A Bank-Weiser type a-posteriori error estimate for a mixed formulation of a nonlinear elliptic problem [1]

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Abstract

We consider the numerical solution, via the mixed finite element method, of a nonlinear elliptic partial differential equation in divergence form with Dirichlet boundary conditions. Besides the *temperature* u and the *flux* $\boldsymbol{\sigma}$, we introduce ∇u as a further unknown, which yields a variational formulation with a twofold saddle point structure. We derive a reliable a-posteriori error estimate that depends on the solution of a local linear boundary value problem, which does not need any equilibrium property for its solvability. In addition, for specific finite element subspaces of Raviart-Thomas type we are able to provide a simpler a-posteriori error estimate that does not require the explicit solution of the local problems. Our approach does not need the exact finite element solution, but any reasonable approximation of it, such as, for instance, the one obtained with a fully discrete Galerkin scheme. In particular, we suggest a scheme that uses quadrature formulas to evaluate all the linear and semi-linear forms involved. Finally, several numerical results illustrate the suitability of the error estimator for the adaptive computation of the corresponding discrete solutions.

Key words: Two-fold saddle point problem, Raviart-Thomas, local problems.

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1 Introduction

It is very well known that the application of adaptive algorithms, based on a-posteriori error estimates, guarantees good convergence behaviour of the finite element solution to boundary value problems. Moreover, these adaptive techniques are particularly necessary for nonlinear problems where no a-priori information about the solution, that would allow to construct convenient meshes, is available. Now, the list of references on the a-posteriori error analysis of the finite element method (FEM) is quite large, and due to our present interest in mixed-FEM, we just mention [3], [4], [9], [13], [22] and [23]. In fact, one of the first works on mixed formulations is due to Verfürth [22], where an a-posteriori error estimator of explicit residual type is obtained for the Stokes problem. Then, the Raviart-Thomas and Brezzi-Douglas-Marini spaces are used in [4] to provide estimators, based on residual evaluations and on the solution of local problems, for linear elliptic partial differential equations of second order. In addition, the Raviart-Thomas spaces are also used in [9] and [13] to derive efficient and reliable residual error estimators. Further, the hierarchical a-posteriori error estimator from [5] is generalized in [1] to mixed formulations.

More recently, the classical Bank-Weiser estimator from [6] (see also the related works in [2] and [3]), which involves the solution of equilibrated local Neumann problems, was extended in [10] to large-strain elasticity, including the incompressible case. This method was also applied in [11] and [16] to obtain implicit and explicit residual error estimators for the combination of both FEM and mixed-FEM with boundary elements. In the case of mixed formulations, the estimator depends on the solution of local Dirichlet problems. The main advantage of the Bank-Weiser approach, though provides reliable estimates, is that one can expect a good effectivity index (see, e.g. [11]).

On the other hand, we showed in [7] that the combination of the Bank-Weiser method with the analysis from [12] can be used to derive an a-posteriori error estimate for the twofold saddle point formulation of a nonlinear problem in plane elasticity. This kind of saddle point problem, also called dual-dual variational formulations, have arised from the application of appropriate dual mixed-FEM to nonlinear boundary value problems in potential theory and elasticity (see, e.g. [15], [17] and the references therein). In particular, in [15] we provided the abstract theory for twofold saddle point problems, including the numerical analysis of fully discrete Galerkin schemes, and applied it to a nonlinear elliptic equation in divergence form appearing in the computation of magnetic fields and in steady heat conduction. However, no a-posteriori error analysis was given there. At this point we remark that the extension to dual-dual formulations of a-posteriori error estimates of explicit residual type or of those based on hierarchical bases, is not available yet.

Therefore, the purpose of this paper is to derive a Bank-Weiser type a-posteriori error estimate, similarly as we did in [7], for the nonlinear boundary value problem studied in [15]. The rest of the paper is presented as follows. In Section 2 we introduce the model problem from [15], write the corresponding variational formulation in the form of a twofold saddle point operator equation, and state the unique solvability of it. Section 3 deals with the a-posteriori error analysis. First, we derive our main Bank-Weiser type a-posteriori error estimate, which depends on the solution of local Dirichlet problems, and then we deduce a simpler estimate, which does not require the explicit solution of these local problems. Next, in Section 4 we recall from [15] a fully discrete Galerkin scheme for the model problem. Finally, some numerical results are presented in Section 5.

2 The model problem

Let Ω be a simply connected and bounded domain in $\mathbb{R}^2$ with Lipschitz continuous boundary $\Gamma := \partial\Omega$. Then, given $f \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma)$ and a function $\boldsymbol{\kappa} : \Omega \times \mathbb{R}^+ \to \mathbb{R}$, we consider the nonlinear boundary value problem with non-homogeneous Dirichlet data: *Find* $u \in H^1(\Omega)$ *such that*

$$\begin{aligned} -\operatorname{div}\left(\boldsymbol{\kappa}(\cdot, |\nabla u|)\,\nabla u\right) &= f \quad \text{in} \quad \Omega\,, \\ u &= g \quad \text{on} \quad \Gamma\,, \end{aligned} \tag{1}$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^2$ and div denotes the divergence operator. This kind of nonlinear elliptic problem in divergence form appears in several applications, such as the computation of the magnetic field of electromagnetic devices and steady heat conduction (see, e.g. [20], [21]).

In what follows, we assume that $\boldsymbol{\kappa} \in C^1(\Omega \times \mathbb{R}^+)$ and that there exist constants $\kappa_0,\, \kappa_1 > 0$, such that for all $(x, \mu) \in \Omega \times \mathbb{R}^+$:

$$\kappa_0 \,\le\, \boldsymbol{\kappa}(x,\mu) \,\le\, \kappa_1\,,$$

$$\kappa_0 \,\le\, \boldsymbol{\kappa}(x,\mu) + \mu\,\frac{\partial}{\partial\mu}\,\boldsymbol{\kappa}(x,\mu) \,\le\, \kappa_1\,, \quad \text{and} \tag{2}$$

$$|\nabla_x\, \boldsymbol{\kappa}(x,\mu)| \,\le\, \kappa_1\,.$$

As in [7] and [15] we are interested in using mixed finite element methods to solve (1). Hence, following those references, we introduce the further unknowns

$\mathbf{t} := \nabla u$ and $\boldsymbol{\sigma} := \boldsymbol{\kappa}(\cdot, |\nabla u|)\, \nabla u$ in Ω, and arrive at the following variational formulation of (1): *Find* $(\mathbf{t}, \boldsymbol{\sigma}, u) \in [L^2(\Omega)]^2 \times H(\mathrm{div};\Omega) \times L^2(\Omega)$ *such that*

$$\begin{aligned}
\int_\Omega \boldsymbol{\kappa}(\cdot, |\mathbf{t}|)\, \mathbf{t} \cdot \mathbf{s}\, dx - \int_\Omega \boldsymbol{\sigma} \cdot \mathbf{s}\, dx \qquad\qquad &= \quad 0\,, \\
-\int_\Omega \mathbf{t} \cdot \boldsymbol{\tau}\, dx \qquad\qquad - \int_\Omega u \,\mathrm{div}\, \boldsymbol{\tau}\, dx &= - < g, \boldsymbol{\tau} \cdot \boldsymbol{\nu} >\,, \\
- \int_\Omega v \,\mathrm{div}\, \boldsymbol{\sigma}\, dx \qquad\qquad &= \int_\Omega f v\, dx\,,
\end{aligned} \tag{3}$$

for all $(\mathbf{s}, \boldsymbol{\tau}, v) \in [L^2(\Omega)]^2 \times H(\mathrm{div};\Omega) \times L^2(\Omega)$, where $< \cdot, \cdot >$ stands for the duality pairing of $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ with respect to the $L^2(\Gamma)$-inner product, and $\boldsymbol{\nu}$ denotes the unit outward normal to Γ.

We recall here that $H(\mathrm{div};\Omega)$ is the space of functions $\boldsymbol{\tau} \in [L^2(\Omega)]^2$ such that $\mathrm{div}\, \boldsymbol{\tau} \in L^2(\Omega)$. It is well known that $H(\mathrm{div};\Omega)$ is a Hilbert space and that for all $\boldsymbol{\tau} \in H(\mathrm{div};\Omega)$, $\boldsymbol{\tau} \cdot \boldsymbol{\nu} \in H^{-1/2}(\Gamma)$ with $||\boldsymbol{\tau} \cdot \boldsymbol{\nu}||_{H^{-1/2}(\Gamma)} \leq ||\boldsymbol{\tau}||_{H(\mathrm{div};\Omega)}$ (see [19] for the proof of these results).

Now, it is easy to see that (3) has the two-fold saddle point structure studied in [15]. In fact, let $X_1 := [L^2(\Omega)]^2$, $M_1 := H(\mathrm{div};\Omega)$, $M := L^2(\Omega)$, $\mathbf{H} := X_1 \times M_1 \times M$, and define the nonlinear operator $\mathbf{A}_1 : X_1 \to X_1'$, the bounded linear operators $\mathbf{B}_1 : X_1 \to M_1'$, $\mathbf{B} : M_1 \to M'$, and the functionals $\mathbf{F}_1 \in X_1'$, $\mathbf{G}_1 \in M_1'$, $\mathbf{G} \in M'$, as follows

$$[\mathbf{A}_1(\mathbf{r}), \mathbf{s}] := \int_\Omega \boldsymbol{\kappa}(\cdot, |\mathbf{r}|)\, \mathbf{r} \cdot \mathbf{s}\, dx\,,$$

$$[\mathbf{B}_1(\mathbf{r}), \boldsymbol{\tau}] := - \int_\Omega \mathbf{r} \cdot \boldsymbol{\tau}\, dx\,,$$

$$[\mathbf{B}(\boldsymbol{\rho}), v] := - \int_\Omega v \,\mathrm{div}\, \boldsymbol{\rho}\, dx\,,$$

$$[\mathbf{F}_1, \mathbf{s}] := 0, \quad [\mathbf{G}_1, \boldsymbol{\tau}] := - < g, \boldsymbol{\tau} \cdot \boldsymbol{\nu} > \quad \text{and} \quad [\mathbf{G}, v] := \int_\Omega f v\, dx$$

for all $(\mathbf{r}, \boldsymbol{\rho})$, $(\mathbf{s}, \boldsymbol{\tau}) \in X_1 \times M_1$ and for all $v \in M$, where $[\cdot, \cdot]$ stands for the duality pairing induced by the corresponding operators. We remark that the first condition in (2) ensures that $\mathbf{A}_1(\mathbf{r}) \in X_1'$ for all $\mathbf{r} \in X_1$, which confirms that $\mathbf{A}_1$ is well defined.

Thus, the variational formulation (3) can be rewritten as: *Find* $(\mathbf{t}, \boldsymbol{\sigma}, u) \in \mathbf{H}$ *such that*

$$\begin{array}{lclcl} [\mathbf{A}_1(\mathbf{t}), \mathbf{s}] & + [\mathbf{B}_1(\mathbf{s}), \boldsymbol{\sigma}] & & = & [\mathbf{F}_1, \mathbf{s}]\,, \\ [\mathbf{B}_1(\mathbf{t}), \boldsymbol{\tau}] & + & [\mathbf{B}(\boldsymbol{\tau}), u] & = & [\mathbf{G}_1, \boldsymbol{\tau}]\,, \\ & [\mathbf{B}(\boldsymbol{\sigma}), v] & & = & [\mathbf{G}, v]\,, \end{array} \tag{4}$$

for all $(\mathbf{s}, \boldsymbol{\tau}, v) \in \mathbf{H}$.

