# The $p$-version of the boundary element method for weakly singular operators on piecewise plane open surfaces

Alexei Bespalov *  Norbert Heuer †

*Dedicated to Professor Wolfgang L. Wendland on the occasion of his 70th birthday.*

**Abstract**

We study piecewise polynomial approximations in negative order Sobolev norms of singularities which are inherent to Neumann data of elliptic problems of second order in polyhedral domains. The worst case of exterior crack problems in three dimensions is included. As an application, we prove an optimal a priori error estimate for the $p$-version of the BEM with weakly singular operators on polyhedral surfaces and piecewise plane open screens.

*Key words*: $p$-version, boundary element method, singularities
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## 1 Introduction and formulation of the problem

In this paper, we prove an optimal error estimate for the $p$-version of the boundary element method (BEM) with weakly singular operators on open and closed piecewise plane surfaces. The energy space, $\tilde{H}^{-1/2}(\Gamma)$, is a Sobolev space of negative order and the problems under consideration have singularities which can be less than $L_2(\Gamma)$ regular. In [4] we considered the case of hypersingular operators where the energy space is $\tilde{H}^{1/2}(\Gamma)$ (for a definition of the Sobolev spaces see §3). There, we generalised known results from Schwab and Suri [10] to situations where solutions are not in $H^1(\Gamma)$. Here, we approximate singular functions in $\tilde{H}^s(\Gamma)$ for negative $s$. We use directional antiderivatives to transform these approximation problems to corresponding ones for singularities in Sobolev spaces of positive order, where techniques analogous to [4, 10] can be used.

*Computational Centre, Far-Eastern Branch of the Russian Academy of Sciences, Khabarovsk, Russia. email: albespalov@yahoo.com. Supported by the London Mathematical Society.
†BICOM, Department of Mathematical Sciences, Brunel University, Uxbridge, Middlesex UB8 3PH, UK. email: norbert.heuer@brunel.ac.uk. Supported by the FONDAP Programme in Applied Mathematics and Fondecyt project no. 1040615, both Chile.

This idea of directional antiderivatives works well for singularities of tensor product form but fails for general functions (cf. Remark 3.13). In a previous paper [3] we studied the case of weakly singular operators on open surfaces with smooth boundary curve (where one has to deal with only one particular edge singularity). In that situation we were able to reduce the approximation problem to a one-dimensional situation where standard derivatives and antiderivatives can be used to map between a scale of Sobolev spaces (see Stephan and Suri [12, Lemma 3.5]). In this paper we consider the general case of singular functions including vertex and edge-vertex singularities. We still use the idea of Stephan and Suri, but have to analyse the full two-dimensional (surface) situation. As mentioned before, having transformed the approximation problems to perform the analysis in Sobolev spaces of positive order, techniques are similar to the ones for hypersingular operators in [4]. However, in order to apply the tool of directional antiderivatives we have to consider model situations on special elements (with small angles at surface vertices). The generalisation to arbitrary elements then uses affine mappings. The analysis of this generalisation is somewhat technical (and details are given in several appendices) since one needs to show that the necessary transformation of singularities does not change their overall behaviour (in the sense of convergence rates of the $p$-version).

An outcome of this paper is that conjectured results [7] on the convergence order of $p$-version BE schemes with weakly singular operators are true. In the particular situation of the Laplacian in the domain exterior to an open screen of the form of a square, our results prove a convergence like $O(p^{-1})$. Here, $p$ denotes the polynomial degree of the ansatz functions. For details and numerical results (confirming this convergence rate) see [7].

In what follows, the analysis applies to open and closed surfaces which must be piecewise plane such that they can be discretised by meshes consisting of triangles and parallelograms. For ease of presentation, however, let us assume that $\Gamma \subset \mathbf{R}^3$ is a plane open surface with polygonal boundary. Then, our model integral equation is

$$Vu(x) := \frac{1}{4\pi} \int_\Gamma \frac{u(y)}{|x-y|} \, dS_y = f(x), \quad x \in \Gamma.$$

It is well known that this equation governs a Dirichlet problem for the Laplacian in the domain exterior to $\Gamma$, with given Dirichlet datum $f$ on $\Gamma$, see [5, 11]. The solution $u$ of the integral equation is the jump across $\Gamma$ of the normal derivative of the solution to the Dirichlet problem. The weak form of this integral equation is: *Find* $u \in \tilde{H}^{-1/2}(\Gamma)$ *such that*

$$\langle Vu, v \rangle = \langle f, v \rangle \quad \forall v \in \tilde{H}^{-1/2}(\Gamma). \tag{1.1}$$

Here, $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $H^{1/2}(\Gamma)$ and $\tilde{H}^{-1/2}(\Gamma)$. The latter space is defined in §3 below.

The remainder of this paper is as follows. In the next section we review regularity results for our model problem which are essential to prove the exact convergence rate of the BEM. We define the scheme of the $p$-version and state our main result (Theorem 2.1) specifying the convergence rate. The subsequent sections give precise details of the approximation analysis. Main result there is a general approximation theorem for singular functions (Theorem 3.7).

First, definitions of Sobolev spaces are recalled at the beginning of §3. Some auxiliary lemmas are collected in §3.1. Our general tool of directional antiderivatives is presented and analysed in §3.2. This tool is then used to analyse edge-, edge-vertex, and vertex singularities in §§3.3, 3.4, and 3.5, respectively. The general approximation theorem is given in §3.6. Detailed proofs of some technical lemmas from §§3.3, 3.4 and 3.5 are postponed to an appendix, see §§4.1, 4.2 and 4.3, respectively.

Throughout the paper, $C$ denotes a generic positive constant which is independent of the polynomial degree $p$.

## 2 The rate of convergence of the $p$-version.

Before presenting and analysing the $p$-version of the BEM let us recall the typical structure of the solution of our model problem for a sufficiently smooth right-hand side function $f$. We use the notation of [4, 10] and refer for more details to [14, 13].

Let $V$ and $E$ denote the sets of vertices and edges of $\Gamma$, respectively. For $v \in V$, let $E(v)$ denote the set of edges with $v$ as an end point. Then, the solution $u$ of (1.1) has the form

$$u = u_{\text{reg}} + \sum_{e \in E} u^e + \sum_{v \in V} u^v + \sum_{v \in V} \sum_{e \in E(v)} u^{ev}, \tag{2.1}$$

where, using local coordinate systems $(r_v, \theta_v)$ and $(x_{e1}, x_{e2})$ with origin $v$, there hold the following representations:

(i) For the regular part there holds $u_{\text{reg}} \in H^k(\Gamma)$, $k > 1/2$.

(ii) The edge singularities $u^e$ have the form

$$u^e = \sum_{j=1}^{m_e} \left( \sum_{s=0}^{s_j^e} b_{js}^e(x_{e1}) |\log x_{e2}|^s \right) x_{e2}^{\gamma_j^e - 1} \chi_1^e(x_{e1}) \chi_2^e(x_{e2}), \tag{2.2}$$

where $\gamma_{j+1}^e \geq \gamma_j^e \geq \frac{1}{2}$, and $m_e$, $s_j^e$ are integers. Here, $\chi_1^e$, $\chi_2^e$ are $C^\infty$ cut-off functions with $\chi_1^e = 1$ in a certain distance to the end points of $e$ and $\chi_1^e = 0$ in a neighbourhood of these vertices. Moreover, for a $\rho_e > 0$, $\chi_2^e = 1$ for $0 \leq x_{e2} \leq \rho_e$ and $\chi_2^e = 0$ for $x_{e2} \geq 2\rho_e$. The functions $b_{js}^e \chi_1^e$ are in $H^m(e)$ for $m$ as large as required.

(iii) The vertex singularities $u^v$ have the form

$$u^v = \chi^v(r_v) \sum_{i=1}^{n_v} \sum_{t=0}^{q_i^v} B_{it}^v |\log r_v|^t r_v^{\lambda_i^v - 1} w_{it}^v(\theta_v), \tag{2.3}$$

where $\lambda_{i+1}^v \geq \lambda_i^v > 0$, $n_v$, $q_i^v \geq 0$ are integers, and $B_{it}^v$ are real numbers. Here, $\chi^v$ is a $C^\infty$ cut-off function with $\chi^v = 1$ for $0 \leq r_v \leq \tau_v$ and $\chi^v = 0$ for $r_v \geq 2\tau_v$ for some $\tau_v > 0$. The functions $w_{it}^v$ are in $H^q(0, \omega_v)$ for $q$ as large as required. Here, $\omega_v$ denotes the interior angle (on $\Gamma$) between the edges meeting at $v$.

(iv) The edge-vertex singularities $u^{ev}$ have the form

$$u^{ev} = u_1^{ev} + u_2^{ev},$$

where

$$u_1^{ev} = \sum_{j=1}^{m_e} \sum_{i=1}^{n_v} \left( \sum_{s=0}^{s_j^e} \sum_{t=0}^{q_i^v} \sum_{l=0}^{s} B_{ijlts}^{ev} |\log x_{e1}|^{s+t-l} |\log x_{e2}|^l \right) x_{e1}^{\lambda_i^v - \gamma_j^e} x_{e2}^{\gamma_j^e - 1} \chi^v(r_v) \chi^{ev}(\theta_v) \tag{2.4}$$

and

$$u_2^{ev} = \sum_{j=1}^{m_e} \sum_{s=0}^{s_j^e} B_{js}^{ev}(r_v) |\log x_{e2}|^s x_{e2}^{\gamma_j^e - 1} \chi^v(r_v) \chi^{ev}(\theta_v) \tag{2.5}$$

with

$$B_{js}^{ev}(r_v) = \sum_{l=0}^{s} B_{jsl}^{ev} |\log r_v|^l. \tag{2.6}$$

Here, $q_i^v$, $s_j^e$, $\lambda_i^v$, $\gamma_j^e$, $\chi^v$ are as above, $B_{ijlts}^{ev}$ are real numbers, and $\chi^{ev}$ is a $C^\infty$ cut-off function with $\chi^{ev} = 1$ for $0 \le \theta_v \le \beta$ and $\chi^{ev} = 0$ for $\frac{3}{2}\beta \le \theta_v \le \omega_v$ for some $0 < \beta \le \min\{\omega_v/2, \pi/8\}$. The functions $B_{jsl}^{ev}$ may be chosen such that

$$B_{js}^{ev}(r_v)\, \chi^v(r_v) \chi^{ev}(\theta_v) = \xi_{js}(x_{e1}, x_{e2})\, \chi_2^e(x_{e2}),$$

where the extension of $\xi_{js}$ by zero onto $\mathbf{R}^{2+} := \{(x_{e1}, x_{e2});\ x_{e2} > 0\}$ lies in $H^m(\mathbf{R}^{2+})$, with $m$ as in (ii). Here, $\chi_2^e$ is a $C^\infty$ cut-off function like in (ii).

Analogously to [4, Remark 2.1] we note the following on the values of the essential parameters $\gamma_1^e$ and $\lambda_1^v$:

**Remark 2.1** *The edge and vertex-edge singularities in* (ii) *and* (iv) *satisfy* $\gamma_1^e \ge 1/2$. *The case* $\gamma_1^e = 1/2$ *is possible for open surfaces and for closed surfaces there holds* $\gamma_1^e > 1/2$. *When* $\gamma_1^e = 1/2$ *then one has to expect that* $u^e$, $u^{ev} \notin L_2(\Gamma)$ *such that no standard approximation theory in* $L_2(\Gamma)$ *is possible (it would not give optimal results, whatsoever). For singular right-hand sides* $f$ *in* (1.1), *it also may occur that* $\gamma_1^e$ *assumes any positive value, which is the minimum requirement to guarantee* $u \in \tilde{H}^{-1/2}(\Gamma)$. *Our analysis will cover this case.*

*For our approximation analysis below, and in order to ensure* $u \in \tilde{H}^{-1/2}(\Gamma)$, *it suffices to require* $\lambda_1^v > -1/2$ *in* (iv). *Note that in* [4], *where we studied the trace of a Neumann problem, the restriction* $\lambda_1^v > 0$ *was necessary to ensure that the trace is in* $\tilde{H}^{1/2}(\Gamma)$. *We do not need this restriction here.*

To introduce the $p$-version of the BEM we discretise $\Gamma$ by a fixed mesh $\{\Gamma_j;\ j = 1, \ldots, J\}$ consisting of triangles and parallelograms. Below we will refer to three different unions of elements. The union of the elements at a node $v$ is denoted by $A_v$, $\bar{A}_v := \cup\{\bar{\Gamma}_j;\ v \in \bar{\Gamma}_j\}$, the union of the elements at one edge $e$ by $A_e$ (the endpoints of $e$ are not included in $e$), $\bar{A}_e := \cup\{\bar{\Gamma}_j;\ \bar{\Gamma}_j \cap e \ne \emptyset\}$, and $A_{ev} := A_v \cap A_e$.

Let $Q = (-1,1)^2$ and $T = \{(x_1, x_2);\ 0 < x_1 < 1,\ 0 < x_2 < x_1\}$ be the reference square and triangle, respectively. For $K = Q$ or $T$, $\mathcal{Q}_p(K)$ denotes the set of polynomials on $K$ of degree $\le p$ in each variable. Moreover, $\mathcal{P}_p(T)$ is the set of polynomials on $T$ of total degree $\le p$. For given $p$ we consider the space of piecewise polynomials on the mesh introduced before,

$$V^p(\Gamma) := \{v \in L_2(\Gamma);\ v|_{\Gamma_j} \circ T_j \in \mathcal{Q}_p(Q) \text{ or } \mathcal{P}_p(T),\ j = 1, \dots, J\}.$$

Here, $T_j$ is an affine mapping with $T_j(K) = \Gamma_j$, $K = Q$ or $T$ as appropriate.

The $p$-version of the BEM then is: *Find $u_p \in V^p(\Gamma)$ such that*

$$\langle V u_p, v\rangle = \langle f, v\rangle \quad \forall v \in V^p(\Gamma). \tag{2.7}$$

The main result of this paper is the following theorem.

**Theorem 2.1** *Let $u \in \tilde{H}^{-1/2}(\Gamma)$ be the solution of* (1.1) *with sufficiently smooth given function $f \in H^{1/2}(\Gamma)$ such that the representation* (2.1)–(2.6) *holds. Let $v_0 \in V$, $e_0 \in E(v_0)$ be such that $\min\{\lambda_1^{v_0} + 1/2, \gamma_1^{e_0}\} = \min_{v\in V, e\in E(v)} \min\{\lambda_1^v + 1/2, \gamma_1^e\}$, with $\lambda_1^v$ and $\gamma_1^e$ being as in* (2.2)–(2.5). *Then denote*

$$\beta = \begin{cases} s_1^{e_0} + q_1^{v_0} + 1/2 & \text{if } \lambda_1^{v_0} = \gamma_1^{e_0} - 1/2, \\ s_1^{e_0} + q_1^{v_0} & \text{otherwise}, \end{cases}$$

*where the numbers $s_1^{e_0}$, $q_1^{v_0}$ are given in* (2.4). *Then the BE approximation $u_p$ defined by* (2.7) *satisfies*

$$\|u - u_p\|_{\tilde{H}^{-1/2}(\Gamma)} \le C\,|\log p|^\beta p^{-2\min\{\gamma_1^{e_0}, \lambda_1^{v_0} + \frac{1}{2}\}},$$

*where $C > 0$ is a constant which is independent of $p$.*

**Proof.** By the quasi-optimal convergence of the BEM (see, e.g., [11]) the proof of the theorem is obtained by using Theorem 3.7 below. □

# 3 Technical details

In this section we give several technical details and prove approximation results for different types of singularities. The outcome is a general approximation theorem (Theorem 3.7) which collects the individual results. In particular, this theorem proves the optimal rate of convergence of the BE scheme (2.7), as stated by Theorem 2.1 before.

First, let us recall the Sobolev norms and spaces that will be used, see [8, 6]. For a domain $\Omega \subset \mathbf{R}^n$ and integer $s$ let $H^s(\Omega)$ be the closure of $C^\infty(\Omega)$ with respect to the norm

$$\|u\|^2_{H^s(\Omega)} = \|u\|^2_{H^{s-1}(\Omega)} + |u|^2_{H^s(\Omega)} \quad (s \ge 1),$$

where

$$|u|^2_{H^s(\Omega)} = \int_\Omega |D^s u(x)|^2\,dx, \quad \text{and} \quad H^0(\Omega) = L_2(\Omega).$$

Here, $|D^s u(x)|^2 = \sum_{|\alpha|=s} |D^\alpha u(x)|^2$ in the usual notation with multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ and with respect to Cartesian coordinates $x = (x_1, \dots, x_n)$. For non-integer $s$, the Sobolev spaces are defined by interpolation. We use the real K-method (see [8]) to define

$$H^s(\Omega) = \Big(L_2(\Omega), H^1(\Omega)\Big)_{s,2} \quad (0 < s < 1)$$

and

$$\tilde{H}^r(\Omega) = \Big(L_2(\Omega), H_0^s(\Omega)\Big)_{\frac{r}{s},2} \quad (1/2 < s \le 1,\ 0 < r < s).$$

Here, $H_0^s(\Omega)$ $(0 < s \le 1)$ is the completion of $C_0^\infty(\Omega)$ in $H^s(\Omega)$ and we identify $H_0^1(\Omega)$ and $\tilde{H}^1(\Omega)$. It is well-known that the norms in $H^s(\Omega)$, $H_0^s(\Omega)$ and $\tilde{H}^s(\Omega)$ are equivalent for $0 < s < 1/2$. For $1/2 < s < 1$, only the norms in $H_0^s(\Omega)$ and $\tilde{H}^s(\Omega)$ are equivalent.

