An Extension of a Theorem of Gehring and Pommerenke

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Abstract

Gehring and Pommerenke have shown that if the Schwarzian derivative Sf of an analytic function f in the unit disk D satisfies $|Sf(z)| \leq 2(1-|z|^2)^{-2}$ then $f(D)$ is a Jordan domain except when $f(D)$ is the image under a Möbius transformation of an infinite parallel strip. The condition $|Sf(z)| \leq 2(1-|z|^2)^{-2}$ is the classical sufficient condition for univalence of Nehari. In this paper we show that the same type of phenomenon established by Gehring and Pommerenke holds for a wider class of univalence criteria of the form $|Sf(z)| \leq p(|z|)$ also introduced by Nehari. These include $|Sf(z)| \leq \pi^2/2$ and $|Sf(z)| \leq 4(1-|z|^2)^{-1}$. We also obtain results on Hölder continuity and quasiconformal extensions.

1 Introduction and Results

Let f be analytic and locally univalent in the unit disc D. It is well known that the size of its Schwarzian derivative $Sf = (f''/f')' - (1/2)(f''/f')^2$ is intimately related to the global univalence of f in D, and to homeomorphic extensions of f to $\bar{\mathbf{C}}$. In important cases this extension will be quasiconformal in $\mathbf{C} \backslash \bar{D}$. A classical instance of this is Nehari's condition

$$|Sf(z)| \leq \frac{2}{(1-|z|^2)^2}\,, \tag{1}$$

which implies the univalence of f in D, [11]. If the stronger inequality

$$|Sf(z)| \leq \frac{2t}{(1-|z|^2)^2} \tag{2}$$

holds for some $0 \leq t < 1$ then $f(D)$ is a quasidisk, and hence f has a quasiconformal extension to the $\bar{\mathbf{C}}$, [1]; see also [6]. A quasidisk is the image of D

under a quasiconformal mapping of $\bar{\mathbf{C}}$. Nehari's univalence criterion was closely studied in [8], where the authors showed that if

$$|Sf(z)| < \frac{2}{(1-|z|^2)^2}$$

then $f(D)$ is a Jordan domain. It follows that f has a homeomorphic extension to the plane. This result was also obtained by Epstein [7] by quite different methods, and in [5] we give a construction of a conformally natural homeomorphic extension which is related to critical points of the Poincaré metric. In [8] the fact that f has a homeomorphic extension follows from the rather surprising phenomenon that if f satisfies (1.1) then $f(D)$ fails to be a Jordan domain in essentially one case. To state the result, let us first introduce the function

$$F_0(z) = \frac{1}{2}\log\frac{1+z}{1-z}\,.$$

Then F_0 satisfies (1), with equality along the real interval, and $F_0(D)$ is an infinite parallel strip. We say that f is Möbius conjugate to F_0 if it is of the form $T_1 \circ F_0 \circ T_2$, with T_1, T_2 Möbius, $T_2(D) = D$. It follows that such an f will also satisfy (1), with equality now along some hyperbolic geodesic. This is a consequence of the chain rule for the Schwarzian

$$S(f \circ g) = (Sf \circ g)(g')^2 + Sg\,,$$

the fact that Möbius transformations have identically vanishing Schwarzians, and that T_2 is a hyperbolic isometry of the disk.

Theorem 1 (Gehring-Pommerenke)*(A) If f is analytic and locally univalent in D with*

$$|Sf(z)| \le \frac{2}{(1-|z|^2)^2}$$

then f is univalent in D and has a spherically continuous extension to the closed disk $\bar{D}$. Either f is Möbius conjugate to F_0 or else $f(D)$ is a Jordan domain. (B) If

$$\limsup_{|z|\to 1}(1-|z|^2)^2|Sf(z)| < 2$$

and if $f(D)$ is a Jordan domain then $f(D)$ is a quasidisk.

The continuous extension to $\bar{D}$ follows from explicit estimates on the modulus of continuity. We have written Theorem 1 so it is roughly in parallel with Theorem 2 below. For the last part of the theorem the authors actually show (Theorem 4 in [8]) that under just the lim sup condition f has a spherically continuous extension to $\bar{D}$ and that there exists a number $m < \infty$ such that f assumes every value at most m times in $\bar{D}$. If $m = 1$ then $f(D)$ is a quasidisk.