Equivalently, let $X := X_1 \times M_1$ and define the nonlinear operator $\mathbf{A} : X \to X'$ and the functional $\mathbf{F} \in X'$ by

$$[\mathbf{A}(\mathbf{r}, \boldsymbol{\rho}), (\mathbf{s}, \boldsymbol{\tau})] := [\mathbf{A}_1(\mathbf{r}), \mathbf{s}] + [\mathbf{B}_1(\mathbf{r}), \boldsymbol{\tau}] + [\mathbf{B}_1(\mathbf{s}), \boldsymbol{\rho}]\,, \tag{5}$$

$$[\mathbf{F}, (\mathbf{s}, \boldsymbol{\tau})] := [\mathbf{F}_1, \mathbf{s}] + [\mathbf{G}_1, \boldsymbol{\tau}]\,,$$

for all $(\mathbf{r}, \boldsymbol{\rho})$, $(\mathbf{s}, \boldsymbol{\tau}) \in X$. Then (4) can be reformulated as: *Find* $(\mathbf{t}, \boldsymbol{\sigma}, u) \in \mathbf{H} := X \times M$ *such that*

$$\begin{array}{lcl} [\mathbf{A}(\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{s}, \boldsymbol{\tau})] & + [\mathbf{B}(\boldsymbol{\tau}), u] & = [\mathbf{F}, (\mathbf{s}, \boldsymbol{\tau})]\,, \\ [\mathbf{B}(\boldsymbol{\sigma}), v] & & = [\mathbf{G}, v]\,, \end{array} \tag{6}$$

for all $(\mathbf{s}, \boldsymbol{\tau}, v) \in \mathbf{H}$.

The solvability of (4) (or (6)) was proved in [15]. It reduces to show that $\mathbf{A}_1$ is strongly monotone and Lipschitz continuous, and that $\mathbf{B}_1$ and $\mathbf{B}$ satisfy appropriate inf-sup conditions. The corresponding result is stated as follows.

Theorem 1 *There exists a unique* $(\mathbf{t}, \boldsymbol{\sigma}, u) \in \mathbf{H}$ *solution of the nonlinear two-fold saddle point formulation* (4). *Moreover, there exists* $C > 0$, *independent of the solution, such that*

$$||(\mathbf{t}, \boldsymbol{\sigma}, u)||_{\mathbf{H}} \leq C \left\{ ||f||_{L^2(\Omega)} + ||g||_{H^{1/2}(\Gamma)} \right\}.$$

Proof. See Theorem 4.1 in [15]. □

3 The a-posteriori error estimate

In what follows, we assume that Γ is a polygonal curve. Then we let $\{\mathcal{T}_h\}_{h\in\mathtt{I}}$ be a regular family of triangulations of Ω, made up of triangles T of diameter h_T, such that $h := \sup\limits_{T\in\mathcal{T}_h} h_T$ and $\bar{\Omega} = \cup\{T : \quad T \in \mathcal{T}_h\}$. Here $\mathtt{I}$ is a at most numerable set of indexes, say $\mathtt{I} := \{h_j\}_{j\in\mathbb{N}}$, with $h_j \geq h_{j+1}\ \forall\, j \in \mathbb{N}$. In addition, let $\mathbf{H}_h := X_{1,h} \times M_{1,h} \times M_h$ be a finite element subspace of $\mathbf{H} := X_1 \times M_1 \times M$, and let $(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \in \mathbf{H}_h$ be a suitable approximation to the solution $(\mathbf{t}, \boldsymbol{\sigma}, u) \in \mathbf{H}$ of the variational formulation (4).

We are interested in deriving an a-posteriori error estimate for the error $||(\mathbf{t}, \boldsymbol{\sigma}, u) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h)||_{\mathbf{H}}$. To this end, following the approach in [10] (see also [11] and [7]), we first let $\hat{\mathbf{A}} : X \to X'$ be any linear and bounded operator such that the corresponding induced bilinear form $\hat{A} : X \times X \to \mathbb{R}$, defined by

$$\hat{A}((\mathbf{r}, \boldsymbol{\rho}), (\mathbf{s}, \boldsymbol{\tau})) \;=\; [\hat{\mathbf{A}}(\mathbf{r}, \boldsymbol{\rho}), (\mathbf{s}, \boldsymbol{\tau})]$$

for all $(\mathbf{r}, \boldsymbol{\rho})$, $(\mathbf{s}, \boldsymbol{\tau}) \in X$, is symmetric and X-elliptic. In particular, we may consider $\hat{\mathbf{A}}$ so that $\hat{A}$ becomes the usual inner product in X, that is

$$[\hat{\mathbf{A}}(\mathbf{r}, \boldsymbol{\rho}), (\mathbf{s}, \boldsymbol{\tau})] \;=\; \langle \mathbf{r}, \mathbf{s}\rangle_{[L^2(\Omega)]^2} + \langle \boldsymbol{\rho}, \boldsymbol{\tau}\rangle_{H(\mathrm{div};\Omega)}\,. \tag{7}$$

Also, let $\hat{\mathbf{A}}_T$ be the linear and bounded operator obtained by restricting $\hat{\mathbf{A}}$ to the triangle $T \in \mathcal{T}_h$. In other words, denoting $X_T := X_{1,T} \times M_{1,T}$ with $X_{1,T} := [L^2(T)]^2$ and $M_{1,T} = H(\mathrm{div}; T)$, and using (7), $\hat{\mathbf{A}}_T : X_T \to X_T'$ is defined by

$$[\hat{\mathbf{A}}_T(\mathbf{r}, \boldsymbol{\rho}), (\mathbf{s}, \boldsymbol{\tau})] := \langle \mathbf{r}, \mathbf{s}\rangle_{[L^2(T)]^2} + \langle \boldsymbol{\rho}, \boldsymbol{\tau}\rangle_{H(\mathrm{div};T)}$$

for all $(\mathbf{r}, \boldsymbol{\rho})$, $(\mathbf{s}, \boldsymbol{\tau}) \in X_T$.

Similarly, let $M_T := L^2(T)$. Then, we denote by $\mathbf{A}_T$ and $\mathbf{B}_T$ the corresponding restrictions to $T \in \mathcal{T}_h$ of the nonlinear operator $\mathbf{A}$ and linear operator $\mathbf{B}$, respectively. This means that $\mathbf{A}_T : X_T \to X_T'$ and $\mathbf{B}_T : M_{1,T} \to M_T'$ are defined as follows:

$$[\mathbf{A}_T(\mathbf{r}, \boldsymbol{\rho}), (\mathbf{s}, \boldsymbol{\tau})] := \int_T \boldsymbol{\kappa}(\cdot, |\mathbf{r}|)\, \mathbf{r}\cdot\mathbf{s}\, dx - \int_T \mathbf{r}\cdot\boldsymbol{\tau}\, dx - \int_T \mathbf{s}\cdot\boldsymbol{\rho}\, dx\,,$$

$$[\mathbf{B}_T(\boldsymbol{\tau}), v] := -\int_T v\, \mathrm{div}\, \boldsymbol{\tau}\, dx\,,$$

for all $(\mathbf{r}, \boldsymbol{\rho})$, $(\mathbf{s}, \boldsymbol{\tau}) \in X_T$ and for all $v \in M_T$.

We now show the following theorem concerning the Ritz projection of the Galerkin error with respect to the operator $\hat{\mathbf{A}}$.

Theorem 2 *Let* $(\bar{\mathbf{t}}, \bar{\boldsymbol{\sigma}})$ *be the unique element in* X *such that*

$$[\hat{\mathbf{A}}(\bar{\mathbf{t}}, \bar{\boldsymbol{\sigma}}), (\mathbf{s}, \boldsymbol{\tau})] = [\mathbf{A}(\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{s}, \boldsymbol{\tau})] - [\mathbf{A}(\mathbf{t}_h, \boldsymbol{\sigma}_h), (\mathbf{s}, \boldsymbol{\tau})] \\ + [\mathbf{B}(\boldsymbol{\tau}), u - u_h] \tag{8}$$

for all $(\mathbf{s}, \boldsymbol{\tau}) \in X$. *In addition, let* $\tilde{\varphi}_h$ *be a function in* $H^1(\Omega)$, *and for each* $T \in \mathcal{T}_h$, *let* $(\hat{\mathbf{t}}_T, \hat{\boldsymbol{\sigma}}_T) \in X_T$ *be the unique solution of the local problem*

$$\begin{aligned}[][\hat{\mathbf{A}}_T(\hat{\mathbf{t}}_T, \hat{\boldsymbol{\sigma}}_T), (\mathbf{s}, \boldsymbol{\tau})] \;&=\; -[\mathbf{A}_T(\mathbf{t}_{h,T}, \boldsymbol{\sigma}_{h,T}), (\mathbf{s}, \boldsymbol{\tau})] \\ -[\mathbf{B}_T(\boldsymbol{\tau}), u_{h,T}] \;-\; \int\limits_{\partial T} \tilde{\varphi}_h\, \boldsymbol{\tau} \cdot \boldsymbol{\nu}\, ds \;&-\; \int\limits_{\partial T \cap \Gamma} (g - \tilde{\varphi}_h)\, \boldsymbol{\tau} \cdot \boldsymbol{\nu}\, ds\end{aligned} \tag{9}$$

for all $(\mathbf{s}, \boldsymbol{\tau}) \in X_T$, *where* $(\mathbf{t}_{h,T}, \boldsymbol{\sigma}_{h,T}, u_{h,T}) := (\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h)\,|_T$. *Then there holds*

$$[\hat{\mathbf{A}}(\bar{\mathbf{t}}, \bar{\boldsymbol{\sigma}}), (\bar{\mathbf{t}}, \bar{\boldsymbol{\sigma}})] \;\le\; \sum_{T \in \mathcal{T}_h} [\hat{\mathbf{A}}_T(\hat{\mathbf{t}}_T, \hat{\boldsymbol{\sigma}}_T), (\hat{\mathbf{t}}_T, \hat{\boldsymbol{\sigma}}_T)]\,.$$

In particular, if $\hat{\mathbf{A}}$ *is given by* (7), *we have*

$$||(\bar{\mathbf{t}}, \bar{\boldsymbol{\sigma}})||_X^2 \;\le\; \sum_{T \in \mathcal{T}_h} ||(\hat{\mathbf{t}}_T, \hat{\boldsymbol{\sigma}}_T)||_{X_T}^2\,.$$

Proof. We first observe that the unique solvability of the local problem (9) follows from the Lax-Milgram Theorem and the fact that the bilinear form induced by $\hat{\mathbf{A}}_T$ is bounded and X_T-elliptic, and that the right hand side of (9) becomes a linear and bounded functional on X_T.

