GCD BOUNDS FOR ANALYTIC FUNCTIONS

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Abstract. We consider the problem giving upper bounds for the counting function of common zeros (i.e. the gcd) of two entire analytic functions in various settings. Applying techniques from analytic number theory and Diophantine approximation in the context of entire and meromorphic functions, we establish bounds of this type covering several aspects that were not well-suited for the previously used methods in the context of this problem.

Contents

1. Introduction

In [5], Bugeaud, Corvaja and Zannier showed that for every $\epsilon > 0$ and for all multiplicatively independent positive integers a, b, the following bound holds as n grows:

$$\gcd(a^n - 1, b^n - 1) \ll_{\epsilon,a,b} \exp(\epsilon n). \tag{1}$$

Corvaja and Zannier generalized (1) by replacing a^n, b^n by multiplicatively independent S-units $u, v \in \mathbb{Z}$, showing that in this case

$$\gcd(u - 1, v - 1) < \max\{|u|, |v|\}^\epsilon \tag{2}$$

holds with finitely many exceptions. See [8] and [9, Proposition 4] for details. These bounds are far from obvious and they have found various applications in number theory, see for instance [15]. The proof of these inequalities requires a very ingenious application of Schmidt's Subspace Theorem [22].

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It was later realized that gcd bounds of this type are basically the same as particular instances of Vojta's conjecture in the case of targets of codimension 2 in the two dimensional torus $\mathbb{G}_m^2$, see [25, 10] for further analysis of this point of view and various generalizations.

The purpose of this paper is to investigate analogues of the previous gcd bounds for analytic and meromorphic functions in various settings.

An analogue of (1) for complex polynomials was established in [1] by linking the gcd problem to number theoretical results on "unlikely intersections"; namely, the gcd bound is deduced from a theorem of Ihara, Serre and Tate on torsion points in algebraic curves in $\mathbb{G}_m^2$. A generalized version of (2) for algebraic function fields of characteristic zero is established in [10] by studying Wronskians, which in particular yields a new proof of the Ihara-Serre-Tate theorem. The case of function fields in positive characteristic is governed by anomalous examples, as studied in detail in [24].

The case of complex holomorphic functions has also been studied. Using techniques of Nevanlinna theory and jet spaces, Noguchi, Winkelmann and Yamanoi [16] prove a very general height inequality for holomorphic maps $F : \mathbb{C} \to S$ into semi-abelian varieties, which in the particular case of $S = \mathbb{G}_m^2$ yields an analogue of (2) of the form:

Let F, G be entire complex holomorphic units (functions with no zeros, i.e., exponentials of entire functions) and suppose that F, G are multiplicatively independent. For every $\epsilon > 0$ we have

$$N(F-1, G-1, r) \leq_{exc} \epsilon \max\{T(F, r), T(G, r)\}. \tag{3}$$

Here, $T(f, r)$ denotes the Nevanlinna height function, $N(f, g, r)$ is an analogue of a gcd: it is the counting function of common zeros of f and g, and the notation $\leq_{exc}$ means the estimate holds except for r in a set of finite Lebesgue measure possibly depending on ϵ. (See Section 2 for the relevant definitions from Nevanlinna theory.)

As mentioned before, our goal is to study analogues of the bounds (1) and (2) for analytic functions. The main open problems that we address in this work are the following:

Problem 1.1. *Can one obtain a gcd bound for general analytic/meromorphic functions (not just exponentials)?*

We give an affirmative answer to this question *on average* (in the aspect of n) for meromorphic functions over a complete algebraically closed field of any characteristic, not just over $\mathbb{C}$. For instance, we obtain

Theorem 1.2. *Let k be an algebraically closed field of any characteristic, complete with respect to a non-trivial absolute value (such as $\mathbb{C}$ or $\mathbb{C}_p$). Let f, g be algebraically independent meromorphic functions on k. For any given $\epsilon > 0$ there is m_ϵ such that for all $m \geq m_\epsilon$ we have*

$$\frac{1}{m} \sum_{m/2 < n \leq m} N(f^n - 1, g^n - 1, r) < \epsilon \max\{mT_f(r), mT_g(r)\}.$$

We actually prove much stronger average bounds of which the previous theorem is a special case, see Section 5.1. Our results have applications analogous to some classical number-theoretical problems (see Section 5.3). The proof of our average gcd bound does not use the second main theorem of Nevanlinna theory in any form, but instead, it is based on an elementary construction coming from algebraic combinatorics, and some estimates from analytic number theory.

From our more precise average bounds we will deduce the following result, applicable in a very general context (see Corollary 5.4):

Theorem 1.3. *Let k be an algebraically closed field of any characteristic, complete for a non-trivial absolute value. Let f, g be algebraically independent meromorphic functions over k. Let $\epsilon > 0$. There is an exceptional set E of positive integers having density zero in $\mathbb{N}$ such that for each $n \in \mathbb{N} \setminus E$ the following bound holds*

$$N(f^n - 1, g^n - 1, r) <_\infty \epsilon \max\{nT_f(r), nT_g(r)\},$$

where the notation "$<_\infty$" means that the inequality holds for r in a certain set $U \subseteq \mathbb{R}_{>0}$ of infinite Lebesgue measure.

We suspect that the difficulty in this aspect of the gcd problem relies in the fact the various versions of the second main theorem in Nevanlinna theory only include "places at infinity" in the proximity function. In order to adapt the successful approach from number fields, it seems that one would need to include vanishing orders (at infinitely many points) in the proximity function.

Problem 1.4. *Can the* max *in* (3) *be replaced by a* min*?*

The point is that when G is small compared to F in the sense that $T_G(r) = o(T_F(r))$, the gcd bound (3) gives no information. Unfortunately, the answer to this question is negative in general (see Section 4.1 for an example). Nevertheless, under suitable hypothesis on F and G one can indeed obtain a positive answer that allows one to get a non-trivial gcd bound even when $T_G(r) = o(T_F(r))$. For instance, we obtain:

Theorem 1.5. *Let F and G be complex holomorphic entire functions with no zeros (i.e., units) such that F and G are multiplicatively independent. Suppose that both F and G have finite order. Then for every $\epsilon > 0$, there is n_ϵ such that for all $n \geq n_\epsilon$ we have*

$$N(F^n - 1, G^n - 1, r) < \epsilon \min\{T(F^n, r), T(G^n, r)\} + O(\log r)$$

for sufficiently large r. Moreover, if we further assume that neither F nor G is the exponential of a linear polynomial and that the pair (F, G) is indecomposable (cf. Section 4) then for every $\epsilon > 0$ we have

$$N(F - 1, G - 1, r) \ll_{F,G,\epsilon} \min\{T(F, r), T(G, r)\}^{\frac{1}{2}+\epsilon} + O(\log r)$$

for sufficiently large r.

See Section 4.1 for stronger versions of this theorem and a discussion on the optimality of the hypotheses. Our proof uses methods originated in Pila's work on the Andre-Oort conjecture, although in this context the methods apply in a more direct way.

It is also possible to remove the "finite order" assumption by introducing a rather mild growth condition (here, $M_f(r)$ denotes the maximum modulus $\max\{|f(z)| : |z| \leq r\}$).

Theorem 1.6. *Let f, g be non-constant entire functions and assume that they are algebraically independent over $\mathbb{Q}$. Suppose that*

$$\lim_{r\to\infty} \frac{\log M_f(12r) \log M_g(12r)}{\min\{M_f(r), M_g(r)\}} = 0.$$

Write $F = \exp(2\pi i f)$ and $G = \exp(2\pi i g)$. Then for every $\epsilon > 0$ we have the bound

$$N(F - 1, G - 1, r) <_\infty \epsilon \min\{T(F, r), T(G, r)\}.$$

See Section 4.2 for the proof and for a discussion on the optimality of the growth condition. Our proof uses methods from transcendental number theory and we will follow the main ideas from [30], suitably extended to estimate *simultaneous* integral values of entire functions.

Problem 1.7. *Can one prove gcd bounds analogous to* (1), (2) *and* (3) *for non-Archimedean analytic functions?*

This question is not just for the sake of generalizing, but actually there is a concrete technical obstruction that prevents one from obtaining such a result just by adapting the known theory over $\mathbb{C}$. The work of [16], in its present form, only applies to the complex setting. In fact, it is a theorem of Cherry [6] that the only analytic maps $\mathbb{A}^1 \to S$ with S a semi-abelian variety over non-Archimedean fields are the constant maps. (In the particular case of $\mathbb{G}_m$ this is the same as the lack of globally convergent non-Archimedean exponential functions.) So, in the non-Archimedean setting one is forced to work with non-units. Nevertheless, our theorems 1.2 and 1.3 already apply in the non-Archimedean setting. In addition, we prove a bound of the form

$$N(F^n - 1, G^n - 1, r) < \left(\frac{1}{2} + \epsilon\right) \max\{T(F^n, r), T(G^n, r)\} \tag{4}$$

for F, G entire non-Archimedean functions in characteristic zero, provided that $n \gg_\epsilon 1$ (see Section 6). We suspect that the bound should hold with 1/2 removed, but it seems that such a version of the gcd inequality for *general* F, G, even in the non-Archimedean setting, is beyond the current techniques.

In addition to our work on the previous main questions, in Section 3 we provide an alternative more elementary proof of the Noguchi-Winkelmann-Yamanoi theorem in the particular case of the gcd bound (3) by adapting the number-theoretical argument from [9] to Nevanlinna theory (which serves as an example of Vojta's dictionary working in the direction "number theory to complex analysis"). Also, in Section 7 we discuss the precise relation of Vojta's conjectures for blow-up surfaces with gcd bounds in the context of analytic functions (see [25] for the corresponding analysis over number fields).

From an informal point of view, large part of this work consists of applying number theory in Nevanlinna theory, both by analogy and by actually applying number theoretical results that escape the classical analogy of Nevanlinna theory and Diophantine approximation. We hope that, beyond the present study of the gcd problem, these connections can serve to extend the usual analogies between value distribution and number theory.

2. Nevanlinna theory

2.1. **Nevanlinna Theory over $\mathbb{C}$.** We will set up some notation and definitions in Nevanlinna theory for complex meromorphic functions and recall some basic results. We refer to [12, Chapter VI] or [20, Chapter 1] for details.

Let f be a meromorphic function and $z \in \mathbb{C}$. Denote by $v_z(f) := \mathrm{ord}_z(f)$.

$$v_z^+(f) := \max\{0, v_z(f)\}, \quad \text{and} \quad v_z^-(f) := -\min\{0, v_z(f)\}.$$

Let $n_f(r,\infty)$ (respectively, $n_f^{(1)}(r,\infty)$) denote the number of poles of f in $\{z : |z| \le r\}$, counting multiplicity (respectively, ignoring multiplicity). The *counting function* and *truncated counting function* of f at ∞ are defined respectively by

$$\begin{aligned} N_f(r,\infty) &:= \int_0^r \frac{n_f(t,\infty) - n_f(0,\infty)}{t} dt + n_f(0,\infty)\log r \\ &= \sum_{0<|z|\le r} v_z^-(f) \log\left|\frac{r}{z}\right| + v_0^-(f)\log r, \end{aligned}$$

and

$$\begin{aligned} N_f^{(1)}(r,\infty) &:= \int_0^r \frac{n_f^{(1)}(t,\infty) - n_f^{(1)}(0,\infty)}{t} dt + n_f^{(1)}(0,\infty)\log r \\ &= \sum_{0<|z|\le r} \min\{1, v_z^-(f)\} \log\left|\frac{r}{z}\right| + \min\{1, v_0^-(f)\}\log r. \end{aligned}$$

Then define the *counting function* $N_f(r,a)$ and the *truncated counting function* $N_f^{(1)}(r,a)$ for $a \in \mathbb{C}$ as

$$N_f(r,a) := N_{1/(f-a)}(r,\infty) \quad \text{and} \quad N_f^{(1)}(r,a) := N_{1/(f-a)}^{(1)}(r,\infty).$$

The *proximity function* $m_f(r,\infty)$ is defined by

$$m_f(r,\infty) := \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi},$$

where $\log^+ x = \max\{0, \log x\}$. For any $a \in \mathbb{C}$, the *proximity function* $m_f(r,a)$ is defined by

$$m_f(r,a) := m_{1/(f-a)}(r,\infty).$$

Finally, the *characteristic function* is defined by

$$T(f,r) := m_f(r,\infty) + N_f(r,\infty).$$

Jensen's formula can be stated as follows.

Theorem 2.1. *Let f be a meromorphic function on $\{z : |z| \le r\}$ which is not the zero function. Then*

$$\int_0^{2\pi} \log |f(re^{i\theta})| \frac{d\theta}{2\pi} = N_f(r,0) - N_f(r,\infty) + \log |c_f|,$$

where c_f is the leading coefficient of f expanded as Laurent series in z, i.e., $f = c_f z^m + \cdots$ with $c_f \neq 0$.

We now recall the main theorems of Nevanlinna theory.

Theorem 2.2 (First Main Theorem). *Let f be a non-constant meromorphic function on $\mathbb{C}$. Then for every $a \in \mathbb{C}$, and any positive real number r,*

$$m_f(r,a) + N_f(r,a) = T(f,r) + O(1),$$

where $O(1)$ is a constant independent of r.

Theorem 2.3 (Truncated Second Main Theorem). *Let f be a non-constant meromorphic function on $\mathbb{C}$, and $a_1, \ldots, a_q$ be distinct elements in $\mathbb{C} \cup \{\infty\}$. Then for $r > 0$,*

$$(q-2)T(f,r) \leq_{exc} \sum_{i=1}^{q} N_f^{(1)}(r,a_i) + O(\log^+ T(f,r)),$$

where $\leq_{exc}$ means the estimate holds except for r in a set of finite Lebesgue measure.

The following notation will be central in our work, as it gives an analogue for the notion of gcd in the context of meromorphic functions. For meromorphic functions F, G write

$$n(F,G,r) := \sum_{|z|\leq r} \min\{v_z^+(F), v_z^+(G)\}$$

and

$$N(F,G,r) := \int_0^r \frac{n(F,G,t) - n(F,G,0)}{t} dt + n(F,G,0)\log r.$$

This will be the counting function of the greatest common divisor of F and G.