For $s \in [-1, 0)$ the spaces are defined by duality:

$$H^s(\Omega) = (\tilde{H}^{-s}(\Omega))', \quad \tilde{H}^s(\Omega) = (H^{-s}(\Omega))'.$$

For integer $k \ge 0$ and $\mu \in [0,1]$ we also consider the spaces of continuously differentiable functions $C^k(\bar{\Omega})$ and $C^{k,\mu}(\bar{\Omega})$ with norms

$$\|u\|_{C^k(\bar{\Omega})} = \sum_{|\alpha| \le k} \sup_{x \in \Omega} |D^\alpha u(x)|$$

and

$$\|u\|_{C^{k,\mu}(\bar{\Omega})} = \|u\|_{C^k(\bar{\Omega})} + \sum_{|\alpha|=k} \sup_{x,y \in \Omega,\, x \ne y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\mu}.$$

An overview of this section is as follows. In §3.1 we collect several technical lemmas. In §3.2 we present a general scheme that is used to deal with the approximation of functions in Sobolev spaces of negative order. Typical edge and edge-vertex singularities are analysed in §§3.3 and 3.4. In §3.5 we study vertex singularities, and the general approximation result for a function which includes all the different types of singularities is given in §3.6.

## 3.1 Auxiliary lemmas

We collect several technical results.

**Lemma 3.1** [3, Lemma 3.1] *Let $\Omega \subset \mathbf{R}^2$ be a Lipschitz domain. If $u \in \tilde{H}^s(\Omega)$ with $0 \le s \le 1$, then for $i = 1, 2$, $\frac{\partial u}{\partial x_i} \in \tilde{H}^{s-1}(\Omega)$, and*

$$\left\| \frac{\partial u}{\partial x_i} \right\|_{\tilde{H}^{s-1}(\Omega)} \le C \|u\|_{\tilde{H}^s(\Omega)},$$

*where $C > 0$ is independent of $u$.*

**Lemma 3.2** [3, Lemma 3.2] *Let $\Omega$, $\Omega_1$ be two Lipschitz domains in $\mathbf{R}^n$, and $\Omega_1 \subset \Omega$. Then, for $0 \le s < 1/2$, there holds*

$$\|u\|_{\tilde{H}^{-s}(\Omega_1)} \le C\|u\|_{\tilde{H}^{-s}(\Omega)} \quad \forall u \in \tilde{H}^{-s}(\Omega), \tag{3.1}$$

*where the constant $C > 0$ is independent of $u$.*

Let $K$ be a triangle or parallelogram. We quote the following two lemmas (restricted to $K$) from [9] (see Theorem 3.8 and Lemma 5.5 in Chapter 2 therein).

**Lemma 3.3** *Let $m > 1$ be real. Let $\mu = m - 1$ if $m < 2$, $\mu < 1$ if $m = 2$, and $\mu = 1$ if $m > 2$. Then $H^m(K) \subset C^{0,\mu}(\bar{K})$, and*

$$\|u\|_{C^{0,\mu}(\bar{K})} \le C\,\|u\|_{H^m(K)}.$$

**Lemma 3.4** *Let $u \in H^s(K)$ for real $s \ge 0$, and $v \in C^{[s]'-1,1}(\bar{K})$, where $[s]'$ denotes the minimal integer such that $s \le [s]'$. Then $uv \in H^s(K)$, and*

$$\|uv\|_{H^s(K)} \le C\,\|u\|_{H^s(K)}\,\|v\|_{C^{[s]'-1,1}(\bar{K})}.$$

Then we use these two lemmas to prove the following statement.

**Lemma 3.5** *Let $u \in \tilde{H}^{-s}(K)$ for real $s \in [0,1]$, and $\varphi \in H^m(K)$ with $m > 2$. Then $u\varphi \in \tilde{H}^{-s}(K)$, and*

$$\|u\varphi\|_{\tilde{H}^{-s}(K)} \le C\,\|u\|_{\tilde{H}^{-s}(K)}\,\|\varphi\|_{H^m(K)}.$$

**Proof.** Let $v \in H^s(K)$ with $s \in [0,1]$. Applying Lemmas 3.3 and 3.4 we conclude that $v\varphi \in H^s(K)$, and

$$\|v\varphi\|_{H^s(K)} \le C\,\|v\|_{H^s(K)}\,\|\varphi\|_{C^{0,1}(\bar{K})} \le C\,\|v\|_{H^s(K)}\,\|\varphi\|_{H^m(K)}. \tag{3.2}$$

Since $u \in \tilde{H}^{-s}(K)$, we use (3.2) to obtain for any $v \in H^s(K)$

$$|(v, u\varphi)_{L_2(K)}| = |(v\varphi, u)_{L_2(K)}| \le \|u\|_{\tilde{H}^{-s}(K)}\,\|v\varphi\|_{H^s(K)} \le C\,\|u\|_{\tilde{H}^{-s}(K)}\,\|v\|_{H^s(K)}\,\|\varphi\|_{H^m(K)}.$$

Hence

$$\|u\varphi\|_{\tilde{H}^{-s}(K)} = \sup_{v \in H^s(K)} \frac{|(v, u\varphi)_{L_2(K)}|}{\|v\|_{H^s(K)}} \le C\|u\|_{\tilde{H}^{-s}(K)}\|\varphi\|_{H^m(K)},$$

which proves the lemma. $\square$

## 3.2 The general scheme of the error analysis

Our analysis of polynomial approximations for all typical singular functions in (2.1) will follow the same scheme described in this section.

Let $K$ be a triangle or parallelogram. The typical situation in the sections below is as follows: given a singular function $u \in \tilde{H}^s(K)$ with $-1 \le s < s_0$, find an approximating polynomial $u_p$ and estimate $(u - u_p)$ in the norm of $\tilde{H}^s(K)$ for any $-1 \le s < s_0$. Here, $s_0 \in (-\frac{1}{2}, 0]$ depends on the regularity of $u$. The main steps of our analysis are as follows.

First, we consider a triangle (or quadrilateral) $\Omega_0$ such that $K \subset \Omega_0$ and define a function $U$ satisfying the following properties:

$$U = 0 \quad \text{on} \quad \partial\Omega_0; \tag{3.3}$$

$$\frac{\partial U(x)}{\partial x_2} = u(x) \quad \text{for} \quad x \in K. \tag{3.4}$$

Then for given $p \ge 2$ we find a polynomial $U_p$ approximating the function $U$ on $\Omega_0$ such that $U_p \in \mathcal{Q}_p(\Omega_0)$, $U_p = 0$ on $\partial\Omega_0$, and

$$\|U - U_p\|_{\tilde{H}^s(\Omega_0)} \le C\, p^{-2(\alpha - s)}\, |\log p|^\beta, \qquad 0 \le s < s_0 + 1, \tag{3.5}$$

where $\alpha > 0$ and $\beta \ge 0$ are independent of $s$ and $p$.

Having (3.5) we can prove the result on the polynomial approximation of the singular function $u \in \tilde{H}^s(K)$ $(-1 \le s < s_0)$.

**Lemma 3.6** *If the function $U$ satisfies properties* (3.3), (3.4), *and if $U_p \in \mathcal{Q}_p(\Omega_0)$, $U_p = 0$ on $\partial\Omega_0$ and inequality* (3.5) *holds, then there exists a polynomial $u_p \in \mathcal{Q}_p(K)$ such that*

$$\|u - u_p\|_{\tilde{H}^s(K)} \le C\, p^{-2(\alpha - 1 - s)}\, |\log p|^\beta, \quad -1 \le s < s_0. \tag{3.6}$$

*Here, $\alpha$ and $\beta$ are the parameters from* (3.5).

**Proof.** Let us define the polynomial $u_p$ as

$$u_p(x) := \frac{\partial U_p(x)}{\partial x_2}, \qquad x \in \Omega_0.$$

Then $u_p \in \mathcal{Q}_p(\Omega_0)$, and recalling (3.4) one has

$$(u - u_p)(x) = \frac{\partial}{\partial x_2}(U - U_p)(x) \quad \text{for} \quad x \in K.$$

Therefore, using Lemma 3.2, Lemma 3.1 and estimate (3.5) we obtain for any fixed $s' \in (1/2, s_0 + 1)$

$$\|u - u_p\|_{\tilde{H}^{s'-1}(K)} \;=\; \left\| \frac{\partial}{\partial x_2}(U - U_p) \right\|_{\tilde{H}^{s'-1}(K)} \le C \left\| \frac{\partial}{\partial x_2}(U - U_p) \right\|_{\tilde{H}^{s'-1}(\Omega_0)}$$

$$\leq \; C\,\|U-U_p\|_{\tilde H^{s'}(\Omega_0)} \leq C\,p^{-2(\alpha-s')}|\log p|^{\beta}. \tag{3.7}$$

Hence we have proved (3.6) for $s = s'-1 \in (-\frac{1}{2}, s_0)$. On the other hand, applying Lemma 3.1 and inequality (3.5) with $s=0$, we have

$$\begin{aligned}\|u-u_p\|_{\tilde H^{-1}(K)} &= \left\|\frac{\partial}{\partial x_2}(U-U_p)\right\|_{\tilde H^{-1}(K)} \leq C\,\|U-U_p\|_{H^0(K)}\\ &\leq \; C\,\|U-U_p\|_{H^0(\Omega_0)} \leq C\,p^{-2\alpha}|\log p|^{\beta}.\end{aligned}$$

Since $-1/2 < s'-1 < s_0$ in (3.7) and $\alpha$, $\beta$ are independent of $s'$, interpolation between $\tilde H^{-1}(K)$ and $\tilde H^{s'-1}(K)$ gives (3.6) for any $s \in [-1,-1/2]$. □

Thus the problem of defining and analysing a polynomial approximation for a singular function $u \in \tilde H^s(K)$ is reduced to the construction of a function $U$ satisfying properties (3.3), (3.4) and the definition of a polynomial approximation $U_p$ of $U$ which satisfies (3.5).

### 3.3 Approximation of edge singularities

Let $K = \Gamma_j$ be one of the elements along an edge $e$, i.e., $\bar K \cap e \neq \emptyset$. We will study polynomial approximations of the edge singularity term $u^e$ given by (2.2) over the element $K$. Without loss of generality we assume that

$$u^e(x_1,x_2) = x_2^{\gamma-1}|\log x_2|^{\beta}\chi_1(x_1)\chi_2(x_2), \tag{3.8}$$

where $\gamma > 0$, $\beta \geq 0$ is integer, $\chi_1 \in H^m(e)$ with $m > 2\gamma+2$, $\chi_1$ vanishes in neighbourhoods of the vertices $v_1$, $v_2 \in \bar e$, and $\chi_2$ is a $C^\infty$ cut-off function satisfying

$$\chi_2(x_2) = 1 \;\text{ for }\; 0 \leq x_2 \leq \rho_e/2 \;\text{ and }\; \chi_2(x_2) = 0 \;\text{ for }\; x_2 \geq \rho_e, \tag{3.9}$$

for some $\rho_e > 0$. Here, for simplicity we write $(x_1,x_2)$ for the local coordinates used in (2.2). Observe that $u^e \in \tilde H^s(K)$ for any $s \in [-1, s_0)$ with $s_0 = \min\{0, \gamma-1/2\} \in (-1/2, 0]$.

Let $\rho_1$, $\rho_2$, and $d$ be real numbers such that the interval $(\rho_1,\rho_2)$ (respectively, the interval $(0,d)$) is the orthogonal projection of the element $K$ onto the coordinate line $x_2=0$ (respectively, $x_1=0$) (see Figure 1). We introduce two more $C^\infty$ cut-off functions $\tilde\chi_1(x_1)$ and $\tilde\chi_2(x_2)$ satisfying

$$\tilde\chi_1(x_1) = 1 \;\text{ for }\; x_1 \in [\rho_1,\rho_2] \;\text{ and }\; \tilde\chi_1(x_1) = 0 \;\text{ for }\; x_1 \in \mathbf{R}\backslash(\rho_1-\varepsilon, \rho_2+\varepsilon), \tag{3.10}$$

$$\tilde\chi_2(x_2) = 1 \;\text{ for }\; 0 \leq x_2 \leq d \;\text{ and }\; \tilde\chi_2(x_2) = 0 \;\text{ for }\; x_2 \geq d+\varepsilon \tag{3.11}$$

with some $\varepsilon > 0$.

Let $Q_1$ be the square $(a_1,a_2)\times(0,d_1)$ with base along the line $x_2=0$, where

$$a_1 < \rho_1-\varepsilon < \rho_2+\varepsilon < a_2, \quad d_1 > d+\varepsilon. \tag{3.12}$$

Then we define the function $U$ as

$$U(x) := \tilde{\chi}_1(x_1)\tilde{\chi}_2(x_2) \int_0^{x_2} u^e(x_1, \xi_2) d\xi_2, \qquad x \in Q_1.$$

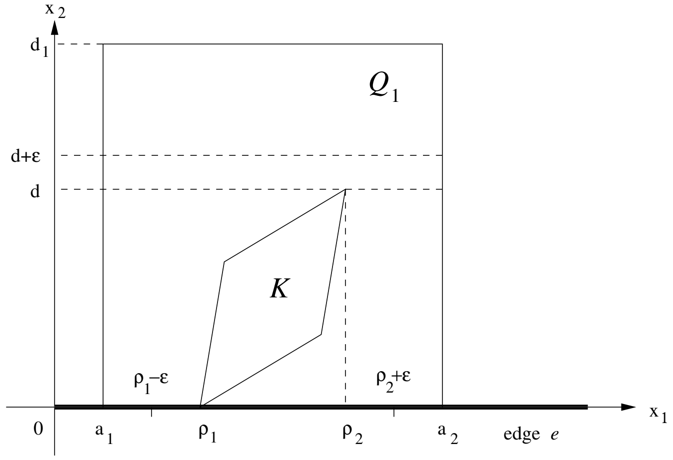


Figure 1: The element $K$ along the edge $e$.

