algebra embeddings of (b).

(ii) The assignment $p \mapsto p(\mathcal{D}^k)$ $(p \in S(V)_{\mathbb{C}})$, together with the W-action (2.8)*, defines a faithful representation $\pi_k : H_k \to \mathrm{End}(C^\infty(V_{reg}))$.*

Proof. Suppose that $\sum_{w\in W} p_w(\mathcal{D}^k)w = 0$ as an endomorphism of $C^\infty(V_{reg})$, with only finitely many $p_w \in S(V)_{\mathbb{C}}$'s non zero. We show that all p_w's are zero. Equation (4.4) implies

$$\sum_{w\in W} p_w(\partial)(wf)|_{C_+} \equiv 0, \qquad f \in C^\infty(V_{reg}). \tag{4.8}$$

Applying (4.8) to functions f of the form $u^{-1}g$ with $u \in W$ and with $g \in C^\infty(V_{reg})$ having support in the fundamental alcove C_+, we conclude that $p_u(\partial) = 0$ as a constant coefficient differential operator on smooth functions in some open ball $D \subset C^+$, hence $p_u = 0$.

The proof of the theorem is now standard: let $\widetilde{H}_k$ be the complex unital associative algebra generated by $v \in V$ and s_a $(a \in I)$ with defining relations as in (b) and (c) (so the vectors $v \in V$ pair-wise commute, the s_a $(a \in I)$ are involutions satisfying the Coxeter relations associated to Σ and I, and the generators satisfy the cross relations from (c)). By Theorem 4.1 and by the paragraph preceding this theorem, the assignment $v \mapsto \mathcal{D}^k_v$, together with the W-action (2.8), uniquely defines an algebra morphism $\pi_k : \widetilde{H}_k \to \mathrm{End}(C^\infty(V_{reg}))$. By the previous paragraph and by the cross relations in $\widetilde{H}_k$ it follows that π_k is injective and that $\widetilde{H}_k \simeq S(V)_{\mathbb{C}} \otimes \mathbb{C}[W]$ as vector spaces (the Poincaré-Birkhoff-Witt Theorem for $\widetilde{H}_k$). Both statements of the theorem are now immediately clear. □

We use the notation M_{π_k} to indicate that a subspace $M \subseteq C^\infty(V_{reg})$ is a W-submodule or H_k-submodule of $C^\infty(V_{reg})$ with respect to the π_k-action.

Remark 4.3. If the values k_a of the multiplicity function k are considered to be independent central variables in the definition of H_k, then H_k is graded by imposing the degree of $w \in W$ to be zero and the degrees of $v \in V$ and k_a to be one. As a graded algebra, H_k is the associated graded algebra of Cherednik's [3] degenerate double affine Hecke algebra $\mathbb{H}_k$, considered as a filtered algebra by the same degree function (the only difference in the definition of $\mathbb{H}_k$ is the cross relation (see Theorem 4.2 (c)), which now is of the form

$$s_a \cdot v - \big(s_a v\big) \cdot s_a = k_a Da(v)$$

for $a \in I$, where $S(V)$ is considered as a W-module algebra with the action of s_0 defined by $s_0 v = s_\varphi(v) + 2\|\varphi\|^{-2}\varphi(v)1 \in S(V)$).

Lemma 4.4. *The center $Z(H_k)$ of H_k contains $S(V)_{\mathbb{C}}^{W_0}$.*

Proof. Observe that the cross relations in H_k (see Theorem 4.2(c)) imply

$$s_a \cdot p - \big(s_{Da}p\big) \cdot s_a = -k_a \frac{\big(s_{Da}p\big) - p}{Da^\vee} \tag{4.9}$$

for $a \in I$ and $p \in S(V)_{\mathbb{C}}$. It follows from (4.9) that $S(V)_{\mathbb{C}}^{W_0} \subseteq Z(H_k)$. □

Remark 4.5. Observe that the subalgebra $H_k^{(0)} \subset H_k$ generated by W_0 and $S(V)_{\mathbb{C}}$ is isomorphic to the degenerate affine Hecke algebra (also known as the graded Hecke algebra), see e.g. [15, 23]. By [23, Prop. 4.5] we have $Z(H_k^{(0)}) = S(V)_{\mathbb{C}}^{W_0}$.

For trivial multiplicity parameters $k \equiv 0$, the operator $p(\mathcal{D}^0)$ ($p \in S(V)_{\mathbb{C}}$) on $C^\infty(V_{reg})$ is the constant-coefficient differential operator $p(\partial)$ on $C^\infty(V_{reg})$. We have the following striking fact when $p \in S(V)_{\mathbb{C}}$ is W_0-invariant.

Corollary 4.6. *For $p \in S(V)_{\mathbb{C}}^{W_0}$ we have $p(\mathcal{D}^k) = p(\partial)$ as operators on $C^\infty(V_{reg})$.*

Proof. Let $p \in S(V)_{\mathbb{C}}^{W_0}$ and $f \in C^\infty(V_{reg})$. By (4.4) we have $p(\mathcal{D}^k)f|_{C_+} = p(\partial)f|_{C_+}$. Let $w \in W$ and $v \in C_+$. By Lemma 4.4 applied twice (once with multiplicity function k, once with $k \equiv 0$), we have

$$\begin{aligned}\big(p(\mathcal{D}^k)f\big)(w^{-1}v) &= \big(p(\mathcal{D}^k)(wf)\big)(v)\\ &= \big(p(\partial)(wf)\big)(v) = \big(p(\partial)f\big)(w^{-1}v),\end{aligned}$$

hence $p(\mathcal{D}^k)f = p(\partial)f$. □

Remark 4.7. The Dunkl operators $\mathcal{D}_v^k$, Theorem 4.1, Theorem 4.2 and Corollary 4.6 have their obvious analogs in the context of finite root systems. In that case, the Dunkl-type operators are

$$\partial_v + \sum_{\alpha \in \Sigma_0^+} k_\alpha \alpha(v) \chi_\alpha(\cdot) s_\alpha, \qquad v \in V$$

realizing, together with the W_0-action (2.8), an action of the degenerate affine Hecke algebra $H_k^{(0)}$ on the space of smooth functions on $V \setminus \bigcup_{\alpha \in \Sigma_0^+} V_\alpha$. For classical root systems these operators were constructed using solutions of classical Yang-Baxter equations and reflection equations in [28], [24] (type A) and [19]. This construction fits into Cherednik's [2] general framework relating root system analogs of r-matrices to (degenerate) affine Hecke algebras and Dunkl operators.

5. Integral-Reflection Operators and the Propagation Operator

Heckman and Opdam [15] clarified the role of the degenerate affine Hecke algebra $H_k^{(0)}$ in Gutkin's [11] work when the underlying root system is finite. It led to an explicit action of $H_k^{(0)}$ as directional derivatives and integral-reflection operators. In this section we extend these results to the present affine set-up. We show that Gutkin's [11] propagation operator intertwines this action with the action π_k which is defined in the previous section in terms of Dunkl-type differential-reflection operators.

The integral-reflection operators $Q_{k,a}$ (see (3.9)) for $a \in \Sigma$ are endomorphisms of $C(V)$ satisfying

$$wQ_{k,a}w^{-1} = Q_{k,w(a)}, \qquad w \in W,\ a \in \Sigma \tag{5.1}$$

with respect to the W-action (2.8) on $C(V)$. We furthermore have

$$Q_{k,a}f|_{V_a} = f|_{V_a}, \qquad a \in \Sigma. \tag{5.2}$$

By [11, Thm. 2.3], the assignment

$$s_a \mapsto Q_k(s_a) := Q_{k,a} \qquad (a \in I) \tag{5.3}$$

extends to a representation Q_k of W on $C(V)$. In particular, for $w \in W$ and any choice of decomposition $w = s_{j_1}s_{j_2}\cdots s_{j_r}$ as a product of simple reflections ($j_l \in \{0, \ldots, n\}$), we have

$$Q_k(w) = Q_k(s_{j_1})Q_k(s_{j_2})\cdots Q_k(s_{j_r}) \tag{5.4}$$

as operators on $C(V)$.

Definition 5.1. *Gutkin's [11] propagation operator T_k is the endomorphism of $C(V)$ defined by*

$$(T_k f)(w^{-1}v) = \big(Q_k(w)f\big)(v), \qquad v \in C_+, \quad w \in W. \tag{5.5}$$

In particular, T_0 is the identity operator on $C(V)$.

A W-submodule $M \subseteq C(V)$ with respect to the Q_k-action will be denoted by M_{Q_k}. By construction the propagation operator $T_k : C(V)_Q \to C(V)_\pi$ is W-equivariant. In fact, by [11, Thm. 2.6] T_k is an isomorphism of W-modules.

Observe that the operators $Q_k(w)$ ($w \in W$) preserve the space $C^\infty(V)$ of complex valued, smooth functions on V. The following result is the affine analog of [15, Thm. 2.1] and [15, Cor. 2.3].

Theorem 5.2. *The assignment $v \mapsto \partial_v$ ($v \in V$), together with the W-action* (5.3) *on $C^\infty(V)$, extends uniquely to a representation $Q_k : H_k \to$* $\mathrm{End}(C^\infty(V))$.

Proof. It suffices to verify the cross relations (see Theorem 4.2(c)), which follow directly from [11, Lem. 2.1]. □

We will also use the notation M_{Q_k} to indicate that a subspace $M \subseteq C^\infty(V)$ is a H_k-submodule with respect to the Q_k-action. Observe that $C^\omega(V)_Q \subseteq C^\infty(V)_Q$ as H_k-submodule.