The purpose of this paper is to observe that Theorem 1 can be extended to a wider class of univalence criteria introduced by Nehari. Let $p(z)$ be analytic and even in D, and satisfy the following three conditions:

(i) $|p(z)| \leq p(|z|)$,

(ii) $(1-x^2)^2 p(x)$ is non-increasing on $(0,1)$,

(iii) the differential equation $u'' + pu = 0$ has a real, non-vanishing solution on $(-1,1)$.

Nehari showed that if

$$|Sf(z)| \leq 2p(|z|) \tag{3}$$

then f is univalent in D, [12], [13]. This result encompasses (1) as well as the conditions

$$|Sf(z)| \leq \frac{\pi^2}{2}\,, \tag{4}$$

$$|Sf(z)| \leq \frac{4}{1-|z|^2}\,, \tag{5}$$

and

$$|Sf(z)| \leq \frac{2s(1-(s-1)|z|^2)}{(1-|z|^2)^2}\,,\ 1 < s < 2. \tag{6}$$

The condition (4) was in Nehari's first paper on the subject [11]. (5) was stated by Pokornyi in [14] and a proof was published by Nehari in [12]. The interpolating criterion (6) was given in [13]. See also [3] for a study of general univalence criteria.

In (4), (5), and (6) the extremals (determined up to a Möbius transformation) are, respectively,

$$F_1(z) = \frac{1}{\pi}\tanh(\pi z)\,,\ \ F_2(z) = \int_0^z \frac{d\zeta}{(1-\zeta^2)^2}\,,\ \ F_3(z) = \int_0^z \frac{d\zeta}{(1-\zeta^2)^s}\,.$$

Let p an analytic function in the disk satisfying the three conditions above, and let

$$F(z) = \int_0^z y^{-2}(\zeta)\,d\zeta,$$

where the function y solves

$$y'' + py = 0\,,\ y(0) = 1\,, y'(0) = 0, \tag{7}$$

in the disk. Then $SF(z) = 2p(z)$, and $F(0) = 0$, $F'(0) = 1$ and $F''(0) = 0$. The functions F_0 through F_3 are also normalized in this way. As defined, F is known only to be meromorphic. We will show that it is always analytic in the disk. By (i), F will satisfy (3) and hence will be univalent by Nehari's theorem. Note that F is real-valued on the real axis.

The univalence theorem in [13] only required p to be a real-valued, continuous function defined on $(-1,1)$, but then there need not be an analytic, univalent extremal. The fact that the analytic extremals F share properties with the logarithmic extremal F_0 is one of the points of this paper. However, to draw a distinction, whereas F_0 becomes infinite at ± 1, the general F need not, and there is an interesting difference between the cases $F(1)$ finite or infinite. In the case when $F(1) < \infty$, for any function f satisfying $|Sf(z)| \leq 2p(|z|)$ one knows from the results in [2] that $f(D)$ is a quasidisk. This includes $F(D)$. We are therefore interested here in the case when $F(1) = \infty$. In this case $F(D)$ clearly fails to be a Jordan domain as F is odd, so $F(-1) = F(1) = \infty$ as a point on the sphere. But this is the only way that $F(D)$ fails to be a Jordan domain. Our main result is thus in several parts.

Theorem 2 *Suppose f is analytic and locally univalent in D with*

$$|Sf(z)| \leq 2p(|z|)$$

and that $F(1) = \infty$.
(A) f is univalent in D and admits a spherically continuous extension to $\bar{D}$. Either f is Möbius conjugate to the extremal F or else $f(D)$ is a Jordan domain.
(B) If

$$\lim_{x \to 1} (1 - x^2)^2 p(x) < 1,$$

and if $f(D)$ is a Jordan domain then $f(D)$ is a quasidisk.
(C) F is univalent on $\bar{D} \backslash \{-1, 1\}$.

After this work was completed we learned of two interesting papers by Steinmetz, [15], [16]. In the second paper the author also considers Nehari's p-criterion and, by different methods than we use here, he obtains Parts (A) and (C) above.

Corollary 1 *If*

$$|Sf(z)| < 2p(|z|)$$

then $f(D)$ is a Jordan domain.