**Remark 3.1** *Observe that $U = 0$ on the line $x_2 = 0$. Moreover, due to (3.10)–(3.12), the function $U$ vanishes in neighbourhoods of the lines $x_1 = a_1$, $x_1 = a_2$, $x_2 = d_1$ (hence $U = 0$ on $\partial Q_1$), and on the element $K$ one has*

$$\frac{\partial U(x)}{\partial x_2} = \frac{\partial}{\partial x_2} \left( \int_0^{x_2} u^e(x_1, \xi_2) d\xi_2 \right) = u^e(x), \qquad x \in K. \tag{3.13}$$

**Lemma 3.7** *There exists a sequence $U_p \in \mathcal{Q}_{2p+2}(Q_1)$, $p = 2, 3, \ldots$, such that $U_p = 0$ on $\partial Q_1$, and for $0 \le s < \min\{1, \gamma + 1/2\}$*

$$\|U - U_p\|_{\tilde{H}^s(Q_1)} \le C\, p^{-2(\gamma+1/2-s)}\, |\log p|^{\beta}. \tag{3.14}$$

**Proof.** Introducing an auxiliary function $\hat{U}$ by

$$\hat{U}(x) := [(x_1 - a_1)(x_1 - a_2)(x_2 - d_1)]^{-1} U(x), \qquad x \in Q_1,$$

and recalling properties of the function $U$ (see Remark 3.1), we see that $\hat{U} = 0$ on $\partial Q_1$. Furthermore, one has

$$\hat{U}(x) \quad = \quad \frac{\tilde{\chi}_1(x_1)\tilde{\chi}_2(x_2)}{(x_1 - a_1)(x_1 - a_2)(x_2 - d_1)} \int_0^{x_2} u^e(x_1, \xi_2) d\xi_2$$

$$= \frac{\chi_1(x_1)\tilde{\chi}_1(x_1)\tilde{\chi}_2(x_2)}{(x_1-a_1)(x_1-a_2)(x_2-d_1)} \int_0^{x_2} \xi_2^{\gamma-1} |\log \xi_2|^\beta \chi_2(\xi_2) d\xi_2$$

and after integration by parts (see Lemma 4.1 for details)

$$\begin{aligned} &= \sum_{k=0}^{\beta} C_k(\gamma,\beta) x_2^\gamma |\log x_2|^k \frac{\chi_1(x_1)\tilde{\chi}_1(x_1)\tilde{\chi}_2(x_2)}{(x_1-a_1)(x_1-a_2)(x_2-d_1)} \chi_2(x_2) \\ &\quad - \frac{\chi_1(x_1)\tilde{\chi}_1(x_1)\tilde{\chi}_2(x_2)}{(x_1-a_1)(x_1-a_2)(x_2-d_1)} \sum_{k=0}^{\beta} C_k(\gamma,\beta) \int_0^{x_2} \xi_2^\gamma |\log \xi_2|^k \chi_2'(\xi_2) d\xi_2 \\ &=: V(x) - W(x), \qquad C_k(\gamma,\beta) = \frac{\beta!}{\gamma^{\beta-k+1} k!}. \end{aligned} \tag{3.15}$$

Here we used the fact that $\gamma > 0$. Observe that $\chi_1(x_1)$, when extended by zero onto $\mathbf{R}$, belongs to $H^m(\mathbf{R})$. Hence

$$\frac{\chi_1(x_1)\tilde{\chi}_1(x_1)\tilde{\chi}_2(x_2)}{(x_1-a_1)(x_1-a_2)(x_2-d_1)} \in H^m(Q_1) \quad \text{for } m > 2\gamma + 2.$$

For the polynomial approximation of $V$ in (3.15) we refer to [4, Theorem 3.2] if $0 < \gamma \le 1/2$ and to [10, Theorem 6.1] if $\gamma > 1/2$: there exists a polynomial $V_p \in \mathcal{Q}_{2p}(Q_1)$ such that $V_p = 0$ on the line $x_2 = 0$ and

$$\|V - V_p\|_{H^s(Q_1)} \le C\, p^{-2(\gamma+1/2-s)}\, |\log p|^\beta, \quad 0 \le s < \min\{1, \gamma + 1/2\}. \tag{3.16}$$

On the other hand, the integral in the expression of $W$ in (3.15) is an analytic function of $x_2$ vanishing in the neighbourhood of zero. Also $\chi_2'(\xi_2) = 0$ for $\xi_2 \in (0, \rho_e/2) \cup (\rho_e, +\infty)$ and therefore, $W \in H_0^m(Q_1)$. Hence the standard approximation result [1, Theorem 4.1] yields a polynomial $W_p \in \mathcal{Q}_{p+1}(Q_1)$ vanishing on $\partial Q_1$ such that

$$\|W - W_p\|_{H^s(Q_1)} \le C\, p^{-(m-1)}\, \|W\|_{H^m(Q_1)}, \quad 0 \le s \le 1. \tag{3.17}$$

Define $\hat{U}_p := V_p - W_p$. Then $\hat{U}_p \in \mathcal{Q}_{2p}(Q_1)$, $\hat{U}_p = 0$ on the line $x_2 = 0$, and by (3.15)–(3.17) one has for $0 \le s < \min\{1, \gamma + 1/2\}$

$$\|\hat{U} - \hat{U}_p\|_{H^s(Q_1)} \le C\, p^{-2(\gamma+1/2-s)}\, |\log p|^\beta + C\, p^{-(m-1)} \le C\, p^{-2(\gamma+1/2-s)}\, |\log p|^\beta \tag{3.18}$$

since $m > 2\gamma + 2 \ge 2\gamma + 2 - 2s$.

Now the polynomial $U_p(x) := (x_1 - a_1)(x_1 - a_2)(x_2 - d_1)\hat{U}_p(x) \in \mathcal{Q}_{2p+2}(Q_1)$ satisfies the conditions of the lemma. In fact, $U_p = 0$ on $\partial Q_1$, and for $s \in [0, \min\{1, \gamma + 1/2\}) \setminus \{1/2\}$ inequality (3.14) is obtained by using (3.18):

$$\|U - U_p\|_{\tilde{H}^s(Q_1)} \le C\, \|U - U_p\|_{H^s(Q_1)} \le C\, \|\hat{U} - \hat{U}_p\|_{H^s(Q_1)}$$

$$\leq \quad C\, p^{-2(\gamma+1/2-s)}\, |\log p|^{\beta}, \quad 0 \leq s < \min\{1, \gamma+1/2\}, \quad s \neq 1/2.$$

Here we used the fact that $(U - U_p) \in H_0^s(Q_1) = \tilde{H}^s(Q_1)$ for the above values of $s$. Estimate (3.14) for $s = 1/2$ then follows by interpolation between $H^0(Q_1)$ and $\tilde{H}^{s'}(Q_1)$ with $1/2 < s' < \min\{1, \gamma+1/2\}$. $\square$

**Remark 3.2** *If $\gamma > 1/2$ in* (3.8)*, then $U \in H_0^1(Q_1)$, and inequality* (3.16) *in the proof of Lemma* 3.7 *remains valid for $s = 1$, cf.* [10, Theorem 6.1]. *Therefore, in this case estimate* (3.14) *holds for any $s \in [0, 1]$.*

Thus given the singular function $u^e$ in (3.8), we have defined the function $U$ vanishing on $\partial Q_1$ and satisfying (3.13). We have also found the polynomial $U_p(x) \in \mathcal{Q}_{2p+2}(Q_1)$ approximating $U$ on $Q_1$. Since $U_p = 0$ on $\partial Q_1$ and inequality (3.14) holds for the error of approximation $(U - U_p)$, the application of Lemma 3.6 with $\Omega_0 = Q_1 \supset K = \Gamma_j$, $s_0 = \min\{0, \gamma - 1/2\}$ and $\alpha = \gamma + 1/2$ gives the following result.

**Theorem 3.1** *Let $u^e$ be given by* (2.2) *on the element $\Gamma_j$. Then there exists a sequence $u_p^e \in \mathcal{Q}_{2p+2}(\Gamma_j)$, $p = 2, 3, \ldots$, such that*

$$\|u^e - u_p^e\|_{\tilde{H}^s(\Gamma_j)} \leq C\, p^{-2(\gamma-1/2-s)}\, |\log p|^{\beta}, \quad -1 \leq s < \min\{0, \gamma - 1/2\}, \tag{3.19}$$

*where $\gamma = \gamma_1^e > 0$ and $\beta = s_1^e \geq 0$ is an integer.*

**Remark 3.3** *If $\gamma_1^e > 1/2$ in* (2.2) *then, due to Remark* 3.2*, estimate* (3.19) *holds for any $s \in [-1, 0]$.*

Using the result of Theorem 3.1 we now study polynomial approximations of the edge-vertex singularities $u_2^{ev}$. To this end, for a given edge $e$ and vertex $v \in \bar{e}$, we consider an element $\Gamma_j \in A_{ev}$ such that $\bar{\Gamma}_j \cap e \neq \emptyset$ and $v \in \bar{\Gamma}_j$ simultaneously.

**Theorem 3.2** *Let $u_2^{ev}$ be given by* (2.5) *on the element $\Gamma_j$. Then there exists a sequence $u_{2,p}^{ev} \in \mathcal{Q}_{3p+2}(\Gamma_j)$, $p = 2, 3, \ldots$, such that*

$$\|u_2^{ev} - u_{2,p}^{ev}\|_{\tilde{H}^{-1/2}(\Gamma_j)} \leq C\, p^{-2\gamma}\, |\log p|^{\beta}, \tag{3.20}$$

*where $\gamma = \gamma_1^e > 0$ and $\beta = s_1^e \geq 0$ is an integer.*

**Proof.** Assume that

$$u_2^{ev}(x_1, x_2) = x_2^{\gamma-1} |\log x_2|^{\beta} \chi_2(x_2) \chi(x_1, x_2), \tag{3.21}$$

where $\gamma > 0$, $\beta \geq 0$ is integer, $\chi_2$ is a $C^\infty$ cut-off function defined by (3.9), and the function $\chi$ extended by zero onto $\mathbf{R}^{2+} := \{(x_1, x_2);\ x_2 > 0\}$ lies in $H^m(\mathbf{R}^{2+})$ with $m > 2\gamma + 2$.

Let us denote $f(x_2) = x_2^{\gamma-1}|\log x_2|^\beta \chi_2(x_2)$, so that $u_2^{ev}(x_1,x_2) = f(x_2)\chi(x_1,x_2)$. The function $f$ has the same form as in (3.8) (with $\chi_1(x_1) \equiv 1$). Therefore, repeating for the function $f$ the arguments which led us to Theorem 3.1, we find a polynomial $f_p \in \mathcal{Q}_{2p+2}(\Gamma_j)$ such that

$$\|f - f_p\|_{\tilde{H}^{-1/2}(\Gamma_j)} \le C\, p^{-2\gamma}\, |\log p|^\beta. \tag{3.22}$$

Moreover, since $f \in \tilde{H}^{-1/2}(\Gamma_j)$,

$$\|f_p\|_{\tilde{H}^{-1/2}(\Gamma_j)} \le C. \tag{3.23}$$

The function $\chi$ (or its extension by zero) lies in $H^m(\Gamma_j)$ with $m > 2\gamma + 2 > 2$. Therefore, using Theorem 3.1 in [2], we find a polynomial $\chi_p \in \mathcal{Q}_p(\Gamma_j)$ satisfying

$$\|\chi - \chi_p\|_{H^s(\Gamma_j)} \le C\, p^{-(m-s)}\, \|\chi\|_{H^m(\Gamma_j)}, \quad 0 \le s \le m. \tag{3.24}$$

Now let us define $u_{2,p}^{ev}(x) := f_p(x)\chi_p(x) \in \mathcal{Q}_{3p+2}(\Gamma_j)$. Then recalling again that $m > 2$ we use Lemma 3.5 and inequalities (3.22)–(3.24) to obtain for a fixed $\varepsilon > 0$

$$\begin{aligned}
\|u_2^{ev} - u_{2,p}^{ev}\|_{\tilde{H}^{-1/2}(\Gamma_j)} &\le \|\chi(f - f_p)\|_{\tilde{H}^{-1/2}(\Gamma_j)} + \|f_p(\chi - \chi_p)\|_{\tilde{H}^{-1/2}(\Gamma_j)} \\
&\le C\|f - f_p\|_{\tilde{H}^{-1/2}(\Gamma_j)}\|\chi\|_{H^m(\Gamma_j)} + C\|f_p\|_{\tilde{H}^{-1/2}(\Gamma_j)}\|\chi - \chi_p\|_{H^{2+\varepsilon}(\Gamma_j)} \\
&\le C\, p^{-2\gamma}\, |\log p|^\beta + C\, p^{-(m-2-\varepsilon)}.
\end{aligned} \tag{3.25}$$

We choose $\varepsilon$ in (3.25) small enough such that $0 < \varepsilon \le m - 2\gamma - 2$. Then $p^{-(m-2-\varepsilon)} \le p^{-2\gamma}$ and estimate (3.20) follows. $\square$

**Remark 3.4** *Polynomial approximations for the function $u_2^{ev}$ given by (2.5) also satisfy the more general estimate*

$$\|u_2^{ev} - u_{2,p}^{ev}\|_{\tilde{H}^s(\Gamma_j)} \le C\, p^{-2(\gamma-1/2-s)}\, |\log p|^\beta, \quad -1 \le s < \min\{0, \gamma - 1/2\}. \tag{3.26}$$

*This fact is established by using the same arguments as in the proof of Theorem 3.2. We assume that the function $\chi$ in (3.21), extended by zero onto $\mathbf{R}^{2+}$, lies in $H^m(\mathbf{R}^{2+})$ with $m > 2\gamma + 3$. Then, instead of (3.25), we have for $-1 \le s < \min\{0, \gamma - 1/2\}$*

$$\begin{aligned}
\|u_2^{ev} - u_{2,p}^{ev}\|_{\tilde{H}^s(\Gamma_j)} &\le C\|f - f_p\|_{\tilde{H}^s(\Gamma_j)}\|\chi\|_{H^m(\Gamma_j)} + C\|f_p\|_{\tilde{H}^s(\Gamma_j)}\|\chi - \chi_p\|_{H^{2+\varepsilon}(\Gamma_j)} \\
&\le C\, p^{-2(\gamma-1/2-s)}\, |\log p|^\beta + C\, p^{-(m-2-\varepsilon)} \le C\, p^{-2(\gamma-1/2-s)}\, |\log p|^\beta.
\end{aligned}$$

*Here we chose $\varepsilon$ such that $0 < \varepsilon \le m - 2\gamma - 3$, since then the estimate $p^{-(m-2-\varepsilon)} \le p^{-2(\gamma-1/2-s)}$ holds for any $s \in [-1, \min\{0, \gamma - 1/2\})$.*

### 3.4 Approximation of edge-vertex singularities

Before analysing the approximation of a general edge-vertex singularity we study a model situation of an element which has an angle less than $\pi/4$ at the vertex, see Figure 2. The corresponding main result is given by Theorem 3.3 below. The general situation is then considered by using affine transformations and proving that such transformations essentially do not alter the singular behaviour of the edge-vertex singularity (in the sense that the convergence order of the $p$-approximation is not affected), see Theorem 3.4.

For the model situation let $Q = (0,1) \times (0,1)$, $T = \{(x_1, x_2) \in Q;\ x_2 < x_1\}$, and let $K \subset T$ be a parallelogram with vertices $(0,0)$, $(a,0)$, $(a\cos\varphi, a\sin\varphi)$, $(a(1+\cos\varphi), a\sin\varphi)$, where $0 < a < 1$, $0 < \varphi < \pi/4$ (see Figure 2). We consider a component of the edge-vertex singularity terms $u_1^{ev}$ over the square $Q$:

$$u(x_1, x_2) = x_1^{\lambda-\gamma} x_2^{\gamma-1} |\log x_1|^{\beta_1} |\log x_2|^{\beta_2} \chi(r)\tilde{\chi}(\theta), \tag{3.27}$$

where $\lambda > -1/2$, $\gamma > 0$, $\beta_i \geq 0$ $(i = 1,2)$ are integers, and $\chi,\ \tilde{\chi} \in C^\infty(\mathbf{R}^+)$ are cut-off functions satisfying

$$\chi(r) = 1 \quad \text{for} \quad 0 \leq r \leq a/4 \quad \text{and} \quad \chi(r) = 0 \quad \text{for} \quad r \geq a/2,$$

$$\tilde{\chi}(\theta) = 1 \quad \text{for} \quad 0 \leq \theta \leq \varphi/3 \quad \text{and} \quad \tilde{\chi}(\theta) = 0 \quad \text{for} \quad \theta \geq \varphi/2.$$

Here, $(r, \theta)$ denote the polar coordinates with origin at $(0,0)$.

Observe that $u \in \tilde{H}^s(K)$ for any $s \in [-1, s_0)$ with $s_0 = \min\{0, \lambda, \gamma - 1/2\} \in (-1/2, 0]$. Now we choose the domain $\Omega_0$ (that appears in the general procedure of Section 3.2) to be the triangle defined before, $\Omega_0 = T \supset K$, and define the function $U$ satisfying properties (3.3), (3.4). To this end, we introduce an auxiliary cut-off function $\tilde{\chi}_1 \in C^\infty(\mathbf{R}^+)$,

$$\tilde{\chi}_1(\theta) = 1 \quad \text{for} \quad 0 \leq \theta \leq \varphi \quad \text{and} \quad \tilde{\chi}_1(\theta) = 0 \quad \text{for} \quad \theta \geq \pi/4, \tag{3.28}$$

and a function $U$,

$$U(x) := \tilde{\chi}_1(\theta) \int_0^{x_2} u(x_1, \xi_2)\, d\xi_2, \quad x \in Q. \tag{3.29}$$

**Remark 3.5** *Observe that $U = 0$ on $\partial T$ and on $[\frac{a}{2}, 1] \times [0,1]$. Moreover, due to* (3.28)*, one has*

$$\frac{\partial U(x)}{\partial x_2} = \frac{\partial}{\partial x_2} \left( \int_0^{x_2} u(x_1, \xi_2)\, d\xi_2 \right) = u(x), \qquad x \in K.$$

In the following lemma we study polynomial approximations of $U$. This result is of central importance to apply the procedure from Section 3.2 which is used to prove Theorem 3.3 below.

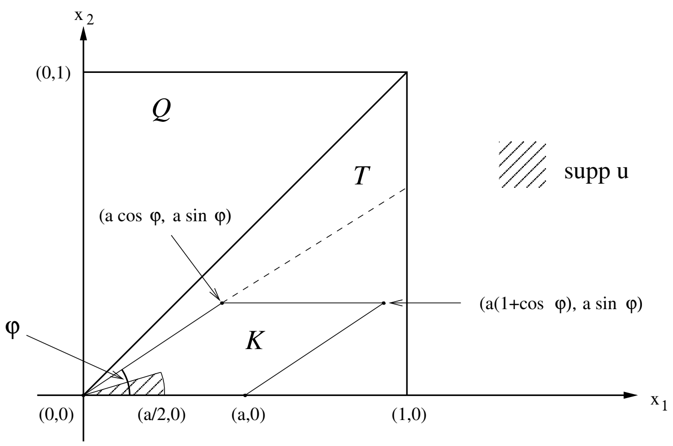


Figure 2: The parallelogram $K$ and support of the function $u$ in (3.27).

