Sel. math., New ser. 14 (2009), 571–605

1022-1824/040571-35, *published online* 14 March 2009
DOI 10.1007/s00029-009-0516-y

**Selecta Mathematica
New Series**

Trigonometric Cherednik algebra at critical level and quantum many-body problems

E. Emsiz, E. M. Opdam and J. V. Stokman

Abstract. For any module over the affine Weyl group we construct a representation of the associated trigonometric Cherednik algebra $A(k)$ at critical level in terms of Dunkl type operators. Under this representation the center of $A(k)$ produces quantum conserved integrals for root system generalizations of quantum spin-particle systems on the circle with delta function interactions. This enables us to translate the spectral problem of such a quantum spin-particle system to questions in the representation theory of $A(k)$. We use this approach to derive the associated Bethe ansatz equations. They are expressed in terms of the normalized intertwiners of $A(k)$.

Mathematics Subject Classification (2000). Primary 20C08; Secondary 81R12.

Keywords. Trigonometric Cherednik algebra, quantum integrable systems, singular potentials.

1. Introduction

The trigonometric Cherednik algebra [4, 5] depends on a root system R, a multiplicity function k, and a level c. For noncritical level $c \neq 0$ it is an indispensable tool in the analysis of quantum Calogero–Moser systems with trigonometric potentials. In this paper we use the trigonometric Cherednik algebra $A(k)$ at critical level $c = 0$ to analyze the root system generalizations of quantum spin-particle systems on the circle with delta function interactions.

The study of one-dimensional quantum spin-particle systems with delta function interactions goes back to Lieb and Liniger [23]. These systems are particularly well studied and have an amazingly rich structure: see, e.g., [23, 37, 26, 36, 35, 12], to name just a few. They have been successfully analyzed by Bethe ansatz methods and by quantum inverse scattering methods. We want to advertise here yet another technique which is based on degenerate Hecke algebras. It allows us to extend the Bethe ansatz techniques to quantum Hamiltonians with delta function potentials along the root hyperplanes of any (affine) root system. This builds on many earlier works (see, e.g., [12, 16, 13, 32, 7, 27, 20, 17, 9]). In special cases the associated quantum system describes one-dimensional quantum spin-particles with pairwise delta function interactions and with boundary reflection terms.

To clarify the interrelations between these techniques we feel that it is instructive to start with a short discussion about the analysis of the quantum spin-particle systems on the circle with delta function interactions using the classical Bethe ansatz method. It goes back to the work of Lieb and Liniger [23] in case of the quantum Bose gas. The extension of these techniques to quantum particles with spin was considered, amongst others, by McGuire [25, 26], Flicker and Lieb [10], Gaudin [11] and Yang [36].

Consider the natural action of $S_n \ltimes \mathbb{Z}^n$ on $\mathbb{R}^n$ by permutations and translations and choose a representation $\rho : S_n \ltimes \mathbb{Z}^n \to \mathrm{GL}_{\mathbb{C}}(M)$ (it encodes the spin of the quantum particles). Consider the quantum Hamiltonian

$$\mathcal{H}_k^M = -\Delta - k \sum_{\substack{1 \le i < j \le n \\ m \in \mathbb{Z}}} \delta(x_i - x_j + m)\rho(s_{i,j;m})$$

with Δ the Laplacian on $\mathbb{R}$, $k \in \mathbb{C}$ a coupling constant, δ Dirac's delta function and $s_{i,j;m} \in S_n \ltimes \mathbb{Z}^n$ the orthogonal reflection in the affine hyperplane

$$\{x \in \mathbb{R}^n \mid x_j - x_i = m\}.$$

With the present choice of notations, coupling constant $k < 0$ (respectively $k > 0$) corresponds to repulsive (respectively attractive) delta function interactions between the quantum particles.

Consider the domain

$$C_+ = \{x \in \mathbb{R}^n \,|\, x_1 > x_2 > \cdots > x_n > x_1 - 1\}$$

in $\mathbb{R}^n$. The Bethe hypothesis in this context (see [23]) is to look for eigenfunctions of $\mathcal{H}_k^M$ that have an expansion in plane waves in each $S_n \ltimes \mathbb{Z}^n$-translate of C_+. The hypothesis is justified by the following result.

Theorem 1.1. *Fix $\lambda \in \mathbb{C}^n$ such that $\lambda_i \neq \lambda_j$ for $i \neq j$. Choose furthermore $m_w \in M$ ($w \in S_n$). There exists a unique M-valued continuous function f_λ on $\mathbb{R}^n$ satisfying*

(i) *f_λ is an eigenfunction of $\mathcal{H}_k^M$ (in the weak sense) with eigenvalue given by $-\lambda^2 := -\sum_{j=1}^n \lambda_j^2$.*
(ii) *On each $S_n \ltimes \mathbb{Z}^n$-translate of C_+, f_λ can be expressed as a sum of plane waves $e^{w\lambda}$ ($w \in S_n$) with coefficients in M.*
(iii) *$f_\lambda|_{C_+} = \sum_{w \in S_n} m_w e^{w\lambda}|_{C_+}$.*

The proof of the theorem includes an explicit recipe how to propagate $\sum_{w \in S_n} m_w e^{w\lambda}|_{C_+}$ to the eigenfunction f_λ of $\mathcal{H}_k^M$. It is based on the reformulation of the spectral problem $\mathcal{H}_k^M f_\lambda = -\lambda^2 f_\lambda$ as a boundary value problem. The uniqueness of f_λ is a subtle point; we discuss it in Section 5.

Fix $\lambda \in \mathbb{C}^n$ generic (to be made precise in the main text) and consider the diagonal $S_n \ltimes \mathbb{Z}^n$-action $(w \cdot f)(x) := \rho(w) f(w^{-1}x)$ on the space of M-valued functions on $\mathbb{R}^n$. The study of the S_n-invariant eigenfunctions f_λ has led to the discovery of the famous Yang–Baxter equation with spectral parameters as follows [25, 36]. Write $s_{i,j} = s_{i,j;0} \in S_n$ and consider the elements

$$Y_{i,j}^k(u) = \frac{u + ks_{i,j}}{u-k}, \qquad 1 \leq i < j \leq n,$$

in the group algebra $\mathbb{C}[S_n]$ of S_n for generic $u \in \mathbb{C}$. The S_n-invariance of the eigenfunction f_λ is equivalent to the plane wave coefficients m_w being of the form $m_w = \rho(J_w^k(\lambda))m$ ($m \in M$) with $J_w^k(\lambda)$ the unique elements in $\mathbb{C}[S_n]$ satisfying $J_1^k(\lambda) = 1$ and

$$J_{s_{i,i+1}w}^k(\lambda) = Y_{i,i+1}^k(\lambda_{w^{-1}(i)} - \lambda_{w^{-1}(i+1)})s_{i,i+1}J_w^k(\lambda) \tag{1.1}$$

for $1 \leq i < n$ and $w \in S_n$. We denote the corresponding S_n-invariant eigenfunction by f_λ^m. The consistency of (1.1) is equivalent to the $Y_{i,j}$ being unitary solutions of the Yang–Baxter equation,

$$\begin{aligned} Y_{i,j}^k(u)Y_{i,l}^k(u+v)Y_{j,l}^k(v) &= Y_{j,l}^k(v)Y_{i,l}^k(u+v)Y_{i,j}^k(u), && 1 \leq i < j < l \leq n, \\ Y_{i,j}^k(-u) &= Y_{i,j}^k(u)^{-1}, && 1 \leq i < j \leq n. \end{aligned}$$

We now translate the requirement that f_λ^m is $\mathbb{Z}^n$-invariant to explicit conditions on $m \in M$. The group $S_n \ltimes \mathbb{Z}^n$ is generated by S_n and one additional element π, which we characterize here by its action on $\mathbb{R}^n$,

$$\pi(x_1, \ldots, x_{n-1}, x_n) = (x_n + 1, x_1, \ldots, x_{n-1}).$$

There exists a unique extension of the $J_w^k(\lambda)$ to elements $J_w^k(\lambda)$ ($w \in S_n \ltimes \mathbb{Z}^n$) in the group algebra $\mathbb{C}[S_n \ltimes \mathbb{Z}^n]$ of $S_n \ltimes \mathbb{Z}^n$ satisfying $J_\pi^k(\lambda) = \pi$ and satisfying the cocycle relation

$$J_{vw}^k(\lambda) = J_v^k((Dw)\lambda)J_w(\lambda), \qquad \forall v, w \in S_n \ltimes \mathbb{Z}^n,$$

where $D : S_n \ltimes \mathbb{Z}^n \to S_n$ is the natural surjective group homomorphism (omitting the translation part). Then f_λ^m is $\mathbb{Z}^n$-invariant if and only if $m \in M$ satisfies the Bethe ansatz equations

$$\rho(J_y^k(\lambda))m = e^{\lambda(y)}m, \qquad \forall y \in \mathbb{Z}^n$$

(see Theorem 5.10). The Bethe ansatz equations (5.7) for special modules M were derived in, e.g., [23, 26, 36, 11, 35].

Our results will show that for root system R of type A, the trigonometric Cherednik algebra $A(k)$ at critical level $c = 0$ enters the analysis of these quantum systems in three closely related ways:

(i) The quantum Hamiltonian $\mathcal{H}_k^M$ can be constructed from a family of commuting Dunkl type differential-reflection operators. These Dunkl operators, together with the diagonal $S_n \ltimes \mathbb{Z}^n$-action, give a presentation of $A(k)$.

(ii) The propagation procedure, which extends a plane wave $\sum_{w \in S_n} m_w e^{w\lambda}|_{C_+}$ to the eigenfunction f_λ of $\mathcal{H}_k^M$, is governed by a representation of $S_n \ltimes \mathbb{Z}^n$ defined in terms of integral-reflection operators. Together with the constant coefficient differential operators it gives another presentation of $A(k)$.

(iii) The cocycle $\{J^k_w(\lambda)\}_{w\in S_n\ltimes\mathbb{Z}^n}$ comes from the action of the normalized intertwiners of $A(k)$ on principal series modules of $A(k)$.

In this paper we discuss (i)–(iii) for arbitrary root systems R. This is subsequently used to apply the Bethe ansatz methods to the root system generalizations of the spin-particle systems on the circle with delta function interactions. It leads in particular to the Bethe ansatz equations for any root system R (see Theorem 5.10).

2. Notations

2.1. Orthogonal reflections in affine hyperplanes

Let V be a Euclidean vector space with scalar product $\langle\cdot,\cdot\rangle$. The linear dual V^* inherits from V the structure of a Euclidean vector space and we also write $\langle\cdot,\cdot\rangle$ for the associated scalar product on V^*. In this and the next subsection B denotes a commutative unital noetherian $\mathbb{R}$-algebra. For any real vector space M we use M_B as a shorthand notation for the B-module $B\otimes_{\mathbb{R}} M$. We extend the scalar products on V and on V^* to B-bilinear forms on V_B and V_B^*, which we still denote by $\langle\cdot,\cdot\rangle$. Let $P(V)$ denote the algebra of real polynomial functions on V. We regularly identify $P_B(V) := P(V)_B$ with the symmetric algebra $S(V_B^*)$ of V_B^* by interpreting $\xi\in V_B^*$ as the B-valued polynomial $v\mapsto\xi(v)$ on V.

Consider the space $\mathrm{Aff}_B(V) := \mathrm{Aff}(V)_B$ of B-valued affine linear functions on V. It is the subspace of $P_B(V)$ consisting of B-valued polynomials of degree ≤ 1 on V. Under the natural identification $P_B(V)\simeq S(V_B^*)$, the affine linear function $\phi\in\mathrm{Aff}_B(V)$ identifies with $\xi+\lambda 1\in V_B^*\oplus B1$ ($\xi\in V_B^*$, $\lambda\in B$) where ξ is the gradient $D\phi$ of ϕ and $\lambda=\phi(0)$, i.e.

$$\phi(v)=\xi(v)+\lambda,\qquad \forall v\in V.$$

The co-vector $\xi^\vee\in V$ associated to $\xi\in V^*\setminus\{0\}$ is defined by

$$\eta(\xi^\vee)=2\frac{\langle\eta,\xi\rangle}{\langle\xi,\xi\rangle},\qquad \forall\eta\in V^*.$$

For $\phi\in\mathrm{Aff}(V)$ with nonzero gradient, the map $s_\phi: V\to V$ defined by

$$s_\phi(v)=v-\phi(v)(D\phi)^\vee,\qquad v\in V,$$

is the orthogonal reflection in the affine hyperplane $V_\phi=\{v\in V\,|\,\phi(v)=0\}$. Observe that s_ϕ is linear if ϕ is linear, in which case we also write s_ϕ for its B-linear extension $s_\phi: V_B\to V_B$.

For $v\in V_B$ we define translation operators $t_v: V_B\to V_B$ by

$$t_v(v')=v+v',\qquad v'\in V_B.$$

Note that $f\circ t_v=t_{f(v)}\circ f$ for B-linear mappings $f: V_B\to V_B$. Furthermore, for $\phi\in\mathrm{Aff}(V)$ with nonzero gradient we have

$$s_\phi=s_{D\phi}t_{\phi(0)(D\phi)^\vee}. \tag{2.1}$$

2.2. Root systems

Let $R \subset V^*$ be a reduced, crystallographic root system in V^*. We assume that R is irreducible when considered as a root system in $\mathrm{span}_{\mathbb{R}}\{R\} \subset V^*$. The associated co-root system is $R^\vee = \{\alpha^\vee\}_{\alpha \in R} \subset V$.

Note that we do not require that R spans V^*. This allows us for instance to consider the root system $R = \{\epsilon_i - \epsilon_j\}_{1 \leq i \neq j \leq n}$ of type A_{n-1} with ambient Euclidean space $V^* = \mathbb{R}^n$, where $\{\epsilon_i\}_{i=1}^n$ is the standard orthonormal basis of $\mathbb{R}^n$.

The Weyl group W is the subgroup of $O(V)$ generated by the orthogonal reflections s_α in the root hyperplanes V_α ($\alpha \in R$), hence V has a canonical $\mathbb{R}[W]$-module structure. By extension of scalars we consider V_B as a $B[W]$-module. In the dual module V_B^* the action of s_α is given explicitly by

$$s_\alpha(\xi) = \xi - \xi(\alpha^\vee)\alpha, \qquad \xi \in V_B^*,$$

for $\alpha \in R$.

We fix a full lattice $X \subset V^*$ satisfying the following two properties:

(i) X contains the root lattice Q of R.
(ii) The lattice $Y \subset V$ dual to X contains the co-root lattice $Q^\vee$ of $R^\vee$.

The conditions (i) and (ii) imply that X and Y are W-invariant. A key example is the root system of type A_{n-1} with $V^* = \mathbb{R}^n$ and $X = \bigoplus_{j=1}^n \mathbb{Z}\epsilon_j$.

Definition 2.1. The *extended affine Weyl group* associated to the above data is $W^{\mathrm{a}} = W \ltimes Y$. The *affine Weyl group* is the normal subgroup $W \ltimes Q^\vee$ of W^{a}.

We will identify W^{a} with the subgroup $\{wt_y\}_{w \in W,\, y \in Y}$ of the group of isometries of V. The *gradient map* $D : W^{\mathrm{a}} \to W$ is the surjective group homomorphism defined by $D(wt_y) = w$ ($w \in W$, $y \in Y$).

Fix $b \in B$. For $w \in W$ and $y \in Y$ we set

$$(wt_y)^{(b)}(v) := (wt_{by})(v), \qquad v \in V_B.$$

This defines a left W^{a}-action on V_B. Observe that $w^{(1)} = w$ and $w^{(0)} = Dw$ for $w \in W^{\mathrm{a}}$. Furthermore,

$$w^{(b)} \circ \lambda_b = \lambda_b \circ w, \qquad w \in W^{\mathrm{a}},$$

where $\lambda_b(v) := bv$ ($v \in V_B$).

By transposition the W^{a}-action $w \mapsto w^{(b)}$ on V_B gives rise to left actions of W^{a} on $P_B(V) \simeq S(V_B^*)$ by B-algebra automorphisms. The subspace $\mathrm{Aff}_B(V)$ of $P_B(V) \simeq S(V_B^*)$ is W^{a}-invariant for all scaling factors $b \in B$ since

$$(wt_{by})(\xi + \lambda 1) = w(\xi) + (\lambda - b\xi(y))1 \tag{2.2}$$

for $w \in W$, $y \in Y$, $\xi \in V_B^*$ and $\lambda \in B$.

Definition 2.2. The subset $R^{\mathrm{a}} = R + \mathbb{Z}1$ of $V^* + \mathbb{R}1$ is the *affine root system* associated to R.

By (2.2),

$$(wt_y)(a) = w(\alpha) + (m - \alpha(y))1, \qquad a = \alpha + m1 \in R^{\mathrm{a}},$$

for $w \in W$ and $y \in Y$. By assumption the roots $\alpha \in R$ are contained in the lattice X, hence $\alpha(y) \in \mathbb{Z}$ for $\alpha \in R$ and $y \in Y$. Thus R^{a} is W^{a}-invariant.