Remark 2.4. *If F, G are holomorphic, then we can define a holomorphic function H as the gcd of F and G by considering the respective Weierstrass factorizations. Then H would be well-defined only up to multiplication by a holomorphic unit. Nevertheless, we would still have $N(H,r) = N(F,G,r)$. So, we prefer to define $N(F,G,r)$ as above without introducing a holomorphic gcd since this approach gives the same result and avoids all choices and ambiguities (and moreover, it works in the meromorphic case as well).*

2.2. **Nevanlinna theory for holomorphic maps.** Let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d. We also use the notation $D(\mathbf{x})$ for a homogeneous polynomial of degree d such that D is its zero locus. The homogeneous polynomial $D(\mathbf{x})$ is only well-defined up to a linear factor, but we fix one choice once and for all. For $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbb{P}^n(\mathbb{C})$, we let

$$\lambda_D(\mathbf{x}) = -\log \frac{|D(\mathbf{x})|}{\max\{|x_0|, \ldots, |x_n|\}^d}. \tag{1}$$

This expression is invariant under scaling of projective coordinates.

Let $\mathbf{f} : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map. Let $\mathbf{f} = (f_0, \ldots, f_n)$ be a reduced representation of $\mathbf{f}$, i.e. $f_0, \ldots, f_n$ are entire functions on $\mathbb{C}$ without common zeros such that for all $z \in \mathbb{C}$ we have $\mathbf{f}(z) = [f_0(z) : \ldots : f_n(z)]$. The Nevanlinna-Cartan *characteristic function* $T_{\mathbf{f}}(r)$ is defined by

$$T_{\mathbf{f}}(r) = \int_0^{2\pi} \log \|\mathbf{f}(re^{i\theta})\| \frac{d\theta}{2\pi},$$

where $\|\mathbf{f}(z)\| = \max\{|f_0(z)|, \ldots, |f_n(z)|\}$. This definition is independent, up to an additive constant, of the choice of the reduced representation of $\mathbf{f}$. The *proximity function* of $\mathbf{f}$ with respect to D is defined by

$$m_{\mathbf{f}}(r,D) = \int_0^{2\pi} \lambda_D(\mathbf{f}(re^{i\theta})) \frac{d\theta}{2\pi}.$$

Let $\mathbf{n}_{\mathbf{f}}(r,D)$ be the number of zeros of $D \circ \mathbf{f}$ in the disk $|z| \leq r$, counting multiplicity. The integrated counting function is defined by

$$N_{\mathbf{f}}(r,D) = \int_0^r \frac{\mathbf{n}_{\mathbf{f}}(t,D) - \mathbf{n}_{\mathbf{f}}(0,D)}{t} dt + \mathbf{n}_{\mathbf{f}}(0,D)\log r.$$

In this context, the First Main Theorem reads as follows.

Theorem 2.5. *Let $\mathbf{f} : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map, and let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d. If $f(\mathbb{C}) \not\subset D$, then for $r > 0$,*

$$dT_{\mathbf{f}}(r) = m_{\mathbf{f}}(r, D) + N_{\mathbf{f}}(r, D) + O(1),$$

where $O(1)$ is bounded independently of r.

The following is a generalized Second Main Theorem due to Vojta in [27]. Although it does not involve truncated counting functions, it permits very fine control of the proximity functions.

Theorem 2.6. *Let $H_1, H_2, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ (not necessarily in general position). $\mathbf{f} : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map whose image does not lie in any proper linear subspace. Then*

$$\int_0^{2\pi} \max_J \sum_{j \in J} \lambda_{H_j}(\mathbf{f}(re^{i\theta})) \frac{d\theta}{2\pi} \leq_{exc} (n+1) T_{\mathbf{f}}(r) + O(\log^+ T_{\mathbf{f}}(r)) + o(\log r), \tag{2}$$

where the maximum is taken over all subsets J of $\{1, \ldots, q\}$ such that the linear forms corresponding to H_j, $j \in J$, are linearly independent.

2.3. **Nevanlinna theory over non-Archimedean fields.** Here we give some basic notation and definitions in non-Archimedean Nevanlinna theory. See [11] for reference. Let $\mathbf{k}$ be an algebraically closed field (of arbitrary characteristic) complete with respect to a non-Archimedean absolute value $|-|$. Let $\mathcal{A}$ be the ring of entire analytic functions on $\mathbf{k}$, that is, functions that can be expressed as a power series over $\mathbf{k}$ with infinite radius of convergence. Given $h(z) = \sum_{j=0}^{\infty} a_j z^j \in \mathcal{A}$ an entire analytic function on $\mathbf{k}$, for each real number $r \geq 0$ we define

$$\begin{aligned} |h|_r &:= \sup_j |a_j| r^j = \sup\{|h(z)| : z \in \mathbf{k} \text{ with } |z| \leq r\} \\ &= \sup\{|h(z)| : z \in \mathbf{k} \text{ with } |z| = r\}. \end{aligned}$$

For $r > 0$, the function $|-|_r$ defines an absolute value on $\mathcal{A}$.

The definition of the counting functions are the same as the complex case. The Poisson-Jensen formula for an entire function h can be stated as

$$N_h(r, 0) = \log |h|_r + O(1).$$

The field of non-Archimedean meromorphic function on $\mathbf{k}$, denoted by $\mathcal{M}$, is the fraction field of $\mathcal{A}$. One can show that a non-Archimedean meromorphic function $f \in \mathcal{M}$ is the quotient of two non-Archimedean entire functions h/g such that h and g do not have common zeros on $\mathbf{k}$. The above notations can be extended as follows:

For $r > 0$ the absolute value $|-|_r$ has a unique extension from to an absolute value on $\mathcal{M}$ and it satisfies

$$|f|_r = \frac{|h|_r}{|g|_r}.$$

The *proximity functions* are given by

$$m_f(r, \infty) = \log^+ |f|_r, \quad m_f(r, a) = \log^+ \frac{1}{|f-a|_r},$$

and the *characteristic function* is

$$T(f, r) = m_f(r, \infty) + N_f(r, \infty).$$

As in the complex case, we have the following version of the first and second main theorems.

Theorem 2.7. *Let f be a non-constant non-Archimedean meromorphic function on $\mathbf{k}$ and let $a \in \mathbf{k}$. Then for $r > 0$*

$$m_f(r, a) + N_f(r, a) = T(f, r) + O(1)$$

where $O(1)$ is bounded independently of r.

Theorem 2.8. *Let f be a non-constant non-Archimedean meromorphic function on $\mathbf{k}$, and $a_1, \ldots, a_q$ be distinct elements in $\mathbf{k} \cup \{\infty\}$. Then for $r > 0$,*

$$(q-2) T(f, r) \leq \sum_{i=1}^{q} N_f^{(1)}(r, a_i) - \log r + O(1).$$

For non-Archimedean meromorphic functions F, G we define the gcd counting functions exactly as in the complex case. Namely, we define

$$n(F,G,r) := \sum_{|z|\le r} \min\{v_z^+(F), v_z^+(G)\}$$

and

$$N(F,G,r) := \int_0^r \frac{n(F,G,t)-n(F,G,0)}{t}dt + n(F,G,0)\log r.$$

3. An alternative proof of a gcd bound for holomorphic units

In this section we work over the complex numbers.

We say that f and g are multiplicatively independent if for all $(m,n) \in \mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}$ we have $f^m \cdot g^n \notin \mathbb{C}$. In this section we give a more elementary proof of the following important special case of a theorem of Noguchi, Winkelmann and Yamanoi [16], as discussed in the introduction.

Theorem 3.1. *Let f, g be entire functions with no zeros and suppose that f and g are multiplicatively independent. Let $\epsilon > 0$. Then we have*

$$N(f-1, g-1, r) \le_{exc} \epsilon \max\{T(f,r), T(g,r)\}.$$

Our proof of Theorem 3.1 breaks into two steps. We first show by Borel Lemma that if two units f and g are multiplicatively independent, then they actually are algebraically independent. Then we adapt the method in [9] from the number field case to the complex setting assuming that f and g are algebraically independent. In this second step, we will need to use Theorem 2.6, a generalized version of the second main theorem by Vojta, as a substitute for the version of Schmidt's subspace theorem used in [9].

We now recall Borel's Lemma (cf. [20, Theorem A.3.3.2]).

Theorem (Borel's Lemma)**.** *Let $f_0, \ldots, f_{n+1}$ be entire functions without zeros, satisfying*

$$f_0 + \ldots + f_n + f_{n+1} = 0.$$

Considering the partition

$$\{0,1,2,\ldots,n+1\} = I_1 \cup I_2 \ldots \cup I_k$$

such that i and j are in the same set I_ℓ if and only if $f_i = c_{ij} f_j$ for some nonzero constant c_{ij}. Then for each ℓ we have

$$\sum_{i\in I_\ell} f_i = 0.$$

Proof of Theorem 3.1. If f and g are algebraically dependent over $\mathbb{C}$, then they satisfy a non-trivial $\mathbb{C}$-linear relation, say

$$\sum_{i=0}^{m}\sum_{j=0}^{u} a_{ij} f^i g^j = 0, \tag{3}$$

where $a_{ij} \in \mathbb{C}$. We may further assume that no proper sub-sums of the left hand side of (3) is zero. Since f and g are entire functions without zeros, Borel's Lemma implies that we have pairs of indices $(i,j) \ne (i',j')$ such that $a_{ij} f^i g^j$ is a constant multiple of some $a_{i'j'} f^{i'} g^{j'}$ appearing in the left hand side of (3), which is not possible since f and g are multiplicatively independent. Therefore, we conclude that f and g are algebraically independent over $\mathbb{C}$.

Next, we will adapt the arguments in [9] into the complex setting. For a positive integer j, we let

$$z_j := \frac{f^j - 1}{g-1}. \tag{4}$$

Note that

$$z_j = z_1 \cdot (f^{j-1} + \cdots + f + 1). \tag{5}$$

Fix a positive integer h and consider the identity

$$\frac{1}{g-1} = \frac{1}{g}\cdot\frac{1}{1-g^{-1}} = \frac{1}{g}\big(1+g^{-1}+\cdots+g^{-h+1}+\frac{g^{-h}}{1-g^{-1}}\big). \tag{6}$$

Fix a second positive integer k. Then for $j\in\{1,\cdots,k\}$ we obtain, on multiplying by f^j-1 in the above identity,

$$z_j = (f^j-1)\cdot\big(g^{-1}+g^{-2}+\cdots+g^{-h}+\frac{g^{-h}}{g-1}\big). \tag{7}$$

We put $M = hk+h+k$; for convenience we write coordinates in $\mathbb{C}^M$ as

$$\mathbf{x} = (x_1,\ldots,x_M) = (z_1,\ldots,z_k,y_{0,1},\ldots,y_{0,h},\ldots,y_{k,1},\ldots,y_{k,h}).$$

In this notation, we choose linear forms $H_i = x_i$ for $1\le i\le M$ and

$$H_{M+j} = z_j + y_{0,1}+\ldots+y_{0,h}-y_{j,1}-\ldots-y_{j,h}$$

for $j=1,\ldots,k$. Let ξ be a gcd of $f-1$ and $g-1$, i.e. $\frac{f-1}{\xi}$ and $\frac{g-1}{\xi}$ are entire with no common zero, and let $\eta=\frac{g-1}{\xi}$. Set

$$F = (z_1\eta,\ldots,z_k\eta,g^{-1}\eta,\ldots,g^{-h}\eta,fg^{-1}\eta,\ldots,fg^{-h}\eta,\ldots,f^kg^{-1}\eta,\ldots,f^kg^{-h}\eta) \tag{8}$$

which is a reduced representation of a holomorphic map from $\mathbb{C}$ to $\mathbb{P}^{M-1}$. Then

$$T_F(r) = \int_0^{2\pi}\log\|F(re^{i\theta})\|\frac{d\theta}{2\pi}. \tag{9}$$

For simplicity of notation, we let $|\xi|_{r,\theta} := |\xi(re^{i\theta})|$ for a meromorphic function ξ and let $\|F\|_{r,\theta} := \|F(re^{i\theta})\|$.

For $r>0$, we denote by $S_r^+ = \{\theta\in[0,2\pi) : |g|_{r,\theta}>1\}$ and $S_r^- = \{\theta\in[0,2\pi) : |g|_{r,\theta}\le 1\}$. For $\theta\in S_r^+$, we have that

$$\begin{aligned}|H_{M+j}(F)|_{r,\theta} &= |\eta|_{r,\theta}\cdot|z_j - f^jg^{-1}-\cdots-f^jg^{-h}+g^{-1}+\cdots+g^{-h}|_{r,\theta}\\ &= |\eta|_{r,\theta}\cdot|f^j-1|_{r,\theta}\frac{|g|_{r,\theta}^{-h}}{|1-g|_{r,\theta}}\\ &\le 2\max(1,|f|_{r,\theta})^j\cdot\frac{1}{|1-g|_{r,\theta}}|g|_{r,\theta}^{-h}\cdot|\eta|_{r,\theta}\end{aligned} \tag{10}$$

for $j=1,\ldots,k$ by (7). Consequently, for $\theta\in S^+$ we have

$$\begin{aligned}\sum_{j=1}^k\lambda_{H_{M+j}}(F(re^{i\theta})) \ge &-\frac{k(k+1)}{2}\log^+|f|_{r,\theta}+k\log|1-g|_{r,\theta}+kh\log|g|_{r,\theta}\\ &+k\log\|F\|_{r,\theta}-k\log|\eta|_{r,\theta}-\frac{k(k+1)}{2}\log 2.\end{aligned} \tag{11}$$

For $\theta\in S_r^-$, we recall that $z_j=(f^j-1)/(g-1)$, so

$$\begin{aligned}|H_j(F)|_{r,\theta} &\le |f^j-1|_{r,\theta}|1-g|_{r,\theta}^{-1}\cdot|\eta|_{r,\theta}\\ &\le 2\max(1,|f|_{r,\theta})^j\cdot|1-g|_{r,\theta}^{-1}\cdot|\eta|_{r,\theta}\end{aligned} \tag{12}$$

for $1\le j\le k$. Therefore,

$$\sum_{j=1}^k\lambda_{H_j}(F(re^{i\theta})) \ge -\frac{k(k+1)}{2}\log^+|f|_{r,\theta}+k\log|1-g|_{r,\theta}+k\log\|F\|_{r,\theta}-k\log|\eta|_{r,\theta}-\frac{k(k+1)}{2}\log 2 \tag{13}$$

for $\theta\in S^-$.

