ON THE COEFFICIENTS OF SMALL UNIVALENT FUNCTIONS

M. CHUAQUI , B. OSGOOD AND CH. POMMERENKE

Abstract. For every $\alpha > 0$ there exists an analytic univalent function $f(z) = a_1 z + a_2 z^2 + \cdots$ satisfying

$$(1 - |z|^2)|f''(z)/f'(z)| \leq \alpha \quad \text{for } |z| < 1$$

such that $|a_n| > n^{c\alpha^2 - 1}$ for infinitely many n.

1. Introduction

Let the function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be analytic in the unit disk $\mathbb{D}$ and let $f'(z) \neq 0$. We assume that

$$(1.1) \qquad (1 - |z|^2)|f''(z)/f'(z)| \leq \alpha \quad \text{for } z \in \mathbb{D}\,.$$

If f us univalent then (1.2) holds with $\alpha = 6$ [7, Prop.1.2]. Conversely, if (1.2) holds with $\alpha \leq 1$ then f is univalent, and if $\alpha < 1$ then $f(\mathbb{D})$ is a quasidisk [1]. A related condition on the Schwarzian Sf is

$$(1.2) \qquad ||Sf|| = \sup_{|z|<1} (1 - |z|^2)^2 |Sf(z)| \leq \alpha\,.$$

If f is univalent then $||Sf|| \leq 6$. Conversely, if $||Sf|| \leq 2$ then f is univalent [6]; this Nehari class has been studied e.g. in [3], [4]. If $f''(0) = 0$ then

$$(1.3) \qquad ||Sf|| \leq \alpha \leq 2 \Rightarrow (1.2) \Rightarrow ||Sf|| \leq \text{const} \cdot \alpha\,.$$

We shall study univalent functions that are small in the sense that $\alpha > 0$ is small in (1.2) or equivalently in (1.3); see (1.4). This does not imply anything about the regularity of the boundary. Indeed there are functions satisfying (1.2) with arbitrarily small α sich that $\partial f(\mathbb{D})$ does not possess a tangent at any point [7, p.190 and p.193/194].

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We need a result of Makarov [5]; explicit bounds were given by Rohde [7, p.191].

Proposition 1: *For small $\alpha > 0$ there exists a univalent function $g(z) = z + \cdots$ such that*

$$(1 - |z|^2)|g''(z)/g'(z)| \leq \alpha/2 \quad \text{for } z \in \mathbb{D}\,, \tag{1.4}$$

and, with some constant $c_0 > 0$,

$$\int_{-\pi}^{\pi} |g'(re^{it})|dt > (1-r)^{-c_0\alpha^2} \quad \text{for } r_0 < r < 1\,. \tag{1.5}$$

In order to obtain g from the function f constructed by Makarov we set $g' = (f')^{\alpha/12}$.

The standard way to obtain an upper bound of the coefficients is to use the elementary estimate

$$|na_n| < \text{const} \int_{-\pi}^{\pi} |f'(re^{it})|dt\,,\ r = 1 - \frac{\text{const}}{n}\,. \tag{1.6}$$

Carleson and Jones have shown that, surprisingly, one does not lose much in (1.7); see [2, Thm. 2] and Proposition 2 below.

If f satisfies (1.2) then [7, Exer. 8.3.4] and (1.7) show that

$$a_n = O\left(n^{\alpha^2/4-1}\right) \quad (n \to \infty)\,. \tag{1.7}$$

We shall prove that this estimate is best possible except for the constant.

Theorem: *For sufficiently small $\alpha > 0$, there exists a univalent function with*

$$(1 - |z|^2)|f''(z)/f'(z)| \leq \alpha \quad \text{for } z \in \mathbb{D}$$

such that, with some constant $c > 0$,

$$|a_n| > n^{c\alpha^2-1} \quad \text{for infinitely many } n\,. \tag{1.8}$$

2. The Carleson-Jones modification

We shall need the following variant of an important theorem of Carleson and Jones [2, Th. 1]. Our variant contains information about ψ''/ψ'.

