# The $p$-version of the boundary element method for hypersingular operators on piecewise plane open surfaces

Alexei Bespalov *  Norbert Heuer †

**Abstract**

We prove an optimal a priori error estimate for the $p$-version of the boundary element method with hypersingular operators on piecewise plane open surfaces. The solutions of problems on open surfaces typically exhibit a singular behavior at the edges and corners of the surface which prevent an approximation analysis in $H^1$. We analyze the approximation by polynomials of typical singular functions in fractional order Sobolev spaces thus giving, as an application, the optimal rate of convergence of the $p$-version of the boundary element method. This paper extends the results of [C. Schwab, M. Suri, *The optimal p-version approximation of singularities on polyhedra in the boundary element method*, SIAM J. Numer. Anal., **33** (1996), pp. 729–759] who only considered closed surfaces where the solution is in $H^1$.

*Key words*: $p$-version, boundary element method, singularities
*AMS Subject Classification*: 41A10, 65N15, 65N38

## 1 Introduction and formulation of the problem

In this paper, we prove the optimal rate of convergence of the $p$-version of the boundary element method (BEM) with hypersingular operators on piecewise plane open surfaces. Even though the $p$-version was developed already in the late eighties (see [10] for the BEM in two dimensions), several problems are still open. For example there are no precise error estimates for the $p$-version with weakly singular operators on open or closed surfaces. The only precise estimates we know of are for hypersingular operators on polyhedral surfaces [8]. The analysis on open surfaces, where stronger singularities appear, was an open problem which is dealt with in this paper.

The BEM with hypersingular operators on open surfaces is used to approximate the solutions to Neumann problems for elliptic operators of second order in exterior domains. Typical solutions possess singularities which behave like $y^{1/2}$ where $y$ denotes the distance to a smooth part of the boundary of the surface. Let us denote the surface by $\Gamma$. Then the above mentioned singularity

*Computational Center, Far-Eastern Branch of the Russian Academy of Sciences, Khabarovsk, Russia. email: albespalov@yahoo.com. Financed by the FONDAP Program in Applied Mathematics, Chile.

†GI$^2$MA, Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile. email: norbert@ing-mat.udec.cl. Supported by the FONDAP Program in Applied Mathematics and the Fondecyt project no. 1010220, both Chile.

is in $H^{1-\varepsilon}(\Gamma)$ for any $\varepsilon > 0$. The energy space of hypersingular operators is $\tilde{H}^{1/2}(\Gamma)$, also known as $H_{00}^{1/2}(\Gamma)$. This space can be defined by interpolation between $L_2(\Gamma)$ and $H_0^1(\Gamma)$. Therefore, in order to study the convergence of the BEM, one has to analyze the approximation in $\tilde{H}^{1/2}(\Gamma)$ of a function which has the regularity $H^{1-\varepsilon}(\Gamma)$, $\varepsilon > 0$. This regularity is not enough to apply the results from [8], where approximation is studied in $H^{1/2}$ through interpolation between $L_2$ and $H^1$ (on a closed surface).

Applying standard $p$-version approximation results which make use of standard Sobolev regularity, see [2], one obtains a sequence of piecewise polynomials $\phi_p$ of degree $p$ which gives an upper bound for the error like

$$\|y^{1/2} - \phi_p\|_{H^s(\Gamma)} \le C p^{s-r} \|y^{1/2}\|_{H^r(\Gamma)}, \quad 0 \le s \le 1.$$

That means, for $s = 1/2$ one obtains a rate of convergence no more than $1/2 - \varepsilon$ ($\varepsilon > 0$) where the appearing constant ($C$ in the above estimate) depends in an unspecified way on $\varepsilon$. This estimate for the rate of convergence is far from being optimal. Better estimates can be obtained by making use of the special singular structure of the functions under consideration, either directly or via weighted spaces which appropriately reflect the singular behavior. Here we follow the direct way as in [8], however avoiding $H^1$-estimates when necessary. Our analysis shows for the above mentioned singularity a convergence for the $p$-version like $p^{-1}$ in the energy norm.

Recently, for problems with radial singularities in two dimensions, optimal upper and lower bounds for the $p$-version (FEM and BEM) have been proved, see [1, 5]. There, it has been shown that Jacobi-weighted Sobolev and Besov spaces are appropriate for the precise approximation analysis. For problems in three dimensions, however, where edge and corner-edge singularities appear, no such theory with weighted spaces is available.

In what follows, the analysis applies to open and closed surfaces which must be piecewise plane such that they can be discretized by meshes consisting of triangles and parallelograms. For ease of presentation, however, let us assume that $\Gamma \subset \mathbf{R}^3$ is a plane open surface with polygonal boundary. Then, our model integral equation is

$$Wu(x) := \frac{1}{4\pi} \frac{\partial}{\partial n_x} \int_\Gamma u(y) \frac{\partial}{\partial n_y} \frac{1}{|x-y|} \, dS_y = f(x), \quad x \in \Gamma.$$

Here, $n$ denotes the unit vector normal to $\Gamma$ in a specified direction. It is well known that this equation governs a Neumann problem for the Laplacian in the domain exterior to $\Gamma$, with given Neumann datum $-f$ on $\Gamma$, see [3, 9]. The solution $u$ of the integral equation is the jump across $\Gamma$ of the solution to the Neumann problem. The weak form of this integral equation is: *Find* $u \in \tilde{H}^{1/2}(\Gamma)$ *such that*

$$\langle Wu, v \rangle = \langle f, v \rangle \quad \forall v \in \tilde{H}^{1/2}(\Gamma). \tag{1.1}$$

Here, $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $H^{-1/2}(\Gamma)$ and $\tilde{H}^{1/2}(\Gamma)$, $H^{-1/2}(\Gamma)$ being the dual space of $\tilde{H}^{1/2}(\Gamma)$. The latter space is defined in §3 below.

An overview of this paper is as follows. In the next section we recall the regularity results for the solution of our model problem and define the approximation spaces for the $p$-version of the BEM. Moreover, we present the main result which gives an asymptotically optimal upper bound for the error in the energy norm (Theorem 2.1). The remainder of the paper deals with the approximation of typical singular functions within a scale of Sobolev norms. First, in §3.1, we collect several auxiliary results. Second, the approximation of singularities on intervals is studied in §3.2. Then, in §§3.3 and 3.4, edge and edge-vertex singularities are studied in detail. Collecting the individual results we present a general approximation theorem for singular functions (Theorem 3.8), in §3.5. As a special case, this yields the optimal convergence in the energy norm of the $p$-version of the BEM (Theorem 2.1).

Throughout the paper, $C$ denotes a generic positive constant which is independent of the polynomial degree $p$.

## 2 The rate of convergence of the $p$-version.

First, let us recall the typical structure of the solution of our model problem for a sufficiently smooth right-hand side function $f$. We follow the notation of [8] and refer for more details to [11].

Let $V$ and $E$ denote the sets of vertices and edges of $\Gamma$, respectively. For $v \in V$, let $E(v)$ denote the set of edges with $v$ as an end point. Then, the solution $u$ of (1.1) has the form

$$u = u_{\rm reg} + \sum_{e\in E} u^e + \sum_{v\in V} u^v + \sum_{v\in V}\sum_{e\in E(v)} u^{ev}, \tag{2.1}$$

where, using local coordinate systems $(r_v, \theta_v)$ and $(x_{e1}, x_{e2})$ with origin $v$, there hold the following representations:

(i) The regular part $u_{\rm reg} \in H^k(\Gamma)$, $k > 3/2$.

(ii) The edge singularities $u^e$ have the form

$$u^e = \sum_{j=1}^{m_e} \left( \sum_{s=0}^{s_j^e} b_{js}^e(x_{e1}) |\log x_{e2}|^s \right) x_{e2}^{\gamma_j^e} \chi_1^e(x_{e1}) \chi_2^e(x_{e2}), \tag{2.2}$$

where $\gamma_{j+1}^e \ge \gamma_j^e \ge \frac{1}{2}$, and $m_e$, $s_j^e$ are integers. Here, $\chi_1^e$, $\chi_2^e$ are $C^\infty$ cut-off functions with $\chi_1^e = 1$ in a certain distance to the end points of $e$ and $\chi_1^e = 0$ in a neighborhood of these vertices. Moreover, for a $\rho_e > 0$, $\chi_2^e = 1$ for $0 \le x_{e2} \le \rho_e$ and $\chi_2^e = 0$ for $x_{e2} \ge 2\rho_e$. The functions $b_{js}^e \chi_1^e \in H^m(e)$ for $m$ as large as required. (Below we need $m > 2\gamma_1^e + 2$.)

(iii) The vertex singularities $u^v$ have the form

$$u^v = \chi^v(r_v) \sum_{i=1}^{n_v} \sum_{t=0}^{q_i^v} B_{it}^v |\log r_v|^t r_v^{\lambda_i^v} w_{it}^v(\theta_v), \tag{2.3}$$

where $\lambda_{i+1}^v \ge \lambda_i^v > 0$, $n_v$, $q_i^v \ge 0$ are integers, and $B_{it}^v$ are real numbers. Here, $\chi^v$ is a $C^\infty$ cut-off function with $\chi^v = 1$ for $0 \le r_v \le \tau_v$ and $\chi^v = 0$ for $r_v \ge 2\tau_v$ for some $\tau_v > 0$. The functions

$w_{it}^v \in H^q(0,\omega_v)$ for $q$ as large as required. Here, $\omega_v$ denotes the interior angle (on $\Gamma$) between the edges meeting at $v$.

(iv) The edge-vertex singularities $u^{ev}$ have the form

$$u^{ev} = u_1^{ev} + u_2^{ev},$$

where

$$u_1^{ev} = \sum_{j=1}^{m_e}\sum_{i=1}^{n_v}\left(\sum_{s=0}^{s_j^e}\sum_{t=0}^{q_i^v}\sum_{l=0}^{s} B_{ijlts}^{ev}|\log x_{e1}|^{s+t-l}|\log x_{e2}|^l\right) x_{e1}^{\lambda_i^v-\gamma_j^e} x_{e2}^{\gamma_j^e}\,\chi^v(r_v)\chi^{ev}(\theta_v) \tag{2.4}$$

and

$$u_2^{ev} = \sum_{j=1}^{m_e}\sum_{s=0}^{s_j^e} B_{js}^{ev}(r_v)|\log x_{e2}|^s x_{e2}^{\gamma_j^e}\,\chi^v(r_v)\chi^{ev}(\theta_v) \tag{2.5}$$

with

$$B_{js}^{ev}(r_v) = \sum_{l=0}^{s} B_{jsl}^{ev}(r_v)|\log r_v|^l. \tag{2.6}$$

Here, $q_i^v$, $s_j^e$, $\lambda_i^v$, $\gamma_j^e$, $\chi^v$ are as above, $B_{ijlts}^{ev}$ are real numbers, and $\chi^{ev}$ is a $C^\infty$ cut-off function with $\chi^{ev}=1$ for $0\le\theta_v\le\beta$ and $\chi^{ev}=0$ for $\frac{3}{2}\beta\le\theta_v\le\omega_v$ for some $0<\beta\le\min\{\omega_v/2,\pi/8\}$. The functions $B_{jsl}^{ev}$ may be chosen such that

$$B_{js}^{ev}(r_v)\,\chi^v(r_v)\chi^{ev}(\theta_v) = \xi_{js}(x_{e1},x_{e2})\,\chi_2^e(x_{e2}),$$

where the extension of $\xi_{js}$ by zero onto $\mathbf{R}^{2+} := \{(x_{e1},x_{e2});\ x_{e2}>0\}$ lies in $H^m(\mathbf{R}^{2+})$, with $m$ as in (ii). Here, $\chi_2^e$ is a $C^\infty$ cut-off function as in (ii).