Consider the space $CB^\omega(V)$ of functions $f \in C(V)$ such that $f|_C$ is the restriction of a (necessarily unique) analytic function on V for all alcoves $C \in \mathcal{C}$ (cf. Remark 2.4). Denote $C^{\omega,(k)}(V)$ for the space of functions $f \in CB^\omega(V)$ satisfying

$$\partial^r_{Db^\vee} f\big(v + 0Db^\vee\big) - \partial^r_{Db^\vee} f\big(v - 0Db^\vee\big) = \big(1 - (-1)^r\big)k_b\partial^{r-1}_{Db^\vee} f\big(v + 0Db^\vee\big) \tag{5.6}$$

for $b \in \Sigma^+$, $v \in V_b$ sub-regular and $r \in \mathbb{Z}_{>0}$. A function $f \in C^{\omega,(k)}(V)$ automatically satisfies the jump conditions (5.6) for $b \in \Sigma^-$, $v \in V_b$ sub-regular and $r \in \mathbb{Z}_{>0}$, hence the space $C^{\omega,(k)}(V)$ is not dependent on the choice of positive roots Σ^+ in Σ. We thus can and will interpret $C^{\omega,(k)}(V)_{\pi_k}$ and $CB^\omega(V)_{\pi_k}$ as W-submodules of $C^\infty(V_{reg})_{\pi_k}$. Observe furthermore that $C^{\omega,(k)}(V)$ is a subspace of the space $C^{1,(k)}(V)$ used in the formulation of the boundary value problems (see Proposition 2.2 and Definition 2.3).

Observe that the propagation operator T_k restricts to a linear map

$$T_k : C^\omega(V) \to CB^\omega(V).$$

We now obtain the following theorem.

Theorem 5.3. *(i) $C^{\omega,(k)}(V)_{\pi_k} \subseteq C^\infty(V_{reg})_{\pi_k}$ is a H_k-submodule.*
(ii) The propagation operator T_k restricts to an isomorphism

$$T_k : C^\omega(V)_{Q_k} \xrightarrow{\sim} C^{\omega,(k)}(V)_{\pi_k}$$

of H_k-modules.

Proof. We first show that T_k restricts to a linear isomorphism $T_k : C^\omega(V) \xrightarrow{\sim} C^{\omega,(k)}(V)$. For this we use the commutation relations

$$s_a \cdot \big(Da^\vee\big)^r - (-1)^r \big(Da^\vee\big)^r \cdot s_a = \big(1-(-1)^r\big)k_a\big(Da^\vee\big)^{r-1}, \qquad a \in I,\ r \in \mathbb{Z}_{>0} \tag{5.7}$$

in H_k, which follows from (4.9) applied to $p = (Da^\vee)^r \in S(V)_{\mathbb{C}}$.

Let $\phi \in C^\omega(V)$ and denote $f = T_k\phi \in CB^\omega(V)$. We show that f satisfies the derivative jumps (5.6) over sub-regular $v \in V_b$ ($b \in \Sigma^+$) for all $r \in \mathbb{Z}_{>0}$. In view of the W-equivariance of the propagation operator T_k, it suffices to derive the derivative jumps for f over sub-regular vectors $v \in V_a \cap \overline{C_+}$ ($a \in I$). Fix $a \in I$, $v \in V_a \cap \overline{C_+}$ sub-regular and $r \in \mathbb{Z}_{>0}$. For $\epsilon > 0$ small we have $v + tDa^\vee = s_a(v - tDa^\vee) \in C_+$ for $0 < t < \epsilon$. Hence

$$\partial^r_{Da^\vee} f(v + 0Da^\vee) = \partial^r_{Da^\vee}\phi(v) = Q_k(s_a)(\partial^r_{Da^\vee}\phi)(v), \tag{5.8}$$

where the second equality follows from (5.2). On the other hand,

$$\partial^r_{Da^\vee} f(v - 0Da^\vee) = (-1)^r\partial^r_{Da^\vee}(s_a f)(v + 0Da^\vee) = (-1)^r\partial^r_{Da^\vee}(Q_k(s_a)\phi)(v). \tag{5.9}$$

Combining (5.8) and (5.9) now yields

$$\begin{aligned}
\partial^r_{Da^\vee} f(v + 0Da^\vee) - \partial^r_{Da^\vee} f(v - 0Da^\vee) &= \big(\big(Q_k(s_a)\partial^r_{Da^\vee} - (-1)^r\partial^r_{Da^\vee}Q_k(s_a)\big)\phi\big)(v)\\
&= \big(1-(-1)^r\big)k_a\partial^{r-1}_{Da^\vee}\phi(v)\\
&= \big(1-(-1)^r\big)k_a\partial^{r-1}_{Da^\vee} f(v + 0Da^\vee),
\end{aligned}$$

where the second equality follows from (the Q_k-image of) (5.7). Thus $f \in C^{\omega,(k)}(V)$.

The map $T_k : C^\omega(V) \to C^{\omega,(k)}(V)$ is clearly injective. We now proceed to prove surjectivity. Let $f \in C^{\omega,(k)}(V)$ and denote ψ for the unique analytic function on V satisfying $\psi|_{C_+} = f|_{C_+}$. The function $g := f - T_k\psi \in C^{\omega,(k)}(V)$ satisfies $g|_{\overline{C_+}} \equiv 0$. Combined with the continuity of g and the derivative jump conditions (5.6) for g, we obtain

$$\big(\partial^r_{Da^\vee} g\big)(v - 0Da^\vee) = 0$$

for $r \in \mathbb{Z}_{\geq 0}$, $a \in I$ and $v \in V_a \cap \overline{C_+}$ sub-regular. Since $g|_C$ ($C \in \mathcal{C}$) has an extension to an analytic function on the whole Euclidean space V, we conclude that $g|_{\overline{C}} \equiv 0$ for the neighboring alcoves $C = s_aC_+$ ($a \in I$) of C_+. Continuing inductively we conclude that $g \equiv 0$ on V, hence $f = T_k\psi$.

It remains to show that the isomorphism

$$T_k : C^\omega(V)_{Q_k} \xrightarrow{\sim} C^{\omega,(k)}(V)_{\pi_k}$$

of W-modules is in fact an isomorphism of H_k-modules. For this it suffices to show that

$$T_k\big(\partial_v f\big)|_{V_{reg}} = \mathcal{D}^k_v(T_k f|_{V_{reg}}) \tag{5.10}$$

for $v \in V$ and $f \in C^\omega(V)$. To prove (5.10) we use the commutation relation

$$w \cdot v = \big((Dw)v\big) \cdot w + \sum_{a \in \Sigma^+ \cap w^{-1}\Sigma^-} k_a Da(v) w s_a \tag{5.11}$$

in H_k, which can be easily proved by induction on the length $l(w)$ of $w \in W$ using the cross relations in H_k (see Theorem 4.2(c)). Fix $w \in W$ and $v' \in C_+$. By (5.11) and Theorem 5.2 we have

$$\begin{aligned} T_k(\partial_v f)(w^{-1}v') &= Q_k(w)(\partial_v f)(v') \\ &= \partial_{(Dw)v}(Q_k(w)f)(v') + \sum_{a \in \Sigma^+ \cap w^{-1}\Sigma^-} k_a Da(v) Q_k(ws_a) f(v') \\ &= \partial_v(T_k f)(w^{-1}v') + \sum_{a \in \Sigma^+ \cap w^{-1}\Sigma^-} k_a Da(v) T_k f(s_a w^{-1}v') \\ &= \mathcal{D}_v^k(T_k f)(w^{-1}v'), \end{aligned}$$

where the last equality follows from (4.3). □

Remark 5.4. The assertion [11, Thm. 2.7] that, in Gutkin's notation, the propagation operator T_k is an automorphism of the W-module CB^∞, seems to be incorrect. In fact, the integral-operators $\mathcal{I}(a)$ ($a \in \Sigma$) do not preserve CB^∞, contrary to the claim in the proof of [11, Thm. 2.7]. In [11], this result is used to link $\mathrm{BVP}_k(\lambda)$ to $E(\lambda)$ (see (3.1)). We will show in Sect. 6 that Theorem 5.3(ii) suffices to provide this link.

Remark 5.5. Theorem 5.3 has an obvious analog in the context of finite root systems (compare with Remark 4.7). In the case of a finite root system of type A, the intertwining properties of the propagation operator with respect to the degenerate affine Hecke algebra actions were considered in [16] and the normal derivative jump conditions of higher order were considered in [12].

Corollary 5.6. *Fix $v \in V$. The Dunkl operator $\mathcal{D}_v^k$ is a linear operator on $C^{\omega,(k)}(V)$ satisfying $\mathcal{D}_v^k\big(T_k f\big) = T_k\big(\partial_v f\big)$ for all $f \in C^\omega(V)$.*

In the following proposition we relate the Dunkl operators $\mathcal{D}_v^k$ to the quantum Hamiltonian $\mathcal{H}_k$ (see (2.4) and (2.5)). Recall that $p_2(\partial) = \Delta$ for the W_0-invariant polynomial $p_2 = \|\cdot\|^2$ on V^*.

Proposition 5.7. *For $f \in C^{\omega,(k)}(V)$ we have*

$$-p_2(\mathcal{D}^k)f = \mathcal{H}_k f \tag{5.12}$$

as distributions on V.

Proof. Fix $f \in C^{\omega,(k)}(V)$, then $p_2(\mathcal{D}^k)f \in C^{\omega,(k)}(V) \subseteq C(V)$ and $p_2(\mathcal{D}^k)f|_{V_{reg}} = \Delta f|_{V_{reg}}$ by Corollary 4.6. Furthermore, f satisfies the first order normal derivative jumps (2.6) over the affine hyperplanes V_a ($a \in \Sigma^+$). The identity (5.12) then follows from a standard argument using Green's identity, cf. (the proof of) Proposition 2.2. □

By Proposition 5.7 it is justified to interpret the quantum Hamiltonian $\mathcal{H}_k$ on $C^{\omega,(k)}(V)$ as the operator $-p_2(\mathcal{D}^k)$ on $C^{\omega,(k)}(V)$. The complete integrability of the quantum system is then directly reflected by the commutativity of the Dunkl operators $\mathcal{D}_v^k$ ($v \in V$). More precisely, the space $C^{\omega,(k)}(V)_\pi^W$ serves as an algebraic model for the Hilbert space of quantum states associated to the bosonic quantum system on $V/Q^\vee$ with Hamiltonian $\mathcal{H}_k = -p_2(\mathcal{D}^k)$. The pair-wise commuting operators $p(\mathcal{D}^k)$ ($p \in S(V)_{\mathbb{C}}^{W_0}$) on $C^{\omega,(k)}(V)_\pi^W$ are the corresponding quantum conserved integrals.