Proposition 2: *Let $n > 1600$ and $\epsilon > 0$ be given and let the analytic function $\varphi(z) = \sum \alpha_k z^k$ satisfy*

$$(1 - |z|^2)|\varphi''(z)/\varphi'(z)| \leq \gamma \leq 6 \quad \text{for } z \in \mathbb{D}\,. \tag{2.1}$$

Then there exists a function $\psi = \sum \beta_k z^k$ such that, with $r = 1 - 1600/n$,

$$|\beta_k| \geq |\alpha_k| r^{k-1} \text{for} 1 \leq k \leq n\,, \tag{2.2}$$

$$(1-|z|^2)|\psi''(z)/\psi'(z)| < \gamma + \epsilon \quad \text{for } z \in \mathbb{D}\,, \tag{2.3}$$

$$|2n\beta_{2n}| > c'\epsilon \int_{-\pi}^{\pi} |\varphi'(re^{it})|dt \tag{2.4}$$

where $c' > 0$ is an absolute constant.

Proof: Let $\delta > 0$. We consider the polynomial

$$p(z) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\sum_{\nu=0}^{2n}\left(1 - \frac{|\nu - n|}{n+1}e^{i(n-\nu)t}\right)\frac{|\varphi'(re^{it})|}{\varphi'(re^{it})}z^{n+\nu}dt$$

of degree $3n$ and for $z \in \mathbb{D}$ define

$$\psi(z) = \frac{1}{r}\varphi(rz) + \frac{\delta}{n}p(z)\varphi'(rz)\,. \tag{2.5}$$

Then (2.2) holds because $p(z)$ contains no powers z^k with $k < n$. Using the properties of the Fejér kernel [8, p. I88], Carleson and Jones [2, p.178] have shown that

$$|\beta_{2n}| \geq \frac{\delta}{4\pi n}\int_{-\pi}^{\pi}|\varphi'(re^{it})|dt - |\alpha_{2n}| \geq \frac{\delta}{8\pi n}\int_{-\pi}^{\pi}|\varphi'(re^{it})|dt\,; \tag{2.6}$$

if the second inequality is false then we simple choose $\psi = \varphi$ instead of (2.6) and (2.2), (2.3) and (2.4) are trivially true. Furthermore, for $z \in \mathbb{D}$

$$|p(z)| \leq 1\ ,\ \ |p'(z)| \leq 3n\,; \tag{2.7}$$

the first estimate follows from (2.5) and then the second from Bernstein's inequality [8, p. II 11].

Now we estimate ψ''/ψ'. Let $c_1, \ldots$ denote suitable absolute constants. It follows from (2.6) that, for $z \in \overline{\mathbb{D}}$,

$$|\psi'(z)| \geq |\varphi'(rz)| - \frac{\delta}{n}|p'(z)\varphi'(rz) + p(z)\varphi''(rz)|$$

$$\geq |\varphi'(rz)|\left(1 - \frac{\delta}{n}\left(3n + \frac{\gamma}{1-r^2}\right)\right) \geq |\varphi'(rz)|(1 - c_1\delta) \tag{2.8}$$

by (2,8) and (2.1). Furthermore

$$|\psi''(z)| \leq |\varphi''(rz)| + \frac{\delta}{n}\left|p''(z)\varphi'(rz) + 2rp'(z)\varphi''(rz) + r^2p(z)\varphi'''(rz)\right|\,. \tag{2.9}$$

Using a standard argument we deduce from (2.8) and (2.1) that

$$|p''(z)| \leq \frac{c_3 n}{1-|z|}\ ,\ \left|\frac{\varphi'''(rz)}{\varphi'(rz)}\right| \leq \frac{c_4\gamma}{(1-r)(1-|rz|)} \leq \frac{c_5\gamma n}{1-|z|}\,.$$

Hence we obtain from (2.10) by (2.1) and (2.8) that

$$(1-|z|^2)\left|\frac{\psi''(z)|}{|\varphi'(rz)|}\right| \le \frac{\gamma+2\delta(c_3+3\gamma+c_5\gamma)}{1-c_1\delta} \le \gamma + c_6\delta\,.$$

Therefore (2.3) holds is we choose $\delta = \epsilon/c_6$, and (2.4) holds by (2.7).

