An a-posteriori error estimate for a linear-nonlinear transmission problem in plane elastostatics

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Abstract. We consider the coupling of dual-mixed finite element and boundary element methods to solve a linear-nonlinear transmission problem in plane hyperelasticity with mixed boundary conditions. Besides the displacement and the stress tensor, we introduce the strain tensor as an additional unknown, which yields a two-fold saddle point operator equation as the corresponding variational formulation. We derive a reliable a-posteriori error estimate that depends on the solution of local Dirichlet problems and on residual terms on the transmission and Neumann boundaries, which are given in a negative order Sobolev norm. Our approach does not need the exact Galerkin solution, but any reasonable approximation of it. In addition, the analysis does not depend on special finite element or boundary element subspaces. However, for certain specific subspaces we are able to provide two fully local a-posteriori error estimates, in which the residual terms are bounded by weighted local L^2-norms. Further, one of the error estimates does not require the explicit solution of the local problems.

Key words. mixed finite elements, boundary elements, a-posteriori error estimator, local problems, hyperelasticity.

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1. Introduction

In order to obtain good convergence behavior of the Galerkin solution of linear and nonlinear boundary value problems, one usually requires to apply adaptive

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algorithms that are based on a-posteriori error estimates. This is particularly necessary for finite elements (FEM), boundary elements (BEM), and the coupling of these methods. Indeed, the list of references is quite extensive and we may refer to [2], [3], [8], [12], [30], and [31] for an overview on the a-posteriori error analysis of the finite element method. Similarly, explicit residual error indicators and estimators based on hierarchical subspace decompositions for boundary integral equations, are given in [17] and [26]. Also, explicit residual error indicators and hierarchical basis estimators for the combination of BEM with FEM and mixed-FEM, are provided in [14], [15], [16], [24], and [27].

More recently, the classical Bank-Weiser estimator from [6] (see also [1] and [2]), which involves the solution of equilibrated local Neumann problems, has been extended to a wider class of boundary value problems, including large strain elasticity, incompressibility, and interface problems (see, e.g. [9], [10], [11], and [22]). In particular, we used this method in [22] to derive an a-posteriori error estimate for the coupling of mixed-FEM and BEM to solve a transmission problem of linear elasticity in $\mathbf{R}^2$ with Dirichlet-Neumann boundary conditions. In this case, the estimator depends on the solution of local Dirichlet problems and on a residual term on the coupling interface. As shown in [10], the main advantage of this approach, though yielding only reliable estimates, is that one can expect a good effectivity index.

On the other hand, we proved in [7] that one can also combine the Bank-Weiser method with ideas from [11] to derive an a-posteriori error estimate for the two-fold saddle point formulation of a nonlinear problem in plane elasticity. It is important to remark that this kind of saddle point problem, also called dual-dual variational formulations, have arisen from the necessity of applying dual-mixed finite element methods to several nonlinear boundary value problems in potential theory and elasticity (see, e.g. [20], [25] and the references therein). In fact, in [20] we combined the dual mixed-FEM with BEM to study the solvability and Galerkin approximations of a linear-nonlinear transmission problem in hyperelasticity. In order to deal with the nonlinearity, we introduced there the strain tensor as an additional unknown, which yielded, precisely, the two-fold saddle point structure. Then, we extended the PEERS space from [5] and provided explicit finite element/boundary element subspaces satisfying all the discrete compatibility conditions. No a-posteriori error analysis was given in [20].

Consequently, the purpose of this work is to follow the analysis in [7] and extend the results from [22] to provide a reliable a-posteriori error estimate for the dual-dual variational formulation of the linear-nonlinear transmission problem studied in [20]. The rest of the paper is organized as follows. In Section 2 we recall the transmission problem from [20], provide the corresponding variational formulation in the form of a two-fold saddle point operator equation, and state the unique solvability of it. Then, Section 3 deals with the a-posteriori error analysis. First, Section 3.1 gives some preliminaries results, including an upper bound for the Galerkin projection of the error. The main a-posteriori error estimate, which depends on the solution of a local Dirichlet problem and boundary residual terms in negative order Sobolev norms, is derived in Section 3.2. Finally, in Section 3.3 we consider the extended PEERS subspace from [20] and provide two fully local a-posteriori error estimates, in which the boundary residual terms are bounded by weighted local L^2-norms. Moreover, one of the error estimates does not require the explicit solution of the local problems.

Throughout the paper C denotes a generic positive constant.

2. The transmission problem

Before stating the transmission problem from [20], we need to provide some notations. In what follows, given any Hilbert space U, U^2 and $U^{2\times 2}$ denote, respectively, the space of vectors and square matrices of order 2 with entries in U. In particular, $\mathbf{R}^{2\times 2}$ is the space of square matrices of order 2 with real entries, $\mathbf{I}_2 := (\delta_{ij})$ is the identity matrix of $\mathbf{R}^{2\times 2}$, and given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij}) \in \mathbf{R}^{2\times 2}$, we use the notations

$$\mathrm{tr}\,(\boldsymbol{\tau}) := \sum_{i=1}^{2} \tau_{ii} \quad , \quad \boldsymbol{\zeta} : \boldsymbol{\tau} := \sum_{i,j=1}^{2} \zeta_{ij}\,\tau_{ij} \quad , \quad \boldsymbol{\tau}^{\mathbf{T}} := (\tau_{ji})\,, \quad \text{and}$$

$$\mathrm{dev}\,\boldsymbol{\tau} := \left(\boldsymbol{\tau} - \frac{1}{2}\mathrm{tr}\,(\boldsymbol{\tau})\mathbf{I}_2\right) : \left(\boldsymbol{\tau} - \frac{1}{2}\mathrm{tr}\,(\boldsymbol{\tau})\mathbf{I}_2\right)\,.$$

Now, let $\Omega_0 \subset \mathbf{R}^2$ be a bounded and simply connected domain with Lipschitz continuous boundary Γ. Also, let Ω be the annular region bounded by Γ and another closed, Lipschitz continuous curve $\Gamma_1 \subset \mathbf{R}^2 \setminus \bar{\Omega}_0$. Further, let Γ_D and Γ_N be two disjoint subsets of Γ_1 such that $\Gamma_1 = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ and $\Gamma_D \neq \emptyset$ (see Figure 2.1 below).

We are interested in determining the displacements and stresses of a material occupying the region $\bar{\Omega}_0 \cup \bar{\Omega}$ which is linear elastic in Ω_0 and hyperelastic in the surrounding annular domain Ω. As usual, the linear elasticity is defined in terms of the well known Hooke law. In addition, for the description of the hyperelasticity we assume the validity of the Hencky-von Mises stress-strain relation as discussed in [28] (see, also, [29] and [32]). Therefore, given displacement fields $\mathbf{u}_0$ and $\mathbf{u}$ in Ω_0 and Ω, respectively, strain fields $\mathbf{e}(\mathbf{u}_0)$, $\mathbf{e}(\mathbf{u})$, and stress fields $\boldsymbol{\sigma}_0$, $\boldsymbol{\sigma}$, there hold the following constitutive equations:

$$\begin{aligned} &\boldsymbol{\sigma}_0 = \lambda_0\,\mathrm{tr}\,\mathbf{e}(\mathbf{u}_0)\mathbf{I}_2 + 2\mu_0\,\mathbf{e}(\mathbf{u}_0) && \text{in } \Omega_0\,, \\ &\boldsymbol{\sigma} = [\kappa - \mu(\mathrm{dev}\,\mathbf{e}(u))]\mathrm{tr}\,\mathbf{e}(\mathbf{u})\,\mathbf{I}_2 + 2\mu(\mathrm{dev}\,\mathbf{e}(u))\,\mathbf{e}(\mathbf{u}) && \text{in } \Omega\,, \end{aligned} \tag{2.1}$$

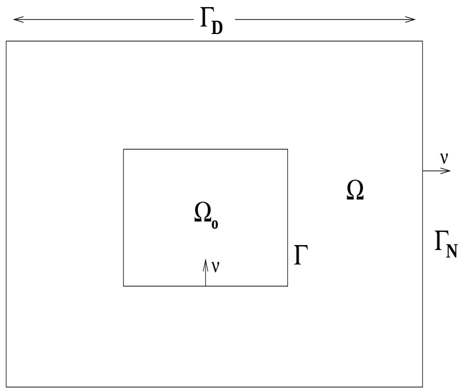

Fig. 2.1. Geometry of the problem.

where λ_0 and μ_0 are the Lamé coefficients for the elastic material, which satisfy $\mu_0 > 0$ and $\lambda_0 + \mu_0 > 0$, κ is a positive constant, called the *bulk modulus*, and $\mu : \mathbf{R}^+ \to \mathbf{R}$ is a nonlinear Lamé function. We assume that $\mu \in C^1(\mathbf{R}^+)$ and that there exist constants μ_1, μ_2 such that

$$\begin{array}{c} 0 < \mu_1 \le \mu(t) \; < \; \kappa \quad \text{and} \\ 0 < \mu_1 \le \mu(t) + 2\, t\, \mu'(t) \le \mu_2 \end{array} \tag{2.2}$$

for all $t \in \mathbf{R}^+$.

Then, given a body load $\mathbf{f} \in [L^2(\Omega)]^2$ and a traction $\mathbf{g} \in [H^{-1/2}(\Gamma_N)]^2$, our linear-nonlinear transmission problem reads: *Find* $\mathbf{u}_0 \in [H^1(\Omega_0)]^2$ *and* $\mathbf{u} \in [H^1(\Omega)]^2$ *such that*

$$\begin{array}{c} \mathbf{div}\, \boldsymbol{\sigma}_0 = 0 \quad \text{in} \quad \Omega_0 \,, \\ \mathbf{u}_0 - \mathbf{u} = 0 \quad \text{and} \quad \boldsymbol{\sigma}_0 \boldsymbol{\nu} - \boldsymbol{\sigma}\boldsymbol{\nu} = 0 \quad \text{on} \quad \Gamma \,, \\ \mathbf{div}\, \boldsymbol{\sigma} = -\,\mathbf{f} \quad \text{in} \quad \Omega \,, \\ \mathbf{u} = 0 \quad \text{on} \quad \Gamma_D \,, \quad \boldsymbol{\sigma}\boldsymbol{\nu} = \mathbf{g} \quad \text{on} \quad \Gamma_N \,, \end{array} \tag{2.3}$$

where $\boldsymbol{\nu}$ is the unit outward normal to $\partial\Omega := \Gamma \cup \Gamma_1$, and $\mathbf{div}$ denotes the usual divergence operator acting along each row of the corresponding tensor.

Problem (2.3) was solved in [20] by using the boundary integral equation method in Ω_0 and a dual-mixed finite element approach in Ω. In order to set the corresponding variational formulation (see eqs. (4.2) and (4.3) in [20]) we first let $\mathcal{N}_\Gamma(\mathbf{e})$ be the restriction to Γ of $\mathcal{N}(\mathbf{e})$, the null space of the strain tensor $\mathbf{e}$, also known as the space of rigid body displacements, and let $\langle \cdot, \cdot \rangle_\Gamma$ stand for the duality between $[H^{-1/2}(\Gamma)]^2$ and $[H^{1/2}(\Gamma)]^2$ with respect to the $[L^2(\Gamma)]^2$-inner product. Then we define the spaces

$$\mathbf{H}(\mathbf{div}\,; \Omega) := \left\{ \boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} : \quad \mathbf{div}\; \boldsymbol{\tau} \in [L^2(\Omega)]^2 \right\} ,$$

$$\mathbf{H_0}(\mathbf{div}\,; \Omega) := \{ \boldsymbol{\tau} \in \mathbf{H}(\mathbf{div}\,; \Omega) : \quad \langle \boldsymbol{\tau}\boldsymbol{\nu}, \boldsymbol{\omega} \rangle_\Gamma = 0 \quad \forall\, \boldsymbol{\omega} \in \mathcal{N}_\Gamma(\mathbf{e}) \, \} \,,$$

$$\mathcal{R} := \left\{ \boldsymbol{\delta} \in [L^2(\Omega)]^{2\times 2} : \quad \boldsymbol{\delta} + \boldsymbol{\delta}^{\mathbf{T}} = 0 \right\} ,$$

$$[\tilde{H}^{1/2}(\Gamma_N)]^2 := \left\{ \mathbf{z}|_{\Gamma_N} : \quad \mathbf{z} \in [H^1(\Omega)]^2 \quad \text{and} \quad \mathbf{z} = 0 \quad \text{on} \quad \Gamma_D \right\} ,$$

and

$$[H_0^{1/2}(\Gamma)]^2 \; := \; [H^{1/2}(\Gamma)]^2 / \mathcal{N}_\Gamma(\mathbf{e}) \, .$$

Next, we introduce the auxiliary unknowns

$$\mathbf{t} := \mathbf{e}(\mathbf{u}) \in [L^2(\Omega)]^{2\times 2} ,$$

$$\boldsymbol{\gamma} := \frac{1}{2} (\nabla \mathbf{u} - (\nabla \mathbf{u})^{\mathbf{T}}) \in \mathcal{R} ,$$

$$\boldsymbol{\phi} := \mathbf{u}|_\Gamma \in [H^{1/2}(\Gamma)]^2 , \quad \text{and} \quad \boldsymbol{\eta} := -\mathbf{u}|_{\Gamma_N} \in [\tilde{H}^{1/2}(\Gamma_N)]^2 ,$$

the nonlinear functions $\hat{\lambda}$, $\hat{\mu} : \mathbf{R}^{2\times 2} \to \mathbf{R}$, with

$$\hat{\mu}(\mathbf{s}) := \mu(\operatorname{dev} \mathbf{s}) \quad \text{and} \quad \hat{\lambda}(\mathbf{s}) := \kappa - \hat{\mu}(\mathbf{s}) \quad \forall\, \mathbf{s} \in \mathbf{R}^{2\times 2} ,$$

and the boundary integral operators $\mathbf{V}$, $\mathbf{K}$, $\mathbf{K}'$ and $\mathbf{W}$ denoting, respectively, the single, double, adjoint of the double, and hypersingular layer potentials of the Lamé system.