We note that the linear forms $H_{k+1},\dots,H_{M+k}$ are linearly independent, so are the linear forms $H_1,\dots,H_M$. Theorem 2.6 implies that

$$\int_{S^-}\sum_{j=1}^{M}\lambda_{H_j}(re^{i\theta})\frac{d\theta}{2\pi}+\int_{S^+}\sum_{j=k+1}^{M+k}\lambda_{H_j}(re^{i\theta})\frac{d\theta}{2\pi}\le_{exc} MT_F(r)+O(\log^+T_F(r))+o(\log r). \tag{14}$$

On the other hand, the left hand side of the above inequality equals

$$\int_{S^-}\sum_{j=1}^{k}\lambda_{H_j}(F(re^{i\theta}))\frac{d\theta}{2\pi}+\int_{S^+}\sum_{j=M+1}^{M+k}\lambda_{H_j}(F(re^{i\theta}))\frac{d\theta}{2\pi}+\sum_{j=k+1}^{M}\int_0^{2\pi}\lambda_{H_j}(F(re^{i\theta}))\frac{d\theta}{2\pi}. \tag{15}$$

For $k+1\le j\le M$, it follows from Theorem 2.5, the First Main Theorem, that

$$\int_0^{2\pi}\lambda_{H_j}(F(re^{i\theta}))\frac{d\theta}{2\pi}=m_F(r,H_i)=T_F(r)-N_F(r,H_i)=T_F(r)-N_\eta(r,0) \tag{16}$$

since f and g are units. To estimate the first two terms of (15), we use (11), (13) and the following identities:

$$\int_0^{2\pi}\log^+|f|_{r,\theta}\frac{d\theta}{2\pi}=m_f(r,\infty)=T(f,r)+O(1)$$
$$\int_{S^+}\log|g|_{r,\theta}\frac{d\theta}{2\pi}=\frac{1}{2\pi}\int_0^{2\pi}\log^+|g|_{r,\theta}d\theta=T(g,r)+O(1)$$
$$\int_0^{2\pi}\log|1-g|_{r,\theta}\frac{d\theta}{2\pi}=N_{g-1}(r,0)-N_{g-1}(r,\infty)+O(1)=N_{g-1}(r,0)+O(1)\quad\text{(by Theorem 2.1)}$$
$$\int_0^{2\pi}\log|\eta|_{r,\theta}\frac{d\theta}{2\pi}=N_\eta(r,0)+O(1)\quad\text{(by Theorem 2.1).}$$

Then

$$\begin{aligned}&\int_{S^-}\sum_{j=1}^{k}\lambda_{H_j}(F(re^{i\theta}))\frac{d\theta}{2\pi}+\int_{S^+}\sum_{j=M+1}^{M+k}\lambda_{H_j}(F(re^{i\theta}))\frac{d\theta}{2\pi}\\ &\ge kT_F(r)-kN_\eta(r,0)-\frac{k(k+1)}{2}T(f,r)+hkT(g,r)+kN_{g-1}(r,0)+O(1)\end{aligned} \tag{17}$$

Consequently, (15) is at least

$$MT_F(r)-MN_\eta(r,0)+kN_{g-1}(r,0)-\frac{k(k+1)}{2}T(f,r)+hkT(g,r)+O(1).$$

Together with (14), $N_{g-1}(r,0)\le T(g-1,r)=T(g,r)+O(1)$ and that

$$N_\eta(r,0)=N_{g-1}(r,0)-N(f-1,g-1,r),$$

we have

$$\begin{aligned}MN(f-1,g-1,r)&\le_{exc}\frac{k(k+1)}{2}T(f,r)-hkT(g,r)+(M-k)N_{g-1}(r,0)+O(\log^+T_F(r))+o(\log r)\\ &\le_{exc}\frac{k(k+1)}{2}T(f,r)+hT(g,r)+O(\log^+T_F(r))+o(\log r).\end{aligned}$$

Let $k>\frac{2}{\epsilon}$ and $h>\frac{k}{\epsilon}$. Then

$$N(f-1,g-1,r)\le_{exc}\epsilon\max\{T(f,r),T(g,r)\},$$

for r sufficiently large. $\square$

4. Bounds for gcd using min instead of max

In this section we again work over the complex numbers.

Let F and G be two entire functions. One can easily derive from Theorem 2.2 that

$$N(F-1,G-1,r) \leq \min\{T(F,r),T(G,r)\}+O(1). \tag{18}$$

Thus, if $T(G,r)=o(T(F,r))$ then the bound given by Theorem 3.1 in the case of holomorphic units is too crude. Our goal in this section is to investigate to what extent one can bound $N(F-1,G-1,r)$ in terms of $\min\{T(F,r),T(G,r)\}$ beyond the trivial bound (18).

First of all, the following example shows that equality can occur in (18), and therefore, some restriction on F and G must be imposed in order to obtain a non-trivial estimate.

Example 4.1. *Let $F=\exp(2\pi iz)$ and $G=\exp(2\pi ip(z))$, where $p(z)$ is a polynomial over $\mathbb{Z}$ of degree $n\geq 2$. It's clear that they are algebraically independent, and hence multiplicatively independent. Moreover, $\exp(2\pi iz)-1$ has zeros if and only if $z\in\mathbb{Z}$ and all zeros are simple. Since $\exp(2\pi ip(z))=1$ for all $z\in\mathbb{Z}$,*

$$N(F-1,G-1,r)=2r+O(\log r)$$

On the other hand,

$$\min\{T(F,r),T(G,r)\}=T(F,r)=2r.$$

Therefore, in this example we get that $N(F-1,G-1,r)$ and $\min\{T(F,r),T(G,r)\}$ are equal, up to a negligible error.

In this example, F is small compared to G in the sense that $T_F(r)=o(T_G(r))$, and therefore the gcd bound in Theorem 3.1 offers no information. In this section we will impose suitable hypothesis on the holomorphic units F and G in order to obtain a non-trivial gcd bound even when $T_G(r)=o(T_F(r))$.

In the first part of this section we assume that F and G are non-constant entire functions of finite order with no zero. In the second part, we impose certain mild growth condition on F and G in order to remove the finite order assumption, at the cost of obtaining a bound valid for all r on a set of infinite measure (as opposed to all r outside a set of finite measure).

4.1. Units of finite order.

We now discuss the case when F and G are entire functions of finite order with no zeros. An entire function of finite order with no zero is of the form $\exp(p)$ where p is a polynomial over $\mathbb{C}$, thus, we may let $F=\exp(f)$ and $G=\exp(g)$ where f and g are non-constant polynomials over $\mathbb{C}$. To simplify the notation, we will write d_p for the degree of a polynomial p.

The next notion will play a central role.

Definition 4.2. *Let f and g be nonconstant entire functions. The pair (f,g) is indecomposable if there exists no entire function h different form a polynomial of degree 1 such that $f=\tilde{f}\circ h$ and $g=\tilde{g}\circ h$ for some entire functions $\tilde{f},\tilde{g}$.*

We will prove:

Theorem 4.3. *Let F and G be complex holomorphic entire functions with no zeros (i.e. units) such that F and G are multiplicatively independent. Suppose that both F and G have finite order. Then for every $\epsilon>0$, there is n_ϵ such that for all $n\geq n_\epsilon$ we have*

$$N(F^n-1,G^n-1,r)<\epsilon\min\{T(F^n,r),T(G^n,r)\}.$$

Moreover, if we further assume that the pair (F,G) is indecomposable and that neither F nor G is the exponential of a linear polynomial, then for every $\epsilon>0$ we have

$$N(F-1,G-1,r)\ll_{F,G,\epsilon}\min\{T(F,r),T(G,r)\}^{\frac{1}{2}+\epsilon}.$$

We will deduce Theorem 4.3 from the next two more precise results.

Theorem 4.4. *Let $f,g\in\mathbb{C}[x]$ be non-constant polynomials and write $F=\exp(f)$ and $G=\exp(g)$. Suppose that we have an indecomposable pair of polynomials $(\tilde{f},\tilde{g})$, both non-linear, such that $f=\tilde{f}\circ h$ and $g=\tilde{g}\circ h$ for some polynomial h. Then for every $\epsilon>0$ we have*

$$N(F-1,G-1,r)\ll_{f,g,\epsilon}\min\{T(F,r),T(G,r)\}^{\epsilon+1/\min(d_{\tilde{f}},d_{\tilde{g}})}.$$

Theorem 4.5. *Let $f, g \in \mathbb{C}[x]$ be non-constant polynomials and write $F = \exp(f)$ and $G = \exp(g)$. Suppose that we have an indecomposable pair of polynomials $(\tilde{f}, \tilde{g})$ such that $f = \tilde{f} \circ h$ and $g = \tilde{g} \circ h$ for some polynomial h of degree $D = d_h$. If $d_{\tilde{f}} = 1$ and $d_{\tilde{g}} = d$, then for all positive integers n we have*

$$N(F^n - 1, G^n - 1, r) \ll_{f,g} n^{1/d} r^D$$

where the implicit constant only depends on f and g. Hence

$$N(F^n - 1, G^n - 1, r) \ll_{f,g} n^{-(1-\frac{1}{d})} \min\{T(F^n, r), T(G^n, r)\}.$$

The proof of Theorem 4.4 uses the following result by Bombieri and Pila [3, Theorem 5], along with the next two lemmas.

Theorem 4.6. *Let C be a geometrically irreducible curve defined by $F(x, y) \in \mathbb{R}[x, y]$ of degree $d_C \geq 2$ and let $M \geq \exp(d_C^6)$. Then the number of integral points on C and inside a square $[-M, M] \times [-M, M]$ does not exceed*

$$4M^{1/d_C} \exp(12\sqrt{d_C \log M \log\log M}).$$

Lemma 4.7. *Let f and g be nonconstant polynomials in $\mathbb{C}[x]$ with degree d_f and d_g respectively. Suppose that the pair (f, g) is indecomposable. Let C be the Zariski closure of the image of the map $\phi = [f : g : 1] : \mathbb{P}^1 \to \mathbb{P}^2$. Then the degree of the curve C is $\max(d_f, d_g)$.*

Moreover, if f, g take infinitely many rational values simultaneously, then there are complex numbers $a \neq 0$ and b such that the polynomials $f(az + b)$ and $g(az + b)$ have rational coefficients.

Proof. Since (f, g) is indecomposable, ϕ is generically one to one so that it is the normalization of C.

The degree of C is the intersection number with any line $L \subseteq \mathbb{P}^2$, and since ϕ is the normalization of C, this number is the degree of the zero divisor of $\phi^* H$ for any linear form H, hence the assertion about the degree of C.

For the second part, let K be a field where f, g have coefficients, so that C is defined over K and $\phi : \mathbb{P}^1_K \to C$ is defined over K. If f, g take infinitely many rational values simultaneously, then C can be defined over $\mathbb{Q}$; we write C_0 for a model of C over $\mathbb{Q}$. Let $\nu : \mathbb{P}^1_{\mathbb{Q}} \to C_0$ be the normalization of C_0 defined over $\mathbb{Q}$. After base change to K we get that $\psi := \phi^{-1} \circ (\nu \otimes Id_K) : \mathbb{P}^1_K \to \mathbb{P}^1_K$ is an automorphism (i.e. a Möbius transformation defined over K) satisfying that $\phi \circ \psi = \nu \times Id_K$. This proves that the pair (f, g) composed with a suitable non-constant Möbius transformation ψ gives a pair of rational functions u, v defined over $\mathbb{Q}$. As f, g are polynomials, we get that either u, v are polynomials or they have exactly one pole (at the same point for both); in the first case we are done (ψ is a linear polynomial), so let's assume that we are in the second case. As u, v are defined over $\mathbb{Q}$, the unique pole occurs at a rational number $q \in \mathbb{Q}$ (by considering the Galois action). Hence ψ has its unique pole at q, and therefore $\psi \circ \left(\frac{1}{z} + q\right)$ is a linear polynomial $az + b$ with the required property, because $f(az + b) = u \circ \left(\frac{1}{z} + q\right)$ is still defined over $\mathbb{Q}$ (similarly for g). □

Lemma 4.8. *Let F and G are holomorphic functions and $h(x) := a_d x^d + \ldots + a_1 x + a_0$ in $\mathbb{C}[x]$ with $a_d \neq 0$. Then for sufficiently large r, we have*

$$N(F \circ h, G \circ h, r) \leq N(F, G, (d + 1) \cdot |h| r^d) + O(\log r), \tag{19}$$

where $|h| := \max\{1, |a_d|, \ldots, |a_0|\}$.

Proof. We first observe that $|h(z)| \leq (d + 1) \cdot |h| \max\{1, |z|\}^d$. Secondly, for $w \in \mathbb{C}$

$$\prod_{z \in h^{-1}(w)} z^{v_z(h-w)} = \frac{(-1)^d}{a_d}(a_0 - w), \quad \text{and} \quad \sum_{z \in h^{-1}(w)} v_z(h - w) = d. \tag{20}$$

Since $v_z(F(h)) = v_w(F)\cdot v_z(h-w)$ for $z\in\mathbb{C}$ and $w=h(z)$,

$$\begin{aligned}
N(F(h),G(h),r) &= \sum_{0<|z|\le r} \min\{v_z(F(h)), v_z(G(h))\}\log|\frac{r}{z}| + \min\{v_0(F(h)), v_0(G(h))\}\log r\\
&= \sum_{0<|z|\le r} v_z(h-w)\cdot \min\{v_w(F), v_w(G)\}\log|\frac{r}{z}| + O(\log r)\\
&\le \sum_{0<|w|\le (d+1)\cdot |h| r^d} \min\{v_w(F), v_w(G)\}\cdot (\sum_{z\in h^{-1}(w)} v_z(h-w)\log|\frac{r}{z}|) + O(\log r)\\
&= \sum_{0<|w|\le (d+1)\cdot |h| r^d} \min\{v_w(F), v_w(G)\}\cdot \log|\frac{a_d\cdot r^d}{w-a_0}| + O(\log r)\\
&\le \sum_{0<|w|\le (d+1)\cdot |h| r^d} \min\{v_w(F), v_w(G)\}\log|\frac{a_d\cdot r^d}{w}| + O(\log r)\\
&\le N(F,G,(d+1)\cdot|h|r^d) + O(\log r)
\end{aligned}$$

for r sufficiently large. □

Proof of Theorem 4.4. By Lemma 4.8 and the fact that $T(\exp(f),r)\asymp r^{d_f}$, it suffices to show that for any given $\epsilon>0$

$$N(\exp(\tilde f)-1,\exp(\tilde g)-1,r) \ll_{\tilde f,\tilde g,\epsilon} r^{1+\epsilon}.$$

So we can assume that the pair (f,g) is indecomposable, and that both f,g are polynomials of degree at least 2. We also let $f=2\pi i f_1$ and $g=2\pi i g_1$. Denote by $d=\max\{d_f,d_g\}$.

