Quasidisks and the Noshiro-Warschawski Criterion

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Abstract

We show that a domain $R \subset \mathbb{H} = \{z : \Re z > 0\}$ has the property that $f(\mathbb{D})$ is a quasidisk for all f for which $f'(\mathbb{D}) \subset R$ if and only if there is a compact $K \subset \mathbb{H}$ such that $rK \cap (\mathbb{H} \setminus R) \neq \varnothing$ for all $r > 0$. This constitutes a refinement of the well known Noshiro-Warschawski univalence criterion.

1. Introduction

It is well known that an analytic function f on $\mathbb{D} = \{z : |z| < 1\}$ for which $f'(\mathbb{D}) \subset \mathbb{H} = \{z : \Re z > 0\}$ is necessarily univalent. This fact, frequently referred to as the Noshiro-Warschawski criterion [7],[11], was also independently discovered by Wolff [12] somewhat later; refinements of this criterion have, moreover, been obtained by several subsequent authors (see, for example [5], [9], [10]). It is an easy matter to see that this criterion is sharp in the sense that if U is any domain which properly contains $\mathbb{H}$, then there are nonunivalent f on $\mathbb{D}$ for which $f'(\mathbb{D}) \subset U$. It is likewise immediate that if $f'(\mathbb{D}) \subset \mathbb{H}$ and f has a continuous extension to $\overline{\mathbb{D}}$, then f is one-to-one on $\overline{\mathbb{D}}$. It has been shown that deeper sharp sufficient conditions for univalence involving bounds on the Schwarzian Sf or f''/f' have strengthened forms which imply that the image is actually a quasidisk. The first such result, due to Ahlfors and Weill [1], was vastly generalized by Gehring and Pommerenke [4], who proved that for a univalence criterion of the form $|Sf(z)| \leq \rho(z)$ on a quasidisk Ω the stronger bound $|Sf(z)| \leq t\rho(z)$, $0 \leq t < 1$ implies that $f(\Omega)$ is also a quasidisk, and also pointed out that a similar statement is true for f''/f'.

When the Noshiro-Warschawski criterion is expressed in the form $|\Im \log f'(z)| \leq \pi/2$ these results prompt one to seek corresponding strengthenings, an obvious (and correct) thought being that $|\Im \log f'(z)| \leq t\pi/2$, $0 \leq t < 1$ should imply that $f(\mathbb{D})$ is a quasidisk. In this paper we consider the problem of determining a much wider family of strengthenings, to which end we completely characterize those domains $R \subset \mathbb{H}$ with the property that $f'(\mathbb{D}) \subset R$ implies that $f(\mathbb{D})$ is a quasidisk by showing that they are precisely the ones for which there exists a compact $K \subset \mathbb{H}$ such that $rK \cap (\mathbb{H} \setminus R) \neq \varnothing$ for all $r > 0$.

2. Preliminaries and Statement of Main Result

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In what follows the symbols $C, C', C_1, C_2, \ldots$ will denote absolute constants; a given such symbol may be used to denote different constants when context precludes confusion. Several definitions are necessary before we give complete and precise statements of the results and pertinent motivating comments. We begin with the following (see [8])

Definition. A domain G is said to be a *John domain* if

(i) G is bounded and simply connected;

(ii) there is a constant C_1 such that for every linear crosscut $[a, b]$ of G

$$\operatorname{diam} H \leq C_1|b-a|$$

holds for one of the two components H of $G \backslash [a, b]$.

The domain G is said to be *linearly connected* if it satisfies (i) and

(iii) there is a constant C_2 such that any two points $w_1, w_2 \in G$ can be joined by a curve $\Gamma \subset G$ for which

$$\operatorname{diam} \Gamma \leq C_2|w_2-w_1| \,.$$

Finally, G is said to be a *quasidisk* if it is a linearly connected John domain.

Definition 1. A closed subset X of $\mathbb{H}$ will be said to have property M_∞ (M_0) if there exists a compact $K \subset \mathbb{H}$ and an $r_1 > 0$ such that for all $r \geq r_1$ $(0 < r \leq r_1)$, $rK \cap X \neq \emptyset$. Furthermore, X will be said to have property M if it has both properties M_∞ and M_0.

As is easily shown (see Proposition 1 at the beginning of Section 3) X has property M if and only if there is a compact $K \subset \mathbb{H}$ such that $rK \cap X \neq \emptyset$ for all $r > 0$; we have however, chosen to break this condition into M_∞ and M_0 for notational reasons as well as to be able to discuss these partial conditions individually.

The main result of this paper is the following

Theorem. *Let $R \subset \mathbb{H}$ be a domain. Then $f(\mathbb{D})$ is a quasidisk for all f analytic in $\mathbb{D}$ for which $f'(\mathbb{D}) \subset R$ if and only if $\mathbb{H} \backslash R$ has property M. Furthermore, the constants in the definition of quasidisk depend only on the set R.*

We mention in passing that the first sentence of this theorem can be stated in the somewhat more general, but clearly equivalent form, as follows. *Let X be a closed subset of $\mathbb{H}$. Then $f(\mathbb{D})$ is a quasidisk for all f for which $f'(\mathbb{D}) \subset \mathbb{H} \backslash X$ if and only if the set $\mathbb{H} \backslash E$ has the property M for each component E of $\mathbb{H} \backslash X$.* In what follows, however, we will work with the theorem as originally stated and R will always denote a subdomain of $\mathbb{H}$. The symbol X will denote a closed subset of $\mathbb{H}$. With reference to Definition 1, the size of X varies inversely with that of K, and the smallest possible K are those consisting of a single point z_0, for which the corresponding sets X are slits at ∞ and 0 in the direction of z_0. The theorem says that $X = \mathbb{H} \backslash R$ can be a much smaller set, such as $\{t^n z_0 : n \in \mathbb{Z}\}, t > 1$, which is easily seen to have property M when K is taken to be the segment $[z_0, tz_0]$. We were led to the theorem by the observation that if $f'(\mathbb{D}) \subset \mathbb{H}$ and f' is C^1 on an open arc $A \subset \partial\mathbb{D}$ for which $f'(A)$ is tangent to $\partial\mathbb{H}$ at 0, then $f(\mathbb{D})$ will have an entrant cusp, and therefore will not be a linearly connected domain. A analogous state of affairs in regard to the John condition occurs if $f(A)$ is tangent to $\partial\mathbb{H}$ at ∞. The idea of the theorem is for $f'(\mathbb{D})$ to omit minimal subsets of $\mathbb{H}$ whose absence prevents the argument of $f'(\zeta)$ from suddenly jumping by π or $-\pi$ at any point on $\partial\mathbb{D}$, thereby keeping $f(\mathbb{D})$ from having such cusps. A quantification of how far away a measurable real-valued function u on $\partial\mathbb{D}$ satisfying $|u| \leq \frac{\pi}{2}$ stays from having such jumps is given in Definition 2 below.

When viewed in light of these comments, it is tempting to believe, as we originally did, that if $\mathbb{H}\backslash R$ has property M_0 alone, then $f'(\mathbb{D}) \subset R$ implies that $f(\mathbb{D})$ is linearly connected and that an analogous statement holds relating M_∞ and the John property (ii). We will show in Section 6 that neither of the suppositions is true and, moreover, that in the theorem the disk cannot be replaced by an arbitrary convex domain.