**Remark 2.1** *(i) For the edge and the vertex-edge singularities in* (ii) *and* (iv) *we specified* $\gamma_1^e \ge 1/2$. *The value* $\gamma_1^e = 1/2$ *may occur for open surfaces whereas, for closed surfaces, there holds* $\gamma_1^e > 1/2$. *In the case* $\gamma_1^e = 1/2$ *one has (if the corresponding factors do not vanish)* $u^e$, $u^{ev} \notin H^1(\Gamma)$. *This is the particular case we have in mind to study in this paper. For singular right-hand sides* $f$ *in* (1.1), *it also may occur that* $\gamma_1^e$ *assumes any positive value, which is the minimum requirement to guarantee* $u \in \tilde H^{1/2}(\Gamma)$. *Our analysis below will cover this least regular case.*

*(ii) For our approximation analysis below, and in order to ensure* $u \in H^{1/2}(\Gamma)$, *it suffices to require* $\lambda_1^v > -1/2$ *in* (iv). *To guarantee* $u \in \tilde H^{1/2}(\Gamma)$ *for open surfaces however, one needs the given restriction* $\lambda_1^v > 0$.

Next we consider the $p$-version of the BEM for the approximate solution of (1.1). In the following $p$ will always specify a polynomial degree. We discretize $\Gamma$ by a fixed mesh $\{\Gamma_j;\ j = 1,\dots,J\}$ consisting of triangles and parallelograms. In the proofs below we will refer to three different unions of elements. The union of the elements at a node $v$ is denoted by $A_v$, $\bar A_v :=$

$\cup\{\bar\Gamma_j;\ v\in\bar\Gamma_j\}$, the union of the elements at one edge $e$ by $A_e$ (the endpoints of $e$ are not included in $e$), $\bar A_e := \cup\{\bar\Gamma_j;\ \bar\Gamma_j\cap e\neq\emptyset\}$, and $A_{ev} := A_v\cap A_e$.

Let $Q=(-1,1)^2$ and $T=\{(x_1,x_2);\ 0<x_1<1,\ 0<x_2<x_1\}$ be the reference square and triangle, respectively. For $K=Q$ or $T$, $\mathcal{Q}_p(K)$ denotes the set of polynomials on $K$ of degree $\le p$ in each variable. Moreover, $\mathcal{P}_p(T)$ is the set of polynomials on $T$ of total degree $\le p$, and $\mathcal{P}_p(I)$ is the set of polynomials of degree $\le p$ on an interval $I\subset\mathbf{R}$. For given $p$, we then consider the space of continuous, piecewise polynomials on the mesh introduced before,

$$V_0^p(\Gamma) := \{v\in C^0(\Gamma);\ v|_{\partial\Gamma}=0,\ v|_{\Gamma_j}\circ T_j\in\mathcal{Q}(Q) \text{ or } \mathcal{P}_p(T),\ j=1,\dots,J\}.$$

Here, $T_j$ is an affine mapping with $T_j(K)=\Gamma_j$, $K=Q$ or $T$ as appropriate. Note that $V_0^p(\Gamma)\subset\tilde H^{1/2}(\Gamma)$. Now, the $p$-version of the BEM is: *Find $u_p\in V_0^p(\Gamma)$ such that*

$$\langle Wu_p, v\rangle = \langle f, v\rangle \quad \forall v\in V_0^p(\Gamma). \tag{2.7}$$

The main result of this paper is the following theorem.

**Theorem 2.1** *Let $u\in\tilde H^{1/2}(\Gamma)$ be the solution of* (1.1) *with sufficiently smooth given function $f\in H^{1/2}(\Gamma)$ such that the representation* (2.1) *holds. Let $v_0\in V$, $e_0\in E(v_0)$ such that $\min\{\lambda_1^{v_0}+1/2,\gamma_1^{e_0}\}=\min_{v\in V,e\in E(v)}\min\{\lambda_1^v+1/2,\gamma_1^e\}$, with $\lambda_1^v$ and $\gamma_1^e$ being as in* (2.2)–(2.5). *Then denote*

$$\beta=\begin{cases} s_1^{e_0}+q_1^{v_0}+1/2 & \text{if } \lambda_1^{v_0}=\gamma_1^{e_0}-1/2,\\ s_1^{e_0}+q_1^{v_0} & \text{otherwise,}\end{cases}$$

*where the numbers $s_1^{e_0}$, $q_1^{v_0}$ are given in* (2.4). *Then the BE approximation $u_p$ defined by* (2.7) *satisfies*

$$\|u-u_p\|_{\tilde H^{1/2}(\Gamma)} \le C\,|\log p|^{\beta} p^{-2\min\{\gamma_1^{e_0},\lambda_1^{v_0}+\frac12\}},$$

*where $C>0$ is a constant which is independent of $p$.*

**Proof.** By the quasi-optimal convergence of the BEM (see, e.g., [9]) the proof of the theorem is obtained by the estimate (3.59) of Theorem 3.8 below. □

# 3 Technical details.

First, let us recall the definition of the Sobolev spaces which will be used below, see [6, 4]. For a domain $\Omega\subset\mathbf{R}^n$ and integer $s$ let $H^s(\Omega)$ be the closure of $C^\infty(\Omega)$ with respect to the norm

$$\|u\|^2_{H^s(\Omega)} = \|u\|^2_{H^{s-1}(\Omega)} + |u|^2_{H^s(\Omega)} \quad (s\ge 1),$$

where

$$|u|^2_{H^s(\Omega)} = \int_\Omega |D^s u(x)|^2\,dx, \quad \text{and} \quad H^0(\Omega)=L_2(\Omega).$$

Here, $|D^s u(x)|^2 = \sum_{|\alpha|=s} |D^\alpha u(x)|^2$ in the usual notation with multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ and with respect to Cartesian coordinates $x = (x_1, \ldots, x_n)$. For non-integer positive $s$ with $s = m + \sigma$ for integer $m \geq 0$ and $0 < \sigma < 1$, the norm in $H^s(\Omega)$ is

$$\|u\|^2_{H^s(\Omega)} = \|u\|^2_{H^m(\Omega)} + |u|^2_{H^s(\Omega)}$$

with semi-norm

$$|u|^2_{H^s(\Omega)} = \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\sigma}}\, dx\, dy.$$

The closure of $C_0^\infty(\Omega)$ with respect to the above norms is denoted by $H_0^s(\Omega)$. For a domain $\Omega$ with Lipschitz boundary $\partial\Omega$, $\tilde{H}^{1/2}(\Omega)$ denotes the space of functions in $H^{1/2}(\Omega)$ whose extensions by zero are in $H^{1/2}(\mathbf{R}^n)$. A norm in this space, which is also known as $H_{00}^{1/2}(\Omega)$, is

$$\|u\|^2_{\tilde{H}^{1/2}(\Omega)} = \|u\|^2_{L_2(\Omega)} + |u|^2_{H^{1/2}(\Omega)} + \int_\Omega \frac{|u(x)|^2}{\operatorname{dist}(x, \partial\Omega)}\, dx.$$

For non-integer $s$, the Sobolev spaces can also defined by interpolation. We use the real K-method and obtain the equivalent definitions (see [6])

$$H^s(\Omega) = \Big(L_2(\Omega), H^1(\Omega)\Big)_{s,2} \quad (0 < s < 1)$$

and

$$\tilde{H}^{1/2}(\Omega) = \Big(L_2(\Omega), H_0^s(\Omega)\Big)_{\frac{1}{2s},2} \quad (1/2 < s \leq 1).$$

For integer $k \geq 0$ and $\mu \in [0,1]$ we also consider the spaces of continuously differentiable functions $C^k(\bar{\Omega})$ and $C^{k,\mu}(\bar{\Omega})$ with norms

$$\|u\|_{C^k(\bar{\Omega})} = \sum_{|\alpha|\leq k} \sup_{x\in\Omega} |D^\alpha u(x)|$$

and

$$\|u\|_{C^{k,\mu}(\bar{\Omega})} = \|u\|_{C^k(\bar{\Omega})} + \sum_{|\alpha|=k} \sup_{x,y\in\Omega,\, x\neq y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\mu}.$$

In the following sections we provide the details of our approximation theory leading to a general approximation theorem (Theorem 3.8) which includes as a special case the upper bound stated by Theorem 2.1 above. First, in §3.1 we collect several technical lemmas. Necessary one-dimensional approximation results are given in §3.2. In §§3.3 and 3.4 we analyze typical edge and edge-vertex singularities, as they appear in the representation of the solution $u$ of (1.1), see (2.2) and (2.4). In this paper we do not study the approximation of vertex singularities of the type (2.3). These functions are in $H^1$ such that we can rely on the results of Schwab and Suri [8]. The general approximation result for a function which includes all the different types of singularities is given in §3.5.

## 3.1 Auxiliary lemmas.

We collect several technical results.

**Lemma 3.1** *Let $\Gamma$ be a Lipschitz domain which is decomposed into a fixed number of non-intersecting Lipschitz subdomains $\Gamma_j$. Then, for $0 \le s \le 1$ with $s \ne 1/2$, there holds*

$$\|u\|_{H^s(\Gamma)} \le C \sum_j \|u\|_{H^s(\Gamma_j)} \quad \forall u \in H^s(\Gamma),$$

*where the constant $C > 0$ is independent of $u$.*

**Proof.** For $s \in (0, 1/2)$ the estimate follows from

$$\|u\|_{H^s(\Gamma)} \le C \sum_j \|u\|_{H_0^s(\Gamma_j)} \quad \forall u \in H^s(\Gamma)$$

by the identity $H_0^s(\Gamma_j) = H^s(\Gamma_j)$ for $0 \le s < 1/2$, see, e.g., [4]. It remains to consider the case $s \in (1/2, 1)$. Without loss of generality we consider two subdomains $\Gamma = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\bar{\Gamma}_1 \cap \bar{\Gamma}_2 = \gamma$. Denoting the space $H^s(\Gamma_1 \times \Gamma_2) := \{(u_1, u_2) \in H^s(\Gamma_1) \times H^s(\Gamma_2);\ (u_1 - u_2)|_\gamma = 0\}$ we define the operator

$$T : \begin{cases} H^s(\Gamma) & \to & H^s(\Gamma_1 \times \Gamma_2), \\ u & \mapsto & (u_1, u_2) := (u|_{\Gamma_1}, u|_{\Gamma_2}). \end{cases}$$

$T$ is linear, bijective and continuous since there holds

$$\|u\|_{H^s(\Gamma_1)} + \|u\|_{H^s(\Gamma_2)} \le C\|u\|_{H^s(\Gamma)} \quad \forall u \in H^s(\Gamma).$$

The space $H^s(\Gamma_1 \times \Gamma_2) \subset H^s(\Gamma_1) \times H^s(\Gamma_2)$ is Banach since it is closed. This holds because it is the kernel of the continuous operator

$$\begin{cases} H^s(\Gamma_1) \times H^s(\Gamma_2) & \to & H^{s-1/2}(\gamma), \\ (u_1, u_2) & \mapsto & (u_1 - u_2)|_\gamma. \end{cases}$$

Therefore, $T$ is continuously invertible and the statement for two subdomains follows. The estimate for more subdomains follows by induction. □

**Lemma 3.2** *Let $f \in H^m(Q)$ for real $m \ge 0$. Then there exists a sequence $f_p \in \mathcal{Q}_p(Q)$, $p = 0, 1, 2, \ldots$, such that for $0 \le s \le m$*

$$\|f - f_p\|_{H^s(Q)} \le C\, p^{-(m-s)}\, \|f\|_{H^m(Q)}.$$

**Proof.** This statement is proved by [2, Lemma 3.1] for integer $s = 0, 1, \ldots, m$, and for non-integer $s$ it follows by interpolation. □

We quote the following two lemmas (restricted to the standard square $Q$) from [7] (see Theorem 3.8 and Lemma 5.5 in Chapter 2 therein).