Then we note from (6) that

$$[\mathbf{A}(\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{s}, \boldsymbol{\tau})] + [\mathbf{B}(\boldsymbol{\tau}), u] = -\int\limits_{\Gamma} g\, \boldsymbol{\tau} \cdot \boldsymbol{\nu}\, ds$$

for all $(\mathbf{s}, \boldsymbol{\tau}) \in X$, and hence

$$[\hat{\mathbf{A}}(\bar{\mathbf{t}}, \bar{\boldsymbol{\sigma}}), (\mathbf{s}, \boldsymbol{\tau})] = -\int\limits_{\Gamma} g\, \boldsymbol{\tau} \cdot \boldsymbol{\nu}\, ds - [\mathbf{A}(\mathbf{t}_h, \boldsymbol{\sigma}_h), (\mathbf{s}, \boldsymbol{\tau})] \\ -[\mathbf{B}(\boldsymbol{\tau}), u_h] \quad \forall\, (\mathbf{s}, \boldsymbol{\tau}) \in X\,. \tag{10}$$

Since the bilinear form induced by $\hat{\mathbf{A}}$ is symmetric and X-elliptic, we have that

$$-\frac{1}{2}[\hat{\mathbf{A}}(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}}),(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}})] = \inf_{(\mathbf{s},\boldsymbol{\tau})\in X}\Big\{\frac{1}{2}[\hat{\mathbf{A}}(\mathbf{s},\boldsymbol{\tau}),(\mathbf{s},\boldsymbol{\tau})] - [\hat{\mathbf{A}}(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}}),(\mathbf{s},\boldsymbol{\tau})]\Big\},$$

which, using (10), yields

$$-\frac{1}{2}\,[\hat{\mathbf{A}}(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}}),(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}})] \;=\; \inf_{(\mathbf{s},\boldsymbol{\tau})\in X}\;\mathbf{J}(\mathbf{s},\boldsymbol{\tau}) \tag{11}$$

with

$$\begin{aligned}\mathbf{J}(\mathbf{s},\boldsymbol{\tau}) :=&\ \frac{1}{2}\,[\hat{\mathbf{A}}(\mathbf{s},\boldsymbol{\tau}),(\mathbf{s},\boldsymbol{\tau})] + \int_\Gamma g\,\boldsymbol{\tau}\cdot\boldsymbol{\nu}\,ds\\ &+[\mathbf{A}(\mathbf{t}_h,\boldsymbol{\sigma}_h),(\mathbf{s},\boldsymbol{\tau})] + [\mathbf{B}(\boldsymbol{\tau}),u_h] \qquad \forall\,(\mathbf{s},\boldsymbol{\tau})\in X\,.\end{aligned} \tag{12}$$

Now, since

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}\tilde{\varphi}_h\,\boldsymbol{\tau}\cdot\boldsymbol{\nu}\,ds = \int_\Gamma \tilde{\varphi}_h\,\boldsymbol{\tau}\cdot\boldsymbol{\nu}\,ds\,,$$

we obtain

$$\begin{aligned}\int_\Gamma g\,\boldsymbol{\tau}\cdot\boldsymbol{\nu}\,ds &= \int_\Gamma \tilde{\varphi}_h\,\boldsymbol{\tau}\cdot\boldsymbol{\nu}\,ds + \int_\Gamma (g-\tilde{\varphi}_h)\,\boldsymbol{\tau}\cdot\boldsymbol{\nu}\,ds\\ &= \sum_{T\in\mathcal{T}_h}\int_{\partial T}\tilde{\varphi}_h\,\boldsymbol{\tau}\cdot\boldsymbol{\nu}\,ds \;+\; \int_\Gamma (g-\tilde{\varphi}_h)\,\boldsymbol{\tau}\cdot\boldsymbol{\nu}\,ds\,.\end{aligned} \tag{13}$$

It follows from (12) and (13) that

$$\mathbf{J}(\mathbf{s},\boldsymbol{\tau}) \;=\; \sum_{T\in\mathcal{T}_h}\mathbf{J}_T(\mathbf{s}_T,\boldsymbol{\tau}_T) \quad \forall\,(\mathbf{s},\boldsymbol{\tau})\in X\,, \tag{14}$$

where $(\mathbf{s}_T,\boldsymbol{\tau}_T)\in X_T$ is the restriction of $(\mathbf{s},\boldsymbol{\tau})$ to T, and

$$\begin{aligned}\mathbf{J}_T(\mathbf{s}_T,\boldsymbol{\tau}_T) :=&\ \frac{1}{2}[\hat{\mathbf{A}}_T(\mathbf{s}_T,\boldsymbol{\tau}_T),(\mathbf{s}_T,\boldsymbol{\tau}_T)] + [\mathbf{A}_T(\mathbf{t}_{h,T},\boldsymbol{\sigma}_{h,T}),(\mathbf{s}_T,\boldsymbol{\tau}_T)]\\ &+[\mathbf{B}_T(\boldsymbol{\tau}_T),u_{h,T}] \;+\; \int_{\partial T}\tilde{\varphi}_h\,\boldsymbol{\tau}_T\cdot\boldsymbol{\nu}\,ds \;+\; \int_{\partial T\cap\Gamma}(g-\tilde{\varphi}_h)\,\boldsymbol{\tau}_T\cdot\boldsymbol{\nu}\,ds.\end{aligned} \tag{15}$$

Now we introduce the *broken* space

$$X^{br} := \{(\mathbf{s},\boldsymbol{\tau})\in[L^2(\Omega)]^2\times[L^2(\Omega)]^2:\ (\mathbf{s}_T,\boldsymbol{\tau}_T)\in X_T\ \forall\,T\in\mathcal{T}_h\}.$$

Then, since $X \subseteq X^{br}$, we can write from (11) and (14)

$$
\begin{aligned}
-\frac{1}{2}\,[\hat{\mathbf{A}}(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}}),(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}})] \;&\geq\; \inf_{(\mathbf{s},\boldsymbol{\tau})\in X^{br}} \sum_{T\in\mathcal{T}_h} \mathbf{J}_T(\mathbf{s}_T,\boldsymbol{\tau}_T) \\
&= \sum_{T\in\mathcal{T}_h} \inf_{(\mathbf{s}_T,\boldsymbol{\tau}_T)\in X_T} \mathbf{J}_T(\mathbf{s}_T,\boldsymbol{\tau}_T)\,,
\end{aligned}
\tag{16}
$$

and, according to (15),

$$
\inf_{(\mathbf{s}_T,\boldsymbol{\tau}_T)\in X_T} \mathbf{J}_T(\mathbf{s}_T,\boldsymbol{\tau}_T) \;=\; -\frac{1}{2}[\hat{\mathbf{A}}_T(\hat{\mathbf{t}}_T,\hat{\boldsymbol{\sigma}}_T),(\hat{\mathbf{t}}_T,\hat{\boldsymbol{\sigma}}_T)] \tag{17}
$$

where $(\hat{\mathbf{t}}_T,\hat{\boldsymbol{\sigma}}_T)\in X_T$ is the unique solution of the local problem (9).

Therefore, (16) and (17) complete the proof of the theorem. $\square$

We remark that the above theorem does not require any side restriction on $\tilde{\varphi}_h$, and hence this function can be chosen arbitrarily in $H^1(\Omega)$. However, we show below that under the assumption that $u \in H^1(\Omega)$, the proposed a-posteriori error estimate becomes efficient up to a term depending on $(u - \tilde{\varphi}_h)$, which yields the concept of *quasi-efficiency*.

The following technical result is also needed.

Lemma 3 *The operator $\mathbf{A}_1$ has continuous first order Gâteaux derivative $D\mathbf{A}_1$, and there exist $\tilde{C}_1$, $\tilde{C}_2 > 0$ such that*

$$
|\,D\mathbf{A}_1(\tilde{\mathbf{r}})(\mathbf{r},\mathbf{s})\,| \;\leq\; \tilde{C}_1\,||\mathbf{r}||_{X_1}\,||\mathbf{s}||_{X_1}
$$

and

$$
D\mathbf{A}_1(\tilde{\mathbf{r}})(\mathbf{s},\mathbf{s}) \;\geq\; \tilde{C}_2\,||\mathbf{s}||_{X_1}^2\,,
$$

for all $\tilde{\mathbf{r}}$, $\mathbf{r}$, $\mathbf{s} \in X_1 := [L^2(\Omega)]^2$.

Proof. The first order Gâteaux derivative $D\mathbf{A}_1$ applies X_1 into $\mathcal{L}(X_1, X_1')$, which means that given $\tilde{\mathbf{r}} := (\tilde{r}_1,\tilde{r}_2) \in X_1$, $D\mathbf{A}_1(\tilde{\mathbf{r}})$ can be identified with a bounded bilinear form on $X_1 \times X_1$ defined by

$$
D\mathbf{A}_1(\tilde{\mathbf{r}})(\mathbf{r},\mathbf{s}) := \lim_{\epsilon\to 0} \left\{ \frac{[\mathbf{A}_1(\tilde{\mathbf{r}}+\epsilon\mathbf{r}),\mathbf{s}] - [\mathbf{A}_1(\tilde{\mathbf{r}}),\mathbf{s}]}{\epsilon} \right\} \tag{18}
$$

for all $(\mathbf{r},\mathbf{s}) \in X_1 \times X_1$. Here, $\mathcal{L}(X_1, X_1')$ stands, as usual, for the space of bounded linear operators acting from X_1 into X_1'.

Indeed, from the definition of $\mathbf{A}_1$ and (18), we obtain

$$D\mathbf{A}_1(\tilde{\mathbf{r}})(\mathbf{r},\mathbf{s}) = \int\limits_\Omega \sum_{i,j=1}^{2} \frac{\partial}{\partial \tilde{r}_j} \left(\boldsymbol{\kappa}(\cdot, |\tilde{\mathbf{r}}|)\, \tilde{r}_i\right) r_j\, s_i\, dx$$

$$= \int\limits_\Omega \sum_{i,j=1}^{2} \left\{ \frac{\partial}{\partial \mu} \boldsymbol{\kappa}(\cdot, |\tilde{\mathbf{r}}|)\, \frac{\tilde{r}_j\, \tilde{r}_i}{|\tilde{\mathbf{r}}|} + \boldsymbol{\kappa}(\cdot, |\tilde{\mathbf{r}}|)\, \delta_{ij} \right\} r_j\, s_i\, dx$$

$$= \int\limits_\Omega \left\{ |\tilde{\mathbf{r}}|\, \frac{\partial}{\partial \mu} \boldsymbol{\kappa}(\cdot, |\tilde{\mathbf{r}}|)\, \frac{(\tilde{\mathbf{r}}\cdot \mathbf{r})\,(\tilde{\mathbf{r}}\cdot \mathbf{s})}{|\tilde{\mathbf{r}}|^2} + \boldsymbol{\kappa}(\cdot, |\tilde{\mathbf{r}}|)\,(\mathbf{r}\cdot\mathbf{s}) \right\} dx \tag{19}$$

for all $\mathbf{r} := (r_1, r_2)$, $\mathbf{s} := (s_1, s_2) \in X_1$.

It follows that

$$|\, D\mathbf{A}_1(\tilde{\mathbf{r}})(\mathbf{r},\mathbf{s})\,| \;\le\; \int\limits_\Omega \left\{ |\tilde{\mathbf{r}}| \left| \frac{\partial}{\partial \mu} \boldsymbol{\kappa}(\cdot, |\tilde{\mathbf{r}}|) \right| + \boldsymbol{\kappa}(\cdot, |\tilde{\mathbf{r}}|) \right\} |\mathbf{r}|\, |\mathbf{s}|\, dx\,,$$

which, using (2), yields

$$|\, D\mathbf{A}_1(\tilde{\mathbf{r}})(\mathbf{r},\mathbf{s})\,| \;\le\; \tilde{C}_1 \int\limits_\Omega |\mathbf{r}|\, |\mathbf{s}|\, dx \;\le\; \tilde{C}_1\, ||\mathbf{r}||_{X_1}\, ||\mathbf{s}||_{X_1}$$

with $\tilde{C}_1 = 3\,\kappa_1$.