6. The Boundary Value Problem Revisited

The operators $p(\mathcal{D}^k)$ $(p \in S(V)_{\mathbb{C}}^{W_0})$ on $C^{\omega,(k)}(V)$ satisfy

$$p(\mathcal{D}^k)f|_{V_{reg}} = p(\partial)f|_{V_{reg}}, \qquad f \in C^{\omega,(k)}(V)$$

by Corollary 4.6. This key observation leads to an explicit connection between the spectral problem of the operators $p(\mathcal{D}^k)$ $(p \in S(V)_{\mathbb{C}}^{W_0})$ and the boundary value problem as formulated in Definition 2.3. We will first do the analysis for the spectral problem of the quantum Hamiltonian $\mathcal{H}_k$ (defined by (2.4) and (2.5)).

For $E \in \mathbb{C}$ we write $\mathcal{E}(E)$ for the space of functions $f \in C^\omega(V)$ satisfying $\Delta f = -Ef$ on V (cf. Example 2.5). By Lemma 4.4, $\mathcal{E}(E)_{Q_k} \subseteq C^\omega(V)_{Q_k}$ is a H_k-submodule. Denote $\mathcal{E}_k(E)$ for the space of functions $f \in CB^\omega(V)$ satisfying $\mathcal{H}_k f = Ef$ as distributions on V (cf. Proposition 2.2).

Theorem 6.1. *Fix $E \in \mathbb{C}$.*
(i) We have

$$\mathcal{E}_k(E) = \{f \in C^{\omega,(k)}(V) \mid p_2(\mathcal{D}^k)f = -Ef\}, \tag{6.1}$$

hence $\mathcal{E}_k(E)_{\pi_k} \subseteq C^{\omega,(k)}(V)_{\pi_k}$ is a H_k-submodule.
(ii) The propagation operator T_k restricts to an isomorphism

$$T_k : \mathcal{E}(E)_{Q_k} \xrightarrow{\sim} \mathcal{E}_k(E)_{\pi_k}$$

of H_k-modules.

Proof. (i) We first show that $\mathcal{E}_k(E) \subseteq C^{\omega,(k)}(V)$. Fix $f \in \mathcal{E}_k(E)$. By Proposition 2.2, $f \in C^{1,(k)}(V) \cap CB^\omega(V)$ and $\Delta f|_{V_{reg}} = -Ef|_{V_{reg}}$. Let ψ be the unique analytic function on V satisfying $\psi|_{C_+} = f|_{C_+}$, then $\psi \in \mathcal{E}(E)$. By Theorem 5.3 and Corollary 4.6 we conclude that $T_k\psi \in C^{\omega,(k)}(V)$ and $\Delta(T_k\psi)|_{V_{reg}} = -E(T_k\psi)|_{V_{reg}}$. Hence

$$g := f - T_k\psi \in C^{1,(k)}(V) \cap CB^\omega(V)$$

satisfies $\Delta g|_{V_{reg}} = -Eg|_{V_{reg}}$ and has the additional property that $g|_{\overline{C_+}} \equiv 0$. Fix $v \in V_a \cap \overline{C_+}$ $(a \in I)$ sub-regular. The nontrivial normal derivative jump condition (2.6) for g at v trivializes since $g|_{\overline{C_+}} \equiv 0$, hence g is continuously differentiable in an open neighborhood U of v. It follows that $g|_U$ is a distribution solution of the (hypo)elliptic constant coefficient differential operator $\Delta + E$ (cf. Example 2.5), hence $g|_U$ is smooth. Since $g|_{\overline{C_+}} \equiv 0$, we conclude that

$$\partial^r_{Da^\vee} g(v - 0Da^\vee) = \partial^r_{Da^\vee} g(v + 0Da^\vee) = 0, \qquad r \in \mathbb{Z}_{\geq 0}.$$

As in the proof of Theorem 5.3 we conclude that $g|_{\overline{s_aC_+}} \equiv 0$ for $a \in I$ (alternatively, this is a direct consequence of Holmgren's Uniqueness Theorem). Continuing inductively, we conclude that $g \equiv 0$ on V. Hence $f = T_k\psi \in C^{\omega,(k)}(V)$.

Formula (6.1) now follows from Proposition 5.7. Since $p_2(\mathcal{D}^k) = \pi_k(p_2)$, Lemma 4.4 implies that $\mathcal{E}_k(E)_{\pi_k} \subseteq C^{\omega,(k)}(V)_{\pi_k}$ is a H_k-submodule.
(ii) This follows directly from Theorem 5.3, (6.1) and the fact that $Q_k(p_2) = p_2(\partial) = \Delta$.
□

We now extend these results to the solution spaces $\mathrm{BVP}_k(\lambda)$ of the boundary value problem (Definition 2.3). For a H_k-module M and $\lambda \in V_{\mathbb{C}}^*$ we define

$$M_\lambda := \{m \in M \mid p \cdot m = p(\lambda)m \quad \forall\, p \in S(V)_{\mathbb{C}}^{W_0}\}, \tag{6.2}$$

which is a H_k-submodule of M in view of Lemma 4.4. By Remark 4.5 the module M_λ consists of the vectors $m \in M$ transforming according to the central character $\lambda \in V_{\mathbb{C}}^*$ for the action of the center of the degenerate affine Hecke algebra $H_k^{(0)} \subseteq H_k$.

Corollary 6.2. *Let $\lambda \in V_{\mathbb{C}}^*$. The space* $\mathrm{BVP}_k(\lambda)$ *is the H_k-submodule $C^{\omega,(k)}(V)_{\pi_k,\lambda}$ of $C^{\omega,(k)}(V)_{\pi_k}$.*

Proof. By Corollary 4.6 and Theorem 5.3 we have

$$C^{\omega,(k)}(V)_{\pi_k,\lambda} = \{f \in C^{\omega,(k)}(V) \mid p(\partial)f|_{V_{reg}} = p(\lambda)f|_{V_{reg}} \quad \forall\, p \in S(V)_{\mathbb{C}}^{W_0}\}, \tag{6.3}$$

hence $C^{\omega,(k)}(V)_{\pi_k,\lambda} \subseteq \mathrm{BVP}_k(\lambda)$. By Proposition 2.2 and Remark 2.4 we have

$$\mathrm{BVP}_k(\lambda) \subseteq \mathcal{E}_k(-p_2(\lambda)).$$

Theorem 6.1 and (6.3) now imply that $\mathrm{BVP}_k(\lambda) \subseteq C^{\omega,(k)}(V)_{\pi_k,\lambda}$. $\square$

Theorem 6.3. *Let $\lambda \in V_{\mathbb{C}}^*$.*

(i) The propagation operator T_k restricts to an isomorphism $T_k\colon E(\lambda)_{Q_k} \xrightarrow{\sim} \mathrm{BVP}_k(\lambda)_{\pi_k}$ of left H_k-modules.
(ii) The map G (3.5) *restricts to an isomorphism $G : E(\lambda)_{Q_k}^W \xrightarrow{\sim} \mathrm{BVP}_k(\lambda)_{\pi_k}^W$.*

Proof. (i) The restriction of the propagation operator T_k to the H_k-module $E(\lambda)_{Q_k} = C^\omega(V)_{Q_k,\lambda}$ defines an isomorphism

$$T_k : E(\lambda)_{Q_k} \xrightarrow{\sim} C^{\omega,(k)}(V)_{\pi_k,\lambda}$$

of H_k-modules in view of Theorem 5.3. Corollary 6.2 now completes the proof.

(ii) This follows from (i) and from the fact that the propagation map T_k acts on $Q_k(W)$-invariant functions in the same way as the map G (3.5). $\square$

As observed in Sect. 3, Theorem 6.3 (ii) can be used to reformulate the main results on the solution space $\mathrm{BVP}_k(\lambda)_\pi^W$ (see Theorem 2.6) to the boundary value problem in terms of the space of invariants $E(\lambda)_Q^W$, where $E(\lambda)$ now is the solution space to the boundary value problem with zero normal derivative jumps over sub-regular vectors. Theorem 3.5 is the resulting reformulation of Theorem 2.6. In order to prove Theorem 3.5 we analyze the space $E(\lambda)_Q^W$ in detail in the following sections.

7. Invariants in $E(\lambda)$

In this section we analyze the sub-space $E(\lambda)_Q^{W_0}$ of W_0-invariants of $E(\lambda)_Q$. First we recall some well known properties of the space $E(\lambda)$ from [30, 15]. For technical purposes it is convenient to introduce the following terminology.