As we have indicated, in order to establish the sufficiency of the condition in the theorem, it is necessary to work with the boundary value function $u(\theta) = u_f(\theta) = \Im\{\log f'(e^{i\theta})\}$, which is measurable and satisfies

$$|u(\theta)| \leq \pi/2 \quad \text{a.e. on } \partial\mathbb{D} \tag{1}$$

(see, e.g., [6], p. 38). For a given such u and a given interval $I = [\alpha, \beta]$ we define

$$\overline{u}(t, I) = u(\frac{\alpha+\beta}{2} + t) - u(\frac{\alpha+\beta}{2} - t)\,,$$

$t \in [-\frac{|I|}{2}, \frac{|I|}{2}]$, so that $\overline{u}$ is in effect twice the odd part of u with respect to the midpoint $\frac{\alpha+\beta}{2}$ of I.

Definition 2. Let $0 < \tau < 1$. A measurable function on $\partial\mathbb{D}$ which satisfies (1) has the property $M^*_\infty(\tau)$ (respectively $M^*_0(\tau)$) if

$$\frac{2}{|I|\pi} \int_0^{|I|/2} \overline{u}(t, I)\, dt \leq 1 - \tau \; (\geq \tau - 1)$$

for all intervals I of length at most τ. Such a function has property $M^*(\tau)$ if it has both of these properties.

For convenience, we will use the notation

$$\mathcal{F}_X = \{g : g(\mathbb{D}) \subset \mathbb{H}\backslash X\}\,.$$

3. Some Auxiliary Propositions

Proposition 1. *If X has property M, then there is some compact $K_1 \subset \{z : |\Im z| < \frac{\pi}{2}\}$ such that*

$$K_1 \cap (t + \log X) \neq \emptyset$$

for all $t \in \mathbb{R}$.

Proof. By definition there is a compact K and positive numbers $r_0 \leq r_1$ such that $rK \cap X \neq \emptyset$ for $0 < r \leq r_0$ and $r > r_1$. Let $z_0 \in X$ and let $K' = \{\rho z_0 : \frac{1}{r_1} \leq \rho \leq \frac{1}{r_0}\}$. Then $rK' \cap X \neq \emptyset$ for $r_0 \leq r \leq r_1$, so that $r(K \cup K') \cap X \neq \emptyset$ for all $r > 0$; that is, $\log(K \cup K') \cap (t + \log X) \neq \emptyset$ for all $t \in \mathbb{R}$. □

Proposition 2. *Let X have property M. Then there exists some $\tau = \tau_X > 0$ such that $u(\theta) = \Im\{\log h(e^{i\theta})\}$ has property $M^*(\tau)$ for all $h \in \mathcal{F}_X$.*

Proof. For given X with property M we show that there is a $\tau = \tau_X$ such that all of the corresponding u have property $M^*_\infty(\tau)$; that they have property $M^*_0(\tau')$ for some $\tau' = \tau'_X$ follows in the same way. Let $K_1 \subset \{z : |\Im z| < \frac{\pi}{2}\}$ be the compact set of Proposition 1.

If no such τ existed then there would be a sequence $\{h_n\}$ in $\mathcal{F}_X$ and a sequence $\{\epsilon_n\}$ with ϵ_n decreasing to 0 such that

$$\frac{1}{\epsilon_n \pi} \int_0^{\epsilon_n} \overline{u}_n(t, I_n)\, dt \;\to 1\,,$$

where $u_n(\theta) = \Im\{\log h_n(e^{i\theta})\}$ and $I_n = (-\frac{\pi}{2} - \epsilon_n, -\frac{\pi}{2} + \epsilon_n)$. Let $T(z) = i\frac{z-i}{z+i}$, which maps the upper half-plane $i\mathbb{H}$ onto $\mathbb{D}$ with $T(0) = -i$. Let $T^{-1}(e^{iI_n}) = (-\epsilon'_n, \epsilon'_n)$, so that ϵ'_n also decreases to 0. Since $T'(0) = 2$, it follows that $\epsilon_n/\epsilon'_n \to 2$ and that if

$$g_n(z) = \log\left(\frac{h_n(T(\epsilon'_n z))}{|h_n(T(\epsilon'_n i))|}\right), \tag{2}$$

then

$$\frac{1}{\pi}\int_0^1 \Im\{g_n(t) - g_n(-t)\}dt \to 1 \tag{3}$$

and

$$g_n(i\mathbb{H}) \not\supset t + K_1 \quad \text{for any } t \in \mathbb{R}\,;\, n \in \mathbb{N}\,. \tag{4}$$

Since $|\Im g_n(t)| \leq \frac{\pi}{2}$, (3) implies that

$$\Im g_n \to \frac{\pi}{2}(\chi_{(0,1)} - \chi_{(-1,0)}) \text{ in } L^1(-1,1)\,. \tag{5}$$

It is well known (see [6]) that if w is analytic in the upper-half plane $i\mathbb{H}$ and has bounded real part there, then $\Re w$ has nontangential boundary values $u(t)$ for almost all $t \in \mathbb{R}$, u is measurable, and in terms of u, w is given by the Poisson integral formula

$$w(z) = \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{1+zt}{(z-t)(1+t^2)}u(t)dt + \Im\{w(i)\}\,.$$

Writing g_n as $\frac{1}{i}(ig_n)$, and taking into account that, by (2), $\Re\{g_n(i)\} = 0$, this formula says that

$$g_n(z) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1+zt}{(z-t)(1+t^2)}\Im g_n(t)\,dt = -\frac{1}{\pi}\int_{-1}^{1} \frac{\Im g_n(t)}{z-t}\,dt + q_n(z)\,,$$

where

$$q_n(z) = -\frac{1}{\pi}\int_{|t|\geq 1} \frac{1+zt}{(z-t)(1+t^2)}\Im g_n(t)dt - \frac{1}{\pi}\int_{-1}^{1} \frac{t}{1+t^2}\Im g_n(t)dt\,.$$

The sequence $\{q_n\}$ is obviously analytic and uniformly bounded on $\frac{1}{2}\mathbb{D}$, and furthermore, $\Im\{q_n(0)\} = 0$. By (5), $g_n(z) - q_n(z)$ tends locally uniformly in $i\mathbb{H}$ to

$$\frac{1}{2}\Big\{\int_{-1}^{0} \frac{1}{z-t}\,dt - \int_0^1 \frac{1}{z-t}\,dt\Big\} = -\log z + \frac{1}{2}\log(z^2-1)\,,$$

where $\Im\{\log z\}$ is taken to lie in $(0, 2\pi)$ for $z \in \mathbb{C}\backslash\{x : x \geq 0\}$. From this together with the fact that $q_n(z)$ is uniformly bounded on $\frac{1}{2}\mathbb{D}$ and $\Im\{q_n(0)\} = 0$, a simple application of the argument principle shows that given any $\epsilon, M > 0$ there exists a $t \in \mathbb{R}$ and an $n \in \mathbb{N}$ such that $g_n(i\mathbb{H})$ contains $t + R_{\epsilon,M}$, where

$$R_{\epsilon,M} = \{z : 0 < \Re z < M, |\Im z| < \frac{\pi}{2} - \epsilon\}\,.$$

But for ϵ sufficiently small and M sufficiently large, $t' + K_1 \subset t + R_{\epsilon,M} \subset g_n(i\mathbb{H})$ for some t', which contradicts (4). $\square$

Next we have

Proposition 3. *There is an absolute constant C and a function $\alpha = \alpha(\tau)$, $0 < \alpha < 1$ with the following property: if g is analytic on $\mathbb{D}$ with $|\Im\{g(z)\}| \le \frac{\pi}{2}$ in $\mathbb{D}$ and is such that $u(e^{i\theta}) = \Im\{g(e^{i\theta})\}$ has property $M^*(\tau)$, then*

$$|g(z) - g(0)| \le \alpha \log \frac{1}{1-|z|} + C \text{ , for all } z \in \mathbb{D} .$$

Proof. It is clearly enough to show this for $z = -iy, y \in (0,1)$. Again, it is easiest to switch attention to the upper half-plane $i\mathbb{H}$ via the transformation $T(z) = i\frac{z-i}{z+i}$, which maps $[0,i] \subset i\mathbb{H}$ onto $[-i,0] \subset \overline{\mathbb{D}}$. If we write $u^*(z) = u(T(z))$, then it is easy to see that

$$|\frac{1}{\beta\pi}\int_0^\beta \overline{u^*}(t)\, dt| \le 1 - \tau_1 \text{ }, 0 < \beta \le 1 \,, \tag{6}$$

where $\tau_1 = \tau_1(\tau) > 0$ and $\overline{u^*}(t) = u^*(t) - u^*(-t)$. Now, $T'(iy) = \frac{2}{(y+1)^2}$, so that

$$\frac{1}{2} \le T'(iy) \le 2 \text{ }, \, 0 \le y \le 1 \,.$$

This implies that it is sufficient to show that for $0 \le y < 1$

$$|g(T(iy)) - g(T(i))| \le \alpha \log \frac{1}{y} + C$$

for some $\alpha \in (0,1)$ and some C.