**Lemma 3.3** *Let $m > 1$ be real. Let $\mu = m - 1$ if $m < 2$, $\mu < 1$ if $m = 2$, and $\mu = 1$ if $m > 2$. Then $H^m(Q) \subset C^{0,\mu}(\bar{Q})$, and*

$$\|u\|_{C^{0,\mu}(\bar{Q})} \le C \, \|u\|_{H^m(Q)}.$$

**Lemma 3.4** *Let $u \in H^m(Q)$ for real $m \ge 0$, and $v \in C^{[m]'-1,1}(\bar{Q})$, where $[m]'$ denotes the minimal integer such that $m \le [m]'$. Then $uv \in H^m(Q)$, and*

$$\|uv\|_{H^m(Q)} \le C \, \|u\|_{H^m(Q)} \, \|v\|_{C^{[m]'-1,1}(\bar{Q})}.$$

**Lemma 3.5** *Let $0 < s < 1$, and assume that*

$$J = \int_I \int_I \frac{\|u(x_1,\cdot) - u(y_1,\cdot)\|^2_{L_2(I)}}{|x_1 - y_1|^{1+2s}} \, dx_1 \, dy_1 + \int_I \int_I \frac{\|u(\cdot,x_2) - u(\cdot,y_2)\|^2_{L_2(I)}}{|x_2 - y_2|^{1+2s}} \, dx_2 \, dy_2 < \infty.$$

*Then*

$$|u|_{H^s(Q)} \le C \, J^{1/2}.$$

**Proof.** Following [7, Chapter 2, Lemma 5.3] we have

$$\begin{aligned}
|u|^2_{H^s(Q)} &\le C \int_Q \int_Q \frac{|u(x_1,x_2) - u(y_1,x_2)|^2}{|x-y|^{2+2s}} \, dx \, dy + C \int_Q \int_Q \frac{|u(y_1,x_2) - u(y_1,y_2)|^2}{|x-y|^{2+2s}} \, dx \, dy \\
&\le C \int_I \int_I \int_I \frac{|u(x_1,x_2) - u(y_1,x_2)|^2}{|x_1 - y_1|^{2s}} \int_I \frac{dy_2}{|x-y|^2} \, dx_1 \, dx_2 \, dy_1 \\
&\quad + C \int_I \int_I \int_I \frac{|u(y_1,x_2) - u(y_1,y_2)|^2}{|x_2 - y_2|^{2s}} \int_I \frac{dx_1}{|x-y|^2} \, dx_2 \, dy_1 \, dy_2 \le C \, J,
\end{aligned}$$

because, for instance,

$$\int_I \frac{dy_2}{|x-y|^2} = \frac{1}{|x_1 - y_1|} \left( \arctan \frac{1 + x_2}{|x_1 - y_1|} + \arctan \frac{1 - x_2}{|x_1 - y_1|} \right) \le \frac{C}{|x_1 - y_1|}.$$

$\square$

**Remark 3.1** *Lemma 3.5 holds analogously for an arbitrary rectangle $Q = [a,b] \times [c,d]$.*

The following lemma is proved in [8, Lemma 9.2].

**Lemma 3.6** *Let $K$ be a triangle or parallelogram, and let $l$ be a side of $K$ with vertices $v_1$, $v_2$. Let $w_p \in \mathcal{P}_p(l)$ be such that $w_p(v_1) = w_p(v_2) = 0$, and $\|w_p\|_{L_2(l)} \le f(p)$. Then there exists $u_p \in \mathcal{Q}_{p+1}(K)$ such that $u_p = w_p$ on $l$, $u_p = 0$ on $\partial K \backslash l$, and for $0 \le s \le 1$*

$$\|u_p\|_{H^s(K)} \le C \, f(p) \, p^{-1+2s}.$$

### 3.2 Approximation results in one dimension.

We start with a polynomial approximation of the singular function in the one-dimensional case.

Let

$$u(x) = (1+x)^\alpha |\log(1+x)|^\beta \chi(x), \quad x \in I = (-1,1), \tag{3.1}$$

where $\alpha > 0$ is real, $\beta \geq 0$ is integer, $\chi \in C^\infty(I)$ is a cut-off function with $\chi(x) = 1$ for $x \in (-1, -1+\delta]$ and $\chi(x) = 0$ for $x \geq -1+2\delta$ $(0 < \delta \leq 1/4)$.

Observe that $u \in H^s(I)$ for any $s \in [0, 1/2+\alpha)$, moreover $u \in \tilde{H}^{1/2}(I)$.

**Theorem 3.1** *Let $u$ be given by (3.1) with $0 < \alpha \leq 1/2$ and integer $\beta \geq 0$. Then there exists a sequence $u_p \in \mathcal{P}_{p+1}(I)$ such that*

$$\|u - u_p\|_{H^s(\tilde{I})} \leq C\, p^{-2(\alpha+1/2-s)}\, |\log p|^\beta, \quad 0 \leq s < 1/2+\alpha, \tag{3.2}$$

$$u_p(-1) = u(-1) = 0,$$

*where $\tilde{I} = [-1, 0]$.*

**Remark 3.2** *If $\alpha > 1/2$, then $u \in H^1(I)$. In this case the approximations of $u$ by polynomials of degree $p$ have been investigated in [8], and the optimal estimate (3.2) has been established for $s = 0, 1$, and then, by interpolation, for any $s \in [0,1]$. For $0 < \alpha \leq 1/2$, $u \notin H^1(I)$, and that approach cannot be used. Therefore Theorem 3.1 extends the result of [8].*

**Remark 3.3** *In [5] the polynomial approximations of function (3.1) with $\alpha > 0$ and integer $\beta \geq 0$ have been investigated, and the following optimal error estimate in the norm of the space $\tilde{H}^{1/2}(I)$ has been proved by studying the approximability of functions in the Jacobi-weighted Sobolev and Besov spaces:*

$$\|u - u_p\|_{\tilde{H}^{1/2}(I)} \leq C\, p^{-2\alpha}\, (1 + \log p)^{\beta^*},$$

*where $u_p \in \mathcal{P}_p(I)$, $u_p(\pm 1) = u(\pm 1) = 0$, $\beta^* = \beta$ if $\alpha$ is non-integer or integer and $\beta = 0$, and $\beta^* = \beta - 1$ if $\alpha$ is an integer and $\beta \geq 1$.*

The proof of Theorem 3.1 follows the guidelines of the proof of Theorem 5.1 in [8]. Let $\xi(x) = 1 + x$, and

$$u_0(x) = u(x)/\xi(x) = (1+x)^{\alpha-1} |\log(1+x)|^\beta \chi(x).$$

Introducing a cut-off function $\omega$ such that

$$\omega \in C^\infty(R), \quad \omega(t) = 0 \ \text{ for } \ t \leq 1, \quad \omega(t) = 1 \ \text{ for } \ t \geq 2, \tag{3.3}$$

we define for a small $\Delta \in (0,1)$

$$\omega^\Delta(x) = \omega\Big(\tfrac{1+x}{\Delta}\Big), \quad \tilde{\omega}^\Delta(x) = 1 - \omega^\Delta(x), \quad x \in I. \tag{3.4}$$

Let us decompose the function $u_0$ as

$$u_0 = u/\xi = u_0\omega^\Delta + u_0\tilde{\omega}^\Delta =: v + w, \tag{3.5}$$

such that $v \in C_0^\infty(I)$ with $\operatorname{supp} v \subset [-1+\Delta, 1)$ and $\operatorname{supp} w \subset [-1, -1+2\Delta]$.

**Lemma 3.7** *Let $\Delta = p^{-2}$. Then there exists a polynomial $u_p \in \mathcal{P}_{p+1}(I)$ such that $u_p(-1) = 0$, and for $0 \le s \le 1$*

$$\|\xi v - u_p\|_{H^s(\tilde{I})} \;\le\; C\, p^{-2(\alpha+1/2-s)}\,|\log p|^{\beta}, \tag{3.6}$$

*where $\tilde{I} = [-1, 0]$.*

**Proof.** Note that, due to the smoothness of $v$, the proofs of Lemmas 5.1–5.5 in [8], using interpolation between $H^0$ and $H^1$, apply. Therefore, here we outline the proof of (3.6) retaining the following notation from [8]: for any function $f$ on $I$ and any $0 < \Delta < 1/4$ we define the function $f_\Delta$ such that

$$f_\Delta(x) = \begin{cases} f(x - 2\Delta) & \text{for } \; x > -1 + 2\Delta, \\ 0 & \text{for } \; x \le -1 + 2\Delta. \end{cases}$$

First we observe (see Lemma 5.1 in [8]) that for any integer $k \ge 0$ there exists a constant $C(k)$ independent of $\Delta$ such that

$$\begin{aligned} \left|\frac{d^k v}{dx^k}\right| \;&\le\; C(k)|\log\Delta|^{\beta}(1+x)^{\alpha-1-k} \quad \text{on} \quad [-1+\Delta, 1), \\ \left|\frac{d^k v}{dx^k}\right| \;&=\; 0 \quad \text{on} \quad (-1, -1+\Delta]. \end{aligned} \tag{3.7}$$

We see that the $i$-th derivative $v^{[i]}$ of the function $v$ also satisfies (3.7) with $\alpha$ replaced by $\alpha - i$. Then applying Lemma 5.4 of [8] to the derivatives $v^{[i]}$ ($0 \le i \le k$, where $k = [\alpha + 1/2]$ is the largest integer $\le \alpha + 1/2$), we find polynomials $v_p^{[i]} \in \mathcal{P}_p(I)$ satisfying (cf. (5.14) in [8])

$$\|\xi_\Delta (v^{[i]} - v_p^{[i]})\|_{H^s(\tilde{I}_\Delta)} \;\le\; C\, p^{-2(\alpha-i+1/2-s)}\,|\log p|^{\beta}, \tag{3.8}$$

where $\tilde{I}_\Delta = (-1 + 2\Delta, 2\Delta)$, $0 \le s \le 1$, $0 \le i \le k$.

Denoting by

$$T_k(v_\Delta) = \sum_{i=0}^{k} (-1)^i \frac{(2\Delta)^i}{i!} v^{[i]} \tag{3.9}$$

the Taylor expansion of $v_\Delta$, one has by [8, Lemma 5.5]

$$\|\xi_\Delta (v_\Delta - T_k(v_\Delta))\|_{H^s(\tilde{I}_\Delta)} \;\le\; C\, p^{-2(\alpha-s+1/2)}\,|\log p|^{\beta}, \tag{3.10}$$

where $\Delta = p^{-2}$, $k = [\alpha + 1/2]$, $0 \le s \le 1$, and the positive constant $C$ does not depend on $p$ and $\Delta$. Now we set

$$z_{p\Delta} = \xi_\Delta \sum_{i=0}^{[\alpha+1/2]} (-1)^i \frac{(2\Delta)^i}{i!} v_p^{[i]}.$$

Then $z_{p\Delta} \in \mathcal{P}_{p+1}(\tilde{I}_\Delta)$, $z_{p\Delta}(-1+2\Delta) = 0$, and

$$\|\xi_\Delta v_\Delta - z_{p\Delta}\|_{H^s(\tilde{I}_\Delta)} \le \left\|\xi_\Delta\Big(v_\Delta - \sum_{i=0}^{[\alpha+1/2]} (-1)^i \frac{(2\Delta)^i}{i!} v^{[i]}\Big)\right\|_{H^s(\tilde{I}_\Delta)} + \sum_{i=0}^{[\alpha+1/2]} \frac{(2\Delta)^i}{i!} \|\xi_\Delta(v^{[i]} - v_p^{[i]})\|_{H^s(\tilde{I}_\Delta)} \le Cp^{-2(\alpha-s+1/2)}|\log p|^\beta \quad (3.11)$$

by using (3.8), (3.10), and the triangle inequality. Translating back to $\tilde{I}$ this gives a polynomial $u_p$ which satisfies the statement of the lemma. $\square$

The function $\xi w$ (see (3.5)) is approximated by zero. In contrast to [8] we have to estimate the error of this approximation directly in the norm of the space $H^s(\tilde{I})$ for $0 \le s < 1/2 + \alpha$.

**Lemma 3.8** *Let $\Delta = p^{-2}$. Then for $0 \le s < 1/2+\alpha$,*

$$\|\xi w\|_{H^s(\tilde{I})} \le C\, p^{-2(\alpha+1/2-s)}\, |\log p|^\beta. \quad (3.12)$$

**Proof.** For $s = 0$ we obviously have using (3.1), (3.3)–(3.5)

$$\|\xi w\|^2_{H^0(\tilde{I})} \le C \int_{-1}^{-1+2\Delta} (1+x)^{2\alpha} |\log(1+x)|^{2\beta}\, dx \le C\Delta^{2\alpha+1}|\log\Delta|^{2\beta} \le Cp^{-4(\alpha+1/2)}|\log p|^{2\beta},$$

where we used the fact that $\alpha > 0$.