Let

$$\mu = \lim_{x \to 1} (1 - x^2)^2 p(x). \tag{8}$$

We will show that $\mu \leq 1$, and that $\mu = 1$ if and only if $p(x) = (1 - x^2)^{-2}$. Both facts will be a consequence of (iii). Thus $\mu = 1$ corresponds to the case treated by Gehring and Pommerenke, while $\mu < 1$ includes (4), (5) and (6). To prove Theorem 2 we will use the fact that

$$\limsup_{|z| \to 1} (1 - |z|^2)^2 |Sf(z)| \leq 2\mu.$$

Next we have two results on Hölder continuity for functions satisfying $|Sf(z)| \leq 2p(|z|)$. Again the case of interest is when $F(1) = \infty$.

Theorem 3 *Suppose $F(1) = \infty$, $\mu < 1$, that f satisfies $|Sf(z)| \le 2p(|z|)$ with $f''(0) = 0$, and that f is not Möbius conjugate to F. Then f is Hölder continuous for any exponent $\alpha > \sqrt{1-\mu}$. If $x = 1$ is a regular singular point of (7) then f is Hölder continuous with exponent $\alpha = \sqrt{1-\mu}$.*

Recall that $x = 1$ is a regular singular point of (7) when $(1-x)^2 p(x)$ is analytic at $x = 1$. Such is ths case for the functions p as in (4), (5) and (6). It follows from (7) that the solution y is concave down, and because of its initial conditions, y is decreasing on $(0,1)$. Hence $\lim_{x\to 1} y(x)$ exists, and the assumption that $F(1) = \infty$ implies that this limit must be 0. The further assumption that $x = 1$ is a regular singular point gives enough information on the order of vanishing of F to improve the Hölder exponent.

The assumption that $f''(0) = 0$ is not restrictive at all since, as we shall see, it can always be achieved by taking a suitable Möbius transformation of f without introducing a pole. The same techniques will also show that the extremal F is locally Hölder continuous on $\partial D \backslash \{-1, 1\}$, in the sense that for each $w \in \partial D \backslash \{-1, 1\}$ there exist $c, \epsilon > 0$ such that

$$|F(z_1) - F(z_2)| \le c|z_1 - z_2|^\alpha$$

for all $z_1, z_2 \in \bar{D} \backslash \{-1, 1\}$, $|z_1 - w|, |z_2 - w| < \epsilon$.

For a similar result on Hölder continuity see [15].

2 Proofs

We begin by showing that the extremal functions are always analytic as a consequence of a general lemma to that effect. Recall that the conditions (i), (ii) and (iii) on p are in force.

Lemma 1 *If f is meromorphic in D with $|Sf(z)| \le 2p(|z|)$ and $f''(0) = 0$ then f is analytic in D.*

Proof: As above, we let y be the solution of the initial value problem (7),

$$y'' + py = 0\,,\ y(0) = 1\,, y'(0) = 0\,,$$

and

$$F(z) = \int_0^z y^{-2}(\zeta) d\zeta\,.$$

First observe that on $(-1, 1)$ the real, even function y cannot vanish. For otherwise it would have at least two zeros there, which then by the Sturm oscillation-comparison theorem would force every solution of the differential equation to vanish at least once in $(-1, 1)$, contradicting condition (iii). Hence F is analytic on a neighborhood of $(-1, 1)$ in D.

Without loss of generality we may next assume that $f(0) = 0, f'(0) = 1$. Then

$$f(z) = \int_0^z v^{-2}(\zeta) d\zeta$$

where

$$v'' + \frac{1}{2}(Sf)v = 0, \quad v(0) = 1,\ v'(0) = 0.$$

Since $|Sf(z)| \leq 2p(|z|)$, it follows from Lemma 2 in [4] that

$$|v(z)| \geq y(|z|)$$

in the largest disk $|z| < r \leq 1$ on which f is analytic. Hence $|f(z)| \leq F(|z|)$ there, which shows that f cannot have a pole in D.

In particular, since $|SF(z)| = 2|p(z)| \leq 2p(|z|)$ by condition (i), we conclude that the extremals themselves satisfy $|F(z)| \leq F(|z|)$ and are analytic in the disk. Then they are also univalent by Nehari's p-theorem.

Remark If f is analytic in D with $f(z) = z + a_2 z^2 + \ldots$ then the function $f^\dagger = f/(1 + a_2 f)$ has $f^\dagger(z) = z + O(z^3)$. If f satisfies $|Sf(z)| \leq 2p(|z|)$ then so does $f^\dagger$. It cannot have a pole in D because, again, it will be subject to the bound $|f^\dagger(z)| \leq F(|z|)$ on the largest disk $|z| < r \leq 1$ on which it is analytic, and F is analytic in all of D. The point is that when the Schwarzian is bounded in this way it is possible to normalize an analytic function to get $f''(0) = 0$ and still be analytic.