Thus, we observe that

$$\mathbf{t} + \boldsymbol{\gamma} \; = \; \nabla \mathbf{u} \quad \text{in} \quad \Omega ,$$

$$\boldsymbol{\sigma} \; = \; \hat{\lambda}(\mathbf{t}) \operatorname{tr}(\mathbf{t}) \, \mathbf{I}_2 + 2 \hat{\mu}(\mathbf{t}) \, \mathbf{t} \quad \text{in} \quad \Omega ,$$

and recall from [20] that there hold

$$\boldsymbol{\phi} = \left(\frac{1}{2} \mathbf{I} + \mathbf{K} \right) \boldsymbol{\phi} \; - \; \mathbf{V}(\boldsymbol{\sigma}\boldsymbol{\nu}) \quad \text{on} \quad \Gamma$$

and

$$\mathbf{W}\boldsymbol{\phi} + \left(\frac{1}{2} \mathbf{I} + \mathbf{K}' \right) (\boldsymbol{\sigma}\boldsymbol{\nu}) = 0 \quad \text{on} \quad \Gamma ,$$

where $\mathbf{I}$ is a generic identity operator.

It is well known that $\mathbf{V} : [H^{-1/2+\epsilon}(\Gamma)]^2 \to [H^{1/2+\epsilon}(\Gamma)]^2$, $\mathbf{K} : [H^{1/2+\epsilon}(\Gamma)]^2 \to [H^{1/2+\epsilon}(\Gamma)]^2$, $\mathbf{K}' : [H^{-1/2+\epsilon}(\Gamma)]^2 \to [H^{-1/2+\epsilon}(\Gamma)]^2$ and $\mathbf{W} : [H^{1/2+\epsilon}(\Gamma)]^2 \to [H^{-1/2+\epsilon}(\Gamma)]^2$ are linear and continuous $\forall\, \epsilon \in [-\frac{1}{2}, \frac{1}{2}]$ (see, e.g., [18]). Also, $\mathbf{K}(\boldsymbol{\omega}) = \frac{\boldsymbol{\omega}}{2}$ and $\mathbf{W}(\boldsymbol{\omega}) = 0 \; \forall\, \boldsymbol{\omega} \in \mathcal{N}_\Gamma(\mathbf{e})$, and $\mathbf{W}$ and $\mathbf{V}$ are symmetric positive definite and positive semi-definite on $[H_0^{1/2}(\Gamma)]^2$ and $[H^{-1/2}(\Gamma)]^2$, respectively (see, e.g., [19]).

Then, denoting

$$X_1 := [L^2(\Omega)]^{2\times 2} \times [H_0^{1/2}(\Gamma)]^2 , \quad M_1 := \mathbf{H_0}(\mathbf{div}\, ; \Omega) ,$$

$$\text{and} \quad M := [L^2(\Omega)]^2 \times \mathcal{R} \times [\tilde{H}^{1/2}(\Gamma_N)]^2 ,$$

the variational formulation of (2.3) can be written as the following two-fold saddle point operator equation (see [21]): *Find* $((\mathbf{t},\boldsymbol{\phi}),\boldsymbol{\sigma},(\mathbf{u},\gamma,\boldsymbol{\eta}))\in\mathbf{H}:=X_1\times M_1\times M$ *such that*

$$\begin{bmatrix}\mathbf{A}_1 & \mathbf{B}_1' & \mathbf{O}\\ \mathbf{B}_1 & -\mathbf{S} & \mathbf{B}'\\ \mathbf{O} & \mathbf{B} & \mathbf{O}\end{bmatrix}\begin{bmatrix}(\mathbf{t},\boldsymbol{\phi})\\ \boldsymbol{\sigma}\\ (\mathbf{u},\gamma,\boldsymbol{\eta})\end{bmatrix}=\begin{bmatrix}\mathbf{O}\\ \mathbf{O}\\ \mathbf{F}\end{bmatrix},\tag{2.4}$$

where $\mathbf{O}$ denotes a generic null operator/functional, and $\mathbf{A}_1:X_1\to X_1'$, $\mathbf{B}_1:X_1\to M_1'$, $\mathbf{B}_1':M_1\to X_1'$, $\mathbf{B}:M_1\to M'$, $\mathbf{B}':M\to M_1'$, $\mathbf{S}:M_1\to M_1'$, and $\mathbf{F}\in M'$, are defined as follows

$$[\mathbf{A}_1(\mathbf{r},\boldsymbol{\varrho}),(\mathbf{s},\boldsymbol{\psi})]:=\int_\Omega\left[\hat\lambda(\mathbf{r})\,\mathrm{tr}\,(\mathbf{r})\,\mathrm{tr}\,(\mathbf{s})+2\,\hat\mu(\mathbf{r})\,\mathbf{r}:\mathbf{s}\right]\,dx+\langle\mathbf{W}\boldsymbol{\varrho},\boldsymbol{\psi}\rangle_\Gamma\,,$$

$$[\mathbf{B}_1(\mathbf{r},\boldsymbol{\varrho}),\boldsymbol{\tau}]:=-\int_\Omega\mathbf{r}:\boldsymbol{\tau}\,dx+\langle\boldsymbol{\tau}\boldsymbol{\nu},\left(\frac{1}{2}\mathbf{I}+\mathbf{K}\right)\boldsymbol{\varrho}\rangle_\Gamma\,,$$

$$[\mathbf{B}_1'(\boldsymbol{\zeta}),(\mathbf{s},\boldsymbol{\psi})]:=[\mathbf{B}_1(\mathbf{s},\boldsymbol{\psi}),\boldsymbol{\zeta}]\,,$$

$$[\mathbf{B}(\boldsymbol{\zeta}),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\xi})]:=-\int_\Omega\mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\zeta}\,dx-\int_\Omega\boldsymbol{\zeta}:\boldsymbol{\delta}\,dx-\langle\boldsymbol{\zeta}\boldsymbol{\nu},\boldsymbol{\xi}\rangle_{\Gamma_N}\,,$$

$$[\mathbf{B}'(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\xi}),\boldsymbol{\tau}]:=[\mathbf{B}(\boldsymbol{\tau}),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\xi})]\,,$$

$$[\mathbf{S}(\boldsymbol{\zeta}),\boldsymbol{\tau}]:=\langle\boldsymbol{\tau}\boldsymbol{\nu},\mathbf{V}(\boldsymbol{\zeta}\boldsymbol{\nu})\rangle_\Gamma\,,$$

and

$$[\mathbf{F},(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\xi})]:=\int_\Omega\mathbf{f}\cdot\mathbf{v}\,dx-\langle\mathbf{g},\boldsymbol{\xi}\rangle_{\Gamma_N}\,,$$

for all $(\mathbf{r},\boldsymbol{\varrho})$, $(\mathbf{s},\boldsymbol{\psi})\in X_1$, $\boldsymbol{\zeta}$, $\boldsymbol{\tau}\in M_1$, and $(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\xi})\in M$. Here and in the following $[\cdot,\cdot]$ denotes the duality induced by the operators appearing in each case, and $\langle\cdot,\cdot\rangle_{\Gamma_N}$ denotes the duality between $[H^{-1/2}(\Gamma_N)]^2$ and $[\tilde H^{1/2}(\Gamma_N)]^2$ with respect to the $[L^2(\Gamma_N)]^2$-inner product.

Now, let $X:=X_1\times M_1:=[L^2(\Omega)]^{2\times2}\times[H_0^{1/2}(\Gamma)]^2\times\mathbf{H_0}(\mathbf{div}\,;\Omega)$ and define $\mathbf{A}:X\to X'$ by

$$\begin{aligned}[\mathbf{A}(\mathbf{r},\boldsymbol{\varrho},\boldsymbol{\zeta}),(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau})]:=&[\mathbf{A}_1(\mathbf{r},\boldsymbol{\varrho}),(\mathbf{s},\boldsymbol{\psi})]+[\mathbf{B}_1(\mathbf{r},\boldsymbol{\varrho}),\boldsymbol{\tau}]\\&+[\mathbf{B}_1(\mathbf{s},\boldsymbol{\psi}),\boldsymbol{\zeta}]-[\mathbf{S}(\boldsymbol{\zeta}),\boldsymbol{\tau}]\,,\end{aligned}$$

for all $(\mathbf{r},\boldsymbol{\varrho},\boldsymbol{\zeta})$, $(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau})\in X$. It is important to remark that $\mathbf{A}_1$, and hence $\mathbf{A}$, is a nonlinear operator.

Then (2.4) can be reformulated as: *Find* $((\mathbf{t},\boldsymbol{\phi},\boldsymbol{\sigma}),(\mathbf{u},\gamma,\boldsymbol{\eta}))\in\mathbf{H}:=X\times M$ *such that*

$$\begin{aligned}[\mathbf{A}(\mathbf{t},\boldsymbol{\phi},\boldsymbol{\sigma}),(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau})]\;+[\mathbf{B}(\boldsymbol{\tau}),(\mathbf{u},\gamma,\boldsymbol{\eta})]&=0\,,\\ [\mathbf{B}(\boldsymbol{\sigma}),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\xi})]&=[\mathbf{F},(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\xi})]\,,\end{aligned}\tag{2.5}$$

for all $((\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau}),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\xi})) \in \mathbf{H}$.

Concerning the solvability of (2.4) (equivalently (2.5)), we have the following result.

Theorem 2.1. *There exists a unique* $((\mathbf{t},\boldsymbol{\phi},\boldsymbol{\sigma}),(\mathbf{u},\boldsymbol{\gamma},\boldsymbol{\eta})) \in \mathbf{H}$ *solution of the nonlinear two-fold saddle point formulation* (2.4)*. Moreover, there exists* $C > 0$*, independent of the solution, such that*

$$||((\mathbf{t},\boldsymbol{\phi},\boldsymbol{\sigma}),(\mathbf{u},\boldsymbol{\gamma},\boldsymbol{\eta}))||_{\mathbf{H}} \leq C \left\{ ||\mathbf{f}||_{[L^2(\Omega)]^2} + ||\mathbf{g}||_{[H^{-1/2}(\Gamma_N)]^2} \right\}.$$

Proof. It reduces to show that $\mathbf{A}_1$ is strongly monotone and Lipschitz continuous, that $\mathbf{B}_1$ and $\mathbf{B}$ satisfy appropriate inf-sup conditions, and that $\mathbf{S}$ is positive semi-definite. For details we refer to Section 4 in [20], and to the abstract results provided in [21]. □

3. The a-posteriori error analysis

3.1. Preliminaries

Throughout this section we assume that Γ and Γ_1 are polygonal curves. Then we let $\{\mathcal{T}_h\}_{h\in\mathtt{I}}$ be a regular family of triangulations of Ω, made up of triangles T of diameter h_T, such that $h := \sup_{T\in\mathcal{T}_h} h_T$ and $\bar{\Omega} = \cup\{T : \quad T \in \mathcal{T}_h\}$. Here $\mathtt{I}$ is a set of indexes, say $\mathtt{I} := \{h_j\}_{j\in\mathrm{N}}$, with $h_j \geq h_{j+1}\ \forall j \in \mathbf{N}$. In addition, let $\mathbf{H}_h := X_{1,h} \times M_{1,h} \times M_h$ be a finite element subspace of $\mathbf{H} := X_1 \times M_1 \times M$, and let $(\mathbf{t}_h,\boldsymbol{\phi}_h,\boldsymbol{\sigma}_h,u_h,\boldsymbol{\gamma}_h,\boldsymbol{\eta}_h) \in \mathbf{H}_h$ be an approximation to the exact solution $(\mathbf{t},\boldsymbol{\phi},\boldsymbol{\sigma},u,\boldsymbol{\gamma},\boldsymbol{\eta}) \in \mathbf{H}$ of the variational formulation (2.4).