**Lemma 3.8** *There exists a sequence* $U_p \in \mathcal{Q}_{p+3}(T)$, $p = 2, 3, \ldots$, *such that* $U_p = 0$ *on* $\partial T$, *and for* $0 \le s < \min\{1, \lambda+1, \gamma+1/2\}$

$$\|U - U_p\|_{\tilde{H}^s(T)} \le C\, p^{-2(\min\{\lambda+1,\gamma+1/2\}-s)}\, |\log p|^{\beta}, \tag{3.30}$$

*where*

$$\beta = \begin{cases} \beta_1 + \beta_2 + 1/2 & \textit{if } \lambda = \gamma - 1/2, \\ \beta_1 + \beta_2 & \textit{otherwise.} \end{cases} \tag{3.31}$$

The proof of Lemma 3.8 has a structure similar to the proof of Theorem 7.1 in [10]. Let

$$\xi(x_1, x_2) := x_2(x_1 - x_2) = x_1 x_2 \frac{x_1 - x_2}{x_1} = x_1 x_2 (1 - \tan\theta), \tag{3.32}$$

and

$$U_0(x_1, x_2) := \frac{U(x_1, x_2)}{\xi(x_1, x_2)} = \frac{\Phi(\theta)}{x_1 x_2} \int_0^{x_2} u(x_1, \xi_2)\, d\xi_2, \tag{3.33}$$

where $\Phi(\theta) = \frac{\tilde{\chi}_1(\theta)}{1-\tan\theta}$. Note that $\Phi \in C^\infty(0, \pi/2)$, $\Phi(0) = 1$, and $\Phi(\theta) = 0$ for $\theta \ge \pi/4$, because the function $\tilde{\chi}_1 \in C^\infty(\mathbf{R}^+)$ satisfies (3.28). Introducing a cut-off function $\omega$ such that

$$\omega \in C^\infty(\mathbf{R}), \quad \omega(z) = 0 \quad \text{for} \quad z \le 1, \quad \omega(z) = 1 \quad \text{for} \quad z \ge 2, \tag{3.34}$$

we define for a small $\Delta \in (0, 1)$

$$\omega^\Delta(x_2) = \omega\Big(\frac{x_2}{\Delta}\Big), \quad \tilde{\omega}^\Delta(x_2) = 1 - \omega^\Delta(x_2), \quad x_2 \ge 0. \tag{3.35}$$

Then we split $U_0$ into a smooth function $v_0$ and a function $w_0$ with small support:

$$U_0(x_1,x_2) = \frac{U(x_1,x_2)}{\xi(x_1,x_2)} \;\; = \;\; U_0(x_1,x_2)\omega^{\Delta}(x_2) + U_0(x_1,x_2)\tilde{\omega}^{\Delta}(x_2)$$
$$=: \;\; v_0(x_1,x_2) + w_0(x_1,x_2). \tag{3.36}$$

In order to approximate the smooth part $v_0$ in (3.36), we will need the following auxiliary result whose proof is given in the appendix (Section 4.2).

**Lemma 3.9** *For any integers $k,l \geq 0$ there exists a positive constant $C(k+l)$ independent of $\Delta$ such that for $(x_1,x_2) \in Q$*

$$\left|\frac{\partial^{k+l} v_0}{\partial x_1^k \partial x_2^l}\right| \leq C(k+l) \begin{cases} 0 \quad for \quad x_2 < \Delta \quad or \quad x_2 > x_1, \\ x_1^{\lambda-\gamma-1-k} x_2^{\gamma-1-l} |\log \Delta|^{\beta_1+\beta_2} \quad otherwise. \end{cases} \tag{3.37}$$

Since $v_0$ satisfies (3.37), the approximation result for this function immediately follows from [10] (see the proof of Theorem 7.1 and Remark 7.1 therein):

**Lemma 3.10** *Let $\Delta = p^{-2}$. If $v_0$ satisfies* (3.37)*, then there exists a sequence $v_p \in \mathcal{Q}_{p+2}(Q)$, $p = 2,3,\ldots$, such that $v_p = 0$ on the lines $x_2 = 0$ and $x_1 = x_2$, and for any $0 \leq s \leq 1$*

$$\|\xi v_0 - v_p\|_{H^s(T)} \leq C\, p^{-2(\min\{\lambda+1,\gamma+1/2\}-s)}\, |\log p|^{\beta_1+\beta_2}, \tag{3.38}$$

*where $T = \{(x_1,x_2);\ 0 < x_1 < 1,\ 0 < x_2 < x_1\}$, and the constant $C > 0$ is independent of $p$.*

The function $w_0$ in (3.36) has small support,

$$\operatorname{supp} w_0 \subset \bar{R}_{\Delta} = \Big\{(x_1,x_2) \in \bar{T};\ x_1 \leq \tfrac{a}{2},\ x_2 \leq 2\Delta\Big\}.$$

We approximate the function $\xi w_0$ by zero and study the error of this approximation in the norm of the space $H^s(T)$.

**Lemma 3.11** *Let $\Delta = p^{-2}$. Then for $0 \leq s < \min\{1, \lambda+1, \gamma+1/2\}$*

$$\|\xi w_0\|_{H^s(T)} \leq C\, p^{-2(\min\{\lambda+1,\gamma+1/2\}-s)}\, |\log p|^{\sigma}, \tag{3.39}$$

*where $\sigma = \beta_1+\beta_2$ if $\lambda < \gamma - 1/2$, $\sigma = \beta_1+\beta_2+1/2$ if $\lambda = \gamma - 1/2$, $\sigma = \beta_2$ otherwise, and $C > 0$ is independent of $p$.*

The proof of this lemma is given in the appendix (Section 4.2).

**Remark 3.6** *If $\lambda > 0$ and $\gamma > 1/2$ in* (3.27) *(i.e., $\min\{2\lambda-1, 2\gamma-2\} > -1$), then $U \in H_0^1(T)$, and by using the same arguments as in the proof of Lemma* 3.11 *it is easy to show that*

$$\left|\frac{\partial(\xi w_0)}{\partial x_i}\right| \leq C x_1^{\lambda-\gamma} x_2^{\gamma-1} |\log x_1|^{\beta_1} |\log x_2|^{\beta_2}, \quad i = 1,2, \quad x \in R_{\Delta},$$

*and*

$$\|\xi w_0\|_{H^1(T)} \leq C \Delta^{\min\{\lambda,\gamma-1/2\}} |\log \Delta|^{\sigma}.$$

*Thus, in this case estimate* (3.39) *holds for any $s \in [0,1]$.*

**Proof of Lemma 3.8.** Let us consider the function $\hat{U}(x) = (1-x_1)^{-1}U(x)$ for $x \in Q$. Then analogously to (3.36) we define functions $v_0$ and $w_0$ such that

$$\hat{U}_0(x) = \hat{U}(x)/\xi(x) = \hat{U}_0(x)\omega^{\Delta}(x_2) + \hat{U}_0(x)\tilde{\omega}^{\Delta}(x_2) =: v_0(x) + w_0(x), \tag{3.40}$$

where $\xi$, $\omega^{\Delta}$, and $\tilde{\omega}^{\Delta}$ are introduced in (3.32) and (3.35).

Recalling Remark 3.5 we conclude that $\hat{U} = 0$ on $\partial T$ and in the rectangle $[\frac{a}{2},1]\times[0,1]$. Since the factor $(1-x_1)^{-1}$ does not alter the character of singular behaviour of $U$, the function $v_0$ satisfies (3.37) and Lemmas 3.10, 3.11 remain valid. The application of Lemma 3.10 gives a polynomial $v_p \in \mathcal{Q}_{p+2}(Q)$ vanishing on the lines $x_2 = 0$ and $x_1 = x_2$. Then using (3.38), (3.39), and decomposition (3.40) we obtain

$$\begin{aligned} \|\hat{U} - v_p\|_{H^s(T)} &\le \|\xi v_0 - v_p\|_{H^s(T)} + \|\xi w_0\|_{H^s(T)} \\ &\le Cp^{-2(\min\{\lambda+1,\gamma+1/2\}-s)}|\log p|^{\beta}, \quad 0 \le s < \min\{1, \lambda+1, \gamma+1/2\}, \end{aligned} \tag{3.41}$$

where $\beta$ is defined by (3.31). Let us define $U_p(x) := (1-x_1)\,v_p(x)$. Then $U_p \in \mathcal{Q}_{p+3}(T)$, $U_p = 0$ on $\partial T$, and estimate (3.41) yields

$$\|U - U_p\|_{H^s(T)} \le C\,\|\hat{U} - v_p\|_{H^s(T)} \le C\,p^{-2(\min\{\lambda+1,\gamma+1/2\}-s)}\,|\log p|^{\beta}. \tag{3.42}$$

Since $U = U_p = 0$ on $\partial T$, $(U - U_p) \in H_0^s(T) = \tilde{H}^s(T)$ for any $s \in [0, \min\{1, \lambda+1, \gamma+\frac{1}{2}\}) \setminus \{\frac{1}{2}\}$, and (3.42) immediately leads to (3.30) for these values of $s$. For $s = \frac{1}{2}$, estimate (3.30) then follows by interpolation between $H^0(T)$ and $\tilde{H}^{s'}(T)$ with $\frac{1}{2} < s' < \min\{1, \lambda+1, \gamma+\frac{1}{2}\}$. $\square$

Thus we conclude that the function $U$ defined by (3.29) and its polynomial approximation $U_p$ satisfy all assumptions of Lemma 3.6 with $\Omega_0 = T \supset K$, $s_0 = \min\{0, \lambda, \gamma - 1/2\}$, $\alpha = \min\{\lambda+1, \gamma+1/2\}$, and error estimate (3.5) being provided by Lemma 3.8. The application of Lemma 3.6 gives the following result.

**Theorem 3.3** *Let $u$ be given by* (3.27) *with $\lambda > -1/2$, $\gamma > 0$, and integers $\beta_i \ge 0$ $(i = 1, 2)$. Then there exists a sequence $u_p \in \mathcal{Q}_{p+3}(K)$, $p = 2, 3, \ldots$, such that*

$$\|u - u_p\|_{\tilde{H}^s(K)} \le C\,p^{-2(\min\{\lambda,\gamma-1/2\}-s)}\,|\log p|^{\beta}, \quad -1 \le s < \min\{0, \lambda, \gamma - 1/2\}, \tag{3.43}$$

*where $\beta$ is defined by* (3.31)*, and $K \subset T$ is a parallelogram with vertices* $(0,0)$, $(a\cos\varphi, a\sin\varphi)$, $(a, 0)$, $(a(1+\cos\varphi), a\sin\varphi)$ *for some $0 < a < 1$ and $0 < \varphi < \pi/4$.*

**Remark 3.7** *If $\lambda > 0$ and $\gamma > 1/2$ in* (3.27)*, then due to the statement in Remark* 3.6*, estimate* (3.30) *holds for any $s \in [0,1]$. Therefore, in this case estimate* (3.43) *is true for any $s \in [-1, 0]$.*

**Remark 3.8** *Note that all the results above for edge-vertex singularities $u_1^{ev}$ remain valid if, instead of* (3.27)*, the function $u$ is defined as $u(x_1,x_2) = x_1^{\lambda-\gamma}x_2^{\gamma-1}|\log x_1|^{\beta_1}|\log x_2|^{\beta_2}f(r,\theta)$, where $f(r,\theta)$ is a sufficiently smooth function vanishing for $r \ge \frac{a}{2}$ and $\theta \ge \frac{\varphi}{2}$.*

Now we consider a general element and prove an approximation result for edge-vertex singularities by applying an affine transformation and using Theorem 3.3. For a given edge $e$ and a vertex $v \in \bar{e}$ let $\Gamma_j$ be an element of $A_{ev}$ such that $\bar{\Gamma}_j \cap e \neq \emptyset$ and $v \in \bar{\Gamma}_j$. We obtain the following main result on the approximation of edge-vertex singularities on a general element (touching the respective edge and vertex).

**Theorem 3.4** *Let $u_1^{ev}$ be given by (2.4) on $\Gamma_j$. Then there exists a sequence $u_{1,p}^{ev} \in \mathcal{Q}_{p+3}(\Gamma_j)$, $p = 2, 3, \ldots$, such that*

$$\|u_1^{ev} - u_{1,p}^{ev}\|_{\tilde{H}^s(\Gamma_j)} \leq C\, p^{-2(\min\{\lambda,\gamma-1/2\}-s)}\, |\log p|^{\beta}, \quad -1 \leq s < \min\{0, \lambda, \gamma - 1/2\}, \tag{3.44}$$

*where $\lambda = \lambda_1^v > -1/2$, $\gamma = \gamma_1^e > 0$, $\beta = q_1^v + s_1^e + 1/2$ if $\lambda_1^v = \gamma_1^e - 1/2$, and $\beta = q_1^v + s_1^e$ otherwise.*

**Proof.** Without loss of generality, we assume that

$$u_1^{ev}(x_{e1}, x_{e2}) = x_{e1}^{\lambda-\gamma} x_{e2}^{\gamma-1} |\log x_{e1}|^{\beta_1} |\log x_{e2}|^{\beta_2} \chi^v(r_v) \chi^{ev}(\theta_v), \tag{3.45}$$

and $\Gamma_j$ is a parallelogram with vertices $(0,0)$, $(b,0)$, $(b\cos\psi, b\sin\psi)$, $(b(1+\cos\psi), b\sin\psi)$, where $b > 0$ is the length of each side of $\Gamma_j$, $\psi \in (0,\pi)$ is the inner angle of $\Gamma_j$ at the vertex $v = (0,0)$. Let $K$ be the parallelogram considered in Theorem 3.3 (see Figure 2). Then $\Gamma_j$ is the image of $K$ under the linear invertible mapping $M$ given by

$$M : \begin{cases} x_{e1} = \frac{b}{a}\left(x_1 + \frac{\cos\psi - \cos\varphi}{\sin\varphi} x_2\right), \\ x_{e2} = \frac{b\sin\psi}{a\sin\varphi}\, x_2. \end{cases} \tag{3.46}$$

If $f$ is a function defined on $\Gamma_j$, then we will denote by $\tilde{f} = f \circ M$ the corresponding function defined on $K$. We may assume that the cut-off functions $\chi^v$, $\chi^{ev}$ in (3.45) are such that $\mathrm{supp}\,(\chi^v \chi^{ev}) \subset [0,1)^2$, and

$$\mathrm{supp}\ (\tilde{\chi}^v \tilde{\chi}^{ev}) \subset S = \left\{(r,\theta);\ 0 \leq r \leq \tfrac{a}{2},\ 0 \leq \theta \leq \tfrac{\varphi}{2}\right\}.$$

We also note that

$$x_{e1} = \frac{b}{a}\left(x_1 + \frac{\cos\psi - \cos\varphi}{\sin\varphi} x_2\right) = \frac{b\,(\sin\varphi + (\cos\psi - \cos\varphi)\tan\theta)}{a\sin\varphi}\, x_1, \tag{3.47}$$

and for $\theta \in [0, \varphi/2]$ one has

$$\begin{aligned} \sin\varphi + (\cos\psi - \cos\varphi)\tan\theta \ &\geq\ \min\{\sin\varphi,\ \sin\varphi + (\cos\psi - \cos\varphi)\tan\tfrac{\varphi}{2}\} \\ &=\ \min\{\sin\varphi,\ \sin\varphi + (\cos\psi + 1)\tan\tfrac{\varphi}{2} - (1 + \cos\varphi)\tan\tfrac{\varphi}{2}\} \\ &=\ \min\{\sin\varphi,\ (\cos\psi + 1)\tan\tfrac{\varphi}{2}\} > 0. \end{aligned}$$

Hence we deduce from (3.45)–(3.47)

$$\tilde{u}_1^{ev}(x_1,x_2) = x_1^{\lambda-\gamma}x_2^{\gamma-1}\sum_{k=0}^{\beta_1}\sum_{l=0}^{\beta_2}|\log x_1|^k|\log x_2|^l f_{k,l}(\theta)(\tilde{\chi}^v\tilde{\chi}^{ev})(r,\theta),$$

where $f_{k,l}(\theta)$ are smooth functions on $S$.

We see that each component of $\tilde{u}_1^{ev}$ has the same form as the function $u$ in (3.27) multiplied by a smooth function $F(r,\theta)$. Therefore, applying Theorem 3.3 (see also Remark 3.8) we find a polynomial approximation $\tilde{u}_{1,p}^{ev}$ for the function $\tilde{u}_1^{ev}$ on $K$. Then the polynomial $u_{1,p}^{ev} = \tilde{u}_{1,p}^{ev}\circ M^{-1}$ satisfies the conditions of the theorem. □

### 3.5 Approximation of vertex singularities

In this section we analyse the approximation of vertex singularities. As before for edge-vertex singularities, we first study a model situation on an element with restricted angle condition (the corresponding result is given by Theorem 3.5). This theorem is then used to prove the analogous result on general elements (Theorem 3.6).