Since $|u^*(t)| \le \frac{\pi}{2}$, we have by the Poisson integral formula that for $0 \le y < 1$,

$$|g(T(iy)) - g(T(i))| \le |\Re\{g(T(iy)) - g(T(i))\}| + \pi \le \left| \frac{1}{\pi}\int_{-\infty}^{\infty} \Re\{\frac{1+iyt}{(iy-t)(1+t^2)}\}u^*(t)\, dt \right| + \pi$$

$$\le \frac{1}{\pi}\left|\int_{-1}^{1} \frac{t(y^2-1)}{(y^2+t^2)(1+t^2)} u^*(t)\, dt\right| + C_1 = \frac{1}{\pi}\left|\int_0^1 \frac{t(y^2-1)}{(y^2+t^2)(1+t^2)}\overline{u^*}(t)\, dt\right| + C_1$$

$$= \frac{1}{\pi}\left|\int_0^1 t(\frac{1}{1+t^2} - \frac{1}{y^2+t^2})\overline{u^*}(t)\, dt\right| + C_1 \le \frac{1}{\pi}\left|\int_0^1 \frac{t}{y^2+t^2}\overline{u^*}(t)\, dt\right| + C_2 \,.$$

If we let $v(s) = \int_0^s \overline{u^*}(t)\, dt$, then (6) implies that $|v(s)| \le \pi s(1-\tau_1)$, $0 \le s \le 1$. Upon integrating by parts we have

$$\int_0^1 \frac{t}{y^2+t^2}\overline{u^*}(t)\, dt = \frac{v(1)}{y^2+1} + \int_0^y \frac{t^2-y^2}{(y^2+t^2)^2}v(t)\, dt + \int_y^1 \frac{t^2-y^2}{(y^2+t^2)^2}v(t)\, dt \,.$$

Now,

$$\left|\int_0^y \frac{t^2-y^2}{(y^2+t^2)^2}v(t)\, dt\right| \le \pi(1-\tau_1)\int_0^y \frac{y^2-t^2}{(y^2+t^2)^2}t\, dt = \frac{\pi}{2}(1-\tau_1)(1-\log 2) \,,$$

and

$$\left|\int_y^1 \frac{t^2-y^2}{(y^2+t^2)^2}v(t)\, dt\right| \le \pi(1-\tau_1)\int_y^1 \frac{t^2-y^2}{(y^2+t^2)^2}t\, dt$$

$$= \pi(1-\tau_1)\{\frac{1}{2}\log(1+y^2) + \frac{(y^2-1)}{2(1+y^2)} - \frac{1}{2}\log 2 - \log y\} \,,$$

so that we have

$$|g(T(iy)) - g(T(i))| \le (1-\tau_1)\log\frac{1}{y} + C_3\,,$$

as desired. □

Before continuing we note several consequences of Proposition 3. In the first place, it follows immediately from this estimate together with Proposition 2 that if X is a closed subset of $\mathbb{H}$ with property M and $f' \in \mathcal{F}_X$, then

$$|f'(z)| \le C' \frac{|f'(0)|}{(1-|z|)^{\alpha}}\,, \tag{7}$$

where $C' = e^C$, and here and in what follows $\alpha(\tau_X)$ is abbreviated as α. Although it is known that (7) implies Hölder continuity (see [4]), we include a derivation of this fact since later on we shall use one of the estimates obtained on the way (see (8) below). If $\zeta = re^{i\phi} \in \mathbb{D}$, the function

$$h(z) = \frac{e^{-i\phi}}{1-r} f(\zeta + (1-r)e^{i\phi}z)$$

satisfies $h'(\mathbb{D}) \subset \mathbb{H}\backslash X$. Then by (7)

$$|h'(z)| \le C' \frac{|f'(\zeta)|}{(1-|z|)^{\alpha}}\,,$$

and therefore if $r \le \rho \le \rho' < 1$,

$$|f(\rho' e^{i\phi}) - f(\rho e^{i\phi})| \le \int_{\frac{\rho-r}{1-r}}^{\frac{\rho'-r}{1-r}} \left|\frac{d}{dt} f(\zeta + (1-r)e^{i\phi}t)\right| dt \le (1-r)C'|f'(\zeta)| \int_{\frac{\rho-r}{1-r}}^{\frac{\rho'-r}{1-r}} \frac{1}{(1-t)^{\alpha}}\, dt$$

$$= \frac{(1-r)C'|f'(\zeta)|}{1-\alpha}\Big((\frac{1-\rho}{1-r})^{1-\alpha} - (\frac{1-\rho'}{1-r})^{1-\alpha}\Big)\,. \tag{8}$$

This implies, in particular, that

$$f(e^{i\phi}) = \lim_{\rho\to 1^-} f(\rho e^{i\phi})$$

exists, and with $r = 0$, $\rho' = 1$ that

$$|f(e^{i\phi}) - f(\rho e^{i\phi})| \le \frac{C'|f'(0)|}{1-\alpha}(1-\rho)^{1-\alpha}\,. \tag{9}$$

Since by (7) we have that

$$|f(\rho e^{i\phi_1}) - f(\rho e^{i\phi_2})| \le \frac{C'|f'(0)|\rho}{(1-\rho)^{\alpha}}|\phi_1 - \phi_2|\,,$$

it follows, with $1-\rho = |\phi_1 - \phi_2| < 1$, that

$$|f(e^{i\phi_1}) - f(e^{i\phi_2})| \le |f(e^{i\phi_1}) - f(\rho e^{i\phi_1})| + |f(\rho e^{i\phi_1}) - f(\rho e^{i\phi_2})| + |f(\rho e^{i\phi_2}) - f(e^{i\phi_2})|$$

$$\le C'|f'(0)|(\frac{2}{1-\alpha} + 1)|\phi_1 - \phi_2|^{1-\alpha}\,, \tag{10}$$

for $|\phi_1 - \phi_2| \le 1$.