We now apply the approach from [9] (see also [10] and [22]) to derive an a-posteriori estimate for the error $||(\mathbf{t},\boldsymbol{\phi},\boldsymbol{\sigma},u,\boldsymbol{\gamma},\boldsymbol{\eta}) - (\mathbf{t}_h,\boldsymbol{\phi}_h,\boldsymbol{\sigma}_h,u_h,\boldsymbol{\gamma}_h,\boldsymbol{\eta}_h)||_{\mathbf{H}}$. For this purpose, we need to choose any linear and bounded operator $\hat{\mathbf{A}} : X \to X'$ such that the corresponding induced bilinear form $\hat{A} : X \times X \to \mathbf{R}$, defined by

$$\hat{A}((\mathbf{r},\boldsymbol{\varrho},\boldsymbol{\zeta}),(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau})) = [\hat{\mathbf{A}}(\mathbf{r},\boldsymbol{\varrho},\boldsymbol{\zeta}),(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau})]$$

for all $(\mathbf{r},\boldsymbol{\varrho},\boldsymbol{\zeta})$, $(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau}) \in X$, is symmetric and X-elliptic.

In the following, for simplicity, we take

$$[\hat{\mathbf{A}}(\mathbf{r},\boldsymbol{\varrho},\boldsymbol{\zeta}),(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau})] := \langle \mathbf{r},\mathbf{s}\rangle_{[L^2(\Omega)]^{2\times2}} + \langle \mathbf{W}\boldsymbol{\varrho},\boldsymbol{\psi}\rangle_\Gamma + \langle \boldsymbol{\zeta},\boldsymbol{\tau}\rangle_{\mathbf{H}(\mathbf{div};\Omega)}, \quad (3.1)$$

where

$$\langle \mathbf{r},\mathbf{s}\rangle_{[L^2(\Omega)]^{2\times2}} := \int_\Omega \mathbf{r} : \mathbf{s}\, dx \quad (3.2)$$

and

$$\langle \boldsymbol{\zeta},\boldsymbol{\tau}\rangle_{\mathbf{H}(\mathbf{div};\Omega)} := \int_\Omega \boldsymbol{\zeta} : \boldsymbol{\tau}\, dx + \int_\Omega \mathbf{div}\,\boldsymbol{\zeta} \cdot \mathbf{div}\,\boldsymbol{\tau}\, dx \quad (3.3)$$

are the usual inner products of $[L^2(\Omega)]^{2\times 2}$ and $\mathbf{H}(\mathbf{div}\,;\Omega)$, respectively.

Then, given $T \in \mathcal{T}_h$ we denote by $\langle \cdot,\cdot\rangle_{[L^2(T)]^{2\times 2}}$ and $\langle \cdot,\cdot\rangle_{\mathbf{H}(\mathbf{div}\,;T)}$ the corresponding restrictions of (3.2) and (3.3) to $[L^2(T)]^{2\times 2}$ and $\mathbf{H}(\mathbf{div}\,;T)$. Further, we let $(\mathbf{t}_{h,T}, \boldsymbol{\sigma}_{h,T}, \mathbf{u}_{h,T}, \boldsymbol{\gamma}_{h,T})$ be the restriction of $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h)$ to T.

The following theorem provides an important upper bound for the Ritz projection of the Galerkin error with respect to the operator $\hat{\mathbf{A}}$.

Theorem 3.1. *Let $(\bar{\mathbf{t}}, \bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\sigma}})$ be the unique element in X such that*

$$\begin{aligned} &[\hat{\mathbf{A}}(\bar{\mathbf{t}}, \bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\sigma}}), (\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})] = [\mathbf{A}(\mathbf{t}, \boldsymbol{\phi}, \boldsymbol{\sigma}), (\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})] \\ &-[\mathbf{A}(\mathbf{t}_h, \boldsymbol{\phi}_h, \boldsymbol{\sigma}_h), (\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})] + [\mathbf{B}(\boldsymbol{\tau}), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\eta}) - (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\eta}_h)] \end{aligned} \tag{3.4}$$

for all $(\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau}) \in X$, and let $\boldsymbol{\varphi}_h \in [H^{1/2}(\cup_{T\in\mathcal{T}_h} \partial T)]^2$ be an approximation to $\mathbf{u}$ on element boundaries with $\boldsymbol{\varphi}_h = 0$ on Γ_D. In addition, for each $T \in \mathcal{T}_h$ define

$$\hat{\mathbf{t}}_T := \boldsymbol{\sigma}_{h,T} - \hat{\lambda}(\mathbf{t}_{h,T})\,\mathrm{tr}\,(\mathbf{t}_{h,T})\,\mathbf{I}_2 - 2\,\hat{\mu}(\mathbf{t}_{h,T})\,\mathbf{t}_{h,T}\,,$$

and let $\hat{\boldsymbol{\sigma}}_T \in \mathbf{H}(\mathbf{div}\,;T)$ be the unique solution of the local problem

$$\langle \hat{\boldsymbol{\sigma}}_T, \boldsymbol{\tau}\rangle_{\mathbf{H}(\mathbf{div}\,;T)} \;=\; F_{h,T}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}(\mathbf{div}\,;T)\,, \tag{3.5}$$

where

$$\begin{aligned} F_{h,T}(\boldsymbol{\tau}) &:= \int_T (\mathbf{t}_{h,T} + \boldsymbol{\gamma}_{h,T}) : \boldsymbol{\tau}\,dx + \int_T \mathbf{u}_{h,T}\cdot \mathbf{div}\,\boldsymbol{\tau}\,dx - \int_{\partial T} \boldsymbol{\varphi}_h \cdot \boldsymbol{\tau}\boldsymbol{\nu}_T\,ds \\ &+\langle \boldsymbol{\tau}\boldsymbol{\nu}, \boldsymbol{\varphi}_h + \boldsymbol{\eta}_h\rangle_{\partial T\cap\Gamma_N} + \langle \boldsymbol{\tau}\boldsymbol{\nu}, \boldsymbol{\varphi}_h + \mathbf{V}(\boldsymbol{\sigma}_h\boldsymbol{\nu}) - \left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right)\boldsymbol{\phi}_h\rangle_{\partial T\cap\Gamma}\,, \end{aligned} \tag{3.6}$$

with $\boldsymbol{\nu}_T$ being the unit outward normal to ∂T. Then there holds

$$\begin{aligned} [\hat{\mathbf{A}}(\bar{\mathbf{t}}, \bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\sigma}}), (\bar{\mathbf{t}}, \bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\sigma}})] \;&\leq\; \sum_{T\in\mathcal{T}_h} ||\hat{\mathbf{t}}_T||^2_{[L^2(T)]^{2\times 2}} \;+\; \sum_{T\in\mathcal{T}_h} ||\hat{\boldsymbol{\sigma}}_T||^2_{\mathbf{H}(\mathbf{div}\,;T)} \\ &+||\,\mathbf{W}^{-1}\,||\;||\mathbf{W}\boldsymbol{\phi}_h + \left(\frac{1}{2}\mathbf{I} + \mathbf{K}'\right)(\boldsymbol{\sigma}_h\boldsymbol{\nu})||^2_{[H^{-1/2}(\Gamma)]^2}\,. \end{aligned} \tag{3.7}$$

Proof. We first observe from (2.5) that

$$[\mathbf{A}(\mathbf{t}, \boldsymbol{\phi}, \boldsymbol{\sigma}), (\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})] + [\mathbf{B}(\boldsymbol{\tau}), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\eta})] \;=\; 0 \quad \forall\,(\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau}) \in X\,,$$

and hence

$$\begin{aligned} &[\hat{\mathbf{A}}(\bar{\mathbf{t}}, \bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\sigma}}), (\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})] = -[\mathbf{A}(\mathbf{t}_h, \boldsymbol{\phi}_h, \boldsymbol{\sigma}_h), (\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})] \\ &-[\mathbf{B}(\boldsymbol{\tau}), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\eta}_h)] \quad \forall\,(\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau}) \in X\,. \end{aligned} \tag{3.8}$$

Since the bilinear form induced by $\hat{\mathbf{A}}$ is symmetric and X-elliptic, we have that

$$- \frac{1}{2} [\hat{\mathbf{A}}(\bar{\mathbf{t}}, \bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\sigma}}), (\bar{\mathbf{t}}, \bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\sigma}})]$$
$$= \min_{(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau})\in X} \left\{ \frac{1}{2} [\hat{\mathbf{A}}(\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau}), (\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})] - [\hat{\mathbf{A}}(\bar{\mathbf{t}}, \bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\sigma}}), (\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})] \right\},$$

which, according to (3.8), yields

$$- \frac{1}{2} [\hat{\mathbf{A}}(\bar{\mathbf{t}}, \bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\sigma}}), (\bar{\mathbf{t}}, \bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\sigma}})] = \min_{(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau})\in X} \mathbf{J}(\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau}), \tag{3.9}$$

with

$$\begin{aligned} \mathbf{J}(\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau}) &:= \frac{1}{2} [\hat{\mathbf{A}}(\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau}), (\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})] \\ &+ [\mathbf{A}(\mathbf{t}_h, \boldsymbol{\phi}_h, \boldsymbol{\sigma}_h), (\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})] + [\mathbf{B}(\boldsymbol{\tau}), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\eta}_h)]. \end{aligned} \tag{3.10}$$

Now, since $\boldsymbol{\varphi}_h = 0$ on Γ_D, we get

$$\sum_{T\in\mathcal{T}_h} \int_{\partial T} \boldsymbol{\varphi}_h \cdot \boldsymbol{\tau}\boldsymbol{\nu}_T \, ds - \int_{\Gamma_N\cup\Gamma} \boldsymbol{\varphi}_h \cdot \boldsymbol{\tau}\boldsymbol{\nu} \, ds = 0. \tag{3.11}$$

Then, adding (3.11) to (3.10) and using the definitions of $\hat{\mathbf{A}}$, $\mathbf{A}$ and $\mathbf{B}$, we obtain

$$\mathbf{J}(\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau}) = \sum_{T\in\mathcal{T}_h} \mathbf{J}_{1,T}(\mathbf{s}_T) + \mathbf{J}_2(\boldsymbol{\psi}) + \sum_{T\in\mathcal{T}_h} \mathbf{J}_{3,T}(\boldsymbol{\tau}_T), \tag{3.12}$$

where $(\mathbf{s}_T, \boldsymbol{\tau}_T)$ is the restriction of $(\mathbf{s}, \boldsymbol{\tau})$ to T,

$$\mathbf{J}_{1,T}(\mathbf{s}_T) := \frac{1}{2} ||\mathbf{s}_T||^2_{[L^2(T)]^{2\times 2}} - \langle \hat{\mathbf{t}}_T, \mathbf{s}_T \rangle_{[L^2(T)]^{2\times 2}}, \tag{3.13}$$

$$\mathbf{J}_2(\boldsymbol{\psi}) := \frac{1}{2} \langle \mathbf{W}\boldsymbol{\psi}, \boldsymbol{\psi} \rangle_\Gamma + \langle \mathbf{W}\boldsymbol{\phi}_h + \left(\frac{1}{2}\mathbf{I} + \mathbf{K}' \right) (\boldsymbol{\sigma}_h\boldsymbol{\nu}), \boldsymbol{\psi} \rangle_\Gamma, \tag{3.14}$$

and

$$\mathbf{J}_{3,T}(\boldsymbol{\tau}_T) := \frac{1}{2} ||\boldsymbol{\tau}_T||^2_{\mathbf{H}(\mathbf{div};T)} - F_{h,T}(\boldsymbol{\tau}_T). \tag{3.15}$$

By replacing (3.12) back into (3.9), and using that $\mathbf{s} \in [L^2(\Omega)]^{2\times 2}$ if and only if $\mathbf{s}_T \in [L^2(T)]^{2\times 2}$ $\forall T \in \mathcal{T}_h$, we can write

$$\begin{aligned} - \frac{1}{2} [\hat{\mathbf{A}}(\bar{\mathbf{t}}, \bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\sigma}}), (\bar{\mathbf{t}}, \bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\sigma}})] &= \sum_{T\in\mathcal{T}_h} \left\{ \min_{\mathbf{s}_T\in[L^2(T)]^{2\times 2}} \mathbf{J}_{1,T}(\mathbf{s}_T) \right\} \\ &+ \min_{\boldsymbol{\psi}\in[H_0^{1/2}(\Gamma)]^2} \mathbf{J}_2(\boldsymbol{\psi}) + \min_{\boldsymbol{\tau}\in\mathbf{H_0}(\mathbf{div};\Omega)} \left\{ \sum_{T\in\mathcal{T}_h} \mathbf{J}_{3,T}(\boldsymbol{\tau}_T) \right\}. \end{aligned} \tag{3.16}$$

Since $\boldsymbol{\sigma}_{h,T}$, $\mathbf{t}_{h,T} \in [L^2(T)]^{2\times 2}$, it follows from (3.13) that

$$\min_{\mathbf{s}_T \in [L^2(T)]^{2\times 2}} \mathbf{J}_{1,T}(\mathbf{s}_T) \;=\; -\,\frac{1}{2}\,||\hat{\mathbf{t}}_T||^2_{[L^2(T)]^{2\times 2}}\,. \tag{3.17}$$

Note that $\hat{\mathbf{t}}_T$ corresponds to the explicit local residual of the nonlinear constitutive equation in Ω (cf. (2.1)).