For the model situation let $\kappa > 1$ and denote $S_\kappa = \{x \in Q;\ \kappa^{-1}x_1 < x_2 < \kappa x_1\}$. Let $K$ be a parallelogram such that $K \subset Q$, $(0,0)$ is a vertex of $K$, the measure of the inner angle of $K$ at this vertex is equal to $\varphi \in (0,\frac{\pi}{2})$, the length of each side of $K$ is equal to $a \in (0,1)$, and $K$ is symmetric with respect to the line $x_1 = x_2$ (see Figure 3). Then

$$K \subset S_{\kappa_0} \quad \text{with} \quad \kappa_0 = \tan\left(\tfrac{\pi}{4}+\tfrac{\varphi}{2}\right).$$

We consider a component of the vertex singularity terms over the square $Q$:

$$u(r,\theta) = r^{\lambda-1}|\log r|^\beta\chi(r)w(\theta), \tag{3.48}$$

where $(r,\theta)$ denote local polar coordinates with origin at $(0,0)$, $\lambda > -1/2$, $\beta \ge 0$ is an integer, $w(\theta)$ is sufficiently smooth, and $\chi$ is a $C^\infty$ cut-off function satisfying

$$\chi(r) = 1 \quad \text{for} \quad 0 \le r \le \delta/2 \quad \text{and} \quad \chi(r) = 0 \quad \text{for} \quad r \ge \delta. \tag{3.49}$$

Here, $\delta \in (0,1)$ is assumed to be small enough. Observing that $u \in \tilde{H}^s(K)$ for any $s \in [-1,s_0)$ with $s_0 = \min\{0,\lambda\} \in (-1/2,0]$, we study polynomial approximations for $u$.

Assume that $0 < \delta < \kappa_0^{-1}$ and choose $\kappa$ such that

$$1 < \kappa_0 < \kappa < \delta^{-1}. \tag{3.50}$$

We introduce a $C^\infty$ cut-off function $\tilde{\chi}$ satisfying

$$\begin{aligned}\tilde{\chi}(\theta) &= 1 \quad \text{for} \quad \arctan\kappa_0^{-1} \le \theta \le \arctan\kappa_0,\\ \tilde{\chi}(\theta) &= 0 \quad \text{for} \quad \theta \le \arctan\kappa^{-1} \quad \text{and} \quad \theta \ge \arctan\kappa.\end{aligned} \tag{3.51}$$

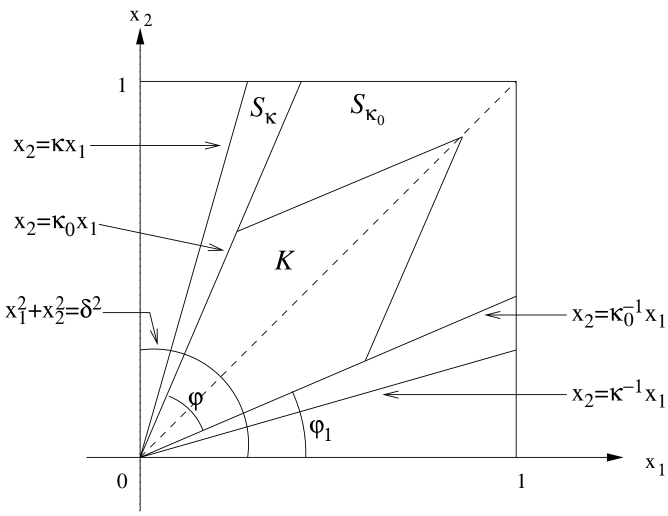


Figure 3: The parallelogram $K$ and the domains $S_{\kappa_0}$, $S_\kappa$.

Then we define

$$U(x) := \tilde{\chi}(\theta) \int_0^{x_2} u(r(x_1, \xi_2), \theta(x_1, \xi_2)) d\xi_2, \quad x \in Q. \tag{3.52}$$

**Remark 3.9** *Observe that* $U = 0$ *in* $[\delta, 1] \times [0, 1]$ *because of* (3.49). *Moreover, due to* (3.50) *and* (3.51)*,* $U = 0$ *on* $\partial S_\kappa$ *and in neighbourhoods of the lines* $x_i = 1$ $(i = 1, 2)$*. We also note that for any* $x \in K$ *there holds*

$$\frac{\partial U(x)}{\partial x_2} = \frac{\partial}{\partial x_2} \left( \int_0^{x_2} u(r(x_1, \xi_2), \theta(x_1, \xi_2)) d\xi_2 \right) = u(r(x), \theta(x)).$$

In the following lemma we study polynomial approximations of $U$. As for the edge-vertex singularities, this result is of central importance to apply the procedure from Section 3.2 which is used to prove Theorem 3.5 below.

**Lemma 3.12** *There exists a sequence* $U_p \in \mathcal{Q}_{p+3}(S_\kappa)$*,* $p = 2, 3, \ldots$*, such that* $U_p = 0$ *on* $\partial S_\kappa$ *and for* $0 \le s < \min\{1, \lambda + 1\}$

$$\|U - U_p\|_{\tilde{H}^s(S_\kappa)} \le C\, p^{-2(\lambda+1-s)}\, |\log p|^\beta. \tag{3.53}$$

For the proof of Lemma 3.12 we use the approach applied first in [1] and developed later in [10] (see, in particular, Theorem 5.1 in [1] and Theorem 8.1 in [10]). Let

$$\xi(x_1, x_2) = (x_1 - \kappa x_2)(\kappa x_1 - x_2) = r^2 \Phi_1(\theta)$$

and

$$U_0(x_1,x_2) = \frac{U(x_1,x_2)}{\xi(x_1,x_2)} = r^{-2}\Phi_2(\theta)\int\limits_0^{x_2} u(r(x_1,\xi_2),\theta(x_1,\xi_2))d\xi_2,$$

where $\Phi_2(\theta) = \tilde{\chi}(\theta)/\Phi_1(\theta)$ is a smooth function vanishing for $\theta \le \arctan\kappa^{-1}$ and for $\theta \ge \arctan\kappa$. Then we introduce a cut-off function $\omega$ by (3.34) and decompose $U_0$ as

$$\begin{aligned} U_0(x_1,x_2) = \frac{U(x_1,x_2)}{\xi(x_1,x_2)} &= U_0(x_1,x_2)\omega^{\Delta}(r) + U_0(x_1,x_2)\tilde{\omega}^{\Delta}(r) \\ &=: v_0(x_1,x_2) + w_0(x_1,x_2), \end{aligned} \tag{3.54}$$

where $\omega^{\Delta}$ and $\tilde{\omega}^{\Delta}$ were defined in (3.35) for a small $\Delta \in (0,1)$.

Thus we have a smooth function $v_0$ vanishing for $0 \le r \le \Delta$ and a function $w_0$ with small support, $\operatorname{supp} w_0 \subset \bar{K}_\Delta = \{x \in \bar{S}_\kappa;\ 0 \le r \le 2\Delta\}$. For the approximation of $v_0$ we will need the following lemma. Its proof is given in the appendix (Section 4.3).

**Lemma 3.13** *Let $k$ and $l$ be non-negative integers. Then there exists a constant $C(k+l)$ independent of $\Delta$ such that for $(x_1,x_2) \in Q$ and for $i = 1,2$*

$$\left|\frac{\partial^{k+l} v_0}{\partial x_1^k \partial x_2^l}\right| \le C(k+l)\begin{cases} 0 & \text{for } \ 0 < r < \Delta, \\ x_i^{\lambda-2-k-l}|\log\Delta|^{\beta} & \text{otherwise.}\end{cases} \tag{3.55}$$

Let us now assume that $v_0$ satisfies (3.55) and not necessarily has the explicit form considered above. The approximation of such functions by polynomials was investigated in [1] when proving Theorem 5.1 therein, and was also studied in [10, Theorem 8.1]. The estimate for the error of this approximation in the norm of $H^1(S_\kappa)$ immediately follows from [1], while [10] gives also the estimate in the $L_2$-norm and then, by interpolation, in the norm of $H^s(S_\kappa)$ with $0 \le s \le 1$. Thus the following statement holds.

**Lemma 3.14** *Let $\Delta = p^{-2}$. If $v_0$ satisfies (3.55), then there exists a sequence $v_p \in \mathcal{Q}_{p+2}(Q)$, $p = 2,3,\ldots$, such that $v_p = 0$ on the lines $x_1 = \kappa x_2$ and $x_2 = \kappa x_1$, and for any $0 \le s \le 1$*

$$\|\xi v_0 - v_p\|_{H^s(S_\kappa)} \le C\,p^{-2(\lambda+1-s)}\,|\log p|^{\beta}. \tag{3.56}$$

Recalling that $\operatorname{supp} w_0 \subset \bar{K}_\Delta = \{x \in \bar{S}_\kappa;\ 0 \le r \le 2\Delta\}$ (see (3.54)) we approximate the function $\xi w_0$ by zero.

**Lemma 3.15** *Let $\Delta = p^{-2}$. Then for $0 \le s < \min\{1, \lambda+1\}$*

$$\|\xi w_0\|_{H^s(S_\kappa)} \le C\,p^{-2(\lambda+1-s)}\,|\log p|^{\beta}. \tag{3.57}$$

A proof of Lemma 3.15 is given in the appendix.

**Remark 3.10** *If $\lambda > 0$ in* (3.48)*, then $U \in H_0^1(S_\kappa)$, and arguing as in the proof of Lemma* 3.15 *one can show that*

$$\left|\frac{\partial(\xi w_0)}{\partial x_i}\right| \le C r^{\lambda-1} |\log r|^\beta, \quad i = 1,2, \quad x \in K_\Delta,$$

*and*

$$\|\xi w_0\|_{H^1(S_\kappa)} \le C\Delta^\lambda |\log \Delta|^\beta.$$

*Therefore, in this case estimate* (3.57) *holds for any $s \in [0,1]$.*

Now we can prove the above formulated result on the approximation of the function $U$ on $S_\kappa$.

**Proof of Lemma 3.12.** Considering the function $\hat{U}(x) := (1-x_1)^{-1}(1-x_2)^{-1}U(x)$ for $x \in Q$ and recalling Remark 3.9 we note that $\hat{U} = 0$ on $\partial S_\kappa$ and in neighbourhoods of the lines $x_i = 1$ $(i = 1,2)$. Then the proof repeats the same steps as the ones in the proof of Lemma 3.8:

i) Analogously to (3.54) we define $v_0$ and $w_0$ such that

$$\hat{U}_0(x) = \hat{U}(x)/\xi(x) = \hat{U}_0(x)\omega^\Delta(r) + \hat{U}_0(x)\tilde{\omega}^\Delta(r) =: v_0(x) + w_0(x). \tag{3.58}$$

ii) Since the factor $(1-x_1)^{-1}(1-x_2)^{-1}$ does not alter the character of the singular behaviour of $U$, the function $v_0$ satisfies (3.55), and Lemmas 3.14, 3.15 are valid. The application of Lemma 3.14 gives a polynomial $v_p \in \mathcal{Q}_{p+2}(Q)$ vanishing on the lines $x_1 = \kappa x_2$ and $x_2 = \kappa x_1$. Then $U_p(x) = (1-x_1)(1-x_2)\,v_p(x) \in \mathcal{Q}_{p+3}(Q)$, $U_p = 0$ on $\partial S_\kappa$, and using (3.56), (3.57), (3.58) we prove the estimate

$$\|U - U_p\|_{H^s(S_\kappa)} \le C p^{-2(\lambda+1-s)} |\log p|^\beta, \qquad 0 \le s < \min\{1, \lambda+1\}. \tag{3.59}$$

iii) Since $U{=}U_p{=}0$ on $\partial S_\kappa$, estimate (3.59) leads to (3.53) for any $s{\in}[0, \min\{1,\lambda{+}1\})\backslash\{\frac{1}{2}\}$, because $(U{-}U_p) \in H_0^s(S_\kappa) = \tilde{H}^s(S_\kappa)$ for these values of $s$. For $s = \frac{1}{2}$ estimate (3.53) then follows by interpolation between $H^0(S_\kappa)$ and $\tilde{H}^{s'}(S_\kappa)$ with $\frac{1}{2} < s' < \min\{1, \lambda+1\}$. $\square$

We use the result of Lemma 3.12 to estimate the approximation error for the typical vertex singularity $u$ given by (3.48).

**Theorem 3.5** *Let $u$ be given by* (3.48) *with $\lambda > -1/2$ and integer $\beta \ge 0$. Then there exists a sequence $u_p \in \mathcal{Q}_{p+3}(K)$, $p = 2,3,\ldots$, such that*

$$\|u - u_p\|_{\tilde{H}^s(K)} \le C\, p^{-2(\lambda-s)}\, |\log p|^\beta, \qquad -1 \le s < \min\{0, \lambda\}, \tag{3.60}$$

*where $K$ is the parallelogram shown in Figure* 3.

**Proof.** Let the function $U$ be defined by (3.52), and let $U_p$ be its polynomial approximation given by Lemma 3.12. Since $U = U_p = 0$ on $\partial S_\kappa$, $\frac{\partial U(x)}{\partial x_2} = u(x)$ for $x \in K \subset S_\kappa$ (see Remark 3.9), and inequality (3.53) holds, the desired statement follows by application of Lemma 3.6 with $\Omega_0 = S_\kappa$, $s_0 = \min\{0, \lambda\}$, and $\alpha = \lambda + 1$. □

**Remark 3.11** *If* $\lambda > 0$ *in* (3.48) *then, due to Remark* 3.10*, inequalities* (3.56), (3.57)*, and hence* (3.53) *are satisfied for* $0 \le s \le 1$*. Therefore, in this case estimate* (3.60) *holds for any* $s \in [-1, 0]$*.*

**Remark 3.12** *Analogously to Remark* 3.8*, the results of Lemma* 3.12 *and Theorem* 3.5 *remain valid if, instead of* (3.48)*, the function* $u$ *has the form* $u(r,\theta) = r^{\lambda-1}|\log r|^\beta f(r,\theta)$*, where* $f(r,\theta)$ *is sufficiently smooth and vanishes for* $r \ge \delta$*.*

Now, using Theorem 3.5, we prove an approximation result for vertex singularities on a general element attaching the vertex. For a given vertex $v$ of $\Gamma$ let $\Gamma_j$ be an element with $v \in \bar{\Gamma}_j$. We then have the following result.

**Theorem 3.6** *Let* $u^v$ *be given by* (2.3) *on* $\Gamma_j$*. Then there exists a sequence* $u^v_p \in \mathcal{Q}_{p+3}(\Gamma_j)$, $p = 2, 3, \ldots$*, such that*

$$\|u^v - u^v_p\|_{\tilde{H}^s(\Gamma_j)} \le C\, p^{-2(\lambda - s)}\, |\log p|^\beta, \quad -1 \le s < \min\{0, \lambda\},$$

*where* $\lambda = \lambda^v_1 > -1/2$ *and* $\beta = q^v_1 \ge 0$ *is integer.*

**Proof.** Without loss of generality we assume that

$$u^v(r_v, \theta_v) = r_v^{\lambda-1}|\log r_v|^\beta \chi^v(r_v) w^v(\theta_v) \tag{3.61}$$

and that $\Gamma_j$ is the parallelogram shown in Figure 4 with all sides having the length $b > 0$. The interior angle at the vertex $v = (0,0)$ is $\psi = \psi_2 - \psi_1 \in (0, \pi)$. Note that $\psi_1 \in [0, \pi)$ is permitted (thus an edge of the element may coincide with part of the boundary of $\Gamma$).

