The *hp*-version of the boundary element method with quasi-uniform meshes for weakly singular operators on surfaces [*]

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Dedicated to Professor Gabriel N. Gatica on the occasion of his 50th birthday.

Abstract

We prove an a priori error estimate for the *hp*-version of the boundary element method with weakly singular operators in three dimensions. The underlying meshes are quasi-uniform. Our model problem is that of the Laplacian exterior to an open surface where the solution has strong singularities which are not L_2-regular. Our results confirm previously conjectured convergence rates in h (the mesh size) and p (the polynomial degrees) and these rates are given explicitly in terms of the exponents of the singular functions. In particular, for sufficiently smooth given data we prove a convergence in the energy norm like $O(h^{1/2}p^{-1})$.

Key words: *hp*-version with quasi-uniform meshes, boundary element method, weakly singular operators, singularities
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1 Introduction

In recent papers we proved several a priori error estimates for the *p*- and the *hp*-version with quasi-uniform meshes of the boundary element method (BEM). The *p*-version of the BEM is a finite element Galerkin method for boundary integral equations where a fixed mesh is used and where the approximation is improved by increasing polynomial degrees. The *hp*-version combines mesh refinements with the increase of polynomial degrees.

We are particularly interested in three-dimensional elliptic problems of second order in domains interior or exterior to polyhedra or exterior to open surfaces. The corresponding boundary

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integral equations thus live on polyhedral or open surfaces and their solutions are irregular at edges and vertices. In this paper we analyse the *hp*-version of the BEM for weakly singular operators that appear when considering Dirichlet-type boundary value problems. For problems in two dimensions (on polygons or open curves) it is long known that the p-version converges twice as fast as the h-version (in terms of numbers of unknowns), see [3] for the finite element method (FEM) and [12] for the BEM. In three dimensions this fact has been confirmed only recently and only partially. For instance, for polyhedra, there are no corresponding a priori error estimates for the *hp*-version of the FEM with quasi-uniform meshes. For results on the p-version see [11].

In three dimensions, Schwab and Suri [16] were the first to analyse the p-version of the BEM, but only for hypersingular operators and on closed surfaces where solutions are in H^1. In [5] we improved and extended those results to the case of open surfaces (where solutions are not in H^1 in general). The case of weakly singular operators (only the p-version) has been dealt with in [7]. In [6] we extended the p-version results for hypersingular operators to the *hp*-version with quasi-uniform meshes. Preliminary results for the *hp*-version and weakly singular operators have been presented in [4]. There, non-optimal estimates (depending on an unspecified parameter ϵ) and only for smooth boundary curves are proved.

In this paper, we fully extend our p-version results from [7] to the *hp*-version with quasi-uniform meshes. We prove that, for singular functions, the p-version converges also for weakly singular operators twice as fast as the h-version. In particular, we prove the conjecture from [13] claiming that for sufficiently smooth given data the *hp*-version on open surfaces converges like $O(h^{1/2}p^{-1})$. Here, h refers to the mesh size and p specifies the polynomial degree. Usually, *hp*-version results are obtained from corresponding p-version results be scaling arguments. However, in the case of weakly singular operators, the energy space is a negative order Sobolev space and corresponding norms are defined by duality. Therefore, technical details are somewhat involved. But more importantly, the energy norm is not scalable under affine transformations. We circumvent this difficulty by considering a specific family of norms which are scalable.

To prove our a priori error estimate we consider the representation of the exact solution to our model problem by a finite number of singular functions plus a smooth remainder. We present exact approximation results for the singularities and prove an *hp*-approximation result on quasi-uniform meshes for smooth functions based on Sobolev regularity (Theorem 4.1). The technique to prove Theorem 4.1 appears to be standard for the h-version but, to our knowledge, the *hp*-result is new.

Let us note that, of course, the *hp*-version with geometrically graded meshes is, at least for standard elliptic problems, more attractive than the *hp*-version with quasi-uniform meshes. Whereas the latter converges at most algebraically fast the former method converges faster than any algebraic order [13]. However, for problems with an oscillating behaviour a uniformly lower bound for p, not being small, can be advantageous to minimise numerical dispersion errors, see [1, 14]. Secondly, in contrast to the *hp*-version with geometric meshes, quasi-uniform *hp*-methods deal with high order polynomial degrees also close to singularities and the required analysis (provided in this paper) is interesting.

Let us present our model problem. We consider a plane open surface $\Gamma \subset \mathbf{R}^3$ with polygonal boundary so that it can be discretised by meshes consisting of triangles and parallelograms. We note that our analysis will apply to open and closed piecewise smooth Lipschitz surfaces but for ease of presentation we consider the geometrically simpler case of a flat surface with polygonal boundary. Our model problem is the variational formulation of the equation with single layer potential V stemming from the Laplacian: *Find* $u \in \tilde{H}^{-1/2}(\Gamma)$ *such that*

$$\langle Vu, v\rangle = \langle f, v\rangle \quad \forall v \in \tilde{H}^{-1/2}(\Gamma). \tag{1.1}$$

Here, $f \in H^{1/2}(\Gamma)$ is a given function,

$$Vu(x) := \frac{1}{4\pi}\int_\Gamma \frac{u(y)}{|x-y|}\, dS_y, \qquad V:\ \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma),$$

is the single layer potential operator of the Laplacian, $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{L_2(\Gamma)}$ denotes the extension of the $L_2(\Gamma)$-inner product by duality, and $\tilde{H}^{-1/2}(\Gamma)$ is the dual space of $H^{1/2}(\Gamma)$. For the definition of $H^{1/2}(\Gamma)$ see Section 3.

The paper is organised as follows. In the next section we define the *hp*-version of the BEM for the approximate solution of our model problem. We review regularity results for the solution to problem (1.1) and formulate the main theorem stating an a priori error estimate for the *hp*-version of the BEM with quasi-uniform meshes. In Section 3 we recall definitions of the Sobolev spaces and their norms, and collect several technical lemmas. Sections 4 and 5 are focused on the approximation analysis of smooth and singular functions in negative order Sobolev norms. In Section 6 the obtained results are combined to prove a general approximation theorem.

Throughout the paper, C denotes a generic positive constant which does not depend on h or p.

2 The *hp*-version of the BEM and an a priori error estimate

For the approximate solution of (1.1) we apply the *hp*-version of the BEM on quasi-uniform meshes. In what follows, $h > 0$ and $p \geq 0$ will always specify the mesh parameter and a polynomial degree, respectively. For any $\Omega \subset \mathbf{R}^n$ we will denote $\rho_\Omega = \sup\{\mathrm{diam}(B);\ B \text{ is a ball in } \Omega\}$.

Let $\mathcal{M} = \{\Delta_h\}$ be a family of meshes $\Delta_h = \{\Gamma_j;\ j = 1,\dots,J\}$ on Γ, where the elements Γ_j are open triangles or parallelograms such that $\bar\Gamma = \cup_{j=1}^J \bar\Gamma_j$. For any $\Gamma_j \in \Delta_h$ we denote $h_j = \mathrm{diam}(\Gamma_j)$. In this paper we consider a family $\mathcal{M}$ of quasi-uniform meshes Δ_h on Γ in the sense that there exist positive constants σ_1, σ_2 independent of $h = \max\limits_j h_j$ such that for any $\Gamma_j \in \Delta_h$ and arbitrary $\Delta_h \in \mathcal{M}$ there holds

$$h \leq \sigma_1 h_j, \qquad h_j \leq \sigma_2 \rho_{\Gamma_j}. \tag{2.1}$$

Let $Q = (-1,1)^2$ and $T = \{(x_1,x_2);\ 0 < x_1 < 1,\ 0 < x_2 < x_1\}$ be the reference square and triangle, respectively. Then for any $\Gamma_j \in \Delta_h$ one has $\Gamma_j = M_j(K)$ where M_j is an affine mapping with Jacobian $|J_j| \simeq h_j^2$ and $K = Q$ or T as appropriate.

Below we will refer to three different unions of elements. The union of the elements at a node v is denoted by A_v, i.e., $\bar{A}_v := \cup\{\bar{\Gamma}_j;\ v \in \bar{\Gamma}_j\}$, the union of the elements at one edge e by A_e (the endpoints of e are not included in e), $\bar{A}_e := \cup\{\bar{\Gamma}_j;\ \bar{\Gamma}_j \cap e \neq \emptyset\}$, and $A_{ev} := A_v \cap A_e$.

Further, $\mathcal{P}^1_p(T)$ denotes the set of polynomials on T of total degree $\leq p$, and $\mathcal{P}^2_p(Q)$ is the set of polynomials on Q of degree $\leq p$ in each variable. Let $K \subset \mathbf{R}^2$ be an arbitrary triangle or parallelogram, and let $K = M(T)$ or $K = M(Q)$ with an invertible affine mapping M. Then by $\mathcal{P}_p(K)$ we denote the set of polynomials v on K such that $v \circ M \in \mathcal{P}^1_p(T)$ if K is a triangle and $v \circ M \in \mathcal{P}^2_p(Q)$ if K is a parallelogram (in particular, we will use this notation for $K = Q$ and $K = T$). For a given non-negative integer p, we then consider the space of piecewise polynomials on the mesh $\Delta_h \in \mathcal{M}$,

$$V^{h,p}(\Gamma) := \{v \in L_2(\Gamma);\ v|_{\Gamma_j} \in \mathcal{P}_p(\Gamma_j),\ j = 1, \ldots, J\}.$$

Note that $V^{h,p}(\Gamma) \subset \tilde{H}^{-1/2}(\Gamma)$. Now, the *hp*-version of the BEM is: *Find* $u_{hp} \in V^{h,p}(\Gamma)$ *such that*

$$\langle V u_{hp}, v\rangle = \langle f, v\rangle \quad \forall v \in V^{h,p}(\Gamma). \tag{2.2}$$

Since the operator V is continuous, symmetric, and positive definite, any boundary element method for problem (1.1) converges quasi-optimally, see [9, 17], i.e., there exists a constant $C > 0$ independent of h and p such that

$$\|u - u_{hp}\|_{\tilde{H}^{-1/2}(\Gamma)} \leq C \inf\{\|u - v\|_{\tilde{H}^{-1/2}(\Gamma)};\ v \in V^{h,p}(\Gamma)\}. \tag{2.3}$$

Before giving our main result stating an a priori error estimate for (2.2) let us recall the typical structure of the solution of our model problem for a sufficiently smooth right-hand side function f. We use the notation of [5, 16] and refer for more details to [19, 20].