With these preliminaries we can now prove

Proposition 4. *If X has property M, and $f' \in \mathcal{F}_X$, then f has a continuous extension to $\overline{\mathbb{D}}$ and $f(e^{i\theta})$ satisfies (9). Furthermore, for such X the class $\{\frac{1}{|f'(0)|}f : f' \in \mathcal{F}_X\}$ is uniformly Hölder continuous on $\overline{\mathbb{D}}$, with the constant and exponent depending only on X.*

Proof. From (9) and (10) it follows that there exists a constant A depending only on X such that $|f(z_1) - f(z_2)| \leq A|f'(0)||z_1 - z_2|^{1-\alpha}$ for $z_1 \in \overline{\mathbb{D}}$ and $z_2 \in \partial\mathbb{D}$. If $z_1, z_2 \in \overline{\mathbb{D}}$ with $|z_1| \leq |z_2|$, then $g(z) = \frac{1}{|z_2|}f(|z_2|z)$ satisfies the same hypothesis as f with $g'(0) = f'(0)$, so that

$$|f(z_1) - f(z_2)| = |f(|z_2|\frac{z_1}{|z_2|}) - f(|z_2|\frac{z_2}{|z_2|})| = |z_2||g(\frac{z_1}{|z_2|}) - g(\frac{z_2}{|z_2|})|$$

$$\leq A|z_2||f'(0)|\left|\frac{z_1}{|z_2|} - \frac{z_2}{|z_2|}\right|^{1-\alpha} = A|z_2|^{\alpha}|f'(0)||z_1 - z_2|^{1-\alpha},$$

from which the desired conclusion follows immediately. □

Next we prove

Proposition 5. *Let $f' \in \mathcal{F}_X$, where X has property M, and let z_0 be the midpoint of the hyperbolic geodesic which joins points $a, b \in \partial\mathbb{D}$. Let E denote the shorter of the arcs of $\partial\mathbb{D}$ joining a and b. Let Q be the curvilinear quadrilateral*

$$Q = [a, |z_0|a] \cup |z_0|E \cup [|z_0|b, b] \cup E.$$

Then there is absolute constant C_4 such that

$$\mathrm{diam} f(Q) \leq \frac{C_4}{1-\alpha}|f'(z_0)|(1 - |z_0|).$$

Proof. First of all,

$$|f'(\zeta)| \leq C_5|f'(z_0)|, \tag{11}$$

for all $\zeta \in |z_0|E$. To see this, we recall that for normalized univalent functions g in $\mathbb{D}$, $|g'(z)| \leq (1+|z|)/(1-|z|)^3$ (see, for example [2]), so that the derivatives of such functions are uniformly bounded on $\frac{1}{2}\mathbb{D}$. Since $|z_0|E$ can be covered be a bounded number of open disks of radius $\frac{1}{2}(1-|z_0|)$, with centers $z_0, z_1, \cdots \in |z_0|E$, (11) follows upon application of the stated fact to the functions $f(z_i + (1-|z_0|)z)/(1-|z_0|)$ in succession. Applying (8) with $\zeta \in |z_0|E$, $\rho = r = |z_0| = |\zeta|$ and $r \leq \rho' \leq 1$, we have that

$$\mathrm{diam} f([\zeta, \zeta/|\zeta|]) \leq \frac{C'C_5}{1-\alpha}|f'(z_0)|(1 - |z_0|) \tag{12}$$

for $\zeta \in |z_0|E$. Since (11) implies that

$$\mathrm{diam} f(|z_0|E) \leq C_6|f'(z_0)||b - a|,$$

and $|b - a| \leq 2(1 - |z_0|)$, we conclude from (12) that

$$\mathrm{diam} f(Q) \leq 2(\frac{C'C_5}{1-\alpha} + C_6)|f'(z_0)|(1 - |z_0|),$$

as desired. □

The final proposition is a very weak form of the Gehring-Hayman theorem (see [3] and [8, p.88]).

Proposition 6. *For each closed $X \subset \mathbb{H}$ with property M there is a constant $C = C_X$ depending solely on X with the following property: let S be the hyperbolic geodesic joining any two points $z_1, z_2 \in \partial\mathbb{D}$. Then for any f with $f' \in \mathcal{F}_X$ and any Jordan arc Γ joining z_1, z_2 in $\mathbb{D}$*

$$\mathrm{diam} f(S) \leq C_X \mathrm{diam} f(\Gamma)\,.$$

Proof. If this were not true, then there would exist a sequence $\{f_n\}$ with $f_n' \in \mathcal{F}_X$ and sequences $\{z_1^{(n)}\}, \{z_2^{(n)}\}, S_n, \Gamma_n$, where $z_1^{(n)}, z_2^{(n)} \in \partial\mathbb{D}$, and S_n and Γ_n are, respectively, the hyperbolic segment and a Jordan arc joining these points in $\mathbb{D}$, such that

$$\frac{\mathrm{diam} f(\Gamma_n)}{\mathrm{diam} f(S_n)} \to 0\,. \tag{13}$$

By a simple compactness argument based on Proposition 4 it follows that $|z_1^{(n)} - z_2^{(n)}| \to 0$ and, after replacing $f_n(z)$ by $e^{-i\phi_n} f(e^{i\phi_n} z)$ with appropriate ϕ_n, we can assume that

$$z_1^{(n)} = -ie^{-i\tau_n} \quad \text{and} \quad z_2^{(n)} = -ie^{i\tau_n}\,, \tag{14}$$

where $\tau_n > 0$ and $\tau_n \to 0$. Let $z_0^{(n)} = -i|z_0^{(n)}|$ be the midpoint of S_n. Let

$$g_n(z) = \frac{f_n((1-|z_0^{(n)}|)z - i) - f_n(z_0^{(n)})}{(1-|z_0^{(n)}|)} \tag{15}$$

on the disk $D_n = \frac{1}{1-|z_0^{(n)}|}(\mathbb{D} + i)$. Obviously, $g_n'(D_n) \subset \mathbb{H}\backslash X$. Let $h_n(z) = g_n(z)/|g_n'(i)|$ and $S_n' = \frac{1}{1-|z_0^{(n)}|}(S_n + i)$. Then it folows from standard bounds for the derivatives of univalent functions (see [2]) and Proposition 4 that there exist positive constants K_1 and K_2 (which depend only on X) such that

$$K_1 \leq \mathrm{diam}\, h_n(S_n') \leq K_2\,, \tag{16}$$

and that for $N \in \mathbb{N}$ and any given $\rho > 0$, the family $\{h_n : n \geq N\}$ is uniformly bounded and equicontinuous in $D_N \cap \rho\mathbb{D}$ with the bound and the modulus of equicontinuity depending only on ρ and X. Thus there is a subsequence, which for convenience we continue to call $\{h_n\}$, which converges locally uniformly in $i\mathbb{H}$ to a function h which has a continuous extension to $i\overline{\mathbb{H}}$.

Let $w_k^{(n)} = \frac{1}{1-|z_0^{(n)}|}(z_k^{(n)} + i)$, $k = 1, 2$ and $\Gamma_n' = \frac{1}{1-|z_0^{(n)}|}(\Gamma_n + i)$. Then $w_k^{(n)} \to -1, 1$, for $k = 1, 2$, respectively. It follows from (13) and (16) that $\mathrm{diam}\, h_n(\Gamma_n') \to 0$, so that $h(-1) = h(1)$. From this and the fact that $h'(i\mathbb{H}) \subset \mathbb{H}$ it follows that $h([-1, 1])$ is a vertical segment.