On the other hand, for $\mathbf{r} = \mathbf{s} \in X_1$, we get from (19)

$$D\mathbf{A}_1(\tilde{\mathbf{r}})(\mathbf{s},\mathbf{s}) = \int\limits_\Omega \left\{ |\tilde{\mathbf{r}}|\, \frac{\partial}{\partial \mu} \boldsymbol{\kappa}(\cdot, |\tilde{\mathbf{r}}|)\, \frac{(\tilde{\mathbf{r}}\cdot\mathbf{s})^2}{|\tilde{\mathbf{r}}|^2} + \boldsymbol{\kappa}(\cdot, |\tilde{\mathbf{r}}|)\, |\mathbf{s}|^2 \right\} dx\,.$$

If $\frac{\partial}{\partial \mu} \boldsymbol{\kappa}(\cdot, |\tilde{\mathbf{r}}|) < 0$, then

$$|\tilde{\mathbf{r}}|\, \frac{\partial}{\partial \mu} \boldsymbol{\kappa}(\cdot, |\tilde{\mathbf{r}}|)\, \frac{(\tilde{\mathbf{r}}\cdot\mathbf{s})^2}{|\tilde{\mathbf{r}}|^2} \;\ge\; |\tilde{\mathbf{r}}|\, \frac{\partial}{\partial \mu} \boldsymbol{\kappa}(\cdot, |\tilde{\mathbf{r}}|)\, |\mathbf{s}|^2$$

and hence, using (2), we get

$$D\mathbf{A}_1(\tilde{\mathbf{r}})(\mathbf{s},\mathbf{s}) \;\ge\; \int\limits_\Omega \left\{ |\tilde{\mathbf{r}}|\, \frac{\partial}{\partial \mu} \boldsymbol{\kappa}(\cdot, |\tilde{\mathbf{r}}|) + \boldsymbol{\kappa}(\cdot, |\tilde{\mathbf{r}}|) \right\} |\mathbf{s}|^2\, dx \;\ge\; \kappa_0\, ||\mathbf{s}||^2_{X_1}\,.$$

If $\frac{\partial}{\partial \mu} \boldsymbol{\kappa}(\cdot, |\tilde{\mathbf{r}}|) \ge 0$, then, because of (2) again, we obtain

$$D\mathbf{A}_1(\tilde{\mathbf{r}})(\mathbf{s},\mathbf{s}) \;\ge\; \int\limits_\Omega \boldsymbol{\kappa}(\cdot, |\tilde{\mathbf{r}}|)\, |\mathbf{s}|^2\, dx \;\ge\; \kappa_0\, ||\mathbf{s}||^2_{X_1}\,.$$

In any case we conclude that

$$D\mathbf{A}_1(\tilde{\mathbf{r}})(\mathbf{s},\mathbf{s}) \,\geq\, \tilde{C}_2\,||\mathbf{s}||^2_{X_1} \quad \forall\,\mathbf{s}\in X_1\,,$$

with $\tilde{C}_2 = \kappa_0$. Finally, the continuity of $D\mathbf{A}_1$ follows from the corresponding property of $\boldsymbol{\kappa}$, which completes the proof. $\square$

We can provide now the main a-posteriori error estimate.

Theorem 4 *Let $\tilde{\varphi}_h \in H^1(\Omega)$, and for each $T \in \mathcal{T}_h$, let $(\hat{\mathbf{t}}_T, \hat{\boldsymbol{\sigma}}_T) \in X_T$ be the unique solution of the local problem*

$$\begin{aligned}[][\hat{\mathbf{A}}_T(\hat{\mathbf{t}}_T, \hat{\boldsymbol{\sigma}}_T), (\mathbf{s},\boldsymbol{\tau})] \,&=\, -\,[\mathbf{A}_T(\mathbf{t}_{h,T}, \boldsymbol{\sigma}_{h,T}), (\mathbf{s},\boldsymbol{\tau})] \\ -\,[\mathbf{B}_T(\boldsymbol{\tau}), u_{h,T}] \,-\, \int_{\partial T} \tilde{\varphi}_h\, \boldsymbol{\tau}\cdot\boldsymbol{\nu}\,ds \,&-\, \int_{\partial T\cap\Gamma} (g - \tilde{\varphi}_h)\,\boldsymbol{\tau}\cdot\boldsymbol{\nu}\,ds\end{aligned}$$

for all $(\mathbf{s},\boldsymbol{\tau}) \in X_T$. Then there exists $C > 0$, independent of h, such that

$$||(\mathbf{t},\boldsymbol{\sigma},u) - (\mathbf{t}_h,\boldsymbol{\sigma}_h,u_h)||_{\mathbf{H}} \,\leq\, C\,\left\{\sum_{T\in\mathcal{T}_h} \boldsymbol{\eta}_T^2\right\}^{1/2}\,,$$

where for each triangle T we define

$$\boldsymbol{\eta}_T^2 \,:=\, [\hat{\mathbf{A}}_T(\hat{\mathbf{t}}_T, \hat{\boldsymbol{\sigma}}_T), (\hat{\mathbf{t}}_T, \hat{\boldsymbol{\sigma}}_T)] \,+\, ||f + \operatorname{div}\boldsymbol{\sigma}_{h,T}||^2_{L^2(T)}\,.$$

Proof. Given $\tilde{\mathbf{r}} \in X_1$, we let $\mathbf{P}_{\tilde{\mathbf{r}}} : \mathbf{H} \to \mathbf{H}'$ be the global linear operator arising from the variational formulation (4) after replacing $[\mathbf{A}_1(\cdot),\cdot]$ by $D\mathbf{A}_1(\tilde{\mathbf{r}})(\cdot,\cdot)$, that is

$$\begin{aligned}[][\mathbf{P}_{\tilde{\mathbf{r}}}(\mathbf{r},\boldsymbol{\rho},w), (\mathbf{s},\boldsymbol{\tau},v)] &:= D\mathbf{A}_1(\tilde{\mathbf{r}})(\mathbf{r},\mathbf{s}) + [\mathbf{B}_1(\mathbf{r}),\boldsymbol{\tau}] \\ &+ [\mathbf{B}_1(\mathbf{s}),\boldsymbol{\rho}] + [\mathbf{B}(\boldsymbol{\rho}),v] + [\mathbf{B}(\boldsymbol{\tau}),w]\,,\end{aligned}$$

for all $(\mathbf{r},\boldsymbol{\rho},w)$, $(\mathbf{s},\boldsymbol{\tau},v) \in \mathbf{H}$.

We note from the previous lemma that $D\mathbf{A}_1(\tilde{\mathbf{r}})(\cdot,\cdot) \,:\, X_1 \times X_1 \to \mathbb{R}$ is a bounded and X_1-elliptic bilinear form, with corresponding constants independent of $\tilde{\mathbf{r}}$. In addition, we recall that the linear operators $\mathbf{B}$ and $\mathbf{B}_1$ satisfy the continuous inf-sup conditions.

Therefore, the linear version of the abstract Theorem 2.4 in [14] (or Theorem 2.3 in [15]) implies that $\mathbf{P}_{\tilde{\mathbf{r}}}$ satisfies a uniform global inf-sup condition. This means that there exists $C_0 > 0$, independent of $\tilde{\mathbf{r}}$, such that

$$
\begin{aligned}
||(\mathbf{r},\boldsymbol{\rho},w)||_{\mathbf{H}} \,\le\, C_0 \sup_{\substack{(\mathbf{s},\boldsymbol{\tau},v)\in\mathbf{H}\\ ||(\mathbf{s},\boldsymbol{\tau},v)||\le 1}} & \Big\{ D\mathbf{A}_1(\tilde{\mathbf{r}})(\mathbf{r},\mathbf{s}) + [\mathbf{B}_1(\mathbf{r}),\boldsymbol{\tau}] \\
& + [\mathbf{B}_1(\mathbf{s}),\boldsymbol{\rho}] + [\mathbf{B}(\boldsymbol{\rho}),v] + [\mathbf{B}(\boldsymbol{\tau}),w] \Big\}
\end{aligned} \tag{20}
$$

for all $(\mathbf{r},\boldsymbol{\rho},w) \in \mathbf{H}$, and for all $\tilde{\mathbf{r}} \in [L^2(\Omega)]^2$.

On the other hand, according to the properties of the Gâteaux derivative of $\mathbf{A}_1$, we know that there exists $\tilde{\mathbf{r}}_h \in X_1$, which is a convex combination of $\mathbf{t}$ and $\mathbf{t}_h$, such that

$$
[\mathbf{A}_1(\mathbf{t}),\mathbf{s}] \;-\; [\mathbf{A}_1(\mathbf{t}_h),\mathbf{s}] \;=\; D\mathbf{A}_1(\tilde{\mathbf{r}}_h)(\mathbf{t}-\mathbf{t}_h,\mathbf{s}) \quad \forall\, \mathbf{s}\in X_1\,. \tag{21}
$$

Thus, applying (20) with $\tilde{\mathbf{r}} = \tilde{\mathbf{r}}_h$ and $(\mathbf{r},\boldsymbol{\rho},w) = (\mathbf{t},\boldsymbol{\sigma},u) - (\mathbf{t}_h,\boldsymbol{\sigma}_h,u_h) \in \mathbf{H}$, we obtain

$$
\begin{aligned}
||(\mathbf{t},\boldsymbol{\sigma},u) - (\mathbf{t}_h,\boldsymbol{\sigma}_h,u_h)||_{\mathbf{H}} \,\le\, C_0 \sup_{\substack{(\mathbf{s},\boldsymbol{\tau},v)\in\mathbf{H}\\ ||(\mathbf{s},\boldsymbol{\tau},v)||\le 1}} \Big\{ D\mathbf{A}_1(\tilde{\mathbf{r}}_h)(\mathbf{t}-\mathbf{t}_h,\mathbf{s}) \\
+ [\mathbf{B}_1(\mathbf{t}-\mathbf{t}_h),\boldsymbol{\tau}] + [\mathbf{B}_1(\mathbf{s}),\boldsymbol{\sigma}-\boldsymbol{\sigma}_h] + [\mathbf{B}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),v] + [\mathbf{B}(\boldsymbol{\tau}),u-u_h] \Big\}
\end{aligned}
$$

which, using (21), the definition of the operator $\mathbf{A}$ (cf. (5)), the Ritz projection $(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}})$ introduced in Theorem 2, and the definition of $\mathbf{B}$, yields

$$
\begin{aligned}
& \frac{1}{C_0}\, ||(\mathbf{t},\boldsymbol{\sigma},u) - (\mathbf{t}_h,\boldsymbol{\sigma}_h,u_h)||_{\mathbf{H}} \,\le \\
& \sup_{\substack{(\mathbf{s},\boldsymbol{\tau},v)\in\mathbf{H}\\ ||(\mathbf{s},\boldsymbol{\tau},v)||\le 1}} \left\{ [\hat{\mathbf{A}}(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}}),(\mathbf{s},\boldsymbol{\tau})] - \int_\Omega v\,\mathrm{div}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\,dx \right\}.
\end{aligned} \tag{22}
$$

Then, the boundedness and ellipticity of $\hat{\mathbf{A}}$, and the estimate from Theorem 2 give

$$
\begin{aligned}
|\,[\hat{\mathbf{A}}(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}}),(\mathbf{s},\boldsymbol{\tau})]\,| \;&\le\; ||\hat{\mathbf{A}}||\;||(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}})||_X\;||(\mathbf{s},\boldsymbol{\tau})||_X \\
&\le\; C \left\{ \sum_{T\in\mathcal{T}_h} [\hat{\mathbf{A}}_T(\hat{\mathbf{t}}_T,\hat{\boldsymbol{\sigma}}_T),(\hat{\mathbf{t}}_T,\hat{\boldsymbol{\sigma}}_T)] \right\}^{1/2}.
\end{aligned} \tag{23}
$$

Finally, since $\mathrm{div}\,\boldsymbol{\sigma} = -f$ in Ω, (22), (23) and Cauchy-Schwarz's inequality complete the proof of the theorem. $\square$

We now show that if $\hat{A}$ is the usual inner product in X (cf. (7)), then the computation of the solution $(\hat{\mathbf{t}}_T, \hat{\boldsymbol{\sigma}}_T)$ of the local problem (9) is simplified. In fact, according to the definitions of A_T and B_T, (9) yields

$$\begin{aligned}
\langle \hat{\mathbf{t}}_T, \mathbf{s} \rangle_{[L^2(T)]^2} + \langle \hat{\boldsymbol{\sigma}}_T, \boldsymbol{\tau} \rangle_{H(\mathrm{div};T)} &= \int_T \{ \boldsymbol{\sigma}_{h,T} - \boldsymbol{\kappa}(\cdot, |\mathbf{t}_{h,T}|)\, \mathbf{t}_{h,T} \} \cdot \mathbf{s}\, dx \\
&+ \int_T \mathbf{t}_{h,T} \cdot \boldsymbol{\tau}\, dx + \int_T u_{h,T} \operatorname{div} \boldsymbol{\tau}\, dx \\
&- \int_{\partial T} \tilde{\varphi}_h\, \boldsymbol{\tau} \cdot \boldsymbol{\nu}\, ds - \int_{\partial T \cap \Gamma} (g - \tilde{\varphi}_h)\, \boldsymbol{\tau} \cdot \boldsymbol{\nu}\, ds
\end{aligned} \tag{24}$$

for all $(\mathbf{s}, \boldsymbol{\tau}) \in X_T := [L^2(T)]^2 \times H(\mathrm{div}; T)$. Thus, taking separately $\boldsymbol{\tau} = 0$ and $\mathbf{s} = 0$ in (24), we obtain

$$\int_T \hat{\mathbf{t}}_T \cdot \mathbf{s}\, dx = \int_T \{ \boldsymbol{\sigma}_{h,T} - \boldsymbol{\kappa}(\cdot, |\mathbf{t}_{h,T}|)\, \mathbf{t}_{h,T} \} \cdot \mathbf{s}\, dx \tag{25}$$

for all $\mathbf{s} \in [L^2(T)]^2$, and

$$\begin{aligned}
\langle \hat{\boldsymbol{\sigma}}_T, \boldsymbol{\tau} \rangle_{H(\mathrm{div};T)} &= \int_T \mathbf{t}_{h,T} \cdot \boldsymbol{\tau}\, dx + \int_T u_{h,T} \operatorname{div} \boldsymbol{\tau}\, dx \\
&- \int_{\partial T} \tilde{\varphi}_h\, \boldsymbol{\tau} \cdot \boldsymbol{\nu}\, ds - \int_{\partial T \cap \Gamma} (g - \tilde{\varphi}_h)\, \boldsymbol{\tau} \cdot \boldsymbol{\nu}\, ds
\end{aligned} \tag{26}$$

for all $\boldsymbol{\tau} \in H(\mathrm{div}; T)$.