By (2.1) we have

$$s_a = s_\alpha t_{m\alpha^\vee}, \qquad a = \alpha + m1 \in R^{\mathrm{a}}.$$

Consequently, the affine Weyl group $W \ltimes Q^\vee$ is generated by the orthogonal reflections s_a in the affine hyperplanes V_a, where a runs over the set R^{a} of affine roots. Note furthermore that $ws_aw^{-1} = s_{wa}$ and $Ds_a = s_{Da}$ for all $a \in R^{\mathrm{a}}$ and $w \in W^{\mathrm{a}}$.

We fix a basis F of R. Write $R^\pm$ for the associated positive and negative roots, and $\theta \in R$ for the highest root of R with respect to F. The basis F of R extends to a basis F^{a} of R^{a} by adding the simple affine root $a_0 := -\theta + 1$ to F. The associated simple reflection s_{a_0} will be denoted by s_0. The *positive* and *negative* affine roots are $R^{\mathrm{a},+} = (R + \mathbb{Z}_{>0}) \cup R^+$ and $R^{\mathrm{a},-} = -R^{\mathrm{a},+}$ respectively. Define the *length* of $w \in W^{\mathrm{a}}$ by

$$l(w) = \#(R^{\mathrm{a},+} \cap w^{-1}(R^{\mathrm{a},-})).$$

The following proposition is well known.

Proposition 2.3. *Set* $\Omega = \{w \in W^{\mathrm{a}} \mid l(w) = 0\}$.

(i) Ω *is an abelian subgroup of* W^{a}*, isomorphic to* $W^{\mathrm{a}}/(W \ltimes Q^\vee) \simeq Y/Q^\vee$.
(ii) $\omega \in \Omega$ *permutes the set* F^{a} *of simple roots.*

3. The trigonometric Cherednik algebra

3.1. The algebra H_L^{a}

Let $L = \mathbb{C}[\mathbf{c}, \mathbf{k}]$ be the polynomial algebra in the indeterminates $\mathbf{c}$ and the $\mathbf{k}_{\mathcal{O}}$, where $\mathcal{O}$ runs over the set of W^{a}-orbits in R^{a}. We write $\mathbf{k}_b = \mathbf{k}_{W^{\mathrm{a}}(b)}$ for $b \in R^{\mathrm{a}}$, and $\mathbf{k}_0 = \mathbf{k}_{a_0}$.

Depending on the root system R and the lattice Y, we thus have one, two or three commuting indeterminates $\mathbf{k}_{\mathcal{O}}$. Concretely, if R is not of type C_n $(n \geq 1)$ then the W^{a}-orbits are of the form $W^{\mathrm{a}}(\beta)$ with β running through a complete set of representatives of the W-orbits of R. The same is true for R of type C_n if $\theta(Y) = \mathbb{Z}$. For R of type C_n $(n \geq 1)$ and $\theta(Y) = 2\mathbb{Z}$ we have two (if $n = 1$) or three (if $n \geq 2$) W^{a}-orbits in R^{a}, namely $W^{\mathrm{a}}(a_0)$ and the $W^{\mathrm{a}}(\beta)$ with $\beta \in R$ representatives of the W-orbits in R (cf. [24]).

Definition 3.1 ([4, 5]). The *trigonometric Cherednik algebra* (also known as the *degenerate double affine Hecke algebra*) is the associative unital L-algebra H_L^{a} satisfying:

(i) H_L^{a} contains $S(V_L^*)$ and $L[W^{\mathrm{a}}]$ as subalgebras.

(ii) The multiplication map defines an isomorphism

$$S(V_L^*) \otimes_L L[W^{\mathrm{a}}] \to H_L^{\mathrm{a}}$$

of L-modules.

(iii) We have the *cross relations*

$$s_a \cdot \xi - s_a^{(\mathbf{c})}(\xi) \cdot s_a = -\mathbf{k}_a \xi(Da^\vee), \qquad \forall a \in F^{\mathrm{a}},\ \forall \xi \in V^*. \tag{3.1}$$

(iv) $\omega \cdot \xi = \omega^{(\mathbf{c})}(\xi) \cdot \omega$ for $\omega \in \Omega$ and $\xi \in V^*$.

Remark 3.2. The cross relations for $\alpha \in F$ read

$$s_\alpha \cdot \xi - s_\alpha(\xi) \cdot s_\alpha = -\mathbf{k}_\alpha \xi(\alpha^\vee).$$

For $a = a_0 = -\theta + 1$ this becomes

$$s_0 \cdot \xi - (s_\theta(\xi) + \mathbf{c}\xi(\theta^\vee)1) \cdot s_0 = \mathbf{k}_0 \xi(\theta^\vee).$$

The existence of H_L^{a} needs proof; it follows from an explicit realization of H_L^{a} due to Cherednik [5], in terms of Dunkl–Cherednik operators. For the sake of completeness we will recall it in the next subsection.

In the remainder of this subsection we consider the trigonometric Cherednik algebra with specialized parameters. For $c \in \mathbb{C}$ and for a W^{a}-invariant function $k : R^{\mathrm{a}} \to \mathbb{C}$ (called a *multiplicity function*), we write $H^{\mathrm{a}}(k,c)$ for the complex associative algebra obtained from H_L^{a} by specializing $\mathbf{c}$ and $\mathbf{k}_a$ to c and k_a, respectively. We call $c \in \mathbb{C}$ the *level* of $H^{\mathrm{a}}(k,c)$. The subalgebra $H(k)$ of $H^{\mathrm{a}}(k,c)$ generated by $S(V_{\mathbb{C}}^*)$ and $\mathbb{C}[W]$ is independent of c. It is the degenerate affine Hecke algebra [8, 24].

By induction on $l(w)$ we have

$$w \cdot \xi = (w^{(c)}(\xi)) \cdot w - \sum_{a \in R^{\mathrm{a},+} \cap w^{-1} R^{\mathrm{a},-}} k_a \xi(Da^\vee) w s_a, \qquad \forall w \in W^{\mathrm{a}},\ \forall \xi \in V_{\mathbb{C}}^*, \tag{3.2}$$

in $H^{\mathrm{a}}(k,c)$ (cf. [30, Prop. 1.1]). The cross relations in $H^{\mathrm{a}}(k,c)$ between the simple reflections s_a ($a \in F^{\mathrm{a}}$) and $p \in S(V_{\mathbb{C}}^*)$ can also be made explicit. For this we first introduce rescaled roots $a^{(c)}$ ($a = \alpha + m1 \in R^{\mathrm{a}}$) by

$$a^{(c)} = \alpha + cm1 \in \mathrm{Aff}(V_{\mathbb{C}}).$$

Observe that $s_a^{(c)} = s_{a^{(c)}}$. We also use the notation $s_{a^{(c)}}$ for the associated action on $S(V_{\mathbb{C}}^*) \simeq P(V_{\mathbb{C}})$ by algebra automorphisms. Observe that $w^{(c)}(a^{(c)}) = (w(a))^{(c)}$ for $w \in W^{\mathrm{a}}$ and $a \in R^{\mathrm{a}}$. With these notations the cross relations (3.1) in $H^{\mathrm{a}}(k,c)$ imply

$$s_a \cdot p - s_{a^{(c)}}(p) \cdot s_a = k_a \left(\frac{s_{a^{(c)}}(p) - p}{a^{(c)}} \right), \qquad \forall a \in F^{\mathrm{a}},\ \forall p \in S(V_{\mathbb{C}}^*). \tag{3.3}$$

It follows from (3.3) that the center of the degenerate affine Hecke algebra $H(k)$ is the subalgebra $S(V_{\mathbb{C}}^*)^W$ of W-invariant elements in the symmetric algebra $S(V_{\mathbb{C}}^*)$ (see [24] for details).

For $c \neq 0$ we have $H^{\mathrm{a}}(k,c) \simeq H^{\mathrm{a}}(k/c,1)$ as algebras, where k/c is the multiplicity that takes value k_a/c at $a \in R^{\mathrm{a}}$. The map $H^{\mathrm{a}}(k,c) \to H^{\mathrm{a}}(k/c,1)$ realizing the algebra isomorphism is determined by $\xi \mapsto c\xi$ and $w \mapsto w$ for $\xi \in V_{\mathbb{C}}^*$ and $w \in W^{\mathrm{a}}$. The center of $H^{\mathrm{a}}(k,c)$ is trivial if $c \neq 0$ (cf. [2, Prop. 1.3.6] for R of type A).

We denote $H^{\mathrm{a}}(k) := H^{\mathrm{a}}(k,0)$ for the trigonometric Cherednik algebra at level 0. Its center is studied in [28]. In this paper we only need the simple observation that $S(V_{\mathbb{C}}^*)^W$ is contained in the center of $H^{\mathrm{a}}(k)$, in view of (3.3).

Remark 3.3. For the root system R of type A the spectrum of the center of $H^{\mathrm{a}}(k)$ is explicitly described in [28]. It is called the *trigonometric Calogero–Moser space.*

In analogy with terminology for affine Lie algebras we call $c = 0$ the *critical level* (cf. [6, 2]). The trigonometric Cherednik algebra $H^{\mathrm{a}}(k)$ at critical level is the main object of study in this paper.

3.2. The Cherednik representation

For completeness, in this subsection we recall the faithful representation of H_L^{a} in terms of Dunkl–Cherednik operators. We start with two convenient lemmas for proving that some $L[W^{\mathrm{a}}]$-module M and a suitable compatible family of L-linear operators on M give rise to an H_L^{a}-module structure on M. The second lemma will be used at a later stage with specialized parameters to construct a representation of the trigonometric Cherednik algebra $H^{\mathrm{a}}(k)$ at critical level.

Lemma 3.4. *Let M be a left $L[W^{\mathrm{a}}]$-module and $N \subseteq M$ an L-submodule which generates M as an $L[W^{\mathrm{a}}]$-module. Suppose furthermore that*

(i) $T_\xi \in \mathrm{End}_L(M)$ *is a family of linear operators depending linearly on $\xi \in V^*$.*
(ii) *The cross relations*

$$\begin{aligned} s_a T_\xi - T_{s_a^{(\mathbf{c})}(\xi)} s_a &= -\mathbf{k}_a \xi(Da^\vee)\mathrm{Id}_M, && a \in F^{\mathrm{a}},\\ \omega T_\xi - T_{\omega^{(\mathbf{c})}(\xi)}\omega &= 0, && \omega \in \Omega, \end{aligned}$$

are satisfied as endomorphisms of M, where $T_{\xi+\lambda 1} := T_\xi + \lambda \mathrm{Id}_M$ for $\xi \in V^*$ and $\lambda \in L$.
(iii) *The kernel of the commutator $[T_\xi, T_\eta]$ contains N for all $\xi, \eta \in V^*$.*

Then the T_ξ ($\xi \in V^*$) pairwise commute as endomorphisms of M. Hence the W^{a}-action on M, together with $\xi \mapsto T_\xi$, turns M into an H_L^{a}-module.

Proof. The lemma is a direct consequence of the identities

$$w[T_\xi, T_\eta] = [T_{(Dw)\xi}, T_{(Dw)\eta}]w, \qquad \forall w \in W^{\mathrm{a}}, \forall \xi, \eta \in V^*, \tag{3.4}$$

in $\mathrm{End}_L(M)$. Formula (3.4) is a consequence of properties (i) and (ii) only. In fact, it suffices to establish (3.4) for $w = s_a$ ($a \in F^{\mathrm{a}}$) and for $w = \omega \in \Omega$, in which case it follows by straightforward computations from (i) and (ii). $\square$

We have the following dual version of Lemma 3.4.

Lemma 3.5. *Let M be a left $L[W^{\mathrm{a}}]$-module. Let $p : M \to N$ be an L-linear map to some L-module N such that $\{0\}$ is the only $L[W^{\mathrm{a}}]$-submodule of M contained* *in the kernel* $\ker(p)$ *of* p*. Assume furthermore the existence of a family of* L*-linear operators* T_ξ *on* M *satisfying conditions* (i) *and* (ii) *of Lemma* 3.4*.*

If the image of the commutator $[T_\xi, T_\eta]$ *is contained in* $\ker(p)$ *for all* $\xi, \eta \in V^*$*, then the* T_ξ $(\xi \in V^*)$ *pairwise commute as endomorphisms of* M*. In this situation the* W^{a}*-action on* M*, together with* $\xi \mapsto T_\xi$*, turns* M *into an* H_L^{a}*-module.*

Proof. Analogous to the proof of Lemma 3.4. □

Lemma 3.4 can be used to verify that H_L^{a} admits a realization in terms of Dunkl–Cherednik operators [5]. In the present set-up it involves some small adjustments since we do not require $\mathbf{k}_0 = \mathbf{k}_\theta$. This relates to the extension to nonreduced root systems from [30].

We write the standard basis of the group algebra $L[Y]$ as $\{e^y\}_{y \in Y}$. The algebra structure is then governed by $e^y e^{y'} = e^{y+y'}$ and $e^0 = 1$. If we interpret $L[Y]$ as the algebra of regular L-valued functions on $V_{\mathbb{C}}^*/2\pi\sqrt{-1}X$, the basis element e^y corresponds to the trigonometic function $\xi \mapsto e^{\xi(y)}$.

Definition 3.6 (cf. [28]). The *trigonometric Weyl algebra* $\mathcal{A}_L$ is the unique unital associative L-algebra satisfying

(i) $\mathcal{A}_L$ contains $S(V_L^*)$ and $L[Y]$ as subalgebras.
(ii) The multiplication map defines an isomorphism $S(V_L^*) \otimes_L L[Y] \to \mathcal{A}_L$ of L-modules.
(iii) The cross relations
$$[\xi, e^y] = \mathbf{c}\xi(y)e^y$$
hold for all $\xi \in V^*$ and $y \in Y$.

The indeterminates $\mathbf{k}_a$ in $\mathcal{A}_L$ are merely dummy parameters. We include them in the definition of $\mathcal{A}_L$ to avoid ground ring extensions at later stages.

The existence of $\mathcal{A}_L$ is immediate, since it can be realized as the L-subalgebra of $\mathrm{End}_L(L[Y])$ generated by $L[Y]$ (viewed as multiplication operators) and by $\mathbf{c}\partial_\xi$ $(\xi \in V_{\mathbb{C}}^*)$, where ∂_ξ is the L-linear derivation $\partial_\xi e^y = \xi(y)e^y$ $(y \in Y)$ of $L[Y]$.

With $\mathbf{k}$ and $\mathbf{c}$ specialized to a fixed multiplicity function $k : R \to \mathbb{C}$ and a level $0 \neq c \in \mathbb{C}$, the associated specialized complex algebra $\mathcal{A}(c)$ is the algebra of differential operators on the compact torus $\sqrt{-1}V^*/2\pi\sqrt{-1}X$ with regular coefficients. For $c = 0$ it is the algebra of regular functions on the cotangent bundle of $\sqrt{-1}V^*/2\pi\sqrt{-1}X$. It inherits the structure of a Poisson algebra from the semiclassical limit of $\mathcal{A}(c)$ as $c \to 0$.

Let $\mathcal{A}_L^{(\delta)}$ be the right localization of $\mathcal{A}_L$ at $\delta := \prod_{\alpha \in R^+}(1 - e^{-2\alpha^\vee})$. It is easy to check that $\mathcal{A}_L^{(\delta)}$ is a ring containing $\mathcal{A}_L$. Write $L[Y]^{(\delta)} = \mathcal{A}_L^{(\delta)} \otimes_{\mathcal{A}_L} L[Y]$. The Weyl group W acts naturally by L-algebra automorphisms on $\mathcal{A}_L$ and $\mathcal{A}_L^{(\delta)}$. We write $\mathcal{A}_L^{(\delta)} \# W$ for the associated smashed product L-algebra. It is isomorphic to $\mathcal{A}_L^{(\delta)} \otimes_L L[W]$ as an L-module. The localized $\mathcal{A}_L^{(\delta)}$-module $L[Y]^{(\delta)}$ is a faithful $\mathcal{A}_L^{(\delta)} \# W$-module. We call it the *basic representation* of $\mathcal{A}_L^{(\delta)} \# W$.

To give Cherednik's realization of H_L^{a} as an L-subalgebra of $\mathcal{A}_L^{(\delta)} \# W$ it is convenient to use the following reparametrization of $\mathbf{k}$. Define $\mathbf{l}_\alpha$ ($\alpha \in R$) by

$$\mathbf{l}_\alpha = \begin{cases} \mathbf{k}_\alpha, & \alpha \notin W\theta, \\ \mathbf{k}_0, & \alpha \in W\theta, \end{cases}$$

and set $\rho(\mathbf{k}) = \frac{1}{2}\sum_{\alpha\in R^+} \mathbf{k}_\alpha \alpha^\vee \in V_L$.

For $\xi \in V^*$ we define the *Dunkl–Cherednik operator* [5, 30] by

$$\mathbf{D}_\xi := \xi + \sum_{\alpha\in R^+} \xi(\alpha^\vee)\left(\frac{\mathbf{k}_\alpha + \mathbf{l}_\alpha e^{-\alpha^\vee}}{1-e^{-2\alpha^\vee}}\right)(1-s_\alpha) - \xi(\rho(\mathbf{k})) \in \mathcal{A}_L^{(\delta)} \# W. \qquad (3.5)$$

Under the basic representation of $\mathcal{A}_L^{(\delta)} \# W$ it restricts to an operator on $L[Y]$.