3. Proof of the theorem

(a) By $c_1, \ldots$ we denote positive absolute constants. Let $0<\alpha<1$ and let g be the function in Proposition 1 of Makarov. Let $0<q_k<1$ and let m_k be a (large) integer. For $z \in D$ we define h_k by

$$h_k'(z) = q_k g'(z^{m_k})\ ,\ h_k(0)=0\,. \tag{3.1}$$

Let $|z| = r < 1$ and write $m = m_k$. We obtain from (3.1) and (1.5) that

$$(1-r^2)\left|\frac{h_k''(z)|}{h_k'(rz)}\right| = (1-r^2)mr^{m-1}\left|\frac{g''(z^m)|}{g'(z^m|}\right| \le \frac{\alpha m r^{m-1}(1-r^2)}{2(1-r^{2m})} < \frac{\alpha}{2}\,. \tag{3.2}$$

If $r \le 1-1/\sqrt{m}$ then $r^m \le (1-1/\sqrt{m})^m < e^{-\sqrt{m}}$ and thus

$$(1-r^2)|h_k''(z)/h_k'(z)| < e^{\sqrt{m}/2} \tag{3.3}$$

provided that $m = m_k$ is large.

It follows from (1.5) ny integration that

$$|g'(z)| \le c_1(1-r)^{-1/2} \tag{3.4}$$

and thus by the maximum principle that $|z^{-1}(g'(z)-1)| \le c_2(1-r)^{-1/2}$. hence, by (3.1),

$$|h_k(z) - q_k z| = q_k\left|\int_0^z [g'(\zeta^m)-1]\,d\zeta\right| \le c_2\int_0^1 s^m(1-s^m)^{-1/2}ds \le \frac{c_3}{m}\,.$$

Choosing $q_k = 1 - c_3 m_k^{-1} - m_k^{-1/8}$ we deduce that

$$|h_k(z)| < q_k + c_3 m_k^{-1} = 1 - m_k^{-1/8} \quad \text{for} z \in D\,. \tag{3.5}$$

(b) We will recursively define integers m_k and n_k with

$$m_k > \max\left[n_k, \left(c_4\alpha^{-1}2^{k+2}\right)^8\right] \quad \text{for } k = 1, 2, \ldots \tag{3.6}$$

(see (3.10) below for c_4) and functions

$$f_k(z) = z + \sum_{n=2}^{\infty} a_{kn} z^n \quad (z \in \mathbb{D}) \tag{3.7}$$

starting with $f_0(z) = z$. We write

$$\eta_k = \max\left(\frac{\alpha}{2}, \sup_{z\in\mathbb{D}}(1-|z|^2)|f_k''(z)/f_k'(z)|\right). \tag{3.8}$$

Suppose that η_j and f_j have already been constructed for $j \le k$, also m_j for $j < k$. Let m_k satisfy (3.6) and define $\varphi_k = f_k \circ h_k$. Then

$$\frac{\varphi_k''(z)}{\varphi_k'(z)} = \frac{h_k''(z)}{h_k'(z)} + h_k'(z)\frac{f_k''(h_k(z))}{f_k'(h_k(z))}$$

and thus, for $|z| = r < 1$,

$$(1-r^2)\left|\frac{\varphi_k''}{\varphi_k'}\right| \le (1-r^2)\left|\frac{h_k''}{h_k'}\right| + \frac{(1-r^2)|h_k'|}{1-|h_k|^2}(1-|h_k|^2)\left|\frac{f_k''(h_k)}{f_k'(h_k)}\right|. \tag{3.9}$$