For $z\in\mathbb{C}$, the condition $\exp(f(z))=1$ is equivalent to the condition $f_1(z)\in\mathbb{Z}$. Moreover, it follows from comparing the Taylor expansions of $f(z)$ and $\exp(f(z))-1$ at $b\in\mathbb{C}$ with $f_1(b)\in\mathbb{Z}$ that $v_b(\exp(f(z))-1) = v_b(f(z)-f(b)) = v_b(f_1(z)-f_1(b))$, and this multiplicity does not exceed the degree of f.

Therefore,

$$n(\exp(f)-1,\exp(g)-1,r) = \sum_{j,k\in\mathbb{Z}} n(f_1-j,g_1-k,r) \le d\cdot \sum_{|z|<r}\delta((f_1(z),g_1(z))\in\mathbb{Z}\times\mathbb{Z}) \tag{21}$$

where the notation $\delta(P)$ means: 1 if the statement P is true, 0 if P is false.

Suppose that there are only finitely many $z\in\mathbb{C}$ such that $f_1(z)\in\mathbb{Z}$ and $g_1(z)\in\mathbb{Z}$. Then (21) implies that

$$N(\exp(f)-1,\exp(g)-1,r)\le O(\log r)$$

proving the result in this case. Therefore, we can assume from now that there are infinitely many $z\in\mathbb{C}$ such that $f_1(z)\in\mathbb{Z}$ and $g_1(z)\in\mathbb{Z}$. By Lemma 4.7 we can assume (after a harmless linear substitution if necessary) that f_1 and g_1 are in fact in $\mathbb{Q}[x]$.

Let C be the Zariski closure of the image of the map $\phi:[f_1:g_1:1]:\mathbb{P}^1\to\mathbb{P}^2$ which is a curve of degree $d>1$ by Lemma 4.7. If $|z|\le r$, then $|f_1(z)|,|g_1(z)|\le cr^d$ where $c=(d+1)\max\{|f_1|,|g_1|\}$. Let us write

$$\mathcal{Z}_C(M) := \{[m:n:1]\in C | (m,n)\in\mathbb{Z}^2\cap[-M,M]\times[-M,M]\}.$$

The map $\phi:\mathbb{P}^1\to C$ is generically 1-to-1, and at all points it is at most d-to-1, so we have the (crude) bound

$$\sum_{|z|<r}\delta((f_1(z),g_1(z))\in\mathbb{Z}\times\mathbb{Z}) \le d\cdot \#\mathcal{Z}_C(cr^d). \tag{22}$$

By Theorem 4.6 and the fact that C has degree $d>1$, we have

$$\#\mathcal{Z}_C(cr^d)\le 4c^{1/d} r\exp(12\sqrt{d\log cr^d\log\log cr^d})$$

for r large. Consequently, $N(\exp(f)-1,\exp(g)-1,r)\ll_{f,g,\epsilon} r^{1+\epsilon}$ for any given $\epsilon>0$. □

Proof of Theorem 4.5. Let n be a positive integer. It will be important to keep uniformity on n, so all the implicit constants in the bounds occurring in this proof will (possibly) depend on f and g, but not on n.

We first consider the case when the pair (f, g) is indecomposable, so that $d_f = 1$ and $d_g = d$. In this case, our aim is to prove that

$$N(\exp(nf) - 1, \exp(ng) - 1, r) \ll_{f,g} n^{1/d} r. \tag{23}$$

After a linear change of variables which only depends on f, we can assume that $f(z) = 2\pi iz$; this change of variables only contribute to the implicit constant $\ll_{f,g}$. Moreover, it will be convenient to write $g = 2\pi iq$ with q a polynomial. With this notation can assume that $q \in \mathbb{Q}[z]$; for otherwise z and $h(z)$ can only take finitely many rational values simultaneously, and we would have

$$N(\exp(nf) - 1, \exp(ng) - 1, r) \ll_{f,g} \log r.$$

We note that

$$\begin{aligned} n(\exp(2\pi inz) - 1, \exp(2\pi inq(z)) - 1, r) &= \sum_{|b|\le r} \sum_{(j,k)\in\mathbb{Z}^2} \min(v_b(nz - j), v_b(nq(z) - k)) \\ &= \sum_{a\in[-nr,nr]\cap\mathbb{Z}} \delta(nq(a/n) \in \mathbb{Z}) \end{aligned}$$

where we used that $v_b(nz - j)$ is either 0 or 1 (the notation $\delta(P)$ means: 1 if P is true, 0 if P is false). Now we estimate the last sum. Let $Q(x, y) \in \mathbb{Z}[x, y]$ be a homogeneous polynomial of total degree d such that the coefficients of Q are coprime and let $D \in \mathbb{Z}$ be such that $q(z) = Q(z, 1)/D$ (Q, D are unique up to sign). Then $Q(x, y) = a_0x^d + a_1x^{d-1}y + \ldots + a_dy^d$ with $a_0 \neq 0 \in \mathbb{Z}$. Then for a, b coprime integers, on has that $\gcd(Q(a, b), b)$ divides a_0. With all these observations and notation we find (the sums run over positive integers):

$$\begin{aligned} \sum_{a\le nr} \delta(nq(a/n) \in \mathbb{Z}) &= \sum_{a\le nr} \delta(Dn^{d-1}|Q(a, n)) \\ &= \sum_{e|n} \sum_{\substack{\gcd(t,e)=1\\ t\le er}} \delta(De^d|nQ(t, e)) \\ &\le \sum_{e|n} \sum_{\substack{\gcd(t,e)=1\\ t\le er}} \delta(e^d|na_0^d) \\ &\le \sum_{e^d|na_0^d} (r + 1)\varphi(e) \\ &= (r + 1) \sum_{e|u(na_0^d)} \varphi(e) \end{aligned}$$

where did the change of variables $e = n/\gcd(a, n)$ and $t = a/\gcd(a, n)$, the symbol $u(M)$ denotes the largest d-th power dividing M, and φ is Euler's totient function. From classic number theory, we know that

$$\sum_{e|A} \varphi(e) = A$$

and certainly we have $u(M) \le M^{1/d}$, so we get

$$\sum_{a\le nr} \delta(nq(a/n) \in \mathbb{Z}) \le (r + 1) \cdot (na_0^d)^{1/d} \ll_{f,g} n^{1/d}(r + 1).$$

A similar estimate holds in the range $a \in [-nr, 1] \cap \mathbb{Z}$, and $a = 0$ contributes at most by 1, therefore for all $r > 0$ we get

$$n(\exp(2\pi inz) - 1, \exp(2\pi inq(z)) - 1, r) - n(\exp(2\pi inz) - 1, \exp(2\pi inq(z)) - 1, 0) \ll_{f,g} n^{1/d}(r + 1).$$

It is important that this estimate holds even for small r uniformly on n because we will integrate it. We also observe that

$$n(\exp(2\pi inz)-1,\exp(2\pi inq(z))-1,r)-n(\exp(2\pi inz)-1,\exp(2\pi inq(z))-1,0)=0 \quad \text{for } 0\le r<1/n.$$

Therefore, for $r>1+\log n$

$$\begin{aligned}
&N(\exp(2\pi inz)-1,\exp(2\pi inq(z))-1,r)\\
&\quad=\int_0^r n(\exp(2\pi inz)-1,\exp(2\pi inq(z))-1,t)-n(\exp(2\pi inz)-1,\exp(2\pi inh(z))-1,0)\frac{dt}{t}\\
&\qquad+n(\exp(2\pi inz)-1,\exp(2\pi inq(z))-1,0)\log r\\
&\quad\ll_{f,g}\int_{1/n}^r \frac{n^{1/d}(t+1)}{t}dt+\log r\\
&\quad\ll_{f,g} n^{1/d}r
\end{aligned}$$

which proves (23).

Finally, let us deduce the general case. The bound

$$N(\exp(nf)-1,\exp(ng)-1,r)\ll_{f,g} n^{1/d}r^D+O(\log r)$$

is obtained from the indecomposable case (that is, from (23)) by using Lemma 4.8.

With the notation of the statement, since $\tilde{f}$ is linear we see that

$$\min\{T(\exp(nf),r),T(\exp(ng),r)\}\asymp_{f,g} nr^D$$

which gives

$$N(\exp(nf)-1,\exp(ng)-1,r)\ll_{f,g} n^{\frac{1}{d}-1}\min\{T(\exp(nf),r),T(\exp(ng),r)\}.$$

□

Finally, we prove Theorem 4.3. We need the following lemma.

Lemma 4.9. *Let f and g be non-constant entire functions such that $af+bg=2\pi ic$ for some $a,b,c\in\mathbb{C}$, not all zeros. If f and g take infinitely many values in $2\pi i\mathbb{Z}$, then $\exp(f)$ and $\exp(g)$ are multiplicatively dependent.*

Proof. As f and g take infinitely same many values in $2\pi i\mathbb{Z}$, a,b,c must be rational with $ab\neq 0$. Taking non-zero integer powers of $\exp(f),\exp(g)$ we can further assume that a,b,c are integers. Consequently, $1=\exp(2\pi ic)=\exp(af+bg)=\exp(af)\exp(bg)$ which shows that $\exp(f)$ and $\exp(g)$ are multiplicatively dependent. □

Proof of Theorem 4.3. By Lemma 4.8 we can assume that the pair (F,G) is indecomposable. By Lemma 4.9 not both F,G are exponential of linear polynomials. If exactly one of them is the exponential of a linear polynomial, then we use Theorem 4.5 to conclude the first bound in the statement of the theorem. If none of F,G is the exponential of a linear polynomial we use Theorem 4.4 to get the second bound in the statement of the theorem, which also implies the first bound. □

4.2. **Min bound for units with growth condition.** Let us recall the notation

$$M_f(r)=\max\{|f(z)|:|z|=r\}$$

for the maximum modulus of an entire function f. This function of r measures the growth of f.

Theorem 4.10. *Let f,g be non-constant entire functions and assume that they are algebraically independent over $\mathbb{Q}$. Suppose that*

$$\lim_{r\to\infty}\frac{\log M_f(12r)\log M_g(12r)}{\min\{M_f(r),M_g(r)\}}=0. \tag{24}$$

Write $F=\exp(2\pi if)$ and $G=\exp(2\pi ig)$. Then for every $\epsilon>0$ we have

$$N(F-1,G-1,r)<_\infty \epsilon\min\{T(F,r),T(G,r)\}, \tag{25}$$

where the notation "$<_\infty$" means that the inequality holds for r in a certain set $U\subseteq\mathbb{R}_{>0}$ of infinite Lebesgue measure.

We remark that the theorem can be stated in an equivalent way by using common integer values of f and g instead of common zeros of $F-1$ and $G-1$. This alternative formulation can be of independent interest for number-theoretical purposes. Also, the constant 12 appearing in the statement is not intended to be optimal.

Before proving Theorem 4.10, let us consider an interesting class of entire functions which clarifies the role of the growth condition in the theorem.

Let $\mathcal{C}$ be the set of entire functions that $\mathbb{C}[z]$-linear combinations of functions of the form $\exp(P(z))$ with $P(z) \in \mathbb{C}[z]$. That is, the elements of $\mathcal{C}$ are the functions of the form

$$f = Q_1 e^{P_1} + \ldots + Q_n e^{P_n}$$

with Q_i, P_i complex polynomials. Thus, $\mathcal{C}$ contains (for instance) all polynomials, $\sin(z)$, $\cos(z)$, $\sinh(z)$ and $\cosh(z)$.

First of all, we observe that Theorem 4.10 can very well be applied to cases when f is a small function of g, such as $f = R(z) \in \mathbb{C}[z]$ and $g = Q_1 e^{P_1} + \ldots + Q_n e^{P_n}$ satisfying that $1 \le \deg P_j < d := \deg R$ for all j, and not all Q_i zero. With these choices we would have

$$\frac{\log M_f(12r) \log M_g(12r)}{\min\{M_f(r), M_g(r)\}} \ll \frac{(\log r) r^{d-1}}{r^d} \to 0$$

which is acceptable for the theorem. Moreover, continuing with the same type of examples, we note that the growth hypothesis is optimal at least for functions in $\mathcal{C}$, in the sense that if we allow $\deg(P_j) \ge d := \deg R$ for some j then we would get examples such as

$$f = z, \quad g = \exp(2\pi i z)$$

(note that this pair is algebraically independent over $\mathbb{Q}$ and indecomposable) for which we have

$$N(F-1, G-1, r) = \min\{T(F,r), T(G,r)\} + O(\log r).$$

The proof of Theorem 4.10 will follow the main idea from [30], adapted to estimate simultaneous integral values of entire functions. We remark that in [17] a similar problem is considered. However, our result does not impose monotonicity conditions on f and g (unlike [17]), although our hypothesis is more restrictive in the aspect of growth conditions.

We will need the following arithmetic lemma.

Lemma 4.11 (Siegel's lemma)**.** *Let $\mathcal{A}$ be an $m \times n$ matrix with integer coefficients, and assume that $m < n$ and that all the entries of $\mathcal{A}$ have absolute value bounded by X. Then there is a non-zero tuple $\mathbf{a} = (a_i)_i \in \mathbb{Z}^n$ in the kernel of $\mathcal{A}$ such that $|a_i| \le (nX)^{m/(n-m)}$ for each $1 \le i \le n$.*

Also, we will use the following standard results from complex analysis.

Lemma 4.12. *Let h be a non-constant entire function. Then for all $r < R$*

$$T(h,r) \le \log^+ M_h \le \frac{R+r}{R-r} T(h,R).$$

Lemma 4.13. *Let h be a non-constant entire function which is not a linear polynomial, and let $1 > \delta > 0$. Then for all large enough r we have*

$$M_h(\delta r) \le \delta M_h(r).$$

Proof. From Cauchy's bound we see that for every C the inequality

$$M_h(r) > Cr$$

holds for infinitely many values of r. Take $C = 1 + M_h(1)$, then there is $r_0 > 1$ with

$$\frac{\log M_h(r_0) - \log M_h(1)}{\log r_0 - \log 1} > 1.$$

By Hadamard's 3-circles theorem we know that $\log M_h(r)$ is a convex function of $\log r$, thus we obtain that for every $r > \max\{r_0, 1/\delta\}$ the following holds

$$\frac{\log M_h(r) - \log M_h(\delta r)}{\log r - \log(\delta r)} > 1$$

which gives what we want. $\square$

Lemma 4.14. *Let X be a discrete set in $\mathbb{C}$ and let $n_X^{(1)}(r) = \#X \cap B(r)$,*

$$\begin{aligned} N_X^{(1)}(r) &:= \int_0^r \frac{n_X^{(1)}(t) - n_X^{(1)}(0)}{t} dt + n_X^{(1)}(0) \log r \\ &= n_X^{(1)}(0) \log r + \sum_{\substack{0<|z|\le r \\ z\in X}} \log \frac{r}{|z|}. \end{aligned}$$

Then for every $0 < \theta < 1$ we have the inequalities

(i) $n_X^{(1)}(r) \ge \frac{1}{\log \theta^{-1}} \left(N_X^{(1)}(r) - N_X^{(1)}(\theta r) \right)$;

(ii) $N_X^{(1)}(r) \ge (\log \theta^{-1}) n_X^{(1)}(\theta r)$ *for* $r > 1/\theta$.