In order to prove (3.12) for $0 < s < 1/2+\alpha$ we consider an affine mapping $M : [0,1] \to [-1, -1+2\Delta]$. Using standard scaling arguments one has

$$\|f\|_{L_2(-1,-1+2\Delta)} \simeq \Delta^{1/2}\|F\|_{L_2(0,1)}$$

for any function $f \in L_2(-1,-1+2\Delta)$ with $F := f \circ M$ and, for $f \in H^1_0(-1,-1+2\Delta)$,

$$|f|_{H^1(-1,-1+2\Delta)} \simeq \Delta^{-1/2}|F|_{H^1(0,1)}.$$

Then by interpolation we find for $0 < s < 1$

$$\|f\|_{\tilde{H}^s(-1,-1+2\Delta)} \simeq \Delta^{1/2-s}\|F\|_{\tilde{H}^s(0,1)}, \quad (3.13)$$

provided that $f \in \tilde{H}^s(-1,-1+2\Delta)$.

Let $0 < s < 1/2+\alpha$. Then by (3.1), (3.3)–(3.5)

$$\|\xi w\|_{H^s(\tilde{I})} \le C\,\|\xi w\|_{\tilde{H}^s(-1,-1+2\Delta)} \le C\,\|(1+x)^\alpha\,|\log(1+x)|^\beta\,\tilde{\omega}^\Delta(x)\|_{\tilde{H}^s(-1,-1+2\Delta)},$$

and applying (3.13) we get

$$\|\xi w\|_{H^s(\tilde{I})} \le C\Delta^{1/2-s}\|(2\Delta t)^\alpha\,|\log(2\Delta t)|^\beta\,(1-\omega(2t))\|_{\tilde{H}^s(0,1)}$$

$$
\begin{aligned}
&\le \; C\Delta^{\alpha+1/2-s}|\log\Delta|^\beta \Big(\|t^\alpha(1-\omega(2t))\|_{\tilde H^s(0,1)} + \|t^\alpha|\log t|^\beta(1-\omega(2t))\|_{\tilde H^s(0,1)}\Big)\\
&\le \; C\Delta^{\alpha+1/2-s}|\log\Delta|^\beta. \qquad (3.14)
\end{aligned}
$$

Taking $\Delta = p^{-2}$ in (3.14) we obtain (3.12) for $0 < s < 1/2+\alpha$. □

**Proof of Theorem 3.1.** The desired statement follows from Lemmas 3.7 and 3.8 making use of decomposition (3.5). □

### 3.3 Approximation of edge singularities.

Let us consider one component of the edge singularity terms $u^e$ over the standard square $Q = I \times I$:

$$
u(x_1,x_2) = (1+x_2)^\gamma|\log(1+x_2)|^\beta\chi(x_2)\tilde\chi(x_1,x_2), \qquad (3.15)
$$

where $0 < \gamma \le 1/2$, $\beta \ge 0$ is integer, $\tilde\chi \in H^m(Q)$ with $m > 2\gamma+2$, and $\chi \in C^\infty(I)$ is a cut-off function with support in $[-1,\delta]$, $0<\delta<1$.

As in the one-dimensional case we see that $u \in H^s(Q)$ for any $s \in [0, 1/2+\gamma)$.

**Theorem 3.2** *Let $u$ be given by* (3.15). *Then there exists a sequence $u_p \in \mathcal{Q}_{2p+2}(Q)$, $p = 1, 2, \ldots,$ such that*

$$
\|u-u_p\|_{H^s(Q)} \;\le\; C\,p^{-2(\gamma+1/2-s)}\,|\log p|^\beta, \quad 0 \le s < 1/2+\gamma, \qquad (3.16)
$$

$$
u_p = 0 \quad \textit{on} \quad \textit{the} \quad \textit{line} \quad x_2 = -1,
$$

*and for any line $l$ in $Q$ which intersects the line $x_2=-1$*

$$
\|u-u_p\|_{L_2(l\cap\bar Q)} \;\le\; C\,p^{-2(\gamma+1/2)}\,|\log p|^\beta. \qquad (3.17)
$$

*The positive constants $C$ in* (3.16) *and* (3.17) *are independent of $p$.*

**Proof.** Let $\varphi(x_2) = (1+x_2)^\gamma|\log(1+x_2)|^\beta\chi(x_2)$. Applying Theorem 3.1 we find a polynomial $\varphi_p \in \mathcal{P}_p(I)$ such that $\varphi_p(-1) = \varphi(-1) = 0$, and

$$
\|\varphi-\varphi_p\|_{H^s(I)} \;\le\; C\,p^{-2(\gamma+1/2-s)}\,|\log p|^\beta, \quad 0 \le s < 1/2+\gamma. \qquad (3.18)
$$

Since $\varphi \in H^s(I)$ for any $s \in [0, 1/2+\gamma)$, one has by (3.18)

$$
\|\varphi_p\|_{H^s(I)} \;\le\; C, \quad 0 \le s < 1/2+\gamma. \qquad (3.19)
$$

Further, let $\tilde\chi_p \in \mathcal{Q}_p(Q)$ be the polynomial approximation to $\tilde\chi$ given by Lemma 3.2. Setting $u_p(x_1,x_2) = \varphi_p(x_2)\tilde\chi_p(x_1,x_2)$ we find $u_p \in \mathcal{Q}_{2p}(Q)$ with $u_p = 0$ for $x_2 = -1$, and

$$
\|u-u_p\|_{H^s(Q)} \le \|\tilde\chi(\varphi-\varphi_p)\|_{H^s(Q)} + \|\varphi_p(\tilde\chi-\tilde\chi_p)\|_{H^s(Q)}, \quad 0 \le s < 1/2+\gamma. \qquad (3.20)
$$

Since $m > 2\gamma+2 > 2$, $\tilde{\chi} \in H^m(Q) \subset C^1(\bar{Q})$, and inequality (3.16) for $s = 0$ immediately follows from (3.18)–(3.20), Lemma 3.2, and the imbedding $H^{1+\varepsilon}(Q) \subset C^0(\bar{Q})$ for any $\varepsilon > 0$.

Now let us consider the case when $1/2 < s < 1/2 + \gamma$.

Since $\tilde{\chi} \in H^m(Q)$ for $m > 2$ and $1/2 < s < 1$, Lemma 3.3 and Lemma 3.4 yield

$$\|\tilde{\chi}(\varphi - \varphi_p)\|_{H^s(Q)} \leq C\|\tilde{\chi}\|_{C^{0,1}(\bar{Q})}\|\varphi - \varphi_p\|_{H^s(Q)} \leq C\|\varphi - \varphi_p\|_{H^s(Q)}. \tag{3.21}$$

Using the expression for the semi-norm in $H^s(Q)$ one has

$$\begin{aligned} |\varphi - \varphi_p|^2_{H^s(Q)} &= \int\limits_Q \int\limits_Q \frac{|(\varphi - \varphi_p)(x_2) - (\varphi - \varphi_p)(y_2)|^2}{((x_1 - y_1)^2 + (x_2 - y_2)^2)^{1+s}}\, dx_1\, dx_2\, dy_1\, dy_2 \\ &\leq \int\limits_I \int\limits_I \frac{|(\varphi - \varphi_p)(x_2) - (\varphi - \varphi_p)(y_2)|^2}{|x_2 - y_2|^{2s}}\, dx_2\, dy_2 \int\limits_I \int\limits_I \frac{dx_1\, dy_1}{(x_1 - y_1)^2 + (x_2 - y_2)^2} \\ &\leq C \int\limits_I \int\limits_I \frac{|(\varphi - \varphi_p)(x_2) - (\varphi - \varphi_p)(y_2)|^2}{|x_2 - y_2|^{1+2s}}\, dx_2\, dy_2 = C\, |\varphi - \varphi_p|^2_{H^s(I)}. \end{aligned}$$

Then we obtain by (3.18) and (3.21)

$$\|\tilde{\chi}(\varphi - \varphi_p)\|_{H^s(Q)} \leq C p^{-2(\gamma+1/2-s)}|\log p|^\beta, \quad 1/2 < s < 1/2 + \gamma. \tag{3.22}$$

For the second term in the right-hand side of (3.20) we again use Lemma 3.3, Lemma 3.4, and then inequality (3.19):

$$\begin{aligned} \|\varphi_p(\tilde{\chi} - \tilde{\chi}_p)\|_{H^s(Q)} &\leq C\, \|\tilde{\chi} - \tilde{\chi}_p\|_{C^{0,1}(\bar{Q})}\|\varphi_p\|_{H^s(Q)} \\ &\leq C\, \|\tilde{\chi} - \tilde{\chi}_p\|_{H^{2+\varepsilon}(Q)}\|\varphi_p\|_{H^s(I)} \leq C\, \|\tilde{\chi} - \tilde{\chi}_p\|_{H^{2+\varepsilon}(Q)}, \quad \varepsilon > 0. \end{aligned}$$

Hence, taking $\varepsilon$ sufficiently small (namely, such that $2 + \varepsilon < m$) we get by Lemma 3.2

$$\|\varphi_p(\tilde{\chi} - \tilde{\chi}_p)\|_{H^s(Q)} \leq C p^{-m+2+\varepsilon}, \quad 1/2 < s < 1/2 + \gamma. \tag{3.23}$$

Then (3.20), (3.22), and (3.23) lead to estimate (3.16) for any $s \in (1/2, 1/2+\gamma)$. Here we choose $\varepsilon$ small enough such that $1 + \varepsilon < 2s$, since then one can estimate $p^{-m+2+\varepsilon} \leq p^{-2\gamma-1+2s}$.

For $0 < s \leq 1/2$ the result follows by interpolation.

In order to prove (3.17) we write analogously to (3.20)

$$\begin{aligned} \|u - u_p\|_{L_2(l\cap\bar{Q})} &\leq \|\tilde{\chi}(\varphi - \varphi_p)\|_{L_2(l\cap\bar{Q})} + \|\varphi_p(\tilde{\chi} - \tilde{\chi}_p)\|_{L_2(l\cap\bar{Q})} \\ &\leq C\left(\|\tilde{\chi}\|_{C^0(\bar{Q})}\|\varphi - \varphi_p\|_{L_2(I)} + \|\tilde{\chi} - \tilde{\chi}_p\|_{C^0(\bar{Q})}\|\varphi_p\|_{L_2(I)}\right). \end{aligned}$$

Hence, similarly to the proof of (3.16) for $s = 0$, estimate (3.17) follows from (3.18), (3.19), and Lemma 3.2, using that $m > 2\gamma + 2$. □

Now, let us consider the union $A_e$ of the elements along an edge $e$. We have the following result.

**Theorem 3.3** *Let $u^e$ be given by (2.2). Then there exists a continuous function $u_p^e$ defined on $A_e$ such that $u_p^e \in \mathcal{Q}_{2p+3}(\Gamma_i)$ for $\Gamma_i \in A_e$, $u_p^e = 0$ on $\partial A_e$,*

$$\|u^e - u_p^e\|_{H^s(A_e)} \leq C\, p^{-2(\gamma+1/2-s)}\, |\log p|^\beta, \quad 0 \leq s < \min\{1, 1/2+\gamma\}, \tag{3.24}$$

*and*

$$\|u^e - u_p^e\|_{\tilde{H}^{1/2}(A_e)} \leq C\, p^{-2\gamma}\, |\log p|^\beta, \tag{3.25}$$

*where $\gamma = \gamma_1^e > 0$ and $\beta = s_1^e \geq 0$ is integer.*

**Proof.** For $\gamma_1^e > 1/2$ estimate (3.24) with $0 \leq s \leq 1$ has been proved in [8]. Estimate (3.25) then follows by interpolation between $H^0(A_e)$ and $H_0^1(A_e)$, because in this case $(u^e - u_p^e) \in H_0^1(A_e)$.