Since $\boldsymbol{\sigma}_{h,T}$, $\mathbf{t}_{h,T} \in [L^2(T)]^2$, it follows from (25) that

$$\hat{\mathbf{t}}_T = \boldsymbol{\sigma}_{h,T} - \boldsymbol{\kappa}(\cdot, |\mathbf{t}_{h,T}|)\, \mathbf{t}_{h,T} \qquad \forall\, T \in \mathcal{T}_h\,, \tag{27}$$

which corresponds to the explicit residual of the nonlinear equation describing the flux $\boldsymbol{\sigma}$.

Then we can state the following result.

Theorem 5 *Let $\tilde{\varphi}_h \in H^1(\Omega)$, and for each $T \in \mathcal{T}_h$, let $\hat{\boldsymbol{\sigma}}_T \in H(\mathrm{div}; T)$ be the unique solution of the local problem*

$$\langle \hat{\boldsymbol{\sigma}}_T, \boldsymbol{\tau} \rangle_{H(\mathrm{div};T)} = \int_T \mathbf{t}_{h,T} \cdot \boldsymbol{\tau}\, dx + \int_T u_{h,T} \operatorname{div} \boldsymbol{\tau}\, dx$$

$$- \int_{\partial T} \tilde{\varphi}_h \, \boldsymbol{\tau} \cdot \boldsymbol{\nu} \, ds \; - \int_{\partial T \cap \Gamma} (g - \tilde{\varphi}_h) \, \boldsymbol{\tau} \cdot \boldsymbol{\nu} \, ds \tag{28}$$

for all $\boldsymbol{\tau} \in H(\mathrm{div}; T)$. Then there exists $C > 0$, independent of h, such that

$$|| \, (\mathbf{t}, \boldsymbol{\sigma}, u) \, - \, (\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \, ||_{\mathbf{H}} \, \leq \, C \left\{ \sum_{T \in \mathcal{T}_h} \hat{\boldsymbol{\eta}}_T^2 \right\}^{1/2} , \tag{29}$$

where

$$\hat{\boldsymbol{\eta}}_T^2 \, := \, || \, \hat{\boldsymbol{\sigma}}_T \, ||^2_{H(\mathrm{div};T)} \, + \, || \, \boldsymbol{\sigma}_{h,T} \, - \, \boldsymbol{\kappa}(\cdot, |\mathbf{t}_{h,T}|) \, \mathbf{t}_{h,T} \, ||^2_{[L^2(T)]^2}$$
$$+ \, ||f + \mathrm{div}\, \boldsymbol{\sigma}_{h,T}||^2_{L^2(T)} \, . \tag{30}$$

Proof. It follows straightforward from Theorems 2, 4 and the previous analysis. $\square$

It is important to observe that, in practice, the infinite dimensional local problem (28) is solved approximately by replacing $H(\mathrm{div}; T)$ by a finite element subspace. Since these local problems are linear, we suggest to use the p or the $h - p$ version for solving them.

The following lemma provides a-priori estimates for the solution of the local problem (28). They will be used to prove the *quasi-efficiency* of the estimate given in Theorem 5, and to derive a fully explicit reliable a-posteriori error estimate.

Lemma 6 *Let $\tilde{\varphi}_h \in H^1(\Omega)$ and for each $T \in \mathcal{T}_h$ let $\hat{\boldsymbol{\sigma}}_T \in H(\mathrm{div}; T)$ be the unique solution of the local problem (28). Then there exists $C > 0$, independent of h and T, such that*

$$||\hat{\boldsymbol{\sigma}}_T||_{H(\mathrm{div};T)} \leq \Big\{ ||\mathbf{t}_{h,T} - \nabla \tilde{\varphi}_h||^2_{[L^2(T)]^2} + ||u_{h,T} - \tilde{\varphi}_h||^2_{L^2(T)}$$
$$+ ||g - \tilde{\varphi}_h||^2_{H^{1/2}(\partial T \cap \Gamma)} \Big\}^{1/2} . \tag{31}$$

In addition, for any $z \in H^1(\Omega)$ such that $z = g$ on Γ, we get

$$||\hat{\boldsymbol{\sigma}}_T||_{H(\mathrm{div};T)} \leq \Big\{ ||\mathbf{t}_{h,T} - \nabla z||^2_{[L^2(T)]^2} + ||u_{h,T} - z||^2_{L^2(T)}$$

$$+||z - \tilde{\varphi}_h||^2_{H^{1/2}(\tilde{\partial T})} \Bigg\}^{1/2}, \tag{32}$$

where $\tilde{\partial T}$ is the part of ∂T not contained in Γ.

Proof. A simple integration by parts gives

$$\int_{\partial T} \tilde{\varphi}_h \, \boldsymbol{\tau} \cdot \boldsymbol{\nu} \, ds = \int_T \nabla \tilde{\varphi}_h \cdot \boldsymbol{\tau} \, dx + \int_T \tilde{\varphi}_h \operatorname{div} \boldsymbol{\tau} \, dx \,, \tag{33}$$

and hence (28) becomes

$$\begin{aligned}\langle \hat{\boldsymbol{\sigma}}_T, \boldsymbol{\tau} \rangle_{H(\operatorname{div};T)} \; &= \int_T (\mathbf{t}_{h,T} - \nabla \tilde{\varphi}_h) \cdot \boldsymbol{\tau} \, dx \; + \int_T (u_{h,T} - \tilde{\varphi}_h) \operatorname{div} \boldsymbol{\tau} \, dx \\ &- \int_{\partial T \cap \Gamma} (g - \tilde{\varphi}_h) \, \boldsymbol{\tau} \cdot \boldsymbol{\nu} \, ds \qquad \forall \, \boldsymbol{\tau} \in H(\operatorname{div}; T) \,, \end{aligned} \tag{34}$$

which yields (31).

Now, given $z \in H^1(\Omega)$ such that $z = g$ on Γ, we can write

$$\begin{aligned} & - \int_{\partial T} \tilde{\varphi}_h \, \boldsymbol{\tau} \cdot \boldsymbol{\nu} \, ds \; - \int_{\partial T \cap \Gamma} (g - \tilde{\varphi}_h) \, \boldsymbol{\tau} \cdot \boldsymbol{\nu} \, ds \\ &= - \int_{\partial T} z \, \boldsymbol{\tau} \cdot \boldsymbol{\nu} \, ds + \int_{\partial T} (z - \tilde{\varphi}_h) \, \boldsymbol{\tau} \cdot \boldsymbol{\nu} \, ds - \int_{\partial T \cap \Gamma} (z - \tilde{\varphi}_h) \, \boldsymbol{\tau} \cdot \boldsymbol{\nu} \, ds \\ &= - \int_T \nabla z \cdot \boldsymbol{\tau} \, dx \; - \int_T z \operatorname{div} \boldsymbol{\tau} \, dx + \int_{\tilde{\partial T}} (z - \tilde{\varphi}_h) \, \boldsymbol{\tau} \cdot \boldsymbol{\nu} \, ds \,, \end{aligned}$$

which, replaced back into (28), leads (32) and concludes the proof. □

We are now in a position to prove the following result which shows that the reliable a-posteriori error estimate given by Theorem 5 is locally (and hence globally) *quasi-efficient* in the sense that it is efficient up to a term depending on the traces of $(u - \tilde{\varphi}_h)$ on the interior element boundaries.

Lemma 7 *Assume that u and $\tilde{\varphi}_h \in H^1(\Omega)$. Then there exists $C > 0$, independent of h, such that*

$$\hat{\boldsymbol{\eta}}_T^2 \; \leq \; C \Bigg\{ \, ||\mathbf{t}_{h,T} - \mathbf{t}||^2_{[L^2(T)]^2} + ||\boldsymbol{\sigma}_{h,T} - \boldsymbol{\sigma}||^2_{H(\operatorname{div};T)}$$

$$+||u_{h,T} - u||^2_{L^2(T)} + ||u - \tilde{\varphi}_h||^2_{H^{1/2}(\partial T)} \Big\}$$

for all $T \in \mathcal{T}_h$, and hence

$$\sum_{T \in \mathcal{T}_h} \hat{\boldsymbol{\eta}}_T^2 \le C \left\{ ||(\mathbf{t}, \boldsymbol{\sigma}, u) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h)||^2_{\mathbf{H}} + \sum_{S \in \mathcal{E}_h(\Omega)} ||u - \tilde{\varphi}_h||^2_{H^{1/2}(S)} \right\},$$

where $\mathcal{E}_h(\Omega)$ is the set of interior edges of $\mathcal{T}_h$.

Proof. Since $\mathbf{t} = \nabla u$ and $f = -\operatorname{div} \boldsymbol{\sigma}$ in Ω, we get from (30) and (32) (with $z = u$) that

$$\hat{\boldsymbol{\eta}}_T^2 \le ||\mathbf{t}_{h,T} - \mathbf{t}||^2_{[L^2(T)]^2} + ||u_{h,T} - u||^2_{L^2(T)} + ||u - \tilde{\varphi}_h||^2_{H^{1/2}(\partial T)}$$

$$+ ||\operatorname{div}(\boldsymbol{\sigma}_{h,T} - \boldsymbol{\sigma})||^2_{L^2(T)} + ||\boldsymbol{\sigma}_{h,T} - \boldsymbol{\kappa}(\cdot, |\mathbf{t}_{h,T}|)\, \mathbf{t}_{h,T}||^2_{[L^2(T)]^2} . \tag{35}$$

Next, since $\boldsymbol{\sigma} = \boldsymbol{\kappa}(\cdot, |\mathbf{t}|)\, \mathbf{t}$ in Ω, we obtain

$$||\boldsymbol{\sigma}_{h,T} - \boldsymbol{\kappa}(\cdot, |\mathbf{t}_{h,T}|)\, \mathbf{t}_{h,T}||_{[L^2(T)]^2} \le ||\boldsymbol{\sigma}_{h,T} - \boldsymbol{\sigma}||_{[L^2(T)]^2}$$

$$+ ||\, \boldsymbol{\kappa}(\cdot, |\mathbf{t}|)\, \mathbf{t} - \boldsymbol{\kappa}(\cdot, |\mathbf{t}_{h,T}|)\, \mathbf{t}_{h,T}\, ||_{[L^2(T)]^2} . \tag{36}$$

But, using the assumptions on $\boldsymbol{\kappa}$ (cf. (2)) and proceeding similarly as in the proof of Lemma 3 (see also Theorem 3.8 in [18]), one can prove that

$$||\, \boldsymbol{\kappa}(\cdot, |\mathbf{t}|)\, \mathbf{t} - \boldsymbol{\kappa}(\cdot, |\mathbf{t}_{h,T}|)\, \mathbf{t}_{h,T}\, ||_{[L^2(T)]^2} \le 3\, \kappa_1\, ||\mathbf{t} - \mathbf{t}_{h,T}||_{[L^2(T)]^2} . \tag{37}$$

Therefore, (35), (36) and (37) imply the local *quasi-efficiency*. Finally, the corresponding global property is obtained by summing up over all the triangles $T \in \mathcal{T}_h$, which completes the proof. $\square$

The above lemma shows that $\tilde{\varphi}_h$ must be chosen so that $\tilde{\varphi}_h|_{\partial T}$ is as close as possible, at least empirically, to $u|_{\partial T}$ for all $T \in \mathcal{T}_h$.