We claim that for each $x \in (-1, 1)$ and each $\epsilon > 0$, $\Gamma_n' \cap (x + \epsilon\mathbb{D}) \neq \varnothing$ for infinitely many n. If this is not true, then there are $x_0 \in (-1, 1), \epsilon > 0$ and $M \in \mathbb{N}$, such that for all $n \geq M$ the set $(x_0 + \epsilon\mathbb{D}) \cap D_n$ lies inside the Jordan curve $\Gamma_n' \cup A_n$, where A_n is the shorter arc of ∂D_n joining $w_1^{(n)}$ to $w_2^{(n)}$. However, since $|h_n'(i)| = 1$, there is some $\delta > 0$ such that $h_n((x_0 + \epsilon\mathbb{D}) \cap D_n)$ contains a disk of radius δ for all $n \geq M$, so that for all such n the image of the interior domain of $\Gamma_n' \cup A_n$ contains such a disk. But this contradicts the argument principle since for all sufficiently large n, $h_n(\Gamma_n' \cup A_n) \subset h([-1, 1]) + (\delta/2)\mathbb{D}$, since $h([-1, 1])$ is a vertical line segment. This establishes the claim, from which it follows immediately that $h([-1, 1]) = \{h(1)\}$ in light of the fact that

$\operatorname{diam} h_n(\Gamma'_n) \to 0$. Thus (by the reflection principle, for example), h is constant. This, however, is a contradiction since $|h'(i)| = 1$. $\square$

4. Sufficiency of Property M

Throughout this section we assume that X has property M and that $f'(\mathbb{D}) \subset \mathbb{H}\backslash X$. We begin by showing that $f(\mathbb{D})$ is a John domain. Let $a, b \in \partial\mathbb{D}$ be such that $[f(a), f(b)]$ is a rectilinear crosscut of $f(\mathbb{D})$. Let S be the non-Euclidean segment joining a to b in $\mathbb{D}$ and let $A = f^{-1}([f(a), f(b)])$. Then Proposition 6 says that

$$\operatorname{diam} f(S) \leq C_X \operatorname{diam} f(A) = C_X |f(b) - f(a)|\,. \tag{17}$$

If z_0 is the midpoint of S, then the 1/4-theorem says that

$$\operatorname{diam} f(S) \geq \frac{1}{4}|f'(z_0)|(1 - |z_0|)\,. \tag{18}$$

Let E, Q be as in Proposition 5. From (17) and (18) together with Proposition 5 we have

$$\begin{aligned}\frac{1}{4}|f'(z_0)|(1-|z_0|) &\leq \operatorname{diam} f(S) \leq C_X|f(b)-f(a)| \\ &\leq C_4 C_X \frac{|f(b)-f(a)||f'(z_0)|(1-|z_0|)}{(1-\alpha(\tau_X))\operatorname{diam} f(Q)}\,,\end{aligned}$$

so that

$$\operatorname{diam} f(Q) \leq \frac{4C_4C_X}{1-\alpha(\tau_X)}|f(b)-f(a)|\,.$$

But since one of the components of $f(\mathbb{D})\backslash[f(a), f(b)]$ has as its boundary $[f(a), f(b)] \cup f(E)$ whose diameter is bounded above by

$$|f(b)-f(a)| + \operatorname{diam} f(Q) \leq \frac{(4C_4C_X+1)|f(b)-f(a)|}{(1-\alpha(\tau_X))}\,,$$

we see that $f(\mathbb{D})$ is indeed a John domain. The constant

$$C'_X = \frac{(4C_4C_X+1)}{(1-\alpha(\tau_X))}$$

in the definition of John domain depends solely on X.

To finish the proof we show that $f(\mathbb{D})$, which is bounded by the Jordan curve $f(\partial\mathbb{D})$ (that f is one-to-one on $\partial\mathbb{D}$ follows trivially from the fact that $f'(\mathbb{D}) \subset \mathbb{H}$), is also linearly connected and that the relevant constant can be taken to depend only on X. We assume that this is not the case and proceed as in the proof of Proposition 6 to conclude that there must be a sequence of functions $\{f_n\}$ with $f'_n \in \mathcal{F}_X$ and sequences $\{z_1^{(n)}\}, \{z_2^{(n)}\}$ with $|z_1^{(n)}| = |z_2^{(n)}| = 1$ and $z_1^{(n)} \neq z_2^{(n)}$ such that

$$\frac{\operatorname{diam} f_n([z_1^{(n)}, z_2^{(n)}])}{|f(z_1^{(n)}) - f(z_2^{(n)})|} \to \infty\,. \tag{19}$$

As in that proof, it follows that $|z_1^{(n)} - z_2^{(n)}| \to 0$ and we may assume that (14) holds. Let $z_0^{(n)}$ be the midpoint of the hyperbolic geodesic joining $z_1^{(n)}$ to $z_2^{(n)}$, and let Q_n be the curvilinear quadrilateral of Proposition 5 with $\{a,b\}=\{z_1^{(n)},z_2^{(n)}\}$. Since $[z_1^{(n)},z_2^{(n)}]\subset Q_n$, it follows from the 1/4-theorem, Proposition 5 and Proposition 6 that

$$\frac{1}{4C_X}|f'(z_0^{(n)})|(1-|z_0^{(n)}|)\le \operatorname{diam} f([z_1^{(n)},z_2^{(n)}])\le \operatorname{diam} f(Q_n)\le C''_X|f'(z_0^{(n)})|(1-|z_0^{(n)}|)\,, \qquad (20)$$

where $C''_X=\frac{C_4}{1-\alpha(\tau_X)}$ depends only on X.

Let g_n be as in (15) on the disk $D_n=\frac{1}{1-|z_0^{(n)}|}(\mathbb{D}+i)$, and $h_n(z)=g_n(z)/|g_n'(i)|$. Then exactly as in the proof of Proposition 6 some subsequence of $\{h_n\}$, which we continue to call $\{h_n\}$, converges locally uniformly in $\mathbb{H}$ to a function h which has a continuous extension to $\overline{\mathbb{H}}$. Since $g_n'(i)=f_n'(-|z_0^{(n)}|i)=f_n'(z_0^{(n)})$, it follows from (19) and (20) that $h(-1)=h(1)$, so that in light of (20) $h([-1,1])$ is a nondegenerate vertical segment. But it then follows that there is some $\zeta\in(-1,1)$ such that h is analytic at ζ and $h'(\zeta)=0$. Let K_1 be the compactum of Proposition 1. Obviously, there are $N\in\mathbb{N}$ and $s\in\mathbb{R}$ such that

$$\log h_n'(D_N\cap(\mathbb{D}+\zeta))\supset K_1+s$$

for some $n\ge N$. However, this implies that

$$\log f_n'(\mathbb{D})\supset K_1+s+\log|f_n(z_0^{(n)})|\,,$$

which contradicts Proposition 1. □

5. Necessity of Property M

In this section we construct mappings which show that if $X=\mathbb{H}\backslash R$ does not have property M_0, then there is an f with $f'\in\mathcal{F}_X$ for which $f(\mathbb{D})$ is not linearly connected. The idea is quite simple: given an X which does not have this property, there is a sequence of disjoint semi-disks of the form $r_n((\mathbb{D}\cap\mathbb{H})+\frac{1}{n})$, with $r_n\to 0$, none of which intersect X; that such a sequence of disks exists follows immediately if we take $K=\overline{\mathbb{D}\cap\mathbb{H}}+\frac{1}{n}$ in the definition of property M_0. These semi-disks can be connected by thin curvilinear strips to form a Jordan domain G. When this is done in the manner that we will indicate in the details to follow, G will come close enough, in a qualitative sense, to containing a neighborhood of $0\in\mathbb{H}$ for the antiderivative f of a one-to-one mapping of $\mathbb{D}$ onto G to map $\mathbb{D}$ onto a domain that is not linearly connected. The construction can be modified in a simple manner to show that if X does not have property M_∞, then there is a corresponding f for which $f(\mathbb{D})$ fails to satisfy (ii) in the definition of quasidisk in Section 2.