Proof. We have

$$\begin{aligned} \sum_{\substack{\theta r<|z|\le r \\ z\in X}} \log \frac{r}{|z|} &= N_X^{(1)}(r) - n_X^{(1)}(0) \log r - \sum_{\substack{0<|z|\le \theta r \\ z\in X}} \log \frac{r}{|z|} \\ &= N_X^{(1)}(r) - N_X^{(1)}(\theta r) - n_X^{(1)}(0) \log \theta^{-1} - \sum_{\substack{0<|z|\le \theta r \\ z\in X}} \log \theta^{-1} \\ &= N_X^{(1)}(r) - N_X^{(1)}(\theta r) - n_X^{(1)}(\theta r) \log \theta^{-1}. \end{aligned}$$

The first inequality now follows from

$$(n_X^{(1)}(r) - n_X^{(1)}(\theta r)) \log \theta^{-1} \ge \sum_{\substack{\theta r<|z|\le r \\ z\in X}} \log \frac{r}{|z|}$$

and the second inequality follows from

$$\sum_{\substack{\theta r<|z|\le r \\ z\in X}} \log \frac{r}{|z|} \ge 0$$

and the fact that $N_X^{(1)}(\theta r) > 0$ for $r > 1/\theta$. $\square$

Lemma 4.15. *Let h be a non-constant entire function with $h(0) = 0$, let $0 < \theta \le 1/4$ and let $H = \exp(2\pi i h)$. Then we have for all large r*

$$\log M_H(\theta r) \le 2\pi M_h(\theta r) \le 96\theta T(H, r).$$

Proof. By Polya [18] (see [7] for the sharp version that we will use) we have for all large r the bound $M_H(r) \ge \exp(M_{2\pi i h}(r/2)/8)$, that is

$$M_{2\pi i h}(r/2) \le 8 \log M_H(r). \tag{26}$$

Thus we have

$$\begin{aligned} \log M_H(\theta r) &\le M_{2\pi i h}(\theta r) \\ &\le 4\theta M_{2\pi i h}(r/4) \qquad \text{(by Lemma 4.13)} \\ &\le 32\theta \log M_H(r/2) \qquad \text{(by (26))} \\ &\le 32\theta \times 3T(H, r). \qquad \text{(by Lemma 4.12)} \end{aligned}$$

$\square$

For the next lemma see Lemma 6.2.1 in [29].

Lemma 4.16. *Let f be entire, let $0 < \rho < R$ and $0 < r < R$. Then*

$$\log M_f(r) \le \log M_f(R) - n_f(0, \rho) \log \frac{R^2 - r\rho}{R(r + \rho)}.$$

The next lemma will allow us to ignore multiplicities in the proof of Theorem 4.10.

Lemma 4.17. *Let f, g be non-constant entire functions and let $F = \exp(2\pi i f), G = \exp(2\pi i g)$. For all $\delta > 0$ we have the bound*

$$N(F-1, G-1, r) \leq_{exc} N^{(1)}(F-1, G-1, r) + \delta \min\{T(F, r), T(G, r)\}.$$

Proof. Let $n \geq 1$ be a large integer and define $F_n = \exp(2\pi i f/n), G_n = \exp(2\pi i g/n)$. Let μ_n be the set of n-th roots of 1. The first and the second main theorem give

$$N_{F_n}(\mu_n, r) - N^{(1)}_{F_n}(\mu_n, r) \leq_{exc} nT(F_n, r) - (n-3)T(F_n, r) = 3T(F_n, r)$$

and similarly for G_n. It follows that

$$\begin{aligned} N(F-1, G-1, r) &- N^{(1)}(F-1, G-1, r) \\ &= N(F_n^n - 1, G_n^n - 1, r) - N^{(1)}(F_n^n - 1, G_n^n - 1, r) \\ &\leq_{exc} 3 \min\{T(F_n, r), T(G_n, r)\} \\ &= \frac{3}{n} \min\{T(F, r), T(G, r)\}. \end{aligned}$$

□

Proof of Theorem 4.10. After substituting z by $z - z_0$ in both f and g we can assume that both f, g take an integer value at $z = 0$, and after adding suitable integers to f, g (not necessarily the same) we can assume without loss of generality that f, g take the value 0 at $z = 0$. Also, in view of Lemma 4.17 it suffices to prove the result using $N^{(1)}(F-1, G-1, r)$ instead of $N(F-1, G-1, r)$.

Let $\epsilon > 0$ and suppose that the result is false. Then we have

$$N^{(1)}(F-1, G-1, r) \geq_{exc} \epsilon \min\{T(F, r), T(G, r)\}. \tag{27}$$

Define

$$\begin{aligned} \mu(r) &= \min\{T(F, r), T(G, r))\} \\ S_r &= \{z \in B(r) : f(z), g(z) \in \mathbb{Z}\} \end{aligned}$$

so that

$$\#S_r = n^{(1)}(F-1, G-1, r).$$

Recalling that $F = \exp(2\pi i f)$ and $f(0) = 0$ and similarly for g, we have for any fixed $0 < \theta < 1/4$

$$\begin{aligned} \#S_r &= n^{(1)}(F-1, G-1, r) \\ &\geq \frac{1}{\log \theta^{-1}} \Big(N^{(1)}(F-1, G-1, r) - N^{(1)}(F-1, G-1, \theta r) \Big) \qquad \text{(by Lemma 4.14)} \\ &\geq_{exc} \frac{1}{\log \theta^{-1}} \left(\epsilon \mu(r) - \log \min\{M_F(\theta r), M_G(\theta r)\} \right) + O_\theta(1) \\ &\geq \frac{\epsilon - 96\theta}{\log \theta^{-1}} \mu(r) + O_\theta(1). \qquad \text{(by Lemma 4.15)} \end{aligned}$$

Let us choose $\theta = \epsilon/100$ so that $\kappa := \frac{\epsilon - 96\theta}{2 \log \theta^{-1}} > 0$ is a positive constant depending only on $\epsilon > 0$ (which is fixed). Thus

$$\#S_r >_{exc} \kappa \mu(r). \tag{28}$$

Define

$$\begin{aligned} h(r) &= 2 \left(\frac{\#S_r \log M_g(r)}{\log M_f(r)} \right)^{1/2} \\ k(r) &= 2 \left(\frac{\#S_r \log M_f(r)}{\log M_g(r)} \right)^{1/2} \end{aligned}$$

then, by (24), the growth condition, and Lemma 4.13 we know that both $h(r), k(r)$ grow to infinity. Moreover we have

$$4\#S_r = h(r)k(r).$$

Let r_0 be a large sufficiently large number (throughout the proof we will impose a finite number of conditions on it) and let $h_0 := h(r_0)$ and $k_0 := k(r_0)$. In view of (28) we can assume that the bound

$$\#S_r > \kappa\mu(r)$$

holds for all $r \geq r_0$, except perhaps on an *open* set $E \subseteq \mathbb{R}_{\geq r_0}$ of measure $\leq 1/2$ (taking r_0 large enough). The natural projection map $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is a covering map and if we endow $\mathbb{R}/\mathbb{Z}$ with the push-forward measure (which agrees with the Lebesgue measure on any fundamental domain $[x, x+1)$) we deduce that $\pi(E)$ has measure $\leq 1/2$. It follows that there is $r' \in [r_0, r_0+1)$ such that $r'+\mathbb{N} = \{r', r'+1, r'+2, \dots\} \cap E = \emptyset$, and we can replace r_0 by r', so that now the bound $\#S_r > \kappa\mu(r)$ holds for all $r \in r_0 + \mathbb{N}$. Making r_0 larger if necessary, we can assume that $h(r) \geq h_0$ and $k(r) \geq k_0$ for all $r \in r_0 + \mathbb{N}$.

By Siegel's lemma there are find integers a_{ij} such that

$$A(z) := \sum_{0\leq i<h_0} \sum_{0\leq j\leq k_0} a_{ij} f(z)^i g(z)^j$$

satisfies:

- $A(z) = 0$ for each $z \in S_{r_0}$,
- $\log |a_{ij}| \leq 2(\#S_{r_0})^{1/2}(\log M_f(r_0) \log M_g(r_0))^{1/2}$, and
- not all the a_{ij} are zero.

In fact, the first item imposes $\#S_{r_0}$ (possibly dependent) linear equations on the a_{ij} with integer coefficients (here we use that f, g are integer valued on S_{r_0}), and there are at least $h_0k_0 = 4\#S_{r_0}$ unknowns, so the logarithm of the bound in Siegel's lemma is

$$\begin{aligned}
&\frac{\#S_{r_0}}{h_0k_0 - \#S_{r_0}} \log(h_0k_0 M_f(r_0)^{h_0} M_g(r_0)^{k_0}) \\
&\qquad\qquad = \frac{1}{3}(\log(h_0k_0) + h_0 \log M_f(r_0) + k_0 \log M_g(r_0)) \\
&\qquad\qquad < 2(\#S_{r_0})^{1/2}(\log M_f(r_0) \log M_g(r_0))^{1/2}.
\end{aligned}$$

By construction A vanishes on S_{r_0}. Also, A is not identically zero because f, g are algebraically independent over $\mathbb{Q}$, so that it makes sense to consider $\log M_A(r)$. We claim that actually for every r we have

(i)$_r$: A vanishes at every point of S_r; and
(ii)$_r$: $\log M_A(r) < 0$.

As in [30] this is done by an inductive argument.

First note that for every r we have that (ii)$_r$ implies (i)$_r$ because if $z_0 \in S_r$ and $A(z_0) \neq 0$ then we have $|A(z_0)| \geq 1$. Moreover, note that it suffices to prove these claims for an unbounded sequence of values of r, and we restrict ourselves to $r \in r_0 + \mathbb{N}$, so that in particular we have $\#S_r \geq \kappa\mu(r)$.

We know that A vanishes at every point of S_{r_0}, that is, we already know (i)$_{r_0}$. For $r \in r_0 + \mathbb{N}$, suppose that (i)$_r$ holds and let us show that (ii)$_{r+1}$ holds (this will complete the induction).

The triangle inequality gives for all $R \geq r_0$

$$\begin{aligned}
\log M_A(R) &\leq \log \left(h_0k_0 \max_{ij}\{|a_{ij}|\} M_f(R)^{h_0} M_g(R)^{k_0} \right) \\
&\leq \log(4\#S_{r_0}) + 2(\#S_{r_0})^{1/2}(\log M_f(r_0) \log M_g(r_0))^{1/2} \\
&\quad + 2(\#S_{r_0})^{1/2} \left((\log M_f(r_0) \log M_g(R))^{1/2} + (\log M_f(R) \log M_g(r_0))^{1/2} \right) \\
&\leq 4(\#S_{r_0})^{1/2} \left((\log M_f(r_0) \log M_g(R))^{1/2} + (\log M_f(R) \log M_g(r_0))^{1/2} \right) \\
&\leq 8(\#S_{r_0})^{1/2}(\log M_f(R) \log M_g(R))^{1/2}
\end{aligned}$$

and since we are assuming (i)$_r$ we have

$$n_A^{(1)}(0, r) \geq \#S_r \geq \kappa\mu(r).$$

Therefore Lemma 4.16 (with $\rho = r+1$ and $R = 3r$ so that $(R^2 - r\rho)/(R(r+\rho)) > 1.3$) gives

$$\begin{aligned}\log M_A(r+1) &\le \log M_A(3r) - n_A^{(1)}(0,r)\log(1.3)\\ &\le 8(\#S_{r_0})^{1/2}(\log M_f(3r)\log M_g(3r))^{1/2} - \log(1.3)\#S_r\\ &\le 8(\#S_r)\left(\left(\frac{\log M_f(3r)\log M_g(3r)}{\#S_r}\right)^{1/2} - \frac{\log(1.3)}{8}\right)\\ &\le 8(\#S_r)\left(\left(\frac{\log M_f(3r)\log M_g(3r)}{\kappa\mu(r)}\right)^{1/2} - \frac{\log(1.3)}{8}\right)\end{aligned}$$

which is negative thanks to our growth hypothesis, since by Lemma 4.15 we have

$$\frac{\log M_f(3r)\log M_g(3r)}{\min\{T(F,r),T(G,r)\}} \le \frac{12\log M_f(3r)\log M_g(3r)}{\pi\min\{M_f(r/4),M_g(r/4)\}}$$

and r_0 is sufficiently large with $r \ge r_0$ (recall that κ only depends on ϵ). This proves $(\mathrm{ii})_{r+1}$, completing the induction.

Therefore A is an entire function with the following properties:

- A is not the zero function
- A has "many" zeros (in the sense that $(\mathrm{i})_r$ holds for all r)
- A is bounded (because $(\mathrm{ii})_r$ holds for all r).

These three properties are contradictory by Liouville's theorem. This contradiction shows that the bound (27) cannot hold, hence proving the result. □

5. Some general gcd bounds on average

The results in this section cover the complex case and also the non-Archimedean case.

5.1. **The main results of average gcd.** From now on, we let $k = \mathbb{C}$ or $\mathbf{k}$, an algebraically closed field complete with respect to a non-Archimedean absolute value $|\ |$ and of arbitrary characteristic.

The lack of exponential functions in the non-Archimedean case forces us to look to the general case when f, g are not assumed to be exponentials. This is more difficult, mainly, because Nevanlinna theory is not well-suited (as far as we know) to consider vanishing orders at given infinitely many points of k as part of the "bad places" (i.e. in the proximity function).