Assume now that $0 < \gamma_1^e \leq 1/2$. Let $K = \Gamma_i \in A_e$, and let $l_j \subset \partial K$ be the sides of $K$ ($j = \overline{1,3}$ or $j = \overline{1,4}$). Denote

$$\tau = \{l_j \subset \partial K;\ \bar{l}_j \cap \bar{e} = \emptyset\}, \qquad \mathcal{A} = \{l_j \subset \partial K;\ \bar{l}_j \cap \bar{e} \text{ contains only a single point}\}.$$

Then on the basis of Theorem 3.2 using exactly the same constructions and arguments as in [8, Theorem 6.2] we find a sequence $u_p^{e,K} \in \mathcal{Q}_{2p+2}(K)$ such that $u_p^{e,K} = 0$ on $(\partial K \cap \bar{e}) \cup \tau$,

$$\begin{aligned} \|u^e - u_p^{e,K}\|_{H^s(K)} &\leq C\, p^{-2(\gamma+1/2-s)}\, |\log p|^\beta, \quad 0 \leq s < 1/2+\gamma, &(3.26)\\ \|u^e - u_p^{e,K}\|_{L_2(l_j)} &\leq C\, p^{-2(\gamma+1/2)}\, |\log p|^\beta \quad \text{for every } l_j \in \mathcal{A}, &(3.27) \end{aligned}$$

where $\gamma = \gamma_1^e$ and $\beta = s_1^e$. Suppose now that $K_1,\ K_2 \in A_e$ and $\bar{K}_1 \cap \bar{K}_2 = l$ is the common edge of the elements $K_1,\ K_2$. Let $u_p^{e,i} \in \mathcal{Q}_{2p+2}(K_i)$, $i = 1, 2$, be the approximations to $u^e$ constructed above and satisfying estimates (3.26), (3.27). Then the jump $w_p = (u_p^{e,1} - u_p^{e,2})|_l \in \mathcal{P}_{2p+2}(l)$ vanishes at the end points of $l$ and, because of (3.27),

$$\|w_p\|_{L_2(l)} \leq \|u^e - u_p^{e,1}\|_{L_2(l)} + \|u^e - u_p^{e,2}\|_{L_2(l)} \leq C\, p^{-2(\gamma+1/2)}\, |\log p|^\beta.$$

Hence, using Lemma 3.6 we find a polynomial $z_p \in \mathcal{Q}_{2p+3}(K_2)$ such that $z_p = w_p$ on $l$, $z_p = 0$ on $\partial K_2 \backslash l$, and for $0 \leq s \leq 1$

$$\|z_p\|_{H^s(K_2)} \leq C p^{-2(\gamma+1-s)} |\log p|^\beta.$$

Further we set

$$\tilde{u}_p^e = u_p^{e,1} \ \text{ on } \ K_1, \qquad \tilde{u}_p^e = u_p^{e,2} + z_p \ \text{ on } \ K_2.$$

Then $\tilde{u}_p^e$ is continuous on $K_1 \cup K_2$ and satisfy estimates (3.26), (3.27) for both elements. In particular, for $0 \leq s < 1/2 + \gamma$, there holds

$$\|u^e - \tilde{u}_p^e\|_{H^s(K_2)} \leq \|u^e - u_p^{e,2}\|_{H^s(K_2)} + \|z_p\|_{H^s(K_2)} \leq C p^{-2(\gamma+1/2-s)} |\log p|^\beta.$$

Using the same arguments as above we adjust the polynomial $u_p^{e,K}$ on the element $K \in A_e \cap A_{ev}$. We construct the function $\tilde{u}_p^e$ satisfying estimates (3.26), (3.27) and vanishing on the side $l \in$

$\mathcal{A} \subset \partial K$ such that $l \cap \operatorname{supp} u^e = \emptyset$. In this case the jump is $w_p = (-u_p^{e,K})|_l \in \mathcal{P}_{2p+2}(l)$, and we set $\tilde{u}_p^e = u_p^{e,K} + z_p$ on $K$. Here, $z_p \in \mathcal{Q}_{2p+3}(K)$ is constructed using Lemma 3.6. Obviously $\tilde{u}_p^e = 0$ on $l$, and estimates (3.26), (3.27) are valid.

Repeating this procedure, we obtain a continuous function $u_p^e$ defined on $A_e$ such that $u_p^e \in \mathcal{Q}_{2p+3}(\Gamma_i)$ for $\Gamma_i \in A_e$, $u_p^e = 0$ on $\partial A_e$, and for $0 \le s < 1/2 + \gamma$

$$\|u^e - u_p^e\|_{H^s(\Gamma_i)} \;\le\; C\, p^{-2(\gamma+1/2-s)}\, |\log p|^\beta, \quad \Gamma_i \in A_e. \tag{3.28}$$

If $s \ne 1/2$, then we obtain (3.24) from (3.28) by estimating

$$\|u^e - u_p^e\|_{H^s(A_e)} \le C \sum_i \|u^e - u_p^e\|_{H^s(\Gamma_i)} \le C p^{-2(\gamma+1/2-s)} |\log p|^\beta.$$

Here, we used Lemma 3.1. Estimate (3.24) for the case $s = 1/2$ then follows by interpolation between $H^0(A_e)$ and $H^{s'}(A_e)$ with $1/2 < s' < 1/2 + \gamma$.

Since $u^e - u_p^e = 0$ on $\partial A_e$, $(u^e - u_p^e) \in H_0^{s'}(A_e)$ for any fixed $s' \in (1/2, 1/2+\gamma)$, and

$$\|u^e - u_p^e\|_{H_0^{s'}(A_e)} \le C\, \|u^e - u_p^e\|_{H^{s'}(A_e)} \le C\, p^{-2(\gamma+1/2-s')}\, |\log p|^\beta, \quad 1/2 < s' < 1/2 + \gamma.$$

Then we get estimate (3.25) by interpolation between $H^0(A_e)$ and $H_0^{s'}(A_e)$. □

### 3.4 Approximation of edge-vertex singularities.

Let us consider a component of the edge-vertex singularity term over the standard square $Q$:

$$u(x_1, x_2) = (1 + x_1)^{\lambda-\gamma} (1 + x_2)^\gamma |\log(1 + x_1)|^{\beta_1} |\log(1 + x_2)|^{\beta_2} \chi(r) \tilde{\chi}(\theta), \tag{3.29}$$

where $\beta_i \ge 0$ $(i = 1, 2)$ are integers, $\gamma > 0$, $\lambda > -1/2$, and $\chi,\ \tilde{\chi} \in C^\infty(R^+)$ are cut-off functions such that

$$\chi(r) = 1 \;\text{ for }\; 0 \le r \le 1/4 \;\text{ and }\; \chi(r) = 0 \;\text{ for }\; r \ge 1/2,$$
$$\tilde{\chi}(\theta) = 1 \;\text{ for }\; 0 \le \theta \le \pi/8 \;\text{ and }\; \tilde{\chi}(\theta) = 0 \;\text{ for }\; \theta \ge 3\pi/16.$$

Here, $(r, \theta)$ denote the polar coordinates with origin at $(-1, -1)$.

Observe that $u \in H^s(Q)$ for $0 \le s < \min\{\lambda + 1, \gamma + 1/2\}$, moreover $u \in \tilde{H}^{1/2}(Q)$.

**Theorem 3.4** *Let $u$ be given by (3.29) on $Q$ with integer $\beta_i \ge 0$ $(i = 1, 2)$ and $\gamma$, $\lambda$ are such that $1/2 < \min\{\lambda + 1, \gamma + 1/2\} \le 1$. Then there exists a sequence $u_p \in \mathcal{Q}_{p+2}(Q)$, $p = 1, 2, \ldots$, and a constant $C > 0$ independent of $p$ such that*

$$\|u - u_p\|_{H^s(\tilde{T})} \;\le\; C\, p^{-2(\min\{\lambda+1,\gamma+1/2\}-s)}\, |\log p|^\beta, \quad 0 \le s < \min\{\lambda + 1, \gamma + 1/2\}, \tag{3.30}$$

*where $\tilde{T} = \{(x_1, x_2);\ -1 < x_1 < 0,\ -1 < x_2 < x_1\}$, and*

$$\beta = \begin{cases} \beta_1 + \beta_2 + 1/2 & \text{if } \lambda = \gamma - 1/2, \\ \beta_1 + \beta_2 & \text{otherwise.} \end{cases} \tag{3.31}$$

**Remark 3.4** *The polynomial approximations for the function $u$ in* (3.29) *with $\gamma > 1/2$ and $\lambda > 0$ have been investigated in* [8]. *Since $u \in H^1(Q)$ in that case, the optimal estimates of the type* (3.30) *have been obtained for $s = 0, 1$ and then, by interpolation, for $s \in [0,1]$. The assumption $1/2 < \min\{\lambda+1, \gamma+1/2\} \le 1$ means that either $-1/2 < \lambda \le 0$ and $\gamma > 0$ or $\lambda > -1/2$ and $0 < \gamma \le 1/2$. In these cases $u \notin H^1(Q)$, and the proof of Theorem* 7.1 *in* [8] *does not apply.*

The proof of Theorem 3.4 is separated into two parts, covered by the next two lemmas. Using the idea of the proof to Theorem 7.1 in [8] we denote

$$\xi(x_1, x_2) = (1 + x_2)(x_1 - x_2),$$

and

$$u_0(x_1, x_2) = \frac{u(x_1, x_2)}{\xi(x_1, x_2)} = (1+x_1)^{\lambda-\gamma-1}(1+x_2)^{\gamma-1}|\log(1+x_1)|^{\beta_1}|\log(1+x_2)|^{\beta_2}\frac{1+x_1}{x_1-x_2}\chi(r)\tilde{\chi}(\theta).$$

Then we introduce a cut-off function $\omega$ by (3.3) and decompose the function $u_0$ as

$$u_0(x_1, x_2) = \frac{u(x_1, x_2)}{\xi(x_1, x_2)} = u_0(x_1, x_2)\omega^{\Delta}(x_2) + u_0(x_1, x_2)\tilde{\omega}^{\Delta}(x_2) =: v(x_1, x_2) + w(x_1, x_2), \quad (3.32)$$

where $\omega^{\Delta}$ and $\tilde{\omega}^{\Delta}$ are defined by (3.4).

We start with the approximation of the smooth part $v$ in (3.32). Since $v \in C^{\infty}(\bar{Q})$ and $v = 0$ for $x_2 > x_1$ and for $-1 \le x_2 \le -1 + \Delta$, the techniques of [8], estimating in $L_2$ and $H^1$ and using interpolation theory, are applicable for this function. We resume the results in the following lemma.

**Lemma 3.9** *Let $\Delta = p^{-2}$. Then there exists a sequence $u_p \in \mathcal{Q}_{p+2}(Q)$, $p = 1, 2, \ldots$, such that*

$$\|\xi v - u_p\|_{H^s(\tilde{T})} \le C\, p^{-2(\min\{\lambda+1,\gamma+1/2\}-s)}\, |\log p|^{\beta_1+\beta_2}, \quad 0 \le s \le 1.$$

*As in Theorem* 3.4, $\tilde{T} = \{(x_1, x_2);\ -1 < x_1 < 0,\ -1 < x_2 < x_1\}$.