Now, let us introduce the *broken* space

$$\mathbf{H}(\mathbf{div}\,;\Omega)^{br} := \{\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} : \quad \boldsymbol{\tau}_T \in \mathbf{H}(\mathbf{div}\,;T) \quad \forall\, T \in \mathcal{T}_h\,\}\,.$$

Then, since $\mathbf{H_0}(\mathbf{div}\,;\Omega) \subseteq \mathbf{H}(\mathbf{div}\,;\Omega)^{br}$, we get

$$\min_{\boldsymbol{\tau}\in \mathbf{H_0}(\mathbf{div}\,;\Omega)} \left\{ \sum_{T\in\mathcal{T}_h} \mathbf{J}_{3,T}(\boldsymbol{\tau}_T) \right\} \;\geq\; \min_{\boldsymbol{\tau}\in \mathbf{H}(\mathbf{div}\,;\Omega)^{br}} \left\{ \sum_{T\in\mathcal{T}_h} \mathbf{J}_{3,T}(\boldsymbol{\tau}_T) \right\}$$

$$=\; \sum_{T\in\mathcal{T}_h} \min_{\boldsymbol{\tau}_T\in \mathbf{H}(\mathbf{div}\,;T)} \mathbf{J}_{3,T}(\boldsymbol{\tau}_T) \;=\; -\,\frac{1}{2}\sum_{T\in\mathcal{T}_h} ||\hat{\boldsymbol{\sigma}}_T||^2_{\mathbf{H}(\mathbf{div}\,;T)}\,, \tag{3.18}$$

where the last equation has made use of (3.15) and the definition of the local problem (3.5). At this point we remark that the unique solvability of (3.5) follows from the Lax-Milgram Theorem and the fact that $F_{h,T}$ becomes a linear and bounded functional on $\mathbf{H}(\mathbf{div}\,;T)$.

On the other hand, the minimizer to $\mathbf{J}_2$ is the unique $\tilde{\boldsymbol{\psi}} \in [H_0^{1/2}(\Gamma)]^2$ (guaranteed by the positive definiteness of $\mathbf{W}$) solution to the equation

$$\langle \mathbf{W}\tilde{\boldsymbol{\psi}}, \boldsymbol{\psi}\rangle_\Gamma \;=\; -\,\langle \mathbf{W}\boldsymbol{\phi}_h \,+\, \left(\frac{1}{2}\mathbf{I} + \mathbf{K}'\right)(\boldsymbol{\sigma}_h\boldsymbol{\nu}), \boldsymbol{\psi}\rangle_\Gamma$$

for all $\boldsymbol{\psi} \in [H_0^{1/2}(\Gamma)]^2$, which yields

$$\min_{\boldsymbol{\psi}\in[H_0^{1/2}(\Gamma)]^2} \mathbf{J}_2(\boldsymbol{\psi}) \;=\; -\,\frac{1}{2}\,\langle \mathbf{W}\tilde{\boldsymbol{\psi}}, \tilde{\boldsymbol{\psi}}\rangle_\Gamma$$

$$\geq\; -\,\frac{1}{2}\,||\mathbf{W}^{-1}||\;||\mathbf{W}\boldsymbol{\phi}_h \,+\, \left(\frac{1}{2}\mathbf{I} + \mathbf{K}'\right)(\boldsymbol{\sigma}_h\boldsymbol{\nu})||^2_{[H^{-1/2}(\Gamma)]^2}\,. \tag{3.19}$$

Finally, (3.16), (3.17), (3.18) and (3.19) imply (3.7) and complete the proof of the theorem. □

The following technical result will also be needed.

Lemma 3.2. *The operator* $\mathbf{A}_1$ *has continuous first order Gâteaux derivative* $D\mathbf{A}_1$*, and there exist* $\tilde{C}_1$, $\tilde{C}_2 > 0$ *such that*

$$|\,D\mathbf{A}_1(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})((\mathbf{r}, \boldsymbol{\varrho}), (\mathbf{s}, \boldsymbol{\psi}))\,| \;\leq\; \tilde{C}_1\,||(\mathbf{r}, \boldsymbol{\varrho})||_{X_1}\,||(\mathbf{s}, \boldsymbol{\psi})||_{X_1}$$

and

$$D\mathbf{A}_1(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})((\mathbf{s}, \boldsymbol{\psi}), (\mathbf{s}, \boldsymbol{\psi})) \;\geq\; \tilde{C}_2\,||(\mathbf{s}, \boldsymbol{\psi})||^2_{X_1}\,,$$

for all $(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})$, $(\mathbf{r}, \boldsymbol{\varrho})$, $(\mathbf{s}, \boldsymbol{\psi}) \in X_1 := [L^2(\Omega)]^{2\times 2} \times [H_0^{1/2}(\Gamma)]^2$.

Proof. Given $i, j \in \{1,2\}$, we let $a_{ij} : \mathbf{R}^{2\times 2} \to \mathbf{R}$ be the nonlinear mapping defined by

$$a_{ij}(\mathbf{r}) := \hat{\lambda}(\mathbf{r}) \operatorname{tr}(\mathbf{r}) \, \delta_{ij} + 2 \, \hat{\mu}(\mathbf{r}) \, r_{ij}$$

for all $\mathbf{r} := (r_{ij}) \in \mathbf{R}^{2\times 2}$. According to the assumptions on κ and μ (cf. (2.2)), one can prove (see [23] or Lemmas 4.1, 4.2, and 4.3 in [4]) that a_{ij} is of class C^1, and that there exist C_1, $C_2 > 0$ such that

$$\left| \frac{\partial}{\partial \tilde{r}_{kl}} \, a_{ij}(\tilde{\mathbf{r}}) \right| \; \leq \; C_1 \tag{3.20}$$

and

$$\sum_{i,j,k,l=1}^{2} \frac{\partial}{\partial \tilde{r}_{kl}} \, a_{ij}(\tilde{\mathbf{r}}) \, s_{kl} \, s_{ij} \; \geq \; C_2 \sum_{i,j=1}^{2} s_{ij}^2 \, , \tag{3.21}$$

for all $\tilde{\mathbf{r}} := (\tilde{r}_{ij})$, $\mathbf{s} := (s_{ij}) \in \mathbf{R}^{2\times 2}$.

Now, for all $(\mathbf{r}, \boldsymbol{\varrho})$, $(\mathbf{s}, \boldsymbol{\psi}) \in X_1$, with $\mathbf{r}(\cdot) = (r_{ij}(\cdot))$ and $\mathbf{s}(\cdot) = (s_{ij}(\cdot))$, we can write

$$[\mathbf{A}_1(\mathbf{r}, \boldsymbol{\varrho}), (\mathbf{s}, \boldsymbol{\psi})] = [\tilde{\mathbf{A}}_1(\mathbf{r}), \mathbf{s}] \; + \; \langle \mathbf{W} \boldsymbol{\varrho}, \boldsymbol{\psi} \rangle_{\Gamma} \, ,$$

where $\tilde{\mathbf{A}}_1 : [L^2(\Omega)]^{2\times 2} \to [L^2(\Omega)]^{2\times 2'}$, the pure nonlinear part of $\mathbf{A}_1$, is defined by

$$[\tilde{\mathbf{A}}_1(\mathbf{r}), \mathbf{s}] := \sum_{i,j=1}^{2} \int_{\Omega} a_{ij}(\mathbf{r}) \, s_{ij} \, dx \, .$$

Then, because of the linearity of $\mathbf{W}$, we obtain

$$D\mathbf{A}_1(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})((\mathbf{r}, \boldsymbol{\varrho}), (\mathbf{s}, \boldsymbol{\psi})) \; = \; D\tilde{\mathbf{A}}_1(\tilde{\mathbf{r}})(\mathbf{r}, \mathbf{s}) + \langle \mathbf{W} \boldsymbol{\varrho}, \boldsymbol{\psi} \rangle_{\Gamma} \, ,$$

with

$$D\tilde{\mathbf{A}}_1(\tilde{\mathbf{r}})(\mathbf{r}, \mathbf{s}) := \int_{\Omega} \sum_{i,j,k,l=1}^{2} \frac{\partial}{\partial \tilde{r}_{kl}} \, a_{ij}(\tilde{\mathbf{r}}) \, r_{kl} \, s_{ij} \, dx \tag{3.22}$$

for all $(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})$, $(\mathbf{r}, \boldsymbol{\varrho})$, $(\mathbf{s}, \boldsymbol{\psi}) \in X_1$.

Therefore, (3.20) and the Cauchy-Schwarz inequality imply

$$\left| D\tilde{\mathbf{A}}_1(\tilde{\mathbf{r}})(\mathbf{r}, \mathbf{s}) \right| \leq C_1 \int_{\Omega} \sum_{i,j,k,l=1}^{2} |r_{kl}| \, |s_{ij}| \, dx \leq \; 4 \, C_1 \, ||\mathbf{r}||_{[L^2(\Omega)]^{2\times 2}} \, ||\mathbf{s}||_{[L^2(\Omega)]^{2\times 2}} \, ,$$

and hence, using the boundedness of $\mathbf{W}$, we get

$$|D\mathbf{A}_1(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})((\mathbf{r}, \boldsymbol{\varrho}), (\mathbf{s}, \boldsymbol{\psi}))| \leq \; \tilde{C}_1 \, ||(\mathbf{r}, \boldsymbol{\varrho})||_{X_1} \, ||(\mathbf{s}, \boldsymbol{\psi})||_{X_1} \tag{3.23}$$

for all $(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})$, $(\mathbf{r}, \boldsymbol{\varrho})$, $(\mathbf{s}, \boldsymbol{\psi}) \in X_1$, where $\tilde{C}_1 := \max\{4C_1, ||\mathbf{W}||\}$.

On the other hand, since $\mathbf{W}$ is positive definite on $[H_0^{1/2}(\Gamma)]^2$, there exists $\alpha_0 > 0$ such that

$$\langle \mathbf{W} \boldsymbol{\psi}, \boldsymbol{\psi} \rangle_{\Gamma} \; \geq \; \alpha_0 \, ||\boldsymbol{\psi}||^2_{[H_0^{1/2}(\Gamma)]^2} \qquad \forall \, \boldsymbol{\psi} \in [H_0^{1/2}(\Gamma)]^2 \, .$$

Then, using (3.21) and (3.22), we deduce that

$$\begin{aligned} D\mathbf{A}_1(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})((\mathbf{s}, \boldsymbol{\psi}), (\mathbf{s}, \boldsymbol{\psi})) &= D\tilde{\mathbf{A}}_1(\tilde{\mathbf{r}})(\mathbf{s}, \mathbf{s}) + \langle \mathbf{W}\boldsymbol{\psi}, \boldsymbol{\psi} \rangle_\Gamma \\ &\geq \tilde{C}_2 \, ||(\mathbf{s}, \boldsymbol{\psi})||_{X_1}^2 \end{aligned} \tag{3.24}$$

for all $(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})$, $(\mathbf{s}, \boldsymbol{\psi}) \in X_1$, where $\tilde{C}_2 := \min\{C_2, \alpha_0\}$. Finally, the continuity of $D\mathbf{A}_1$ is a consequence of (3.22) and the fact that the nonlinear mappings a_{ij} are of class C^1. □

We remark from the previous lemma that, for all $(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}}) \in X_1$, $D\mathbf{A}_1(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})$ can be identified with the bilinear form on $X_1 \times X_1$ defined by (3.22), which, in virtue of (3.23) and (3.24), is uniformly bounded and X_1-elliptic, respectively. In addition, using the conditions satisfied by $D\mathbf{A}_1$ and $D\tilde{\mathbf{A}}_1$ one can easily show that $\mathbf{A}_1$ and $\tilde{\mathbf{A}}_1$ are strongly monotone and Lipschitz continuous. Moreover, (3.20) and (3.21) also yield these properties for $\tilde{\mathbf{A}}_{1,T}$, the restriction of $\tilde{\mathbf{A}}_1$ to a triangle $T \in \mathcal{T}_h$, with constants independent of T.

3.2. The main a-posteriori error estimate

The results given in the previous subsection allow us to prove now the following main theorem.