It turns out that if we care about the *average* behavior as we vary n in the expression

$$N(F^n - 1, G^n - 1, r),$$

then a non-trivial max bound can be achieved in full generality (that is, without assuming that F, G are units). In particular, it will follow that a very sharp bound holds for all but a negligible set of exponents n. To state this result in a precise way, let us define for any given x and $0 < \Delta < x$ the quantity

$$\mathrm{Avg}_{x,\Delta}(f^n - 1, g^n - 1, r) := \frac{1}{\Delta}\sum_{x-\Delta<n\le x} N(f^n - 1, g^n - 1, r)$$

which is nothing but the average size of $N(f^n - 1, g^n - 1, r)$ for $n \in (x-\Delta, x]$ (that is, for n "Δ-close" to x); we will investigate this quantity for given x and Δ, as $r \to \infty$.

One's hope is that for multiplicatively independent f, g we have that for every $\epsilon > 0$, if $n \gg_\epsilon 1$ then

$$N(f^n - 1, g^n - 1) <_{exc} \epsilon\max\{nT(f,r), nT(g,r)\}.$$

However, in this generality, such a bound seems beyond the existing techniques. We will give further discussion of this conjectured inequality in Section 7.

The average version should then state that, for every $\epsilon > 0$, if $x \gg_\epsilon 1$ and if Δ is not so small (so that we can actually take advantage of averaging, then

$$\mathrm{Avg}_{x,\Delta}(f^n - 1, g^n - 1, r) < \epsilon\max\{xT(f,r), xT(g,r)\}$$

for r away from a set of finite measure (possibly depending on ϵ and x). In fact we prove a much stronger and general average result:

Theorem 5.1. *Let $k=\mathbb{C}$ or $\mathbf{k}$, an algebraically closed field complete with respect to a non-Archimedean absolute value of arbitrary characteristic. Let f,g be algebraically independent meromorphic functions on k. For any given $x\geq e$, any $y\in[e,x)$ and any $\Delta\in[1,x)$ we have*

$$\mathrm{Avg}_{x,\Delta}(f^n-1,g^n-1,r)<\left(\frac{14(y\log y)^{1.5}}{x}+\frac{2\sqrt{2}\cdot\Delta^{0.5}}{y}+\frac{2\sqrt{2}}{\Delta^{0.5}}\right)\max\{xT(f,r),xT(g,r)\}+O_x(1).$$

Before discussing the proof of this theorem, let us present some corollaries that optimize different aspects of this general bound.

The next bound is for averages over intervals of fixed length, for large exponent. It is nontrivial already for the average of $N(f^n-1,g^n-1,r)$ over nine consecutive values of the exponent n (taking $\Delta=9$).

Corollary 5.2. *Let $k=\mathbb{C}$ or $\mathbf{k}$, an algebraically closed field complete with respect to a non-Archimedean absolute value of arbitrary characteristic. Let f,g be algebraically independent meromorphic functions on k. Let Δ be a given positive integer. For any given $\epsilon>0$, we have that all $n\gg_{\epsilon,\Delta}1$ satisfy*

$$\frac{1}{\Delta}\sum_{j=1}^{\Delta}N(f^{n+1-j}-1,g^{n+1-j}-1,r)<\left(\frac{2\sqrt{2}}{\Delta^{0.5}}+\epsilon\right)\max\{nT(f,r),nT(g,r)\}+O(1).$$

Proof. Choose $y=x^{2/5}$ and $x=n$. This actually gives $\epsilon=\epsilon(n)=O_\tau(1/n^{2/5-\tau})$ for any $\tau>0$. □

The next bound is for the average over the dyadic interval $n\in(x/2,x]$ (taking $\Delta=x/2$).

Corollary 5.3. *Let $k=\mathbb{C}$ or $\mathbf{k}$, an algebraically closed field complete with respect to a non-Archimedean absolute value of arbitrary characteristic. Let f,g be algebraically independent meromorphic functions on k. For each $x\geq 6$ we have the bound*

$$\frac{1}{x}\sum_{x/2<n\leq x}N(f^n-1,g^n-1,r)<\frac{6+7(\log x)^{1.5}}{x^{1/10}}\max\{xT(f,r),xT(g,r)\}+O_x(1).$$

Proof. Take $\Delta=x/2$ and $y=x^{3/5}$; as $x\geq 6$ we have $y\geq e$. The quantity in parenthesis in the main theorem becomes

$$14\cdot(3/5)^{1.5}\frac{(\log x)^{1.5}}{x^{1/10}}+\frac{2}{x^{1/10}}+\frac{4}{x^{0.5}}<\frac{6+7(\log x)^{1.5}}{x^{1/10}}.$$

□

Note that the factor $\epsilon(x):=(6+7(\log x)^{1.5})/x^{1/10}$ is as small as we want provided that x is large. Thus, this corollary is an improved version of Theorem 1.2 from the introduction, and it gives the following improved version of Theorem 1.3 as a consequence:

Corollary 5.4. *Let $k=\mathbb{C}$ or $\mathbf{k}$, an algebraically closed field complete with respect to a non-Archimedean absolute value of arbitrary characteristic. Let $\epsilon>0$. Let E_ϵ be set of positive integers n for which the following fails*

$$(*)\qquad N(f^n-1,g^n-1,r)<_\infty \epsilon\max\{nT(f,r),nT(g,r)\}.$$

Then for any positive τ there is a uniform constant $K_{\epsilon,\tau}$ depending only on the numbers ϵ and τ (independent of f,g) such that for all large x

$$\#E_\epsilon\cap[1,x]<K_{\epsilon,\tau}x^{\frac{9}{10}+\tau}.$$

In particular, E_ϵ has density zero in $\mathbb{N}$, which means that most n satisfy $(*)$.

Proof. Let x_0 be a sufficiently large number and let $x>x_0$ be arbitrary. Since f and g are nonconatant meromorphic functions, the constant $O_x(1)$ in Corollary 5.3 is bounded by $\max\{T_f(r),T_g(r)\}$ for sufficiently large r. Therefore, we have

$$\sum_{x/2<n\leq x}N(f^n-1,g^n-1,r)<(6+7(\log x)^{1.5})x^{9/10}\max\{xT(f,r),xT(g,r)\}\tag{29}$$

for $r\gg 1$. On the other hand, for each $n\in E_\epsilon$

$$N(f^n-1,g^n-1,r)\geq_{exc}\epsilon\max\{nT(f,r),nT(g,r)\}.\tag{30}$$

Let $m_{x,\epsilon} = \#E_\epsilon \cap (x/2, x]$. Then (30) implies that

$$\sum_{x/2<n\leq x} N(f^n-1, g^n-1, r) \geq_{exc} \epsilon m_{x,\epsilon} \max\{\frac{x}{2}T(f,r), \frac{x}{2}T(g,r)\}. \tag{31}$$

It follows from (29) and (31) that

$$m_{x,\epsilon} < \frac{1}{\epsilon}(12 + 14(\log x)^{1.5})x^{9/10}.$$

Consequently,

$$\begin{aligned}\#E_\epsilon \cap [1,x] &\leq x_0 + \sum_{i=0}^{\log_2 x} m_{x/2^i,\epsilon} \\ &\leq x_0 + \frac{1}{\epsilon}(12+14(\log x)^{1.5})x^{9/10}\sum_{i=0}^{\infty}\frac{1}{2^{9i/10}}\end{aligned}$$

from which the result follows. □

5.2. **Proof of the average bound.** The proof of Theorem 5.1 does not use the second main theorem, only the first main theorem (on $\mathbb{P}^2$).

We need the following elementary construction from algebraic combinatorics (see [26] for a discussion on some striking applications of it).

Lemma 5.5. *Let k be a field and let S be a set of M points in k^2. There is a non-zero polynomial $P(X,Y) \in k[X,Y]$ of degree $d \leq \sqrt{2M}$ which vanishes at every point of S.*

Proof. The vector space $V_d = k[X,Y]_{\leq d}$ of bivariate polynomials of degree $\leq d$ has dimension

$$\binom{d+2}{2} = \frac{(d+2)(d+1)}{2}.$$

Evaluation at points in S defines a linear map $L : V_d \to k^M$. For M small relative to d this map will have non-trivial kernel (which is what we want); more precisely, there is non-trivial kernel as soon as $M < (d+1)(d+2)/2$. Taking $d = \lfloor\sqrt{2M}\rfloor$ we see that the last requirement is achieved. □

Before proving Theorem 5.1, we also need an arithmetic lemma that will allow us to count roots of unity. We write φ for Euler's totient function, μ for Möbius function, ω for the function counting the number of distinct prime divisors, and $[a,b]$ for the lcm.

Lemma 5.6. *For all $X > e$ we have*

$$\sum_{[a,b]\leq X} 1 < 22X(\log X)^3$$

where the sum runs over all ordered pairs of positive integers (a,b) whose lcm it at most X.

Proof. Let $\ell(n)$ be the number of ordered pairs of positive integers (a,b) with $[a,b] = n$. For any such pair (a,b) we have a unique way to write $n = uvg$ with u,v coprime and $a = ug, b = vg$ (in fact, g is the gcd of a,b). Thus ℓ can be expressed as a Dirichlet convolution

$$\ell(n) = \sum_{g|n}\sum_{\substack{(u,v)=1\\ uv=n/g}} 1 = \sum_{g|n} 2^{\omega(n/g)} = (1 * 2^\omega)(n).$$

Let $\zeta(s)$ be the Riemann zeta function. Since the Dirichlet series of 2^ω is $\zeta(s)^2/\zeta(2s)$ (see [14, Exercise 1.2.6]), the previous identity gives that for $\Re(s) > 1$ we have

$$\sum_{n\geq 1}\frac{\ell(n)}{n^s} = \frac{\zeta(s)^3}{\zeta(2s)}$$

which has a pole of order 3 at $s = 1$. Choose $s = 1 + 1/\log X$, then, on the one hand we have

$$\sum_{n\geq 1}\frac{\ell(n)}{n^{1+1/\log X}} > e^{-1}\sum_{n=1}^{X}\ell(n)/n > \frac{1}{eX}\sum_{n\leq X}\ell(n)$$

and on the other hand, one knows that $\zeta(\sigma) < 2/(\sigma-1)$ for all real $1<\sigma\le 2$, so we have

$$\frac{\zeta(1+1/\log X)^3}{\zeta(2+2/\log X)} < \zeta(1+1/\log X)^3 < 8(\log X)^3.$$

The result follows, since the sum in the statement is precisely $\sum_{n\le X}\ell(n)$. □

We remark that the sum is actually asymptotic to $(\pi^2/6)X(\log X)^2$ as can be seen from the Tauberian theorem. We prefer our bound which is valid for all $X>e$, not just for large enough X.

Proof of Theorem 5.1. We write μ_n for the set of n-th roots of 1. Fix $x\ge 6$ and $1\le\Delta<x$, and consider the set

$$S_x = \bigcup_{x-\Delta<n\le x} \mu_n\times\mu_n \subseteq k^2.$$

Then

$$\#S_x \le \sum_{x-\Delta<n\le x} n^2 \le x^2\Delta.$$

For $p\in S_x$ we write $p=(\zeta_p,\xi_p)$ and we let a_p (resp. b_p) be the multiplicative order of ζ_p (resp. ξ_p).

For any given y with $e<y<x$, let us split the set S_x as

$$S_x^- := \{p\in S_x : [a_p,b_p]\le y\},\quad S_x^+ := \{p\in S_x : [a_p,b_p]> y\}.$$

Note that by Lemma 5.6 (and using the trivial bound $\varphi(n)\le n$) we have

$$\#S_x^- \le \sum_{[a,b]\le y}\varphi(a)\varphi(b)\le 22y^3(\log y)^3$$

and also, we have

$$\#S_x^+ \le \#S_x \le x^2\Delta.$$

By Lemma 5.5, there are algebraic curves $C^-,C^+\subseteq\mathbb{P}^2$ defined over k, of degrees

$$d^-\le 7(y\log y)^{1.5},\quad d^+\le\sqrt{2}\cdot x\Delta^{0.5}$$

respectively, and passing through all points in S_x^- and S_x^+ respectively – we are identifying a fixed affine chart in $\mathbb{P}^2$ with k^2. Consider the analytic map

$$F=[f:g:1]:\mathbb{A}^1\to\mathbb{P}^2.$$

This map has Zariski dense image because f,g are algebraically independent. Hence, the image of F is not contained in C^-, nor C^+. We also note that it follows from the definition of characteristic functions that

$$T_F(r)\le 2_k\max\{T(f,r),T(g,r)\},$$

where $2_k=2$ if $k=\mathbb{C}$ and $2_k=1$ if k is non-Archimedean. We want to relate $\mathrm{Avg}_{x,\Delta}(f^n-1,g^n-1,r)$ to $N_F(C^-,r)$ and $N_F(C^+,r)$. To clarify the discussion, let us briefly explain the strategy: if the sets $\mu_n\times\mu_n$ were disjoint as n varies in $(x-\Delta,x]$, then we would see that each point in S_x is counted in exactly one summand of $\mathrm{Avg}_{x,\Delta}(f^n-1,g^n-1,r)$, hence we would have

$$(?)\qquad \Delta\mathrm{Avg}_{x,\Delta}(f^n-1,g^n-1,r)\le N_F(C^-,r)+N_F(C^+,r).$$

Unfortunately, the sets $\mu_n\times\mu_n$ are not disjoint as n varies, and we need to take repetitions into account. So, while it is not clear that the bound (?) holds, we will nevertheless prove a weaker version of it which suffices for our purposes.