Let V and E denote the sets of vertices and edges of Γ, respectively. For $v \in V$, let $E(v)$ denote the set of edges with v as an end point. Then, the solution u of (1.1) has the form

$$u = u_{\text{reg}} + \sum_{e\in E} u^e + \sum_{v\in V} u^v + \sum_{v\in V}\sum_{e\in E(v)} u^{ev}, \tag{2.4}$$

where, using local polar and Cartesian coordinate systems (r_v, θ_v) and (x_{e1}, x_{e2}) with origin v, there hold the following representations:

(i) The regular part $u_{\text{reg}} \in H^k(\Gamma)$, $k > 0$.

(ii) The edge singularities u^e have the form

$$u^e = \sum_{j=1}^{m_e} \left(\sum_{s=0}^{s^e_j} b^e_{js}(x_{e1}) |\log x_{e2}|^s \right) x_{e2}^{\gamma^e_j - 1}\, \chi^e_1(x_{e1}) \chi^e_2(x_{e2}), \tag{2.5}$$

where $\gamma^e_{j+1} \geq \gamma^e_j \geq \frac{1}{2}$, and m_e, s^e_j are integers. Here, χ^e_1, χ^e_2 are C^∞ cut-off functions with $\chi^e_1 = 1$ in a certain distance to the end points of e and $\chi^e_1 = 0$ in a neighbourhood of these vertices. Moreover, $\chi^e_2 = 1$ for $0 \leq x_{e2} \leq \delta_e$ and $\chi^e_2 = 0$ for $x_{e2} \geq 2\delta_e$ with some $\delta_e \in (0, \frac{1}{2})$. The functions $b^e_{js}\chi^e_1$ are in $H^m(e)$ for m as large as required.

(iii) The vertex singularities u^v have the form

$$u^v = \chi^v(r_v) \sum_{i=1}^{n_v} \sum_{t=0}^{q_i^v} B_{it}^v |\log r_v|^t r_v^{\lambda_i^v - 1} w_{it}^v(\theta_v), \tag{2.6}$$

where $\lambda_{i+1}^v \ge \lambda_i^v > 0$, n_v, $q_i^v \ge 0$ are integers, and B_{it}^v are real numbers. Here, χ^v is a C^∞ cut-off function with $\chi^v = 1$ for $0 \le r_v \le \tau_v$ and $\chi^v = 0$ for $r_v \ge 2\tau_v$ with some $\tau_v \in (0, \frac{1}{2})$. The functions w_{it}^v are in $H^q(0, \omega_v)$ for q as large as required. Here, ω_v denotes the interior angle (on Γ) between the edges meeting at v.

(iv) The edge-vertex singularities u^{ev} have the form

$$u^{ev} = u_1^{ev} + u_2^{ev},$$

where

$$u_1^{ev} = \sum_{j=1}^{m_e} \sum_{i=1}^{n_v} \left(\sum_{s=0}^{s_j^e} \sum_{t=0}^{q_i^v} \sum_{l=0}^{s} B_{ijlts}^{ev} |\log x_{e1}|^{s+t-l} |\log x_{e2}|^l \right) x_{e1}^{\lambda_i^v - \gamma_j^e} x_{e2}^{\gamma_j^e - 1} \, \chi^v(r_v) \chi^{ev}(\theta_v) \tag{2.7}$$

and

$$u_2^{ev} = \sum_{j=1}^{m_e} \sum_{s=0}^{s_j^e} B_{js}^{ev}(r_v) |\log x_{e2}|^s x_{e2}^{\gamma_j^e - 1} \, \chi^v(r_v) \chi^{ev}(\theta_v) \tag{2.8}$$

with

$$B_{js}^{ev}(r_v) = \sum_{l=0}^{s} B_{jsl}^{ev}(r_v) |\log r_v|^l. \tag{2.9}$$

Here, q_i^v, s_j^e, λ_i^v, γ_j^e, χ^v are as above, B_{ijlts}^{ev} are real numbers, and χ^{ev} is a C^∞ cut-off function with $\chi^{ev} = 1$ for $0 \le \theta_v \le \beta_v$ and $\chi^{ev} = 0$ for $\frac{3}{2}\beta_v \le \theta_v \le \omega_v$ for some $\beta_v \in (0, \min\{\omega_v/2, \pi/8\}]$. The functions B_{jsl}^{ev} may be chosen such that

$$B_{js}^{ev}(r_v) \, \chi^v(r_v) \chi^{ev}(\theta_v) = \chi_{js}(x_{e1}, x_{e2}) \, \chi_2^e(x_{e2}), \tag{2.10}$$

where the extension of χ_{js} by zero onto $\mathbf{R}^{2+} := \{(x_{e1}, x_{e2});\ x_{e2} > 0\}$ lies in $H^m(\mathbf{R}^{2+})$ for m as large as required. Here, χ_2^e is a C^∞ cut-off function as in (ii).

The following theorem is the main result of this paper.

Theorem 2.1 *Let* $u \in \tilde{H}^{-1/2}(\Gamma)$ *be the solution of* (1.1) *with sufficiently smooth given function* $f \in H^{1/2}(\Gamma)$ *such that representation* (2.4)–(2.10) *holds. Let* $v_0 \in V$, $e_0 \in E(v_0)$ *be such that* $\min\{\lambda_1^{v_0} + 1/2, \gamma_1^{e_0}\} = \min_{v \in V, e \in E(v)} \min\{\lambda_1^v + 1/2, \gamma_1^e\}$, *with* λ_1^v *and* γ_1^e *being as in* (2.5)–(2.8). *Then, for any* $h > 0$ *and every* $p \ge \min\{\lambda_1^{v_0} - 1, \gamma_1^{e_0} - 3/2\}$, *the BE approximation* u_{hp} *defined by* (2.2) *satisfies*

$$\|u - u_{hp}\|_{\tilde{H}^{-1/2}(\Gamma)} \le C \, h^{\min\{\lambda_1^{v_0} + 1/2, \gamma_1^{e_0}\}} \, (p+1)^{-2\min\{\lambda_1^{v_0} + 1/2, \gamma_1^{e_0}\}} \left(1 + \log \tfrac{p+1}{h}\right)^{\beta + \nu}, \tag{2.11}$$

where

$$\beta = \begin{cases} q_1^{v_0} + s_1^{e_0} + \frac{1}{2} & \text{if } \lambda_1^{v_0} = \gamma_1^{e_0} - \frac{1}{2}, \\ q_1^{v_0} + s_1^{e_0} & \text{otherwise,} \end{cases} \tag{2.12}$$

for numbers $q_1^{v_0}$, $s_1^{e_0}$ *as given in* (2.7), *and*

$$\nu = \begin{cases} \frac{1}{2} & \text{if } p = \min\{\lambda_1^{v_0} - 1,\, \gamma_1^{e_0} - \frac{3}{2}\}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.13}$$

If $0 \le p < \min\{\lambda_1^{v_0} - 1, \gamma_1^{e_0} - 3/2\}$, *then for any* $h > 0$ *there holds*

$$\|u - u_{hp}\|_{\tilde{H}^{-1/2}(\Gamma)} \le C\, h^{p+3/2}. \tag{2.14}$$

The positive constants C *in* (2.11) *and* (2.14) *are independent of* h *and* p.

Proof. Considering enough singularity terms in representation (2.4)–(2.8) we obtain a sufficiently high regularity for the function $u_{\text{reg}} \in H^k(\Gamma)$. Then, due to the quasi-optimal convergence (2.3) of the BEM, the assertion immediately follows from Theorem 6.1 below. □

3 Preliminaries

First of all, let us recall the Sobolev spaces and norms that will be used, see [15]. For a domain $\Omega \subset \mathbf{R}^n$ and an integer s, let $H^s(\Omega)$ be the closure of $C^\infty(\Omega)$ with respect to the norm

$$\|u\|^2_{H^s(\Omega)} = \|u\|^2_{H^{s-1}(\Omega)} + |u|^2_{H^s(\Omega)} \quad (s \ge 1),$$

where

$$|u|^2_{H^s(\Omega)} = \int_\Omega |D^s u(x)|^2\, dx, \quad \text{and} \quad H^0(\Omega) = L_2(\Omega).$$

Here, $|D^s u(x)|^2 = \sum_{|\alpha|=s} |D^\alpha u(x)|^2$ in the usual notation with multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ and with respect to Cartesian coordinates $x = (x_1, \ldots, x_n)$. For a positive non-integer $s = m + \sigma$ with integer $m \ge 0$ and $0 < \sigma < 1$, the norm in $H^s(\Omega)$ is

$$\|u\|^2_{H^s(\Omega)} = \|u\|^2_{H^m(\Omega)} + |u|^2_{H^s(\Omega)}$$

with semi-norm

$$|u|^2_{H^s(\Omega)} = \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\sigma}}\, dx\, dy.$$

The Sobolev spaces $\tilde{H}^s(\Omega)$ for $s \in (0,1)$ and for a bounded Lipschitz domain Ω are defined by interpolation. We use the real K-method of interpolation (see [15]) to define

$$\tilde{H}^s(\Omega) = \Big(L_2(\Omega), H_0^t(\Omega)\Big)_{\frac{s}{t},2} \quad (1/2 < t \le 1,\ 0 < s < t).$$

Here, $H_0^t(\Omega)$ $(0 < t \le 1)$ is the completion of $C_0^\infty(\Omega)$ in $H^t(\Omega)$ and we identify $H_0^1(\Omega)$ and $\tilde{H}^1(\Omega)$. Note that the Sobolev spaces $H^s(\Omega)$ also satisfy the interpolation property

$$H^s(\Omega) = \Big(L_2(\Omega), H^1(\Omega)\Big)_{s,2} \quad (0 < s < 1).$$

Furthermore, the semi-norm $|\cdot|_{H^1(\Omega)}$ defines the norm on $\tilde{H}^1(\Omega)$ due to Poincaré's inequality.