To end this section, we set an appropriate choice of $\tilde{\varphi}_h$ for the particular case in which the finite element subspace M_h is given by piecewise constant functions (see (45) below for details). In fact, we define $\tilde{\varphi}_h \in C(\bar{\Omega})$ as the unique continuous function such that $\tilde{\varphi}_{h,T} := \tilde{\varphi}_h|_T$ is a polynomial of degree ≤ 1, for all $T \in \mathcal{T}_h$, $\tilde{\varphi}_h(\bar{x}) = g(\bar{x})$, for each vertex $\bar{x}$ of the triangulation $\mathcal{T}_h$ lying on Γ, and $\tilde{\varphi}_h(\bar{x})$ is the weighted average (according to the areas of the triangles) of the constant values of u_h on all the triangles $T \in \mathcal{T}_h$ to which $\bar{x}$ belongs, for each vertex $\bar{x}$ of the triangulation $\mathcal{T}_h$ lying in Ω.

Since $\tilde{\varphi}_h \in H^1(\Omega)$, we can use it to provide a simpler a-posteriori error estimate, which does not require the explicit solution of the local problem (28). More precisely, assuming that $X_{1,h}$ is also given by piecewise constant functions (see (43) below), we can establish the following result.

Theorem 8 *Let $X_{1,h}$ and M_h be given by* (43) *and* (45)*, respectively, and let $\tilde{\varphi}_h \in H^1(\Omega)$ be as indicated above. Then there exists $C > 0$, independent of h, such that*

$$|| \, (\mathbf{t}, \boldsymbol{\sigma}, u) \, - \, (\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \, ||_{\mathbf{H}} \, \le \, C \left\{ \sum_{T \in \mathcal{T}_h} \tilde{\boldsymbol{\eta}}_T^2 \right\}^{1/2} , \tag{38}$$

where for each triangle $T \in \mathcal{T}_h$ we define

$$\begin{aligned} \tilde{\boldsymbol{\eta}}_T^2 \, := \, & h_T^2 \, |\mathbf{t}_{h,T} - \nabla \tilde{\varphi}_{h,T}|^2 + h_T^2 \sum_{j=1}^{3} (\, u_{h,T} - \tilde{\varphi}_h(\bar{x}_{j,T}) \,)^2 \, + \, ||g - \tilde{\varphi}_h||^2_{H^{1/2}(\partial T \cap \Gamma)} \\ & + ||\boldsymbol{\sigma}_{h,T} \, - \, \boldsymbol{\kappa}(\cdot, |\mathbf{t}_{h,T}|) \, \mathbf{t}_{h,T}||^2_{[L^2(T)]^2} + ||f + \operatorname{div} \boldsymbol{\sigma}_{h,T}||^2_{L^2(T)} \, , \end{aligned} \tag{39}$$

with $\{\bar{x}_{j,T}\}_{j=1}^3$ being the vertices of T.

Proof. It follows from Theorem 5 (cf. (30)) and the estimate (31) that

$$|| \, (\mathbf{t}, \boldsymbol{\sigma}, u) \, - \, (\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \, ||_{\mathbf{H}} \, \le \, C \left\{ \sum_{T \in \mathcal{T}_h} \bar{\boldsymbol{\eta}}_T^2 \right\}^{1/2} , \tag{40}$$

where

$$\begin{aligned} \bar{\boldsymbol{\eta}}_T^2 \, := \, & ||\mathbf{t}_{h,T} - \nabla \tilde{\varphi}_{h,T}||^2_{[L^2(T)]^2} + ||u_{h,T} - \tilde{\varphi}_h||^2_{L^2(T)} + ||g - \tilde{\varphi}_h||^2_{H^{1/2}(\partial T \cap \Gamma)} \\ & + ||\boldsymbol{\sigma}_{h,T} \, - \, \boldsymbol{\kappa}(\cdot, |\mathbf{t}_{h,T}|) \, \mathbf{t}_{h,T}||^2_{[L^2(T)]^2} + ||f + \operatorname{div} \boldsymbol{\sigma}_{h,T}||^2_{L^2(T)} \, . \end{aligned}$$

Since $\mathbf{t}_{h,T}$ and $\nabla \tilde{\varphi}_{h,T}$ are constant vectors, and the area of T is bounded above by h_T^2, we easily get

$$||\mathbf{t}_{h,T} - \nabla \tilde{\varphi}_{h,T}||^2_{[L^2(T)]^2} \, \le \, h_T^2 \, |\mathbf{t}_{h,T} - \nabla \tilde{\varphi}_{h,T}|^2 \, . \tag{41}$$

Next, since $(u_{h,T} - \tilde{\varphi}_{h,T})$ is a polynomial of degree ≤ 1, we deduce that

$$\begin{aligned} ||u_{h,T} - \tilde{\varphi}_h||^2_{L^2(T)} \, & \le \, h_T^2 \max_{j \in \{1,2,3\}} (\, u_{h,T} - \tilde{\varphi}_h(\bar{x}_{j,T}) \,)^2 \\ & \le \, h_T^2 \sum_{j=1}^{3} (\, u_{h,T} - \tilde{\varphi}_h(\bar{x}_{j,T}) \,)^2 \, . \end{aligned} \tag{42}$$

In this way, (40), (41) and (42) yield (38) and complete the proof. $\square$

We end this section by remarking that, for implementation purposes, the $H^{1/2}(\partial T \cap \Gamma)$-norms can be bounded by using the interpolation theorem. In other words, assuming that $g \in H^1(\partial T \cap \Gamma)$, we can write

$$||g - \tilde{\varphi}_h||^2_{H^{1/2}(\partial T\cap\Gamma)} \leq ||g - \tilde{\varphi}_h||^2_{L^2(\partial T\cap\Gamma)} \; ||g - \tilde{\varphi}_h||^2_{H^1(\partial T\cap\Gamma)} \,.$$

4 A fully discrete Galerkin scheme

We now recall the main aspects of the fully discrete Galerkin scheme for (4) that was introduced and analyzed in [15]. We consider the canonical triangle with vertices $\hat{P}_1 = (0,0)^T$, $\hat{P}_2 = (1,0)^T$ and $\hat{P}_3 = (0,1)^T$ as a reference triangle $\hat{T}$, and introduce the family of bijective affine mappings $\{F_T\}_{T\in\mathcal{T}_h}$, such that $F_T(\hat{T}) = T$. It is well known that $F_T(\hat{x}) = B_T\,\hat{x} + b_T$ for all $\hat{x} := (\hat{x}_1, \hat{x}_2)^T \in \hat{T}$, where B_T, a square matrix of order 2, and $b_T \in \mathbb{R}^2$, depend only on the vertices of T.

Now, given the reference triangle $\hat{T}$ and a triangle $T \in \mathcal{T}_h$, we let $\mathbb{P}_0(\hat{T})$ and $\mathbb{P}_0(T)$ be the spaces of constant functions defined on $\hat{T}$ and T, respectively. In addition, we consider the lowest order Raviart-Thomas spaces

$$\mathcal{RT}_0(T) := \{\, \boldsymbol{\tau} : \quad \boldsymbol{\tau} = |\det(B_T)|^{-1}\, B_T\, \hat{\boldsymbol{\tau}} \circ F_T^{-1}\,,\; \hat{\boldsymbol{\tau}} \in \mathcal{RT}_0(\hat{T}) \,\}\,,$$

where

$$\mathcal{RT}_0(\hat{T}) := \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} \right\}.$$

Then, following [15], we define the finite element subspace $\mathbf{H}_h := X_{1,h} \times M_{1,h} \times M_h$ of $\mathbf{H}$, where:

$$X_{1,h} := \left\{ \mathbf{s}_h \in [L^2(\Omega)]^2 : \quad \mathbf{s}_h|_T \in [\mathbb{P}_0(T)]^2 \quad \forall\, T \in \mathcal{T}_h \right\}, \tag{43}$$

$$M_{1,h} := \{ \boldsymbol{\tau}_h \in H(\text{div};\Omega) : \quad \boldsymbol{\tau}_h|_T \in \mathcal{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h \}\,, \tag{44}$$

and

$$M_h := \{\, v_h \in L^2(\Omega) : \quad v_h|_T \in \mathbb{P}_0(T) \quad \forall\, T \in \mathcal{T}_h \,\}\,. \tag{45}$$

In what follows we associate to each scalar function $\varphi : T \to \mathbb{R}$ the function $\hat{\varphi} := \varphi \circ F_T : \hat{T} \to \mathbb{R}$, and conversely. Then, we introduce a quadrature formula on $\hat{T}$:

$$\hat{Q}(\hat{\varphi}) := \sum_{j=1}^{N} \hat{w}_j \, \hat{\varphi}(\hat{z}_j) \simeq \int_{\hat{T}} \hat{\varphi} \, d\hat{x} \,, \tag{46}$$

where $\{\hat{z}_1, \hat{z}_2, ..., \hat{z}_N\}$ is a set of nodes on $\hat{T}$ and $\{\hat{w}_1, \hat{w}_2, ..., \hat{w}_N\}$ is the corresponding set of weights. We assume that (46) is exact for constant functions $\hat{\varphi} \in \mathbb{P}_0(\hat{T})$, which means that $\sum_{j=1}^{N} \hat{w}_j = \frac{1}{2}$. According to (46), the quadrature formula on each $T \in \mathcal{T}_h$ is defined by

$$Q_T(\varphi) := |\det(B_T)| \, \hat{Q}(\hat{\varphi}) \; = \; 2\,|T| \, \hat{Q}(\hat{\varphi}) \simeq \int_T \varphi \, dx \tag{47}$$

for all $\varphi \in C^0(T)$.

Similarly, we introduce a quadrature formula on $\tilde{e} := [0, 1]$:

$$\tilde{Q}(\tilde{\varphi}) := \sum_{j=1}^{m} \tilde{w}_j \tilde{\varphi}(\tilde{z}_j) \simeq \int_{\tilde{e}} \tilde{\varphi}(\tilde{t}) \, d\tilde{t} \quad \forall \, \tilde{\varphi} \in C^0(\tilde{e}) \,, \tag{48}$$

where $\{\tilde{z}_1, \tilde{z}_2, ..., \tilde{z}_m\}$ is a set of nodes in $\tilde{e}$ and $\{\tilde{w}_1, \tilde{w}_2, ..., \tilde{w}_m\}$ is the corresponding set of weights. We assume that (48) is exact for linear functions $\tilde{\varphi} \in \mathbb{P}_1(\tilde{e})$. Thus, given an edge e of a triangle $T \in \mathcal{T}_h$, the associated quadrature formula is defined by

$$Q_e(\varphi) := |e| \, \tilde{Q}(\tilde{\varphi}) \simeq \int_e \varphi(t) \, dt \,, \tag{49}$$

with $\tilde{\varphi} := \varphi \, \circ \, F_T$.