If $K$ is the parallelogram considered in Theorem 3.5 (see Figure 3), then $\Gamma_j$ is the image of $K$ under the linear invertible mapping $M$ given by

$$M : \begin{cases} x_{e1} = A_1 x_1 + B_1 x_2, \\ x_{e2} = A_2 x_1 + B_2 x_2, \end{cases} \tag{3.62}$$

where

$$A_1 = \frac{b\,(\cos\psi_1 \cos\varphi_1 - \cos\psi_2 \sin\varphi_1)}{a \cos 2\varphi_1}, \qquad B_1 = \frac{b\,(\cos\psi_2 \cos\varphi_1 - \cos\psi_1 \sin\varphi_1)}{a \cos 2\varphi_1},$$

$$A_2 = \frac{b\,(\sin\psi_1 \cos\varphi_1 - \sin\psi_2 \sin\varphi_1)}{a \cos 2\varphi_1}, \qquad B_2 = \frac{b\,(\sin\psi_2 \cos\varphi_1 - \sin\psi_1 \sin\varphi_1)}{a \cos 2\varphi_1}$$

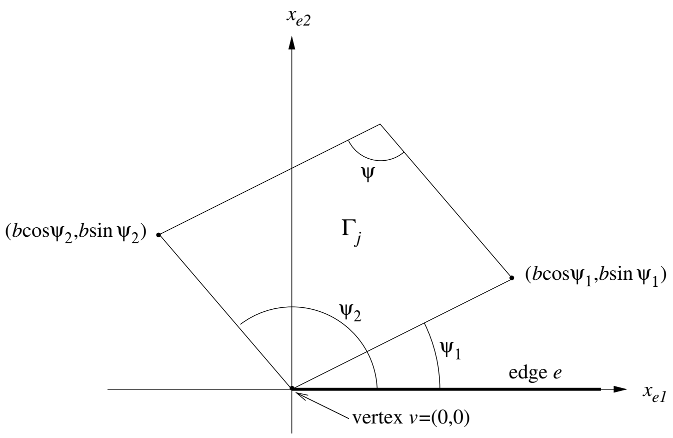


Figure 4: The element $\Gamma_j$ at the vertex $v$.

with $\varphi_1 = \frac{\pi}{4} - \frac{\varphi}{2}$. Hence

$$r_v^2 = x_{e1}^2 + x_{e2}^2 = (A_1^2 + A_2^2)x_1^2 + (B_1^2 + B_2^2)x_2^2 + 2(A_1B_1 + A_2B_2)x_1x_2,$$

and recalling that $\psi_2 - \psi_1 = \psi$, $2\varphi_1 = \frac{\pi}{2} - \varphi$, we obtain by simple calculations

$$\begin{aligned} r_v^2 &= \frac{b^2}{a^2\sin^2\varphi}\Big((1-\cos\psi\cos\varphi)(x_1^2+x_2^2) + 2(\cos\psi - \cos\varphi)x_1x_2\Big) \\ &= \frac{b^2}{a^2\sin^2\varphi}\Big(1 - \cos\psi\cos\varphi + (\cos\psi - \cos\varphi)\sin 2\theta\Big)r^2. \end{aligned} \tag{3.63}$$

Note that for any $\theta \in [0, 2\pi)$ one has

$$\begin{aligned} &1 - \cos\psi\cos\varphi + (\cos\psi - \cos\varphi)\sin 2\theta \\ &\quad\geq \min\{(1-\cos\psi)(1+\cos\varphi),\ (1+\cos\psi)(1-\cos\varphi)\} > 0. \end{aligned} \tag{3.64}$$

Now for any function $f$ defined on $\Gamma_j$ we denote by $\tilde{f} = f \circ M$ the corresponding function defined on $K$. We may assume that the cut-off function $\chi^v$ in (3.61) is such that $\operatorname{supp}\chi^v \subset [0,1)$, and $\operatorname{supp}\tilde{\chi}^v \subset S_1 = \{(r,\theta);\ 0 \leq r \leq \delta,\ 0 \leq \theta < 2\pi\}$. Therefore, we deduce from (3.61)–(3.64) that

$$\tilde{u}^v(r,\theta) = r^{\lambda-1}\sum_{k=0}^{\beta} |\log r|^k f_k(r,\theta),$$

where $f_k(r,\theta)$ are smooth functions vanishing for $r \geq \delta$.

We see that the function $\tilde{u}^v$ has the same form as given in Remark 3.12. Applying Theorem 3.5 to this function, we find a polynomial approximation $\tilde{u}_p^v$ on $K$ whose transform $u_p^v = \tilde{u}_p^v \circ M^{-1}$ satisfies the conditions of the theorem. □

### 3.6 The general approximation result

Collecting the results of the previous sections, we are now able to estimate the error of approximation of the function $u$ given by (2.1)–(2.6).

**Theorem 3.7** *Let the function $u$ be given by* (2.1)–(2.6) *on $\Gamma$. Also, let $v_0 \in V$, $e_0 \in E(v_0)$ be such that* $\min\{\lambda_1^{v_0} + 1/2, \gamma_1^{e_0}\} = \min_{v\in V, e\in E(v)} \min\{\lambda_1^v + 1/2, \gamma_1^e\}$*, with $\lambda_1^v$ and $\gamma_1^e$ being as in* (2.2)–(2.5)*. Then, for every $p = 2, 3, \ldots$, there exists a function $u_p \in V^p$ such that*

$$\|u - u_p\|_{\tilde{H}^s(\Gamma)} \le C \max\left\{p^{-k},\, p^{-2(\min\{\lambda_1^{v_0}, \gamma_1^{e_0} - 1/2\} - s)}\, |\log p|^{\beta}\right\} \tag{3.65}$$

*for $-1 \le s < \min\{0, \lambda_1^{v_0}, \gamma_1^{e_0} - 1/2\}$, and $\beta$ is defined as in* (3.31) *(in terms of $q_1^{v_0}$ and $s_1^{e_0}$).*

**Proof.** We use Theorem 3.1, Theorem 3.2 together with Remark 3.4, Theorems 3.4 and 3.6 to find piecewise polynomials $u_p^e$, $u_{2,p}^{ev}$, $u_{1,p}^{ev}$, and $u_p^v$ defined on $A_e$, $A_{ev}$, $A_{ev}$, and $A_v$, respectively. Extending $u_p^e$, $u_{2,p}^{ev}$, $u_{1,p}^{ev}$ and $u_p^v$ by zero onto the remaining parts of $\Gamma$, we see that $\|u^e - u_p^e\|_{\tilde{H}^s(\Gamma)}$, $\|u_2^{ev} - u_{2,p}^{ev}\|_{\tilde{H}^s(\Gamma)}$, $\|u_1^{ev} - u_{1,p}^{ev}\|_{\tilde{H}^s(\Gamma)}$ and $\|u^v - u_p^v\|_{\tilde{H}^s(\Gamma)}$ are bounded as in (3.19), (3.26), (3.44) and (3.60), respectively. For the regular part $u_{\text{reg}}$ of $u$ in (2.1) we use a standard $L_2$ approximation result giving a piecewise polynomial $u_{\text{reg},p}$ which satisfies

$$\|u_{\text{reg}} - u_{\text{reg},p}\|_{\tilde{H}^s(\Gamma)} \le \|u_{\text{reg}} - u_{\text{reg},p}\|_{L_2(\Gamma)} \le C\, p^{-k} \|u_{\text{reg}}\|_{H^k(\Gamma)}, \quad -1 \le s \le 0.$$

Making use of the regularity as given by the parameters in (2.2)–(2.6), applying the triangle inequality, and combining all the estimates, we obtain (3.65). $\square$

**Remark 3.13** *In the proof of Theorem* 3.7 *we used standard $L_2$ approximation and the trivial inclusion $L_2(\Gamma) \subset \tilde{H}^s(\Gamma)$ $(-1 \le s < 0)$ to estimate the approximation error for the regular part of $u$. This individual estimate is not optimal but does not influence the optimality of the combined estimate when considering enough singularity terms to obtain a sufficiently high regularity for $u_{\text{reg}}$. We do not know of any technique to directly estimate approximation errors in negative order Sobolev norms, and the general technique of directional antiderivatives presented in* §3.2 *does not work for functions whose regularity is known only in Sobolev spaces.*

# 4 Appendix

## 4.1 Edge singularities

We proof a technical result which is used in Section 3.3.

**Lemma 4.1** *Let $\gamma > 0$, $\beta \geq 0$ an integer and let the function $\chi_2$ be defined by* (3.9)*. Then*

$$\int_0^{x_2} \xi_2^{\gamma-1} |\log \xi_2|^\beta \chi_2(\xi_2) d\xi_2$$
$$= \sum_{k=0}^{\beta} C_k(\gamma,\beta) x_2^\gamma |\log x_2|^k \chi_2(x_2) - \sum_{k=0}^{\beta} C_k(\gamma,\beta) \int_0^{x_2} \xi_2^\gamma |\log \xi_2|^k \chi_2'(\xi_2) d\xi_2$$

*with* $C_k(\gamma,\beta) = \frac{\beta!}{\gamma^{\beta-k+1}k!}$.

**Proof.** We define

$$J = \int_0^{x_2} \xi_2^{\gamma-1} |\log \xi_2|^\beta \chi_2(\xi_2) d\xi_2$$

and perform integration by parts.

First let us assume that $x_2 \in (0,1)$. Then for any $\xi_2 \in (0, x_2)$ one has $|\log \xi_2| = -\log \xi_2$. Hence

$$\begin{aligned}
J &= (-1)^\beta \int_0^{x_2} \xi_2^{\gamma-1} (\log \xi_2)^\beta \chi_2(\xi_2) d\xi_2 \\
&= \left| \begin{array}{ll} \bar{u} = \chi_2(\xi_2), & d\bar{v} = \xi_2^{\gamma-1}(\log \xi_2)^\beta d\xi_2; \\ d\bar{u} = \chi_2'(\xi_2) d\xi_2, & \bar{v} = \xi_2^\gamma \sum_{k=0}^{\beta} \frac{(-1)^{\beta+k}\beta!}{\gamma^{\beta-k+1}k!} (\log \xi_2)^k = (-1)^\beta \xi_2^\gamma \sum_{k=0}^{\beta} C_k(\gamma,\beta) |\log \xi_2|^k, \\ C_k(\gamma,\beta) = \frac{\beta!}{\gamma^{\beta-k+1}k!} & \end{array} \right| \\
&= \xi_2^\gamma \chi_2(\xi_2) \sum_{k=0}^{\beta} C_k(\gamma,\beta) |\log \xi_2|^k \Big|_0^{x_2} - \sum_{k=0}^{\beta} C_k(\gamma,\beta) \int_0^{x_2} \xi_2^\gamma |\log \xi_2|^k \chi_2'(\xi_2) d\xi_2 \\
&= \sum_{k=0}^{\beta} C_k(\gamma,\beta) x_2^\gamma |\log x_2|^k \chi_2(x_2) - \sum_{k=0}^{\beta} C_k(\gamma,\beta) \int_0^{x_2} \xi_2^\gamma |\log \xi_2|^k \chi_2'(\xi_2) d\xi_2; \qquad (4.1)
\end{aligned}$$

here we used the fact that $\gamma > 0$.

Now suppose that $x_2 \geq 1$. Since $\chi_2(\xi_2) = 0$ for $\xi_2 \geq \rho_e$, we have for sufficiently small $\rho_e$ (in particular, we assume here that $\rho_e < 1$, so that $\rho_e < 1 \leq x_2$)

$$J = \int_0^{\rho_e} \xi_2^{\gamma-1} (\log \xi_2)^\beta \chi_2(\xi_2) d\xi_2$$

and analogously as above

$$= \quad \xi_2^\gamma \chi_2(\xi_2) \sum_{k=0}^{\beta} C_k(\gamma,\beta) |\log \xi_2|^k \Big|_0^{\rho_e} - \sum_{k=0}^{\beta} C_k(\gamma,\beta) \int_0^{\rho_e} \xi_2^\gamma |\log \xi_2|^k \chi_2'(\xi_2) d\xi_2. \qquad (4.2)$$

Recalling again that $\chi_2(\rho_e) = \chi_2(x_2) = 0$ and $\chi_2'(\xi_2) = 0$ for $\xi_2 > \rho_e$, we rewrite (4.2) as

$$J = \sum_{k=0}^{\beta} C_k(\gamma,\beta) x_2^\gamma |\log x_2|^k \chi_2(x_2) - \sum_{k=0}^{\beta} C_k(\gamma,\beta) \int_0^{x_2} \xi_2^\gamma |\log \xi_2|^k \chi_2'(\xi_2) d\xi_2, \quad x_2 \geq 1.$$

Therefore we conclude that equality (4.1) holds for any $x_2 \geq 0$. $\square$

### 4.2 Edge-vertex singularities

In this section we give detailed proofs for technical results stated in Section 3.4. The notation of that section is used here.

We will use the following inequalities:

$$\begin{aligned} &\left|\frac{\partial r}{\partial x_1}\right| = |\cos\theta| \leq 1, \quad \left|\frac{\partial r}{\partial x_2}\right| = |\sin\theta| \leq 1, \\ &\left|\frac{\partial \theta}{\partial x_1}\right| = \left|\frac{\sin\theta}{r}\right| = \frac{|\sin\theta\,\cos\theta|}{x_1} \leq \frac{1}{x_1}, \quad \left|\frac{\partial \theta}{\partial x_2}\right| = \left|\frac{\cos\theta}{r}\right| = \frac{|\sin\theta\,\cos\theta|}{x_2} \leq \frac{1}{x_2}. \end{aligned} \qquad (4.3)$$

Furthermore, for any integer $k \geq 1$, we derive by (3.34), (3.35)

$$\begin{aligned} \left|\frac{\partial^k \omega^\Delta(x_2)}{\partial x_2^k}\right| = \left|\frac{\partial^k \tilde{\omega}^\Delta(x_2)}{\partial x_2^k}\right| &= \begin{cases} 0 & \text{for } 0 < x_2 < \Delta \text{ or } x_2 > 2\Delta, \\ \left|\omega^{(k)}\left(\frac{x_2}{\Delta}\right)\right| \left(\frac{1}{\Delta}\right)^k & \text{for } \Delta \leq x_2 \leq 2\Delta \end{cases} \\ &\leq C\, x_2^{-k} \qquad \text{for } x_2 > 0. \end{aligned} \qquad (4.4)$$

We will also need estimates for derivatives of the function $u$ given by (3.27),

$$u(x_1,x_2) = x_1^{\lambda-\gamma} x_2^{\gamma-1} |\log x_1|^{\beta_1} |\log x_2|^{\beta_2} \chi(r)\tilde{\chi}(\theta),$$

on the triangle $T$. Since $\chi,\ \tilde{\chi} \in C^\infty(\mathbf{R}^+)$, one has for $(x_1,x_2) \in T$ and for any $\xi_2 \in [0,x_2]$

$$|u(x_1,\xi_2)| \leq C x_1^{\lambda-\gamma} \xi_2^{\gamma-1} |\log x_1|^{\beta_1} |\log \xi_2|^{\beta_2}, \qquad (4.5)$$

and

$$\left|\frac{\partial u(x_1,\xi_2)}{\partial x_1}\right| \quad = \quad \xi_2^{\gamma-1} |\log \xi_2|^{\beta_2} \left|\frac{\partial}{\partial x_1}\Big(x_1^{\lambda-\gamma} |\log x_1|^{\beta_1} \chi(r(x_1,\xi_2)) \tilde{\chi}(\theta(x_1,\xi_2))\Big)\right|$$

$$
\begin{aligned}
&\leq C\xi_2^{\gamma-1}|\log \xi_2|^{\beta_2}\Big[x_1^{\lambda-\gamma-1}|\log x_1|^{\beta_1}+\beta_1 x_1^{\lambda-\gamma-1}|\log x_1|^{\beta_1-1} \\
&\quad +\Big|x_1^{\lambda-\gamma}|\log x_1|^{\beta_1}\chi'(r)\tfrac{\partial r}{\partial x_1}\tilde{\chi}(\theta)\Big|+\Big|x_1^{\lambda-\gamma}|\log x_1|^{\beta_1}\chi(r)\tilde{\chi}'(\theta)\tfrac{\partial \theta}{\partial x_1}\Big|\Big] \\
&\leq C\xi_2^{\gamma-1}|\log \xi_2|^{\beta_2}x_1^{\lambda-\gamma-1}\max\Big\{\beta_1|\log x_1|^{\beta_1-1},|\log x_1|^{\beta_1}\Big\} \\
&\leq C\xi_2^{\gamma-1}|\log \xi_2|^{\beta_2}x_1^{\lambda-\gamma-1}\max\Big\{1,|\log x_1|^{\beta_1}\Big\}.
\end{aligned}
$$

Here we applied inequalities (4.3) and used the fact that $x_1 \in (0,1)$. Repeating this procedure we obtain

$$
\left|\frac{\partial^k u(x_1,\xi_2)}{\partial x_1^k}\right| \leq C\xi_2^{\gamma-1}|\log \xi_2|^{\beta_2}x_1^{\lambda-\gamma-k}\max\Big\{1,|\log x_1|^{\beta_1}\Big\}, \quad \xi_2\in[0,x_2], \quad k\geq 0, \tag{4.6}
$$

and, by similar arguments,

$$
\left|\frac{\partial^{k+l} u(x_1,x_2)}{\partial x_1^k\partial x_2^l}\right| \leq Cx_1^{\lambda-\gamma-k}x_2^{\gamma-1-l}\max\Big\{1,|\log x_1|^{\beta_1}\Big\}\max\Big\{1,|\log x_2|^{\beta_2}\Big\}, \quad k,l\geq 0. \tag{4.7}
$$

**Proof of Lemma 3.9.** Using (3.33) and (3.36) we write

$$
\begin{aligned}
\frac{\partial^{k+l}v_0}{\partial x_1^k\partial x_2^l} = \sum_{\substack{k_1+k_2=k\\k_1,k_2\geq 0}} C(k_1,k_2)\sum_{\substack{l_1+l_2=l\\l_1,l_2\geq 0}} C(l_1,l_2)\frac{\partial^{k_1+l_1}}{\partial x_1^{k_1}\partial x_2^{l_1}}\left(\frac{\Phi(\theta)\omega^\Delta(x_2)}{x_1x_2}\right)& \\
\times\frac{\partial^{l_2}}{\partial x_2^{l_2}}\left(\int_0^{x_2}\frac{\partial^{k_2}u(x_1,\xi_2)}{\partial x_1^{k_2}}d\xi_2\right).&
\end{aligned} \tag{4.8}
$$