Theorem 3.3. *Let $\boldsymbol{\varphi}_h \in [H^{1/2}(\cup_{T\in\mathcal{T}_h} \partial T)]^2$ be an approximation to $\mathbf{u}$ on element boundaries with $\boldsymbol{\varphi}_h = 0$ on Γ_D. In addition, for each $T \in \mathcal{T}_h$ define*

$$\hat{\mathbf{t}}_T := \boldsymbol{\sigma}_{h,T} - \hat{\lambda}(\mathbf{t}_{h,T}) \, \mathrm{tr}\,(\mathbf{t}_{h,T}) \, \mathbf{I}_2 - 2\,\hat{\mu}(\mathbf{t}_{h,T}) \, \mathbf{t}_{h,T} \,,$$

and let $\hat{\boldsymbol{\sigma}}_T \in \mathbf{H}(\mathbf{div}\,; T)$ be the unique solution of the local problem (3.5)*. Then there exists $C > 0$, independent of h, such that*

$$||(\mathbf{t}, \boldsymbol{\phi}, \boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\eta}) - (\mathbf{t}_h, \boldsymbol{\phi}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\eta}_h)||_{\mathbf{H}} \leq C \left\{ \sum_{T\in\mathcal{T}_h} \boldsymbol{\theta}_T^2 + R_\Gamma^2 + R_{\Gamma_N}^2 \right\}^{1/2},$$

where

$$\begin{aligned} \boldsymbol{\theta}_T^2 \; := \; \Big\{ \, & ||\hat{\mathbf{t}}_T||_{[L^2(T)]^{2\times 2}}^2 \, + \, ||\hat{\boldsymbol{\sigma}}_T||_{\mathbf{H}(\mathbf{div}\,;T)}^2 \\ & + ||\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_{h,T}||_{[L^2(T)]^2}^2 \, + \, ||\boldsymbol{\sigma}_{h,T} - \boldsymbol{\sigma}_{h,T}^{\mathbf{T}}||_{[L^2(T)]^{2\times 2}}^2 \Big\}, \end{aligned}$$

$$R_\Gamma := ||\mathbf{W}\boldsymbol{\phi}_h + \left(\frac{1}{2}\mathbf{I} + \mathbf{K}'\right)(\boldsymbol{\sigma}_h \boldsymbol{\nu})||_{[H^{-1/2}(\Gamma)]^2} \,,$$

and

$$R_{\Gamma_N} := ||\boldsymbol{\sigma}_h \boldsymbol{\nu} - \mathbf{g}||_{[H^{-1/2}(\Gamma_N)]^2} \,.$$

Proof. Let $(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}}) \in X_1$ and let $\mathbf{L}_{(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})} : \mathbf{H} \to \mathbf{H}'$ be the linear operator arising from the variational formulation (2.4) (or (2.5)) after replacing $[\mathbf{A}_1(\cdot, \cdot), (\cdot, \cdot)]$ by $D\mathbf{A}_1(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})((\cdot, \cdot), (\cdot, \cdot))$.

Using that $D\mathbf{A}_1(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})$ is a uniformly bounded and X_1-elliptic bilinear form, that $\mathbf{B}$ and $\mathbf{B}_1$ satisfy the continuous inf-sup conditions, and that $\mathbf{S}$ is positive semi-definite, we conclude, by applying Theorem 2.3 from [21], that $\mathbf{L}_{(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})}$ satisfies a uniform global inf-sup condition. This means that there exists $C_0 > 0$, independent of $(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})$, such that

$$
\begin{aligned}
||(\tilde{\mathbf{t}}, \tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\sigma}}, \tilde{\mathbf{u}}, \tilde{\gamma}, \tilde{\boldsymbol{\eta}})||_{\mathbf{H}} \leq C_0 \sup_{\substack{(\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\xi}) \in \mathbf{H} \\ ||(\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\xi})|| \leq 1}} & \Big\{ D\mathbf{A}_1(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})((\tilde{\mathbf{t}}, \tilde{\boldsymbol{\phi}}), (\mathbf{s}, \boldsymbol{\psi})) \\
& + [\mathbf{B}_1(\tilde{\mathbf{t}}, \tilde{\boldsymbol{\phi}}), \boldsymbol{\tau}] + [\mathbf{B}_1(\mathbf{s}, \boldsymbol{\psi}), \tilde{\boldsymbol{\sigma}}] - [\mathbf{S}(\tilde{\boldsymbol{\sigma}}), \boldsymbol{\tau}] \\
& + [\mathbf{B}(\tilde{\boldsymbol{\sigma}}), (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\xi})] + [\mathbf{B}(\boldsymbol{\tau}), (\tilde{\mathbf{u}}, \tilde{\gamma}, \tilde{\boldsymbol{\eta}})] \Big\}
\end{aligned}
\tag{3.25}
$$

for all $(\tilde{\mathbf{t}}, \tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\sigma}}, \tilde{\mathbf{u}}, \tilde{\gamma}, \tilde{\boldsymbol{\eta}}) \in \mathbf{H}$ and for all $(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}}) \in X_1$.

Now, since $D\mathbf{A}_1$ is continuous, we deduce that there exists $(\tilde{\mathbf{r}}_h, \tilde{\boldsymbol{\varrho}}_h) \in X_1$, which is a convex combination of $(\mathbf{t}, \boldsymbol{\phi})$ and $(\mathbf{t}_h, \boldsymbol{\phi}_h)$, such that

$$
[\mathbf{A}_1(\mathbf{t}, \boldsymbol{\phi}) - \mathbf{A}_1(\mathbf{t}_h, \boldsymbol{\phi}_h), (\mathbf{s}, \boldsymbol{\psi})] = D\mathbf{A}_1(\tilde{\mathbf{r}}_h, \tilde{\boldsymbol{\varrho}}_h)((\mathbf{t} - \mathbf{t}_h, \boldsymbol{\phi} - \boldsymbol{\phi}_h), (\mathbf{s}, \boldsymbol{\psi})) \tag{3.26}
$$

for all $(\mathbf{s}, \boldsymbol{\psi}) \in X_1$.

Then, applying (3.25) with $(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}}) = (\tilde{\mathbf{r}}_h, \tilde{\boldsymbol{\varrho}}_h)$ and $(\tilde{\mathbf{t}}, \tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\sigma}}, \tilde{\mathbf{u}}, \tilde{\gamma}, \tilde{\boldsymbol{\eta}})$ given by the error, and using (3.26), the definition of the operator $\mathbf{A}$, the Ritz projection $(\bar{\mathbf{t}}, \bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\sigma}})$ introduced in Theorem 3.1, the second equation of (2.5), and the definitions of $\mathbf{B}$ and $\mathbf{F}$, we arrive to

$$
\begin{aligned}
& \frac{1}{C_0} ||(\mathbf{t}, \boldsymbol{\phi}, \boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\eta}) - (\mathbf{t}_h, \boldsymbol{\phi}_{,h} \, \boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\eta}_h)||_{\mathbf{H}} \\
\leq & \sup_{\substack{(\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\xi}) \in \mathbf{H} \\ ||(\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\xi})|| \leq 1}} \Big\{ [\hat{\mathbf{A}}(\bar{\mathbf{t}}, \bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\sigma}}), (\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})] + \int_\Omega (\mathbf{f} + \mathbf{div}\, \boldsymbol{\sigma}_h) \cdot \mathbf{v} \, dx \\
& + \langle \boldsymbol{\sigma}_h \boldsymbol{\nu} - \mathbf{g}, \boldsymbol{\xi} \rangle_{\Gamma_N} + \int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\delta} \, dx \Big\} .
\end{aligned}
\tag{3.27}
$$

Next, the boundedness and ellipticity of $\hat{\mathbf{A}}$, and the estimate from Theorem 3.1, imply

$$
|[\hat{\mathbf{A}}(\bar{\mathbf{t}}, \bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\sigma}}), (\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})]| \leq ||\hat{\mathbf{A}}|| \; ||(\bar{\mathbf{t}}, \bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\sigma}})||_X \, ||(\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})||_X
$$

$$\leq C \left\{ \sum_{T \in \mathcal{T}_h} (||\hat{\mathbf{t}}_T||^2_{[L^2(T)]^{2\times 2}} + ||\hat{\boldsymbol{\sigma}}_T||^2_{\mathbf{H}(\mathbf{div}\,;T)}) + R_\Gamma^2 \right\}^{1/2} ||(\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})||_X \quad (3.28)$$

for all $(\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau}) \in X$.

On the other hand, it is easily seen that

$$\left| \int_\Omega (\mathbf{f} + \mathbf{div}\, \boldsymbol{\sigma}_h) \cdot \mathbf{v}\, dx \right|$$

$$\leq \left\{ \sum_{T \in \mathcal{T}_h} ||\mathbf{f} + \mathbf{div}\, \boldsymbol{\sigma}_{h,T}||^2_{[L^2(T)]^2} \right\}^{1/2} ||\mathbf{v}||_{[L^2(\Omega)]^2}\,, \quad (3.29)$$

$$|\,\langle \boldsymbol{\sigma}_h \boldsymbol{\nu} - \mathbf{g}, \boldsymbol{\xi} \rangle_{\Gamma_N}\,| \;\leq\; ||\boldsymbol{\sigma}_h \boldsymbol{\nu} - \mathbf{g}||_{[H^{-1/2}(\Gamma_N)]^2}\, ||\boldsymbol{\xi}||_{[\tilde{H}^{1/2}(\Gamma_N)]^2}\,, \quad (3.30)$$

and

$$\left| \int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\delta}\, dx \right| = \left| \frac{1}{2} \int_\Omega (\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathbf{T}}) : \boldsymbol{\delta}\, dx \right|$$

$$\leq \frac{1}{2} \left\{ \sum_{T \in \mathcal{T}_h} ||\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathbf{T}}||^2_{[L^2(T)]^{2\times 2}} \right\}^{1/2} ||\boldsymbol{\delta}||_{[L^2(\Omega)]^{2\times 2}}\,, \quad (3.31)$$

for all $\mathbf{v} \in [L^2(\Omega)]^2$, $\boldsymbol{\xi} \in [\tilde{H}^{1/2}(\Gamma_N)]^2$, $\boldsymbol{\delta} \in \mathcal{R}$. Consequently, replacing (3.28), (3.29), (3.30) and (3.31) back into (3.27), we obtain the a-posteriori error estimate and conclude the proof of the theorem. □

We now comment on the importance of choosing $\boldsymbol{\varphi}_h$ as close as possible to the trace of $\mathbf{u}$ on $\cup_{T \in \mathcal{T}_h} \partial T$. In fact, if $\mathbf{u} \in [H^1(\Omega)]^2$ and $\boldsymbol{\varphi}_h|_{\partial T} = \mathbf{u}|_{\partial T}\ \forall\ \in \mathcal{T}_h$, we get

$$\int_{\partial T} \boldsymbol{\varphi}_h \cdot \boldsymbol{\tau} \boldsymbol{\nu}_T\, ds = \int_{\partial T} \mathbf{u} \cdot \boldsymbol{\tau} \boldsymbol{\nu}_T\, ds = \int_T \nabla \mathbf{u} : \boldsymbol{\tau}\, dx + \int_T \mathbf{u} \cdot \mathbf{div}\, \boldsymbol{\tau}\, dx\,,$$

and hence the local problem (3.5) becomes

$$\langle \hat{\boldsymbol{\sigma}}_T, \boldsymbol{\tau} \rangle_{\mathbf{H}(\mathbf{div}\,;T)} \;=\; F_{h,T}(\boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathbf{H}(\mathbf{div}\,;T)\,, \quad (3.32)$$

with

$$F_{h,T}(\boldsymbol{\tau}) = \int_T (\mathbf{t}_{h,T} + \boldsymbol{\gamma}_{h,T} - \nabla \mathbf{u}) : \boldsymbol{\tau}\, dx + \int_T (\mathbf{u}_{h,T} - \mathbf{u}) \cdot \mathbf{div}\, \boldsymbol{\tau}\, dx$$

$$+ \langle \boldsymbol{\tau} \boldsymbol{\nu}, \mathbf{u} + \boldsymbol{\eta}_h \rangle_{\partial T \cap \Gamma_N} \;+\; \langle \boldsymbol{\tau} \boldsymbol{\nu}, \mathbf{u} + \mathbf{V}(\boldsymbol{\sigma}_h \boldsymbol{\nu}) - \left(\frac{1}{2} \mathbf{I} + \mathbf{K} \right) \boldsymbol{\phi}_h \rangle_{\partial T \cap \Gamma}\,. \quad (3.33)$$

This shows that $\hat{\boldsymbol{\sigma}}_T \in \mathbf{H}(\mathbf{div}\,;T)$, the Riesz representant of the functional $F_{h,T} \in \mathbf{H}(\mathbf{div}\,;T)'$, depends directly on the actual errors $(\mathbf{t}_{h,T} + \boldsymbol{\gamma}_{h,T} - \nabla \mathbf{u}) \in [L^2(T)]^{2\times 2}$, $(\mathbf{u}_{h,T} - \mathbf{u}) \in [L^2(T)]^2$, $(\mathbf{u} + \boldsymbol{\eta}_h) \in [\tilde{H}^{1/2}(\partial T \cap \Gamma_N)]^2$, and

$\left(\mathbf{u} + \mathbf{V}(\boldsymbol{\sigma}_h \boldsymbol{\nu}) - \left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right) \boldsymbol{\phi}_h\right) \in [H^{1/2}(\partial T \cap \Gamma)]^2$, where $[\tilde{H}^{1/2}(\partial T \cap \Gamma_N)]^2 := \{\boldsymbol{\delta}|_{\partial T \cap \Gamma_N} : \quad \boldsymbol{\delta} \in [\tilde{H}^{1/2}(\Gamma_N)]^2\}$.