For $s \in [-1, 0)$ the Sobolev spaces and their norms are defined by duality with $L_2(\Omega) = H^0(\Omega) = \tilde{H}^0(\Omega)$ as pivot space:

$$H^s(\Omega) = (\tilde{H}^{-s}(\Omega))', \quad \tilde{H}^s(\Omega) = (H^{-s}(\Omega))',$$

$$\|u\|_{H^s(\Omega)} = \sup_{0 \ne v \in \tilde{H}^{-s}(\Omega)} \frac{|\langle u, v\rangle_{L_2(\Omega)}|}{\|v\|_{\tilde{H}^{-s}(\Omega)}}, \quad \|u\|_{\tilde{H}^s(\Omega)} = \sup_{0 \ne v \in H^{-s}(\Omega)} \frac{|\langle u, v\rangle_{L_2(\Omega)}|}{\|v\|_{H^{-s}(\Omega)}},$$

where

$$\langle u, v\rangle_{L_2(\Omega)} = \int_\Omega u(x)v(x)dx.$$

Let us recall the following estimates for the above norms from [2, Theorem 4.1] (see also [18, Lemma 3.2], where these estimates are given for the case of complex interpolation). Let Ω be partitioned into non-overlapping Lipschitz subdomains $\Omega_1, \dots, \Omega_N$. Then for $s \in [-1, 1]$ there hold

$$\sum_{j=1}^N \|u|_{\Omega_j}\|^2_{H^s(\Omega_j)} \le \|u\|^2_{H^s(\Omega)} \quad \forall u \in H^s(\Omega) \tag{3.1}$$

and

$$\|u\|^2_{\tilde{H}^s(\Omega)} \le \sum_{j=1}^N \|u|_{\Omega_j}\|^2_{\tilde{H}^s(\Omega_j)} \quad \forall u \in \tilde{H}^s(\Omega) \text{ with } u|_{\Omega_j} \in \tilde{H}^s(\Omega_j) \text{ for } j = 1, \dots, N. \tag{3.2}$$

Remark 3.1 *We have introduced the Sobolev spaces $H^s(\Omega)$ for any real $s \ge -1$ and note that estimate* (3.1) *is valid for all these values of s (see* [2]*).*

The scaling properties of the norms $\|\cdot\|_{H^s(\Omega)}$ and $\|\cdot\|_{\tilde{H}^s(\Omega)}$ for $s \in [-1, 1]$ are formulated in the following lemma (cf. [2, Lemma 4.3]).

Lemma 3.1 *Let K^h and K be two open subsets of $\mathbf{R}^n$ such that $K^h = M(K)$ under an invertible affine mapping M. Assume that* $\operatorname{diam} K^h \simeq \rho_{K^h} \simeq h$ *and* $\operatorname{diam} K \simeq \rho_K \simeq 1$. *Let u and $\hat{u}$ be the functions defined on K^h and K, respectively, such that $\hat{u} = u \circ M$ and $u = \hat{u} \circ M^{-1}$. Then, for any positive integer m,*

$$|u|_{H^m(K^h)} \simeq h^{\frac{n}{2}-m} |\hat{u}|_{H^m(K)} \tag{3.3}$$

if $\hat{u} \in H^m(K)$. Moreover, for $s \in [0, 1]$ there hold

$$C_1 h^{\frac{n}{2}} \|\hat{u}\|_{H^s(K)} \le \|u\|_{H^s(K^h)} \le C_2 h^{\frac{n}{2}-s} \|\hat{u}\|_{H^s(K)} \tag{3.4}$$

if $\hat{u} \in H^s(K)$;

$$\|u\|_{\tilde{H}^s(K^h)} \simeq h^{\frac{n}{2}-s}\|\hat{u}\|_{\tilde{H}^s(K)} \tag{3.5}$$

if $\hat{u} \in \tilde{H}^s(K)$;

$$C_1 h^{\frac{n}{2}+s}\|\hat{u}\|_{\tilde{H}^{-s}(K)} \le \|u\|_{\tilde{H}^{-s}(K^h)} \le C_2 h^{\frac{n}{2}}\|\hat{u}\|_{\tilde{H}^{-s}(K)} \tag{3.6}$$

if $\hat{u} \in \tilde{H}^{-s}(K)$; *and*

$$\|u\|_{H^{-s}(K^h)} \simeq h^{\frac{n}{2}+s}\|\hat{u}\|_{H^{-s}(K)} \tag{3.7}$$

if $\hat{u} \in H^{-s}(K)$.

Proof. The equivalence (3.3) is valid due to [8, Theorem 3.1.2]. This gives (3.4) for $s = 0, 1$. The equivalence (3.5) for $s = 0, 1$ is also deduced from (3.3), because $|\cdot|_{H^1(K)}$ defines the norm on $\tilde{H}^1(K)$. For $s \in (0, 1)$, one obtains (3.4), (3.5) by interpolation, and for $s \in [0, 1]$, estimates (3.6), (3.7) then follow by duality because $\langle u, v\rangle_{L_2(K^h)} \simeq h^n \langle \hat{u}, \hat{v}\rangle_{L_2(K)}$. □

Inequalities (3.4) and (3.6) in Lemma 3.1 show that the norms $\|\cdot\|_{H^s(\Omega)}$ and $\|\cdot\|_{\tilde{H}^{-s}(\Omega)}$ defined above are not scalable for $s \in (0, 1]$. Therefore, following [10], we consider for a generic subdomain $\omega \subset \Omega$ another family of norms $\|\cdot\|_{H_h^s(\omega)}$ and $\|\cdot\|_{\tilde{H}_h^s(\omega)}$ ($s \in [-1, 1]$) which are scalable. Let

$$\|u\|_{H_h^0(\omega)} = \|u\|_{\tilde{H}_h^0(\omega)} = \|u\|_{L_2(\omega)},$$

$$\|u\|^2_{H_h^1(\omega)} = \operatorname{diam}(\omega)^{-2}\|u\|^2_{L_2(\omega)} + |u|^2_{H^1(\omega)} \quad \text{and} \quad \|u\|^2_{\tilde{H}_h^1(\omega)} = |u|^2_{H^1(\omega)}.$$

Then, analogously as for traditional norms, the norms $\|\cdot\|_{H_h^s(\omega)}$ and $\|\cdot\|_{\tilde{H}_h^s(\omega)}$ for $s \in (0, 1)$ are defined by interpolation and for $s \in [-1, 0)$ by duality arguments. Note that the index h does not refer to the diameter of ω, it is rather an index to indicate the scalability of the norms under affine transformations of ω onto a reference subdomain (element). This fact is formulated in Lemma 3.2 below. In order to prove the analogs of estimates (3.1), (3.2) for these scalable norms, one needs some additional assumptions (see Lemma 3.3).

Lemma 3.2 [10, Lemma 3.1] *Let* K^h, $K \subset \mathbf{R}^2$ *satisfy the assumptions of Lemma* 3.1. *Then using the notation of Lemma* 3.1 *there hold for real* $s \in [-1, 1]$

$$\|u\|_{H_h^s(K^h)} \simeq h^{1-s}\|\hat{u}\|_{H^s(K)}$$

if $\hat{u} \in H^s(K)$, *and*

$$\|u\|_{\tilde{H}_h^s(K^h)} \simeq h^{1-s}\|\hat{u}\|_{\tilde{H}^s(K)}$$

if $\hat{u} \in \tilde{H}^s(K)$.

Both equivalences are uniform for $h > 0$.

Lemma 3.3 [10, Lemma 3.2] *Let* $\Omega \subset \mathbf{R}^2$ *be partitioned into shape regular convex polygonal subdomains* Ω_j *(*$j = 1, \dots, N$*) which are affine transformations of a fixed set of polygons. Then, for all* $u \in H^s(\Omega)$, $s \in [0,1]$, *with* $\int_{\Omega_j} u\,dx = 0$ *(*$j = 1, \dots, N$*) there holds*

$$\sum_{j=1}^{N} \|u|_{\Omega_j}\|^2_{H^s_h(\Omega_j)} \le C\|u\|^2_{H^s(\Omega)}. \tag{3.8}$$

Moreover, for all $u \in \tilde{H}^s(\Omega)$, $s \in [-1,0]$, *with* $u|_{\Omega_j} \in \tilde{H}^s(\Omega_j)$ *and* $\int_{\Omega_j} u\,dx = 0$ *(*$j = 1, \dots, N$*) there holds*

$$\|u\|^2_{\tilde{H}^s(\Omega)} \le C \sum_{j=1}^{N} \|u|_{\Omega_j}\|^2_{\tilde{H}^s_h(\Omega_j)}. \tag{3.9}$$

The positive constants C *in* (3.8) *and* (3.9) *are independent of* u *and* N.