If $\theta(Y) = \mathbb{Z}$ then $a_0 \in W^{\mathrm{a}}\theta$, hence $\mathbf{k}_0 = \mathbf{k}_\theta$ and $\mathbf{l}_\alpha = \mathbf{k}_\alpha$ for all $\alpha \in R$. In this case

$$\mathbf{D}_\xi = \xi + \sum_{\alpha\in R^+} \mathbf{k}_\alpha \xi(\alpha^\vee)\frac{1}{1-e^{-\alpha^\vee}}(1-s_\alpha) - \xi(\rho(\mathbf{k})),$$

which is the Dunkl–Cherednik operator associated to the reduced root system R (see [5]). If $\theta(Y) = 2\mathbb{Z}$ then R is of type C_n for some $n \geq 1$. In this case $\mathbf{D}_\xi$ is the Dunkl–Cherednik operator associated to the nonreduced root system of type BC_n (see [30]).

We now give Cherednik's well known realization of H_L^{a} as a subalgebra of $\mathcal{A}_L^{(\delta)} \# W$. It ensures that the L-algebra H_L^{a}, as defined in Definition 3.1, exists.

Theorem 3.7 ([5]). *The assignments*

$$\begin{aligned} \xi &\mapsto \mathbf{D}_\xi, && \xi \in V^*, \\ w &\mapsto w, && w \in W, \\ t_y &\mapsto e^y, && y \in Y, \end{aligned} \qquad (3.6)$$

uniquely extend to an injective L-algebra homomorphism $H_L^{\mathrm{a}} \to \mathcal{A}_L^{(\delta)} \# W$.

Proof. We sketch a proof based on Lemma 3.4. Composing with the basic representation of $\mathcal{A}_L^{(\delta)} \# W$ we view the right hand sides of (3.6) as elements in $\mathrm{End}_L(L[Y])$. The verification that the assignments extend to an L-algebra homomorphism then reduces to the cross relations by Lemma 3.4 applied to the $L[W^{\mathrm{a}}]$-module $L[Y]$, $T_\xi = \mathbf{D}_\xi$ ($\xi \in V^*$) and $N = Le^0$. The cross relations can be verified by direct computations (cf. [31]). Injectivity follows by a standard argument. □

We write $p(\mathbf{D})$ for the element in $\mathcal{A}_L^{(\delta)} \# W$ corresponding to $p \in S(V_L^*)$.

If we specialize $\mathbf{k}$ to a multiplicity function $k : R \to \mathbb{C}$ and the level $\mathbf{c}$ to a noncritical value $0 \neq c \in \mathbb{C}$, Theorem 3.7 gives rise to the faithful Cherednik representation of $H^{\mathrm{a}}(k,c)$ on $\mathbb{C}[Y]$ in which $\xi \in V_{\mathbb{C}}^*$ acts by the Dunkl–Cherednik operator

$$D_\xi := c\partial_\xi + \sum_{\alpha\in R^+} \xi(\alpha^\vee)\left(\frac{k_\alpha + l_\alpha e^{-\alpha^\vee}}{1-e^{-2\alpha^\vee}}\right)(1-s_\alpha) - \xi(\rho(k)) \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}[Y])$$

(with the obvious meaning of l_α and $\rho(k)$). The corresponding differential-reflection operators $p(D)$ restrict to endomorphisms of $\mathbb{C}[Y]^W$ if $p \in S(V_{\mathbb{C}}^*)^W$, in which case it acts as a differential operator. The resulting commuting differential operators $\{p(D)\}_{p\in S(V_{\mathbb{C}}^*)^W}$ on $\mathbb{C}[Y]^W$ are, up to a gauge factor, the conserved quantum integrals of the quantum trigonometric Calogero–Moser system associated to R (respectively the nonreduced root system of type BC_n) if $\theta(Y) = \mathbb{Z}$ (respectively $\theta(Y) = 2\mathbb{Z}$); see [29] and [18, Part I, §2.2]. Specialized at critical level $c = 0$, the $\{p(D)\}_{p\in S(V_{\mathbb{C}}^*)^W}$ relate to the classical conserved integrals of the trigonometric Calogero–Moser system.

At critical level $c = 0$ various other representations of $H^{\mathrm{a}}(k)$ are known that involve Dunkl type operators (see e.g. [4, 6, 9]). In [4, 6] Dunkl operators with infinite reflection terms are used (this relates to quantum elliptic Calogero–Moser systems). In [9] representations of $H^{\mathrm{a}}(k)$ are considered that involve Dunkl type operators involving jumps over the affine root hyperplanes. This relates to quantum integrable Calogero–Moser type systems with delta function potentials. In Section 4 we generalize the latter representations. This allows us to include quantum spin-particle systems with delta function potentials in the present framework. In these cases the action of the commutative subgroup Y of W^{a} is by translations. This is a key difference from Theorem 3.7, where Y acts by multiplication operators.

3.3. Intertwiners

The algebra $H^{\mathrm{a}}(k,c)$ gives rise to a large supply of nontrivial W^{a}-cocycles. The construction uses the normalized intertwiners associated to $H^{\mathrm{a}}(k,c)$.

Let $H^{\mathrm{a}}(k,c)_{\mathrm{loc}}$ be the localized trigonometric Cherednik algebra obtained by right-adjoining the inverses of $a^{(c)} + k_a$ to $H^{\mathrm{a}}(k,c)$ for all $a \in R^{\mathrm{a}}$. Write $H^{\mathrm{a}}(k,c)_{\mathrm{loc}}^\times$ for the group of units in $H^{\mathrm{a}}(k,c)_{\mathrm{loc}}$.

Proposition 3.8 ([5]). *There exists a unique group homomorphism $W^{\mathrm{a}} \to H^{\mathrm{a}}(k,c)_{\mathrm{loc}}^\times$, denoted by $w \mapsto I_w^{k,c}$, satisfying*

$$I_{s_a}^{k,c} = (s_a \cdot a^{(c)} + k_a) \cdot (a^{(c)} - k_a)^{-1}, \qquad a \in F^{\mathrm{a}},$$
$$I_\omega^{k,c} = \omega, \qquad \omega \in \Omega.$$

Furthermore,

$$I_w^{k,c} \cdot p = w^{(c)}(p) \cdot I_w^{k,c}, \qquad \forall w \in W^{\mathrm{a}}, \forall p \in S(V_{\mathbb{C}}^*),$$

in $H^{\mathrm{a}}(k,c)_{\mathrm{loc}}$.

The $I_w^{k,c}$ ($w \in W^{\mathrm{a}}$) are called the *normalized intertwiners* associated to $H^{\mathrm{a}}(k,c)$. Observe that the $I_w^{k,c}$ for $w \in W$ are independent of the level c.

Consider the set $\mathcal{S}_{k,c}$ consisting of $t \in V_{\mathbb{C}}$ satisfying $a^{(c)}(t) \neq k_a$ for all $a \in R^{\mathrm{a}}$. Note that $S_{k,c}$ is invariant for the action $t \mapsto w^{(c)}(t)$ of W^{a} on $V_{\mathbb{C}}$. For $t \in \mathcal{S}_{k,c}$ consider the character

$$\chi_t : S(V_{\mathbb{C}}^*)_{\mathrm{loc}} \to \mathbb{C}, \qquad p \mapsto p(t),$$

where $S(V_{\mathbb{C}}^*)_{\mathrm{loc}}$ is the localized algebra obtained by adjoining the inverses of a^c+k_a to $S(V_{\mathbb{C}}^*)_{\mathrm{loc}}$ for all $a \in R^{\mathrm{a}}$ (which canonically is a subalgebra of $H^{\mathrm{a}}(k,c)_{\mathrm{loc}}$). The map $w \mapsto w \otimes_{\chi_t} 1$ for $w \in W^{\mathrm{a}}$ gives a vector space identification between the group algebra $\mathbb{C}[W^{\mathrm{a}}]$ and the principal $H^{\mathrm{a}}(k,c)_{\mathrm{loc}}$-module $M(t) := \mathrm{Ind}_{S(V_{\mathbb{C}}^*)_{\mathrm{loc}}}^{H^{\mathrm{a}}(k,c)_{\mathrm{loc}}}(\chi_t)$. For $w \in W^{\mathrm{a}}$ and $t \in \mathcal{S}_{k,c}$ write $I_w^{k,c}(t)$ for the element in the group algebra $\mathbb{C}[W^{\mathrm{a}}]$ associated to $I_w^{k,c} \otimes_{\chi_t} 1 \in M(\lambda)$. In other words, $I_w^{k,c}(t) = \sum_{v \in W^{\mathrm{a}}} p_v^w(t) v$ if the normalized intertwiner $I_w^{k,c}$ expands as $I_w^{k,c} = \sum_{v \in W^{\mathrm{a}}} v \cdot p_v^w$ ($p_v^w \in S(V_{\mathbb{C}}^*)_{\mathrm{loc}}$) in $H^{\mathrm{a}}(k,c)_{\mathrm{loc}}$. The $I_w^{k,c}(t)$ satisfy (and are uniquely characterized by)

$$\begin{aligned} I_{s_a}^{k,c}(t) &= \frac{a^{(c)}(t)s_a + k_a}{a^{(c)}(t) - k_a}, \\ I_{\omega}^{k,c}(t) &= \omega, \\ I_{\sigma\tau}^{k,c}(t) &= I_{\sigma}^{k,c}(\tau^{(c)}(t)) I_{\tau}^{k,c}(t) \end{aligned} \tag{3.7}$$

for $a \in F^{\mathrm{a}}$, $\omega \in \Omega$ and $\sigma, \tau \in W^{\mathrm{a}}$.

Remark 3.9. The cocycle $\{I_w^{k,c}(t)\}_{w \in W^{\mathrm{a}}}$ gives rise to unitary solutions of generalized Yang–Baxter equations with spectral parameters (see [3] for the general theory) and plays a key role in the description of the Bethe ansatz equations associated to quantum spin-particle systems with delta function interactions. For the latter application one is forced to consider the cocycle at critical level $c = 0$. The basic example is for R of type A_{n-1} and $X = \bigoplus_{j=1}^n \mathbb{Z}\epsilon_j$, in which case we have discussed these observations in detail in the introduction. One of the main goals in the present paper is to give similar interpretations of the cocycle $\{I_w^{k,0}(t)\}_{w \in W^{\mathrm{a}}}$ for arbitrary affine Weyl groups W^{a}.

Define

$$i_w^{k,c}(t) = \chi(I_w^{k,c}(t)), \qquad w \in W^{\mathrm{a}},$$

where $\chi : \mathbb{C}[W^{\mathrm{a}}] \to \mathbb{C}$ is the algebra homomorphism mapping w to 1 for all $w \in W^{\mathrm{a}}$. By induction on the length $l(w)$ of $w \in W^{\mathrm{a}}$ we have

$$i_w^{k,c}(t) = \prod_{a \in R^{\mathrm{a},+} \cap w^{-1} R^{\mathrm{a},-}} \frac{a^{(c)}(t) + k_a}{a^{(c)}(t) - k_a}, \qquad w \in W^{\mathrm{a}}. \tag{3.8}$$

In particular,

$$i_w^{k,c}(t) = \prod_{\alpha \in R^+ \cap w^{-1} R^-} \frac{\alpha(t) + k_\alpha}{\alpha(t) - k_\alpha}, \qquad w \in W,$$

which is independent of c. At critical level $c = 0$ we write $I_w^k(t) = I_w^{k,0}(t)$ and $i_w^k(t) = i_w^{k,0}(t)$. We furthermore set $I_y^k(t) = I_{t_y}^k(t)$ and $i_y^k(t) = i_{t_y}^k(t)$ for $y \in Y$. Let $\mathbb{C}[W^{\mathrm{a}}]^\times$ be the group of units of $\mathbb{C}[W^{\mathrm{a}}]$.

Proposition 3.10. *Let $t \in \mathcal{S}_k := \mathcal{S}_{k,0}$.*

(i) *The map $y \mapsto I_y^k(t)$ defines a group homomorphism $Y \to \mathbb{C}[W^{\mathrm{a}}]^\times$.*

(ii) *Suppose that* $\theta(Y)=\mathbb{Z}$. *Then*

$$i_y^k(t)=\prod_{\alpha\in R^+}\left(\frac{\alpha(t)+k_\alpha}{\alpha(t)-k_\alpha}\right)^{\alpha(y)} \tag{3.9}$$

for $y\in Y$.

(iii) *Suppose that* $\theta(Y)=2\mathbb{Z}$. *Then*

$$i_y^k(t)=\prod_{\substack{\alpha\in R^+\\ \alpha\notin W\theta}}\left(\frac{\alpha(t)+k_\alpha}{\alpha(t)-k_\alpha}\right)^{\alpha(y)}\prod_{\substack{\beta\in R^+\\ \beta\in W\theta}}\left(\frac{\beta(t)+k_\theta}{\beta(t)-k_\theta}\right)^{\beta(y)/2}\left(\frac{\beta(t)+k_0}{\beta(t)-k_0}\right)^{\beta(y)/2} \tag{3.10}$$

for $y\in Y$.

Proof. (i). This follows from the cocycle property of $I_w^k(t)$ (the last identity of (3.7)) since translations t_y ($y\in Y$) act trivially under the action $w\mapsto w^{(0)}=Dw$ of W^{a} on $V_{\mathbb{C}}$.

(ii) & (iii). For $y\in Y$ we have

$$R^{\mathrm{a},+}\cap t_y^{-1}R^{\mathrm{a},-}=\{\alpha+m1\mid \alpha\in R,\, B(\alpha)\le m<B(\alpha)+\alpha(y)\}$$

where $B(\alpha)=1$ if $\alpha\in R^-$ and $B(a)=0$ if $\alpha\in R^+$. Using the convention that $\prod_{r=l}^m c_r=1$ if $l>m$ we can thus write

$$i_y^k(t)=\prod_{\alpha\in R^+}\prod_{m=0}^{\alpha(y)-1}\frac{\alpha(t)+k_{\alpha+m1}}{\alpha(t)-k_{\alpha+m1}}\prod_{m=1}^{-\alpha(y)}\frac{\alpha(t)-k_{-\alpha+m1}}{\alpha(t)+k_{-\alpha+m1}}. \tag{3.11}$$

If $\theta(Y)=\mathbb{Z}$ then $k_a=k_{Da}$ for all $a\in R^{\mathrm{a}}$ and (3.11) reduces to (3.9).

If $\theta(Y)=2\mathbb{Z}$ then $k_a=k_{Da}$ for $a\in R^{\mathrm{a}}$ with $Da\notin W\theta$. Furthermore, for $\alpha\in W\theta$ we have $k_{\alpha+(2m)1}=k_\theta$ and $k_{\alpha+(2m+1)1}=k_0$ for $m\in\mathbb{Z}$. Formula (3.11) then becomes (3.10) after straightforward computations. □

4. Representations of the trigonometric Cherednik algebra at critical level

We fix a W^{a}-invariant multiplicity function $k: R^{\mathrm{a}}\to\mathbb{C}$ throughout this section.

4.1. The algebra $A(k)$

In the next subsection we give the representation of the trigonometric Cherednik algebra $H^{\mathrm{a}}(k)$ at critical level in terms of vector-valued Dunkl type operators. To avoid the use of twisted affine root systems it is convenient to work with an adjusted presentation of $H^{\mathrm{a}}(k)$, which we now give first.

Definition 4.1. Let $A(k)$ be the unital associative algebra over $\mathbb{C}$ satisfying:

(i) $A(k)$ contains $S(V_{\mathbb{C}})$ and $\mathbb{C}[W^{\mathrm{a}}]$ as subalgebras.

(ii) The multiplication map defines an isomorphism $S(V_{\mathbb{C}})\otimes_{\mathbb{C}}\mathbb{C}[W^{\mathrm{a}}]\to A(k)$.

(iii) The cross relations

$$s_a \cdot v - s_{Da}(v) \cdot s_a = -k_a Da(v), \qquad \forall a \in F^{\mathrm{a}},$$

hold for all $v \in V_{\mathbb{C}}$.

(iv) $\omega \cdot v = (D\omega)(v) \cdot \omega$ for $\omega \in \Omega$ and $v \in V_{\mathbb{C}}$.

The cross relations (iii) may be replaced by

$$s_a \cdot p - s_{Da}(p) \cdot s_a = k_a \Delta_{Da}(p), \qquad \forall a \in F^{\mathrm{a}}, \forall p \in S(V_{\mathbb{C}}), \tag{4.1}$$

where the divided difference operator $\Delta_\alpha : S(V_{\mathbb{C}}) \to S(V_{\mathbb{C}})$ ($\alpha \in R$) is given by

$$\Delta_\alpha(p) = \frac{s_\alpha(p) - p}{\alpha^\vee}, \qquad p \in S(V_{\mathbb{C}}).$$

Induction on the length of $w \in W^{\mathrm{a}}$ also proves the commutation relations

$$w \cdot v = ((Dw)v) \cdot w - \sum_{a \in R^{\mathrm{a},+} \cap w^{-1} R^{\mathrm{a},-}} k_a (Da)(v) w s_a, \qquad \forall w \in W^{\mathrm{a}}, \forall v \in V, \tag{4.2}$$

in $A(k)$ (cf. (3.2)). The algebra $A(k)$ is the trigonometric Cherednik algebra at critical level, as follows from the following lemma.