Moreover, it follows from (3.32) and (3.33) that

$$\begin{aligned} ||\hat{\boldsymbol{\sigma}}_T||_{\mathbf{H}(\mathbf{div}\,;T)} \;\le\; \Big\{ & ||\mathbf{t}_{h,T} \,+\, \boldsymbol{\gamma}_{h,T} - \nabla \mathbf{u}||^2_{[L^2(T)]^{2\times 2}} \\ & + ||\mathbf{u}_{h,T} - \mathbf{u}||^2_{[L^2(T)]^2} \,+\, ||\mathbf{u} + \boldsymbol{\eta}_h||^2_{[\tilde{H}^{1/2}(\partial T \cap \Gamma_N)]^2} \\ & + ||\mathbf{u} + \mathbf{V}(\boldsymbol{\sigma}_h \boldsymbol{\nu}) - \left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right) \boldsymbol{\phi}_h||^2_{[H^{1/2}(\partial T \cap \Gamma)]^2} \Big\}^{1/2} . \end{aligned} \tag{3.34}$$

The above analysis motivates the following lemma, which shows that the efficiency of the a-posteriori error estimate given by Theorem 3.3 depends directly on how well $\boldsymbol{\varphi}_h$ approximates the trace of $\mathbf{u}$ on the element boundaries.

Lemma 3.4. *Assume that* $\mathbf{u} \in [H^1(\Omega)]^2$ *and define* $\boldsymbol{\varphi}_h|_{\partial T} = \mathbf{u}|_{\partial T}$ $\forall\, T \in \mathcal{T}_h$. *Then there exists* $C > 0$*, independent of* h*, such that for all* $T \in \mathcal{T}_h$

$$\begin{aligned} \boldsymbol{\theta}_T^2 \;\le\; C \Big\{ & ||\mathbf{t}_{h,T} - \mathbf{t}||^2_{[L^2(T)]^{2\times 2}} + ||\boldsymbol{\sigma}_{h,T} - \boldsymbol{\sigma}||^2_{\mathbf{H}(\mathbf{div}\,;T)} + ||\boldsymbol{\gamma}_{h,T} - \boldsymbol{\gamma}||^2_{[L^2(T)]^{2\times 2}} \\ & + ||\mathbf{u}_{h,T} - \mathbf{u}||^2_{[L^2(T)]^2} \,+\, ||\boldsymbol{\eta}_h - \boldsymbol{\eta}||^2_{[\tilde{H}^{1/2}(\partial T \cap \Gamma_N)]^2} \\ & + \left|\left| \left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right)(\boldsymbol{\phi}_h - \boldsymbol{\phi}) \right|\right|^2_{[H^{1/2}(\partial T \cap \Gamma)]^2} + ||\mathbf{V}(\boldsymbol{\sigma}_h \boldsymbol{\nu} - \boldsymbol{\sigma}\boldsymbol{\nu})||^2_{[H^{1/2}(\partial T \cap \Gamma)]^2} \Big\} . \end{aligned} \tag{3.35}$$

In addition, there exists $\tilde{C} > 0$*, independent of* h*, such that*

$$\left\{ \sum_{T \in \mathcal{T}_h} \boldsymbol{\theta}_T^2 + R_\Gamma^2 + R_{\Gamma_N}^2 \right\}^{1/2} \;\le\; \tilde{C}\, ||\vec{\mathbf{t}} - \vec{\mathbf{t}}_h||_{\mathbf{H}} , \tag{3.36}$$

where $\vec{\mathbf{t}} := (\mathbf{t}, \boldsymbol{\phi}, \boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\eta})$*,* $\vec{\mathbf{t}}_h := (\mathbf{t}_h, \boldsymbol{\phi}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\eta}_h)$*, and* $\boldsymbol{\theta}_T$*,* R_Γ*,* R_{Γ_N} *are defined in Theorem* 3.3*.*

Proof. By taking $\mathbf{s} = \boldsymbol{\psi} = 0$ in the first equation of (2.5), and then integrating by parts in Ω, we get

$$\begin{aligned} & \int_\Omega [\nabla \mathbf{u} - (\mathbf{t} + \boldsymbol{\gamma})] : \boldsymbol{\tau}\, dx \;-\; \langle \boldsymbol{\tau}\boldsymbol{\nu}, \mathbf{u} + \boldsymbol{\eta} \rangle_{\Gamma_N} \\ & + \langle \boldsymbol{\tau}\boldsymbol{\nu}, \left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right) \boldsymbol{\phi} - \mathbf{V}(\boldsymbol{\sigma}\boldsymbol{\nu}) - \mathbf{u} \rangle_\Gamma \;=\; 0 \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H_0}(\mathbf{div}\,;\Omega) . \end{aligned}$$

It follows that

$$\nabla \mathbf{u} = \mathbf{t} + \boldsymbol{\gamma} \quad \text{in} \quad \Omega , \qquad \mathbf{u} = -\boldsymbol{\eta} \quad \text{on} \quad \Gamma_N ,$$

and

$$\mathbf{u} = \left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right)\boldsymbol{\phi} - \mathbf{V}(\boldsymbol{\sigma\nu}) \quad \text{on} \quad \Gamma\,,$$

whence (3.34) yields

$$\begin{aligned} ||\hat{\boldsymbol{\sigma}}_T||^2_{\mathbf{H}(\mathbf{div}\,;T)} \;\le\; 2\Bigg\{ & ||\mathbf{t}_{h,T} - \mathbf{t}||^2_{[L^2(T)]^{2\times 2}} \,+\, ||\boldsymbol{\gamma}_{h,T} - \boldsymbol{\gamma}||^2_{[L^2(T)]^{2\times 2}} \\ & + ||\mathbf{u}_{h,T} - \mathbf{u}||^2_{[L^2(T)]^2} \,+\, ||\boldsymbol{\eta}_h - \boldsymbol{\eta}||^2_{[\tilde{H}^{1/2}(\partial T\cap\Gamma_N)]^2} \qquad (3.37) \\ + \left|\left|\left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right)(\boldsymbol{\phi}_h - \boldsymbol{\phi})\right|\right|^2_{[H^{1/2}(\partial T\cap\Gamma)]^2} & + ||\mathbf{V}(\boldsymbol{\sigma}_h\boldsymbol{\nu} - \boldsymbol{\sigma\nu})||^2_{[H^{1/2}(\partial T\cap\Gamma)]^2} \Bigg\}\,. \end{aligned}$$

Now, taking $\boldsymbol{\delta} = \boldsymbol{\xi} = 0$ in the second equation of (2.5), and $\boldsymbol{\psi} = \boldsymbol{\tau} = 0$ in the first one, we obtain, respectively,

$$-\mathbf{div}\,\boldsymbol{\sigma} = \mathbf{f} \quad \text{and} \quad \boldsymbol{\sigma} = \hat{\lambda}(\mathbf{t})\,\mathrm{tr}\,(\mathbf{t})\,\mathbf{I}_2 + 2\,\hat{\mu}(\mathbf{t})\,\mathbf{t} \quad \text{in} \quad \Omega\,.$$

Then we have

$$||\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_{h,T}||_{[L^2(T)]^2} \;=\; ||\mathbf{div}\,(\boldsymbol{\sigma}_{h,T} - \boldsymbol{\sigma})||_{[L^2(T)]^2}\,, \qquad (3.38)$$

and also

$$||\hat{\mathbf{t}}_T||_{[L^2(T)]^{2\times 2}} \;\le\; ||\boldsymbol{\sigma}_{h,T} - \boldsymbol{\sigma}||_{[L^2(T)]^{2\times 2}} \,+\, ||\tilde{\mathbf{A}}_{1,T}(\mathbf{t}_{h,T}) \,-\, \tilde{\mathbf{A}}_{1,T}(\mathbf{t})||_{[L^2(T)]^{2\times 2'}}\,,$$

where $\tilde{\mathbf{A}}_{1,T} : [L^2(T)]^{2\times 2} \to [L^2(T)]^{2\times 2'}$ is the restriction to $T \in \mathcal{T}_h$ of $\tilde{\mathbf{A}}_1$ (cf. proof of Lemma 3.2). For the definition of $\hat{\mathbf{t}}_T$ see Theorem 3.3.

Since $\tilde{\mathbf{A}}_{1,T}$ is Lipschitz continuous, we deduce that there exists $C > 0$, independent of h and T, such that

$$||\hat{\mathbf{t}}_T||_{[L^2(T)]^{2\times 2}} \;\le\; C\,\left\{ ||\boldsymbol{\sigma}_{h,T} - \boldsymbol{\sigma}||_{[L^2(T)]^{2\times 2}} \,+\, ||\mathbf{t}_{h,T} - \mathbf{t}||_{[L^2(T)]^{2\times 2}} \right\}\,. \qquad (3.39)$$

Next, taking $\mathbf{v} = \boldsymbol{\xi} = 0$ in the second equation of (2.5) we get $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathbf{T}}$ in Ω, with which one can prove that

$$||\boldsymbol{\sigma}_{h,T} \,-\, \boldsymbol{\sigma}^{\mathbf{T}}_{h,T}||_{[L^2(T)]^{2\times 2}} \;\le\; 2\,||\boldsymbol{\sigma}_{h,T} \,-\, \boldsymbol{\sigma}||_{[L^2(T)]^{2\times 2}}\,. \qquad (3.40)$$

Thus, (3.37), (3.38), (3.39), (3.40), and the definition of $\boldsymbol{\theta}_T^2$, yield (3.35).

On the other hand, we also find from the variational formulation (2.5) that

$$\mathbf{W}\boldsymbol{\phi} + \left(\frac{1}{2}\mathbf{I} + \mathbf{K}'\right)(\boldsymbol{\sigma\nu}) = 0 \quad \text{on} \quad \Gamma, \quad \text{and} \quad \boldsymbol{\sigma\nu} = \mathbf{g} \quad \text{on} \quad \Gamma_N\,,$$

which, using the continuity of the boundary integral operators $\mathbf{W}$ and $\mathbf{K}'$, imply that

$$R_\Gamma^2 \,+\, R_{\Gamma_N}^2 \;\le\; C\,\left\{ ||\boldsymbol{\phi}_h - \boldsymbol{\phi}||^2_{[H^{1/2}(\Gamma)]^2} \,+\, ||\boldsymbol{\sigma}_h - \boldsymbol{\sigma}||^2_{\mathbf{H}(\mathbf{div}\,;\Omega)} \right\}\,. \qquad (3.41)$$

Next, according to the definition of the Sobolev norm of order 1/2, we see that

$$\sum_{T\in\mathcal{T}_h} ||\boldsymbol{\eta}_h - \boldsymbol{\eta}||^2_{[\tilde{H}^{1/2}(\partial T\cap\Gamma_N)]^2} \leq C\, ||\boldsymbol{\eta}_h - \boldsymbol{\eta}||^2_{[\tilde{H}^{1/2}(\Gamma_N)]^2}, \tag{3.42}$$

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} \left|\left|\left(\frac{1}{2}\mathbf{I}+\mathbf{K}\right)(\boldsymbol{\phi}_h - \boldsymbol{\phi})\right|\right|^2_{[H^{1/2}(\partial T\cap\Gamma)]^2} &\leq C\,\left|\left|\left(\frac{1}{2}\mathbf{I}+\mathbf{K}\right)(\boldsymbol{\phi}_h - \boldsymbol{\phi})\right|\right|^2_{[H^{1/2}(\Gamma)]^2}\\ &\leq\; C\,||\boldsymbol{\phi}_h - \boldsymbol{\phi}||^2_{[H^{1/2}(\Gamma)]^2},\end{aligned} \tag{3.43}$$

and

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} ||\mathbf{V}(\boldsymbol{\sigma}_h\boldsymbol{\nu} - \boldsymbol{\sigma}\boldsymbol{\nu})||^2_{[H^{1/2}(\partial T\cap\Gamma)]^2} &\leq C\,||\mathbf{V}(\boldsymbol{\sigma}_h\boldsymbol{\nu} - \boldsymbol{\sigma}\boldsymbol{\nu})||^2_{[H^{1/2}(\Gamma)]^2}\\ &\leq\; C\;||\boldsymbol{\sigma}_h\;-\;\boldsymbol{\sigma}||^2_{\mathbf{H}(\mathbf{div}\,;\Omega)},\end{aligned} \tag{3.44}$$

where the boundedness of $\mathbf{K}$ and $\mathbf{V}$ has also been used for the derivation of (3.43) and (3.44).

Finally, the efficiency estimate (3.36) is a simple consequence of (3.35), (3.41), (3.42), (3.43) and (3.44). This finishes the proof. □

Before ending this subsection we remark that the a-posteriori error estimate provided by Theorem 3.3 includes the non-local terms R_Γ and R_{Γ_N}, which, though of explicit residual type, are defined in terms of negative order Sobolev norms. In addition, it is also important to observe that the local problem (3.5) holds in a space of infinite dimension, and hence it could only be solved approximately by replacing $\mathbf{H}(\mathbf{div}\,;T)$ by a finite dimensional subspace. In particular, since that problem is linear, we suggest to apply the p or the $h-p$ version to approximate $\hat{\boldsymbol{\sigma}}_T$ (see, e.g., [1]).