We will also need the following auxiliary statement.

Lemma 3.4 *Let* $\Omega^h \subset \mathbf{R}^2$ *be a polygonal domain such that* $\operatorname{diam} \Omega^h \simeq \rho_{\Omega^h} \simeq h$. *Then for all* $v \in \tilde{H}^{-s}(\Omega^h)$ *with* $s \in [0,1]$, *there holds*

$$\left\| v - \frac{1}{|\Omega^h|} \int_{\Omega^h} v\,dx \right\|_{\tilde{H}^{-s}_h(\Omega^h)} \le C\|v\|_{\tilde{H}^{-s}_h(\Omega^h)} \tag{3.10}$$

with a positive constant C *independent of* v *and* h.

Proof. Denote $\bar{v} := \frac{1}{|\Omega^h|} \int_{\Omega^h} v\,dx$. Then for $s \in [0,1]$

$$\|\bar{v}\|_{\tilde{H}^{-s}_h(\Omega^h)} = \frac{1}{|\Omega^h|} \left| \int_{\Omega^h} v\,dx \right| \|1\|_{\tilde{H}^{-s}_h(\Omega^h)} \le \frac{1}{|\Omega^h|} \|v\|_{\tilde{H}^{-s}_h(\Omega^h)} \|1\|_{H^s_h(\Omega^h)} \|1\|_{\tilde{H}^{-s}_h(\Omega^h)}. \tag{3.11}$$

Since $|\Omega^h| \simeq h^2$ and, due to Lemma 3.2,

$$\|1\|_{H^s_h(\Omega^h)} \simeq h^{1-s}, \qquad \|1\|_{\tilde{H}^{-s}_h(\Omega^h)} \simeq h^{1+s},$$

we deduce from (3.11)

$$\|\bar{v}\|_{\tilde{H}^{-s}_h(\Omega^h)} \le C\|v\|_{\tilde{H}^{-s}_h(\Omega^h)}.$$

Then (3.10) follows by using the triangle inequality. □

4 Auxiliary approximation results

In this section we formulate several results regarding the approximation of smooth and singular functions in negative order Sobolev norms.

For the *hp*-approximation of smooth functions on quasi-uniform meshes we prove the following statement, which gives an estimate for the approximation error in the $\tilde{H}^s(\Gamma)$-norm, $s \in [-1, 0]$.

Theorem 4.1 *Let* $m \geq 0$. *Then for any* $u \in H^m(\Gamma)$ *there exists* $u_{hp} \in V^{h,p}(\Gamma)$ *such that for* $s \in [-1, 0]$

$$\|u - u_{hp}\|_{\tilde{H}^s(\Gamma)} \leq C h^{\mu - s}(p+1)^{s-m}\|u\|_{H^m(\Gamma)}, \tag{4.1}$$

where $\mu = \min\{m, p+1\}$.

Proof. In view of the bound (3.2) one needs to find a piecewise polynomial u_{hp} such that for any element $\Gamma_j \subset \Delta_h$ there holds

$$\|(u - u_{hp})|_{\Gamma_j}\|^2_{\tilde{H}^s(\Gamma_j)} \leq C h^{2(\mu - s)}(p+1)^{2(s-m)}\|u|_{\Gamma_j}\|^2_{H^m(\Gamma_j)} \tag{4.2}$$

with $s \in [-1, 0]$ and $\mu = \min\{m, p+1\}$.
(i) First we prove that for any $v \in H^m(\Gamma_j)$ there exists a polynomial $v_p \in \mathcal{P}_p(\Gamma_j)$ such that

$$\|v - v_p\|^2_{L_2(\Gamma_j)} \leq C h^{2\mu}(p+1)^{-2m}\|v\|^2_{H^m(\Gamma_j)}. \tag{4.3}$$

Let $K^h = \Gamma_j \in \Delta_h$ and $K = Q$ (or $K = T$) such that $K^h = M_j(K)$ under affine mapping M_j. Then, due to Lemma 4.1 of [3], there exists a family of operators $\{\hat{\pi}_p\}$, $p = 0, 1, 2, \ldots$, $\hat{\pi}_p : H^m(K) \to \mathcal{P}_p(K)$ such that for any $\hat{v} \in H^m(K)$

$$\|\hat{v} - \hat{\pi}_p \hat{v}\|_{L_2(K)} \leq C(p+1)^{-m}\|\hat{v}\|_{H^m(K)}, \quad m \geq 0.$$

Moreover, if $\hat{v} \in \mathcal{P}_p(K)$, then $\hat{\pi}_p \hat{v} = \hat{v}$.

On the other hand, if $v \in H^m(K^h)$, then $\hat{v} = v \circ M_j \in H^m(K)$ and one has (see, e.g., [3, Lemma 4.4])

$$\inf_{\hat{\varphi} \in \mathcal{P}_p(K)} \|\hat{v} - \hat{\varphi}\|_{H^m(K)} \leq C h^{\min\{m, p+1\} - 1}\|v\|_{H^m(K^h)}.$$

These two results yield (for details, see [3, Lemma 4.5])

$$\|\hat{v} - \hat{\pi}_p \hat{v}\|_{L_2(K)} \leq C h^{\min\{m, p+1\} - 1}(p+1)^{-m}\|v\|_{H^m(K^h)}, \quad m \geq 0.$$

Let $v_p := (\hat{\pi}_p \hat{v}) \circ M_j^{-1} = \hat{v}_p \circ M_j^{-1}$. Then $v_p \in \mathcal{P}_p(\Gamma_j)$ and making use of Lemma 3.2 we deduce that

$$\|v - v_p\|_{L_2(\Gamma_j)} \leq C h^1 \|\hat{v} - \hat{v}_p\|_{L_2(K)} \leq C h^{\mu}(p+1)^{-m}\|v\|_{H^m(\Gamma_j)}.$$

This proves (4.3).

(ii) Now, for given $u \in H^m(\Gamma)$ let $u_{hp} \in V^{h,p}(\Gamma)$ be defined piecewise, on the elements Γ_j, by the $L_2(\Gamma_j)$-projection onto $\mathcal{P}_p(\Gamma_j)$. Then, by (4.3) and using the minimising property of the L_2-projection, we find the estimate

$$\|(u - u_{hp})|_{\Gamma_j}\|_{L_2(\Gamma_j)} \leq Ch^{\mu}(p+1)^{-m}\|u\|_{H^m(\Gamma_j)}. \tag{4.4}$$

Now, by the definition of the $\tilde{H}^s(\Gamma_j)$-norm and by the orthogonality of the $L_2(\Gamma_j)$-projection, we find that

$$\begin{aligned} \|(u - u_{hp})|_{\Gamma_j}\|_{\tilde{H}^s(\Gamma_j)} &= \sup_{v \in H^{-s}(\Gamma_j)\setminus\{0\}} \frac{\langle u - u_{hp}, v\rangle_{L_2(\Gamma_j)}}{\|v\|_{H^{-s}(\Gamma_j)}} \\ &= \sup_{v \in H^{-s}(\Gamma_j)\setminus\{0\}} \inf_{v_p \in \mathcal{P}_p(\Gamma_j)} \frac{\langle u - u_{hp}, v - v_p\rangle_{L_2(\Gamma_j)}}{\|v\|_{H^{-s}(\Gamma_j)}} \\ &\leq \|u - u_{hp}\|_{L_2(\Gamma_j)} \sup_{v \in H^{-s}(\Gamma_j)\setminus\{0\}} \inf_{v_p \in \mathcal{P}_p(\Gamma_j)} \frac{\|v - v_p\|_{L_2(\Gamma_j)}}{\|v\|_{H^{-s}(\Gamma_j)}}. \end{aligned} \tag{4.5}$$

By (4.3) there holds

$$\sup_{v \in H^{-s}(\Gamma_j)\setminus\{0\}} \inf_{v_p \in \mathcal{P}_p(\Gamma_j)} \frac{\|v - v_p\|_{L_2(\Gamma_j)}}{\|v\|_{H^{-s}(\Gamma_j)}} \leq Ch^{\min\{-s,p+1\}}(p+1)^s = Ch^{-s}(p+1)^s.$$

Therefore, (4.5) together with (4.4) proves that

$$\|(u - u_{hp})|_{\Gamma_j}\|^2_{\tilde{H}^s(\Gamma_j)} \leq Ch^{2\mu}(p+1)^{-2m}\|u\|^2_{H^m(\Gamma_j)}\, h^{-2s}(p+1)^{2s}.$$

This is (4.2).

To finish the proof of the theorem it remains to combine inequalities (4.2) over all the elements of the mesh and to apply (3.2). □

Now let us recall some known results regarding the approximation of singularities by polynomials of arbitrary degree in negative order Sobolev spaces on triangles (parallelograms) of fixed size. In the propositions below $K \subset \mathbf{R}^2$ will always denote a triangle or parallelogram such that $\operatorname{diam} K \simeq \rho_K \simeq 1$. The particular location of K in $\mathbf{R}^2$ will be additionally specified in each proposition. We will consider three types of singular functions on K which correspond to the vertex singularity (see (2.6)) and to the edge-vertex singularities of both types (see (2.7)–(2.10)):

$$u_1(x) = r^{\lambda-1}|\log r|^{\beta}\chi(r)w(\theta), \tag{4.6}$$

$$u_2(x) = x_1^{\lambda-\gamma}x_2^{\gamma-1}|\log x_1|^{\beta_1}|\log x_2|^{\beta_2}\chi(r)\tilde{\chi}(\theta), \tag{4.7}$$

$$u_3(x) = x_2^{\gamma-1}|\log x_2|^{\beta}\chi_1(x_1,x_2)\chi_2(x_2), \tag{4.8}$$

where $\lambda > -\frac{1}{2}$ and $\gamma > 0$ are real numbers, β, β_1, $\beta_2 \geq 0$ are integers, (r,θ) are polar coordinates in $\mathbf{R}^2$, χ, $\tilde{\chi}$, χ_2 are C^∞ cut-off functions satisfying

$$\operatorname{supp}\chi \subset [0,\tau_0], \quad \operatorname{supp}\tilde{\chi} \subset [0,\beta_0], \quad \operatorname{supp}\chi_2 \subset [0,\delta_0]$$

for some $\tau_0, \beta_0, \delta_0 > 0$, and the functions w, χ_1 are sufficiently smooth.