Actually, using the arguments in [4] one can prove sharp distortion theorems for functions satisfying the hypotheses of Lemma 1. If G is the solution of $SG = -2p$ with $G(0) = 0$, $G'(0) = 1$ and $G''(0) = 0$ then

$$G'(|z|) \leq |f'(z)| \leq F'(|z|),$$

$$G(|z|) \leq |f(z)| \leq F(|z|).$$

If equality holds at any point other than the origin in any of the inequalities then f is equal to the corrresponding function F or G. We will not prove these facts here, nor will we make any use of them.

Note also that if $F(1) < \infty$ then $F(D)$, and hence $f(D)$, will be *bounded*. Even when $F(1) = \infty$, $f(D)$ will be bounded as long as f is not Möbius conjugate to F. We show this in Lemma 4, below.

Next, recall that $\mu = \lim_{x \to 1}(1 - x^2)^2 p(x)$. The following lemma makes Theorem 1 applicable to the proof of Theorem 2.

Lemma 2 *$\mu \leq 1$ and $\mu = 1$ if and only if $p(x) = (1 - x^2)^{-2}$.*

Proof: Suppose first that $\mu > 1$. Then $p(x) \geq \mu(1 - x^2)^{-2}$, which we shall show implies that the solution y of

$$y'' + py = 0\,,\ y(0) = 1\,, y'(0) = 0$$

vanishes somewhere on $(-1,1)$. This will contradict (iii). Let v be the solution on $(-1,1)$ of

$$v'' + \frac{\mu}{(1-x^2)^2} v = 0\,,\ v(0) = 1\,, v'(0) = 0\,.$$

The function v is given by

$$v(x) = \sqrt{1-x^2}\cos\left(\frac{\eta}{2}\log\frac{1+x}{1-x}\right)$$

where $\eta = \sqrt{\mu - 1}$, see [10], p. 492. In particular, v vanishes on $(-1,1)$ (infinitely often). A standard application of the Sturm comparison theorem shows that $v \geq y$ as long as $y > 0$ on a centered interval about the origin. It follows that y must vanish somewhere on $(-1,1)$ as well.

Hence $\mu \leq 1$, and if $p(x) = (1-x^2)^{-2}$ then obviously $\mu = 1$. Suppose then that $\mu = 1$. This time let $v(x) = \sqrt{1-x^2}$ be the solution of

$$v'' + \frac{1}{(1-x^2)^2} v = 0\,, v(0) = 1\,, v'(0) = 0\,,$$

so that

$$F_0(x) = \frac{1}{2}\log\frac{1+x}{1-x} = \int_0^x v^{-2}(t)\,dt.$$

Since y is positive, the comparison theorem gives $v \geq y$. We let

$$F(x) = \int_0^x y^{-2}(t)\,dt$$

as before, and put $H = F^{-1}$. Since $F_0(-1,1) = \mathbf{R}$ already, then $v \geq y$ implies that F takes $(-1,1)$ onto $\mathbf{R}$ too. (Note that in this lemma we are not assuming at the outset that $F(1) = \infty$.) Let

$$\varphi(s) = \frac{v(H(s))}{y(H(s))}\,, \tag{1}$$

where $s \in \mathbf{R}$. This function is defined so that

$$(F_0 \circ H)(s) = \int_0^s \varphi^{-2}(t)\,dt\,.$$

A straightforward computation shows that

$$\varphi''(s) = \left(p(x) - \frac{1}{(1-x^2)^2}\right) y^4(x)\varphi(s)\,,\ x = H(s).$$

Hence φ is convex, as $p(x) \geq (1-x^2)^{-2}$. In addition, $\varphi(0) = 1$, $\varphi'(0) = 0$. Suppose $p(x) \not\equiv (1-x^2)^{-2}$, say $p(x_0) > (1-x_0^2)^{-2}$ for some $x_0 > 0$. Because $s = 0$ gives an absolute minimum of φ it follows from the convexity that

$$\varphi(s) \geq a + b(s - s_0)\,,\ x_0 = H(s_0),$$

for some constants a, b with $b > 0$. Therefore

$$\int_0^\infty \varphi^{-2}(s)ds < \infty\,,$$

which contradicts the fact that $(F_0 \circ H)(\infty) = \infty$. This completes the proof of the lemma.