Now, given $\mathbf{s}_h \in X_{1,h}$, $\boldsymbol{\tau}_h \in M_{1,h}$ and $v_h \in M_h$ we denote, respectively, by $\mathbf{s}_{h,T}$, $\boldsymbol{\tau}_{h,T}$ and $v_{h,T}$ the corresponding restrictions to the triangle $T \in \mathcal{T}_h$.

Next, we introduce the discrete operators $\mathbf{A}_{1,h} : X_{1,h} \to X'_{1,h}$, $\mathbf{B}_{1,h} : X_{1,h} \to M'_{1,h}$ and $\mathbf{B}_h : M_{1,h} \to M'_h$, as follows:

$$[\mathbf{A}_{1,h}(\mathbf{r}_h), \mathbf{s}_h] := \sum_{T \in \mathcal{T}_h} Q_T(\boldsymbol{\kappa}(\cdot, |\mathbf{r}_{h,T}|)) \, \mathbf{r}_{h,T} \cdot \mathbf{s}_{h,T} \tag{50}$$

for all $\mathbf{r}_h$, $\mathbf{s}_h \in X_{1,h}$,

$$[\mathbf{B}_{1,h}(\mathbf{r}_h), \boldsymbol{\tau}_h] := - \sum_{T \in \mathcal{T}_h} Q_T(\mathbf{r}_{h,T} \cdot \boldsymbol{\tau}_{h,T}) \tag{51}$$

for all $\mathbf{r}_h \in X_{1,h}$, $\boldsymbol{\tau}_h \in M_{1,h}$, and

$$[\mathbf{B}_h(\boldsymbol{\tau}_h), v_h] := - \sum_{T \in \mathcal{T}_h} Q_T(v_{h,T} \operatorname{div} \boldsymbol{\tau}_{h,T}) \tag{52}$$

for all $\boldsymbol{\tau}_h \in M_{1,h}$, $v_h \in M_h$.

It is easy to see that $\mathbf{A}_{1,h}$, $\mathbf{B}_{1,h}$ and $\mathbf{B}_h$ are well defined in the sense that they do map, respectively, $X_{1,h}$, $X_{1,h}$ and $M_{1,h}$, into the indicated dual spaces.

Finally, assuming also that $g \in W^{2,\infty}(\Gamma)$ and $f \in W^{1,\infty}(\Omega)$, we can define the discrete functionals $\mathbf{G}_{1,h} \in M'_{1,h}$ and $\mathbf{G}_h \in M'_h$, as follows:

$$[\mathbf{G}_{1,h}, \boldsymbol{\tau}_h] := - \sum_{e \subseteq \Gamma} Q_e(g)\, \boldsymbol{\tau}_h \cdot \boldsymbol{\nu}|_e \tag{53}$$

for all $\boldsymbol{\tau}_h \in M_{1,h}$, and

$$[\mathbf{G}_h, v_h] := \sum_{T \in \mathcal{T}_h} Q_T(f)\, v_{h,T} \tag{54}$$

for all $v_h \in M_h$.

Therefore, the fully discrete Galerkin scheme associated with (4) reads: *Find* $(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \in \mathbf{H}_h$ *such that*

$$\begin{array}{lcccl} [\mathbf{A}_{1,h}(\mathbf{t}_h), \mathbf{s}_h] & +[\mathbf{B}_{1,h}(\mathbf{s}_h), \boldsymbol{\sigma}_h] & & = & 0\,, \\[2ex] [\mathbf{B}_{1,h}(\mathbf{t}_h), \boldsymbol{\tau}_h] & + & [\mathbf{B}_h(\boldsymbol{\tau}_h), u_h] & = & [\mathbf{G}_{1,h}, \boldsymbol{\tau}_h]\,, \\[2ex] & [\mathbf{B}_h(\boldsymbol{\sigma}_h), v_h] & & = & [\mathbf{G}_h, v_h]\,, \end{array} \tag{55}$$

for all $(\mathbf{s}_h, \boldsymbol{\tau}_h, v_h) \in \mathbf{H}_h$.

The solvability and convergence properties of (55) were also provided in [15]. More precisely, we have the following results.

Theorem 9 *Assume that $f \in W^{1,\infty}(\Omega)$ and that $g \in W^{2,\infty}(\Gamma)$. Then, there exists a unique $(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \in \mathbf{H}_h$ solution of* (55)*. Moreover, there exists $C_1 > 0$, independent of h, such that*

$$||(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h)||_{\mathbf{H}} \leq C_1 \left\{ ||g||_{W^{2,\infty}(\Gamma)} + ||f||_{W^{1,\infty}(\Omega)} \right\}$$

and

$$||(\mathbf{t}, \boldsymbol{\sigma}, u) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h)||_{\mathbf{H}} \leq C_1\, h\, \{1 + ||(\mathbf{t}, \boldsymbol{\sigma}, u)||_{\mathbf{H}}\} + C_1 \inf_{(\mathbf{s}_h, \boldsymbol{\tau}_h, v_h) \in \mathbf{H}_h} ||(\mathbf{t}, \boldsymbol{\sigma}, u) - (\mathbf{s}_h, \boldsymbol{\tau}_h, v_h)||_{\mathbf{H}}\,.$$

In addition, if $\mathbf{t} \in [H^1(\Omega)]^2$, $\boldsymbol{\sigma} \in [H^1(\Omega)]^2$, $\operatorname{div} \boldsymbol{\sigma} \in H^1(\Omega)$ *and* $u \in H^1(\Omega)$, *there exists* $C_2 > 0$, *independent of* h, *such that the following rate of convergence holds*

$$||(\mathbf{t}, \boldsymbol{\sigma}, u) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h)||_{\mathbf{H}} \leq C_2\, h \left\{1 + ||\mathbf{t}||_{[H^1(\Omega)]^2} + ||\boldsymbol{\sigma}||_{[H^1(\Omega)]^2} + ||\operatorname{div} \boldsymbol{\sigma}||_{H^1(\Omega)} + ||u||_{H^1(\Omega)}\right\}.$$

Proof. See Section 4.3 (Theorems 4.4, 4.7 and 4.8) in [15]. □

5 Numerical results

We consider three examples to demonstrate the quality of the error estimate given in (39). In all the examples we choose $\Omega = (0,1)\times(0,1)$ and $\boldsymbol{\kappa}(\cdot, s) = 2 + 1/(1+s)$. It is clear that $\boldsymbol{\kappa}$ satisfy the assumption (2) with $\kappa_0 = 1$ and $\kappa_1 = 3$. We take the right hand side functions f and g in (1) such that $u(x_1, x_2) = 1/(x_1 + x_2 + 0.1)$, in Example 1, $u(x_1, x_2) = ((x_1 - 0.5)^2 + (x_2 - 0.5)^2)^{3/4}$, in Example 2 and $u(x_1, x_2) = ((x_1 - 0.5)^2 + (x_2 - 0.5)^2 + 0.01)^{1/4}$, in Example 3.

In the Example 1 the most critical point is $(0,0)$, for the Example 2 the unknowns $\mathbf{t} = \nabla u$ and $\boldsymbol{\sigma} = \boldsymbol{\kappa}(\cdot, |\nabla u|)\nabla u$ have a singularity at $(0.5, 0.5)$. The same point is critical for the Example 3. We note that our refinement strategy is able to recognize these critical points and generates adapted meshes with refinements towards these points.

For each example we draw the initial mesh, an intermediate mesh obtained by using our adaptive scheme and the final refined mesh (Figs. 1, 3 and 5). We also include the total number of degrees of freedom for each mesh. In Figs. 2, 4 and 6 we plot the total error E, where

$$E := \{\|u - u_h\|^2_{L^2(\Omega)} + \|\mathbf{t} - \mathbf{t}_h\|^2_{L^2(\Omega)^2} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{H(div;\Omega)}\}^{1/2}\,,$$

for both a uniform refinement scheme and an adaptive method. Finally, in Tables 1, 2 and 3 we give the errors for each particular variable, the total error, the error estimate η and the global effectivity index when the number of degrees of freedom increases.

The nonlinear Galerkin systems are solved by the Newton method where we stop the iteration when the error of the residual is less than 10^{-6}. The approximation errors shown in Tables 1, 2 and 3, are calculated on each triangle using a 7 points quadrature rule on each of the triangles.

The adapted meshes are constructed using the metric governed mesh generator BL2D developed at INRIA–Rocquencourt (see [8] for details).

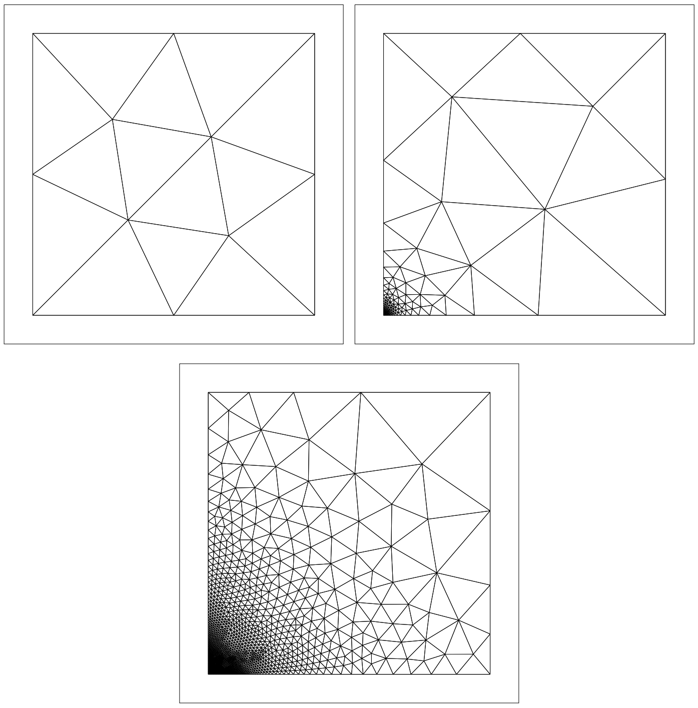

Fig. 1. Initial (14 elements, 67 d.o.f.), intermediate (220 elements, 1011 d.o.f.) and final adapted (3801 elements, 17187 d.o.f.) meshes for Example 1.

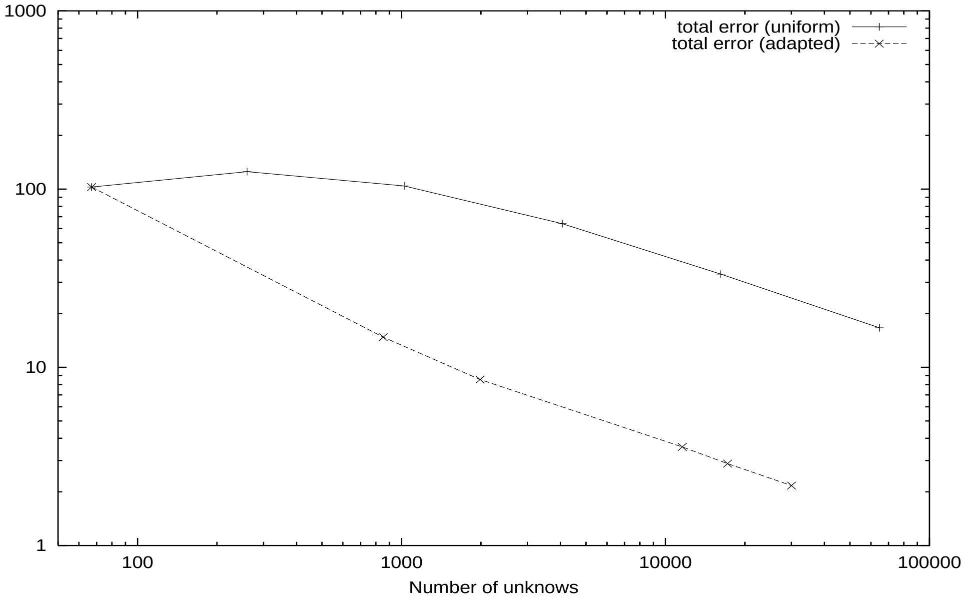

Fig. 2. Total error for Example 1.