Proposition 4.1 *Let* $K \subset \mathbf{R}^2$ *and suppose that the origin* O *is a vertex of* K*. Let* u_1 *be given by* (4.6) *and assume that* $\operatorname{supp}\chi \subset [0,\tau_0]$ *for* $0 < \tau_0 < \rho_K$*. Then there exists a sequence* $u_{1,p} \in \mathcal{P}_p(K),\ p = 0,1,2,\ldots,$ *such that for* $-1 \le s < \min\{0,\lambda\}$

$$\|u_1 - u_{1,p}\|_{\tilde{H}^s(K)} \le C\,(p+1)^{-2(\lambda-s)}\,(1+\log(p+1))^{\beta}. \tag{4.9}$$

Proof. If $p = 0$, then we set $u_{1,p} = 0$ on K, and (4.9) is valid. For $p \ge 1$, the assertion follows from [7, Theorem 3.6] by adjusting the constant C. $\square$

Proposition 4.2 [7, Theorem 3.4] *Let* $K \subset \mathbf{R}^{2+}$*. Suppose that the origin* O *is a vertex of* K *and one of the other vertices of* K *lies on the right semi-axis* Ox_1*. Let* u_2 *be given by* (4.7) *and assume that* $\operatorname{supp} u_2 \subset \bar{S}_0 = \{(r,\theta);\ 0 \le r \le \tau_0,\ 0 \le \theta \le \beta_0 < \frac{\pi}{4}\} \subset \bar{K}$*. Then there exists a sequence* $u_{2,p} \in \mathcal{P}_p(K),\ p = 0,1,2,\ldots,$ *such that for* $-1 \le s < \min\{0,\lambda,\gamma-\frac{1}{2}\}$

$$\|u_2 - u_{2,p}\|_{\tilde{H}^s(K)} \le C\,(p+1)^{-2(\min\{\lambda,\gamma-1/2\}-s)}\,(1+\log(p+1))^{\beta_1+\beta_2+\sigma}, \tag{4.10}$$

where $\sigma = \frac{1}{2}$ *if* $\lambda = \gamma - \frac{1}{2}$*, and* $\sigma = 0$ *otherwise.*

Proposition 4.3 *Let* $K \subset \mathbf{R}^{2+}$ *and suppose that at least one vertex of* K *lies on the axis* Ox_1*. Let* u_3 *be given by* (4.8) *with* $\chi_1 \in H^m(K),\ m > 2\gamma + 3,$ *and assume that* $\operatorname{supp}\chi_2 \subset [0,\delta_0]$ *for* $0 < \delta_0 < \rho_K$*. Then there exists a sequence* $u_{3,p} \in \mathcal{P}_p(K),\ p = 0,1,2,\ldots,$ *such that for* $-1 \le s < \min\{0,\gamma-\frac{1}{2}\}$

$$\|u_3 - u_{3,p}\|_{\tilde{H}^s(K)} \le C\,(p+1)^{-2(\gamma-1/2-s)}\,(1+\log(p+1))^{\beta}\,\|\chi_1\|_{H^m(K)}. \tag{4.11}$$

Proof. For $s = -\frac{1}{2}$, this statement follows from [7, Theorem 3.2]. As shown in [7, Remark 3.4], the general estimate (4.11) for $-1 \le s < \min\{0,\gamma-\frac{1}{2}\}$ also holds. $\square$

5 The *hp*-approximation of singularities

We will use the results of Propositions 4.1–4.3 to estimate the errors of piecewise polynomial approximations of the singular functions u^e, u^v, u_1^{ev}, and u_2^{ev} (see (2.5)–(2.8)) on quasi-uniform meshes. For each singular function we prove an error estimate in terms of both the mesh size h and the polynomial degree p.

5.1 Approximation of the edge-vertex singularity u_1^{ev}

Let $e \in E$ be an edge of Γ with vertices v, w. By l_v and l_w we will denote the edges of ∂A_e such that $\bar{l}_v \cap \bar{e} = \{v\}$ and $\bar{l}_w \cap \bar{e} = \{w\}$.

Let us consider the cut-off functions χ^v and χ^{ev} which appear in the expression for the edge-vertex singularity u_1^{ev} (see (2.7)). We adjust the supports of these cut-off functions as follows:

$$\operatorname{supp}\chi^v \subset [0, 2\tau_v] \ \text{ with } \ 0 < \tau_v < \min\{\tfrac{1}{4}\operatorname{dist}\{v, w\}, \tfrac{1}{2}\},$$
$$\operatorname{supp}\chi^{ev} \subset [0, \tfrac{3}{2}\beta_v] \ \text{ with } \ 0 < \beta_v \le \min\{\tfrac{1}{2}\theta_0, \tfrac{1}{2}\omega_v, \tfrac{\pi}{8}\},$$

where θ_0 is the minimal angle of the elements in the mesh Δ_h. Then u_1^{ev} vanishes outside the sector $S = \{(r_v, \theta_v);\ 0 < r_v < 2\tau_v,\ 0 < \theta_v < \frac{3}{2}\beta_v\}$, in particular, $u_1^{ev} = 0$ on $l_v \cup l_w$. Note that these conclusions also hold for the edge-vertex singularity u_2^{ev} given by (2.8).

Theorem 5.1 *Let $u = u_1^{ev}$ be given by (2.7). Then there exists $u_{hp} \in V^{h,p}(\Gamma)$ with $p \ge \min\{\lambda - 1, \gamma - \frac{3}{2}\}$ such that for $s \in [-1, \min\{0, \lambda, \gamma - \frac{1}{2}\})$,*

$$\|u - u_{hp}\|_{\tilde{H}^s(\Gamma)} \le C\, h^{\min\{\lambda,\gamma-1/2\}-s}\, (p+1)^{-2(\min\{\lambda,\gamma-1/2\}-s)} \left(1 + \log\tfrac{p+1}{h}\right)^{\beta+\nu}, \tag{5.1}$$

where $\lambda = \lambda_1^v > -\frac{1}{2}$, $\gamma = \gamma_1^e > 0$,

$$\beta = \begin{cases} q_1^v + s_1^e + \frac{1}{2} & \text{if } \lambda_1^v = \gamma_1^e - \frac{1}{2}, \\ q_1^v + s_1^e & \text{otherwise,} \end{cases}$$

and

$$\nu = \begin{cases} \frac{1}{2} & \text{if } p = \min\{\lambda - 1,\ \gamma - \frac{3}{2}\}, \\ 0 & \text{otherwise.} \end{cases}$$

If $0 \le p < \min\{\lambda - 1, \gamma - \frac{3}{2}\}$, then there exists $u_{hp} \in V^{h,p}(\Gamma)$ satisfying for $s \in [-1, 0]$

$$\|u - u_{hp}\|_{\tilde{H}^s(\Gamma)} \le C\, h^{p+1-s}. \tag{5.2}$$

Proof. For simplicity we consider the singular function

$$u(x_1, x_2) = x_1^{\lambda-\gamma} x_2^{\gamma-1} |\log x_1|^{\beta_1} |\log x_2|^{\beta_2} \chi^v(r) \chi^{ev}(\theta), \tag{5.3}$$

where $\lambda = \lambda_1^v > -\frac{1}{2}$, $\gamma = \gamma_1^e > 0$, and β_1, $\beta_2 \ge 0$ are integers.

Let us introduce an auxiliary cut-off function $\chi_2 \in C^\infty(\mathbf{R}^+)$ such that for some $\delta \in (0, 1)$

$$\chi_2(t) = 1 \ \text{ for } 0 \le t \le \delta/2 \quad \text{and} \quad \chi_2(t) = 0 \ \text{ for } t \ge \delta.$$

Denote $h_0 = (\sigma_1 \sigma_2)^{-1} h$, where σ_1, σ_2 are the same as in (2.1). We decompose the function u in (5.3) as

$$\begin{aligned} u &= u\chi^v(r/h_0) + u(1 - \chi^v(r/h_0))\chi_2(x_2/h_0) + u(1 - \chi^v(r/h_0))(1 - \chi_2(x_2/h_0)) \\ &=: \varphi_1 + \varphi_2 + \varphi_3. \end{aligned} \tag{5.4}$$

We will approximate the functions φ_i $(i = 1, 2, 3)$ in (5.4) separately.

Approximation of φ_1. Due to the adjustment of the supports of the cut-off functions χ^v and χ^{ev}, there holds $\operatorname{supp}\varphi_1 \subset \bar{K}^h$, where $K^h = \Gamma_1 \subset A_{ev}$ is the element touching the edge e and the vertex v. Moreover, if $K \subset \mathbf{R}^{2+}$ denotes a triangle or parallelogram such that $K^h = M(K)$, where

$$M : x_i = h\hat{x}_i, \quad i = 1, 2, \quad x \in K^h, \quad \hat{x} \in K, \tag{5.5}$$

then K satisfies the assumptions of Proposition 4.2.