**Proof.** For any function $f$ on $Q$ and any $0 < \Delta < 1/4$ let us define the function $f_{\Delta}$ such that

$$f_{\Delta}(x_1, x_2) = \begin{cases} f(x_1 - 2\Delta, x_2 - 2\Delta) & \text{for } (x_1, x_2) \in Q, \quad x_1 > -1 + 2\Delta, \quad x_2 > -1 + 2\Delta, \\ 0 & \text{otherwise.} \end{cases}$$

Let us denote by $v^{[i]}$ the $i$-th directional derivative of $v$ along the direction $x_2 = x_1$, i.e.,

$$v^{[i]} = \left(\frac{1}{\sqrt{2}}\right)^i \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^i v.$$

By [8, Lemma 7.1] we conclude that for any integers $k,\ l \geq 0$ there exists a constant $C(k+l)$ independent of $\Delta$ such that for $(x_1, x_2) \in Q$

$$\left|\frac{\partial^{k+l} v}{\partial x_1^k \partial x_2^l}\right| \leq C(k+l) \begin{cases} 0 \ \text{ for } \ x_2 < -1+\Delta \ \text{ or } \ x_2 > x_1, \\ |\log \Delta|^{\beta_1+\beta_2}(1+x_1)^{\lambda-\gamma-1-k}(1+x_2)^{\gamma-1-l} \ \text{ otherwise.} \end{cases} \tag{3.33}$$

Moreover, since $x_2 \leq x_1$ on $\operatorname{supp} v$, it can be verified that directional derivatives $v^{[i]}$ satisfy (3.33) with $\lambda$ and $\gamma$ being replaced by $\lambda - i$ and $\gamma - i$, respectively. Hence, the application of Lemma 7.6 of [8] to $v^{[i]}$ with $0 \leq i \leq k$ (here, $k = [\lambda+1]$ is the largest integer less than or equal to $\lambda+1$, so that $\lambda - k < 0$) gives polynomials $v_p^{[i]} \in \mathcal{Q}_p(Q)$ such that (cf. (7.29) in [8])

$$\|\xi_\Delta(v^{[i]} - v_p^{[i]})\|_{H^s(\tilde{T}_\Delta)} \ \leq \ C\, p^{-2(\min\{\lambda+1,\gamma+1/2\}-i-s)}\, |\log p|^{\beta_1+\beta_2}, \tag{3.34}$$

where $\tilde{T}_\Delta = \{(x_1+2\Delta, x_2+2\Delta);\ (x_1,x_2) \in \tilde{T}\}$, and $0 \leq s \leq 1$.

Let $T_k(v_\Delta)$ be the Taylor expansion defined by (3.9). Then Lemma 7.7 of [8] yields

$$\|\xi_\Delta(v_\Delta - T_k(v_\Delta))\|_{H^s(\tilde{T}_\Delta)} \ \leq \ C\, p^{-2(\min\{\lambda+1,\gamma+1/2\}-s)}\, |\log p|^{\beta_1+\beta_2}, \tag{3.35}$$

where $\Delta = p^{-2},\ k = [\lambda+1]$, and $0 \leq s \leq 1$.

Now we define a polynomial $z_{p\Delta} = \xi_\Delta \sum\limits_{i=0}^{[\lambda+1]} (-1)^i \frac{(2\Delta)^i}{i!} v_p^{[i]}$. We see that $z_{p\Delta} \in \mathcal{Q}_{p+2}(\tilde{T}_\Delta)$. Then making use of (3.34), (3.35), and the triangle inequality we obtain similarly to (3.11)

$$\|\xi_\Delta v_\Delta - z_{p\Delta}\|_{H^s(\tilde{T}_\Delta)} \leq C p^{-2(\min\{\lambda+1,\gamma+1/2\}-s)} |\log p|^{\beta_1+\beta_2}, \quad 0 \leq s \leq 1.$$

The translation to the triangle $\tilde{T}$ proves the statement of the lemma. □

The function $w$ in (3.32) has a small support, $\operatorname{supp} w \subset R_\Delta = \{(x_1,x_2) \in \tilde{T};\ x_1 \leq -1/2,\ x_2 \leq -1+2\Delta\}$. We show that the function $\xi w$ being approximated by zero leads to the same estimate as in (3.30).

**Lemma 3.10** *Let $\Delta = p^{-2}$. Then*

$$\|\xi w\|_{H^s(\tilde{T})} \ \leq \ C\, p^{-2(\min\{\lambda+1,\gamma+1/2\}-s)}\, |\log p|^{\beta}, \quad 0 \leq s < \min\{\lambda+1, \gamma+1/2\}, \tag{3.36}$$

*where $\beta$ is defined by* (3.31), and $C > 0$ is independent of $p$.

**Proof.** Let us introduce a new parameter $\Delta' = (1+\sqrt{2})\Delta$. Assuming that $\Delta$ is sufficiently small we divide $R_\Delta = (ABEF)$ as (see Figure 1)

$$R_\Delta = R_\Delta^1 \cup R_\Delta^2 \cup R_\Delta^3,$$

where $R_\Delta^1$ is the triangle $ABC$, $R_\Delta^2$ is the trapezoid $ACDG$, $R_\Delta^3$ is the rectangle $DEFG$, $C = (-1+\Delta', -1+2\Delta)$, $D = (-1+2\Delta', -1+2\Delta)$, and $G = (-1+2\Delta', -1)$.

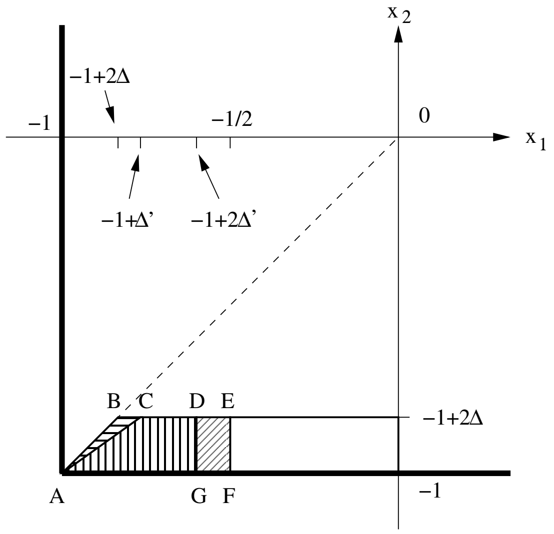


Figure 1: Decomposition for the proof of Lemma 3.10.

Observe that $w = 0$ on $R^1_\Delta$, because $\widehat{CAG} > \frac{3\pi}{16}$. Hence $\operatorname{supp} w \subset \bar{R}^2_\Delta \cup \bar{R}^3_\Delta$.
Let $0 \leq s < \min\{\lambda+1, \gamma+1/2\}$ and $s \neq 1/2$. Then denoting

$$w_0(x_1, x_2) = (1+x_1)^{\lambda-\gamma}(1+x_2)^{\gamma}|\log(1+x_1)|^{\beta_1}|\log(1+x_2)|^{\beta_2}\tilde{\omega}^{\Delta}(x_2),$$

we have by Lemma 3.1

$$\|\xi w\|^2_{H^s(\tilde{T})} \leq C\Big[\|\xi w\|^2_{H^s(R^2_\Delta)} + \|\xi w\|^2_{H^s(R^3_\Delta)}\Big] \leq C\Big[\|w_0\tilde{\chi}\|^2_{H^s(R^2_\Delta)} + \|w_0\|^2_{H^s(R^3_\Delta)}\Big], \tag{3.37}$$

because $\chi(r) \in C^1(\bar{Q})$, and $\tilde{\chi}(\theta) = 1$ on $R^3_\Delta$.

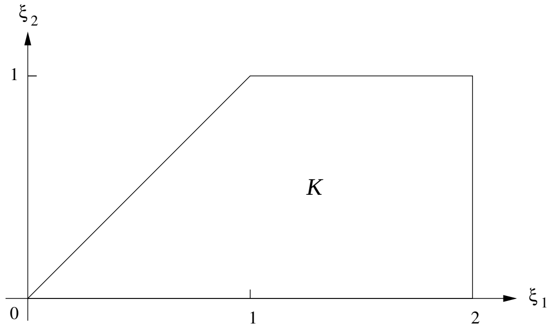


Figure 2: The trapezoid $K$ for the proof of Lemma 3.10.

In order to estimate the first term on the right-hand side of (3.37) we consider the trapezoid $K$ in the plane $O'\xi_1\xi_2$ (see Figure 2). Using the mapping $M : K \to R^2_\Delta$ with

$$M : \begin{cases} x_1 = -1 + \Delta'\xi_1, \\ x_2 = -1 + 2\Delta\xi_2, \end{cases}$$

there holds

$$\|f\|_{L_2(R^2_\Delta)} \simeq \Delta \|F\|_{L_2(K)}$$

with $F := f \circ M$ if $f \in L_2(R^2_\Delta)$, and

$$\|f\|_{H^1(R^2_\Delta)} \le C\|F\|_{H^1(K)}$$

if $f \in H^1(R^2_\Delta)$. Therefore, for $0 < \tilde{s} < 1$, one has by interpolation

$$\|f\|_{H^{\tilde{s}}(R^2_\Delta)} \le C\Delta^{1-\tilde{s}}\|F\|_{H^{\tilde{s}}(K)}, \tag{3.38}$$

provided that $f \in H^{\tilde{s}}(R^2_\Delta)$. Since for $(x_1, x_2) \in R^2_\Delta$ and corresponding $(\xi_1, \xi_2) \in K$

$$\begin{aligned} & w_0(x_1,x_2)\tilde{\chi}(\theta(x_1,x_2)) \\ &= (\Delta'\xi_1)^{\lambda-\gamma}(2\Delta\xi_2)^{\gamma}|\log \Delta'\xi_1|^{\beta_1}|\log 2\Delta\xi_2|^{\beta_2}(1-\omega(2\xi_2))\tilde{\chi}\Big(\arctan \tfrac{2\Delta\xi_2}{\Delta'\xi_1}\Big) \\ &= C\Delta^{\lambda}\xi_1^{\lambda-\gamma}\xi_2^{\gamma}|\log \Delta'\xi_1|^{\beta_1}|\log 2\Delta\xi_2|^{\beta_2}(1-\omega(2\xi_2))\tilde{\chi}\Big(\arctan \tfrac{2\xi_2}{(1+\sqrt{2})\xi_1}\Big), \end{aligned}$$

we use (3.38) with $\tilde{s} = s$ to obtain for sufficiently small $\Delta$

$$\begin{aligned} & \|w_0\tilde{\chi}\|_{H^s(R^2_\Delta)} \\ &\le C\Delta^{\lambda+1-s}\Big\|\xi_1^{\lambda-\gamma}\xi_2^{\gamma}|\log \Delta'\xi_1|^{\beta_1}|\log 2\Delta\xi_2|^{\beta_2}(1-\omega(2\xi_2))\tilde{\chi}\Big(\arctan \frac{2\xi_2}{(1+\sqrt{2})\xi_1}\Big)\Big\|_{H^s(K)} \\ &\le C\Delta^{\lambda+1-s}|\log\Delta|^{\beta_1+\beta_2}, \end{aligned} \tag{3.39}$$

where $C > 0$ is independent of $\Delta$.

Now we prove the estimate for $\|w_0\|^2_{H^s(R^3_\Delta)} = \|w_0\|^2_{L_2(R^3_\Delta)} + |w_0|^2_{H^s(R^3_\Delta)}$. For $\|w_0\|^2_{L_2(R^3_\Delta)}$ one immediately has (hereafter, $\tilde{x}_i = 1 + x_i,\ i = 1, 2$)

$$\begin{aligned} \|w_0\|^2_{L_2(R^3_\Delta)} &\le C\int_{2\Delta'}^{\frac12}\int_0^{2\Delta} \tilde{x}_1^{2(\lambda-\gamma)}\tilde{x}_2^{2\gamma}|\log\tilde{x}_1|^{2\beta_1}|\log\tilde{x}_2|^{2\beta_2}\,d\tilde{x}_2\,d\tilde{x}_1 \\ &\le C\Delta^{2\gamma+1}|\log\Delta|^{2\beta_2}\int_{2\Delta'}^{\frac12}\tilde{x}_1^{2(\lambda-\gamma)}|\log\tilde{x}_1|^{2\beta_1}\,d\tilde{x}_1 \\ &\le C\Delta^{2\gamma+1}|\log\Delta|^{2\beta_2}\begin{cases} \Delta^{2(\lambda-\gamma)+1}|\log\Delta|^{2\beta_1} & \text{if} \quad \lambda < \gamma - 1/2, \\ |\log\Delta|^{2\beta_1+1} & \text{if} \quad \lambda = \gamma - 1/2, \\ 1 & \text{if} \quad \lambda > \gamma - 1/2, \end{cases} \end{aligned} \tag{3.40}$$

provided that $\Delta$ is sufficiently small.