Definition 7.1. *Let J be a subset of the simple roots I_0. The spectral parameter $\lambda \in V_{\mathbb{C}}^*$ is called J-standard if $\lambda \in V^* \oplus i\overline{V_+^*}$ and if the isotropic sub-group of λ in W_0 is the standard parabolic sub-group $W_{0,J}$ generated by the simple reflections s_α ($\alpha \in J$).*

Lemma 7.2. *Let $\lambda \in V_{\mathbb{C}}^*$. The W_0-orbit of λ contains a J-standard spectral parameter for some subset $J \subseteq I_0$.*

Proof. Taking a W_0-translate of λ we may assume that $\lambda = \mu + i\nu$ with $\mu \in V^*$ and $\nu \in \overline{V_+^*}$. The isotropy group of ν in W_0 is a standard parabolic sub-group $W_{0,K} \subset W_0$ for some subset $K \subseteq I_0$. Write $V^* = V_K^* \oplus (V_K^*)^\perp$ with $V_K^* = \mathrm{span}_{\mathbb{R}}\{\alpha \mid \alpha \in K\}$ and $(V_K^*)^\perp$ its orthocomplement in V^*. Set

$$V_{K,+}^* = \{\xi \in V_K^* \mid \xi(\alpha^\vee) > 0 \quad \forall \alpha \in K\},$$

which we view as the fundamental chamber for the action of the standard parabolic sub-group $W_{0,K}$ on V_K^*. Taking a $W_{0,K}$-translate of λ we may assume that $\lambda = \mu + \mu' + i\nu$ with $\mu \in \overline{V_{K,+}^*}$, $\mu' \in (V_K^*)^\perp$, and $\nu \in \overline{V_+^*}$ as before. The isotropy sub-group of λ in W_0 then equals the isotropy sub-group of μ in $W_{0,K}$, which is a standard parabolic sub-group $W_{0,J}$ for some subset $J \subseteq K$ since $\mu \in \overline{V_{K,+}^*}$. □

Observe that a J-standard spectral parameter λ is regular if and only if $J = \emptyset$. Note furthermore that the module $E(\lambda)$ ($\lambda \in V_{\mathbb{C}}^*$) only depends on the orbit $W_0\lambda$. When analyzing the module $E(\lambda)$, we thus may assume without loss of generality that λ is J-standard for some subset $J \subseteq I_0$. In particular, we will now assume this condition for the remainder of this section.

For $j \in \mathbb{Z}_{\geq 0}$ we denote $P^{(j)}(V)_{\mathbb{C}}$ (respectively $P^{(\leq j)}(V)_{\mathbb{C}}$) for the homogeneous polynomials $p \in P(V)_{\mathbb{C}}$ of degree j (respectively the polynomials $p \in P(V)_{\mathbb{C}}$ of degree $\leq j$). The W_0-action (2.8) on $P(V)_{\mathbb{C}}$ respects the natural grading $P(V)_{\mathbb{C}} = \bigoplus_{j=0}^\infty P^{(j)}(V)_{\mathbb{C}}$. Furthermore,

$$E_J(0) = \{f \in P(V)_{\mathbb{C}} \mid p(\partial)f = p(0)f \quad \forall p \in S(V)^{W_{0,J}}\}$$

is a graded $W_{0,J}$-submodule of $P(V)_{\mathbb{C}}$, isomorphic to the regular representation of $W_{0,J}$ (see e.g. [30, Thm. 1.2] and references therein). We write $E_J^{(j)}(0) = E_J(0) \cap P^{(j)}(V)_{\mathbb{C}}$ and $E_J^{(\leq j)}(0) = E_J(0) \cap P^{(\leq j)}(V)_{\mathbb{C}}$.

Denote by W_0^J the minimal coset representatives of $W_0/W_{0,J}$. Steinberg [30] established the decomposition

$$E(\lambda) = \bigoplus_{u \in W_0^J} u\big(E_J(0)e^\lambda\big). \tag{7.1}$$

It follows from (7.1) that $E(\lambda)$, viewed as a W_0-module by the action (2.8), is isomorphic to the regular representation of W_0. Furthermore, we have $E(\lambda) = \bigoplus_{j=0}^\infty E^{(j)}(\lambda)$ with $E^{(j)}(\lambda)$ the W_0-submodule

$$E^{(j)}(\lambda) = \bigoplus_{u \in W_0^J} u\big(E_J^{(j)}(0)e^\lambda\big).$$

We denote $E^{(\leq j)}(\lambda) = \bigoplus_{r=0}^j E^{(r)}(\lambda)$.

Representations of the finite group W_0 do not admit nontrivial continuous deformations, hence $E(\lambda)_Q$ is isomorphic to the regular representation of W_0 for an arbitrary multiplicity function k. In particular, $E(\lambda)_Q^{W_0}$ is one-dimensional for all spectral values $\lambda \in V_{\mathbb{C}}^*$. In fact, by (5.2) the function

$$\psi_\lambda^k = \frac{1}{\#W_0} \sum_{w \in W_0} Q_k(w) e^\lambda \tag{7.2}$$

satisfies $\psi_\lambda^k(0) = 1$ and spans $E(\lambda)_Q^{W_0}$. On the other hand, by (7.1) there exist unique polynomials $p_u^\lambda \in E_J(0)$ ($u \in W_0^J$) such that

$$\psi_\lambda^k(v) = \sum_{u \in W_0^J} p_u^\lambda(u^{-1}v) e^{u\lambda(v)}, \qquad v \in V. \tag{7.3}$$

By (7.1) we have

$$E(\lambda) = \bigoplus_{w \in W_0} \mathbb{C} e^{w\lambda}, \qquad \lambda \in V_{\mathbb{C}}^* \text{ regular}, \tag{7.4}$$

so the polynomials p_w^λ ($w \in W_0$) are constants for regular λ. In fact, from e.g. [8] and [15, Sect. 2] we have

$$\psi_\lambda^k(v) = \frac{1}{\#W_0} \sum_{w \in W_0} \widetilde{c}_k(w\lambda) e^{w\lambda(v)}, \qquad \lambda \in V_{\mathbb{C}}^* \text{ regular}, \tag{7.5}$$

where the c-function $\widetilde{c}_k$ is given by (2.9). In the remainder of the paper it will actually be more convenient to work with the regularized c-function

$$c_k(\mu) := \prod_{\substack{\alpha \in \Sigma_0^+ \\ \mu(\alpha^\vee) \neq 0}} \frac{\mu(\alpha^\vee) + k_\alpha}{\mu(\alpha^\vee)}, \qquad \mu \in V_{\mathbb{C}}^* \tag{7.6}$$

which is equal to $\widetilde{c}_k(\mu)$ for regular μ. We can then write

$$p_w^\lambda = \frac{1}{\#W_0} c_k(w\lambda), \qquad \lambda \in V_{\mathbb{C}}^* \text{ regular}.$$

For singular λ an explicit expression for $p_u^\lambda \in E_J(0)$ ($u \in W_0^J$) is not known. For our purposes it suffices to have explicit expressions for the highest and the next to highest homogeneous components of p_u^λ, which we will now proceed to derive.

We denote $\Sigma_0^J \subseteq \Sigma_0$ for the parabolic root sub-system associated to the simple roots $J \subseteq I_0$. We write N_J for the cardinality of the corresponding set $\Sigma_0^{J,+} := \Sigma_0^J \cap \Sigma_0^+$ of positive roots in Σ_0^J and

$$\delta_J = \frac{1}{2} \sum_{\alpha \in \Sigma_0^{J,+}} \alpha \in V^*.$$

Recall that the minimal coset representatives W_0^J of $W_0/W_{0,J}$ can be characterized by

$$W_0^J = \{u \in W_0 \mid u(\Sigma_0^{J,+}) \subseteq \Sigma_0^+\}.$$

The following lemma now gives a derivational expression for p_u^λ ($u \in W_0^J$).

Lemma 7.3. *Let $\lambda \in V_{\mathbb{C}}^*$ be J-standard. For $u \in W_0^J$ we have*

$$p_u^\lambda = K_J^{-1} \frac{d^{N_J}}{dt^{N_J}}\bigg|_{t=0} \left(\sum_{v \in W_{0,J}} d_u(t) e_{uv}(t)(-1)^{l(v)} e^{tv\delta_J} \right)$$

with coefficients

$$d_u(t) = \prod_{\alpha \in \Sigma_0^+ \setminus u(\Sigma_0^{J,+})} \big(u\delta_J(\alpha^\vee)t + u\lambda(\alpha^\vee)\big)^{-1}, \quad e_{uv}(t) = \prod_{\alpha \in \Sigma_0^+} (uv\delta_J(\alpha^\vee)t + u\lambda(\alpha^\vee) + k_\alpha)$$

and with strictly positive constant $K_J = N_J!\#W_0 \prod_{\alpha \in \Sigma_0^{J,+}} \delta_J(\alpha^\vee)$.

Proof. By (7.2), $\psi_\mu^k(v')$ $(v' \in V)$ depends analytically on the spectral parameter $\mu \in V_{\mathbb{C}}^*$. In particular, $\psi_{\lambda_t}^k(v')$ with $\lambda_t := \lambda + t\delta_J \in V_{\mathbb{C}}^*$ depends analytically on $t \in \mathbb{C}$, and we have the (point-wise) limit

$$\lim_{t \to 0} \psi_{\lambda_t}^k = \psi_\lambda^k. \tag{7.7}$$

For $\epsilon > 0$ we write

$$U_\epsilon^0 = \{t \in \mathbb{C} \mid 0 < |t| < \epsilon\}, \qquad U_\epsilon = \{t \in \mathbb{C} \mid |t| < \epsilon\}.$$

There exists an $\epsilon > 0$ such that λ_t is regular for $t \in U_\epsilon^0$, hence

$$\psi_{\lambda_t}^k = \frac{1}{\#W_0} \sum_{w \in W_0} \left(\prod_{\alpha \in \Sigma_0^+} \frac{w\lambda_t(\alpha^\vee) + k_\alpha}{w\lambda_t(\alpha^\vee)} \right) e^{w\lambda_t}, \qquad t \in U_\epsilon^0$$

by (7.5). Splitting the sum into a double sum $w = uv$ with $u \in W_0^J$ and $v \in W_{0,J}$ and using

$$\begin{aligned} \prod_{\alpha \in \Sigma_0^+} uv\lambda_t(\alpha^\vee) &= (-1)^{l(u)+l(v)} t^{N_J} \prod_{\alpha \in \Sigma_0^{J,+}} \delta_J(\alpha^\vee) \prod_{\beta \in \Sigma_0^+ \setminus \Sigma_0^{J,+}} \lambda_t(\beta^\vee) \\ &= (-1)^{l(v)} t^{N_J} \prod_{\alpha \in \Sigma_0^{J,+}} \delta_J(\alpha^\vee) \prod_{\beta \in \Sigma_0^+ \setminus u(\Sigma_0^{J,+})} u\lambda_t(\beta^\vee), \end{aligned}$$

we obtain

$$t^{N_J} \psi_{\lambda_t}^k = K_J^{-1} N_J! \sum_{u \in W_0^J} \sum_{v \in W_{0,J}} d_u(t) e_{uv}(t)(-1)^{l(v)} e^{tuv\delta_J + u\lambda} \tag{7.8}$$

as analytic functions in $t \in U_\epsilon$ (note that $d_u(t)$ is analytic at $t \in U_\epsilon$). By (7.7), ψ_λ^k is the N_J^{th} term in the power series expansion of (7.8) at $t = 0$, which yields the desired result. □

Define the strictly positive constant C_J^k by

$$C_J^k = \frac{1}{\#W_0} \prod_{\alpha \in \Sigma_0^{J,+}} \frac{k_\alpha}{\delta_J(\alpha^\vee)}.$$

The highest and next to highest homogeneous terms of $p_u^\lambda \in E_J(0)$ ($u \in W_0^J$) can now be explicitly computed as follows.