For $h < \frac{1}{2}$ one has

$$\hat{\varphi}_1(\hat{x}) = \varphi_1(h\hat{x}_1, h\hat{x}_2) = h^{\lambda-1} \sum_{k_1=0}^{\beta_1} \sum_{k_2=0}^{\beta_2} \binom{\beta_1}{k_1} \binom{\beta_2}{k_2} |\log h|^{k_1+k_2} \hat{f}_{\beta_1-k_1,\beta_2-k_2}(\hat{x}),$$

where

$$\hat{f}_{k_1,k_2}(\hat{x}) = \hat{x}_1^{\lambda-\gamma} \hat{x}_2^{\gamma-1} |\log \hat{x}_1|^{k_1} |\log \hat{x}_2|^{k_2} \chi^v(\sigma_1\sigma_2\hat{r}) \chi^{ev}(\hat{\theta}),$$

$\hat{r} = (\hat{x}_1^2 + \hat{x}_2^2)^{1/2}$, $\hat{\theta} = \arctan(\hat{x}_2/\hat{x}_1)$, $k_i = 0, \ldots, \beta_i$ $(i = 1, 2)$.

By Proposition 4.2, for each pair (k_1, k_2) there exists a polynomial $\hat{g}_{k_1,k_2} \in \mathcal{P}_p(K)$ approximating $\hat{f}_{k_1,k_2}$ on K and satisfying for $-1 \le s < \min\{0, \lambda, \gamma - \frac{1}{2}\}$

$$\|\hat{f}_{k_1,k_2} - \hat{g}_{k_1,k_2}\|_{\tilde{H}^s(K)} \le C\,(p+1)^{-2(\min\{\lambda,\gamma-1/2\}-s)}\,(1 + \log(p+1))^{k_1+k_2+\sigma}.$$

Hence, setting

$$\hat{\psi}_1(\hat{x}) := h^{\lambda-1} \sum_{k_1=0}^{\beta_1} \sum_{k_2=0}^{\beta_2} \binom{\beta_1}{k_1} \binom{\beta_2}{k_2} |\log h|^{k_1+k_2} \hat{g}_{\beta_1-k_1,\beta_2-k_2}(\hat{x}),$$

we estimate

$$\begin{aligned}
&\|\hat{\varphi}_1 - \hat{\psi}_1\|_{\tilde{H}^s(K)} \\
&\quad\le\ h^{\lambda-1}(p+1)^{-2(\min\{\lambda,\gamma-1/2\}-s)}(1+\log(p+1))^{\sigma} \ \times \\
&\qquad\times \sum_{k_1,k_2=0}^{\beta_1,\beta_2} \binom{\beta_1}{k_1}\binom{\beta_2}{k_2} |\log h|^{k_1+k_2} C(k_1,k_2)(1+\log(p+1))^{\beta_1-k_1+\beta_2-k_2} \\
&\quad\le\ C(\beta_1,\beta_2) h^{\lambda-1}(p+1)^{-2(\min\{\lambda,\gamma-1/2\}-s)} \Bigl(1 + \log\tfrac{p+1}{h}\Bigr)^{\beta_1+\beta_2} (1+\log(p+1))^{\sigma}.
\end{aligned} \tag{5.6}$$

Let $\psi_1 := \hat{\psi}_1 \circ M^{-1}$ on $K^h = \Gamma_1$. Then $\psi_1 \in \mathcal{P}_p(\Gamma_1)$ and making use of Lemma 3.2 we deduce from (5.6)

$$\|\varphi_1 - \psi_1\|_{\tilde{H}^s_h(\Gamma_1)} \le C h^{\lambda-s}(p+1)^{-2(\min\{\lambda,\gamma-1/2\}-s)} \Bigl(1 + \log\tfrac{p+1}{h}\Bigr)^{\beta_1+\beta_2} (1+\log(p+1))^{\sigma}, \tag{5.7}$$

where $-1 \le s < \min\{0, \lambda, \gamma - \frac{1}{2}\}$, $\sigma = \frac{1}{2}$ if $\lambda = \gamma - \frac{1}{2}$, and $\sigma = 0$ otherwise.

Approximation of φ_2. The function φ_2 in (5.4) has a singular behaviour only with respect to x_2 and has a small support, $\operatorname{supp}\varphi_2 \subset (\bar{A}_e \cap \bar{R}_1^h)$, where $R_1^h = \{(r,\theta);\ \tau_v h_0 < r < 2\tau_v,\ 0 < \theta < \frac{3}{2}\beta_v\}$. Let us write φ_2 as

$$\varphi_2(x_1,x_2) = x_2^{\gamma-1} |\log x_2|^{\beta_2} \chi_1(x_1,x_2) \chi_2(x_2/h_0), \tag{5.8}$$

where

$$\chi_1(x_1,x_2) := x_1^{\lambda-\gamma} |\log x_1|^{\beta_1} \chi^v(r) \chi^{ev}(\theta) (1 - \chi^v(r/h_0)).$$

Note that $\chi_1 \in C^\infty(A_e)$, $\operatorname{supp}\chi_1 \subset \bar{R}_1^h$, in particular, $\chi_1 = 0$ on the edges $l_v, l_w \subset \partial A_e$. Moreover, for any integer $t \geq 0$ there holds

$$|\chi_1|_{H^t(A_e)} \leq C \log^{\beta_1}(1/h) h^{1/2-t} \begin{cases} h^{\lambda-\gamma+1/2} & \text{if } \lambda < \gamma - 1/2, \\ \log^{1/2}(1/h) & \text{if } \lambda = \gamma - 1/2, \\ 1 & \text{if } \lambda > \gamma - 1/2, \end{cases} \tag{5.9}$$

see [6, proof of Theorem 5.1]. To approximate the function φ_2 given by (5.8), we consider an element $K^h = \Gamma_j \subset A_e$. Let $K \subset \mathbf{R}^{2+}$ be a triangle or parallelogram such that $K^h = M(K)$, where M is defined by (5.5). Then at least one vertex of K lies on the axis $O\hat{x}_1$ and

$$\hat{\varphi}_2(\hat{x}) = \varphi_2(h\hat{x}_1, h\hat{x}_2) = h^{\gamma-1} \sum_{k=0}^{\beta_2} \binom{\beta_2}{k} |\log h|^k \hat{f}_{\beta_2-k}(\hat{x}),$$

where

$$\hat{f}_k(\hat{x}) = \hat{x}_2^{\gamma-1} |\log \hat{x}_2|^k \hat{\chi}_1(\hat{x}) \chi_2(\sigma_1\sigma_2\hat{x}_2),$$

$\hat{\chi}_1(\hat{x}) = \chi_1(h\hat{x}_1, h\hat{x}_2)$, $k = 0, 1, \ldots, \beta_2$. Applying Proposition 4.3 to each function $\hat{f}_k$ ($k = 0, 1, \ldots, \beta_2$) we find polynomials $\hat{g}_k \in \mathcal{P}_p(K)$ such that for $-1 \leq s < \min\{0, \gamma - \frac{1}{2}\}$ and for any integer $m > 2\gamma + 3$

$$\|\hat{f}_k - \hat{g}_k\|_{\tilde{H}^s(K)} \leq C\,(p+1)^{-2(\gamma-1/2-s)}\,(1 + \log(p+1))^k\,\|\hat{\chi}_1\|_{H^m(K)}.$$

Hence, setting

$$\hat{\psi}_2(\hat{x}) := h^{\gamma-1} \sum_{k=0}^{\beta_2} \binom{\beta_2}{k} |\log h|^k \hat{g}_{\beta_2-k}(\hat{x}), \quad \psi_{2,j} := \hat{\psi}_2 \circ M^{-1} \in \mathcal{P}_p(\Gamma_j)$$

and applying Lemma 3.2 and Lemma 3.1 we estimate for any element $\Gamma_j \subset A_e$

$$\begin{aligned} \|\varphi_2 - \psi_{2,j}\|_{\tilde{H}_h^s(\Gamma_j)} &\simeq h^{1-s} \|\hat{\varphi}_2 - \hat{\psi}_2\|_{\tilde{H}^s(K)} \\ &\leq C h^{\gamma-s} \sum_{k=0}^{\beta_2} \binom{\beta_2}{k} |\log h|^k \|\hat{f}_{\beta_2-k} - \hat{g}_{\beta_2-k}\|_{\tilde{H}^s(K)} \\ &\leq C h^{\gamma-s} (p+1)^{-2(\gamma-1/2-s)} \Big(1 + \log \tfrac{p+1}{h}\Big)^{\beta_2} \|\hat{\chi}_1\|_{H^m(K)} \\ &\leq C h^{\gamma-s} (p+1)^{-2(\gamma-1/2-s)} \Big(1 + \log \tfrac{p+1}{h}\Big)^{\beta_2} \Big(\sum_{t=0}^{m} h^{2(t-1)} |\chi_1|^2_{H^t(\Gamma_j)} \Big)^{\frac{1}{2}}. \end{aligned} \tag{5.10}$$