First suppose that $0 \le r \le 1-1/\sqrt{m_k}$. Since $h_k(\mathbb{D}) \subset \mathbb{D}$ we have $(1-r^2)|h_k'|/(1-|h_k|^2) \le 1$ and thus, by (3.9), (3.2) and (3.8),

$$(1-r^2)\left|\frac{\varphi_k''}{\varphi_k'}\right| \le e^{-\sqrt{m_k}/2} + \eta_k\,.$$

Now suppose that $1-1/\sqrt{m_k} < r < 1$. Then, by (3.9), (3.2), (3.4) and (3.5),

$$(1-r^2)\left|\frac{\varphi_k''}{\varphi_k'}\right| < \frac{\alpha}{2} + \frac{c_1(1-r^2)m_k^{1/8}}{(1-r^{m_k})^{1/2}}\eta_k \le \frac{\alpha}{2} + c_4 m_k^{-1/8}\eta_k\,. \tag{3.10}$$

If m_k is sufficiently large we therefore obtain, by (3.8) and (3.6),

$$(1-r^2)|\varphi_k''(z)/\varphi_k'(z)| < \eta_k + \alpha 2^{-k-2} \quad \text{for } |z| = r < 1\,. \tag{3.11}$$

This finally determines m_k.

Now let f_{k+1} be the Carleson-Jones modification of φ_k with $\epsilon = \alpha 2^{-k-2}$; see Proposition 2. Since $\eta_0 = \alpha/2$ we obtain from (3.8) and (3.11) that, for $z \in \mathbb{D}$,

$$(1-|z|^2)\left|\frac{f_{k+1}''(z)}{f_{k+1}'(z)}\right| \le \eta_{k+1} < \frac{\alpha}{2} + 2\sum_{j=0}^{k}\alpha 2^{-j-2} < \alpha\,. \tag{3.12}$$

Finally we apply Proposition 1. We choose $n_{k+1} > 2n_k$ so large that (see (3.7), (2.4) and (1.6))

$$\begin{aligned}\left|n_{k+1}a_{k+1,n_{k+1}}\right| &> \frac{c'\alpha}{2^{k+2}}\int_{-\pi}^{\pi}\left|\varphi_k'\left(\left(1-\frac{3200}{n_{k+1}}\right)e^{it}\right)\right| dt \\ &> \frac{c_5\alpha}{2^{k+2}} n_{k+1}^{c_0\alpha^2} > n_{k+1}^{c_0\alpha^2}\,. \end{aligned} \tag{3.13}$$

This concludes our recursive construction.

(c) Since $m_k > n_k$ by (3.6), it follows from (3.1) and (3.7) that

$$\varphi_k(z) = f_k(h_k(z)) = \sum_{n=1}^{n_k} q_k^n a_{kn} z^n + O(z^{n_k+1}) \quad (z \to 0)\,.$$

The coefficients of the Carleson-Jones modification f_{k+1} therefore satisfy

$$|a_{k+1,n}| \geq q_k |a_{k,n}| \left(1 - \frac{3200}{n_{k+1}}\right) \quad \text{for } 1 \leq n \leq n_k$$

by (2.2). Using that $q_k = 1 - c_3 m_k^{-1} - m_k^{-1/8} > 1 - c_6 2^{-k}$ by (3.6), we therefore obtain

$$|a_{k+1,n}| > |a_{j,n}| \prod_{\nu=j}^{k} \left[\left(1 - \frac{c_6}{2^\nu}\right)^n \left(1 - \frac{3200}{n_{\nu+1}}\right)^{n-1} \right]$$

for $k \geq j$ and $n \leq n_j$. Hence, by (3.13) for $k = j-1$,

$$|a_{k+1,n_j}| > c_7 n_j^{c_0 \alpha^2/2 - 1} \quad \text{for} k \geq j \tag{3.14}$$

because $n_{\nu+1} > 2n_\nu$.

We select a convergent subsequence from $/f_k)$. Its limit f satisfies (1.2) by (3.12) and satisfies (1.) by (3.14).

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