Proposition 7.4. *Let $\lambda \in V_{\mathbb{C}}^*$ be J-standard and $u \in W_0^J$.*

(i) The highest homogeneous term h_u^λ of $p_u^\lambda \in E_J(0)$ is of degree N_J and is explicitly given by

$$h_u^\lambda = C_J^k c_k(u\lambda) \prod_{\alpha \in \Sigma_0^{J,+}} \alpha.$$

(ii) Suppose that λ is singular (i.e. $J \neq \emptyset$). The next to highest homogeneous term n_u^λ of $p_u^\lambda \in E_J(0)$ is

$$n_u^\lambda = \partial_{u^{-1}\rho_{u\lambda}^k}\left(h_u^\lambda\right) = C_J^k c_k(u\lambda) \sum_{\beta \in \Sigma_0^{J,+}} u\beta(\rho_{u\lambda}^k) \prod_{\alpha \in \Sigma_0^{J,+}\setminus\{\beta\}} \alpha$$

with

$$\rho_\mu^k = \sum_{\alpha \in \Sigma_0^+} \frac{\alpha^\vee}{\mu(\alpha^\vee) + k_\alpha} \in V_{\mathbb{C}}. \tag{7.9}$$

Remark 7.5. The formula for n_u^λ should be read as an identity between analytic functions in $k_\alpha > 0$ (the possible singularities are easily seen to be removable).

Proof. (i) Observe that $e_{uv}(0) = e_u(0)$ is independent of $v \in W_{0,J}$, and

$$d_u(0)e_u(0) = c_k(u\lambda) \prod_{\alpha \in \Sigma_0^{J,+}} k_\alpha.$$

Combined with Lemma 7.3 we conclude that the highest homogeneous term h_u^λ of p_u^λ is given by

$$\begin{aligned} h_u^\lambda &= \frac{C_J^k}{N_J!} c_k(u\lambda) \frac{d^{N_J}}{dt^{N_J}}\bigg|_{t=0} \sum_{v \in W_{0,J}} (-1)^{l(v)} e^{tv\delta_J} \\ &= \frac{C_J^k}{N_J!} c_k(u\lambda) \sum_{v \in W_{0,J}} (-1)^{l(v)} \big(v\delta_J\big)^{N_J}. \end{aligned} \tag{7.10}$$

On the other hand, by the Weyl denominator formula for Σ_0^J we have

$$\frac{d^{N_J}}{dt^{N_J}}\bigg|_{t=0} \sum_{v \in W_{0,J}} (-1)^{l(v)} e^{tv\delta_J} = \frac{d^{N_J}}{dt^{N_J}}\bigg|_{t=0} e^{t\delta_J} \prod_{\alpha \in \Sigma_0^{J,+}} \left(1 - e^{-t\alpha}\right) = N_J! \prod_{\alpha \in \Sigma_0^{J,+}} \alpha.$$

Combined with the first equality in (7.10) we obtain the desired expression for h_u^λ.
(ii) The next to highest homogeneous term n_u^λ of p_u^λ is

$$n_u^\lambda = \frac{N_J}{K_J}\left\{ d_u'(0)e_u(0)\sum_{v\in W_{0,J}}(-1)^{l(v)}\big(v\delta_J\big)^{N_J-1} + d_u(0)\sum_{v\in W_{0,J}}(-1)^{l(v)}e_{uv}'(0)\big(v\delta_J\big)^{N_J-1}\right\}$$

in view of Lemma 7.3, where the prime denotes the t-derivative. The first $W_{0,J}$-sum in this expression is identically zero since it is a $W_{0,J}$-alternating polynomial of degree $< N_J$. By a direct calculation the remaining expression can be rewritten as

$$n_u^\lambda = \frac{C_J^k}{(N_J-1)!}c_k(u\lambda)\sum_{v\in W_{0,J}}(-1)^{l(v)}(v\delta_J)(u^{-1}\rho_{u\lambda}^k)\big(v\delta_J\big)^{N_J-1}.$$

The desired expression for n_u^λ now follows from (7.10). $\square$

8. The Bethe Ansatz Equations

In this section we show that $E(\lambda)_Q^W \neq \{0\}$ implies that the spectral parameter λ is a purely imaginary solution of the Bethe ansatz equations (2.10).

From the results of the previous section it is clear that $E(\lambda)_Q^W$ is one-dimensional or zero-dimensional. In fact it is one-dimensional if and only if $Q_k(a_0)\psi_\lambda^k = \psi_\lambda^k$, in which case we have

$$E(\lambda)_Q^W = E(\lambda)_Q^{W_0} = \mathrm{span}_{\mathbb{C}}\{\psi_\lambda^k\}.$$

It is convenient to reformulate these observations in terms of

$$\mathcal{J}_k = \partial_{\varphi^\vee}Q_k(a_0) + k_\varphi \tag{8.1}$$

(viewed as an operator on e.g. $C^\infty(V)$ or $E(\lambda)$), which satisfies the elementary commutation relations

$$\mathcal{J}_k\partial_v = \partial_{s_\varphi v}\mathcal{J}_k, \qquad \forall\, v \in V$$

(the operator $\mathcal{J}_k$ can be defined on the level of the algebra H_k as the element $\varphi^\vee\cdot s_0 + k_\varphi \in H_k$, in which case it is the analog of the affine intertwiner from [4] and [27, Sect. 4]). The equality $Q_k(a_0)\psi_\lambda^k = \psi_\lambda^k$ clearly implies $\mathcal{J}_k\psi_\lambda^k = (\partial_{\varphi^\vee} + k_\varphi)\psi_\lambda^k$.

Lemma 8.1. *If λ is regular, then $\mathcal{J}_k\psi_\lambda^k = (\partial_{\varphi^\vee}+k_\varphi)\psi_\lambda^k$ implies $Q_k(a_0)\psi_\lambda^k = \psi_\lambda^k$.*

Proof. By (7.4) we have a unique expansion

$$Q_k(a_0)\psi_\lambda^k - \psi_\lambda^k = \sum_{w\in W_0} d_w e^{w\lambda}$$

with $d_w \in \mathbb{C}$. We conclude from the equality $\mathcal{J}_k\psi_\lambda^k = (\partial_{\varphi^\vee}+k_\varphi)\psi_\lambda^k$ that $w\lambda(\varphi^\vee)d_w = 0$ for all $w \in W_0$. Since λ is regular, this implies $d_w = 0$ for all $w \in W_0$. $\square$

For $p \in P(V)_{\mathbb{C}} \simeq S(V^*)_{\mathbb{C}}$ we write $p(\partial^\mu)$ for the associated constant coefficient differential operator acting on smooth functions in $\mu \in V_{\mathbb{C}}^*$.

Lemma 8.2. *Let $p \in P(V)_{\mathbb{C}} \simeq S(V^*)_{\mathbb{C}}$. For $w \in W_0$ we have*

$$\mathcal{J}_k\big(p(w^{-1}\cdot)e^{w\mu}\big)(v) = -p(\partial^\mu)\big((\mu(w^{-1}\varphi^\vee)+k_\varphi)e^{\mu(w^{-1}\varphi^\vee)}e^{\mu(w^{-1}s_\varphi v)}\big),$$
$$\big(\partial_{\varphi^\vee}+k_\varphi\big)\big(p(w^{-1}\cdot)e^{w\mu}\big)(v) = p(\partial^\mu)\big((\mu(w^{-1}\varphi^\vee)+k_\varphi)e^{\mu(w^{-1}v)}\big),$$

where we view the left hand sides as functions in $v \in V$ and the right hand sides as functions in $\mu \in V_{\mathbb{C}}^*$. In particular,

$$\mathcal{J}_k\big(P^{(\le j)}(V)_{\mathbb{C}}\, e^\mu\big) \subseteq P^{(\le j)}(V)_{\mathbb{C}}\, e^{s_\varphi\mu}, \quad \big(\partial_{\varphi^\vee}+k_\varphi\big)\big(P^{(\le j)}(V)_{\mathbb{C}}\, e^\mu\big) \subseteq P^{(\le j)}(V)_{\mathbb{C}}\, e^\mu$$

for $j \in \mathbb{Z}_{\ge 0}$ and $\mu \in V_{\mathbb{C}}^*$.