Lemma 4.2. *Let $k^\vee$ be the multiplicity function $k^\vee_a = 2k_a/\langle Da, Da\rangle$ ($a \in R^{\mathrm{a}}$). The assignments $v \mapsto \langle v, \cdot\rangle \in V^*_{\mathbb{C}}$ and $w \mapsto w$ for $v \in V_{\mathbb{C}}$ and $w \in W^{\mathrm{a}}$ uniquely extend to a unital algebra isomorphism $A(k^\vee) \to H^{\mathrm{a}}(k)$.*

Proof. Straightforward check. □

It is convenient to alter the notations for the cocycles $I^k_w(t) \in \mathbb{C}[W^{\mathrm{a}}]$ ($w \in W^{\mathrm{a}}$) accordingly. This results in the following definitions and formulas.

Let $C_k \subset V^*_{\mathbb{C}}$ be the W-invariant set of vectors $\lambda \in V^*_{\mathbb{C}}$ satisfying $\lambda(Da^\vee) \neq k_a$ for all $a \in R^{\mathrm{a}}$. For $\lambda \in C_k$ there exists a unique $J^k_w(\lambda) \in \mathbb{C}[W^{\mathrm{a}}]$ ($w \in W^{\mathrm{a}}$) satisfying

$$\begin{aligned} J^k_{s_a}(\lambda) &= \frac{\lambda(Da^\vee)s_a + k_a}{\lambda(Da^\vee) - k_a}, && a \in F^{\mathrm{a}},\\ J^k_\omega(\lambda) &= \omega, && \omega \in \Omega,\\ J^k_{\sigma\tau}(\lambda) &= J^k_\sigma((D\tau)\lambda) J^k_\tau(\lambda), && \sigma, \tau \in W^{\mathrm{a}}. \end{aligned}$$

Setting $j^k_w(\lambda) = \chi(J^k_w(\lambda))$ ($w \in W^{\mathrm{a}}$) and $j^k_y(\lambda) = j^k_{t_y}(\lambda)$ ($y \in Y$), we have

$$j^k_w(\lambda) = \prod_{\alpha \in R^+ \cap w^{-1} R^-} \frac{\lambda(\alpha^\vee) + k_\alpha}{\lambda(\alpha^\vee) - k_\alpha}, \qquad w \in W, \tag{4.3}$$

and

$$j^k_y(\lambda) = \prod_{\substack{\alpha \in R^+ \\ \alpha \notin W\theta}} \left(\frac{\lambda(\alpha^\vee) + k_\alpha}{\lambda(\alpha^\vee) - k_\alpha}\right)^{\alpha(y)} \prod_{\substack{\beta \in R^+ \\ \beta \in W\theta}} \left(\frac{\lambda(\beta^\vee) + k_\theta}{\lambda(\beta^\vee) - k_\theta}\right)^{\beta(y)/2} \left(\frac{\lambda(\beta^\vee) + k_0}{\lambda(\beta^\vee) - k_0}\right)^{\beta(y)/2} \tag{4.4}$$

for $y \in Y$. If $\theta(Y) = \mathbb{Z}$ then the latter formula simplifies to

$$j_y^k(\lambda) = \prod_{\alpha \in R^+} \left(\frac{\lambda(\alpha^\vee) + k_\alpha}{\lambda(\alpha^\vee) - k_\alpha} \right)^{\alpha(y)}, \qquad y \in Y.$$

4.2. The Dunkl type operators

For a complex associative algebra A we write Mod_A for the category of complex left A-modules. The category $\mathrm{Mod}_{\mathbb{C}[W^{\mathrm{a}}]}$ is a tensor category with unit object the trivial W^{a}-module, which we denote by $\mathbb{I}$. In this subsection we define an explicit functor $F_{\mathrm{dr}}^k : \mathrm{Mod}_{\mathbb{C}[W^{\mathrm{a}}]} \to \mathrm{Mod}_{A(k)}$ using vector-valued Dunkl type operators (the subscript "dr" stands for "differential-reflection"). It extends results from the paper [9], in which the $A(k)$-module $F_{\mathrm{dr}}^k(\mathbb{I})$ was constructed. The Dunkl type differential-reflection operators will have the special feature that at the chamber $w^{-1}C_+$ ($w \in W^{\mathrm{a}}$) the number of occurring reflection terms is equal to the "distance" $l(w)$ of $w^{-1}C_+$ to the fundamental chamber C_+. Special cases and other examples of such differential-reflection operators were considered in [32, 27, 20, 19, 9].

Recall that $V_a = a^{-1}(0)$ is the affine root hyperplane of the affine root $a \in R^{\mathrm{a}}$. Write $V_{\mathrm{reg}} = V \setminus \bigcup_{a \in R^{\mathrm{a},+}} V_a$ for the set of regular elements in V. It is well known that

$$V_{\mathrm{reg}} = \bigcup_{w \in W \ltimes Q^\vee} w(C_+)$$

(disjoint union), with $C_+ \subset V_{\mathrm{reg}}$ given by

$$C_+ = \{v \in V \mid a(v) > 0, \, \forall a \in F^{\mathrm{a}}\}.$$

Furthermore, the subgroup Ω of length zero elements in W^{a} permutes F^{a}, hence it acts on C_+. In particular, W^{a} permutes the connected components $\mathcal{C} = \{w(C_+) \mid w \in W \ltimes Q^\vee\}$ of V_{reg}. We call $C \in \mathcal{C}$ a *chamber*, and C_+ the *fundamental chamber*.

Let $C^\omega(V)$ be the space of complex-valued, real analytic functions on V. For a complex left W^{a}-module M we now define a suitable space of M-valued functions on V which are real analytic on V_{reg}, but which are "fuzzy" on the affine root hyperplanes, in the sense that we do not specify their values on the affine root hyperplanes (cf. Remark 4.4(ii)).

Definition 4.3. Let M be a complex left W^{a}-module. We write $B^\omega(V; M)$ for the complex vector space of functions $f : V_{\mathrm{reg}} \to M$ satisfying, for all $C \in \mathcal{C}$, $f|_C = f_C|_C$ for some $f_C \in C^\omega(V) \otimes_{\mathbb{C}} M$ (algebraic tensor product).

Remark 4.4. (i) The map $f \mapsto (f_C)_{C \in \mathcal{C}}$ defines a complex linear isomorphism $B^\omega(V; M) \to \prod_{C \in \mathcal{C}} (C^\omega(V) \otimes M)$. We will use this identification without further reference.

(ii) A function $f \in B^\omega(V; M)$ can be interpreted as a multi M-valued function on V by defining

$$f(v) = \{f_C(v)\}_{C \in \mathcal{C}: \, v \in \overline{C}}$$

for any $v \in V$, where $\overline{C}$ is the closure of the chamber C in the Euclidean space V.

The space $B^\omega(V;M)$ is a W^{a}-module by

$$(w\cdot f)(v)=w(f(w^{-1}(v))),\qquad w\in W^{\mathrm{a}},\ f\in B^\omega(V;M),\ v\in V_{\mathrm{reg}}.$$

We denote the action by a dot to avoid confusion with the W^{a}-action on M. Viewing the f_C's as M-valued functions on V, the action can be expressed as $(w\cdot f)_C(v)=w(f_{w^{-1}C}(w^{-1}v))$ for $w\in W^{\mathrm{a}}$, $C\in\mathcal{C}$ and $v\in V$. Alternatively it can be expressed as $(w\cdot f)_C=(w\otimes w)f_{w^{-1}C}$, viewed as an identity in $C^\omega(V)\otimes_{\mathbb{C}}M$. Here we use the natural W^{a}-action on $C^\omega(V)$, given by $(wg)(v):=g(w^{-1}v)$ for $g\in C^\omega(V)$, $w\in W^{\mathrm{a}}$ and $v\in V$.

Let $\mathcal{I}:\mathbb{R}^\times\to\{0,1\}$ be the indicator function of $\mathbb{R}_{<0}$ and write $\mathcal{I}_a(v):=\mathcal{I}(a(v))$ for $a\in R^{\mathrm{a},+}$ and $v\in V_{\mathrm{reg}}$. Then

$$\mathcal{I}_a|_{w^{-1}C_+}\equiv\begin{cases}1 & \text{if } wa\in R^{\mathrm{a},-},\\ 0 & \text{if } wa\in R^{\mathrm{a},+}.\end{cases}$$

In particular, for $w\in W^{\mathrm{a}}$ we have $\mathcal{I}_a|_{w^{-1}C_+}\equiv 1$ only if a is a positive affine root from the finite set $R^{\mathrm{a},+}\cap w^{-1}R^{\mathrm{a},-}$. This ensures that the Dunkl type operator

$$\mathcal{D}_v^{k,M}f=\partial_v f-\sum_{a\in R^{\mathrm{a},+}}k_a(Da)(v)\mathcal{I}_a(\cdot)(s_a\cdot f),\qquad f\in B^\omega(V;M),\tag{4.5}$$

for $v\in V$ defines a well defined linear operator on $B^\omega(V;M)$, where

$$(\partial_v f)(v')=\frac{d}{dt}\bigg|_{t=0}f(v'+tv)$$

for $v'\in V_{\mathrm{reg}}$ is the directional derivative of f in the direction $v\in V$. Indeed, on a fixed chamber $w^{-1}C_+\in\mathcal{C}$ ($w\in W^{\mathrm{a}}$) the formula (4.5) for the Dunkl operator gives

$$(\mathcal{D}_v^{k,M}f)_{w^{-1}C_+}=\partial_v f_{w^{-1}C_+}-\sum_{a\in R^{\mathrm{a},+}\cap w^{-1}R^{\mathrm{a},-}}k_a(Da)(v)(s_a\otimes s_a)f_{s_aw^{-1}C_+}\tag{4.6}$$

as an identity in $C^\omega(V)\otimes_{\mathbb{C}}M$.

Theorem 4.5. *Let M be a W^{a}-module and k a W^{a}-invariant multiplicity function on R^{a}. The assignments*

$$v\mapsto\mathcal{D}_v^{k,M},\qquad v\in V,$$
$$w\mapsto w\,\cdot,\qquad w\in W^{\mathrm{a}},$$

uniquely extend to an algebra homomorphism $\pi_{k,M}:A(k)\to\mathrm{End}_{\mathbb{C}}(B^\omega(V;M))$.

Proof. We apply Lemma 3.5 (with adjusted notations and specialized parameters) to $B^\omega(V;M)$, considered as a W^{a}-module by the dot-action.

A direct computation (cf. [9, Thm. 4.1]) shows that the $\mathcal{D}_v^{k,M}$ ($v\in V$) satisfy the $A(k)$ type cross relations with respect to the dot-action of W^{a} on $B^\omega(V;M)$. It thus remains to construct an appropriate complex vector space N and a linear map $p:B^\omega(V;M)\to N$ satisfying the conditions of Lemma 3.5. We take $N=$

$C^\omega(V)\otimes_{\mathbb{C}} M$ and p the linear map $p((f_C)_{C\in\mathcal{C}}) = f_{C_+}$. Clearly the only W^{a}-submodule of $B^\omega(V;M)$ that is contained in $\ker(p)$ is $\{0\}$. Furthermore,

$$p(\mathcal{D}_v^{k,M} f) = \partial_v p(f), \qquad \forall f\in B^\omega(V;M),$$

by (4.6), hence the image of $[\mathcal{D}_v^{k,M}, \mathcal{D}_{v'}^{k,M}]$ is contained in $\ker(p)$. Thus Lemma 3.5 can be applied. It yields the desired result. □

Remark 4.6. (i) The theorem reduces to [9, Thm. 4.2] when $Y = Q^\vee$, $k_0 = k_\theta$ and $M = \mathbb{I}$.

(ii) The representation $\pi_{k,\mathbb{I}}$ is faithful (cf. the proof of [9, Thm. 4.2]).

Corollary 4.7. (i) *The assignment $M \mapsto B^\omega(V;M)$ defines a covariant functor $F^k_{\mathrm{dr}} : \mathrm{Mod}_{\mathbb{C}[W^{\mathrm{a}}]} \to \mathrm{Mod}_{A(k)}$ (with the obvious definition on morphisms).*

(ii) *For $p\in S(V_{\mathbb{C}})^W$ we have $\pi_{k,M}(p) = p(\partial)\otimes \mathrm{Id}_M$, where $p(\partial)$ is the constant-coefficient differential operator associated to p.*

Proof. (i) Clear.

(ii) This is analogous to the proof of [9, Cor. 4.6]. □

4.3. Integral-reflection operators

In this subsection we define another explicit functor $F^k_{\mathrm{ir}} : \mathrm{Mod}_{\mathbb{C}[W^{\mathrm{a}}]} \to \mathrm{Mod}_{A(k)}$ using integral-reflection operators [16, 13] (the suscript "ir" stands for "integral-reflection"). It again extends results from the paper [9], in which the $A(k)$-module $F^k_{\mathrm{ir}}(\mathbb{I})$ was constructed.

Let M be a complex left W^{a}-module. The linear dual $M^* = \mathrm{Hom}_{\mathbb{C}}(M;\mathbb{C})$ is a left W^{a}-module by $(w\psi)(m) = \psi(w^{-1}m)$ for $w\in W^{\mathrm{a}}$, $\psi\in M^*$ and $m\in M$. Consider the $A(k)$-module $\mathrm{Ind}_{\mathbb{C}[W^{\mathrm{a}}]}^{A(k)}(M^*)$. As complex vector spaces, we have

$$\mathrm{Ind}_{\mathbb{C}[W^{\mathrm{a}}]}^{A(k)}(M^*) \simeq S(V_{\mathbb{C}})\otimes_{\mathbb{C}} M^*.$$

Expressing the action through the linear isomorphism we get the following explicit $A(k)$-action on $S(V_{\mathbb{C}})\otimes_{\mathbb{C}} M^*$:

$$\begin{aligned} s_a(p\otimes\psi) &= s_{Da}(p)\otimes s_a\psi + k_a\Delta_{Da}(p)\otimes\psi, && a\in F^{\mathrm{a}},\\ \omega(p\otimes\psi) &= (D\omega)(p)\otimes\omega\psi, && \omega\in\Omega,\\ r(p\otimes\psi) &= (rp)\otimes\psi, && r\in S(V_{\mathbb{C}}),\end{aligned}\tag{4.7}$$

for $p\in S(V_{\mathbb{C}})$ and $\psi\in M^*$. We now endow the linear dual $(S(V_{\mathbb{C}})\otimes M^*)^*$ with the structure of left $A(k)$-module using the following simple lemma.

Lemma 4.8. *For a complex left $A(k)$-module N, the linear dual N^* is a left $A(k)$-module by*

$$(X\psi)(n) = \psi(X^\dagger n), \qquad \psi\in N^*, X\in A(k), n\in N,$$

where $X\mapsto X^\dagger$ is the unique unital complex linear anti-algebra involution of $A(k)$ satisfying $w^\dagger = w^{-1}$ ($w\in W^{\mathrm{a}}$) and $v^\dagger = v$ ($v\in V$).

Next we rewrite the $A(k)$-action on a suitable subspace of $(S(V^*) \otimes_{\mathbb{C}} M^*)^*$ in terms of integral-reflection operators.

For $a \in R^{\mathrm{a}}$ we define the integral operator $I(a)$ on $C^{\omega}(V)$ by

$$(I(a)f)(v) = \int_0^{a(v)} f(v - tDa^{\vee})\,dt, \qquad f \in C^{\omega}(V),\ v \in V. \tag{4.8}$$

With respect to the natural action $(wf)(v) = f(w^{-1}(v))$ of $w \in W^{\mathrm{a}}$ on $f \in C^{\omega}(V)$ the integral operators satisfy

$$wI(a)w^{-1} = I(w(a)), \qquad \forall w \in W^{\mathrm{a}}, \forall a \in R^{\mathrm{a}}. \tag{4.9}$$

The integral operators $I(\alpha)$ are adjoint to the divided difference operator $-\Delta_\alpha$ $(\alpha \in R)$ in the following sense.

Lemma 4.9 ([14]). *Let $(\cdot,\cdot) : S(V_{\mathbb{C}}) \times C^{\omega}(V) \to \mathbb{C}$ be the nondegenerate complex bilinear form defined by*

$$(p, f) = (p(\partial)f)(0), \qquad p \in S(V_{\mathbb{C}}),\ f \in C^{\omega}(V).$$

Then

$$(\Delta_\alpha(p), f) = -(p, I(\alpha)f)$$

for $\alpha \in R$, $p \in S(V_{\mathbb{C}})$ and $f \in C^{\omega}(V)$.

We obtain the following immediate consequence.