We begin with some general considerations regarding the attachment of a semi-disk to a domain G by a thin curved strip, as alluded to in the preceding paragraph. Let G be a Jordan domain with piecewise smooth boundary for which

$$\min\{\Re z: z\in\overline{G}\}=\Delta>0\,.$$

Let $z_0\in G$, $\epsilon>0$ and let $r>0$ be so small that $r(1+\epsilon)<\Delta$. Let S be a simple arc which joins a point of ∂G at which ∂G is analytic to the point $r(1+\epsilon)$ in the half-plane $\Re z>r(1+\epsilon)$ but outside of G. Let

$$G_\delta=r((\mathbb{D}\cap\mathbb{H})+\epsilon)\cup G\cup(S+\delta\mathbb{D})\,.$$

Obviously, there is a $\delta_0>0$ such that G_δ is a Jordan domain lying to the right of $\Re z=r\epsilon$ for $\delta\in(0,\delta_0]$; we consider only such δ. Let h_δ denote the canonical mapping of $\mathbb{D}$ onto G_δ with

$h_\delta(0) = z_0$ and $h'_\delta(0) > 0$. In reference to this mapping let a_δ, ζ_δ, b_δ be the points of $\partial\mathbb{D}$ which correspond to the points $r(i+\epsilon)$, $r\epsilon$, $r(-i+\epsilon)$. By the reflection principle h_δ is analytic on the open arc E_δ of $\partial\mathbb{D}$ which contains ζ_δ and has endpoints a_δ, b_δ. We write

$$\alpha_\delta = \arg\frac{\zeta_\delta}{a_\delta} \quad \text{and} \quad \arg\frac{b_\delta}{\zeta_\delta} = \xi_\delta\alpha_\delta\,.$$

Let P_+ and P_- be the set of $\delta \in (0, \delta_0]$ for which $\xi_\delta \geq 1$ and $\xi_\delta \leq 1$, respectively. Assume that $0 \in \overline{P_+}$, the contrary case being the same except for minor notational differences. We claim that

$$\frac{r}{4\alpha_\delta} \leq |h'_\delta(\zeta_\delta)| \leq \frac{2\pi r}{\alpha_\delta}\ \text{, for } \delta \in P_+\,, \tag{21}$$

and

$$\alpha_\delta \to 0 \quad \text{as } \delta \to 0 \text{ in } P_+\,. \tag{22}$$

To see this let $\mathbb{D}_1 = \mathbb{C}\backslash(\partial\mathbb{D}\backslash E_\delta)$, let $\beta_\delta = |a_\delta - \zeta_\delta|$, that is, the minimum distance of ζ_δ to $\partial\mathbb{D}_1$, and let h_δ now stand for the continuation of the original h_δ to all of $\mathbb{D}_1$ by reflection. Note that because all of G_δ lies to the right of the line $\Re z = r\epsilon$, h_δ is univalent in $\mathbb{D}_1$. Then g_δ defined by

$$g_\delta(z) \ = \frac{1}{r}h_\delta(\beta_\delta z + \zeta_\delta) - \epsilon$$

is analytic in $\mathbb{D}$ and satisfies $g_\delta(0) = 0$. Furthermore, g_δ is continuous and one-to-one on $\overline{\mathbb{D}}$. Now,

$$\frac{a_\delta - \zeta_\delta}{\beta_\delta} \in \partial\mathbb{D} \quad \text{and} \quad g_\delta(\frac{a_\delta - \zeta_\delta}{\beta_\delta}) = i\,.$$

Upon applying the 1/4-theorem to g_δ we conclude that

$$|\frac{r}{\beta_\delta h'_\delta(\zeta_\delta)}| \geq \frac{1}{4}\,. \tag{23}$$

Now, the inverse g_δ^{-1} of g_δ is analytic and univalent in $\mathbb{D}$ and maps $\mathbb{D}$ onto a subdomain of $\frac{1}{\beta_\delta}(\mathbb{D}_1 - \zeta_\delta)$ which does not contain the point $\frac{a_\delta - \zeta_\delta}{\beta_\delta}$, which has modulus 1. Application of the 1/4-theorem to g_δ^{-1} then shows that

$$|\frac{\beta_\delta h'_\delta(\zeta_\delta)}{r}| \geq \frac{1}{4}\,. \tag{24}$$

The desired conclusion (21) follows from (23) and (24) since $1 \leq \frac{\alpha_\delta}{\beta_\delta} \leq \frac{\pi}{2}$.

If h_0 is the mapping of $\mathbb{D}$ onto G with $h_0(0) = z_0$ and $h'(0) > 0$ then it follows immediately by the Caratheodory kernel convergence theorem that $h_\delta \to h_0$ locally uniformly in $\mathbb{D}$ as $\delta \to 0$. From this in turn it follows that if $\Gamma \subset \partial G$ is an open analytic arc not containing the point joined by S to $r(1+\epsilon)$ and which corresponds under h_0 to $E \subset \partial\mathbb{D}$, then for any compact subarc E' of E, $h_\delta \to h_0$ uniformly in a neighborhood of E' also. To see this one simply observes that the inverses of the h_δ converge uniformly in a neighborhood of any compact subarc of Γ. With this one concludes that if ∂G is analytic at the point joined by S to $r(1+\epsilon)$, then

$$\Lambda(h_\delta^{-1}(\partial G_\delta\backslash\partial G)) \to 0 \text{ as } \delta \to 0\,,$$

where Λ denotes length. This clearly implies (22).

Now let $\{r_n\}$ be a positive sequence tending to 0 for which $r_{n+1} < \frac{r_n}{2n}$ and let

$$D_n = r_n((\mathbb{D} \cap \mathbb{H}) + \frac{1}{n}),$$

so that each of these semi-disks lies to the left of its predecessor. Let S_n be a smooth simple curve which joins the point

$$r_n(\frac{1+i}{\sqrt{2}} + \frac{1}{n}) \in \partial D_n$$

to the rightmost point $r_{n+1}(1 + \frac{1}{n+1})$ of ∂D_{n+1} in the strip

$$\{z : r_n(1 + \frac{1}{n}) > \Re z > r_{n+1}(1 + \frac{1}{n+1})\}.$$

Then a straightforward recursive construction based on the foregoing considerations shows that for fixed $z_0 \in D_1$ there is a sequence $\{\delta_n\}$ of positive numbers such that

$$G = \bigcup_{n=1}^{\infty} \{D_n \cup (S_n + \delta_n \mathbb{D})\}$$

is a Jordan domain, and, because of (21), is such that if E_n, with endpoints a_n and b_n, is the arc of $\partial\mathbb{D}$ corresponding to the diameter segment of ∂D_n under the canonical mapping h of $\mathbb{D}$ onto G with $h(0) = z_0$ and $h'(0) > 0$, and if ζ_n is the preimage under h of the midpoint $\frac{r_n}{n}$ of the diameter segment of ∂D_n, then

$$\frac{1}{4\pi} \min\{\arg \frac{\zeta_n}{a_n}, \arg \frac{b_n}{\zeta_n}\} \le \frac{r_n}{|h'(\zeta_n)|} \le 8 \min\{\arg \frac{\zeta_n}{a_n}, \arg \frac{b_n}{\zeta_n}\}. \tag{25}$$

If X does not have property M_0, then there clearly exist sequences $\{r_n\}$, $\{S_n\}$ and $\{\delta_n\}$ such that the corresponding domain G just constructed is contained in $R = \mathbb{H} \backslash X$. (Recall that R is connected.) Without loss of generality we may assume that the set N of n for which

$$\min\{\arg \frac{\zeta_n}{a_n}, \arg \frac{b_n}{\zeta_n}\} = \arg \frac{\zeta_n}{a_n}$$

is infinite; the contrary case is handled in exactly the same manner. Fix $n \in N$ and as above let $\alpha_n = \arg \frac{\zeta_n}{a_n}$. We now use the same transformation that we used in the proof of Proposition 6 (see (14), et seq.). Let $f' = h$ and let

$$f_n(z) = \frac{f((\alpha_n z - i)i\zeta_n) - f(\zeta_n)}{\alpha_n r_n}\ ,\ z \in \frac{\mathbb{D} + i}{\alpha_n}.$$

Then $f_n(0) = 0$ and $\frac{1}{8} \le |f_n''(0)| \le 4\pi$ by (25), and f_n' is regular on an arc of length 2 of $\frac{\mathbb{D}+i}{\alpha_n}$ centered at 0 (it maps this arc onto an interval of the diameter segment of $\frac{i\zeta_n}{r_n}\partial D_n$). From this it follows that some subsequence of $\{f_n' : n \in N\}$ converges uniformly to a function on $i\mathbb{H}$ which has nonvanishing derivative and which maps $(-1, 1)$ onto an interval of $\mathbb{R}$ containing 0. This clearly shows that $f(\mathbb{D})$ is not linearly connected.