According to the above, the purpose of the next subsection is to bound the contributions R_Γ, R_{Γ_N}, and $||\hat{\boldsymbol{\sigma}}_T||_{\mathbf{H}(\mathbf{div}\,;T)}$ by easily computable expressions.

3.3. Fully local a-posteriori error estimates

We first need to specify the finite dimensional subspace $\mathbf{H}_h$ to which the approximate solution $(\mathbf{t}_h, \boldsymbol{\phi}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\eta}_h)$ belongs. To this end we consider the same finite element and boundary element spaces from [20] (see also [7]), which are a natural extension of the PEERS space from [5]. In fact, let $X_{1,h} := X^{\mathbf{t}}_{1,h}\times X^{\boldsymbol{\phi}}_{1,h}$, $M_{1,h}$, and $M_h := M^{\mathbf{u}}_h\times\mathcal{R}_h\times M^{\boldsymbol{\eta}}_h$ denote the corresponding subspaces to approximate $(\mathbf{t},\boldsymbol{\phi})$, $\boldsymbol{\sigma}$, and $(\mathbf{u},\boldsymbol{\gamma},\boldsymbol{\eta})$, respectively. Further, let $\{\Gamma_1, ..., \Gamma_m\}$ and $\{\tilde{\Gamma}_1, ..., \tilde{\Gamma}_n\}$ be the partitions on Γ and Γ_N, respectively, induced by the triangulation $\mathcal{T}_h$. Then we define

$$X^{\mathbf{t}}_{1,h} := \{\text{piecewise RT of order } 0 + \text{rotations of cubic bubble functions}\},$$

$$X_{1,h}^{\boldsymbol{\phi}} := [H^{1/2}(\Gamma)]_h^2 / \mathcal{N}_\Gamma(\mathbf{e})$$

or, equivalently

$$X_{1,h}^{\boldsymbol{\phi}} := \{ \boldsymbol{\psi}_h \in [H^{1/2}(\Gamma)]_h^2 : \quad \langle \boldsymbol{\omega}, \boldsymbol{\psi}_h \rangle_\Gamma = 0 \quad \forall\, \boldsymbol{\omega} \in \mathcal{N}_\Gamma(\mathbf{e}) \} \,,$$

where

$$[H^{1/2}(\Gamma)]_h^2 := \{ \boldsymbol{\psi}_h \in [C(\Gamma)]^2 : \quad \boldsymbol{\psi}_h|_{\Gamma_j} \in [\mathbf{P}_1(\Gamma_j)]^2 \quad \forall\, j \in \{1, ..., m\} \} \,,$$

$$M_{1,h} := \{\text{RT of order } 0 + \text{rotations of cubic bubble functions}\} \cap \mathbf{H_0}(\mathbf{div}\,; \Omega) \,,$$

$$M_h^{\mathbf{u}} := \{ \mathbf{v}_h \in [L^2(\Omega)]^2 : \quad \mathbf{v}_h|_T \in [\mathbf{P}_0(T)]^2 \quad \forall\, T \in \mathcal{T}_h \} \,,$$

$$\mathcal{R}_h := \left\{ \begin{bmatrix} 0 & \delta \\ -\delta & 0 \end{bmatrix} \in [H^1(\Omega)]^{2\times 2} : \quad \delta|_T \in \mathbf{P}_1(T) \;\forall\, T \in \mathcal{T}_h \right\} \,,$$

and

$$M_h^{\boldsymbol{\eta}} := \{ \boldsymbol{\xi}_h \in [\tilde{H}^{1/2}(\Gamma_N)]^2 : \quad \boldsymbol{\xi}_h|_{\tilde{\Gamma}_j} \in [\mathbf{P}_1(\tilde{\Gamma}_j)]^2 \quad \forall\, j \in \{1, ..., n\} \} \,.$$

Hereafter, given integers $l \geq 0$, $k \in \{1, 2\}$ and a subset S of $\mathbf{R}^k$, $\mathbf{P}_l(S)$ stands for the space of polynomials in k variables defined in S of total degree at most l.

Therefore, the Galerkin scheme associated to the variational formulation (2.5) reads: *Find* $((\mathbf{t}_h, \boldsymbol{\phi}_h), \boldsymbol{\sigma}_h, (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\eta}_h)) \in \mathbf{H}_h := X_{1,h} \times M_{1,h} \times M_h$ *such that*

$$\begin{aligned} [\mathbf{A}(\mathbf{t}_h, \boldsymbol{\phi}_h, \boldsymbol{\sigma}_h), (\mathbf{s}_h, \boldsymbol{\psi}_h, \boldsymbol{\tau}_h)] \,+\, [\mathbf{B}(\boldsymbol{\tau}_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\eta}_h)] = 0 \,, \\ [\mathbf{B}(\boldsymbol{\sigma}_h), (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\xi}_h)] \,=\, [\mathbf{F}, (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\xi}_h)] \,, \end{aligned} \tag{3.45}$$

for all $((\mathbf{s}_h, \boldsymbol{\psi}_h), \boldsymbol{\tau}_h, (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\xi}_h)) \in \mathbf{H}_h$.

We remark that the existence of a unique $((\mathbf{t}_h, \boldsymbol{\phi}_h), \boldsymbol{\sigma}_h, (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\eta}_h)) \in \mathbf{H}_h$ solution to (3.45), and the corresponding error analysis, were provided in [20] for the particular case $\Gamma_N = \emptyset$. Hence, we do not give any further details here and just assume, from now on, unique solvability of the discrete scheme (3.45).

Lemma 3.5. *Assume that the Neumann data* $\mathbf{g} \in [L^2(\Gamma_N)]^2$*. Then there exists* $C > 0$*, independent of* h*, such that*

$$R_\Gamma^2 \,\leq\, C\, \log[1 + C_h(\Gamma)] \sum_{j=1}^{m} h_j\, ||\mathbf{W}\boldsymbol{\phi}_h + \left(\frac{1}{2}\mathbf{I} + \mathbf{K}'\right)(\boldsymbol{\sigma}_h \boldsymbol{\nu})||^2_{[L^2(\Gamma_j)]^2} \,, \tag{3.46}$$

and

$$R_{\Gamma_N}^2 \,\leq\, C\, \log[1 + C_h(\Gamma_N)] \sum_{j=1}^{n} \tilde{h}_j\, ||\boldsymbol{\sigma}_h \boldsymbol{\nu} \,-\, \mathbf{g}||^2_{[L^2(\tilde{\Gamma}_j)]^2} \,, \tag{3.47}$$

where h_j = *length of* Γ_j, $\tilde{h}_j$ = *length of* $\tilde{\Gamma}_j$,

$$C_h(\Gamma) := \max \left\{ \frac{h_i}{h_j} : \quad |i-j| = 1\,, \quad i,j \in \{1,...,m\} \right\} ,$$

and

$$C_h(\Gamma_N) := \max \left\{ \frac{\tilde{h}_i}{\tilde{h}_j} : \quad |i-j| = 1\,, \quad i,j \in \{1,...,n\} \right\} .$$

Proof. We proceed similarly as in Corollary 1 of [22]. First, the definitions of the subspaces $X^{\boldsymbol{\phi}}_{1,h}$ and $M_{1,h}$ guarantee that $\boldsymbol{\phi}_h \in [H^1(\Gamma)]^2$, $(\boldsymbol{\sigma}_h\boldsymbol{\nu})|_\Gamma \in [L^2(\Gamma)]^2$, and $(\boldsymbol{\sigma}_h\boldsymbol{\nu})|_{\Gamma_N} \in [L^2(\Gamma_N)]^2$. Hence, the mapping properties of $\mathbf{W}$ and $\mathbf{K}'$ imply that $\left(\mathbf{W}\boldsymbol{\phi}_h + \left(\frac{1}{2}\mathbf{I} + \mathbf{K}'\right)(\boldsymbol{\sigma}_h\boldsymbol{\nu})\right) \in [L^2(\Gamma)]^2$. Also, taking $\mathbf{s}_h = \boldsymbol{\tau}_h = 0$ in the first equation of (3.45), we get

$$\langle \mathbf{W}\boldsymbol{\phi}_h + \left(\frac{1}{2}\mathbf{I} + \mathbf{K}'\right)(\boldsymbol{\sigma}_h\boldsymbol{\nu}), \boldsymbol{\psi}_h\rangle_\Gamma = 0 \quad \forall\, \boldsymbol{\psi}_h \in X^{\boldsymbol{\phi}}_{1,h} \,.$$

Next, given $\tilde{\boldsymbol{\psi}}_h \in [H^{1/2}(\Gamma)]^2_h$ we let $\boldsymbol{\psi}_h \in X^{\boldsymbol{\phi}}_{1,h}$ be such that $(\tilde{\boldsymbol{\psi}}_h - \boldsymbol{\psi}_h) \in \mathcal{N}_\Gamma(\mathbf{e})$. Since $\langle \mathbf{W}\boldsymbol{\phi}_h, \boldsymbol{\omega}\rangle_\Gamma = \langle \mathbf{W}(\boldsymbol{\omega}), \boldsymbol{\phi}_h\rangle_\Gamma = 0$ and $\langle\left(\frac{1}{2}\mathbf{I} + \mathbf{K}'\right)(\boldsymbol{\sigma}_h\boldsymbol{\nu}), \boldsymbol{\omega}\rangle_\Gamma = \langle \boldsymbol{\sigma}_h\boldsymbol{\nu}, \left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right)(\boldsymbol{\omega})\rangle_\Gamma = \langle \boldsymbol{\sigma}_h\boldsymbol{\nu}, \boldsymbol{\omega}\rangle_\Gamma = 0$ for all $\boldsymbol{\omega} \in \mathcal{N}_\Gamma(\mathbf{e})$, we deduce that $\mathbf{W}\boldsymbol{\phi}_h + \left(\frac{1}{2}\mathbf{I} + \mathbf{K}'\right)(\boldsymbol{\sigma}_h\boldsymbol{\nu})$ is $[L^2(\Gamma)]^2$-orthogonal to $[H^{1/2}(\Gamma)]^2_h$. Thus, the estimate (3.46) follows from a straightforward application of Theorem 2 in [13].

On the other hand, taking $\mathbf{v}_h = \boldsymbol{\delta}_h = 0$ in the second equation of (3.45), we find that $(\boldsymbol{\sigma}_h\boldsymbol{\nu} - \mathbf{g})$ is $[L^2(\Gamma_N)]^2$-orthogonal to $M^{\boldsymbol{\eta}}_h$, and therefore, the estimate (3.47) is also a consequence of Theorem 2 in [13]. □

As a corollary of Theorem 3.3 and Lemma 3.5 we can establish the following a-posteriori error estimate.

Theorem 3.6. *Let* $\boldsymbol{\varphi}_h \in [H^{1/2}(\cup_{T\in\mathcal{T}_h} \partial T)]^2$ *be an approximation to* $\mathbf{u}$ *on element boundaries with* $\boldsymbol{\varphi}_h = 0$ *on* Γ_D*, and assume that* $\mathbf{g} \in [L^2(\Gamma_N)]^2$*. In addition, for each* $T \in \mathcal{T}_h$ *define*

$$\hat{\mathbf{t}}_T := \boldsymbol{\sigma}_{h,T} - \hat{\lambda}(\mathbf{t}_{h,T})\,\mathrm{tr}\,(\mathbf{t}_{h,T})\,\mathbf{I}_2 - 2\,\hat{\mu}(\mathbf{t}_{h,T})\,\mathbf{t}_{h,T}\,,$$

and let $\hat{\boldsymbol{\sigma}}_T \in \mathbf{H}(\mathbf{div}\,;T)$ *be the unique solution of the local problem* (3.5)*. Then there exists* $C > 0$*, independent of* h*, such that*

$$||(\mathbf{t}, \boldsymbol{\phi}, \boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\eta}) - (\mathbf{t}_h, \boldsymbol{\phi}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\eta}_h)||_{\mathbf{H}} \leq C \left\{ \sum_{T\in\mathcal{T}_h} \hat{\boldsymbol{\theta}}^2_T \right\}^{1/2} ,$$

where

$$\hat{\boldsymbol{\theta}}_T^2 := ||\hat{\mathbf{t}}_T||^2_{[L^2(T)]^{2\times 2}} + ||\hat{\boldsymbol{\sigma}}_T||^2_{\mathbf{H}(\mathbf{div}\,;T)} + ||\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_{h,T}||^2_{[L^2(T)]^2}$$

$$+ ||\boldsymbol{\sigma}_{h,T} - \boldsymbol{\sigma}_{h,T}^{\mathbf{T}}||^2_{[L^2(T)]^{2\times 2}} + \log[1 + C_h(\Gamma_N)]\, h_T\, ||\boldsymbol{\sigma}_h\boldsymbol{\nu} - \mathbf{g}||^2_{[L^2(\partial T\cap\Gamma_N)]^2}$$

$$+ \log[1 + C_h(\Gamma)]\, h_T\, ||\mathbf{W}\boldsymbol{\phi}_h + \left(\frac{1}{2}\mathbf{I} + \mathbf{K}'\right)(\boldsymbol{\sigma}_h\boldsymbol{\nu})||^2_{[L^2(\partial T\cap\Gamma)]^2}\,.$$

Proof. It follows from Theorem 3.3 and Lemma 3.5 by noting that h_j (resp. $\tilde{h}_j$) is smaller than h_T, where $T \in \mathcal{T}_h$ is the triangle having Γ_j (resp. $\tilde{\Gamma}_j$) as a boundary edge. □

We remark that the upper bounds for R_Γ and R_{Γ_N} given by Lemma 3.5 are also lower bounds (with different constants) as long as the partitions on Γ and Γ_N are quasi-uniform (see Theorem 3 in [13]).