Note that by the definition of $v_0$ there holds

$$
\frac{\partial^{k+l}v_0}{\partial x_1^k\partial x_2^l}=0 \ \text{ outside the triangle } \ T_\Delta=\Big\{(x_1,x_2)\in T;\ x_1<\tfrac{a}{2},\ x_2>\Delta\Big\}.
$$

Suppose now that $x\in T_\Delta$. Since $\Phi(\theta)=\tilde{\chi}_1(\theta)/(1-\tan\theta)$ is smooth we obtain with (4.3)

$$
\left|\frac{\partial^{|\alpha|}\Phi}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}\right| \leq C\,x_1^{-\alpha_1}\,x_2^{-\alpha_2}, \quad \alpha_1,\alpha_2\geq 0, \quad |\alpha|=\alpha_1+\alpha_2. \tag{4.9}
$$

Derivatives of $\omega^\Delta(x_2)$ for $x_2\geq\Delta$ satisfy estimates (4.4). Hence

$$
\left|\frac{\partial^{k+l}}{\partial x_1^k\partial x_2^l}\Big(\omega^\Delta(x_2)x_1^{-1}x_2^{-1}\Big)\right| \leq C\,x_1^{-1-k}x_2^{-1-l} \quad \text{for } \ k,l\geq 0,
$$

and by (4.9) we find

$$\left|\frac{\partial^{k_1+l_1}}{\partial x_1^{k_1}\partial x_2^{l_1}}\left(\frac{\Phi(\theta)\omega^{\Delta}(x_2)}{x_1x_2}\right)\right| \le C\, x_1^{-1-k_1}\, x_2^{-1-l_1}, \qquad k_1, l_1 \ge 0. \tag{4.10}$$

Derivatives of the function $u$ on $T_\Delta$ are estimated by using (4.6), (4.7):

$$\left|\frac{\partial^k u(x_1,\xi_2)}{\partial x_1^k}\right| \le C\xi_2^{\gamma-1}|\log\xi_2|^{\beta_2}x_1^{\lambda-\gamma-k}|\log\Delta|^{\beta_1}, \quad \xi_2\in[0,x_2], \quad k\ge 0,$$

and

$$\left|\frac{\partial^{k+l}u(x_1,x_2)}{\partial x_1^k\partial x_2^l}\right| \le Cx_1^{\lambda-\gamma-k}x_2^{\gamma-1-l}|\log\Delta|^{\beta_1+\beta_2}, \quad k,l\ge 0,$$

because $\Delta < x_2 < x_1 < 1$ with sufficiently small $\Delta > 0$. Therefore,

$$\begin{aligned}
\left|\int_0^{x_2}\frac{\partial^{k_2}u(x_1,\xi_2)}{\partial x_1^{k_2}}d\xi_2\right| &\le C\, x_1^{\lambda-\gamma-k_2}|\log\Delta|^{\beta_1}\int_0^{x_2}\xi_2^{\gamma-1}|\log\xi_2|^{\beta_2}d\xi_2 \\
&\le C\, x_1^{\lambda-\gamma-k_2}x_2^{\gamma}|\log\Delta|^{\beta_1+\beta_2}, \quad k_2\ge 0, \\
\left|\frac{\partial^{l_2}}{\partial x_2^{l_2}}\Big(\int_0^{x_2}\frac{\partial^{k_2}u(x_1,\xi_2)}{\partial x_1^{k_2}}d\xi_2\Big)\right| &= \left|\frac{\partial^{k_2+l_2-1}u(x_1,x_2)}{\partial x_1^{k_2}\partial x_2^{l_2-1}}\right| \\
&\le Cx_1^{\lambda-\gamma-k_2}x_2^{\gamma-l_2}|\log\Delta|^{\beta_1+\beta_2}, \ k_2{\ge}0, \ l_2{\ge}1.
\end{aligned} \tag{4.11}$$

Then the desired estimate in (3.37) is derived by using representation (4.8) and inequalities (4.10), (4.11). □

**Proof of Lemma 3.11.** According to equalities (3.29) and (3.36) one has

$$\xi(x)w_0(x) = U(x)\tilde\omega^{\Delta}(x_2) = \tilde\chi_1(\theta)\tilde\omega^{\Delta}(x_2)\int_0^{x_2}u(x_1,\xi_2)d\xi_2, \qquad x\in T.$$

Then using inequality (4.5) we estimate the norm $\|\xi w_0\|_{L_2(T)}$ for sufficiently small $\Delta > 0$:

$$\begin{aligned}
\|\xi w_0\|^2_{L_2(T)} &= \|\xi w_0\|^2_{L_2(R_\Delta)} \le C\int_0^{2\Delta}\int_{x_2}^{a/2}\left(\int_0^{x_2}|u(x_1,\xi_2)|d\xi_2\right)^2 dx_1\,dx_2 \\
&\le C\int_0^{2\Delta}\int_{x_2}^{a/2}x_1^{2(\lambda-\gamma)}|\log x_1|^{2\beta_1}\left(\int_0^{x_2}\xi_2^{\gamma-1}|\log\xi_2|^{\beta_2}d\xi_2\right)^2 dx_1\,dx_2
\end{aligned}$$

$$
\begin{aligned}
&\le C\int_0^{2\Delta}\int_{x_2}^{a/2} x_1^{2(\lambda-\gamma)}x_2^{2\gamma}|\log x_1|^{2\beta_1}|\log x_2|^{2\beta_2}\,dx_1\,dx_2\\
&\le C\begin{cases}\int_0^{2\Delta} x_2^{2\gamma}|\log x_2|^{2\beta_2}x_2^{2(\lambda-\gamma)+1}|\log x_2|^{2\beta_1}\,dx_2 & \text{if}\quad \lambda<\gamma-1/2,\\ \int_0^{2\Delta} x_2^{2\gamma}|\log x_2|^{2\beta_2}|\log x_2|^{2\beta_1+1}\,dx_2 & \text{if}\quad \lambda=\gamma-1/2,\\ \int_0^{2\Delta} x_2^{2\gamma}|\log x_2|^{2\beta_2}\,dx_2 & \text{if}\quad \lambda>\gamma-1/2\end{cases}\\
&\le C\Delta^{\min\{2\lambda+2,2\gamma+1\}}|\log\Delta|^{2\sigma},\quad \min\{\lambda+1,\gamma+1/2\}>0,
\end{aligned}\tag{4.12}
$$

where $\sigma=\beta_1+\beta_2$ if $\lambda<\gamma-1/2$, $\sigma=\beta_1+\beta_2+1/2$ if $\lambda=\gamma-1/2$, and $\sigma=\beta_2$ otherwise.

For $0<s<\min\{1,\lambda+1,\gamma+1/2\}$ we have

$$
\|\xi w_0\|^2_{H^s(T)}=\int_0^\infty t^{-2s}K^2(t,\xi w_0)\frac{dt}{t},\tag{4.13}
$$

where

$$
K^2(t,\xi w_0)=\inf_{\xi w_0=w_1+w_2}\left(\|w_1\|^2_{L_2(T)}+t^2\|w_2\|^2_{H^1(T)}\right).
$$

Let us define for any $t\in(0,\Delta)$

$$
\omega_t(x_2)=\omega\left(\tfrac{x_2}{t}\right),\quad \tilde\omega_t(x_2)=1-\omega_t(x_2),\quad x_2\ge 0,\tag{4.14}
$$

where $\omega$ is as in (3.34). Then by (4.13) we have

$$
\|\xi w_0\|^2_{H^s(T)}\le\int_0^\Delta t^{-2s-1}\left(\|\xi w_0\tilde\omega_t\|^2_{L_2(T)}+t^2\|\xi w_0\omega_t\|^2_{H^1(T)}\right)dt+\int_\Delta^\infty t^{-2s-1}\|\xi w_0\|^2_{L_2(T)}dt.\tag{4.15}
$$

We estimate the norms on the right-hand side of (4.15). Since $\tilde\omega_t(x_2)=0$ for $x_2\ge 2t$, we use the same arguments as in (4.12) to obtain

$$
\begin{aligned}
\|\xi w_0\tilde\omega_t\|^2_{L_2(T)} &= \left\|\tilde\chi_1(\theta)\tilde\omega^\Delta(x_2)\tilde\omega_t(x_2)\int_0^{x_2}u(x_1,\xi_2)d\xi_2\right\|^2_{L_2(R_\Delta)}\\
&\le C\int_0^{2t}\int_{x_2}^{a/2}\left(\int_0^{x_2}|u(x_1,\xi_2)|d\xi_2\right)^2dx_1\,dx_2\le Ct^{\min\{2\lambda+2,2\gamma+1\}}|\log t|^{2\sigma}.
\end{aligned}\tag{4.16}
$$

In order to prove the upper bound for the norm $\|\xi w_0\omega_t\|_{H^1(T)}$ we estimate derivatives of this function on $T$. Since $\omega_t(x_2)=0$ for $0\le x_2\le t$, the function $\xi w_0\omega_t$ vanishes outside the domain $R^1_\Delta=\{(x_1,x_2)\in T;\ x_1<\frac{a}{2},\ t<x_2<2\Delta\}$.

Let $x \in R^1_\Delta$. Then

$$\begin{aligned}
\left|\frac{\partial}{\partial x_1}(\xi(x)w_0(x)\omega_t(x_2))\right| &\leq C\left|\frac{\partial}{\partial x_1}(\xi(x)w_0(x))\right| = C\left|\frac{\partial}{\partial x_1}\left(\tilde{\chi}_1(\theta)\tilde{\omega}^\Delta(x_2)\int\limits_0^{x_2} u(x_1,\xi_2)d\xi_2\right)\right| \\
&\leq C\left(\left|\frac{\partial\tilde{\chi}_1}{\partial\theta}\right|\left|\frac{\partial\theta}{\partial x_1}\right|\int\limits_0^{x_2}|u(x_1,\xi_2)|d\xi_2 + \int\limits_0^{x_2}\left|\frac{\partial u(x_1,\xi_2)}{\partial x_1}\right|d\xi_2\right),
\end{aligned}$$

and applying inequalities (4.3), (4.5), (4.6) we have

$$\begin{aligned}
\left|\frac{\partial(\xi w_0\omega_t)}{\partial x_1}\right| &\leq Cx_1^{\lambda-\gamma-1}|\log x_1|^{\beta_1}\int\limits_0^{x_2}\xi_2^{\gamma-1}|\log\xi_2|^{\beta_2}d\xi_2 \\
&\leq Cx_1^{\lambda-\gamma-1}x_2^{\gamma}|\log x_1|^{\beta_1}|\log x_2|^{\beta_2} \leq Cx_1^{\lambda-\gamma}x_2^{\gamma-1}|\log x_1|^{\beta_1}|\log x_2|^{\beta_2}. \quad (4.17)
\end{aligned}$$

Here we also used the fact that $x_2 < x_1 < \frac{a}{2} < 1$ on $R^1_\Delta$.

Derivatives of the function $\omega_t(x_2)$ satisfy estimates similar to (4.4). Therefore, using (4.3)–(4.5), we find

$$\begin{aligned}
&\left|\frac{\partial}{\partial x_2}(\xi(x)w_0(x)\omega_t(x_2))\right| = C\left|\frac{\partial}{\partial x_2}\left(\tilde{\chi}_1(\theta)\tilde{\omega}^\Delta(x_2)\omega_t(x_2)\int\limits_0^{x_2} u(x_1,\xi_2)d\xi_2\right)\right| \\
&\leq C\left(\left|\frac{\partial\tilde{\chi}_1}{\partial\theta}\right|\left|\frac{\partial\theta}{\partial x_2}\right| + \left|\frac{\partial\tilde{\omega}^\Delta}{\partial x_2}\right| + \left|\frac{\partial\omega_t}{\partial x_2}\right|\right)\int\limits_0^{x_2}|u(x_1,\xi_2)|d\xi_2 + C|u(x_1,x_2)| \\
&\leq C\left(x_2^{-1}x_1^{\lambda-\gamma}|\log x_1|^{\beta_1}\int\limits_0^{x_2}\xi_2^{\gamma-1}|\log\xi_2|^{\beta_2}d\xi_2 + x_1^{\lambda-\gamma}x_2^{\gamma-1}|\log x_1|^{\beta_1}|\log x_2|^{\beta_2}\right) \\
&\leq Cx_1^{\lambda-\gamma}x_2^{\gamma-1}|\log x_1|^{\beta_1}|\log x_2|^{\beta_2}. \quad (4.18)
\end{aligned}$$

Since $\xi w_0\omega_t$ vanishes on $\partial T$ and outside $R^1_\Delta$, there holds

$$\|\xi w_0\omega_t\|^2_{H^1(T)} \leq C|\xi w_0\omega_t|^2_{H^1(T)} \leq C|\xi w_0\omega_t|^2_{H^1(R^1_\Delta)}.$$

Hence we deduce from (4.17), (4.18)

$$\begin{aligned}
\|\xi w_0\omega_t\|^2_{H^1(T)} &\leq C\int\limits_t^{2\Delta}\int\limits_{x_2}^{a/2} x_1^{2(\lambda-\gamma)}x_2^{2(\gamma-1)}|\log x_1|^{2\beta_1}|\log x_2|^{2\beta_2}dx_1dx_2 \\
&\leq C\int\limits_t^{2\Delta} x_2^{\min\{2\lambda-1,2\gamma-2\}}|\log x_2|^{2\sigma}dx_2
\end{aligned}$$

$$\leq \quad C \begin{cases} t^{\min\{2\lambda, 2\gamma-1\}} |\log t|^{2\sigma} & \text{if} \quad \min\{2\lambda-1, 2\gamma-2\} < -1, \\ \Delta^{\min\{2\lambda, 2\gamma-1\}} |\log \Delta|^{2\sigma} & \text{if} \quad \min\{2\lambda-1, 2\gamma-2\} > -1, \end{cases} \tag{4.19}$$

where $\sigma$ is the same as in (4.12).

If $\min\{2\lambda-1, 2\gamma-2\} = -1$, then we introduce a small $\varepsilon$ such that $0 < \varepsilon < 2-2s$ and estimate the norm $\|\xi w_0 \omega_t\|_{H^1(T)}$ as follows

$$\|\xi w_0 \omega_t\|^2_{H^1(T)} \leq C \int\limits_t^{2\Delta} x_2^{-1} |\log x_2|^{2\sigma} dx_2 \leq C |\log t|^{2\sigma} \int\limits_t^{2\Delta} x_2^{-1-\varepsilon} x_2^{\varepsilon} dx_2 \leq C \Delta^{\varepsilon} t^{-\varepsilon} |\log t|^{2\sigma}. \tag{4.20}$$

Now using estimates (4.12), (4.16), and (4.19) for the norms we obtain by (4.15) for $0 < s < \min\{1, \lambda+1, \gamma+1/2\}$

$$\begin{aligned} \|\xi w_0\|^2_{H^s(T)} &\leq C \int\limits_0^{\Delta} t^{-2s-1} t^{\min\{2\lambda+2, 2\gamma+1\}} |\log t|^{2\sigma} dt + C \Delta^{\min\{2\lambda+2, 2\gamma+1\}} |\log \Delta|^{2\sigma} \int\limits_{\Delta}^{\infty} t^{-2s-1} dt \\ &\leq C \Delta^{\min\{2\lambda+2, 2\gamma+1\}-2s} |\log \Delta|^{2\sigma} \qquad \text{if} \ \min\{2\lambda-1, 2\gamma-2\} < -1, \end{aligned} \tag{4.21}$$

$$\begin{aligned} \|\xi w_0\|^2_{H^s(T)} &\leq C \int\limits_0^{\Delta} t^{-2s-1} t^{\min\{2\lambda+2, 2\gamma+1\}} |\log t|^{2\sigma} dt + C \Delta^{\min\{2\lambda, 2\gamma-1\}} |\log \Delta|^{2\sigma} \int\limits_0^{\Delta} t^{-2s-1}\, t^2 dt \\ &\quad + C \Delta^{\min\{2\lambda+2, 2\gamma+1\}} |\log \Delta|^{2\sigma} \int\limits_{\Delta}^{\infty} t^{-2s-1} dt \\ &\leq C \Delta^{\min\{2\lambda+2, 2\gamma+1\}-2s} |\log \Delta|^{2\sigma} \qquad \text{if} \ \min\{2\lambda-1, 2\gamma-2\} > -1. \end{aligned} \tag{4.22}$$

In the case when $\min\{2\lambda-1, 2\gamma-2\} = -1$ we proceed similarly and use estimate (4.20) instead of (4.19). Then recalling that $0 < \varepsilon < 2-2s$ we have for $0 < s < 1$

$$\begin{aligned} \|\xi w_0\|^2_{H^s(T)} &\leq C \int\limits_0^{\Delta} t^{-2s-1} \Big(t^2 + t^2 \Delta^{\varepsilon} t^{-\varepsilon}\Big) |\log t|^{2\sigma} dt + C \Delta^2 |\log \Delta|^{2\sigma} \int\limits_{\Delta}^{\infty} t^{-2s-1} dt \\ &\leq C \left( \int\limits_0^{\Delta} t^{-2s+1} |\log t|^{2\sigma} dt + \Delta^{\varepsilon} \int\limits_0^{\Delta} t^{-2s+1-\varepsilon} |\log t|^{2\sigma} dt + \Delta^{2-2s} |\log \Delta|^{2\sigma} \right) \\ &\leq C \Delta^{2-2s} |\log \Delta|^{2\sigma} \qquad\qquad \text{if} \ \min\{2\lambda-1, 2\gamma-2\} = -1. \end{aligned} \tag{4.23}$$

Combining (4.12) and (4.21)–(4.23) we conclude that for any $\lambda$ and $\gamma$ such that $\min\{\lambda+1, \gamma+1/2\} > 0$ there holds

$$\|\xi w_0\|_{H^s(T)} \leq C \Delta^{\min\{\lambda+1, \gamma+1/2\}-s} |\log \Delta|^{\sigma} \qquad 0 \leq s < \min\{1, \lambda+1, \gamma+1/2\}, \tag{4.24}$$

where $\sigma$ is the same as in (4.12), and $C > 0$ is independent of $\Delta$.