Finally, we see what happens if X does not have property M_∞. In this case, $1/X$ does not have property M_0, and we consider the function h corresponding to $1/X$ defined in the preceding paragraph but one. Let $g'(z) = 1/h(z)$. The argument used in the immediately preceding paragraph then produces a function on $i\mathbb{H}$ whose derivative is of the form $1/k(z)$, where $k(z)$ maps $(-1, 1)$

onto and interval containing 0. From this it follows that $g(\mathbb{D})$ is not a John domain. It should be noted that by making the δ_n small enough one can insure that for the g so constructed $g(\mathbb{D})$ is a non-John Jordan domain, and furthermore, that if $X = \mathbb{H}\backslash R$ has neither of the properties M_0 or M_∞, then by a combination of the constructions one can obtain a g with $g'(\mathbb{D}) \subset R$ for which $g(\mathbb{D})$ is a Jordan domain which is neither linearly connected nor has the John property. $\square$

6. Additional Counterexamples

In this section we construct a function f on $\mathbb{D}$ for which $f'(\mathbb{D}) \subset \mathbb{H} \cap \mathbb{D}$ but for which $f(\mathbb{D})$ is not a John domain; this shows that if X has property M_∞ alone $f'(\mathbb{D}) \subset \mathbb{H}\backslash X$ does not imply that the image is a John domain. To see how to construct such a function, let $0 < \delta < 1$ and consider

$$q(z,\delta) = \log(z-\delta) + \log(z+\delta) - \log z - \frac{1}{2}(\log(z-1) + \log(z+1))\,,$$

where we are using the principal branch of $\log z$, that is, the branch whose imaginary part is in $(0,\pi)$ for $z \in i\mathbb{H}$. A simple calculation shows that the boundary values of $\Im\{q(x,\delta)\}$ on $i\mathbb{H}$ are given by the following combination of characteristic functions

$$\omega(x,\delta) = \frac{\pi}{2}\{\chi_{(0,\delta)}(x) - \chi_{(-\delta,0)}(x) + \chi_{(-1,-\delta)}(x) - \chi_{(\delta,1)}(x)\}\,,$$

so that $e^{q(z,\delta)} = \dfrac{z^2-\delta^2}{z\sqrt{z^2-1}}$ maps $i\mathbb{H}$ onto $\mathbb{H}$. It is also a straightforward matter to see that if f is an antiderivative of $e^{q(z,\delta)}$ on $i\mathbb{H}$, then as x moves along $\mathbb{R}$ rightward starting at $-\infty$, $f(x)$ moves horizontally until one reaches -1, where it makes a left turn, moves vertically up to some point $f(-\delta) = f(-1) + ib$, then reverses direction and moves vertically downward to $f(0) = f(-1) - i\infty$, then vertically upward to $f(1) + ib$, then vertically downward to $f(1)$, and finally rightward once again to ∞. The image $f(i\mathbb{H})$ thus has an infinite downward protuberance pushed into the upper half-plane. The idea of the construction is to smooth out $q(z,\delta)$, in essence by replacing it with $q(z + i\delta^{3/2}, \delta)$, so as to keep the resulting exponential bounded. The image domain so obtained will have a thin downward protuberance emanating from the top a even deeper indentation. Then, on the basis of this construction of a function on $i\mathbb{H}$ one manufactures a mapping of $\mathbb{D}$ with the same behavior repeated with thinner and thinner projections in such a way that the condition (ii) in the definition of quasidisk is violated. The following paragraphs give the analytic details of this construction.

Throughout the following development x and z will represent real and complex variables, respectively, and $0 < \delta \le \frac{1}{4}$. We begin by analyzing the behavior of $q(z + i\delta^{3/2}, \delta)$, for which purpose we consider

$$\begin{aligned}\xi(x,\delta) &= \Re\{\log((x+i\delta^{3/2})^2 - \delta^2) - \log(x + i\delta^{3/2})\} \\ &= A(x,\delta) + B(x,\delta)\,,\end{aligned}$$

where

$$A(x,\delta) = \frac{1}{2}\log((x^2 - \delta^3 - \delta^2)^2 + 4\delta^3 x^2)$$

and

$$B(x,\delta) = -\frac{1}{2}\log(x^2 + \delta^3)\,.$$

For $|x| \le \delta$ we have

$$A(x,\delta) \le \frac{1}{2}\log((3\delta^2)^2 + 4\delta^5) \le \frac{1}{2}\log(13\delta^4) \le \log 4 + 2\log\delta\,,$$

and

$$B(x,\delta) \le -\frac{1}{2}\log\delta^3 = -\frac{3}{2}\log\delta\,.$$

Thus, for $|x| \le \delta$

$$\xi(x,\delta) \le \log 4 + \frac{1}{2}\log\delta \le \log 4\,.$$

For $1 > |x| \ge \delta$ we have

$$A(x,\delta) \le \frac{1}{2}\log((3x^2)^2 + 4|x|^5) \le \log 4 + 2\log|x|\,,$$

and

$$B(x,\delta) \le -\frac{1}{2}\log x^2 = -\log|x|\,,$$

so that

$$\xi(x,\delta) \le \log 4\ ,\ \text{for } |x| < 1\ ,\ 0 < \delta \le \frac{1}{4}\,.$$

Since for $|x| \le \frac{1}{2}$ and $0 < \delta \le \frac{1}{4}$

$$-\frac{1}{2}\Re\{\log((x+i\delta^{3/2})^2 - 1)\} \le \log 2\,,$$

we have that

$$\Re\{q(x+i\delta^{3/2},\delta)\} = \xi(x,\delta) - \frac{1}{2}\Re\{\log((x+i\delta^{3/2})^2-1)\} \le \log 4 + \log 2 \le \log 8 \qquad (26)$$

for all such x, δ.

Let $\eta \in C^\infty(\mathbb{R})$ satisfy $0 \le \eta(x) \le 1$, $\eta(x) = 1$ for $|x| \le 5/8$ and $\eta(x) = 0$ for $|x| \ge 3/4$. Let

$$q_1(z,\delta) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1+zt}{(z-t)(1+t^2)}\eta(t)\Im q(t+i\delta^{3/2},\delta)\,dt\,.$$

Then we claim that the following are true:

(i) $\Im q_1(x,\delta) = 0$ for $|x| \ge 3/4$;

(ii) $q_1(z,\delta) - q(z+i\delta^{3/2},\delta)$ is analytic and bounded for $z \in \frac{1}{2}\mathbb{D}$ uniformly in δ;

(iii) $q_1(z,\delta)$ is bounded in $i\mathbb{H}\backslash(\frac{1}{2}\mathbb{D})$ uniformly in δ;

(iv) $q_1(z,\delta) \to 0$ uniformly in δ as $z \to \infty$ in $i\mathbb{H}$;

(v) $\Re\{q_1(z,\delta)\}$ is uniformly bounded above in $i\mathbb{H}$.