Let $s \neq 0$. For the semi-norm $|w_0|_{H^s(R^3_\Delta)}$ we first use Lemma 3.5 to obtain

$$
\begin{aligned}
&|w_0|^2_{H^s(R^3_\Delta)} \leq \\
&C\Bigg[\int_{2\Delta'}^{\frac{1}{2}}\int_{2\Delta'}^{\frac{1}{2}} \frac{\|\tilde{x}_1^{\lambda-\gamma}\tilde{\eta}^\gamma|\log\tilde{x}_1|^{\beta_1}|\log\tilde{\eta}|^{\beta_2}\tilde{\omega}^\Delta(\tilde{\eta}) - \tilde{y}_1^{\lambda-\gamma}\tilde{\eta}^\gamma|\log\tilde{y}_1|^{\beta_1}|\log\tilde{\eta}|^{\beta_2}\tilde{\omega}^\Delta(\tilde{\eta})\|^2_{L_2(0,2\Delta)}}{|\tilde{x}_1-\tilde{y}_1|^{1+2s}}\, d\tilde{x}_1\, d\tilde{y}_1 \\
&+\int_0^{2\Delta}\int_0^{2\Delta} \frac{\|\tilde{\xi}^{\lambda-\gamma}\tilde{x}_2^\gamma|\log\tilde{\xi}|^{\beta_1}|\log\tilde{x}_2|^{\beta_2}\tilde{\omega}^\Delta(\tilde{x}_2) - \tilde{\xi}^{\lambda-\gamma}\tilde{y}_2^\gamma|\log\tilde{\xi}|^{\beta_1}|\log\tilde{y}_2|^{\beta_2}\tilde{\omega}^\Delta(\tilde{y}_2)\|^2_{L_2(2\Delta',1/2)}}{|\tilde{x}_2-\tilde{y}_2|^{1+2s}}\, d\tilde{x}_2\, d\tilde{y}_2\Bigg] \\
&\quad = \quad C\Bigg[\left\|\tilde{\eta}^\gamma|\log\tilde{\eta}|^{\beta_2}\tilde{\omega}^\Delta(\tilde{\eta})\right\|^2_{L_2(0,2\Delta)}\left|\tilde{x}_1^{\lambda-\gamma}|\log\tilde{x}_1|^{\beta_1}\right|^2_{H^s(2\Delta',1/2)} \\
&\qquad\quad +\left\|\tilde{\xi}^{\lambda-\gamma}|\log\tilde{\xi}|^{\beta_1}\right\|^2_{L_2(2\Delta',1/2)}\left|\tilde{x}_2^\gamma|\log\tilde{x}_2|^{\beta_2}\tilde{\omega}^\Delta(\tilde{x}_2)\right|^2_{H^s(0,2\Delta)}\Bigg]. \qquad (3.41)
\end{aligned}
$$

Then we estimate each norm and semi-norm on the right-hand side of (3.41). Since $\tilde{\omega}^\Delta \in C^\infty(R)$ and $\tilde{\eta}^\gamma|\log\tilde{\eta}|^{\beta_2}$ is an increasing function on $(0, 2\Delta)$, we get

$$\|\tilde{\eta}^\gamma|\log\tilde{\eta}|^{\beta_2}\tilde{\omega}^\Delta(\tilde{\eta})\|^2_{L_2(0,2\Delta)} \leq C\Delta^{2\gamma+1}|\log\Delta|^{2\beta_2}. \qquad (3.42)$$

The function $\varphi(\tilde{x}_1) = \tilde{x}_1^{\lambda-\gamma}|\log\tilde{x}_1|^{\beta_1} \in H^1(2\Delta', 1/2)$. Therefore, we obtain the estimate for $|\varphi|_{H^s(2\Delta',1/2)}$ by interpolation. One has

$$\|\varphi\|^2_{L_2(2\Delta',1/2)} \leq \begin{cases} C\Delta^{2(\lambda-\gamma)+1}|\log\Delta|^{2\beta_1} & \text{if} \quad \lambda < \gamma - 1/2, \\ C|\log\Delta|^{2\beta_1+1} & \text{if} \quad \lambda = \gamma - 1/2, \\ C & \text{if} \quad \lambda > \gamma - 1/2, \end{cases} \qquad (3.43)$$

and

$$
\begin{aligned}
|\varphi|^2_{H^1(2\Delta',1/2)} &\leq C\int_{2\Delta'}^{\frac{1}{2}} \tilde{x}_1^{2(\lambda-\gamma-1)}|\log\tilde{x}_1|^{2\beta_1}\, d\tilde{x}_1 \\
&\leq \begin{cases} C\Delta^{2(\lambda-\gamma)-1}|\log\Delta|^{2\beta_1} & \text{if} \quad \lambda < \gamma + 1/2, \\ C|\log\Delta|^{2\beta_1+1} & \text{if} \quad \lambda = \gamma + 1/2, \\ C & \text{if} \quad \lambda > \gamma + 1/2. \end{cases} \qquad (3.44)
\end{aligned}
$$

Hence we have five cases.
(i) If $\lambda < \gamma - 1/2$, then by (3.43) and (3.44)

$$\|\varphi\|^2_{L_2(2\Delta',1/2)} \leq C\Delta^{2(\lambda-\gamma)+1}|\log\Delta|^{2\beta_1},$$

$$\|\varphi\|^2_{H^1(2\Delta',1/2)} \leq C|\log\Delta|^{2\beta_1}\left[\Delta^{2(\lambda-\gamma)+1} + \Delta^{2(\lambda-\gamma)-1}\right] \leq C\Delta^{2(\lambda-\gamma)-1}|\log\Delta|^{2\beta_1};$$

then by interpolation

$$|\varphi|^2_{H^s(2\Delta',1/2)} \le \|\varphi\|^2_{H^s(2\Delta',1/2)} \le C\Delta^{2(\lambda-\gamma)+1-2s}|\log\Delta|^{2\beta_1} \quad (0<s<1). \tag{3.45}$$

We proceed analogously in the other cases:
(ii) If $\lambda=\gamma-1/2$, then

$$\|\varphi\|^2_{L_2(2\Delta',1/2)} \le C|\log\Delta|^{2\beta_1+1},$$

$$\|\varphi\|^2_{H^1(2\Delta',1/2)} \le C|\log\Delta|^{2\beta_1+1}\left[1+\Delta^{2(\lambda-\gamma)-1}|\log\Delta|^{-1}\right] \le C\Delta^{2(\lambda-\gamma)-1}|\log\Delta|^{2\beta_1+1},$$

$$|\varphi|^2_{H^s(2\Delta',1/2)} \le C\Delta^{2s(\lambda-\gamma)-s}|\log\Delta|^{2\beta_1+1}. \tag{3.46}$$

(iii) If $\gamma-1/2<\lambda<\gamma+1/2$, then

$$\|\varphi\|^2_{L_2(2\Delta',1/2)} \le C,$$

$$\|\varphi\|^2_{H^1(2\Delta',1/2)} \le C\left[1+\Delta^{2(\lambda-\gamma)-1}|\log\Delta|^{2\beta_1}\right] \le C\Delta^{2(\lambda-\gamma)-1}|\log\Delta|^{2\beta_1},$$

$$|\varphi|^2_{H^s(2\Delta',1/2)} \le C\Delta^{2s(\lambda-\gamma)-s}|\log\Delta|^{2\beta_1 s}. \tag{3.47}$$

(iv) If $\lambda=\gamma+1/2$, then

$$\|\varphi\|^2_{L_2(2\Delta',1/2)} \le C, \qquad \|\varphi\|^2_{H^1(2\Delta',1/2)} \le C\left[1+|\log\Delta|^{2\beta_1+1}\right] \le C|\log\Delta|^{2\beta_1+1},$$

$$|\varphi|^2_{H^s(2\Delta',1/2)} \le C|\log\Delta|^{2s(\beta_1+1/2)}. \tag{3.48}$$

(v) If $\lambda>\gamma+1/2$, then

$$\|\varphi\|^2_{L_2(2\Delta',1/2)} \le C, \qquad \|\varphi\|^2_{H^1(2\Delta',1/2)} \le C,$$

$$|\varphi|^2_{H^s(2\Delta',1/2)} \le C. \tag{3.49}$$

We continue estimating the other terms on the right-hand side of (3.41).
The norm $\|\tilde\xi^{\lambda-\gamma}|\log\tilde\xi|^{\beta_1}\|^2_{L_2(2\Delta',1/2)}$ is bounded as in (3.43), and by (3.14) we get

$$\left|\tilde x_2^{\gamma}|\log\tilde x_2|^{\beta_2}\tilde\omega^{\Delta}(\tilde x_2)\right|^2_{H^s(0,2\Delta)} \le C\Delta^{2\gamma+1-2s}|\log\Delta|^{2\beta_2}. \tag{3.50}$$

Using estimates (3.42), (3.43), (3.45)–(3.50) for the norms and semi-norms we obtain by (3.41):
(i) if $\lambda<\gamma-1/2$, then

$$|w_0|^2_{H^s(R^3_\Delta)} \le C\Delta^{2\lambda+2-2s}|\log\Delta|^{2(\beta_1+\beta_2)};$$

(ii) if $\lambda=\gamma-1/2$, then

$$|w_0|^2_{H^s(R^3_\Delta)} \le C\Delta^{2\gamma+1-2s}|\log\Delta|^{2(\beta_1+\beta_2)+1};$$

(iii) if $\gamma - 1/2 < \lambda < \gamma + 1/2$, then

$$|w_0|^2_{H^s(R^3_\Delta)} \le C\Delta^{2\gamma+1-2s}|\log\Delta|^{2\beta_2}\left[\Delta^{2s(\lambda-\gamma)+s}|\log\Delta|^{2\beta_1 s}+1\right] \le C\Delta^{2\gamma+1-2s}|\log\Delta|^{2\beta_2};$$

(iv) if $\lambda = \gamma + 1/2$, then

$$|w_0|^2_{H^s(R^3_\Delta)} \le C\Delta^{2\gamma+1-2s}|\log\Delta|^{2\beta_2}\left[\Delta^{2s}|\log\Delta|^{2s(\beta_1+1/2)}+1\right] \le C\Delta^{2\gamma+1-2s}|\log\Delta|^{2\beta_2};$$

(v) if $\lambda > \gamma + 1/2$, then

$$|w_0|^2_{H^s(R^3_\Delta)} \le C\Delta^{2\gamma+1-2s}|\log\Delta|^{2\beta_2}\left[\Delta^{2s}+1\right] \le C\Delta^{2\gamma+1-2s}|\log\Delta|^{2\beta_2}.$$

Combining these results with (3.40) we get the inequality for the norm

$$\|w_0\|^2_{H^s(R^3_\Delta)} \le C\Delta^{2\min\{\lambda+1,\gamma+1/2\}-2s}|\log\Delta|^{2\sigma}, \tag{3.51}$$

where $\sigma = \beta_1 + \beta_2$ if $\lambda < \gamma - 1/2$, $\sigma = \beta_1 + \beta_2 + 1/2$ if $\lambda = \gamma - 1/2$, and $\sigma = \beta_2$ otherwise.

Finally, taking $\Delta = p^{-2}$ in (3.39), (3.51) and using (3.37) we derive estimate (3.36) for any $s \in [0, \min\{\lambda+1, \gamma+1/2\}) \setminus \{1/2\}$. For $s = 1/2$ this result follows by interpolation between $H^0(\tilde T)$ and $H^{s'}(\tilde T)$ with $1/2 < s' < \min\{\lambda+1, \gamma+1/2\}$. □

**Remark 3.5** *Since $w \in H_0^{s'}(\tilde T)$ for $1/2 < s' < \min\{\lambda+1,\gamma+1/2\}$, by interpolation between $H^0(\tilde T)$ and $H_0^{s'}(\tilde T)$ we obtain the estimate*

$$\|\xi w\|_{\tilde H^{1/2}(\tilde T)} \le C\,p^{-2\min\{\lambda+1/2,\gamma\}}\,|\log p|^{\beta},$$

*where $\beta$ is defined by* (3.31).