We begin the proof of Theorem 2. According to Lemma 2, $\mu = 1$ is taken care of by Theorem 1, so we may now assume that $\mu < 1$. Let f satisfy $|Sf(z)| \leq 2p(|z|)$. By Nehari's theorem f is univalent, and since

$$\limsup_{|z|\to 1}(1-|z|^2)^2|Sf(z)| \leq 2\mu < 2\,,$$

another application of Theorem 1 (more properly, the remarks following Theorem 1) implies that f admits a spherically continuous extension to $\bar{D}$. Assuming also that $F(1) = \infty$, in order to complete the proofs of Parts (A) and (B) of Theorem 2 it suffices to show that either f is Möbius conjugate to F or else f is 1:1 on ∂D, in which case $f(D)$ will actually be a quasidisk. For this we need an observation due to Nehari [12]. We state it here as a separate lemma.

Lemma 3 *Let $z_1, z_2 \in \partial D$, $z_1 \neq z_2$, and let γ be the hyperbolic geodesic in D joining z_1 and z_2. Then there exists a Möbius selfmap T of D such that:*
(a) $T(-1,1) = \gamma$,
(b) $|S(f\circ T)(x)| \leq 2p(|x|)$ for all $x \in (-1,1)$.

Proof: If z_1, z_2 lie on a diameter then T can be chosen to be a rotation. If not, using a rotation we may assume that the points z_1, z_2 lie in the upper half-plane and that γ is symmetric with respect to the imaginary axis. Then for suitable $0 < \rho < 1$,

$$T(z) = \frac{z + i\rho}{1 - i\rho z}$$

takes $(-1,1)$ to γ. We also have

$$S(f\circ T)(z) = Sf(Tz)(T'z)^2$$

hence

$$\begin{aligned}(1-|z|^2)^2|S(f\circ T)(z)| &= (1-|z|^2)^2|(Sf)(Tz)||T'z|^2 \\ &= |(Sf)(Tz)|(1-|Tz|^2)^2 \\ &\leq 2(1-|Tz|^2)^2 p(|Tz|)\,.\end{aligned}$$

Therefore it suffices to show that $|x| \leq |Tx|$ for $x \in (-1,1)$. But

$$|Tx|^2 = \frac{x^2+\rho^2}{1+\rho^2x^2} > x^2\,.$$

Returning to the proof of Theorem 2, suppose $f(z_1) = f(z_2)$ for distinct points on ∂D. Let $g = f \circ T$ with T as in Lemma 3. Then $g(1) = g(-1)$, and by taking a Möbius transformation of g, we may assume that this common value is ∞. An affine change allows us to normalize further so that $g(0) = 0$. We write

$$g(z) = \int_0^z v^{-2}(\zeta) d\zeta \,,$$

where

$$v'' + \frac{1}{2}(Sg)v = 0 \,.$$

As in the proof of Lemma 2, (1), we let

$$\varphi(s) = \frac{v(H(s))}{y(H(s))} \,, \ H = F^{-1}. \tag{2}$$

This is defined on $F(-1,1) = \mathbf{R}$, and

$$(g \circ G)(s) = \int_0^s \varphi^{-2}(t) \, dt \,.$$

The chain rule for the Schwarzian yields

$$S(g \circ G)(s) = (Sg(z) - SF(z))(G'(s))^2, \ z = G(s).$$

Hence for $s \in \mathbf{R}$, $\mathrm{Re}\{S(g \circ G)(s)\} \leq 0$, and a direct calculation gives

$$|\varphi(s)|'' = q(s)\varphi(s)$$

where

$$q(s) = -\frac{1}{2}\mathrm{Re}\{S(g \circ G)(s)\} + \left(\frac{1}{2}\mathrm{Im}\left\{ \frac{(g \circ G)''}{(g \circ G)'}(s) \right\} \right)^2 \geq 0$$

(see [8]). We conclude that $|\varphi|$ is convex on $\mathbf{R}$. Unless it is constant, it will be bounded below by a non-horizontal line, which, as in the prooof of Lemma 2, will imply that either $(g \circ G)(\infty)$ or $(g \circ G)(-\infty)$ is finite. This contradicts the fact that $g(1) = g(-1) = \infty$. For this last double equality to happen the function $|\varphi|$ must be constant on $\mathbf{R}$. But then $q(s) \equiv 0$, which implies that

$$\frac{1}{2}\mathrm{Im}\left\{ \frac{(g \circ G)''}{(g \circ G)'}(s) \right\} \equiv 0.$$