To end this section, we set an appropriate choice of $\boldsymbol{\varphi}_h$. Indeed, we first define $\tilde{\boldsymbol{\varphi}}_h \in [C(\bar{\Omega})]^2$ as the unique function satisfying the following conditions:

a) $\tilde{\boldsymbol{\varphi}}_{h,T} := \tilde{\boldsymbol{\varphi}}_h|_T \in [\mathbf{P}_1(T)]^2$ for all $T \in \mathcal{T}_h$.
b) $\tilde{\boldsymbol{\varphi}}_h(\bar{\mathbf{x}}) = (0,0)^{\mathbf{T}}$ for each vertex $\bar{\mathbf{x}}$ of $\mathcal{T}_h$ lying on $\bar{\Gamma}_D$.
c) $\tilde{\boldsymbol{\varphi}}_h(\bar{\mathbf{x}})$ is the average of the constant values of $\mathbf{u}_h$ on all the triangles $T \in \mathcal{T}_h$ to which $\bar{\mathbf{x}}$ belongs, for any other vertex $\bar{\mathbf{x}}$ of $\mathcal{T}_h$.

Then, since $\tilde{\boldsymbol{\varphi}}_h \in [H^1(\Omega)]^2$ we just put

$$\boldsymbol{\varphi}_h|_{\partial T} = \tilde{\boldsymbol{\varphi}}_h|_{\partial T} \qquad \forall\, T \in \mathcal{T}_h\,. \tag{3.48}$$

Furthermore, we can also use $\tilde{\boldsymbol{\varphi}}_h$ to derive a simpler fully local a-posteriori error estimate, which does not require the explicit solution of the problem (3.5). More precisely, we have the following result.

Theorem 3.7. *Let $\tilde{\boldsymbol{\varphi}}_h \in [H^1(\Omega)]^2$ be as indicated above, assume that $\mathbf{g} \in [L^2(\Gamma_N)]^2$, and for each $T \in \mathcal{T}_h$ define*

$$\hat{\mathbf{t}}_T := \boldsymbol{\sigma}_{h,T} - \hat{\lambda}(\mathbf{t}_{h,T})\,\mathrm{tr}\,(\mathbf{t}_{h,T})\,\mathbf{I}_2 - 2\,\hat{\mu}(\mathbf{t}_{h,T})\,\mathbf{t}_{h,T}\,.$$

In addition, let $\{\bar{\mathbf{x}}_{k,T}\}_{k=1}^3$ be the vertices of $T \in \mathcal{T}_h$, and assume, without loss of generality, that $\bar{\mathbf{x}}_{1,T}$, $\bar{\mathbf{x}}_{2,T} \in \Gamma_N$ when $\partial T \cap \Gamma_N$ is not empty. Then there exists $C > 0$, independent of h, such that

$$||(\mathbf{t},\boldsymbol{\phi},\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\gamma},\boldsymbol{\eta}) - (\mathbf{t}_h,\boldsymbol{\phi}_h,\boldsymbol{\sigma}_h,\mathbf{u}_h,\boldsymbol{\gamma}_h,\boldsymbol{\eta}_h)||_{\mathbf{H}}$$

$$\leq C \left\{ \sum_{T\in\mathcal{T}_h} \left(\tilde{\boldsymbol{\theta}}_T^2 + \tilde{\boldsymbol{\theta}}_{T,\Gamma}^2 + \tilde{\boldsymbol{\theta}}_{T,\Gamma_N}^2 \right) \right\}^{1/2}, \tag{3.49}$$

where

$$\tilde{\boldsymbol{\theta}}_T^2 := \left\{ ||\hat{\mathbf{t}}_T||^2_{[L^2(T)]^{2\times 2}} + \left|\left|\mathbf{t}_{h,T} - \frac{1}{2}\left[\nabla\tilde{\boldsymbol{\varphi}}_{h,T} + (\nabla\tilde{\boldsymbol{\varphi}}_{h,T})^{\mathbf{T}}\right]\right|\right|^2_{[L^2(T)]^{2\times 2}} \right.$$

$$+ h_T^2 \sum_{k=1}^{3} \left|\left| \boldsymbol{\gamma}_{h,T}(\bar{\mathbf{x}}_{k,T}) - \frac{1}{2}\left[\nabla\tilde{\boldsymbol{\varphi}}_{h,T} - (\nabla\tilde{\boldsymbol{\varphi}}_{h,T})^{\mathbf{T}}\right] \right|\right|^2_{\mathrm{R}^{2\times 2}}$$

$$+ h_T^2 \sum_{k=1}^{3} ||\mathbf{u}_{h,T} - \tilde{\boldsymbol{\varphi}}_{h,T}(\bar{\mathbf{x}}_{k,T})||^2_{\mathrm{R}^2}$$

$$\left. + ||\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_{h,T}||^2_{[L^2(T)]^2} + ||\boldsymbol{\sigma}_{h,T} - \boldsymbol{\sigma}^{\mathbf{T}}_{h,T}||^2_{[L^2(T)]^{2\times 2}} \right\},$$

$$\tilde{\boldsymbol{\theta}}^2_{T,\Gamma} := \left\{ \left|\left| \tilde{\boldsymbol{\varphi}}_{h,T} + \mathbf{V}(\boldsymbol{\sigma}_h\boldsymbol{\nu}) - \left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right)\boldsymbol{\phi}_h \right|\right|^2_{[H^{1/2}(\partial T\cap\Gamma)]^2} \right.$$

$$\left. + \log[1 + C_h(\Gamma)]\, h_T\, ||\mathbf{W}\boldsymbol{\phi}_h + \left(\frac{1}{2}\mathbf{I} + \mathbf{K}'\right)(\boldsymbol{\sigma}_h\boldsymbol{\nu})||^2_{[L^2(\partial T\cap\Gamma)]^2} \right\},$$

and

$$\tilde{\boldsymbol{\theta}}^2_{T,\Gamma_N} := p_N(T)\left\{ h_T \sum_{k=1}^{2} ||(\tilde{\boldsymbol{\varphi}}_{h,T} + \boldsymbol{\eta}_h)(\bar{\mathbf{x}}_{k,T})||^2_{\mathrm{R}^2} + ||(\tilde{\boldsymbol{\varphi}}_{h,T} + \boldsymbol{\eta}_h)(\bar{\mathbf{x}}_{2,T} - \bar{\mathbf{x}}_{1,T})||^2_{\mathrm{R}^2} \right.$$

$$\left. + \log[1 + C_h(\Gamma_N)]\, h_T\, ||\boldsymbol{\sigma}_h\boldsymbol{\nu} - \mathbf{g}||^2_{[L^2(\partial T\cap\Gamma_N)]^2} \right\},$$

with $p_N(T) = 1$ *if* $\partial T \cap \Gamma_N$ *is not empty, and* 0 *otherwise.*

Proof. Because of Theorem 3.6, it suffices to bound $||\hat{\boldsymbol{\sigma}}_T||^2_{\mathbf{H}(\mathbf{div}\,;T)}$ for the present choice of $\boldsymbol{\varphi}_h$. In fact, according to the analysis before Lemma 3.4, and replacing $\mathbf{u}$ by $\tilde{\boldsymbol{\varphi}}_h$ in (3.34), we obtain

$$||\hat{\boldsymbol{\sigma}}_T||^2_{\mathbf{H}(\mathbf{div}\,;T)} \le ||\mathbf{t}_{h,T} + \boldsymbol{\gamma}_{h,T} - \nabla\tilde{\boldsymbol{\varphi}}_{h,T}||^2_{[L^2(T)]^{2\times 2}} + ||\mathbf{u}_{h,T} - \tilde{\boldsymbol{\varphi}}_{h,T}||^2_{[L^2(T)]^2}$$

$$+ ||\tilde{\boldsymbol{\varphi}}_{h,T} + \boldsymbol{\eta}_h||^2_{[\tilde{H}^{1/2}(\partial T\cap\Gamma_N)]^2}$$

$$+ \left|\left| \tilde{\boldsymbol{\varphi}}_{h,T} + \mathbf{V}(\boldsymbol{\sigma}_h\boldsymbol{\nu}) - \left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right)\boldsymbol{\phi}_h \right|\right|^2_{[H^{1/2}(\partial T\cap\Gamma)]^2}. \tag{3.50}$$

Also, it is easy to see that

$$||\mathbf{t}_{h,T} + \boldsymbol{\gamma}_{h,T} - \nabla\tilde{\boldsymbol{\varphi}}_{h,T}||^2_{[L^2(T)]^{2\times 2}} \le 2\left\{ \left|\left| \mathbf{t}_{h,T} - \frac{1}{2}[\nabla\tilde{\boldsymbol{\varphi}}_{h,T} + (\nabla\tilde{\boldsymbol{\varphi}}_{h,T})^{\mathbf{T}}] \right|\right|^2_{[L^2(T)]^{2\times 2}} \right.$$

$$+ \left\| \boldsymbol{\gamma}_{h,T} - \frac{1}{2}[\nabla \tilde{\boldsymbol{\varphi}}_{h,T} - (\nabla \tilde{\boldsymbol{\varphi}}_{h,T})^{\mathbf{T}}] \right\|^2_{[L^2(T)]^{2\times 2}} \Big\} . \tag{3.51}$$

Next, since $\tilde{\boldsymbol{\varphi}}_{h,T} \in [\mathbf{P}_1(T)]^2$, $\nabla \tilde{\boldsymbol{\varphi}}_{h,T} \in [\mathbf{P}_0(T)]^{2\times 2}$, $\boldsymbol{\gamma}_{h,T} \in [\mathbf{P}_1(T)]^{2\times 2}$, and the area of T is bounded by h_T^2, we deduce that

$$||\mathbf{u}_{h,T} - \tilde{\boldsymbol{\varphi}}_{h,T}||^2_{[L^2(T)]^2} \le h_T^2 \sum_{k=1}^{3} ||\mathbf{u}_{h,T} - \tilde{\boldsymbol{\varphi}}_{h,T}(\bar{\mathbf{x}}_{k,T})||^2_{\mathrm{R}^2} , \tag{3.52}$$

and

$$\left\| \boldsymbol{\gamma}_{h,T} - \frac{1}{2}[\nabla \tilde{\boldsymbol{\varphi}}_{h,T} - (\nabla \tilde{\boldsymbol{\varphi}}_{h,T})^{\mathbf{T}}] \right\|^2_{[L^2(T)]^{2\times 2}}$$

$$\le h_T^2 \sum_{k=1}^{3} \left\| \boldsymbol{\gamma}_{h,T}(\bar{\mathbf{x}}_{k,T}) - \frac{1}{2}[\nabla \tilde{\boldsymbol{\varphi}}_{h,T} - (\nabla \tilde{\boldsymbol{\varphi}}_{h,T})^{\mathbf{T}}] \right\|^2_{\mathrm{R}^{2\times 2}} . \tag{3.53}$$

On the other hand, since $(\tilde{\boldsymbol{\varphi}}_{h,T} + \boldsymbol{\eta}_h)|_{\partial T \cap \Gamma_N} \in [\mathbf{P}_1(\partial T \cap \Gamma_N)]^2$, we find that

$$||\tilde{\boldsymbol{\varphi}}_{h,T} + \boldsymbol{\eta}_h||^2_{[\tilde{H}^{1/2}(\partial T \cap \Gamma_N)]^2} \le h_T \sum_{k=1}^{2} ||(\tilde{\boldsymbol{\varphi}}_{h,T} + \boldsymbol{\eta}_h)(\bar{\mathbf{x}}_{k,T})||^2_{\mathrm{R}^2}$$

$$+ ||(\tilde{\boldsymbol{\varphi}}_{h,T} + \boldsymbol{\eta}_h)(\bar{\mathbf{x}}_{2,T} - \bar{\mathbf{x}}_{1,T})||^2_{\mathrm{R}^2} . \tag{3.54}$$

In this way, (3.50), (3.51), (3.52), (3.53), (3.54) and the estimate from Theorem 3.6 yield (3.49) and complete the proof. □

Remark 3.8. For implementations, the $[H^{1/2}(\partial T \cap \Gamma)]^2$-norms in the terms $\tilde{\boldsymbol{\theta}}_{T,\Gamma}$ of the previous theorem can be bounded by using the interpolation theorem:

$$||\cdot||^2_{[H^{1/2}(\partial T \cap \Gamma)]^2} \le ||\cdot||_{[L^2(\partial T \cap \Gamma)]^2} \, ||\cdot||_{[H^1(\partial T \cap \Gamma)]^2} .$$

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