Corollary 4.10. *Let M be a left W^{a}-module. Consider $C^{\omega}(V) \otimes_{\mathbb{C}} M$ as a linear subspace of $(S(V_{\mathbb{C}}) \otimes_{\mathbb{C}} M^*)^*$ by interpreting $f \otimes m \in C^{\omega}(V) \otimes_{\mathbb{C}} M$ as the linear functional*

$$p \otimes \psi \mapsto (p, f)\psi(m), \qquad p \in S(V_{\mathbb{C}}),\ \psi \in M^*.$$

Then $C^{\omega}(V) \otimes_{\mathbb{C}} M$ is an $A(k)$-submodule of $(S(V_{\mathbb{C}}) \otimes_{\mathbb{C}} M^*)^*$. The corresponding left $A(k)$-action on $C^{\omega}(V) \otimes_{\mathbb{C}} M$ is explicitly given by

$$\begin{aligned} r &\mapsto r(\partial) \otimes \mathrm{Id}_M, && r \in S(V_{\mathbb{C}}),\\ s_a &\mapsto s_{Da} \otimes s_a - k_a I(Da) \otimes \mathrm{Id}_M, && a \in F^{\mathrm{a}},\\ \omega &\mapsto D\omega \otimes \omega, && \omega \in \Omega. \end{aligned}$$

Proof. Chasing the actions, the proof easily reduces to Lemma 4.9 and the obvious identities

$$\begin{aligned} (rp, f) &= (p, r(\partial)f), && r \in S(V_{\mathbb{C}}),\\ (w(p), f) &= (p, w^{-1}(f)), && w \in W, \end{aligned}$$

for $p \in S(V_{\mathbb{C}})$ and $f \in C^{\omega}(V)$. $\square$

Note that the action of W^{a} on the first tensor leg $C^{\omega}(V)$ of the $A(k)$-module $C^{\omega}(V) \otimes_{\mathbb{C}} M$ is the pull-back action of W under the gradient map D (the commutative subgroup Y of W^{a} acts trivially). We now upgrade it to the standard action of W^{a} on $C^{\omega}(V)$.

Theorem 4.11. *Let M be a complex left W^{a}-module and k a multiplicity function on R^{a}. The assignments*

$$\begin{aligned} v &\mapsto \partial_v \otimes \mathrm{Id}_M, && v \in V,\\ s_a &\mapsto s_a \otimes s_a - k_a I(a) \otimes \mathrm{Id}_M, && a \in F^{\mathrm{a}},\\ \omega &\mapsto \omega \otimes \omega, && \omega \in \Omega, \end{aligned}$$

uniquely extend to an algebra homomorphism $Q_{k,M} : A(k) \to \mathrm{End}_{\mathbb{C}}(C^{\omega}(V) \otimes_{\mathbb{C}} M)$.

Proof. Write $Q : A(k) \to \mathrm{End}_{\mathbb{C}}(C^{\omega}(V) \otimes_{\mathbb{C}} M)$ for the representation map associated to the $A(k)$-action of Corollary 4.10. Since the explicit assignment $Q_{k,M}$ in the statement of the theorem satisfies $Q_{k,M}(v) = Q(v)$ and $Q_{k,M}(s_\alpha) = Q(s_\alpha)$ for $v \in V$ and $\alpha \in F$, it remains to verify the following identities:

(i) $Q_{k,M}(s_0)^2 = \mathrm{Id}$.

(ii) If $(s_\alpha s_0)^m = 1$ in W^{a} for some $\alpha \in F$ and $m \in \mathbb{N}$, then

$$(Q_{k,M}(s_\alpha)Q_{k,M}(s_0))^m = \mathrm{Id}.$$

(iii) $Q_{k,M}(\omega)Q_{k,M}(s_a) = Q_{k,M}(s_{\omega(a)})Q_{k,M}(\omega)$ for $a \in F^{\mathrm{a}}$ and $\omega \in \Omega$.

(iv) The cross relations

$$Q_{k,M}(s_0)(\partial_v \otimes \mathrm{Id}_M) - (\partial_{s_\theta(v)} \otimes \mathrm{Id}_M)Q_{k,M}(s_0) = k_0\theta(v)$$

hold for $v \in V$.

(v) $Q_{k,M}(\omega)(\partial_v \otimes \mathrm{Id}_M) = (\partial_{D\omega(v)} \otimes \mathrm{Id}_M)Q_{k,M}(\omega)$ for $\omega \in \Omega$ and $v \in V$.

The identities (i)–(iii) show that $Q_{k,M}$ defines a W^{a}-action on $C^{\omega}(V) \otimes_{\mathbb{C}} M$. The identities (iv) and (v) ensure that also all the necessary cross relations are satisfied in order for $Q_{k,M}$ to extend to an algebra homomorphism $Q_{k,M} : A(k) \to \mathrm{End}_{\mathbb{C}}(C^{\omega}(V) \otimes_{\mathbb{C}} M)$.

The proofs of (i)–(v) are much facilitated by the simple observation that

$$Q_{k,M}(s_0) = (t_u \otimes \mathrm{Id}_M)Q(s_0)(t_u^{-1} \otimes \mathrm{Id}_M) \tag{4.10}$$

for any $u \in V$ such that $\theta(u) = 1$ (in which case $t_u(-\theta) = -\theta + 1 = a_0$). By (4.10) and Corollary 4.10 the identities (i), (iv) and (v) are immediate. For the braid relation (ii), observe that $(s_\alpha s_0)^m = 1$ in W^{a} for some $\alpha \in F$ and $m \in \mathbb{N}$ implies that α is not a scalar multiple of θ in V^*. Hence there exists a vector $u \in V$ such that $\alpha(u) = 0$ and $\theta(u) = 1$. In this case we have, besides (4.10),

$$Q_{k,M}(s_\alpha) = (t_u \otimes \mathrm{Id}_M)Q(s_\alpha)(t_u^{-1} \otimes \mathrm{Id}_M).$$

Hence $(Q_{k,M}(s_\alpha)Q_{k,M}(s_0))^m = \mathrm{Id}$ follows by conjugating the corresponding valid identity for Q by $t_u \otimes \mathrm{Id}_M$.

The identities (iii) can be checked by a direct computation. □

Remark 4.12. (i) For $k_0 = k_\theta$, $Y = Q^\vee$ and $M = \mathbb{I}$, Theorem 4.11 reduces to [9, Thm. 5.2].

(ii) The assignment $M \mapsto C^{\omega}(V) \otimes_{\mathbb{C}} M$ defines a covariant functor $F^k_{\mathrm{ir}} : \mathrm{Mod}_{\mathbb{C}[W^{\mathrm{a}}]} \to \mathrm{Mod}_{A(k)}$ (with the obvious definition on morphisms).

Let $\mathrm{Forg}^k : \mathrm{Mod}_{A(k)} \to \mathrm{Mod}_{\mathbb{C}[W^{\mathrm{a}}]}$ be the forgetful functor. In the following proposition we compare the space $F_{\mathrm{ir}}^k(M)^{W^{\mathrm{a}}}$ of W^{a}-invariants of the W^{a}-module $\mathrm{Forg}^k(F_{\mathrm{ir}}^k(M))$ with the space $(F_{\mathrm{ir}}^k(\mathbb{I}) \otimes M)^{W^{\mathrm{a}}}$ of W^{a}-invariants of the tensor product W^{a}-module $\mathrm{Forg}^k(F_{\mathrm{ir}}^k(\mathbb{I})) \otimes M$:

Proposition 4.13. *Let M be a left W^{a}-module. Then*

$$F_{\mathrm{ir}}^k(M)^{W^{\mathrm{a}}} = (F_{\mathrm{ir}}^k(\mathbb{I}) \otimes M)^{W^{\mathrm{a}}}.$$

Proof. It is convenient to set up some notation first. We write π_M for the representation map of M. The space $C^\omega(V)$ will be considered with respect to two different left W^{a}-actions. The first is the regular W^{a}-action

$$(L(w)g)(v) = g(w^{-1}v), \qquad g \in C^\omega(V),\, v \in V,\, w \in W^{\mathrm{a}}. \tag{4.11}$$

The resulting W^{a}-module will still be denoted by $C^\omega(V)$. The other W^{a}-action is $Q_{k,\mathbb{I}}|_{W^{\mathrm{a}}}$, in which case we denote the W^{a}-module by $\mathrm{Forg}^k(F_{\mathrm{ir}}^k(\mathbb{I}))$. We omit the forgetful functor from the notation from now on.

Observe that both W^{a}-modules $F_{\mathrm{ir}}^k(M)$ and $F_{\mathrm{ir}}^k(\mathbb{I}) \otimes M$ have the same underlying vector space $C^\omega(V) \otimes_{\mathbb{C}} M$. Their subspaces of Ω-invariants coincide trivially.

Fix $a \in F^{\mathrm{a}}$. For a left W^{a}-module N we write N_+ (respectively N_-) for the space of s_a-invariants (respectively s_a-antiinvariants) in N. It suffices to show that

$$F_{\mathrm{ir}}^k(M)_+ = (F_{\mathrm{ir}}^k(\mathbb{I}) \otimes M)_+. \tag{4.12}$$

By Theorem 4.11 we have

$$Q_{k,M}(s_a) = Q_{k,\mathbb{I}}(s_a) \otimes \pi_M(s_a) + R_M(s_a)$$

with $R_M(s_a) = k_a I(a) \otimes (\pi_M(s_a) - \mathrm{Id}_M)$. We claim that

$$\begin{aligned} &R_M(s_a)((F_{\mathrm{ir}}^k(\mathbb{I}) \otimes M)_+) = \{0\}, \\ &R_M(s_a)((F_{\mathrm{ir}}^k(\mathbb{I}) \otimes M)_-) \subseteq (F_{\mathrm{ir}}^k(\mathbb{I}) \otimes M)_+. \end{aligned} \tag{4.13}$$

To prove (4.13), we first note that

$$I(a)(C^\omega(V)_-) = \{0\}, \qquad I(a)(C^\omega(V)_+) \subseteq C^\omega(V)_-$$

(see, e.g., [9, Lemma 3.4] for the first part). This lifts to the (anti)invariants with respect to the k-dependent actions,

$$I(a)(F_{\mathrm{ir}}^k(\mathbb{I})_-) = \{0\}, \qquad I(a)(F_{\mathrm{ir}}^k(\mathbb{I})_+) \subseteq F_{\mathrm{ir}}^k(\mathbb{I})_-,$$

since $Q_{k,\mathbb{I}}(s_a)g = \pm g$ implies $k_a I(a) g = (L(s_a) \mp \mathrm{Id})g$. This in turn implies (4.13).

We are now ready to prove (4.12). The inclusion $\supseteq$ of (4.12) is an immediate consequence of (4.13). For the converse inclusion we take $f \in F_{\mathrm{ir}}^k(M)_+$ and write $f = f_+ + f_-$ with $f_\pm \in (F_{\mathrm{ir}}^k(\mathbb{I}) \otimes M)_\pm$. By (4.13) we have

$$f = Q_{k,M}(s_a)f = (f_+ + R_M(s_a)f_-) - f_-,$$

hence $f_+ = f_+ + R_M(s_a)f_-$ and $f_- = -f_-$, again by (4.13). We conclude that $f_- = 0$ and $f = f_+ \in (F_{\mathrm{ir}}^k(\mathbb{I}) \otimes M)_+$. □

4.4. The propagation transformation

Let M be a left W^{a}-module. The W^{a}-action on $C^\omega(V)\otimes_{\mathbb{C}} M$ in terms of integral-reflection operators (Theorem 4.11) can be used to propagate a plane wave attached to the fundamental chamber C_+ to a common eigenfunction of the Dunkl operators $\mathcal{D}_v^{k,M}$ ($v\in V$). This idea goes back to Gutkin and Sutherland [16, 13] and was further explored in [9, §5] for $M=\mathbb{I}$. In the present set-up it will give rise to a natural transformation $T^k: F_{\mathrm{ir}}^k\to F_{\mathrm{dr}}^k$ between the two functors $F_{\mathrm{ir}}^k, F_{\mathrm{dr}}^k:\mathrm{Mod}_{\mathbb{C}[W^{\mathrm{a}}]}\to \mathrm{Mod}_{A(k)}$ (see Remark 4.12(ii) and Corollary 4.7(i)).

We now first define $T^{k,M}$ for a given W^{a}-module M as a complex linear map.

Lemma 4.14. *Let M be a left W^{a}-module. There exists a unique linear map*

$$T^{k,M}: C^\omega(V)\otimes_{\mathbb{C}} M\to B^\omega(V;M)$$

satisfying $(T^{k,M}f)_{C_+}=f$ and $T^{k,M}(Q_{k,M}(w)f)=w\cdot(T^{k,M}f)$ for all $w\in W^{\mathrm{a}}$.

Proof. The required properties of $T^{k,M}$ can equivalently be formulated as

$$(T^{k,M}f)_{w^{-1}C_+}=(w^{-1}\otimes w^{-1})(Q_{k,M}(w)f),\qquad \forall w\in W^{\mathrm{a}},$$

in $C^\omega(V)\otimes_{\mathbb{C}} M$. The existence and uniqueness are now immediate. □

By construction the $T^{k,M}$ defines a natural transformation

$$T^k:\mathrm{Forg}^k\circ F_{\mathrm{ir}}^k\to \mathrm{Forg}^k\circ F_{\mathrm{dr}}^k.$$

We call T^k the *propagation transformation.*

Proposition 4.15. *T^k defines a natural transformation $T^k: F_{\mathrm{ir}}^k\to F_{\mathrm{dr}}^k$.*

Proof. It suffices to show that

$$T^{k,M}\circ\partial_v=\mathcal{D}_v^{k,M}\circ T^{k,M}$$

for all $v\in V$. This follows from (4.2) by a direct computation. □

For M a W^{a}-module the corresponding morphism $T^{k,M}: F_{\mathrm{ir}}^k(M)\to F_{\mathrm{dr}}^k(M)$ in $\mathrm{Mod}_{A(k)}$ is a monomorphism. We denote its image by $C^{\omega,k}(V;M)$ and write $F_{\mathrm{dr}}^{\omega,k}:\mathrm{Mod}_{\mathbb{C}[W^{\mathrm{a}}]}\to\mathrm{Mod}_{A(k)}$ for the associated functor. On objects it is given by $F_{\mathrm{dr}}^{\omega,k}(M)=C^{\omega,k}(V;M)$. The propagation transformation T^k defines an equivalence $T^k: F_{\mathrm{ir}}^k\xrightarrow{\sim} F_{\mathrm{dr}}^{\omega,k}$ between the two functors $F_{\mathrm{ir}}^k, F_{\mathrm{dr}}^{\omega,k}:\mathrm{Mod}_{\mathbb{C}[W^{\mathrm{a}}]}\to\mathrm{Mod}_{A(k)}$.

We now characterize $C^{\omega,k}(V;M)$ as a subspace of $B^\omega(V;M)$ in terms of derivative jump conditions over affine root hyperplanes.

For a chamber $C=wC_+$ ($w\in W^{\mathrm{a}}$) we write $R_C^a=\{wa\mid a\in F^{\mathrm{a}}\}$. The $V_b\cap\overline{C}$ ($b\in R_C^a$) are the walls of the chamber C. In particular, if $b\in R_C^a$ then C and s_bC are adjacent chambers with common wall $V_b\cap\overline{C}$, and $Db^\vee$ is a vector normal to $V_b\cap\overline{C}$ which points towards C.

Proposition 4.16. *Let M be a W^{a}-module. Then $C^{\omega,k}(V;M)$ is the space of functions $f \in B^{\omega}(V;M)$ satisfying, for all $C \in \mathcal{C}$, $b \in R^{\mathrm{a}}_C$ and $v \in V_b \cap \overline{C}$,*

$$(p(\partial)f_C)(v) - (p(\partial)f_{s_bC})(v) = k_b s_b[((\Delta_{Db}p)(\partial)f_C)(v)] \tag{4.14}$$

for all $p \in S(V_{\mathbb{C}})$. Here we use the s_b-action on M on the right hand side of (4.14).

Proof. The proof is essentially the same as the proof of [9, Thm. 5.3], which deals with the case $Y = Q^{\vee}$, $k_0 = k_\theta$ and $M = \mathbb{I}$. We thus only give a sketch of the proof in the present set-up.

We call a vector $v \in V$ *subregular* if it lies on exactly one affine root hyperplane V_a ($a \in R^{\mathrm{a},+}$). The $Q_{k,M}$-image of the cross relations (4.1) imply that $f \in C^{\omega,k}(V;M)$ satisfies the jump conditions (4.14) for subregular $v \in V_b \cap \overline{C}$. By continuity they then hold for all $v \in V_b \cap \overline{C}$.

Suppose on the other hand that $f \in B^{\omega}(V;M)$ satisfies the jump conditions (4.14) for all $p \in S(V_{\mathbb{C}})$. For $p = 1$ the jump conditions (4.14) are an elaborate way of saying that f is a continuous M-valued function on V (cf. Remark 4.4(ii)). The jump conditions (4.14) also imply that f satisfies normal derivative jump conditions of all orders over the walls (see Remark 4.18). Since the f_C's are real analytic it follows that f is uniquely determined by its restriction to the fundamental chamber C_+. Hence $T^{k,M}(f_{C_+}) = f \in C^{\omega,k}(V;M)$. □

Corollary 4.17. *If $f \in C^{\omega,k}(V;M)$ then for all $C \in \mathcal{C}$, $b \in R^{a}_C$ and $v \in V_b \cap \overline{C}$,*

$$(p(\partial)f_C)(v) = (p(\partial)f_{s_bC})(v)$$

for all $p = \sum_{m \geq 0}(Db^{\vee})^{2m}p_m \in S(V_{\mathbb{C}})$ with $p_m \in S(V_{Db})_{\mathbb{C}}$.

Remark 4.18. (i) On the right hand side of the jump conditions (4.14) we may replace f_C by f_{s_bC} (or by $\frac{1}{2}(f_C + f_{s_bC})$).