On the other hand, for $s \in \mathbf{R}$,

$$\frac{|\varphi|'}{|\varphi|}(s) = -\frac{1}{2}\mathrm{Re}\left\{ \frac{(g \circ G)''}{(g \circ G)'}(s) \right\} ,$$

and we conclude that

$$\frac{(g \circ G)''}{(g \circ G)'}(s) \equiv 0$$

on $\mathbf{R}$, hence everywhere on $F(D)$. It follows that $g \circ G$ is an affine transformation and therefore g is Möbius conjugate to F. This finishes the proof of Parts (A) and (B).

To prove part (C) we proceed similarly. Suppose z_1, z_2 are distinct points on ∂D such that $F(z_1) = F(z_2)$. Let T_2 be the Möbius transformation provided by Lemma 3, and let T_1 be a second Möbius transformation such that $T_1(F(z_1)) = T_1(F(z_2)) = \infty$. We conclude from Part (A) that $T_1 \circ F \circ T_2$ is of the form $T \circ F$, T Möbius. By taking Schwarzian derivatives we obtain

$$S(T_1 \circ F \circ T_2) = S(F \circ T_2) = (SF) \circ T_2 (T_2')^2 = S(T \circ F) = SF\,,$$

or

$$p(z) = p(T_2(z))(T_2' z)^2.$$

Hence for $z = x_0$ real

$$\begin{aligned}(1 - x_0^2)^2 p(x_0) &= |(1 - x_0^2)^2 p(x_0)| \\ &= |(1 - x_0^2)^2 p(T_2(x_0))(T_2'(x_0))^2| \\ &= (1 - |T_2(x_0)|^2)^2 |p(T_2(x_0))|\,.\end{aligned}$$

Next, we saw in the proof of Lemma 3 that unless T_2 is a rotation, $|T_2(x_0)| > |x_0|$, which by the monotonicity property (ii) implies that $(1 - x^2)^2 p(x)$ is constant for $|x_0| \leq x \leq |T_2(x_0)|$. Thus $(1 - x^2)^2 p(x)$ is constant on $(-1, 1)$ and therefore everywhere. In other words,

$$p(z) = \frac{\mu}{(1 - z^2)^2}\,.$$

In this case the extremal F can be computed explicitly [4]:

$$F(z) = \frac{1}{\eta}\frac{(1+z)^\eta - (1-z)^\eta}{(1+z)^\eta + (1-z)^\eta}\,,$$

where $\eta = \sqrt{1 - \mu}$. This function satisfies the Ahlfors-Weill condition (2) and $F(D)$ is a bounded quasidisk. In particular, $F(1) \neq \infty$ and F is not of the form considered here.

The remaining case is when T_2 is a rotation, $z \mapsto cz$, with $|c| = 1$. The equation (2.5) yields

$$p(z) = c^2 p(cz)$$

which evaluated at $z = 0$ gives $p(0) = 0$ or $c^2 = 1$. If $p(0) = 0$ then $p(z) \equiv 0$, and all maps are Möbius. If $c^2 = 1$ then $c = \pm 1$, which implies that the points z_1, z_2 were ± 1 to begin with. This finishes the proof of Theorem 2.

Corollary 1 is a direct consequence of Theorem 2. We now prove Theorem 3 on Hölder continuity. For this we first require an extension of Lemma 1.

Lemma 4 *Suppose* $F(1) = \infty$. *If* $|Sf(z)| \leq 2p(|z|)$, $f''(0) = 0$ *and* f *is not Möbius conjugate to* F, *then* f *is bounded on* $\bar{D}$.

Proof: If $|f(w)| = \infty$ for some $w \in \partial D$ then the function $|\varphi(s)|$ defined in (2) would have to be constant on a half line, and hence $S(f \circ F^{-1}) \equiv 0$ there. Thus $S(f \circ F^{-1}) \equiv 0$ on all of $F(D)$, so $f \circ F^{-1}$ is a Möbius transformation (in fact the identity), a contradiction.