Proof. Observe that

$$\big(p(w^{-1}\cdot)e^{w\mu}\big)(v) = p(\partial^\mu)\big(e^{\mu(w^{-1}v)}\big),$$

and $p(\partial^\mu)$ (acting on $\mu \in V_{\mathbb{C}}^*$) clearly commutes with $\mathcal{J}_k$ and $(\partial_{\varphi^\vee}+k_\varphi)$ (which act on $v \in V$). Thus it suffices to prove the lemma for $p \equiv 1$, in which case the second formula is trivial. To prove the first formula for $p \equiv 1$ we may assume without loss of generality that $w = e$ is the unit element of W_0. Suppose that $\mu \in V_{\mathbb{C}}^*$ is regular. A direct computation using the definition (3.9) of $Q_k(a_0)$ as an integral-reflection operator yields

$$Q_k(a_0)e^\mu = -\frac{k_\varphi}{\mu(\varphi^\vee)}e^\mu + \left(\frac{\mu(\varphi^\vee)+k_\varphi}{\mu(\varphi^\vee)}\right)e^{\mu(\varphi^\vee)}e^{s_\varphi\mu},$$

hence

$$\mathcal{J}_k(e^\mu) = -\big(\mu(\varphi^\vee)+k_\varphi\big)e^{\mu(\varphi^\vee)}e^{s_\varphi\mu}.$$

In the latter formula the regularity constraint on μ can be removed by continuity. □

We denote $\pi_\lambda^{(j)} : E(\lambda) \to E^{(j)}(\lambda)$ for the projection onto $E^{(j)}(\lambda)$ along the decomposition $E(\lambda) = \bigoplus_{r=0}^\infty E^{(r)}(\lambda)$. Observe that

$$\mathrm{Id}_{E(\lambda)} = \sum_{j=0}^{N_J} \pi_\lambda^{(j)} \tag{8.2}$$

if λ is J-standard in view of Proposition 7.4 (i). In this section we consider the constraint on λ such that

$$\pi_\lambda^{(j)}\big(\mathcal{J}_k\psi_\lambda^k\big) = \pi_\lambda^{(j)}\big((\partial_{\varphi^\vee}+k_\varphi)\psi_\lambda^k\big) \tag{8.3}$$

for the highest degree component $j = N_J$.

The map $u \mapsto u^J$, where $u^J \in W_0^J$ is obtained from the unique decomposition

$$s_\varphi u = u^J u_J, \qquad u^J \in W_0^J,\ u_J \in W_{0,J}, \tag{8.4}$$

defines an involution on W_0^J. Observe that

$$(u^J)_J = (u_J)^{-1}, \qquad u \in W_0^J. \tag{8.5}$$

Recall that c_k denotes the regularized c-function (7.6).

Lemma 8.3. *Suppose that $\lambda \in V_{\mathbb{C}}^*$ is J-standard.*
(i) Equation (8.3) *for $j = N_J$ holds if and only if λ satisfies the equations*

$$c_k(s_\varphi u\lambda)(u\lambda(\varphi^\vee) - k_\varphi)e^{-u\lambda(\varphi^\vee)}(-1)^{l(u_J)} = c_k(u\lambda)(u\lambda(\varphi^\vee)+k_\varphi), \quad \forall\, u \in W_0^J. \tag{8.6}$$

(ii) For $u \in W_0^J$ and for multiplicity functions k such that $c_k(u\lambda) \neq 0$, we have

$$\frac{c_k(s_\varphi u\lambda)}{c_k(u\lambda)} = (-1)^{l(u_J)} \prod_{\alpha \in \Sigma_0^+ \cap s_\varphi \Sigma_0^-} \frac{u\lambda(\alpha^\vee) - k_\alpha}{u\lambda(\alpha^\vee) + k_\alpha}.$$

Proof. (i) By (7.3), Lemma 8.2 and Proposition 7.4(i) we have

$$\begin{aligned}\pi_\lambda^{(N_J)}(\mathcal{J}_k \psi_\lambda^k) &= -C_J^k \sum_{u \in W_0^J} c_k(u\lambda)(u\lambda(\varphi^\vee) + k_\varphi)e^{u\lambda(\varphi^\vee)} e^{s_\varphi u\lambda} \prod_{\alpha \in \Sigma_0^{J,+}} s_\varphi u\alpha,\\ \pi_\lambda^{(N_J)}\big((\partial_{\varphi^\vee} + k_\varphi)\psi_\lambda^k\big) &= C_J^k \sum_{u \in W_0^J} c_k(u\lambda)(u\lambda(\varphi^\vee) + k_\varphi)e^{u\lambda} \prod_{\alpha \in \Sigma_0^{J,+}} u\alpha.\end{aligned} \tag{8.7}$$

The proof now follows by equating the coefficients of $e^{u\lambda}\prod_{\alpha\in\Sigma_0^{J,+}} u\alpha$ ($u \in W_0^J$) in (8.7) using (8.4).
(ii) We first compare the denominators of $c_k(u\lambda)$ and $c_k(s_\varphi u\lambda) = c_k(u^J\lambda)$. If $\mu \in V_{\mathbb{C}}^*$ is regular then

$$\begin{aligned}\prod_{\alpha \in \Sigma_0^+ \setminus u^J \Sigma_0^{J,+}} u^J\mu(\alpha^\vee) &= \prod_{\alpha \in \Sigma_0^+} u^J\mu(\alpha^\vee) \prod_{\beta \in u u_J^{-1} \Sigma_0^{J,+}} (u u_J^{-1}\mu(\beta^\vee))^{-1}\\ &= (-1)^{l(u_J)} \prod_{\alpha \in \Sigma_0^+} s_\varphi u u_J^{-1}\mu(\alpha^\vee) \prod_{\beta \in u\Sigma_0^{J,+}} (u u_J^{-1}\mu(\beta^\vee))^{-1}\\ &= (-1)^{l(u_J)+1} \prod_{\alpha \in \Sigma_0^+ \setminus u\Sigma_0^{J,+}} u u_J^{-1}\mu(\alpha^\vee).\end{aligned}$$

Taking the limit $\mu \to \lambda$ we obtain

$$\prod_{\alpha \in \Sigma_0^+ \setminus u^J \Sigma_0^{J,+}} u^J\lambda(\alpha^\vee) = (-1)^{l(u_J)+1} \prod_{\alpha \in \Sigma_0^+ \setminus u\Sigma_0^{J,+}} u\lambda(\alpha^\vee).$$

A similar (and easier) computation leads to the comparative formula

$$\prod_{\alpha \in \Sigma_0^+ \setminus u^J \Sigma_0^{J,+}} \big(u^J\lambda(\alpha^\vee) + k_\alpha\big) = -\left(\prod_{\beta \in \Sigma_0^+ \cap s_\varphi \Sigma_0^-} \frac{u\lambda(\beta^\vee) - k_\beta}{u\lambda(\beta^\vee) + k_\beta}\right) \prod_{\alpha \in \Sigma_0^+ \setminus u\Sigma_0^{J,+}} \big(u\lambda(\alpha^\vee) + k_\alpha\big)$$

for the numerators of $c_k(u\lambda)$ and $c_k(u^J\lambda)$. Combining both formulas leads to the desired result. □

Recall from Sect. 2 that BAE_k is the set of purely imaginary solutions of the Bethe ansatz equations (2.10).

Proposition 8.4. *Suppose that $\lambda \in V_{\mathbb{C}}^*$ is J-standard. Equation* (8.3) *for $j = N_J$ holds if and only if $\lambda \in \mathrm{BAE}_k$.*

Proof. We first show that λ is purely imaginary if λ satisfies Eq. (8.6). Let $\mu = u\lambda$ ($u \in W_0^J$) be the element in the W_0-orbit of λ having its real part in $\overline{V_+^*}$. Then $c_k(\mu) \neq 0$ since the multiplicity function k is strictly positive, hence (8.6) and Lemma 8.3(ii) imply

$$e^{\mu(\varphi^\vee)} = \frac{\mu(\varphi^\vee) - k_\varphi}{\mu(\varphi^\vee) + k_\varphi} \prod_{\alpha \in \Sigma_0^+ \cap s_\varphi \Sigma_0^-} \frac{\mu(\alpha^\vee) - k_\alpha}{\mu(\alpha^\vee) + k_\alpha}. \tag{8.8}$$

The modulus of the left-hand (respectively right-hand side) of (8.8) is ≥ 1 (respectively ≤ 1) since the real part of μ is in $\overline{V_+^*}$ and the multiplicity function k is strictly positive. Thus $|e^{\mu(\varphi^\vee)}| = 1$, implying that $\mu(\varphi^\vee)$ is purely imaginary. Since $\varphi^\vee = \sum_{j=1}^n m_j a_j^\vee$ with m_j strictly positive integers and since the real part of μ lies in $\overline{V_+^*}$, we conclude that $\mu(a_j^\vee)$ is purely imaginary for all co-roots $a_j^\vee$ ($j = 1, \ldots, n$). This implies $\mu \in iV^*$, hence $\lambda \in iV^*$.

Combined with Lemma 8.3(i) it follows that λ satisfies (8.3) for $j = N_J$ if and only if λ is a purely imaginary solution of Eqs. (8.6). For purely imaginary λ we have $c_k(u\lambda) \neq 0$ for all $u \in W_0^J$ due to the strict positivity of the multiplicity function k. The proof now follows from Lemma 8.3(ii) and Remark 2.7. $\square$

As an immediate result we obtain the following "regular part" of Theorem 3.5.

Corollary 8.5. *Suppose that $\lambda \in V_{\mathbb{C}}^*$ is regular. The space $E(\lambda)_Q^W$ is zero-dimensional or one-dimensional. It is one-dimensional if and only if $\lambda \in \mathrm{BAE}_k$. In that case $E(\lambda)_Q^W$ is spanned by ψ_λ^k* (3.11).

Proof. By the observations at the beginning of the section it suffices to show that $E(\lambda)_Q^W \neq \{0\}$ iff $\lambda \in \mathrm{BAE}_k$.

