Spheres in $\mathbb{R}^n$

Let c_n be the volume in $\mathbb{R}^n$ bounded by

$$x_1^2 + \cdots + x_n^2 = 1\,.$$

It is easy to se that

$$c_{n+1} = 2c_n \int_0^1 (1-x^2)^{n/2} dx\,, \tag{1}$$

which implies that $\lim_{n\to\infty} c_n = 0$.

We want to study the behavior of the center of gravity of the region bounded by a hemisphsere H_n as $n \to \infty$. In $\mathbb{R}^{n+1}$) the region is given by

$$x_1^2 + \cdots + x_n^2 + x_{n+1}^2 \le 1\,, \quad x_{n+1} \ge 0\,,$$

and the center of gravity will be

$$G_n = (0, \ldots, 0, h)\,, \quad h = h(n) > 0\,.$$

Intuition suggests $h(n+1) < h(n)$, and we will furthermore show that

Lemma 1: The center of gravity G_n tends to the origin as $n \to \infty$, that is,

$$\lim_{n\to\infty} h(n) = 0\,.$$

Proof: Let$\epsilon > 0$. We will show first that there exists n_0 so that $h(n) < \epsilon$ if $n \ge n_0$. Let $A = A(n,\epsilon)$ be the subregion of H_n where $x_{n+1} \ge \epsilon$, and let $B = B(n,\epsilon)$ be its complement in H_n. Then

$$\mathrm{Vol}(A) = c_n \int_\epsilon^1 (1-x^2)^{n/2} dx \le c_n \left[1-\epsilon^2\right]^{n/2}\,,$$

while

$$\mathrm{Vol}(B) = c_n \int_0^\epsilon (1-x^2)^{n/2} dx \ge c_n(\epsilon/2)\left[1-(\epsilon/2)^2\right]^{n/2}\,.$$

The second bound follows from the fact that B contains a "disk" of height $\epsilon/2$ and radius $\sqrt{1-(\epsilon/2)^2}$. We will show that if n is sufficiently large, then the region A produces a momentum with respect to the hyperplane $x_{n+1} = \epsilon$ that is smaller than that momentum of the region B. This shows that for such large values of n, the center of gravity must lie below ϵ.

The momentum produced by A is $M_A = \mathrm{Vol}(A) \cdot [a(n) - h(n)]$, where $a(n)$ represents the position on its axis of symmetry of the center of gravity of A. It follows that

$$M_A \le \mathrm{Vol}(A) \le c_n \left[1-\epsilon^2\right]^{n/2}\,. \tag{3}$$

The momentum M_B is larger than that of the aforementioned disk, that is,

$$M_B \geq c_n \frac{\epsilon}{2} \left[1 - (\epsilon/2)^2\right]^{n/2} \cdot \frac{3\epsilon}{4} \,. \tag{4}$$

It follows from (3) and (4) that, for ϵ fixed, $M_B > M_A$ for all large n,

Lemma 2: *The heights $h(n)$ are decreasing with n.*

Proof: The expression $h = h(n)$ is determined by the equation:

$$\int_0^h (h - x)(1 - x^2)^{n/2} dx = \int_h^1 (x - h)(1 - x^2)^{n/2} dx \,. \tag{5}$$

Let $\alpha(s), \beta(s)$ be defined by

$$\alpha(s) = \int_s^1 (x - s)(1 - x^2)^{n/2} dx \quad , \quad \beta(s) = \int_0^s (s - x)(1 - x^2)^{n/2} dx \,.$$

It is easy to see that

$$\alpha'(s) = -\int_s^1 (1 - x^2)^{n/2} dx < 0 \quad , \quad \beta'(s) = \int_0^s (1 - x^2)^{n/2} dx > 0 \,,$$

and that

$$\alpha(0) > 0 \,,\ \alpha(1) = 0 \,,\ \beta(0) = 0 \,,\ \beta(1) > 0 \,.$$

With this, the value $h = h(n)$ for which (5) hold is unique. On the other hand,

$$\frac{\partial \alpha}{\partial n} = \frac{1}{2} \int_s^1 (x - s)(1 - x^2)^{n/2} \log(1 - x^2)\, dx$$

y

$$\frac{\partial \beta}{\partial n} = \frac{1}{2} \int_0^s (s - x)(1 - x^2)^{n/2} \log(1 - x^2)\, dx \,.$$

Since (5) holds, one can see that

$$\int_0^h (h - x)(1 - x^2)^{n/2} \log(1 - x^2)\, dx > \int_h^1 (x - h)(1 - x^2)^{n/2} \log(1 - x^2)\, dx \,,$$

es decir,

$$\frac{\partial \beta}{\partial n}(h) > \frac{\partial \alpha}{\partial n}(h) \,.$$

This ensure that the point of intersection of the graphs of α y β moves to the left as n increases.