(ii) Take $C \in \mathcal{C}$, $b \in R^{a}_C$ and $v \in V_b \cap \overline{C}$. Recall that the vector $Db^{\vee}$ is normal to the wall $V_b \cap \overline{C}$ of C and points towards the chamber C. Then (4.14) for $p = (Db^{\vee})^r$ ($r \in \mathbb{N}$) becomes the following rth order normal derivative jump condition of f over the wall $V_b \cap \overline{C}$ at v:

$$(\partial^r_{Db^{\vee}} f_C)(v) - (\partial^r_{Db^{\vee}} f_{s_bC})(v) = ((-1)^r - 1)k_b s_b[(\partial^{r-1}_{Db^{\vee}} f_C)(v)]. \tag{4.15}$$

For $r = 1$ it reduces to

$$(\partial_{Db^{\vee}} f_C)(v) - (\partial_{Db^{\vee}} f_{s_bC})(v) = -2k_b s_b(f_C(v)). \tag{4.16}$$

Continuity and the higher order normal derivative jump conditions (4.15) suffice to characterize $C^{\omega}(V;M)$ as a subspace of $B^{\omega}(V;M)$ (see the proof of Proposition 4.16).

4.5. Relation to quantum many-body problems

In this subsection we fix a left W^{a}-module M. By Corollary 4.7(ii) the constant coefficient differential operators $\pi_{k,M}(p) = p(\partial)$ ($p \in S(V_{\mathbb{C}})^W$) act on $C^{\omega,k}(V;M)$. They can be interpreted as quantum conserved integrals of a quantum system with delta function potentials as follows.

Denote $C_c^\infty(V)$ for the smooth, compactly supported, complex-valued functions on V. Write dv for the Euclidean volume measure on V. We also write dv for the induced volume measure on the affine root hyperplanes V_b ($b \in R^{\mathrm{a},+}$). We have a linear map $\iota : B^\omega(V;M) \to \mathrm{Hom}_{\mathbb{C}}(C_c^\infty(V);M)$ defined by

$$(\iota f)(\phi) := \int_V f(v)\phi(v)\,dv, \qquad f \in B^\omega(V;M),\ \phi \in C_c^\infty(V), \tag{4.17}$$

where f is interpreted as a multi M-valued function on V (see Remark 4.4(ii)). This in particular allows us to view $\pi_{k,M}(p)f$ for $f \in B^\omega(V;M)$ and for $p \in S(V_{\mathbb{C}})$ as the M-valued distribution $\iota(\pi_{k,M}(p)f)$. We occasionally omit ι if it is clear that the weak interpretation is meant.

Write $C(V;M)$ for the space of continuous M-valued functions on V; it consists of functions $f : V \to M$ such that for all $v \in V$ there exists an open neighborhood U of v in V such that $f|_U \in C(U) \otimes_{\mathbb{C}} M$. Note that $C^{\omega,k}(V;M) \subset CB^\omega(V;M)$, where $CB^\omega(V;M) := C(V;M) \cap B^\omega(V;M)$. Write $\|\cdot\|$ for the norm on the Euclidean space V. We write

$$\mathcal{H}_k^M = -\Delta - \sum_{b \in R^a} \frac{k_b}{\|Db^\vee\|}\delta(b(\cdot))s_b \tag{4.18}$$

for the linear map $\mathcal{H}_k^M : C(V;M) \to \mathrm{Hom}_{\mathbb{C}}(C_c^\infty(V);M)$ defined by

$$(\mathcal{H}_k^M f)(\phi) = -\int_V f(v)(\Delta\phi)(v)\,dv - \sum_{b \in R^{\mathrm{a}}} \frac{k_b}{\|Db^\vee\|}\int_{V_b} s_b(f(v))\phi(v)\,dv$$

for $f \in C(V;M)$ and $\phi \in C_c^\infty(V)$, where s_b only acts on M.

We also write $\|\cdot\|$ for the norm of the Euclidean space V^* and interpret the W-invariant polynomial $\|\cdot\|^2$ on V^* as an element in $S(V_{\mathbb{C}})^W$ in the usual way. Observe that $\|\partial\|^2$ is the Laplacian Δ on V. Note furthermore that $\pi_{k,M}(\|\cdot\|^2) = \Delta$ by Corollary 4.7(ii). The following result directly implies a reformulation of the spectral problem for $\mathcal{H}_k^M$ as a boundary value problem (cf. Subsection 5.1).

Proposition 4.19. *Let M be a left W^{a}-module and $f \in CB^\omega(V;M)$. Then*

$$-\iota(\Delta f) = \mathcal{H}_k^M f$$

if and only if f satisfies the derivative jump conditions (4.14) *for $p \in S(V_{\mathbb{C}})$ of degree one.*

Proof. Let $\phi \in C_c^\infty(V)$ and $f \in CB^\omega(V;M)$. Then

$$\iota(\Delta f)(\phi) = \sum_{C \in \mathcal{C}} \int_C (\Delta f)(v)\phi(v)\,dv$$

(only finitely many terms contribute to the sum). Green's identity allows us to rewrite the right hand side as

$$\int_V f(v)(\Delta\phi)(v)\,dv + \sum_L \int_L ((\partial_{n^L} f_{C_{n^L}})(v) - (\partial_{n^L} f_{C'_{n^L}})(v))\phi(v)\,dv.$$

Here the sum runs over all the walls L, $n^L \in V$ is a unit normal vector to L, and C_{n^L} (respectively C'_{n^L}) is the chamber with wall L such that n^L is pointing away from (respectively towards) the chamber.

On the other hand, $f \in CB^\omega(V; M)$ satisfies the jump conditions (4.14) for $p \in S(V_\mathbb{C})$ of degree one iff, for a given wall $L = V_b \cap \overline{C}$ ($C \in \mathcal{C}$, $b \in R_C^a$),

$$(\partial_{n^L} f_{C_{n^L}})(v) - (\partial_{n^L} f_{C'_{n^L}})(v) = \frac{2k_b}{\|Db^\vee\|} s_b(f(v))$$

for $v \in L = V_b \cap \overline{C}$ (see (4.16)). The result now follows directly. □

It is natural to consider $\mathcal{H}_k^M$ as the quantum Hamiltonian of a quantum physical system. Particular cases of these quantum systems have been extensively studied: see, e.g., [23, 37, 26, 36, 12, 16, 13, 15, 27, 17, 9], to name just a few. Most studies in the literature deal with the root system R of type A, in which case the quantum system describes one-dimensional quantum spin-particles with pairwise delta function interactions. The assumption $k_a < 0$ (respectively $k_a > 0$) then corresponds to repulsive (respectively attractive) delta function interactions. For root systems R of classical type the quantum system relates to one-dimensional quantum spin-particles with pairwise delta function interactions and boundary reflection terms.

5. Spectral theory of the quantum many-body problem

Let N be a left $A(k)$-module and $\lambda \in V_\mathbb{C}^*$. Recall that $S(V_\mathbb{C})^W$ is part of the center of $A(k)$. We write $N_\lambda \subseteq N$ for the $A(k)$-submodule

$$N_\lambda = \{n \in N \mid p \cdot n = p(\lambda)n,\ \forall p \in S(V_\mathbb{C})^W\}.$$

We call N_λ the $A(k)$-submodule of N with *central character* λ. In this section we study the $A(k)$-submodule $F_{\mathrm{ir}}^k(M)_\lambda \simeq F_{\mathrm{dr}}^{\omega,k}(M)_\lambda$.

5.1. The spectral problem

For $\lambda \in V_\mathbb{C}^*$ we write

$$E(\lambda) = \{f \in C^\omega(V) \mid p(\partial)f = p(\lambda)f,\ \forall p \in S(V_\mathbb{C})^W\}. \tag{5.1}$$

Viewed as a W-module with the natural W-action (cf. (4.11)), $E(\lambda)$ is isomorphic to the regular W-representation (see [34]). If $\lambda(\alpha^\vee) \neq 0$ for all $\alpha \in R$ then $E(\lambda) = \bigoplus_{w \in W} \mathbb{C}e^{w\lambda}$ with e^λ the complex plane wave $v \mapsto e^{\lambda(v)}$.

We fix a left W^{a}-module M. The subalgebra $S(V_{\mathbb{C}}) \subset A(k)$ acts by constant coefficient differential operators on the $A(k)$-module $F_{\mathrm{ir}}^k(M)$, hence

$$F_{\mathrm{ir}}^k(M)_\lambda = E(\lambda) \otimes_{\mathbb{C}} M$$

as vector spaces. Consider now the $A(k)$-modules $F_{\mathrm{dr}}^{\omega,k}(M)_\lambda \subset F_{\mathrm{dr}}^k(M)_\lambda$. By Corollary 4.7(ii),

$$F_{\mathrm{dr}}^k(M)_\lambda = \{f \in B^\omega(V; M) \mid f_C \in E(\lambda) \otimes_{\mathbb{C}} M, \forall C \in \mathcal{C}\}$$

as vector spaces. In Subsection 4.4 we characterized $F_{\mathrm{dr}}^{\omega,k}(M)$ as a vector subspace of $F_{\mathrm{dr}}^k(M)$ in terms of derivative jump conditions of arbitrary order over the affine root hyperplanes. Restricted to the submodules of central character λ, the derivative jump conditions of order ≤ 1 suffice:

Proposition 5.1. *Suppose $f \in F_{\mathrm{dr}}^k(M)_\lambda$ satisfies the derivative jump conditions (4.14) for $p \in S(V_{\mathbb{C}})$ of degree ≤ 1. Then $f \in F_{\mathrm{dr}}^{\omega,k}(M)_\lambda$.*

Proof. Choose $f \in F_{\mathrm{dr}}^k(M)_\lambda$ satisfying (4.14) for $p \in S(V_{\mathbb{C}})$ of degree ≤ 1. Then

$$\psi := T^{k,M}(f_{C_+}) - f \in F_{\mathrm{dr}}^k(M)_\lambda$$

also satisfies (4.14) for $p \in S(V_{\mathbb{C}})$ of degree ≤ 1, and $\psi_{C_+} \equiv 0$. It suffices to show that $\psi = 0$. Recall that the neighboring chambers of a given chamber $C \in \mathcal{C}$ are given by $s_b C$ ($b \in R_C^a$). To prove that $\psi = 0$ it suffices to show that $\psi_C \equiv 0$ implies $\psi_{s_b C} \equiv 0$ ($b \in R_C^a$).

Choose a chamber $C \in \mathcal{C}$ such that $\psi_C \equiv 0$ and choose an affine root $b \in R_C^a$. Write L_b for the set of subregular vectors v on the wall $V_b \cap \overline{C}$. The subset $U = C \cup s_b C \cup L_b$ of V is open and pathwise connected. Since ψ satisfies (4.14) for $p = 1$ we see that $\psi|_U \in C(U) \otimes M$. We choose an arbitrary linear functional $\chi \in M^*$ and write $\Psi = \chi(\psi|_U) \in C(U)$. It suffices to show that $\Psi \equiv 0$.

We have $\Psi|_{C \cup L_b} \equiv 0$ by assumption. On the other hand, $\Psi|_{s_b C \cup L_b}$ is the restriction to $s_b C \cup L_b$ of some $g \in E(\lambda)$, since $\psi \in F_{\mathrm{dr}}^k(M)_\lambda$. In particular, $\Delta\Psi = \|\lambda\|^2\Psi$ on $C \cup s_b C$. Since ψ satisfies the jump conditions (4.14) for p of degree one, all directional derivatives of Ψ at the $v \in L_b$ vanish. Hence Ψ is a weak eigenfunction of Δ on U with eigenvalue $\|\lambda\|^2$. This forces Ψ to be smooth on U. Consequently, g is zero on an open neighborhood of $v \in L_b$. Hence $g \equiv 0$ and $\Psi \equiv 0$, as desired. □

Define $CB^\omega(V; M)_\lambda = C(V; M) \cap F_{\mathrm{dr}}^k(M)_\lambda$ for $\lambda \in V_{\mathbb{C}}^*$. Recall that $\|\lambda\|^2$ is the evaluation of the polynomial $\|\cdot\|^2 \in S(V_{\mathbb{C}})^W$ at λ.

Theorem 5.2. *Let $f \in CB^\omega(V; M)_\lambda$. Then $\mathcal{H}_k^M f = -\|\lambda\|^2 f$ weakly if and only if $f = T^{k,M}(f_{C_+}) \in F_{\mathrm{dr}}^{\omega,k}(M)_\lambda$.*

Proof. Suppose $f \in CB^\omega(V; M)_\lambda$ satisfies $\mathcal{H}_k^M f = -\|\lambda\|^2 f$ weakly. By Proposition 4.19, f satisfies the jump conditions (4.14) for $p \in S(V_{\mathbb{C}})$ of degree ≤ 1. Hence

$f = T^{k,M}(f_{C_+}) \in F_{\mathrm{dr}}^{\omega,k}(M)$ by Proposition 5.1. For the converse, if $f \in F_{\mathrm{dr}}^{\omega,k}(M)_\lambda$ then

$$\mathcal{H}_k^M f = -\Delta f = -\|\lambda\|^2 f$$

weakly, where the first equality is due to Proposition 4.19. □

These results lead to the following concrete procedure to produce solutions to the spectral problem of the quantum many-body problem (it is analogous to the usual Bethe ansatz methods for the one-dimensional quantum Bose gas with delta function interactions and its root system generalizations; see, e.g., [23, 26, 12, 16, 13] and the introduction): pick $g \in E(\lambda) \otimes_{\mathbb{C}} M$ (which is a solution to the spectral problem for the free Hamiltonians $p(\partial)$ ($p \in S(V_{\mathbb{C}})^W$)). Then Theorem 5.2 shows that $f = (f_C)_{C \in \mathcal{C}} = T^{k,M} g$ is the unique solution of $\mathcal{H}_k^M f = -\|\lambda\|^2 f$ such that $f_{C_+} = g$ and such that the $f_C \in E(\lambda) \otimes_{\mathbb{C}} M$ satisfy the derivative jump conditions (4.14) for $b \in V_b \cap \overline{C}$ and for $p \in S(V_{\mathbb{C}})$ of degree ≤ 1. For R of type A, this gives Theorem 1.1 from the introduction. Furthermore, f satisfies the derivative jump conditions (4.14) for all $p \in S(V_{\mathbb{C}})$, and the explicit formula for the propagation operator $T^{k,M}$ in terms of integral reflection operators tells us how to explicitly construct the f_C:

$$f_{w^{-1}C_+} = (w^{-1} \otimes w^{-1}) Q_{k,M}(w) g, \qquad \forall w \in W^{\mathrm{a}}.$$

This is not the end of the story though! For $R \subset \mathbb{R}^n$ of type A_{n-1} with $X = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$ the quantum Hamiltonian $\mathcal{H}_k^M$ can be interpreted weakly on V/Y. If we take furthermore $M = P^{\otimes n}$ for some complex vector space P with the S_n action the permutations of the tensor entries and with the trivial action of Y, $\mathcal{H}_k^M$ represents the quantum Hamiltonian of a quantum system describing n quantum spin-particles on the circle $S^1 = \mathbb{R}/\mathbb{Z}$ with internal spins (the quantum spin state space of the quantum particles is P) and with pairwise delta function interactions. The associated spectral problem amounts to analyzing the subspace $F_{\mathrm{dr}}^{\omega,k}(M)_\lambda^Y$ of Y-translation invariant functions in $F_{\mathrm{dr}}^{\omega,k}(M)_\lambda$. In this case the class of functions $g \in E(\lambda) \otimes_{\mathbb{C}} M$ such that the associated $f = T^{k,M} g \in F_{\mathrm{dr}}^{\omega,k}(M)_\lambda$ is Y-translation invariant is more subtle to characterize.

In the present general set-up we will give various characterizations of the subspace $F_{\mathrm{dr}}^{\omega,k}(M)_\lambda^{W^{\mathrm{a}}}$ of $W^{\mathrm{a}} = W \ltimes Y$-invariants in $F_{\mathrm{dr}}^{\omega,k}(M)_\lambda$ (in the context of the previous paragraph this corresponds to the bosonic Y-translation invariant theory). The following type of characterization is commonly used in the physics literature on one-dimensional quantum spin-particle systems with delta function interaction (see, e.g., [23, 26, 36, 22, 15, 7, 1]).

Proposition 5.3. *Let $f \in F_{\mathrm{dr}}^{\omega,k}(M)_\lambda$. Then f is W^{a}-invariant if and only if $g := f_{C_+} \in E(\lambda) \otimes_{\mathbb{C}} M$ satisfies, for all $a \in F^{\mathrm{a}}$, $v \in V_a \cap \overline{C_+}$ and $\omega \in \Omega$,*

$$\begin{aligned} s_a(g(v)) &= g(v), \\ (\mathrm{Id}_M + s_a)((\partial_{Da^\vee} g)(v)) &= -2k_a g(v), \\ (\omega \otimes \omega) g &= g. \end{aligned} \tag{5.2}$$

Proof. Suppose that f is W^{a}-invariant. Then $f_{s_aC_+} = (s_a \otimes s_a) f_{C_+}$ $(a \in F^{\mathrm{a}})$. For $b \in F^{\mathrm{a}}$ and $v \in V_b \cap \overline{C_+}$ the jump condition (4.14) with $p = 1$ becomes the first line of (5.2). The jump condition (4.16) for $b \in F^{\mathrm{a}}$ becomes the second line of (5.2). Let $\omega \in \Omega$. Since $\omega(C_+) = C_+$ we have $f_{C_+} = (\omega \otimes \omega) f_{C_+}$, which is the third line of (5.2).