Approximation of φ_1 and φ_2 on Γ. Using the polynomial approximations $\psi_1(x)$, $x \in \Gamma_1 \subset A_{ev}$ and $\psi_{2,j}(x)$, $x \in \Gamma_j \subset A_e$, constructed above, we define piecewise polynomial functions ϕ_1 and ϕ_2 as follows (below, Γ_j is an arbitrary element of the mesh Δ_h):

$$\phi_1|_{\Gamma_j} \;:=\; \begin{cases} \psi_1 + \frac{1}{|\Gamma_1|}\int_{\Gamma_1}(\varphi_1 - \psi_1)\,dx & \text{if } \Gamma_j = \Gamma_1 \subset A_{ev}, \\ 0 & \text{if } \Gamma_j \neq \Gamma_1, \end{cases}$$

$$\phi_2|_{\Gamma_j} \;:=\; \begin{cases} \psi_{2,j} + \frac{1}{|\Gamma_j|}\int_{\Gamma_j}(\varphi_2 - \psi_{2,j})\,dx & \text{if } \Gamma_j \subset A_e, \\ 0 & \text{if } \Gamma_j \subset (\Gamma\backslash A_e). \end{cases}$$

Then $\phi_i \in V^{h,p}(\Gamma)$, $i = 1, 2$, and for any element $\Gamma_j \in \Delta_h$ there holds

$$\int_{\Gamma_j} (\varphi_i - \phi_i)\,dx = 0, \quad i = 1, 2.$$

Therefore, applying Lemma 3.3 and Lemma 3.4 we estimate by (5.7)

$$\begin{aligned} \|\varphi_1 - \phi_1\|_{\tilde{H}^s(\Gamma)} &\le C\Big(\sum_j \|(\varphi_1 - \phi_1)|_{\Gamma_j}\|^2_{\tilde{H}^s_h(\Gamma_j)}\Big)^{1/2} = C\,\|(\varphi_1 - \phi_1)|_{\Gamma_1}\|_{\tilde{H}^s_h(\Gamma_1)} \\ &= C\left\|(\varphi_1 - \psi_1) - \frac{1}{|\Gamma_1|}\int_{\Gamma_1}(\varphi_1 - \psi_1)\,dx\right\|_{\tilde{H}^s_h(\Gamma_1)} \le C\,\|\varphi_1 - \psi_1\|_{\tilde{H}^s_h(\Gamma_1)} \\ &\le Ch^{\lambda - s}(p+1)^{-2(\min\{\lambda,\gamma-1/2\}-s)}\Big(1 + \log\tfrac{p+1}{h}\Big)^{\beta_1+\beta_2}(1 + \log(p+1))^{\sigma}, \quad (5.11) \end{aligned}$$

where $-1 \le s < \min\{0, \lambda, \gamma - \frac{1}{2}\}$ and σ is the same as in (5.7).

Analogously, using (5.10) we obtain for $-1 \le s < \min\{0, \gamma - \frac{1}{2}\}$ and for integer $m > 2\gamma + 3$

$$\begin{aligned} \|\varphi_2 - \phi_2\|^2_{\tilde{H}^s(\Gamma)} &\le C\sum_{j:\,\Gamma_j \subset A_e} \|(\varphi_2 - \phi_2)|_{\Gamma_j}\|^2_{\tilde{H}^s_h(\Gamma_j)} \le C\sum_{j:\,\Gamma_j \subset A_e} \|\varphi_2 - \psi_{2,j}\|^2_{\tilde{H}^s_h(\Gamma_j)} \\ &\le Ch^{2(\gamma - s)}(p+1)^{-4(\gamma-1/2-s)}\Big(1 + \log\tfrac{p+1}{h}\Big)^{2\beta_2}\sum_{t=0}^{m} h^{2(t-1)}|\chi_1|^2_{H^t(A_e)}. \quad (5.12) \end{aligned}$$

Hence, making use of estimates (5.9) for the semi-norms of χ_1, we find

$$\|\varphi_2 - \phi_2\|_{\tilde{H}^s(\Gamma)} \le Ch^{\min\{\lambda,\gamma-1/2\}-s}(p+1)^{-2(\gamma-1/2-s)}\Big(1 + \log\tfrac{p+1}{h}\Big)^{\beta_2}(\log(1/h))^{\beta_1+\sigma}, \quad (5.13)$$

where $-1 \le s < \min\{0, \gamma - \frac{1}{2}\}$ and σ is the same as in (5.7).

Approximation of φ_3. Now we approximate the smooth function φ_3 in (5.4). Note that $\varphi_3 \in C_0^\infty(\Gamma)$. Moreover, using the results of [6] (see the proof of Theorem 5.1 therein) we can estimate the norm of φ_3. In fact, making use of estimate (5.15) in [6] with λ and γ replaced by $\lambda - 1$ and $\gamma - 1$, respectively, we have for any integer $m \ge \min\{\lambda, \gamma - \frac{1}{2}\}$

$$\|\varphi_3\|_{H^m(\Gamma)} \le Ch^{\min\{\lambda,\gamma-1/2\}-m}(\log(1/h))^{\beta_1+\beta_2+\sigma+\nu},$$

where σ is the same as in (5.7), $\nu = \frac{1}{2}$ if $m = \min\{\lambda, \gamma - \frac{1}{2}\}$, and $\nu = 0$ if $m > \min\{\lambda, \gamma - \frac{1}{2}\}$.

Therefore, applying Theorem 4.1, we find $\phi_3 \in V^{h,p}(\Gamma)$ such that for $s \in [-1, 0]$

$$\|\varphi_3 - \phi_3\|_{\tilde{H}^s(\Gamma)} \leq C h^{\mu - s + \min\{\lambda, \gamma - 1/2\} - m} (p+1)^{s-m} (\log(1/h))^{\beta_1 + \beta_2 + \sigma + \nu}, \tag{5.14}$$

where $m \geq \min\{\lambda, \gamma - \frac{1}{2}\}$, $m \geq 0$, and $\mu = \min\{m, p+1\}$.

If $p > 2\min\{\lambda + \frac{1}{2}, \gamma\} - 1$, we select an integer m satisfying

$$2\min\{\lambda + \tfrac{1}{2}, \gamma\} < m \leq p + 1.$$

Then $\mu = m > \min\{\lambda, \gamma - \frac{1}{2}\}$ and $(p+1)^{s-m} \leq (p+1)^{-2(\min\{\lambda, \gamma - 1/2\} - s)}$ for any $s \in [-1, 0]$.

If $\min\{\lambda, \gamma - \frac{1}{2}\} - 1 < p \leq 2\min\{\lambda + \frac{1}{2}, \gamma\} - 1$ (i.e., p is bounded), we choose an integer m such that

$$\max\left\{0, \min\{\lambda, \gamma - \tfrac{1}{2}\}\right\} < m \leq p + 1,$$

and if $p = \min\{\lambda, \gamma - \frac{1}{2}\} - 1$, then we take $m = \min\{\lambda, \gamma - \frac{1}{2}\} = p + 1$. In both these cases $\mu = m \geq \min\{\lambda, \gamma - \frac{1}{2}\}$ and $(p+1)^{s-m} \leq C(\lambda, \gamma)\,(p+1)^{-2(\min\{\lambda, \gamma - 1/2\} - s)}$ for any $s \in [-1, 0]$.

Thus, for any $p \geq \min\{\lambda, \gamma - \frac{1}{2}\} - 1 = \min\{\lambda - 1, \gamma - \frac{3}{2}\}$, selecting m as indicated above we find by (5.14)

$$\|\varphi_3 - \phi_3\|_{\tilde{H}^s(\Gamma)} \leq C h^{\min\{\lambda, \gamma - 1/2\} - s} (p+1)^{-2(\min\{\lambda, \gamma - 1/2\} - s)} (\log(1/h))^{\beta_1 + \beta_2 + \sigma + \nu}, \tag{5.15}$$

where $s \in [-1, 0]$, σ is the same as in (5.7), $\nu = \frac{1}{2}$ if $p = \min\{\lambda - 1, \gamma - \frac{3}{2}\}$, and $\nu = 0$ if $p > \min\{\lambda - 1, \gamma - \frac{3}{2}\}$.

Approximation of $u = \varphi_1 + \varphi_2 + \varphi_3$. Let us define $u_{hp} := \phi_1 + \phi_2 + \phi_3 \in V^{h,p}(\Gamma)$. Then combining estimates (5.11), (5.13), and (5.15) we obtain (5.1).

It remains to consider the case $0 \leq p < \min\{\lambda - 1, \gamma - \frac{3}{2}\}$. In this case one does not need decomposition (5.4). Since $u \in H^m(\Gamma)$ with $1 \leq m < \min\{\lambda, \gamma - \frac{1}{2}\}$, we apply Theorem 4.1 to find $u_{hp} \in V^{h,p}(\Gamma)$ satisfying for $s \in [-1, 0]$

$$\|u - u_{hp}\|_{\tilde{H}^s(\Gamma)} \leq C h^{\min\{m, p+1\} - s} \|u\|_{H^m(\Gamma)}.$$

Hence, selecting $m \in [p + 1, \min\{\lambda, \gamma - \frac{1}{2}\})$ we obtain (5.2). □

5.2 Approximation of the singular functions u_2^{ev} and u^v

In this sub-section we study the approximation of the edge-vertex singularity u_2^{ev} and the vertex singularity u^v. The proofs of the two theorems below are analogous to the proof of Theorem 5.1, they use the same idea and similar arguments relying on the corresponding p-version results of [7] and some technical results from [6]. That is why we sketch both proofs omitting inessential details.

First, let us consider the edge-vertex singularity u_2^{ev} given by (2.8), (2.10).