For the Hölder continuity in the first part of Theorem 3, let $\delta > 0$ be such that $\mu + 2\delta < 1$, and let $\epsilon > 0$ be small enough so that

$$|Sf(z)| \leq \frac{2(\mu + \delta)}{(1 - |z|^2)^2}$$

for all $1 - \epsilon \leq |z| < 1$. Let $w \in \partial D$. Gehring and Pommerenke produced a conformal mapping ψ of D onto a circular wedge $\Omega \subset D$, with an arc of ∂D centered at w as one if its sides, and such that $1 - \epsilon \leq |z|$ for all $z \in \Omega$. Furthernore, if the wedge is sufficiently narrow then

$$|S(f \circ \psi)(\zeta)| \leq \frac{2(\mu + 2\delta)}{(1 - |\zeta|^2)^2}$$

for all $\zeta \in D$.

Let T be a Möbius transformation such that the map $g = T \circ f \circ \psi$ has $g(0) = 0, g'(0) = 1$ and $g''(0) = 0$. Corollary 1 in [4] implies that g is Hölder continuous with exponent

$$\alpha = \sqrt{1 - (\mu + 2\delta)}\,. \tag{2.7}$$

We want to conclude from here that f is Hölder continuous in Ω with the same exponent. The reflection principle implies that ψ is analytic in a neighborhood of $\psi^{-1}(w)$, hence it suffices to show that $f_1 = f \circ \psi$ is Hölder continuous. We write $g = (af_1 + b)/(cf_1 + d)$, $ad - bc = 1$, or

$$f_1 = \frac{dg - b}{ag - c}\,. \tag{3}$$

But f_1 is bounded on $\bar{D}$ by Lemma 4, which shows that c/a is not in the closure of $g(D)$. It follows from (3) that f_1 is Hölder continuous in Ω as well. To conclude the Hölder continuity everywhere just observe that a finite number of wedges Ω cover a neighborhood in D of ∂D.

Next, suppose that $x = 1$ is a regular singular point of (7). To improve the Hölder exponent for f we need the following lemma on the order of vanishing of the solution at 1.

Lemma 5 *Suppose $x = 1$ is a regular singular point of (7). Then the solution y of (7) satisfies*

$$y(x) \sim (1-x)^{\beta} \quad , \text{ as } x \to 1 \, ,$$

where $\beta = (1 + \sqrt{1-\mu})/2$.

Proof: The possible orders of vanishing at $x = 1$ of the solutions of $u'' + pu = 0$ are given by the roots of the inditial equation

$$m^2 - m + \frac{\mu}{4} = 0 \, ,$$

which are

$$m_1 = \frac{1 + \sqrt{1-\mu}}{2} \quad , \quad m_2 = \frac{1 - \sqrt{1-\mu}}{2} \, ,$$

(see, e.g., [9]). Notice that $0 \leq m_2 < 1/2 < m_1 \leq 1$. Since $F(1) = \int_0^1 y^{-2}(x)dx = \infty$ we conclude that $y(1)$ vanishes to order m_1.

To finish the proof of Theorem 3 we go back to the proof of Theorem 2. There we saw that $f(D)$ was a Jordan domain precisely when the convex function $|\varphi(s)|$ was non-constant. Thus for $s \geq s_0$ there exist constants a, b with $b > 0$ such that

$$|\varphi(s)| \geq a + b(s - s_0) \, .$$

Hence

$$|(f \circ F^{-1})'(s)| \leq \frac{1}{(a + b(s - s_0))^2}$$

or

$$|f'(x)| \leq \frac{F'(x)}{(a + b(F(x) - s_0))^2} \, , \quad x = F(s) \, .$$

It follows that

$$|f'(x)| = O(1-x)^{\sqrt{1-\mu}\ -1} \, , \quad x \to 1 \, .$$

The same argument applied to $f(e^{i\theta} z)$ gives

$$|f'(z)| = O(1 - |z|)^{\sqrt{1-\mu}\ -1} \, , \quad |z| \to 1 \, .$$

Now a standard technique of integrating along hyperbolic geodesics (see e.g. [8] or [4]) gives the desired conclusion.

Finally, since the extension of F to $\bar{D}$ is finite on $\partial D \backslash \{-1, 1\}$, the same proof as before gives the local Hölder continuity of F in $\partial D \backslash \{-1, 1\}$.

Notice also that the Hölder continuity is Lipschitz when $\mu = 0$, such as in (1.5) and (1.6). When $F(1) < \infty$ the second part of Theorem 3 was obtained in [2] (Theorems 2, 3).

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