Conversely, let $f \in F_{\mathrm{dr}}^{\omega,k}(M)_\lambda$ with $g := f_{C_+}$ satisfying (5.2). Then $g \in E(\lambda) \otimes_{\mathbb{C}} M$ and $f = T^{k,M} g$. Define $h = (h_C)_{C \in \mathcal{C}} \in F_{\mathrm{dr}}^{k}(M)_\lambda$ by

$$h_{w^{-1}C_+} = (w^{-1} \otimes w^{-1}) g, \qquad \forall w \in W^{\mathrm{a}}.$$

This is well defined by the third line of (5.2). By construction h is W^{a}-invariant and $h_{C_+} = g = f_{C_+}$. We now show that h satisfies the derivative jump conditions (4.14) for $p \in S(V_{\mathbb{C}})$ of degree ≤ 1. By the W^{a}-invariance of h it suffices to verify the jump conditions for $C = C_+$ the fundamental chamber, $b \in F^{\mathrm{a}}$ and $v \in V_b \cap \overline{C_+}$. The jump conditions then hold for $p = 1$ in view of the first line of (5.2). For p of degree one it suffices to check the jump conditions for $p = Db^\vee$ (see (4.16)), which is a direct consequence of the second line of (5.2).

By Proposition 5.1 we conclude that $h \in F_{\mathrm{dr}}^{\omega,k}(M)_\lambda$, hence $h = T^{k,M}(h_{C_+}) = T^{k,M}(g) = f$. Thus f is W^{a}-invariant. □

Remark 5.4. For $M = \mathbb{I}$ the conditions (5.2) for $g \in E(\lambda)$ simplify to $(\partial_{Da^\vee} g)(v) = -k_a g(v)$ and $\omega g = g$ for $a \in F^{\mathrm{a}}$, $v \in V_a \cap \overline{C_+}$ and $\omega \in \Omega$.

5.2. The Bethe ansatz equations

Proposition 5.3 gives a simple criterion to ensure the W^{a}-invariance of $f = T^{k,M} g \in F_{\mathrm{dr}}^{\omega,k}(M)_\lambda$. A different characterization is $Q_{k,M}(w) g = g$ for all $w \in W^{\mathrm{a}}$. We reformulate this now in explicit Bethe ansatz equations for the coefficients in the plane wave expansion of $g \in E(\lambda) \otimes_{\mathbb{C}} M$.

We write $C_k^{\mathrm{reg}} \subset V_{\mathbb{C}}^*$ for the W-invariant set

$$C_k^{\mathrm{reg}} = \{\lambda \in V_{\mathbb{C}}^* \mid 0 \neq \lambda(Da^\vee) \neq k_a, \ \forall a \in R^{\mathrm{a}}\}.$$

We assume throughout this subsection that $\lambda \in C_k^{\mathrm{reg}}$ unless explicitly specified otherwise. For such a spectral value λ we then have, besides the plane wave basis $\{e^{w\lambda}\}_{w \in W}$ of $E(\lambda)$, the cocycle $\{J_w^k(\lambda)\}_{w \in W^{\mathrm{a}}}$ in $\mathbb{C}[W^{\mathrm{a}}]^\times$ at our disposal. We furthermore fix a left W^{a}-module M in the remainder of this subsection.

The following lemma will be useful in the computations.

Lemma 5.5. *Let $\mu \in V_{\mathbb{C}}^*$ satisfying $\mu(\alpha^\vee) \neq 0$ for all $\alpha \in R$. For $a \in F^{\mathrm{a}}$ and $m \in M$ we have*

$$Q_{k,M}(s_a)(e^\mu \otimes m) = e^{-a(0)\mu(Da^\vee)} e^{s_{Da}\mu} \otimes s_a m + k_a \left(\frac{e^{-a(0)\mu(Da^\vee)} e^{s_{Da}\mu} - e^\mu}{\mu(Da^\vee)} \right) \otimes m.$$

Proof. Since $Q_{k,M}(s_a) = s_a \otimes s_a - k_a I(a) \otimes \mathrm{Id}_M$ the lemma follows from a direct computation using

$$I(a)(e^\mu) = \frac{e^\mu - e^{-a(0)\mu(Da^\vee)} e^{s_{Da}\mu}}{\mu(Da^\vee)}.$$

□

We now first consider the subspace $F_{\mathrm{ir}}^k(M)_\lambda^W$ of $Q_{k,M}(W)$-invariant elements in $F_{\mathrm{ir}}^k(M)_\lambda$. Recall the normalized intertwiners $J_w^k(\lambda) \in \mathbb{C}[W^{\mathrm{a}}]$ for $w \in W^{\mathrm{a}}$ from Subsection 4.1.

Write 1 for the unit element of W.

Proposition 5.6. *Let $m_w \in M$ $(w \in W)$. Then*

$$\sum_{w\in W} e^{w\lambda} \otimes m_w \in F_{\mathrm{ir}}^k(M)_\lambda^W \tag{5.3}$$

if and only if $m_w = J_w^k(\lambda)m_1$ for all $w \in W$.

Proof. We denote the left hand side of (5.3) by f_λ. Clearly $f_\lambda \in F_{\mathrm{ir}}^k(M)_\lambda$ and f_λ is W-invariant if and only if

$$Q_{k,M}(s_\alpha)f_\lambda = f_\lambda, \qquad \forall \alpha \in F.$$

In the latter equations we expand both sides as sums of the plane waves $e^{w\lambda}$ $(w \in W)$ with coefficients in M, using Lemma 5.5 for the left hand side. Comparing coefficients we find that $f_\lambda \in F_{\mathrm{ir}}^k(M)_\lambda^W$ if and only if

$$((w\lambda)(\alpha^\vee) - k_\alpha)m_{s_\alpha w} = ((w\lambda)(\alpha^\vee)s_\alpha + k_\alpha)m_w, \qquad \forall \alpha \in F, \forall w \in W.$$

This can be rewritten as $m_{s_\alpha w} = J_{s_\alpha}^k(w\lambda)m_w$ for all $\alpha \in F$ and $w \in W$. By the cocycle property of $J_w^k(\lambda)$ this in turn is equivalent to $m_w = J_w^k(\lambda)m_1$ for all $w \in W$. □

Corollary 5.7. *The assignment*

$$m \mapsto \psi_\lambda^m := \sum_{w\in W} e^{w\lambda} \otimes J_w^k(\lambda)m$$

defines a complex linear isomorphism $M \to F_{\mathrm{ir}}^k(M)_\lambda^W$. Furthermore,

$$c_k(\lambda)\psi_\lambda^m = \sum_{w\in W} Q_{k,M}(w)(e^\lambda \otimes m) \tag{5.4}$$

with $c_k(\lambda)$ the c-function

$$c_k(\lambda) = \prod_{\alpha\in R^+} \frac{\lambda(\alpha^\vee) - k_\alpha}{\lambda(\alpha^\vee)}.$$

Proof. The first statement is immediate from the previous proposition. To prove the second statement we write g_λ for the right hand side of (5.4). Then $g_\lambda \in F_{\mathrm{ir}}^k(M)_\lambda^W$, hence $g_\lambda = \psi_\lambda^{m'}$ for a unique $m' \in M$. By Lemma 5.5 one easily deduces that

$$J_{w_0}^k(\lambda)m' = c_k(\lambda)J_{w_0}^k(\lambda)m$$

for the longest Weyl group element $w_0 \in W$ with respect to the basis F of R. Hence $m' = c_k(\lambda)m$, which concludes the proof. □

Remark 5.8. (i) The proofs of Proposition 5.6 and Corollary 5.7 are based on the methods from [30, Prop. 1.4].

(ii) For $M = \mathbb{I}$ we write ψ_λ for ψ_λ^m with $m = 1$. Since $j_w^k(\lambda) = c_k(\lambda)^{-1}c_k(w\lambda)$ for $w \in W$ (see (4.3)) we get

$$\psi_\lambda = \frac{1}{c_k(\lambda)} \sum_{w\in W} c_k(w\lambda)e^{w\lambda},$$

in accordance with [12, 30, 17].

We next analyze when $\psi_\lambda^m \in F_{\mathrm{ir}}^k(M)_\lambda^W$ is W^{a}-invariant. We start with the following preliminary lemma.

Lemma 5.9. *For $m \in M$ we have*

$$Q_{k,M}(s_0)\psi_\lambda^m = \psi_\lambda^m \tag{5.5}$$

if and only if

$$J_{s_0w}^k(\lambda)m = e^{-(w\lambda)(\theta^\vee)}J_{s_\theta w}^k(\lambda)m, \qquad \forall w \in W. \tag{5.6}$$

Proof. Substitute the plane wave expansion $\psi_\lambda^m = \sum_{w\in W} e^{w\lambda} \otimes J_w^k(\lambda)m$ in the equality $Q_{k,M}(s_0)\psi_\lambda^m = \psi_\lambda^m$ and expand again both sides as sums of plane waves, using Lemma 5.5 for the left hand side. Equating the coefficients gives the equivalence between (5.5) and (5.6) (cf. the proof of Proposition 5.6). □

Theorem 5.10 (Bethe ansatz equations). *Let $\lambda \in C_k^{\mathrm{reg}}$ and $m \in M$. We have $\psi_\lambda^m \in F_{\mathrm{ir}}^k(M)_\lambda^{W^{\mathrm{a}}}$ if and only if*

$$J_y^k(\lambda)m = e^{\lambda(y)}m, \qquad \forall y \in Y. \tag{5.7}$$

Proof. Since $s_0 = s_\theta t_{-\theta^\vee}$, the cocycle condition for $J_w^k(\lambda)$ ($w \in W^{\mathrm{a}}$) gives

$$J_{s_0w}^k(\lambda) = J_{s_\theta w}^k(\lambda)J_{-w^{-1}(\theta^\vee)}^k(\lambda), \qquad \forall w \in W.$$

By Lemma 5.9 we conclude that $\psi_\lambda^m \in F_{\mathrm{ir}}^k(M)_\lambda^{W\ltimes Q^\vee}$ if and only if

$$J_y^k(\lambda)m = e^{\lambda(y)}m \tag{5.8}$$

for $y \in Q^\vee$ of the form $y = -w^{-1}(\theta^\vee)$ ($w \in W$). The co-root lattice $Q^\vee$ is generated by $W\theta^\vee$ and $y \mapsto J_y^k(\lambda)$ defines a group homomorphism $Y \to \mathbb{C}[W^{\mathrm{a}}]^\times$, hence the theorem holds when $Y = Q^\vee$.

Fix $\omega \in \Omega$ and write $\omega = t_y\sigma$ with $y \in Y$ and $\sigma \in W$. For $w \in W$ we then have

$$\omega(e^{w\lambda}) = e^{-(\sigma w\lambda)(y)}e^{\sigma w\lambda}, \qquad \omega J_w^k(\lambda)m = J_{\omega w}^k(\lambda)m = J_{t_y\sigma w}^k(\lambda)m,$$

where the first identity is in $C^\omega(V)$ (with the standard W^{a}-action). Using the plane wave expansion $\psi_\lambda^m = \sum_{w\in W} e^{w\lambda} \otimes J_w^k(\lambda)m$ we conclude that

$$Q_{k,M}(\omega)\psi_\lambda^m = \psi_\lambda^m$$

if and only if

$$J_{t_yw}^k(\lambda)m = e^{(w\lambda)(y)}J_w^k(\lambda)m, \qquad \forall w \in W.$$

By the cocycle condition for $J^k_w(\lambda)$ ($w \in W^{\mathrm{a}}$), this in turn is equivalent to

$$J^k_{wy}(\lambda)m = e^{\lambda(wy)}m, \qquad \forall w \in W.$$

If $\mathcal{S}$ denotes the set of $y \in Y$ for which there exists an $\omega \in \Omega$ of the form $\omega = t_y\sigma$ ($\sigma \in W$), then we conclude that $\psi^m_\lambda \in F^k_{\mathrm{ir}}(M)^{W^{\mathrm{a}}}_\lambda$ if and only if (5.8) holds for all y in the sublattice L of Y generated by $Q^\vee$ and the W-orbit of $\mathcal{S}$. Since $L = Y$, this concludes the proof of the theorem. $\square$

For nontrivial W^{a}-modules M various special instances of the Bethe ansatz equations (5.7) can be found in the literature: see, e.g., [26, 36, 35, 7]. For $M = \mathbb{I}$ the theorem becomes the following statement.

Corollary 5.11.

$$\psi_\lambda = \frac{1}{c_k(\lambda)} \sum_{w \in W} c_k(w\lambda)e^{w\lambda} \in E(\lambda)$$

is $Q_{k,\mathbb{I}}(W^{\mathrm{a}})$-invariant if and only if the spectral parameter $\lambda \in C^{\mathrm{reg}}_k$ satisfies the Bethe ansatz equations

$$j^k_y(\lambda) = e^{\lambda(y)}, \qquad \forall y \in Y,$$

with $j^k_y(\lambda)$ given by (4.4).

Corollary 5.11 generalizes the Bethe ansatz equations from [9], which dealt with the special case $k_0 = k_\theta$ and $Y = Q^\vee$. For root systems R of classical type it goes back to [23, 12].

5.3. Solutions to the Bethe ansatz equations

For $M = \mathbb{I}$, $k_0 = k_\theta$ and $Y = Q^\vee$, the set of spectral values λ solving the Bethe ansatz equations (5.7) has been analyzed in detail. For R the root system of type A it goes back to [37]. The solutions are parametrized by the extrema of a family of strictly convex functions. Such phenomena happen in various other quantum integrable models, such as the Gaudin model [33].

We now analyze the Bethe ansatz equations (5.7) for $M = \mathrm{Fun}_{\mathbb{C}}(W^{\mathrm{a}})$, the complex-valued functions on W^{a}, viewed as a left W^{a}-module by the left regular action $(L(w)f)(w') = f(w^{-1}w')$ for $w, w' \in W^{\mathrm{a}}$ and $f \in \mathrm{Fun}_{\mathbb{C}}(W^{\mathrm{a}})$. We write $\mathrm{Fun}^R_{\mathbb{C}}(W^{\mathrm{a}})$ for the vector space $\mathrm{Fun}_{\mathbb{C}}(W^{\mathrm{a}})$ endowed with the left W^{a}-action $(R(w)f)(w') = f(w'w)$.

Consider for $\lambda \in V^*_{\mathbb{C}}$ the $A(k)$-module $F^k_{\mathrm{ir}}(\mathbb{I})_\lambda$ as a W^{a}-module by restricting the representation map $Q_{k,\mathbb{I}}$ to W^{a}. We view the complex linear dual $F^k_{\mathrm{ir}}(\mathbb{I})^*_\lambda$ as a W^{a}-module with respect to the associated contragredient action. We have a complex linear map $\Xi_\lambda : F^k_{\mathrm{ir}}(\mathbb{I})^*_\lambda \otimes F^k_{\mathrm{ir}}(\mathbb{I})_\lambda \to \mathrm{Fun}_{\mathbb{C}}(W^{\mathrm{a}})$ defined by

$$\Xi_\lambda(g \otimes v) := g(Q_{k,\mathbb{I}}(\cdot)v).$$

It intertwines the $W^{\mathrm{a}} \times W^{\mathrm{a}}$-action on $F^k_{\mathrm{ir}}(\mathbb{I})^*_\lambda \otimes F^k_{\mathrm{ir}}(\mathbb{I})_\lambda$ with the $W^{\mathrm{a}} \times W^{\mathrm{a}}$-action $L \times R$ on $\mathrm{Fun}_{\mathbb{C}}(W^{\mathrm{a}})$. In particular, Ξ_λ is a W^{a}-module morphism with respect to the conjugation action $(wf)(w') = f(w^{-1}w'w)$ on $\mathrm{Fun}_{\mathbb{C}}(W^{\mathrm{a}})$.

Write $\epsilon_\lambda : \mathbb{I} \to F_{\mathrm{ir}}^k(\mathbb{I})_\lambda \otimes F_{\mathrm{ir}}^k(\mathbb{I})_\lambda^*$ for the co-evaluation map. It is the W^{a}-module morphism defined by $\epsilon_\lambda(1) = \sum_i v_i \otimes v_i^*$, where $\{v_i\}_i$ is a basis of $F_{\mathrm{ir}}^k(\mathbb{I})_\lambda$ and $\{v_i^*\}_i$ is the corresponding dual basis.

Theorem 5.12. *Let $\lambda \in V_{\mathbb{C}}^*$. The mapping*

$$g \mapsto (\mathrm{Id}_{F_{\mathrm{ir}}^k(\mathbb{I})_\lambda} \otimes \Xi_\lambda)(\epsilon_\lambda \otimes \mathrm{Id}_{F_{\mathrm{ir}}^k(\mathbb{I})_\lambda})g$$

defines an isomorphism

$$\Psi_\lambda : F_{\mathrm{ir}}^k(\mathbb{I})_\lambda \to F_{\mathrm{ir}}^k(\mathrm{Fun}_{\mathbb{C}}(W^{\mathrm{a}}))_\lambda^{W^{\mathrm{a}}}$$

of left W^{a}-modules, where $F_{\mathrm{ir}}^k(\mathrm{Fun}_{\mathbb{C}}(W^{\mathrm{a}}))_\lambda^{W^{\mathrm{a}}}$ is regarded as a W^{a}-submodule of $E(\lambda) \otimes_{\mathbb{C}} \mathrm{Fun}_{\mathbb{C}}^R(W^{\mathrm{a}})$ with respect to the action $w \mapsto \mathrm{Id} \otimes R(w)$.