**Proof of Theorem 3.4.** The statement of the theorem follows from Lemmas 3.9 and 3.10 by using (3.32). □

Now, for a given edge $e$ and a vertex $v \in \bar e$ we consider an element $K \in A_{ev}$ such that $\bar K \cap e \ne \emptyset$ and $v \in \bar K$ simultaneously.

**Theorem 3.5** *Let $u_1^{ev}$ be given by* (2.4). *Then there exists a sequence $u_{1,p}^{ev} \in \mathcal{Q}_{p+4}(K)$, $p = 1, 2, \ldots$, such that $u_{1,p}^{ev} = 0$ on $\partial K$,*

$$\|u_1^{ev} - u_{1,p}^{ev}\|_{H^s(K)} \le C\,p^{-2(\min\{\lambda+1,\gamma+1/2\}-s)}\,|\log p|^{\beta}, \quad 0 \le s < \min\{1, \lambda+1, \gamma+1/2\}, \tag{3.52}$$

*and*

$$\|u_1^{ev} - u_{1,p}^{ev}\|_{\tilde H^{1/2}(K)} \le C\,p^{-2\min\{\lambda+1/2,\gamma\}}\,|\log p|^{\beta}, \tag{3.53}$$

*where $\lambda = \lambda_1^v > -1/2$, $\gamma = \gamma_1^e > 0$, $\beta = q_1^v + s_1^e + 1/2$ if $\lambda_1^v = \gamma_1^e - 1/2$, and $\beta = q_1^v + s_1^e$ otherwise.*

**Proof.** First we note that estimate (3.52) with $0 \le s \le 1$ has been proved in [8] for $\lambda_1^v > 0$ and $\gamma_1^e > 1/2$. Estimate (3.53) then follows by interpolation between $H^0(K)$ and $H_0^1(K)$, because in this case $(u_1^{ev} - u_{1,p}^{ev}) \in H_0^1(K)$.

Assume now that $1/2 < \min\{\lambda+1, \gamma+1/2\} \le 1$. Then, applying the result of Theorem 3.4 and repeating the arguments in the proof of Theorem 7.2 in [8] we find a polynomial $u_{1,p}^{ev} \in \mathcal{Q}_{p+4}(K)$ for $p = 1, 2, \ldots$, such that $u_{1,p}^{ev} = 0$ on $\partial K$ and inequality (3.52) holds. Since $(u_1^{ev} - u_{1,p}^{ev}) \in H_0^{s'}(K)$ for $1/2 < s' < \min\{\lambda+1, \gamma+1/2\}$, estimate (3.53) follows by interpolation between $H^0(K)$ and $H_0^{s'}(K)$. □

## 3.5 The general approximation result.

Before giving the general approximation theorem we recall results for vertex singularities and regular functions.

Let $v$ be a vertex of $\Gamma$ and $A_v$ be the union of the elements $\Gamma_j$ such that $v \in \bar{\Gamma}_j$. We consider the corresponding vertex singularity term $u^v$ given by (2.3). Since $u^v \in H^1(A_v)$, we can simply quote the following result from [8, Theorems 8.2, 9.1].

**Theorem 3.6** *Let $u^v$ be given by* (2.3). *Then there exists a continuous function $u_p^v$ defined on $A_v$ such that $u_p^v \in \mathcal{Q}_{p+5}(\Gamma_i)$ for $\Gamma_i \in A_v$, $u_p^v = 0$ on $\partial A_v$, and for $0 \le s \le 1$*

$$\|u^v - u_p^v\|_{H^s(A_v)} \;\le\; C\,p^{-2(\lambda+1)+2s}\,|\log p|^\beta, \tag{3.54}$$

*where $\lambda = \lambda_1^v > 0$ and $\beta = q_1^v \ge 0$ is integer.*

**Remark 3.6** *Observing that $(u^v - u_p^v) \in H_0^1(A_v)$, we get by interpolation between $H^0(A_v)$ and $H_0^1(A_v)$*

$$\|u^v - u_p^v\|_{\tilde{H}^{1/2}(A_v)} \;\le\; C\,p^{-2\lambda_1^v - 1}\,|\log p|^{q_1^v}. \tag{3.55}$$

For the approximation of the regular part $u_{\text{reg}}$ of the solution by the $p$-version we use the standard result [2, Theorem 4.1].

**Theorem 3.7** *If $u_{\text{reg}} \in H^k(\Gamma)$, $k > 3/2$, then for $0 \le s \le 1$,*

$$\inf_{\psi \in V_0^p} \|u_{\text{reg}} - \psi\|_{H^s(\Gamma)} \;\le\; C\,p^{-(k-1)}\,\|u_{\text{reg}}\|_{H^k(\Gamma)}, \tag{3.56}$$

*where $C$ is a positive constant independent of $u_{\text{reg}}$ and $p$.*

**Remark 3.7** *Since for our model problem $u_{\text{reg}} = 0$ on $\partial\Gamma$ and $V_0^p \subset H_0^1(\Gamma)$, one also has*

$$\inf_{\psi \in V_0^p} \|u_{\text{reg}} - \psi\|_{\tilde{H}^{1/2}(\Gamma)} \;\le\; C\,p^{-(k-1)}\,\|u_{\text{reg}}\|_{H^k(\Gamma)}. \tag{3.57}$$

**Remark 3.8** *In general, estimates* (3.56) *and* (3.57) *above are not optimal. In fact, by Theorem* 4.1 *in* [2] *we have*

$$\inf_{\psi\in V_0^p} \|u_{\rm reg}-\psi\|_{H^1(\Gamma)} \;\le\; C\,p^{-(k-1)}\,\|u_{\rm reg}\|_{H^k(\Gamma)},$$

*provided that* $u_{\rm reg}\in H^k(\Gamma)$ *for* $k>3/2$.

*Using the same arguments as in the proof of the mentioned theorem for the norm* $\|\cdot\|_{L_2(\Gamma)}$, *one can also obtain the estimate*

$$\inf_{\psi\in V_0^p} \|u_{\rm reg}-\psi\|_{L_2(\Gamma)} \;\le\; C\,p^{-k}\,\|u_{\rm reg}\|_{H^k(\Gamma)}.$$

*Then by interpolation we derive for* $u_{\rm reg}\in H^k(\Gamma)$ *with* $k>3/2$ *(see also* [10, Theorem 3.1]*):*

$$\inf_{\psi\in V_0^p} \|u_{\rm reg}-\psi\|_{H^s(\Gamma)} \;\le\; C\,p^{-(k-s)}\,\|u_{\rm reg}\|_{H^k(\Gamma)},\quad 0\le s\le 1,$$

$$\inf_{\psi\in V_0^p} \|u_{\rm reg}-\psi\|_{\tilde H^{1/2}(\Gamma)} \;\le\; C\,p^{-(k-1/2)}\,\|u_{\rm reg}\|_{H^k(\Gamma)}.$$

Now we are able to estimate the error of approximation of the function $u$ given by (2.1)–(2.5).

**Theorem 3.8** *Let the function* $u$ *be given by* (2.1)–(2.5) *on* $\Gamma$ *where, as mentioned in Remark* 2.1(i), *any value* $\gamma_1^e>0$ *is permitted. Also, let* $v_0\in V$, $e_0\in E(v_0)$ *be such that* $\min\{\lambda_1^{v_0}+1/2,\gamma_1^{e_0}\}=\min_{v\in V,e\in E(v)}\min\{\lambda_1^v+1/2,\gamma_1^e\}$, *with* $\lambda_1^v$ *and* $\gamma_1^e$ *being as in* (2.2)–(2.5). *Then, for every* $p=1,2,\ldots$, *there exists a function* $u_p\in V_0^p$ *such that*

$$\|u-u_p\|_{H^s(\Gamma)} \;\le\; C\,\max\left\{p^{-(k-1)},\,p^{-2(\min\{\lambda_1^{e_0}+1,\gamma_1^{v_0}+1/2\}-s)}\,|\log p|^{\beta}\right\} \tag{3.58}$$

*and*

$$\|u-u_p\|_{\tilde H^{1/2}(\Gamma)} \;\le\; C\,p^{-2\min\{\lambda_1^{e_0}+1/2,\gamma_1^{v_0}\}}\,|\log p|^{\beta} \tag{3.59}$$

*for* $0\le s<\min\{1,\lambda_1^{e_0}+1,\gamma_1^{v_0}+1/2\}$, *and* $\beta$ *is defined as in* (3.31) *(in terms of* $q_1^{v_0}$ *and* $s_1^{e_0}$*).*

**Proof.** We use Theorems 3.3, 3.5, and 3.6 to find continuous functions (piecewise polynomials) $u_p^e$, $u_{1,p}^{ev}$, and $u_p^v$ defined on $A_e$, $A_{ev}$, and $A_v$ respectively, and vanishing on the boundaries of the corresponding domains. The approximation $u_{2,p}^{ev}$ to the function $u_2^{ev}$ is constructed in exactly the same way as the approximation $u_p^e$ for $u^e$ (see Theorem 3.3). Extending $u_p^e$, $u_{1,p}^{ev}$, $u_{2,p}^{ev}$, and $u_p^v$ by zero onto the remaining parts of $\Gamma$, we see that $\|u^e-u_p^e\|_{H^s(\Gamma)}$, $\|u_1^{ev}-u_{1,p}^{ev}\|_{H^s(\Gamma)}$, and $\|u^v-u_p^v\|_{H^s(\Gamma)}$ are bounded as in (3.24), (3.52), and (3.54), respectively. The bound for $\|u_2^{ev}-u_{2,p}^{ev}\|_{H^s(\Gamma)}$ is the same as for $\|u^e-u_p^e\|_{H^s(\Gamma)}$, and the estimates analogous to (3.25), (3.53), and (3.55) are valid as well. Then making use of Theorem 3.7 (see also Remark 3.7) and the triangle inequality, we obtain estimates (3.58) and (3.59). $\square$

**Acknowledgement.**

This work was accomplished while A. Bespalov was a visiting postdoctoral researcher at the Departamento de Ingeniería Matemática of the Universidad de Concepción, Chile. This author wishes to express his gratitude to the colleagues of that Department for their hospitality and the stimulating research atmosphere.

## References

[1] I. Babuška and B. Q. Guo, *Optimal estimates for lower and upper bounds of approximation errors in the p-version of the finite element method in two dimensions*, Numer. Math., 85 (2000), pp. 219–255.

[2] I. Babuška and M. Suri, *The optimal convergence rate of the p-version of the finite element method*, SIAM J. Numer. Anal., 24 (1987), pp. 750–776.

[3] M. Costabel, *Boundary integral operators on Lipschitz domains: Elementary results*, SIAM J. Math. Anal., 19 (1988), pp. 613–626.

[4] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman Publishing Inc., Boston, 1985.

[5] B. Q. Guo and N. Heuer, *The optimal rate of convergence of the p-version of the boundary element method in two dimensions*, Preprint 02-19, Departamento de Ingeniería Matemática, Universidad de Concepción, Chile, 2002.

[6] J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications I*, Springer-Verlag, New York, 1972.

[7] J. Nečas, *Les Méthodes Directes en Théorie des Équations Elliptiques*, Academia, Prague, 1967.

[8] C. Schwab and M. Suri, *The optimal p-version approximation of singularities on polyhedra in the boundary element method*, SIAM J. Numer. Anal., 33 (1996), pp. 729–759.

[9] E. P. Stephan, *Boundary integral equations for screen problems in* $\mathbf{R}^3$, Integral Equations Operator Theory, 10 (1987), pp. 257–263.

[10] E. P. Stephan and M. Suri, *On the convergence of the p-version of the boundary element Galerkin method*, Math. Comp., 52 (1989), pp. 31–48.

[11] T. von Petersdorff and E. P. Stephan, *Regularity of mixed boundary value problems in* $\mathbf{R}^3$ *and boundary element methods on graded meshes*, Math. Methods Appl. Sci., 12 (1990), pp. 229–249.