Now we return to the proof. For $p\in S_x$ we let

$$\rho(p,x)=\#\{n\in(x-\Delta,x]:p\in\mu_n\times\mu_n\}.$$

The values of $n\in(x-\Delta,x]$ for which $p\in\mu_n\times\mu_n$ are precisely those $n\in(x-\Delta,x]$ divisible by $[a_p,b_p]$ (the lcm), hence

$$\rho(p,x)=\left\lceil\frac{x}{[a_p,b_p]}\right\rceil-\left\lceil\frac{x-\Delta}{[a_p,b_p])}\right\rceil\le\frac{\Delta}{[a_p,b_p]}+1.$$

It follows that

$$\rho(p,x) \leq \begin{cases} \Delta & \text{if } p \in S_x^- \quad \text{trivial upper bound using } \#\mathbb{Z} \cap (x-\Delta, x], \\ \frac{\Delta}{y} + 1 & \text{if } p \in S_x^+ \quad \text{by previous estimate and definition of } S_x^+. \end{cases}$$

Now let us bound $\mathrm{Avg}_{x,\Delta}(f^n - 1, g^n - 1, r)$ in terms of $N_F(C^-, r)$ and $N_F(C^+, r)$. First we note that

$$\begin{aligned} \Delta \mathrm{Avg}_{x,\Delta}(f^n-1, g^n-1, r) &= \sum_{x-\Delta < n \leq x} \sum_{(\zeta,\xi) \in \mu_n \times \mu_n} N(f-\zeta, g-\xi, r) \\ &= \sum_{p \in S_x} \rho(p,x) N(f-\zeta_p, g-\xi_p, r) \\ &= \sum_{p \in S_x^-} \rho(p,x) N(f-\zeta_p, g-\xi_p, r) + \sum_{p \in S_x^+} \rho(p,x) N(f-\zeta_p, g-\xi_p, r). \end{aligned}$$

For S_x^- and using the fact that C^- passes through every point in S_x^-, we have

$$\begin{aligned} \sum_{p \in S_x^-} \rho(p,x) N(f-\zeta_p, g-\xi_p, r) &\leq \Delta \sum_{p \in S_x^-} N(f-\zeta_p, g-\xi_p, r) \\ &\leq \Delta N_F(C^-, r). \end{aligned}$$

Simiarly, for S_x^+ we have

$$\begin{aligned} \sum_{p \in S_x^+} \rho(p,x) N(f-\zeta_p, g-\xi_p, r) &\leq \left(\frac{\Delta}{y} + 1\right) \sum_{p \in S_x^+} N(f-\zeta_p, g-\xi_p, r) \\ &\leq \left(\frac{\Delta}{y} + 1\right) N_F(C^+, r). \end{aligned}$$

Putting these bounds together, we find our substitute for the inequality (?):

$$\Delta \mathrm{Avg}_{x,\Delta}(f^n-1, g^n-1, r) \leq \Delta N_F(C^-, r) + \left(\frac{\Delta}{y} + 1\right) N_F(C^+, r).$$

Now, by the First Main Theorem (Theorem 2.5) and recalling that the degree of C^- is $d^- \leq 7(y \log y)^{1.5}$ we get

$$\begin{aligned} N_F(C^-, r) &\leq d^- T_F(r) + O_x(1) \\ &< 7(y \log y)^{1.5} T_F(r) + O_x(1) \\ &\leq 14(y \log y)^{1.5} \max\{T(f,r), T(g,r)\} + O_x(1). \end{aligned}$$

Similarly, using $d^+ \leq \sqrt{2} \cdot x\Delta^{0.5}$ we get

$$\begin{aligned} N_F(C^+, r) &\leq d^+ T_F(r) + O_x(1) \\ &< \sqrt{2} \cdot x\Delta^{0.5} T_F(r) + O_x(1) \\ &\leq 2\sqrt{2} \cdot x\Delta^{0.5} \max\{T(f,r), T(g,r)\} + O_x(1). \end{aligned}$$

We therefore find

$$\Delta \mathrm{Avg}_{x,\Delta}(f^n-1, g^n-1, r) \leq \left(14\Delta(y \log y)^{1.5} + 2\sqrt{2} \cdot x\Delta^{0.5}\left(\frac{\Delta}{y} + 1\right)\right) \max\{T(f,r), T(g,r)\} + O_x(1)$$

which is the same as

$$\mathrm{Avg}_{x,\Delta}(f^n-1, g^n-1, r) \leq \left(\frac{14(y \log y)^{1.5}}{x} + \frac{2\sqrt{2} \cdot \Delta^{0.5}}{y} + \frac{2\sqrt{2}}{\Delta^{0.5}}\right) \max\{xT(f,r), xT(g,r)\} + O_x(1).$$

$\square$

5.3. **Application: an analogue of a conjecture of Pisot.** In this section we illustrate the applicability of the results obtained from our average bounds in the previous section.

A special case of Pisot's conjecture states that given integers $a, b > 1$, if $a^n - 1$ divides $b^n - 1$ for all large positive integer n, then b is a power of a, see [19] for a solution of the original problem, and see [31] for an overview of the existing improvements. The following result can be viewed as an analogue in the complex and non-Archimedean situation. (Moreover, we will later prove a stronger version of this result in the non-Archimedean case of characteristic zero, see Section 6.2.)

Proposition 5.7. *Let $k = \mathbb{C}$ or $\mathbf{k}$, an algebraically closed field of characteristic zero, complete with respect to a non-Archimedean absolute value. Let f, g be transcendental entire functions with $T(f,r) \asymp T(g,r)$ (i.e. of the same order of magnitude). Suppose that there is a set $\mathcal{P}$ of primes with positive upper density in the primes such that for all $p \in \mathcal{P}$ we have $g^p - 1 | f^p - 1$. Then f, g are multiplicatively dependent and in fact $f^n = g^m$ for some positive integers m and n.*

Proof. We will prove the complex case to simplify the notation; the non-Archimedean case follows similarly.

Assume that f and g are multiplicatively independent.

If f and g are algebraically dependent entire functions, then the image of the analytic curve $F(z) := (f(z), g(z)) \to \mathbb{C} \times \mathbb{C}$ is contained in an algebraic curve $C \subset \mathbb{C} \times \mathbb{C}$. As f, g are multiplicatively independent, the curve C can only contain a finite number of pairs of roots of unity thanks to a theorem of Tate, Serre and Ihara (cf. [13, Chapter 8, Theorem 6.1]).

It follows that $f^p - 1$ and $g^p - 1$ have no common zeros if p is a sufficiently large prime, which is a contradiction. Thus we are left with the case when f and g are algebraically independent.

Suppose now that f, g are algebraically independent. Let $C > 1$ be such that $T(f,r) < CT(g,r)$ for all r large. Then Corollary 5.4 with $\epsilon = 1/(2C)$ gives the existence of infinitely many primes $p \in S := \mathcal{P} \setminus E_\epsilon$ (here we use the assumption that there is $\sigma > 0$ with $\#\mathcal{P} \cap [1, x] > \sigma x / \log x$ infinitely often, and the precise information on the size of E_ϵ provided by the corollary). For any such $p \in S$ satisfying $g^p - 1 | f^p - 1$ we then have

$$N_{g^p-1}(r, 0) = N(g^p - 1, f^p - 1, r) <_\infty \frac{1}{2C} \max\{pT(f,r), pT(g,r)\}.$$

Using Theorem 2.3, we have

$$(p-3)T(g,r) <_{exc} N_{g^p-1}(r, 0).$$

Hence,

$$(p-3)T(g,r) <_\infty \frac{p}{2C} \max\{T(f,r), T(g,r)\} < \frac{p}{2} T(g,r).$$

As soon as $p \geq 7$ we get a contradiction. □

6. More on the non-Archimedean case

A correspondence between non-Archimedean Nevanlinna theory and certain Diophantine statements over the integers $\mathbb{Z}$ or the rational numbers $\mathbb{Q}$ was studied in [2]. The main observation is that $\mathbb{Z}$ has only one archimedean place and the proximity function of non-Archimedean entire functions behave similarly. In particular, the characteristic functions of entire functions on $\mathbf{k}$ enjoy several useful properties as following.

Proposition 6.1. *Let f and g be a non-constant entire functions on $\mathbf{k}$. Then for r sufficiently large, we have*

(i) $T(fg, r) = T(f,r) + T(g,r)$,
(ii) $T(f+g, r) \leq \max\{T(f,r), T(g,r)\}$,
(iii) $T(f,r) = N_f(r,a) + O(1)$, *for* $a \in \mathbf{k}$.

We note that (i) and (iii) fail in the complex setting, while (ii) only holds in a weaker form over the complex numbers.

Proof. Since f has no pole in $\mathbf{k}$, $T(f,r) = m_f(r, \infty) = \log^+ |f|_r = \log |f|_r$ for r sufficiently large. Then the first two assertion follows from the fact that $|fg|_r = |f|_r \cdot |g|_r$ and $|f+g|_r \leq \max\{|f|_r, |g|_r\}$. The last assertion follows from Theorem 2.7 and that $|f-a|_r$ tends to infinity as r is sufficiently large. □

We will be also using the following refinement the truncated second main theorem for diagonal equations of non-Archimedean entire functions, which can be obtained via the linear algebra arguments in [4]. (See also [21] for the complex case.)

Theorem 6.2. *Let $\mathbf{k}$ be an algebraically closed field complete with respect to a non-Archimedean absolute value of characteristic zero. Let $\mathbf{f} = [f_0 : f_1 : \cdots : f_n] : \mathbf{k} \to \mathbb{P}^n(\mathbf{k})$ be a non-Archimedean holomorphic map, with $f_0, \cdots, f_n$ non-Archimedean entire and no common zero. Let $f_{n+1} = -\sum_{i=1}^n f_i$. If $\sum_{i\in I} f_i \neq 0$ for any proper subset I of $\{0, 1, \cdots, n+1\}$, then*

$$T_{\mathbf{f}}(r) := \log \max\{|f_0|_r, \ldots, |f_n|_r\} \leq \sum_{j=0}^{n+1} N_{f_j}^{(n)}(r, 0) - \log^+ r$$

for $r > 0$.

6.1. gcd bounds for non-Archimedean entire functions.

Theorem 6.3. *Let f, g be multiplicatively independent non-Archimedean entire functions f and g on $\mathbf{k}$ of characteristic zero. Then*

$$N(f^n - 1, g^n - 1, r) \leq \frac{n+2}{2n} \max\{T(f^n, r), T(g^n, r)\} - \frac{1}{2} \log r + O(1)$$

for sufficiently large r

We note that the proof of the theorem is analogous to [15] where the gcd of $2^n - 1$ and $3^n - 1$ is estimated under the *abc* conjecture.

Proof. Let h be an entire function on $\mathbf{k}$ such that

$$u := \frac{f^n - 1}{h} \quad \text{and} \quad v := \frac{g^n - 1}{h} \tag{32}$$

are entire functions with no common zero on $\mathbf{k}$, i.e. h is a gcd of $f^n - 1$ and $g^n - 1$. We note that such h exists and is unique up to a constant factor and

$$N(f^n - 1, g^n - 1, r) = N_h(r, 0). \tag{33}$$

We also let α be a gcd of f and g and write $f = \alpha f_0$ and $g = \alpha g_0$. Then (32) yields

$$v f_0^n - u g_0^n = \frac{v - u}{\alpha^n}. \tag{34}$$

We note that $\frac{v-u}{\alpha^n}$ is an entire function. It is clear from (32) that $v f_0^n$ and $u g_0^n$ are not constant and have no common zeros. Applying Theorem 6.2 for (34), we have

$$\begin{aligned}
&\max\{T(v f_0^n, r), T(u g_0^n, r)\} \\
&\leq N_u^{(1)}(r, 0) + N_v^{(1)}(r, 0) + N_{\frac{v-u}{\alpha^n}}^{(1)}(r, 0) + N_{f_0}^{(1)}(r, 0) + N_{g_0}^{(1)}(r, 0) - \log r + O(1) \\
&\leq T(u, r) + T(v, r) + \max\{T(u, r), T(v, r)\} - nT(\alpha, r) + T(f_0, r) + T(g_0, r) - \log r + O(1)
\end{aligned}$$

for r sufficiently large, where the last inequality follows from Theorem 2.7 and that Proposition 6.1. Since u, v, f_0 and g_0 are entire functions on $\mathbf{k}$,

$$T(v f_0^n, r) = T(v, r) + nT(f_0, r) \quad \text{and} \quad T(u g_0^n, r) = T(u, r) + nT(g_0, r)$$

for r sufficiently large. Therefore, we get

$$(n-1)T(f_0, r) \leq T(u, r) + \max\{T(u, r), T(v, r)\} - nT(\alpha, r) + T(g_0, r) - \log r + O(1), \tag{35}$$

and

$$(n-1)T(g_0, r) \leq T(v, r) + \max\{T(u, r), T(v, r)\} - nT(\alpha, r) + T(f_0, r) - \log r + O(1), \tag{36}$$

for r sufficiently large. On the other hand, since $uh = f^n - 1$ and $vh = g^n - 1$,

$$|u|_r \cdot |h|_r = |f^n - 1|_r = |f^n|_r = nT(f, r) \quad \text{and} \quad |v|_r \cdot |h|_r = |g^n - 1|_r = |g^n|_r = nT_g(r)$$

for r sufficiently large. Therefore, if $T(f,r) \geq T(g,r)$ for a fixed r which is sufficiently large, then $T(u,r) \geq T(v,r)$. In this case, (35) implies that

$$T(u,r) \geq \frac{n-2}{2}T(f_0,r) + \frac{n}{2}T(\alpha,r) + \frac{1}{2}\log r + O(1) \geq \frac{n-2}{2}T(f,r) + \frac{1}{2}\log r + O(1)$$

for r sufficiently large. Consequently,

$$\begin{aligned} T(h,r) = nT(f,r) - T(u,r) &\leq \frac{n+2}{2}T(f,r) - \frac{1}{2}\log r + O(1) \\ &= \frac{n+2}{2n}\max\{T(f^n,r), T(g^n,r)\} - \frac{1}{2}\log r + O(1) \end{aligned}$$

for r sufficiently large.

□

6.2. **Divisible sequences revisited: the non-Archimedean case.** We will derive a stronger version of Proposition 5.7. The main tool is the version of the truncated second main theorem stated in Theorem 6.2.