Statement (i) is obvious from the defintion of q_1. Statement (ii) follows from the Poisson integral formula since the boundary values of $\Im\{q_1(z,\delta) - q(z+i\delta^{3/2},\delta)\}$ vanish on $[-\frac{5}{8},\frac{5}{8}]$ and are uniformly bounded in δ on the rest of $\mathbb{R}$ and $\Re\{q_1(i,\delta) - q(i+i\delta^{3/2},\delta)\}$ is uniformly bounded in δ. To see that (iii) is true we break the expression for $q_1(z,\delta)$ into two integrals, $I_1 + I_2$, the first having as its domain of integration $[-\frac{3}{8},\frac{3}{8}]$ and the second $\mathbb{R}\backslash[-\frac{3}{8},\frac{3}{8}]$. The first integral has the desired properties since $[-\frac{3}{8},\frac{3}{8}]$ is disjoint from $\overline{i\mathbb{H}\backslash(\frac{1}{2}\mathbb{D})}$. That I_2 is bounded follows (by integration by parts) from the uniform boundedness of the derivative of $\eta(t)\Im\{q(t+i\delta^{3/2},\delta)$ outside of $[-\frac{3}{8},\frac{3}{8}]$. Statement (iv) follows from the uniform boundedness of $\eta(t)\Im\{q(t+i\delta^{3/2},\delta)$ and the fact that it vanishes outside of $[-1,1]$. Finally, (v) follows from (ii) and (iii) together with (26).

We now proceed to examine the behavior of the antiderivatives of $e^{q_1(z,\delta)}$. It follows from (ii) that if $w = z + i\delta^{3/2}$, then

$$s(z,\delta) = q_1(z,\delta) - \log(w^2 - \delta^2) + \log w = q_1(z,\delta) - q(w,\delta) - \frac{1}{2}\log(w^2-1)$$

is analytic for $z \in \frac{1}{2}\mathbb{D}$. Let $c \in (0,1)$ be fixed and let $a_0 = a_0(\delta) = e^{s(0,\delta)} = |a_0|/i$, the last equality holding since the definition of $s(z,\delta)$ implies that $\Im\{s(0,\delta)\} = -\frac{\pi}{2}$. Since $s(z,\delta)$ is analytic and uniformly bounded in $\frac{1}{2}\mathbb{D}$, $e^{s(z,\delta)} = a_0 + O(z)$. Since for $z \in \mathbb{R}$, $|z| \le |w|$, we have (with integration being performed along the real axis) that

$$\int_{-c\delta}^{c\delta} e^{q_1(z,\delta)}dz = \int_{-c\delta}^{c\delta} \frac{w^2-\delta^2}{w}(a_0 + O(z))dz = a_0\int_{-c\delta}^{c\delta}\frac{w^2-\delta^2}{w}dz + \int_{-c\delta}^{c\delta}(w - \frac{\delta^2}{w})O(z)dz$$

$$= a_0\int_{-c\delta}^{c\delta}\frac{w^2-\delta^2}{w}dz + O(\delta^3)\,.$$

Now,

$$a_0\int_{-c\delta}^{c\delta}\frac{w^2-\delta^2}{w}dz = a_0(2c\delta^{5/2}i - \delta^2\log(-\frac{c+i\sqrt{\delta}}{c-i\sqrt{\delta}}) = a_0 i\pi\delta^2 + O(\delta^{5/2}) = |a_0|\pi\delta^2 + O(\delta^{5/2})\,,$$

so that

$$\int_{-c\delta}^{c\delta} e^{q_1(z,\delta)}dz \sim |a_0|i\pi\delta^2 \quad \text{as } \delta \to 0\,. \tag{27}$$

Note that the imaginary part of $\log(-\frac{c+i\sqrt{\delta}}{c-i\sqrt{\delta}})$ is indeed negative, as direct examination of the relevant part of the integral reveals. On the other hand, we see in the same manner that

$$\int_0^{\pm c\delta} e^{q_1(z,\delta)}dz = \int_0^{\pm c\delta}\frac{w^2-\delta^2}{w}(a_0 + O(z))dz$$

$$= a_0\left(\frac{1}{2}(\delta^{3/2}i \pm c\delta)^2 - \frac{1}{2}(\delta^{3/2}i)^2 - \delta^2\log\frac{\delta^{3/2}i \pm c\delta}{\delta^{3/2}i}\right) + O(\delta^3) \sim \frac{|a_0|}{2i}\delta^2\log\delta \quad \text{as } \delta \to 0\,. \tag{28}$$

Because $|\Im q(z,\delta)| \le \frac{\pi}{2}$ in $i\mathbb{H}$, the same bound holds for $\Im q_1(z,\delta)$, so that $\Re e^{q_1(z,\delta)} > 0$ in $i\mathbb{H}$. Relations (27) and (28) thus tell us that the image of $(-c\delta, c\delta)$ under the antiderivatives of $e^{q_1(z,\delta)}$ is a downward pointing arm of length and width asymptotic to $|a_0|\delta^2\log\frac{1}{\delta}$ and $|a_0|\pi\delta^2$, respectively. This arm hangs down from the top a deep indentation.

To finish the construction we have merely to put into the image a collection of such projections corresponding to a sequence of δ's tending to 0, which we do as follows. In accordance with (v) let K be an upper bound for the real part of $q_1(z,\delta)$. We define

$$p(z,\delta,\rho,x) = q_1(\frac{z-x}{\rho},\delta)\,.$$

It is now a simple matter to see that for any function $\gamma(z)$ which is analytic in $i\overline{\mathbb{H}}$ and for which $\gamma(0) \ne 0$ and any sequence $\{\delta_k\}$ of positive numbers approaching 0, there are sequences $\{\rho_k\}$ and $\{x_k\}$ approaching 0 such that $S(z) = \sum_k p(z,\delta_k,\rho_k,x_k)$ satisfies

$$|e^{S(z)}| \le e^{2K} \quad \text{and} \quad |\Im S(z)| \le \frac{\pi}{2}\,,$$

(the first by (iv) and (v), and the second by (i) and is such that for

$$Q(z,\gamma) = \int_i^z e^{S(\zeta)}\gamma(\zeta)d\zeta\,,$$

the image $Q(\mathbb{D}\cap\mathbb{H},\gamma)$ fails to be a John domain (in light of the estimates (27) and (28) and the comment about the image of the antiderivatives of $e^{q_1(z,\delta)}$ immediately following the latter). As previously, we let $T(z) = i\frac{z-i}{z+i}$, which maps $i\mathbb{H}$ onto $\mathbb{D}$, and consider

$$P(z) = \int_0^z e^{S(T^{-1}(\zeta))}d\zeta = \int_i^{T^{-1}(z)} e^{S(w)}T'(w)dw = Q(T^{-1}(z),\gamma)\,,$$

with $\gamma = T'$. By the foregoing $P(\mathbb{D}) = Q(\mathbb{H},\gamma)$ is not a John domain, even though $P'(\mathbb{D}) \subset i\mathbb{H}\cap(e^{2K}\mathbb{D})$.

Virtually the same construction yields a function f for which $f'(\mathbb{D}) \subset \mathbb{H}\backslash(r\mathbb{D})$, but such that $f(\mathbb{D})$ violates condition (iii) in the defintion of a quasidisk in Section 2. Indeed, one simply has to replace q and q_1 by their negatives throughout, so that what were thiner and thiner spikes now become narrower and narrower indentations. Moreover, by choosing sufficiently small ρ_k's one can ensure that $f' \in H^1(\mathbb{D})$, so that $f(\mathbb{D})$ is a Jordan domain.

We finish by mentioning that there are bounded convex domains D such that the theorem is not true when $\mathbb{D}$ is replaced by D. Indeed, let $0 \in \partial D$ and $D \subset \{z : -\alpha < \arg z < \alpha\} = S$, where $\alpha < \frac{\pi}{2}$. Then $1/D \subset S$, so that $f(z) = \log z$ satisfies $f'(D) \subset \mathbb{H}\backslash X$, where $X = \mathbb{R}e^{i\frac{\pi+2\alpha}{4}}$. Obviously, X satisfies the hypothesis of the theorem but $f(D)$ does not satisfy the John condition (ii) of the definition of quasidisk.

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