Taking $\Delta = p^{-2}$ and using (4.24) we obtain estimate (3.39). □

### 4.3 Vertex singularities

In this section we give detailed proofs for technical results stated in Section 3.5. The notation of that section is used here.

We will need estimates for the function $u$ given by (3.48) and for its derivatives over the domain $S_\kappa$. Let us denote $u(x_1, \xi_2) = u(r(x_1, \xi_2), \theta(x_1, \xi_2))$ for $\xi_2 \in [0, x_2]$.

We recall that for any $x \in S_\kappa$ there holds $\kappa^{-1} x_1 < x_2 < \kappa x_1$. Then

$$x_i \le r(x_1, x_2) = (x_1^2 + x_2^2)^{1/2} \le C(\kappa) x_i \quad \text{for} \quad i = 1, 2, \tag{4.25}$$

and for any $\xi_2 \in [0, x_2]$ one has

$$x_1 \le r(x_1, \xi_2) = (x_1^2 + \xi_2^2)^{1/2} \le r(x_1, x_2) \le C(\kappa) x_1.$$

Therefore,

$$|u(x_1, \xi_2)| \le C x_1^{\lambda - 1} |\log x_1|^\beta \quad \text{for any } \xi_2 \in [0, x_2], \tag{4.26}$$

and

$$\begin{aligned}
\left| \frac{\partial u(x_1, \xi_2)}{\partial x_1} \right| &\le C \Bigg[ r^{\lambda-2}(x_1, \xi_2) \left| \tfrac{\partial r}{\partial x_1} \right| |\log r(x_1, \xi_2)|^\beta + \beta r^{\lambda-2}(x_1, \xi_2) \left| \tfrac{\partial r}{\partial x_1} \right| |\log r(x_1, \xi_2)|^{\beta-1} + \\
&\qquad + r^{\lambda-1}(x_1, \xi_2) |\log r(x_1, \xi_2)|^\beta \left( \left| \chi'(r) \tfrac{\partial r}{\partial x_1} \right| + \left| w'(\theta) \tfrac{\partial \theta}{\partial x_1} \right| \right) \Bigg] \\
&\le C \left[ x_1^{\lambda-2} |\log x_1|^\beta + \beta x_1^{\lambda-2} |\log x_1|^{\beta-1} + x_1^{\lambda-1} |\log x_1|^\beta + x_1^{\lambda-2} |\log x_1|^\beta \right] \\
&\le C x_1^{\lambda-2} \max \left\{ 1, |\log x_1|^\beta \right\},
\end{aligned}$$

because $\chi$ and $w$ are smooth. Repeating this procedure we obtain

$$\left| \frac{\partial^k u(x_1, \xi_2)}{\partial x_1^k} \right| \le C x_1^{\lambda-1-k} \max \left\{ 1, |\log x_1|^\beta \right\}, \quad \xi_2 \in [0, x_2], \quad k \ge 0. \tag{4.27}$$

Using similar arguments and inequalities (4.25) we find

$$\left| \frac{\partial^{k+l} u(x_1, x_2)}{\partial x_1^k \partial x_2^l} \right| \le C x_i^{\lambda-1-k-l} \max \left\{ 1, |\log x_i|^\beta \right\}, \quad i = 1, 2, \quad k, l \ge 0. \tag{4.28}$$

**Proof of Lemma 3.13.** For derivatives of $v_0$ we write

$$\begin{aligned}
\frac{\partial^{k+l} v_0}{\partial x_1^k \partial x_2^l} = \sum_{\substack{k_1+k_2=k \\ k_1, k_2 \ge 0}} C(k_1, k_2) \sum_{\substack{l_1+l_2=l \\ l_1, l_2 \ge 0}} & C(l_1, l_2) \frac{\partial^{k_1+l_1}}{\partial x_1^{k_1} \partial x_2^{l_1}} \left( \Phi_2(\theta) \omega^\Delta(r) r^{-2} \right) \\
& \times \frac{\partial^{l_2}}{\partial x_2^{l_2}} \left( \int_0^{x_2} \frac{\partial^{k_2} u(r(x_1, \xi_2), \theta(x_1, \xi_2))}{\partial x_1^{k_2}} d\xi_2 \right).
\end{aligned} \tag{4.29}$$

Observe that by the definition of $v_0$ we have

$$\frac{\partial^{k+l} v_0}{\partial x_1^k \partial x_2^l} = 0 \quad \text{for } 0 < r < \Delta \text{ and outside } S_\kappa.$$

Therefore, let us assume that $x \in S_\kappa$ and $r(x) > \Delta$ in the following. Then $x_i > \frac{\Delta}{\sqrt{1+\kappa^2}}$, and for sufficiently small $\Delta > 0$ one has

$$\max\{1, |\log x_i|\} \le C|\log \Delta|, \qquad i = 1, 2. \tag{4.30}$$

Since $\Phi_2 = \tilde{\chi}/\Phi_1$ is smooth and the derivatives of $\omega^\Delta(r)$ satisfy estimates (4.4) with $x_2$ replaced by $r$, we obtain

$$\left|\frac{\partial^{k+l}}{\partial r^k \partial \theta^l}\Big(\Phi_2(\theta)\omega^\Delta(r) r^{-2}\Big)\right| \le C \sum_{\substack{k_1+k_2=k\\ k_1,k_2\ge 0}} \left|\frac{\partial^{k_1}\omega^\Delta(r)}{\partial r^{k_1}}\right| \left|\frac{\partial^{k_2} r^{-2}}{\partial r^{k_2}}\right| \le C\, r^{-2-k}, \quad k, l \ge 0.$$

Then we find by using (4.3) and (4.25)

$$\left|\frac{\partial^{k_1+l_1}}{\partial x_1^{k_1}\partial x_2^{l_1}}\Big(\Phi_2(\theta)\omega^\Delta(r) r^{-2}\Big)\right| \le C\, x_i^{-2-k_1-l_1}, \quad i = 1, 2, \quad k_1, l_1 \ge 0. \tag{4.31}$$

Estimates for the derivatives of the function $u$ follow from inequalities (4.27), (4.28), and (4.30):

$$\begin{aligned}
\left|\frac{\partial^k u(r(x_1,\xi_2),\theta(x_1,\xi_2))}{\partial x_1^k}\right| &\le C\, x_1^{\lambda-1-k}|\log\Delta|^\beta, \quad \xi_2 \in [0, x_2], \quad k \ge 0,\\
\left|\frac{\partial^{k+l} u(r(x_1,x_2),\theta(x_1,x_2))}{\partial x_1^k \partial x_2^l}\right| &\le C\, x_i^{\lambda-1-k-l}|\log\Delta|^\beta, \quad i = 1, 2, \quad k, l \ge 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|\int_0^{x_2} \frac{\partial^{k_2} u(r(x_1,\xi_2),\theta(x_1,\xi_2))}{\partial x_1^{k_2}} d\xi_2\right| &\le C \int_0^{x_2} x_1^{\lambda-1-k_2}|\log\Delta|^\beta d\xi_2\\
&\le C\, x_i^{\lambda-k_2}|\log\Delta|^\beta, \quad k_2 \ge 0,\\
\left|\frac{\partial^{l_2}}{\partial x_2^{l_2}}\left(\int_0^{x_2} \frac{\partial^{k_2} u(r(x_1,\xi_2),\theta(x_1,\xi_2))}{\partial x_1^{k_2}} d\xi_2\right)\right| &= \left|\frac{\partial^{k_2+l_2-1} u(r(x_1,x_2),\theta(x_1,x_2))}{\partial x_1^{k_2}\partial x_2^{l_2-1}}\right|\\
&\le C\, x_i^{\lambda-k_2-l_2}|\log\Delta|^\beta, \quad k_2 \ge 0, \ l_2 \ge 1
\end{aligned} \tag{4.32}$$

for $i = 1, 2$. Then representation (4.29) and estimates (4.31), (4.32) give the required bound in (3.55). $\square$

**Proof of Lemma 3.15.** According to equality (3.52) and decomposition (3.54) we have

$$\xi(x)w_0(x) = U(x)\tilde{\omega}^{\Delta}(r) = \tilde{\chi}(\theta)\tilde{\omega}^{\Delta}(r)\int_0^{x_2} u(r(x_1,\xi_2),\theta(x_1,\xi_2))d\xi_2, \qquad x \in S_\kappa.$$

Then for sufficiently small $\Delta > 0$ we obtain, by using (4.26), (hereafter, $\theta_1 = \arctan\kappa^{-1}$, $\theta_2 = \arctan\kappa$, and $u(x_1,\xi_2) = u(r(x_1,\xi_2),\theta(x_1,\xi_2))$)

$$\begin{aligned}
\|\xi w_0\|^2_{L_2(S_\kappa)} &= \|\xi w_0\|^2_{L_2(K_\Delta)} \le C\int_0^{2\Delta}\int_{\theta_1}^{\theta_2}\left(\int_0^{x_2}|u(x_1,\xi_2)|d\xi_2\right)^2 r\,d\theta dr \\
&\le C\int_0^{2\Delta}\int_{\theta_1}^{\theta_2} x_1^{2\lambda-2}|\log x_1|^{2\beta}\,x_2^2\,r\,d\theta dr \\
&\le C\int_0^{2\Delta} r^{2\lambda+1}|\log r|^{2\beta}dr \le C\Delta^{2\lambda+2}|\log\Delta|^{2\beta}, \quad \lambda > -1,
\end{aligned} \tag{4.33}$$

where $C > 0$ is independent of $\Delta$. Let $0 < s < \min\{1, \lambda+1\}$. Then

$$\begin{aligned}
\|\xi w_0\|^2_{H^s(S_\kappa)} &= \int_0^\infty t^{-2s}\inf_{\xi w_0 = w_1+w_2}\left(\|w_1\|^2_{L_2(S_\kappa)} + t^2\,\|w_2\|^2_{H^1(S_\kappa)}\right)\frac{dt}{t} \\
&\le \int_0^\Delta t^{-2s-1}\left(\|\xi w_0\tilde{\omega}_t\|^2_{L_2(S_\kappa)} + t^2\,\|\xi w_0\omega_t\|^2_{H^1(S_\kappa)}\right)dt + \int_\Delta^\infty t^{-2s-1}\|\xi w_0\|^2_{L_2(S_\kappa)}dt,
\end{aligned} \tag{4.34}$$

where $\omega_t$ and $\tilde{\omega}_t$ are defined by (4.14) for any $t \in (0,\Delta)$.

Now we estimate the norms on the right-hand side of (4.34). Since $\tilde{\omega}_t(r) = 0$ for $r \ge 2t$, we use the same arguments as in (4.33) to obtain

$$\begin{aligned}
\|\xi w_0\tilde{\omega}_t\|^2_{L_2(S_\kappa)} &= \left\|\tilde{\chi}(\theta)\tilde{\omega}^{\Delta}(r)\tilde{\omega}_t(r)\int_0^{x_2} u(x_1,\xi_2)d\xi_2\right\|^2_{L_2(K_\Delta)} \\
&\le C\int_0^{2t}\int_{\theta_1}^{\theta_2}\left(\int_0^{x_2}|u(x_1,\xi_2)|d\xi_2\right)^2 r\,d\theta dr \le Ct^{2\lambda+2}|\log t|^{2\beta}.
\end{aligned} \tag{4.35}$$

In order to estimate the norm $\|\xi w_0\omega_t\|_{H^1(S_\kappa)}$, we note that $\xi w_0\omega_t = 0$ outside the domain $K^1_\Delta = \{(x_1,x_2)\in S_\kappa;\ t<r<2\Delta\}$ because $\omega_t(r)=0$ for $0\le r\le t$. Let $x\in K^1_\Delta$. Then

$$\begin{aligned}
\left|\frac{\partial(\xi w_0\omega_t)}{\partial x_1}\right| &= \left|\frac{\partial}{\partial x_1}\left(\tilde\chi(\theta)\tilde\omega^\Delta(r)\omega_t(r)\int\limits_0^{x_2} u(x_1,\xi_2)d\xi_2\right)\right| \\
&\le C\left(\left|\frac{\partial\tilde\chi}{\partial\theta}\right|\left|\frac{\partial\theta}{\partial x_1}\right| + \left|\frac{\partial\tilde\omega^\Delta}{\partial r}\right|\left|\frac{\partial r}{\partial x_1}\right| + \left|\frac{\partial\omega_t}{\partial r}\right|\left|\frac{\partial r}{\partial x_1}\right|\right)\int\limits_0^{x_2}|u(x_1,\xi_2)|d\xi_2 + C\int\limits_0^{x_2}\left|\frac{\partial u(x_1,\xi_2)}{\partial x_1}\right|d\xi_2 \\
&\le C\left(r^{-1}x_1^{\lambda-1}|\log x_1|^\beta x_2 + x_1^{\lambda-2}|\log x_1|^\beta x_2\right) \le Cr^{\lambda-1}|\log r|^\beta. \qquad (4.36)
\end{aligned}$$

Here we applied inequalities (4.3), (4.26), (4.27) and also used the fact that the derivatives of $\tilde\omega^\Delta(r)$ and $\omega_t(r)$ satisfy estimates (4.4) with $x_2$ replaced by $r$. Similarly, using (4.3), (4.4), and (4.26) we find

$$\left|\frac{\partial(\xi w_0\omega_t)}{\partial x_2}\right| \le C\left(r^{-1}\int\limits_0^{x_2}|u(x_1,\xi_2)|d\xi_2 + |u(x_1,x_2)|\right) \le Cr^{\lambda-1}|\log r|^\beta. \qquad (4.37)$$

Since $\xi w_0\omega_t$ vanishes on $\partial S_\kappa$ and outside $K^1_\Delta$, we deduce from (4.36), (4.37) that

$$\begin{aligned}
\|\xi w_0\omega_t\|^2_{H^1(S_\kappa)} &\le C|\xi w_0\omega_t|^2_{H^1(K^1_\Delta)} \le C\int\limits_t^{2\Delta}\int\limits_{\theta_1}^{\theta_2} r^{2\lambda-2}|\log r|^{2\beta}\, r\, d\theta dr \\
&\le C\int\limits_t^{2\Delta} r^{2\lambda-1}|\log r|^{2\beta}dr \le C\begin{cases} t^{2\lambda}|\log t|^{2\beta} & \text{if}\quad \lambda<0, \\ \Delta^\varepsilon t^{-\varepsilon}|\log t|^{2\beta} & \text{if}\quad \lambda=0, \\ \Delta^{2\lambda}|\log\Delta|^{2\beta} & \text{if}\quad \lambda>0. \end{cases} \qquad (4.38)
\end{aligned}$$

Here, for $\lambda=0$, we introduced a small $\varepsilon$ such that $0<\varepsilon<2-2s$, cf. (4.20).

Using estimates (4.33), (4.35), (4.38) for the norms on the right-hand side of (4.34) and repeating the same arguments as in (4.21)–(4.23), we obtain

$$\|\xi w_0\|^2_{H^s(S_\kappa)} \le C\Delta^{2\lambda+2-2s}|\log\Delta|^{2\beta}, \qquad 0<s<\min\{1,\lambda+1\},\quad \lambda>-1, \qquad (4.39)$$

where $C>0$ is independent of $\Delta$.

Taking $\Delta = p^{-2}$ and using (4.33), (4.39) we prove estimate (3.57). □

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