Proposition 6.4. *Let* $\mathbf{k}$ *be an algebraically closed field complete with respect to a non-Archimedean absolute value of zero characteristic. Let* f, g *be transcendental entire functions with* $T_f(r) \asymp T_g(r)$ *(i.e. of the same order of magnitude). Suppose that* $g^n - 1 | f^n - 1$ *for infinitely many integers. Then* f, g *are multiplicatively dependent and in fact* $f^n = g^m$ *for some positive integers* m *and* n*.*

Proof. For a positive integer n, we let

$$q(n) := \frac{f^n - 1}{g^n - 1}. \tag{37}$$

Assume that there exists a positive integer c such that $cT_g(r) > T_f(r)$ for $r \gg 1$. Note that

$$\begin{aligned} (g^{cn} - 1)q(n) &= (g^n - 1)q(n)(1 + g + \cdots + g^{(c-1)n}) && (38) \\ &= (f^n - 1)(1 + g^n + \cdots + g^{(c-1)n}). && (39) \end{aligned}$$

We rewrite this equation as follows.

$$g^{cn}q(n) + \sum_{j=0}^{c-1} g^{jn} - \sum_{j=0}^{c-1} f^n g^{jn} = q(n). \tag{40}$$

Let $F(n) = (g^{cn}q(n), 1, g^n, \cdots, g^{(c-1)n}, -f^n, -f^n g^n, \cdots, -f^n g^{(c-1)n})$. We note that f and g are entire functions and $q(n)$ is entire if $f^n - 1$ is divisible by $g^n - 1$. Moreover, F is a reduced representation when $q(n)$ in entire. Suppose that no proper subsum of the left hand side of (40) is zero. Then Theorem 6.2 implies that for $r > 0$

$$\begin{aligned} T_{F(n)}(r) &\leq N^{(2c)}_{g^{cn}q(n)}(r,0) + \sum_{j=0}^{c-1} N^{(2c)}_{g^{jn}}(r,0) + \sum_{j=0}^{c-1} N^{(2c)}_{f^n g^{jn}}(r,0) + N^{(2c)}_{q(n)}(r,0) - \log^+ r \\ &\leq 2N_{q(n)}(r,0) + (4c^2 + 2c)N_g(r,0) + 2c^2 N_f(r,0) - \log^+ r \\ &\leq 2T(q(n),r) + (4c^2 + 2c)T(g,r) + 2c^2 T(f,r) - \log^+ r \\ &\leq 2T(q(n),r) + (2c^3 + 4c^2 + 2c)T(g,r) - \log^+ r. \end{aligned}$$

We note that $T(g^{cn}q(n),r) \leq T_{F(n)}(r) + O(1)$ and

$$T(g^{cn}q(n),r) = T(g^{cn},r) + T(q(n),r) + O(1)$$

Then the previous equation becomes

$$T(g^{cn},r) - T(q(n),r) \leq (2c^3 + 4c^2 + 2c)T(g,r) - \log^+ r + O(1).$$

Again, since

$$T(q(n),r) = T(f^n - 1,r) - T(g^n - 1,r) + O(1) = T(f^n,r) - T(g^n,r) + O(1),$$

and $T(g^{cn}, r) \geq T(f^n, r)$, we obtain

$$T(g^n, r) \leq (2c^3 + 4c^2 + 2c)T(g, r) - \log^+ r + O(1).$$

This is impossible if $n > 2c^3 + 4c^2 + 2c$.

It remains to consider when some proper subsum of the left hand side of (40) is zero, and it leads to the following two equations with no vanishing proper subsum:

$$\sum_{i \in I_1} g^{in} - \sum_{j \in J_1} f^n g^{jn} = 0, \tag{41}$$

or

$$g^{cn} q(n) + \sum_{i \in I_2} g^{in} - \sum_{j \in J_2} f^n g^{jn} = 0, \tag{42}$$

where I_i and J_i, $i = 1, 2$, are subsets of $\{0, 1, \cdots, c-1\}$. We note that neither I_1 nor J_1 is empty as f and g are nonconstant entire functions. Furthermore, $|I_1| + |J_1| \geq 3$ since f and g are multiplicatively independent. One can show that (41) is impossible if $n \geq 2c^3 + 4c^2$ by Theorem 6.2. Since the arguments are similar to the previous one, we will omit the proof.

We now consider when (42) happens. It is clear that we can exclude the case that $|I_2| = 1$ and $|J_2| = 0$ since $q(n)$ is entire and $c > i$ for $i \in I_2$. If $|I_2| = 0$ and $|J_2| = 1$, then (42) gives $g^{(c-j)n} q(n) = f^n$, where $0 \leq j \leq c - 1$. Since $q(n)$ and f have no common zero, it implies that $q(n)$ is a constant, i.e. $f^n - 1 = ag^n - a$ for some $a \in \mathbf{k}^*$. Therefore, $ag^{(c-j)n} = ag^n - a + 1$ which is impossible. We can now assume that $|I_2| + |J_2| \geq 2$. Similar to the previous argument, we can show that

$$\begin{aligned} T(g^n, r) + T(q(n), r) &\leq N^{(2c)}_{g^{cn} q(n)}(r, 0) + \sum_{j=0}^{c-1} N^{(2c)}_{g^{jn}}(r, 0) + \sum_{j=0}^{c-1} N^{(2c)}_{f^n g^{jn}}(r, 0) - \log^+ r \\ &\leq N_{q(n)}(r, 0) + (4c^2 + 2c) N_g(r, 0) + 2c^2 N_f(r, 0) - \log^+ r \\ &\leq T(q(n), r) + (4c^2 + 2c) T(g, r) + 2c^2 T(f, r) - \log^+ r \\ &\leq T(q(n), r) + (2c^3 + 4c^2 + 2c) T(g, r) - \log^+ r. \end{aligned}$$

This is impossible if $n > 2c^3 + 4c^2 + 2c$. □

7. Expected gcd bounds under Vojta's conjectures

Let us recall the following conjecture of Vojta. (cf. [28, Chapter 15]).

Conjecture 7.1 (Vojta)**.** *Let X be a complex smooth projective variety. Let A be an ample divisor on X, let D be a normal crossing divisor on X and let K_X be a canonical divisor on X. Then for every $\epsilon > 0$ there exists a proper Zariski closed subset $Z = Z(X, D, A, \epsilon)$ such that the inequality*

$$m_{f,D}(r) + T_{f,K_X}(r) \leq_{exc} \epsilon T_{f,A}(r) \tag{43}$$

holds for all non constant holomorphic curves $f : \mathbb{C} \to X$ whose image is not contained in Z.

The purpose of this section is to clarify what should be expected for gcd bounds in the holomorphic setting, at least under the previous conjecture. See also [25] where Silverman shows how Vojta's conjectures on heights of rational points imply very precise gcd bounds in a number of settings of arithmetic relevance. This final section can be seen as the holomorphic counterpart of Silverman's arguments, and it gives a conditional answer to some of the questions posed in the introduction of this work. We will only consider the complex case; if the reader is willing to assume a non-Archimedean version of Vojta's conjecture, then similar conditional results can be obtained for non-Archimedean meromorphic functions.

Proposition 7.2. *Let f_1 and f_2 be algebraically independent complex meromorphic functions. Assume that Vojta's conjecture is true (for $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at a single point). Then for any $\epsilon > 0$, the following* *inequality holds*

$$N(f_1-1,f_2-1,r)\le_{exc}\epsilon\max\{T_{f_1}(r),T_{f_2}(r)\}$$
$$+\frac{1}{1+\epsilon/4}\sum_{i=1}^{2}(N_{f_i,0}(r)+N_{f_i,\infty}(r))+O(1).$$

We will need the following lemma.

Lemma 7.3. *Let $X=\mathbb{P}^1\times\mathbb{P}^1$, let $\pi:V\to X$ be the blow-up of X at a point $P\in X$, and let E be the exceptional divisor. Let K_X be a canonical divisor of X. Then $-\pi^*K_X-E$ is an ample divisor on V.*

Proof. This is a straightforward computation, we include it for the convenience of the reader. Via the Segre embedding we can identify X with a smooth quadric in $\mathbb{P}^3$. If H is a hyperplane on $\mathbb{P}^3$ then $-2H|_X$ is a canonical divisor on X.

We use the Nakai-Moishezon criterion to see that $-\pi^*K_X-E$ is ample. First, note that K_X can be taken so that its support does not contain P, hence

$$(-\pi^*K_X-E)^2=(\pi^*K_X)^2+E^2=4-1=3>0$$

Let C be an irreducible curve on V, we want to see that $C.(-\pi^*K_X-E)>0$. This is clear if $C=E$ so we can assume that $C\neq E$, so that $C':=\pi(C)$ is a curve on X. Let $\mu_P(C')$ be the multiplicity of C' at P, then for any hyperplane $H\subseteq\mathbb{P}^3$ passing through P we have

$$C.E=\mu_P(C')\le(C'.H)_{\mathbb{P}^3}=(C'.H|_X)_X<(C'.-K_X)_X=(C.-\pi^*K_X)_V$$

where the intersection numbers are taken in $\mathbb{P}^3$, X and V as indicated. This proves what we want. (In fact, the same argument shows that $-\pi^*K_X-\theta E$ is ample whenever $0<\theta<2$.) ☐

Proof of Proposition 7.2. Let $X=\mathbb{P}^1\times\mathbb{P}^1$ and take $P=(1,1)\in X$ (identifying $\mathbb{A}^1$ with a fixed affine chart of $\mathbb{P}^1$). Let $\pi:V\to X$ be the blow-up of X at P and let E be the exceptional divisor.

The divisor

$$K_X=-(\infty\times\mathbb{P}^1)-(0\times\mathbb{P}^1)-(\mathbb{P}^1\times\infty)-(\mathbb{P}^1\times 0)$$

is a canonical divisor on X, and therefore we have the following canonical divisor on V

$$K_V:=\pi^*K_X+E. \tag{44}$$

By the previous lemma, the divisor $A:=-K_V$ is an ample divisor of V.

Let us first consider the case when f_1,f_2 are algebraically independent.

Assume Vojta's conjecture on V and let $D=-\pi^*K_X$, which has normal crossings. Let $f=(f_1,f_2):\mathbb{C}\to X$, and let $\tilde f:\mathbb{C}\to V$ be the lift of f such that $\pi\circ\tilde f=f$. The image of f is Zariski dense in X because f_1,f_2 are algebraically independent. Then for $\epsilon>0$, Vojta's conjecture gives

$$m_{\tilde f,-\pi^*K_X}(r)+T_{\tilde f,K_V}(r)\le_{\text{exc}}\epsilon T_{\tilde f,A}(r).$$

By (44) and the definition of A, this implies

$$m_{\tilde f,-\pi^*K_X}(r)+T_{\tilde f,\pi^*K_X}(r)+T_{\tilde f,E}(r)\le_{exc}-\epsilon T_{\tilde f,E}(r)-\epsilon T_{\tilde f,\pi^*K_X}(r)+O(1).$$

Hence,

$$(1+\epsilon)\,T_{\tilde f,E}(r)\le_{exc}N_{\tilde f,-\pi^*K_X}(r)+\epsilon T_{\tilde f,-\pi^*K_X}(r)+O(1).$$

By functoriality properties of the counting function and by the bound $N_{\tilde f,E}(r)\le T_{\tilde f,E}(r)$, we conclude

$$\begin{aligned}(1+\epsilon)\,N(f_1-1,f_2-1,r)&=(1+\epsilon)\,N_{\tilde f,E}(r)\\&\le_{exc}N_{f,-K_X}(r)+\epsilon T_{f,-K_X}(r)+O(1)\\&=\sum_{i=1}^{2}(N_{f_i,0}(r)+N_{f_i,\infty}(r))+2\epsilon(T_{f_1}(r)+T_{f_2}(r))+O(1).\end{aligned}$$

Consequently, we have

$$N(f_1-1,f_2-1,r) \le_{exc} \frac{4\epsilon}{1+\epsilon}\max\{T_{f_1}(r),T_{f_2}(r)\} + \frac{1}{1+\epsilon}\sum_{i=1}^{2}(N_{f_i,0}(r)+N_{f_i,\infty}(r))+O(1)$$

which proves the result f_1, f_2 (after replacing ϵ by $\epsilon'/4$).

□

Let us observe that a similar argument gives the following result for $N(f_1^n-1,f_2^n-1,r)$:

Proposition 7.4. *Let f_1 and f_2 be multiplicatively independent meromorphic functions. Assume that Vojta's conjecture is true (for $\mathbb{P}^1\times\mathbb{P}^1$ blown up along a set of finitely many points of $\mathbb{P}^1\times\mathbb{P}^1$). Then for any $\epsilon>0$, there exists n_0 such that for all $n\ge n_0$*

$$N(f_1^n-1,f_2^n-1,r) \le_{exc} \epsilon\max\{T_{f_1^n}(r),T_{f_2^n}(r)\}.$$

Before going into the proof, let us remark the fact that our Corollary 5.4 (see also Theorem 1.3) can be seen as an unconditional partial result towards this conjectural bound.

Proof. Let us briefly indicate the necessary modifications to the previous argument. If f_1, f_2 are algebraically dependent (and multiplicatively independent as in the statement) the required bound can be proved in an even stronger form by using the result of Tate, Serre and Ihara in the same way as it was applied in Section 5.3. We leave the details to the interested reader. So we can assume that f_1, f_2 are algebraically independent.

Firstly, fix a positive integer n. We let V be the blow-up of $X=\mathbb{P}^1\times\mathbb{P}^2$ at the n^2 points of the set $\mu_n\times\mu_n\subseteq X$, where μ_n is the set of n-th roots of 1. Let E be the exceptional divisor (which consists of n^2 disjoint copies of $\mathbb{P}^1$). One can show that for some sufficiently large M (depending on n), the divisor $-\pi^*K_X-\frac{1}{M}E$ is ample; see Section 4 of [25]. The same computations as done before give us

$$\begin{aligned} N(f_1^n-1,f_2^n-1,r) &\le \frac{4\epsilon}{n(1+\frac{\epsilon}{M})}\max\{nT_{f_1}(r),nT_{f_2}(r)\} + \frac{1}{n(1+\frac{\epsilon}{M})}\sum_{i=1}^{2}(N_{f_i^n,0}(r)+N_{f_i^n,\infty}(r))+O(1)\\ &\le \frac{4\epsilon}{n(1+\frac{\epsilon}{M})}\max\{nT_{f_1}(r),nT_{f_2}(r)\} + \frac{4}{n(1+\frac{\epsilon}{M})}\max\{nT_{f_1}(r),nT_{f_2}(r)\}+O(1)\\ &= \frac{4(1+\epsilon)}{n(1+\frac{\epsilon}{M})}\max\{nT_{f_1}(r),nT_{f_2}(r)\}+O(1). \end{aligned}$$

From here, the result follows. □

One can also formulate a "truncated" version of Vojta's conjecture, where the bound (43) is replaced by the sharper bound

$$T_{f,D+K_X}(r) \le_{exc} N^{(1)}_{f,D}(r)+\epsilon T_{f,A}(r) \tag{45}$$

(this implies (43) by the first main theorem), see [28, Chapter 23]. Assuming this refined conjecture, one deduces a version of Proposition 7.2 with the stronger bound

$$N(f_1-1,f_2-1,r) \le_{exc} \epsilon\max\{T_{f_1}(r),T_{f_2}(r)\} + \frac{1}{1+\epsilon/4}\sum_{i=1}^{2}(N^{(1)}_{f_i,0}(r)+N^{(1)}_{f_i,\infty}(r))+O(1).$$

Finally, let us comment that this stronger bound (which is obtained under a stronger conjecture) implies at once the bound obtained in Proposition 7.4. Nevertheless, Proposition 7.4 only assumes the non-truncated version of Vojta's conjecture.

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