Proof. By construction Ψ_λ defines a complex linear map

$$\Psi_\lambda : F_{\mathrm{ir}}^k(\mathbb{I})_\lambda \to (F_{\mathrm{ir}}^k(\mathbb{I})_\lambda \otimes \mathrm{Fun}_{\mathbb{C}}(W^{\mathrm{a}}))^{W^{\mathrm{a}}}.$$

By Proposition 4.13 we have

$$(F_{\mathrm{ir}}^k(\mathbb{I})_\lambda \otimes \mathrm{Fun}_{\mathbb{C}}(W^{\mathrm{a}}))^{W^{\mathrm{a}}} = F_{\mathrm{ir}}^k(\mathrm{Fun}_{\mathbb{C}}(W^{\mathrm{a}}))_\lambda^{W^{\mathrm{a}}}. \tag{5.9}$$

The resulting linear map

$$\Psi_\lambda : F_{\mathrm{ir}}^k(\mathbb{I})_\lambda \to F_{\mathrm{ir}}^k(\mathrm{Fun}_{\mathbb{C}}(W^{\mathrm{a}}))_\lambda^{W^{\mathrm{a}}}$$

is easily seen to be a morphism of left W^{a}-modules. It remains to show that Ψ_λ is an isomorphism. Define the linear mapping $\Phi_\lambda : F_{\mathrm{ir}}^k(\mathrm{Fun}_{\mathbb{C}}(W^{\mathrm{a}}))_\lambda^{W^{\mathrm{a}}} \to F_{\mathrm{ir}}^k(\mathbb{I})_\lambda$ by

$$\Phi_\lambda\Big(\sum_j g_j \otimes h_j\Big) = \sum_j h_j(1) g_j,$$

where 1 is the unit element of W^{a}. Clearly Φ_λ is a left inverse of Ψ_λ. By the alternative characterization (5.9) we have, for $\sum_j g_j \otimes h_j \in F_{\mathrm{ir}}^k(\mathrm{Fun}_{\mathbb{C}}(W^{\mathrm{a}}))_\lambda^{W^{\mathrm{a}}}$,

$$\sum_j h_j(w) g_j = \sum_j h_j(1) Q_{k,\mathbb{I}}(w) g_j, \qquad \forall w \in W^{\mathrm{a}},$$

which implies that Φ_λ is also the right inverse of Ψ_λ. □

Let $\lambda \in C_k^{\mathrm{reg}}$. Theorem 5.12 gives rise to the following explicit description of the solutions of the Bethe ansatz equations (5.7) for $M = \mathrm{Fun}_{\mathbb{C}}(W^{\mathrm{a}})$.

For a common eigenfunction of the scalar Dunkl type operators, $f = T^{k,\mathbb{I}}(g) \in F_{\mathrm{dr}}^{\omega,k}(\mathbb{I})_\lambda = T^{k,\mathbb{I}}(F_{\mathrm{ir}}^k(\mathbb{I})_\lambda)$, we have

$$f(u^{-1}\,\cdot)|_{C_+} = \sum_{w \in W} c_{g,w}(u) e^{w\lambda}|_{C_+}, \qquad u \in W^{\mathrm{a}},$$

with $c_{g,w} \in \mathrm{Fun}_{\mathbb{C}}(W^{\mathrm{a}})$ determined by

$$Q_{k,\mathbb{I}}(u) g = \sum_{w \in W} c_{g,w}(u) e^{w\lambda}, \qquad \forall u \in W^{\mathrm{a}}.$$

Corollary 5.13. *For $\lambda \in C_k^{\mathrm{reg}}$ we have*

$$\Psi_\lambda(g) = \sum_{w \in W} e^{w\lambda} \otimes c_{g,w}, \qquad \forall g \in F_{\mathrm{ir}}^k(\mathbb{I})_\lambda.$$

In particular, $c_{g,w} = L(J_w^k(\lambda))c_{g,1}$ for $w \in W$ and $g \in F_{\mathrm{ir}}^k(\mathbb{I})_\lambda$, and the functions $c_{g,1} \in \mathrm{Fun}_{\mathbb{C}}(W^{\mathrm{a}})$ ($g \in F_{\mathrm{ir}}^k(\mathbb{I})_\lambda$) give all the solutions of the Bethe ansatz equations (5.7) *for the W^{a}-module $M = \mathrm{Fun}_{\mathbb{C}}(W^{\mathrm{a}})$.*

Remark 5.14. Corollary 5.13 should be compared to the observation of Yang [36] (see also [35, 22, 21]) that an arbitrary eigenfunction of the quantum Hamiltonian of the one-dimensional quantum particle system with delta function potentials gives rise to a symmetric eigenfunction of the quantum Hamiltonian of an associated quantum spin-particle model.

The principal series modules of W^{a} are defined as follows. For $t \in V_{\mathbb{C}}^*$ the principal series module of W^{a} with central character t is the vector space $M(t) = \bigoplus_{w \in W} \mathbb{C}v_w(t)$ with action given by

$$\begin{aligned} uv_w(t) &= v_{uw}(t), & u &\in W,\\ yv_w(t) &= e^{t(w^{-1}y)}v_w(t), & y &\in Y, \end{aligned}$$

for $w \in W$. Observe that $M(t)$ is unitarizable iff $t \in \sqrt{-1}V^*$, in which case the corresponding scalar product is given by $(v_u(t), v_w(t)) := \delta_{u,w}$ for $u, w \in W$ (conjugate linear in the second component).

Yang [36] derived and studied the Bethe ansatz equations (5.7) for a root system R of type A and for M the principal series module $M(0)$. More generally, a natural problem is to describe the solutions of the Bethe ansatz equations (5.7) for the principal series module $M(t)$. We make a modest start here with the following observation.

Proposition 5.15. *Fix $\lambda \in C_k^{\mathrm{reg}}$ and let M be a unitary W^{a}-module. Assume that $k_a \in \mathbb{R}_{<0}$ for all $a \in R^{\mathrm{a}}$ (which corresponds to repulsive delta function interactions in the physical interpretation). If there exists an $0 \neq m \in M$ satisfying the Bethe ansatz equations* (5.7)*, then $\lambda \in \sqrt{-1}V^*$.*

Proof. We assume that $\lambda = \mu + \sqrt{-1}\nu \in C_k^{\mathrm{reg}}$ with $\mu, \nu \in V^*$ and $\mu(\alpha^\vee) \geq 0$ for all $\alpha \in R^+$. It suffices to prove the theorem under these additional assumptions; indeed, if $0 \neq m \in M$ satisfies (5.7) then $0 \neq n := J_w^k(\lambda)m \in M$ ($w \in W$) satisfies

$$J_y^k(w\lambda)n = e^{(w\lambda)(y)}n, \qquad \forall y \in Y.$$

Fix $k < 0$ and $u \in \mathbb{C}$ with $\mathrm{Re}(u) \geq 0$. Write $(\cdot,\cdot)$ for the scalar product on M, and $\|\cdot\|$ for the corresponding norm. For a simple affine root $a \in F^{\mathrm{a}}$ and $0 \neq n \in M$ we then have

$$\left\| \left(\frac{us_a + k}{u - k} \right) n \right\| = \left| \frac{u + k}{u - k} \right| \|n\|,$$

which is $\leq \|n\|$ and $= \|n\|$ iff $\mathrm{Re}(u) = 0$.

We are first going to apply this estimate to show that $J^k_{\theta^\vee}(\lambda)$ has operator norm ≤ 1. Write $s_\theta = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_r}$ $(\alpha_j \in F)$ for a reduced expression of $s_\theta \in W$. Then

$$R^+ \cap s_\theta R^- = \{\beta_1, \ldots, \beta_r\}, \qquad \beta_j = s_{\alpha_r}s_{\alpha_{r-1}}\cdots s_{\alpha_{j+1}}(\alpha_j).$$

Since $t_{\theta^\vee} = s_0 s_\theta$ the cocycle condition gives

$$J^k_{\theta^\vee}(\lambda) = \left(\frac{\lambda(\theta^\vee)s_0 + k_0}{\lambda(\theta^\vee) - k_0}\right)\left(\frac{\lambda(\beta_1)s_{\alpha_1} + k_{\alpha_1}}{\lambda(\beta_1) - k_{\alpha_1}}\right)\cdots\left(\frac{\lambda(\beta_r)s_{\alpha_r} + k_{\alpha_r}}{\lambda(\beta_r) - k_{\alpha_r}}\right).$$

The previous paragraph thus implies that $\|J^k_{\theta^\vee}(\lambda)\| \leq 1$.

Let $0 \neq m \in M$ be a solution of the Bethe ansatz equations (5.7). Then

$$|e^{\lambda(\theta^\vee)}| = \frac{\|J^k_{\theta^\vee}(\lambda)m\|}{\|m\|} \leq 1.$$

Since $\lambda = \mu + \sqrt{-1}\nu$ with $\mu(\alpha^\vee) \geq 0$ for all $\alpha \in R^+$ we obtain $\mu(\theta^\vee) = 0$. We have $\theta^\vee = \sum_{\alpha\in F} n_\alpha \alpha^\vee$ with $n_\alpha \geq 1$, hence $\mu(\alpha^\vee) = 0$ for all $\alpha \in F$. In particular, $\mu \in V^*$ is W-invariant and

$$\|J_w(\lambda)n\| = \|J_w(\sqrt{-1}\nu)n\| = \|n\|, \qquad \forall w \in W, \forall n \in M.$$

Let now $\omega \in \Omega$ and write $\omega = \sigma t_y$ with $\sigma \in W$ and $y \in Y$. Then

$$\|m\| = \|\omega m\| = \|J^k_\omega(\lambda)m\| = \|J^k_\sigma(\lambda)J^k_y(\lambda)m\| = |e^{\lambda(y)}|\,\|m\|,$$

hence $\mu(y) = 0$. As in the proof of Theorem 5.10 we conclude that $\mu(Y) = 0$, hence $\mu = 0$. □

Acknowledgements

Emsiz was supported in part by the Fondo Nacional de Desarrollo Científico y Tecnológico (FONDECYT) grant #3080006. Stokman is supported by the Netherlands Organization for Scientific Research (NWO) in the VIDI-project "Symmetry and modularity in exactly solvable models". He thanks Pavel Etingof for useful discussions.

References

[1] S. Albeverio, S.-M. Fei and P. Kurasov. Many body problems with "spin"-related contact interactions. *Rep. Math. Phys.* **47** (2001), 157–166.

[2] T. Arakawa, T. Suzuki and A. Tsuchiya. Degenerate double affine Hecke algebra and conformal field theory. In: *Topological Field Theory, Primitive Forms and Related Topics* (Kyoto, 1996), Progr. Math. 160, Birkhäuser Boston, Boston, MA, 1998, 1–34.

[3] I. Cherednik. Quantum Knizhnik–Zamolodchikov equations and affine root systems. *Comm. Math. Phys.* **150** (1992), 109–136.

[4] I. Cherednik. Elliptic quantum many-body problem and double affine Knizhnik–Zamolodchikov equation. *Comm. Math. Phys.* **169** (1995), 441–461.

[5] I. Cherednik. Inverse Harish-Chandra transform and difference operators. *Int. Math. Res. Notices* **1997**, no. 15, 733–750.

[6] I. Cherednik. *Double Affine Hecke Algebras*. London Math. Soc. Lecture Note Ser. 319, Cambridge Univ. Press, Cambridge, 2005.

[7] N. Crampé and C. A. S. Young. Bethe equations for a $\mathfrak{g}_2$ model. *J. Phys. A* **39** (2006), L315–L143.

[8] V. G. Drinfeld. Degenerate affine Hecke algebras and Yangians. *Functional Anal. Appl.* **20** (1986), 62–64.

[9] E. Emsiz, E. M. Opdam and J. V. Stokman. Periodic integrable systems with delta-potentials. *Comm. Math. Phys.* **264** (2006), 191–225.

[10] M. Flicker and E. H. Lieb. Delta-function Fermi gas with two-spin deviates. *Phys. Rev.* **161** (1967), 179–188.

[11] M. Gaudin. Un système à une dimension de fermions en interaction. *Phys. Lett.* **24A** (1967), 55–56.

[12] M. Gaudin. *La fonction d'onde de Bethe*. Collection du Commissariat à l'Énergie Atomique: Série Scientifique, Masson, Paris, 1983.

[13] E. Gutkin. Integrable systems with delta-potential. *Duke Math. J.* **49** (1982), 1–21.

[14] E. Gutkin. Operator calculi associated with reflection groups. *Duke Math. J.* **55** (1987), 1–18.

[15] C. H. Gu and C. N. Yang. A one-dimensional N fermion problem with factorized S-matrix. *Comm. Math. Phys.* **122** (1989), 105–116.

[16] E. Gutkin and B. Sutherland. Completely integrable systems and groups generated by reflections. *Proc. Nat. Acad. Sci. USA* **76** (1979), 6057–6059.

[17] G. J. Heckman and E. M. Opdam. Yang's system of particles and Hecke algebras. *Ann. of Math.* (2) **145** (1997), 139–173.

[18] G. J. Heckman and H. Schlichtkrull. *Harmonic Analysis and Special Functions on Symmetric Spaces*. Perspectives Math. 16, Academic Press, San Diego, CA, 1994.

[19] K. Hikami. Notes on the δ-function interacting gas. Intertwining operator in the degenerate affine Hecke algebra. *J. Phys. A* **31** (1998), L85–L91.

[20] K. Hikami and Y. Komori. Nonlinear Schrödinger model with boundary, integrability and scattering matrix based on the degenerate affine Hecke algebra. *Int. J. Modern Phys. A* **12** (1997), 5397–5410.

[21] A. Lascoux, B. Leclerc and J.-Y. Thibon. Flag varieties and the Yang–Baxter equation. *Lett. Math. Phys.* **40** (1997), 75–90.

[22] B. Z. Li, S. Q. Lu and F. C. Pu. Completeness of Bethe type eigenfunctions for the 1D N-body system with δ-function interactions. *Phys. Lett.* **110A** (1985), 65–67.

[23] E. H. Lieb and W. Liniger. Exact analysis of an interacting Bose gas. I. The general solution and the ground state. *Phys. Rev.* (2) **130** (1963), 1605–1616.

[24] G. Lusztig. Affine Hecke algebras and their graded version. *J. Amer. Math. Soc.* **2** (1989), 599–635.

[25] J. B. McGuire. Study of exactly soluble one-dimensional N-body problems. *J. Math. Phys.* **5** (1964), 622–626.

[26] J. B. McGuire. Interacting fermions in one dimension. I. Repulsive potential. *J. Math. Phys.* **6** (1965), 432–439.

[27] S. Murakami and M. Wadati. Connection between Yangian symmetry and the quantum inverse scattering method. *J. Phys. A* **29** (1996), 7903–7915.

[28] A. Oblomkov. Double affine Hecke algebras and Calogero–Moser spaces. *Represent. Theory* **8** (2004), 243–266.

[29] E. M. Opdam. Root systems and hypergeometric functions IV. *Compos. Math.* **67** (1988), 191–206.

[30] E. M. Opdam. Harmonic analysis for certain representations of graded Hecke algebras. *Acta Math.* **175** (1995), 75–121.

[31] E. M. Opdam. *Lecture Notes on Dunkl Operators for Real and Complex Reflection Groups.* MSJ Memoirs 8, Math. Soc. Japan, 2000.

[32] A. P. Polychronakos. Exchange operator formalism for integrable systems of particles. *Phys. Rev. Lett.* **69** (1992), 703–705.

[33] N. Reshetikhin and A. Varchenko. Quasiclassical asymptotics of solutions to the KZ equations. In: *Geometry, Topology and Physics*, Conf. Proc. Lecture Notes Geom. Topology IV, Int. Press, Cambridge, MA, 1995, 293–322.

[34] R. Steinberg. Differential equations invariant under finite reflection groups. *Trans. Amer. Math. Soc.* **112** (1964), 392–400.

[35] B. Sutherland. Further results for the many-body problem in one dimension. *Phys. Rev. Lett.* **20** (1968), 98–100.

[36] C. N. Yang. Some exact results for the many-body problem in one dimension with repulsive delta-function interaction. *Phys. Rev. Lett.* **19** (1967), 1312–1315.

[37] C. N. Yang and C. P. Yang. Thermodynamics of a one-dimensional system of bosons with repulsive delta-function interaction. *J. Math. Phys.* **10** (1969), 1115–1122.

E. Emsiz
Instituto de Matemática y Física
Universidad de Talca
Casilla 747, Talca, Chile
e-mail: `eemsiz@inst-mat.utalca.cl`

E. M. Opdam
KdV Institute for Mathematics
Universiteit van Amsterdam
Plantage Muidergracht 24
1018 TV Amsterdam, The Netherlands
e-mail: `e.m.opdam@uva.nl`

J. V. Stokman
KdV Institute for Mathematics
Universiteit van Amsterdam
Plantage Muidergracht 24
1018 TV Amsterdam, The Netherlands
e-mail: `j.v.stokman@uva.nl`