Since $\mathrm{BAE}_k \subset iV^*$ is a W_0-invariant subset and $E(\lambda)_Q^W$ only depends on the W_0-orbit of λ, we may assume without loss of generality that λ is $\emptyset$-standard. If $E(\lambda)_Q^W \neq \{0\}$ then (8.3) holds, hence $\lambda \in \mathrm{BAE}_k$ by Proposition 8.4. Conversely, suppose that $\lambda \in \mathrm{BAE}_k$. Since λ is regular we have $\mathrm{Id}_{E(\lambda)} = \pi_\lambda^{(0)}$ by (8.2), hence $\mathcal{J}_k \psi_\lambda^k = \big(\partial_{\varphi^\vee} + k_\varphi\big)\psi_\lambda^k$ by Proposition 8.4. By Lemma 8.1 this implies $Q_k(a_0)\psi_\lambda^k = \psi_\lambda^k$, hence $0 \neq \psi_\lambda^k \in E(\lambda)_Q^W$. $\square$

9. The Master Function

In this section we prove Proposition 2.9, which yields a parametrization of the set BAE_k of purely imaginary solutions of the Bethe ansatz equations (2.10) by the weight lattice P.

We first rewrite the Bethe ansatz equations (2.10) in logarithmic form. By a direct computation using the elementary identity

$$e^{-2i\arctan(x)} = \frac{1 - ix}{1 + ix} \qquad (x \in \mathbb{R})$$

the Bethe ansatz equations (2.10) for $\lambda \in iV^*$ can be rewritten as

$$-i\lambda(w\varphi^\vee) + \sum_{\alpha\in\Sigma_0} \arctan\left(\frac{-i\lambda(\alpha^\vee)}{k_\alpha}\right)\alpha(w\varphi^\vee) = 0 \quad \text{modulo } 2\pi\mathbb{Z} \tag{9.1}$$

for all $w \in W_0$. On the other hand, for $\mu \in P$ the gradient of the master function $S_k(\mu,\cdot) : V^* \to \mathbb{R}$ (see (2.14)) is determined by

$$\big(\partial_\xi S_k(\mu,\cdot)\big)(\eta) = \langle \eta - 2\pi\mu + \sum_{\alpha\in\Sigma_0} \arctan\left(\frac{\eta(\alpha^\vee)}{k_\alpha}\right)\alpha, \xi\rangle, \qquad \xi,\eta \in V^*. \tag{9.2}$$

Comparing (9.1) and (9.2) yields the following result.

Lemma 9.1. *We have $\lambda \in \mathrm{BAE}_k$ if and only if $\lambda = i\eta$ with $\eta \in V^*$ an extremal vector of the master function $S_k(\mu,\cdot)$ for some $\mu \in P$.*

Proof. Σ_0 is an irreducible root system in V^*, hence $\{w\varphi \mid w \in W_0\}$ spans V^*. Thus $\eta \in V^*$ is an extremal vector of $S_k(\mu,\cdot)$ if and only if $\big(\partial_{w\varphi} S_k(\mu,\cdot)\big)(\eta) = 0$ for all $w \in W_0$, which by (9.2) is equivalent to

$$\eta(w\varphi^\vee) + \sum_{\alpha\in\Sigma_0} \alpha(w\varphi^\vee)\arctan\left(\frac{\eta(\alpha^\vee)}{k_\alpha}\right) = 2\pi\mu(w\varphi^\vee)$$

for all $w \in W_0$. Comparing to (9.1), the proof now follows from (2.3). □

We thus need to analyze the extrema of the master function $S_k(\mu,\cdot)$ at a given weight $\mu \in P$. Observe that the Hessian $B_\xi^k : V^* \times V^* \to \mathbb{R}$ of $S_k(\mu,\cdot)$ at $\xi \in V^*$ is independent of μ, and is given explicitly by

$$\begin{aligned} B_\xi^k(\eta,\eta') &= \big(\partial_\eta\partial_{\eta'} S_k(\mu,\cdot)\big)(\xi) \\ &= \langle \eta,\eta'\rangle + \frac{1}{2}\sum_{\alpha\in\Sigma_0} k_\alpha\|\alpha\|^2 \frac{\eta(\alpha^\vee)\eta'(\alpha^\vee)}{k_\alpha^2 + \xi(\alpha^\vee)^2}, \qquad \eta,\eta' \in V^*. \end{aligned} \tag{9.3}$$

By the strict positivity of the multiplicity function k, it follows from (9.3) that the Hessian B_ξ^k is positive definite for all $\xi \in V^*$, hence $S_k(\mu,\cdot)$ is strictly convex. Furthermore, for all $\mu \in P$,

$$S_k(\mu,\xi) \geq \frac{\|\xi\|^2}{2} - 2\pi\langle\mu,\xi\rangle \to \infty, \qquad \|\xi\| \to \infty,$$

hence $S_k(\mu,\cdot)$ has a unique extremum $\widehat{\mu}_k \in V^*$, which is a global minimum. It now follows from (9.2) that $\widehat{\mu}_k$ ($\mu \in P$) is uniquely determined by the equation

$$\widehat{\mu}_k + \sigma_{\widehat{\mu}_k}^k = 2\pi\mu \tag{9.4}$$

in V^*, where $\sigma_\lambda^k \in V^*$ ($\lambda \in V^*$) is defined by

$$\sigma_\lambda^k = \sum_{\alpha\in\Sigma_0} \arctan\left(\frac{\lambda(\alpha^\vee)}{k_\alpha}\right)\alpha.$$

Combined with Lemma 9.1 it now follows that the map $\mu \mapsto i\widehat{\mu}_k$ is a bijection from the weight lattice P onto BAE_k. The W_0-equivariance of this map is immediate from the equivariance property

$$\big(\partial_{w\xi} S_k(w\mu,\cdot)\big)(w\eta) = \big(\partial_\xi S_k(\mu,\cdot)\big)(\eta), \qquad \forall\, w \in W_0$$

for $\xi,\eta \in V^*$ and $\mu \in P$. This completes the proof of Proposition 2.9.

10. Moment Gaps

In this section we prove Proposition 2.10, which yields estimates for the location of the deformed weight $\widehat{\mu} = \widehat{\mu}_k$ compared to the parametrizing weight $\mu \in P$. In view of (9.2) and Lemma 9.1, the deformed weight $\widehat{\mu} \in V^*$ $(\mu \in P)$ is the unique solution of (9.4).

The following lemma establishes the necessary bounds for σ_λ^k.

Lemma 10.1. *For $\lambda \in \overline{V_+^*}$,*

$$0 \leq \sigma_\lambda^k(\beta^\vee) \leq \frac{h_k}{n}\lambda(\beta^\vee), \qquad \forall\, \beta \in \Sigma_0^+$$

with $h_k = 2\sum_{\alpha\in\Sigma_0} k_\alpha^{-1}$.

Proof. Fix $\lambda \in \overline{V_+^*}$ and $\beta \in \Sigma_0^+$. Let Σ_0^β be the set of roots $\alpha \in \Sigma_0$ satisfying $\alpha(\beta^\vee) > 0$, then

$$\sigma_\lambda^k(\beta^\vee) = \sum_{\alpha\in\Sigma_0^\beta} \left\{\arctan\left(\frac{\lambda(\alpha^\vee)}{k_\alpha}\right) - \arctan\left(\frac{\lambda(s_\beta\alpha^\vee)}{k_\alpha}\right)\right\}\alpha(\beta^\vee). \tag{10.1}$$

Each term in this sum is positive, hence $\sigma_\lambda^k(\beta^\vee) \geq 0$.

For the second inequality, we use the estimate for $\alpha \in \Sigma_0^\beta$,

$$\arctan\left(\frac{\lambda(\alpha^\vee)}{k_\alpha}\right) - \arctan\left(\frac{\lambda(s_\beta(\alpha^\vee))}{k_\alpha}\right) = \int_{\lambda(s_\beta(\alpha^\vee))/k_\alpha}^{\lambda(\alpha^\vee)/k_\alpha} \frac{dx}{1+x^2} \leq \frac{\lambda(\beta^\vee)\beta(\alpha^\vee)}{k_\alpha},$$

leading to

$$\sigma_\lambda^k(\beta^\vee) \leq \lambda(\beta^\vee)\sum_{\alpha\in\Sigma_0^\beta}\frac{\beta(\alpha^\vee)\alpha(\beta^\vee)}{k_\alpha} = \frac{\lambda(\beta^\vee)}{2}\sum_{\alpha\in\Sigma_0}\frac{\beta(\alpha^\vee)\alpha(\beta^\vee)}{k_\alpha} \tag{10.2}$$

in view of (10.1). Now note that

$$\xi \mapsto \sum_{\alpha\in\Sigma_0} k_\alpha^{-1}\xi(\alpha^\vee)\alpha$$

defines a W_0-equivariant linear map $V^* \to V^*$. By Schur's lemma it equals $C_k\mathrm{Id}_{V^*}$ for some constant $C_k \in \mathbb{C}$. To determine C_k explicitly we fix a basis $\{e_j\}_{j=1}^n$ of V and we denote $\{\epsilon_j\}_{j=1}^n$ for the corresponding dual basis of V^*. Then

$$C_k n = \sum_{j=1}^n \sum_{\alpha\in\Sigma_0} k_\alpha^{-1}\epsilon_j(\alpha^\vee)\alpha(e_j) = h_k$$

with $h_k = 2\sum_{\alpha\in\Sigma_0} k_\alpha^{-1}$. Combined with (10.2) we obtain $\sigma_\lambda^k(\beta^\vee) \leq \frac{h_k}{n}\lambda(\beta^\vee)$. $\square$

Corollary 10.2. *Let $\mu \in P$. We have $\widehat{\mu}_k \in \overline{V_+^*}$ if and only if $\mu \in P^+$.*

Proof. Let $\mu \in P$ and suppose that $\widehat{\mu}_k \in \overline{V_+^*}$. Then for all $\beta \in \Sigma_0^+$,

$$2\pi\mu(\beta^\vee) = \widehat{\mu}_k(\beta^\vee) + \sigma^k_{\widehat{\mu}_k}(\beta^\vee) \geq 0$$

by Lemma 10.1, hence $\mu \in P^+$.