Table 1
Behavior of the errors for the individual variables, the total error, the error estimate η and the effectivity index for Example 1.

N	$\|u - u_h\|$	$\|\mathbf{t} - \mathbf{t}_h\|$	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|$	E	η	Effectivity
67	1.1233	3.3923	102.6925	102.7485	122.1896	1.1892
260	0.4565	2.6035	125.1086	125.1356	127.0025	1.0149
1024	0.1814	1.8992	104.0565	104.0738	104.2936	1.0021
4064	0.0783	1.1946	63.9294	63.9406	63.9939	1.0008
16192	0.0368	0.6584	33.3451	33.3516	33.3775	1.0007
64640	0.0180	0.3396	16.6508	16.6543	16.6751	1.0012

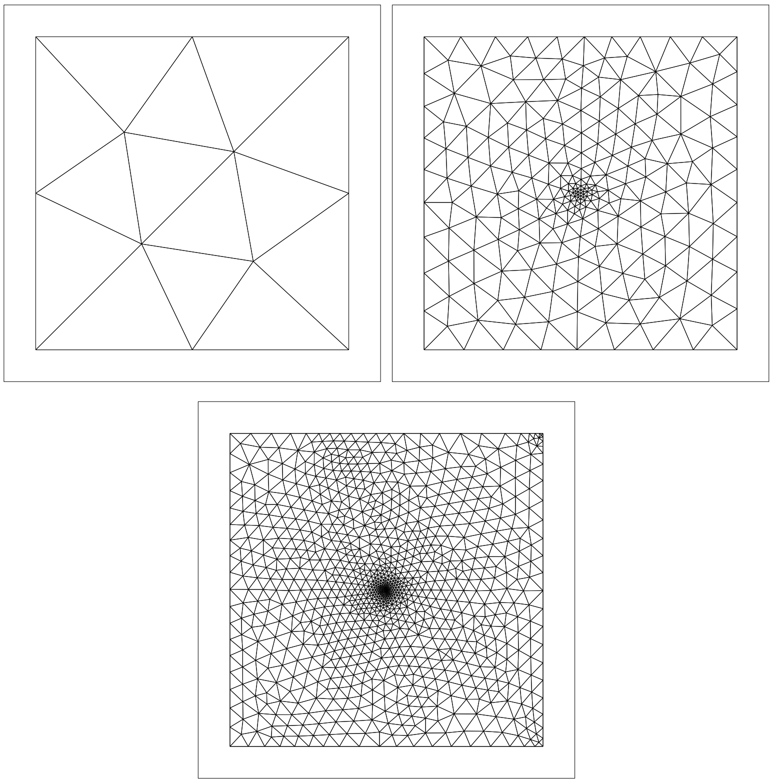

Fig. 3. Initial (14 elements, 67 d.o.f.), intermediate (177 elements, 806 d.o.f.) and final adapted (2079 elements, 9396 d.o.f.) meshes for Example 2.

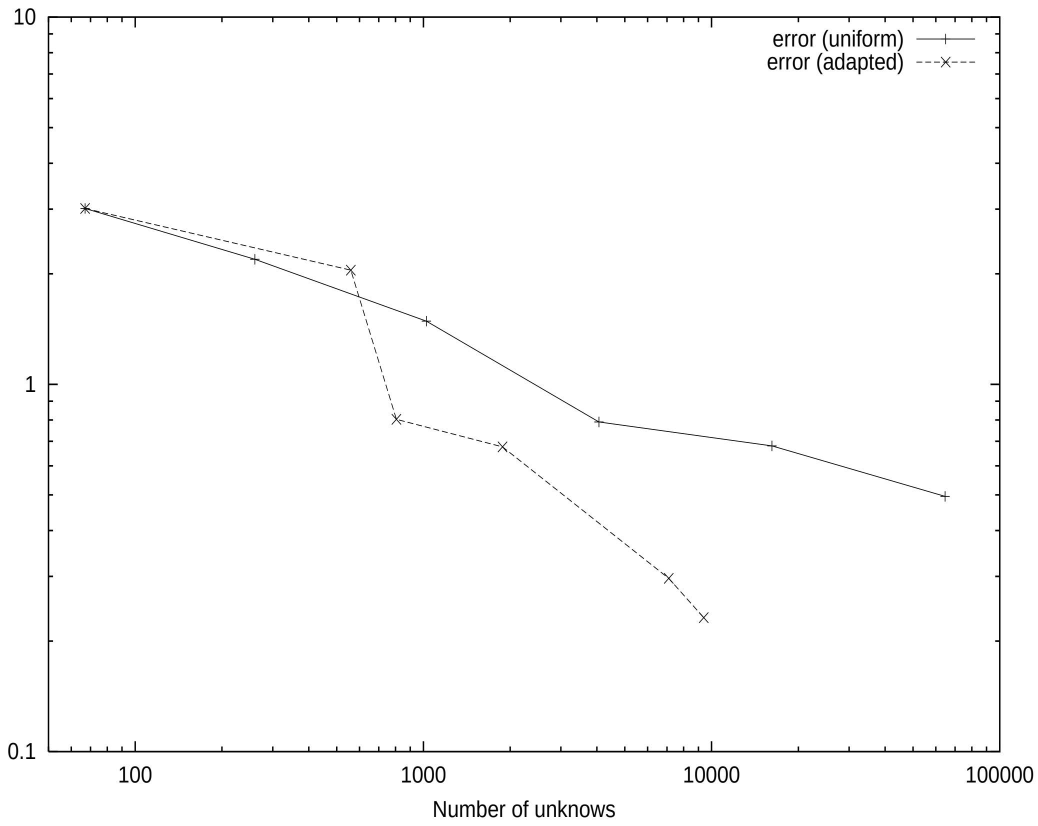

Fig. 4. Total error for Example 2.

Table 2
Behavior of the errors for the individual variables, the total error, the error estimate η and the effectivity index for Example 2.

N	$\|u - u_h\|$	$\|\mathbf{t} - \mathbf{t}_h\|$	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|$	E	η	Effectivity
67	0.0970	0.2699	2.9974	3.0111	3.1277	1.0387
260	0.0450	0.1401	2.1857	2.1906	2.2301	1.0180
1024	0.0210	0.0695	1.4844	1.4862	1.5087	1.0151
4064	0.0102	0.0345	0.7892	0.7900	0.8067	1.0211
16192	0.0051	0.0173	0.6794	0.6797	0.6879	1.0120
64640	0.0025	0.0086	0.4952	0.4953	0.5004	1.0120

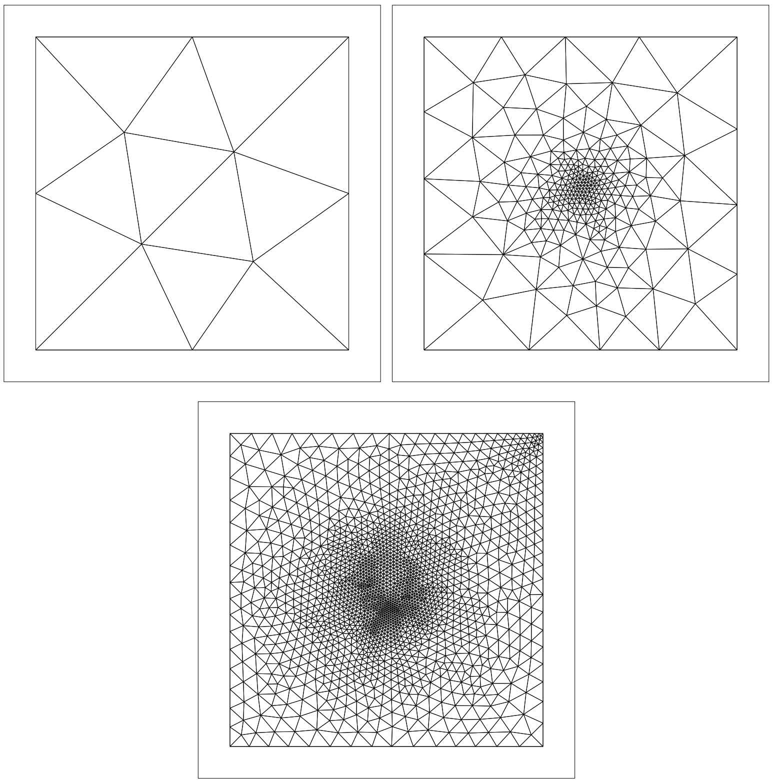

Fig. 5. Initial (14 elements, 67 d.o.f.), intermediate (516 elements, 2330 d.o.f.) and final adapted (3758 elements, 16950 d.o.f.) meshes for Example 3.

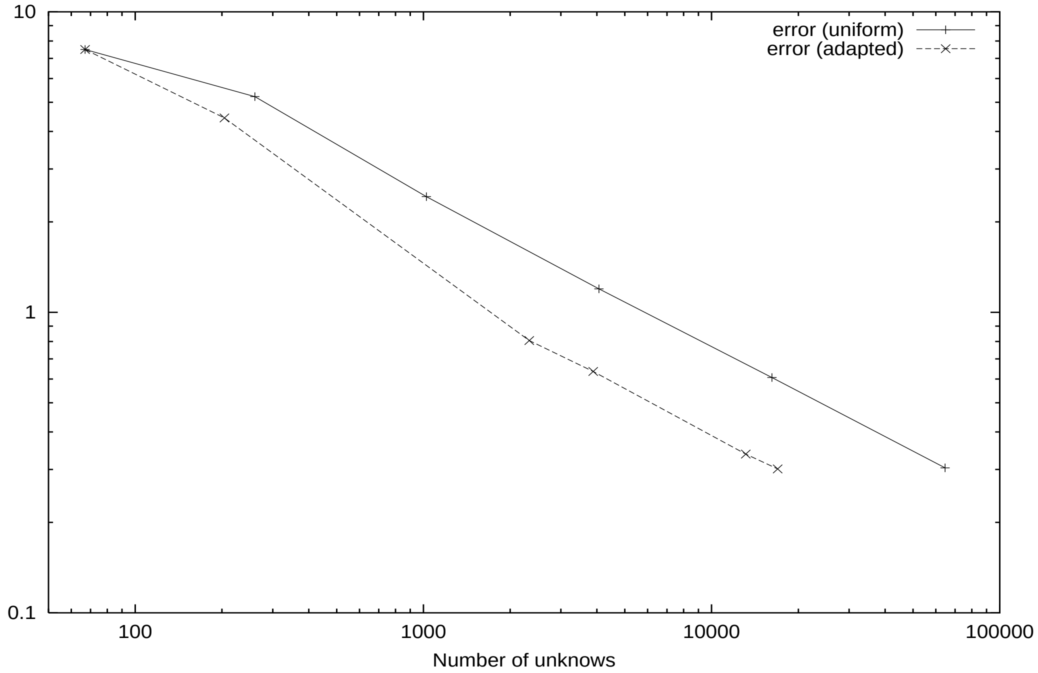

Fig. 6. Total error for Example 3.

Table 3
Behavior of the errors for the individual variables, the total error, the error estimate η and the effectivity index for Example 3.

N	$\|u - u_h\|$	$\|\mathbf{t} - \mathbf{t}_h\|$	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|$	E	η	Effectivity
67	0.0667	0.3422	7.4867	7.4948	7.5201	1.0033
260	0.0382	0.1975	5.2175	5.2214	5.2216	1.0000
1024	0.0167	0.0937	2.4256	2.4275	2.4362	1.0035
4064	0.0083	0.0469	1.1958	1.1967	1.2038	1.0059
16192	0.0041	0.0235	0.6062	0.6067	0.6125	1.0095
64640	0.0020	0.0117	0.3034	0.3036	0.3088	1.0171

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