Conversely, suppose that $\mu \in P^+$ and let $w \in W_0$ such that $w\widehat{\mu}_k \in \overline{V_+^*}$. By Proposition 2.9 this implies $\widehat{w\mu}_k \in \overline{V_+^*}$. By the previous paragraph we conclude that $w\mu \in P^+$. On the other hand $P^+ \cap W_0\mu = \{\mu\}$, hence $w\mu = \mu \in P^+$ and $\widehat{\mu}_k = \widehat{w\mu}_k \in \overline{V_+^*}$. □

Proposition 2.10 is now a direct consequence of Corollary 10.2 and Lemma 10.1.

11. The Pauli Principle

In this section we complete the proof of Theorem 3.5 (and hence also of Theorem 2.6). In view of Proposition 8.4 and Corollary 8.5 it suffices to show the following root system analog of the Pauli principle.

Proposition 11.1. *If $\lambda \in \mathrm{BAE}_k$ is singular then $E(\lambda)_Q^W = \{0\}$.*

For the proof of Proposition 11.1 we may assume without loss of generality that $\lambda \in \mathrm{BAE}_k$ is J-standard (in particular, $\lambda \in i\overline{V_+^*}$). We write $V_J^* \subseteq V^*$ for the real sub-space spanned by the subset J of simple roots. Its complement in V is defined by

$$V_J^\perp = \{v \in V \mid \xi(v) = 0 \quad \forall \xi \in V_J^*\}.$$

Observe that $V_J^\perp = V$ iff $J = \emptyset$ iff λ is regular.

Consider the linear map $K_\lambda^k : V \to V$ defined by

$$K_\lambda^k(v) = v + \sum_{\alpha \in \Sigma_0} \frac{k_\alpha \alpha(v)\alpha^\vee}{k_\alpha^2 - \lambda(\alpha^\vee)^2}, \qquad v \in V.$$

Lemma 11.2. *Let $\lambda \in iV^*$ be a singular J-standard solution of the Bethe ansatz equations* (2.10)*. Then λ satisfies the constraint*

$$\pi_\lambda^{(N_J-1)}\big(\mathcal{J}_k\psi_\lambda^k\big) = \pi_\lambda^{(N_J-1)}\big((\partial_{\varphi^\vee} + k_\varphi)\psi_\lambda^k\big) \tag{11.1}$$

iff $K_\lambda^k(V) \subseteq V_J^\perp$.

Proof. Fix a singular J-standard solution $\lambda \in i\overline{V_+^*}$ of the Bethe ansatz equations (2.10) (in particular $J \neq \emptyset$). By a similar computation as in the proof of Proposition 8.4 we obtain from (7.3), Lemma 8.2 and Proposition 7.4,

$$\pi_\lambda^{(N_J-1)}\big((\partial_{\varphi^\vee} + k_\varphi)\psi_\lambda^k\big) = C_J^k \sum_{u \in W_0^J} c_k(u\lambda) \sum_{\beta \in \Sigma_0^{J,+}} u\beta(a_{u\lambda})e^{u\lambda} \prod_{\alpha \in \Sigma_0^{J,+}\setminus\{\beta\}} u\alpha,$$

$$\pi_\lambda^{(N_J-1)}\big(\mathcal{J}_k\psi_\lambda\big) = C_J^k \sum_{u \in W_0^J} c_k(u\lambda)e^{-u^J\lambda(\varphi^\vee)} \sum_{\beta \in \Sigma_0^{J,+}} u\beta(b_{u^J\lambda})e^{u^J\lambda} \prod_{\alpha \in \Sigma_0^{J,+}\setminus\{\beta\}} u^J u_J\alpha$$

with vectors $a_\mu, b_\mu \in V_{\mathbb{C}}$ ($\mu \in V_{\mathbb{C}}^*$) given by

$$a_\mu = (\mu(\varphi^\vee) + k_\varphi)\rho_\mu^k + \varphi^\vee,$$
$$b_\mu = \big(\mu(\varphi^\vee) - k_\varphi\big)\big(\rho_{s_\varphi\mu}^k + \varphi^\vee\big) - \varphi^\vee,$$

where we have used the involution on W_0^J defined by (8.4), as well as (8.5). For $u \in W_0^J$ we have

$$\begin{aligned}\sum_{\beta\in\Sigma_0^{J,+}} u\beta(b_{u^J\lambda}) \prod_{\alpha\in\Sigma_0^{J,+}\setminus\{\beta\}} u^J u_J\alpha &= (-1)^{l(u_J)}\left(\sum_{\beta\in\Sigma_0^{J,+}} \frac{u\beta(b_{u^J\lambda})}{u^J u_J\beta}\right)\prod_{\alpha\in\Sigma_0^{J,+}} u^J\alpha \\ &= \frac{1}{2}(-1)^{l(u_J)}\left(\sum_{\beta\in\Sigma_0^J} \frac{uu_J^{-1}\beta(b_{u^J\lambda})}{u^J\beta}\right)\prod_{\alpha\in\Sigma_0^{J,+}} u^J\alpha \\ &= (-1)^{l(u_J)} \sum_{\beta\in\Sigma_0^{J,+}} uu_J^{-1}\beta(b_{u^J\lambda}) \prod_{\alpha\in\Sigma_0^{J,+}\setminus\{\beta\}} u^J\alpha.\end{aligned}$$

Consequently (11.1) is equivalent to

$$c_k(u\lambda)u\beta(a_{u\lambda}) = (-1)^{l(u_J)}c_k(u^J\lambda)e^{-u\lambda(\varphi^\vee)}s_\varphi u\beta(b_{u\lambda}), \qquad \forall\, u \in W_0^J,\ \forall\, \beta \in \Sigma_0^{J,+}.$$

Since λ is a solution of the Bethe ansatz equations (see (8.6) for the convenient equivalent form of the Bethe ansatz equations) this is equivalent to

$$\big(u\lambda(\varphi^\vee) - k_\varphi\big)a_{u\lambda} - \big(u\lambda(\varphi^\vee) + k_\varphi\big)s_\varphi b_{u\lambda} \in u(V_J^\perp), \qquad \forall\, u \in W_0^J. \tag{11.2}$$

Note that (11.2) only depends on the coset $uW_{0,J}$ ($u \in W_0^J$). Using the explicit expressions for $a_{u\lambda}$ and $b_{u\lambda}$ we can rewrite (11.2) as

$$\big(w^{-1}\rho_{w\lambda}^k - w^{-1}s_\varphi\rho_{s_\varphi w\lambda}^k\big) + \left(\frac{w\lambda(\varphi^\vee)^2 - k_\varphi^2 - 2k_\varphi}{w\lambda(\varphi)^2 - k_\varphi^2}\right) w^{-1}\varphi^\vee \in V_J^\perp, \quad \forall\, w\in W_0. \tag{11.3}$$

We match (11.3) to the desired condition $K_\lambda^k(V) \subseteq V_J^\perp$ as follows. Since Σ_0 is an irreducible root system in V^*, the condition $K_\lambda^k(V) \subseteq V_J^\perp$ is equivalent to $K_\lambda^k(w^{-1}\varphi^\vee) \in V_J^\perp$ for all $w \in W_0$, which in turn is equivalent to (11.3) if

$$K_\lambda^k(w^{-1}\varphi^\vee) = \big(w^{-1}\rho_{w\lambda}^k - w^{-1}s_\varphi\rho_{s_\varphi w\lambda}^k\big) + \left(\frac{w\lambda(\varphi^\vee)^2 - k_\varphi^2 - 2k_\varphi}{w\lambda(\varphi)^2 - k_\varphi^2}\right) w^{-1}\varphi^\vee \tag{11.4}$$

for all $w \in W_0$. To prove (11.4) we first observe that

$$s_\varphi\rho_{s_\varphi w\lambda}^k = \rho_{w\lambda}^k - 2\sum_{\alpha\in\Sigma_0^+\cap s_\varphi\Sigma_0^-} \frac{k_\alpha\alpha^\vee}{k_\alpha^2 - w\lambda(\alpha^\vee)^2}$$

by the explicit expression (7.9) for ρ^k_μ. Using (2.13) this can be rewritten as

$$w^{-1}\rho^k_{w\lambda} - w^{-1}s_\varphi\rho^k_{s_\varphi w\lambda} = 2\frac{k_\varphi w^{-1}\varphi^\vee}{w\lambda(\varphi^\vee)^2 - k_\varphi^2} + 2\sum_{\alpha\in\Sigma_0^+}\frac{k_\alpha\alpha(\varphi^\vee)w^{-1}\alpha^\vee}{k_\alpha^2 - w\lambda(\alpha^\vee)^2}.$$

The second term can be rewritten as

$$\begin{aligned}2\sum_{\alpha\in\Sigma_0^+}\frac{k_\alpha\alpha(\varphi^\vee)w^{-1}\alpha^\vee}{k_\alpha^2 - w\lambda(\alpha^\vee)^2} &= \sum_{\alpha\in\Sigma_0}\frac{k_\alpha\alpha(\varphi^\vee)w^{-1}\alpha^\vee}{k_\alpha^2 - w\lambda(\alpha^\vee)^2}\\ &= \sum_{\alpha\in\Sigma_0}\frac{k_\alpha\alpha(w^{-1}\varphi^\vee)}{k_\alpha^2 - \lambda(\alpha^\vee)^2}\\ &= K^k_\lambda(w^{-1}\varphi^\vee) - w^{-1}\varphi^\vee.\end{aligned}$$

Combining the latter two formulas yields (11.4). □

It follows from (9.3) that

$$B^k_{-i\lambda}(\eta_v, \eta_{v'}) = \langle K^k_\lambda(v), v'\rangle, \qquad v, v' \in V$$

with $\eta_v = \langle v, \cdot\rangle \in V^*$ and $B^k_{-i\lambda}$ the Hessian of the master function S_k at $-i\lambda \in V^*$. Since $B^k_{-i\lambda}$ is positive definite, $K^k_\lambda : V \xrightarrow{\sim} V$ is a linear isomorphism. Proposition 11.1 thus is an immediate consequence of Lemma 11.2.

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