Theorem 5.2 *Let* $u = u_2^{ev}$ *be given by* (2.8), (2.10). *Then there exists* $u_{hp} \in V^{h,p}(\Gamma)$ *with* $p \ge \gamma - \frac{3}{2}$ *such that for* $s \in [-1, \min\{0, \gamma - \frac{1}{2}\})$,

$$\|u - u_{hp}\|_{\tilde{H}^s(\Gamma)} \le C\, h^{\gamma-1/2-s}\,(p+1)^{-2(\gamma-1/2-s)}\left(1 + \log\tfrac{p+1}{h}\right)^{\beta+\nu}, \tag{5.16}$$

where $\gamma = \gamma_1^e > 0$, $\beta = s_1^e \ge 0$ *is integer,* $\nu = \frac{1}{2}$ *if* $p = \gamma - \frac{3}{2}$, *and* $\nu = 0$ *otherwise.*

If $0 \le p < \gamma - \frac{3}{2}$, *then there exists* $u_{hp} \in V^{h,p}(\Gamma)$ *satisfying for* $s \in [-1, 0]$

$$\|u - u_{hp}\|_{\tilde{H}^s(\Gamma)} \le C\, h^{p+1-s}. \tag{5.17}$$

Proof. Let

$$u(x_1, x_2) = x_2^{\gamma-1}|\log x_2|^{\beta}\chi_1(x_1, x_2)\chi_2^e(x_2), \tag{5.18}$$

where $\gamma = \gamma_1^e > 0$, $\beta \ge 0$ is integer, $\chi_2^e \in C^\infty(\mathbf{R}^+)$ is the same as in (2.5), $\chi_1 \in H^m(\Gamma)$ with m as large as required. We decompose u as

$$u = u\chi_2^e(x_2/h_0) + u(1 - \chi_2^e(x_2/h_0)) =: \varphi_2 + \varphi_3, \quad h_0 = (\sigma_1\sigma_2)^{-1}h. \tag{5.19}$$

The singular part φ_2 of decomposition (5.19) has the same form as in (5.8) with $\beta_2 = \beta$ and with an arbitrary function $\chi_1 \in H^m(\Gamma)$. Therefore, there exists $\phi_2 \in V^{h,p}(\Gamma)$ satisfying for $-1 \le s < \min\{0, \gamma - \frac{1}{2}\}$ and for any integer $k > 2\gamma + 3$ (cf. estimate (5.12))

$$\|\varphi_2 - \phi_2\|^2_{\tilde{H}^s(\Gamma)} \le Ch^{2(\gamma-s)}(p+1)^{-4(\gamma-1/2-s)}\left(1 + \log\tfrac{p+1}{h}\right)^{2\beta}\sum_{t=0}^{k} h^{2(t-1)}|\chi_1|^2_{H^t(A_e)}. \tag{5.20}$$

Since $\operatorname{meas}(A_e) \simeq h$ and $\chi_1 \in H^m(\Gamma)$ with sufficiently large m, we estimate

$$\sum_{t=0}^{k} h^{2(t-1)}\,|\chi_1|^2_{H^t(A_e)} \le Ch^{-2}\,\|\chi_1\|^2_{C^k(\bar{A}_e)}\operatorname{meas}(A_e) \le Ch^{-1}\,\|\chi_1\|^2_{H^m(\Gamma)}.$$

Then we obtain by (5.20)

$$\|\varphi_2 - \phi_2\|_{\tilde{H}^s(\Gamma)} \le Ch^{\gamma-1/2-s}(p+1)^{-2(\gamma-1/2-s)}\left(1 + \log\tfrac{p+1}{h}\right)^{\beta}, \quad s \in [-1, \min\{0, \gamma - \tfrac{1}{2}\}). \tag{5.21}$$

To approximate the smooth part $\varphi_3 \in H^m(\Gamma)$ of decomposition (5.19) we apply Theorem 4.1: there exists $\phi_3 \in V^{h,p}(\Gamma)$ such that for $s \in [-1, 0]$

$$\|\varphi_3 - \phi_3\|_{\tilde{H}^s(\Gamma)} \le Ch^{\mu-s}(p+1)^{s-k}\|\varphi_3\|_{H^k(\Gamma)}, \tag{5.22}$$

where $k \in [0, m]$ and $\mu = \min\{k, p+1\}$.

Recalling the definition of χ_2^e (see (2.5)), we find by simple calculations

$$\|\varphi_3\|^2_{H^k(\Gamma)} \le C(\log(1/h))^{2\beta}\int_{h_0\delta_e}^{2\delta_e} x_2^{2(\gamma-1-k)}dx_2.$$

Hence, for any integer k satisfying $\max\{0, \gamma - \frac{1}{2}\} \le k \le m$, we obtain by (5.22)

$$\|\varphi_3 - \phi_3\|_{\tilde{H}^s(\Gamma)} \le C h^{\gamma - 1/2 - k + \mu - s} (p+1)^{s-k} (\log(1/h))^{\beta+\nu}, \quad s \in [-1, 0], \tag{5.23}$$

where $\mu = \min\{k, p+1\}$, $\nu = \frac{1}{2}$ if $k = \gamma - \frac{1}{2}$, and $\nu = 0$ if $k > \gamma - \frac{1}{2}$.

If $p \ge \gamma - \frac{3}{2}$, then similarly as in the proof of Theorem 5.1 we select an integer k such that $\mu = k$ in (5.23) and $(p+1)^{s-k} \le C(\gamma)(p+1)^{-2(\gamma-1/2-s)}$ for any $s \in [-1, 0]$. Then combination of (5.21) and (5.23) gives (5.16) with $u_{hp} := \phi_2 + \phi_3 \in V^{h,p}(\Gamma)$.

If $0 \le p < \gamma - \frac{3}{2}$, then $u \in H^k(\Gamma)$ with $1 \le k < \gamma - \frac{1}{2}$. In this case we apply Theorem 4.1 directly to the function u: there exists $u_{hp} \in V^{h,p}(\Gamma)$ such that for $s \in [-1, 0]$

$$\|u - u_{hp}\|_{\tilde{H}^s(\Gamma)} \le C h^{\min\{k, p+1\} - s} \|u\|_{H^k(\Gamma)}.$$

Hence, selecting $k \in [p+1, \gamma - \frac{1}{2})$ we prove (5.17). □

Now, let v be a vertex of Γ and let A_v be the union of elements Γ_j with $v \in \bar{\Gamma}_j$.

Theorem 5.3 *Let $u = u^v$ be given by* (2.6). *Then there exists $u_{hp} \in V^{h,p}(\Gamma)$ with $p \ge \lambda - 1$ such that for $-1 \le s \le \min\{0, \lambda\}$,*

$$\|u - u_{hp}\|_{\tilde{H}^s(\Gamma)} \;\le\; C\, h^{\lambda - s}\, (p+1)^{-2(\lambda - s)} \left(1 + \log \tfrac{p+1}{h}\right)^{\beta+\nu}, \tag{5.24}$$

where $\lambda = \lambda_1^v > -\frac{1}{2}$, $\beta = q_1^v \ge 0$ is integer, $\nu = \frac{1}{2}$ if $p = \lambda - 1$, and $\nu = 0$ otherwise.

If $0 \le p < \lambda - 1$, then there exists $u_{hp} \in V^{h,p}(\Gamma)$ satisfying for $s \in [-1, 0]$

$$\|u - u_{hp}\|_{\tilde{H}^s(\Gamma)} \;\le\; C\, h^{p+1-s}. \tag{5.25}$$

Proof. Let

$$u = r^{\lambda-1} |\log r|^{\beta} \chi^v(r) w(\theta),$$

where $\lambda = \lambda_1^v > -\frac{1}{2}$, $\beta \ge 0$ is integer, χ^v is the same as in (2.6), $w \in H^m(0, \omega_v)$, ω_v denotes the interior angle on Γ at v, and m is as large as required.

We decompose u as $u = \varphi_1 + \varphi_2$, where

$$\varphi_1 := u \chi^v(r/h_0), \quad \varphi_2 := u(1 - \chi^v(r/h_0)), \quad h_0 = (\sigma_1\sigma_2)^{-1} h. \tag{5.26}$$

The singular function φ_1 has a small support, $\operatorname{supp}\varphi_1 \subset \bar{A}_v$. Let $K^h = \Gamma_j \subset A_v$ and let $K \subset \mathbf{R}^2$ be a triangle or parallelogram such that $K^h = M(K)$, where M is defined by (5.5). Then $O = (0, 0)$ is a vertex of K and for $h < \frac{1}{2}$ we have

$$\hat{\varphi}_1(\hat{x}) = \varphi_1(h\hat{x}_1, h\hat{x}_2) = h^{\lambda-1} \hat{r}^{\lambda-1} \sum_{k=0}^{\beta} \binom{\beta}{k} |\log h|^k |\log \hat{r}|^{\beta-k} \chi^v(\sigma_1\sigma_2\hat{r}) w(\hat{\theta}).$$

Applying Proposition 4.1 to each function $\hat{r}^{\lambda-1}|\log\hat{r}|^k\chi^v(\sigma_1\sigma_2\hat{r})w(\hat{\theta})$, $k=0,\ldots,\beta$, we find a polynomial $\hat{\psi}_1\in\mathcal{P}_p(K)$ such that for $-1\le s\le\min\{0,\lambda\}$

$$\|\hat{\varphi}_1-\hat{\psi}_1\|_{\tilde{H}^s(K)}\le C\,h^{\lambda-1}\,(p+1)^{-2(\lambda-s)}\left(1+\log\tfrac{p+1}{h}\right)^{\beta}.$$

Hence, setting $\psi_{1,j}:=\hat{\psi}_1\circ M^{-1}\in\mathcal{P}_p(\Gamma_j)$ and applying Lemma 3.2 we estimate

$$\|\varphi_1-\psi_{1,j}\|_{\tilde{H}^s_h(\Gamma_j)}\le C\,h^{\lambda-s}\,(p+1)^{-2(\lambda-s)}\left(1+\log\tfrac{p+1}{h}\right)^{\beta}. \tag{5.27}$$

Now we define a piecewise polynomial ϕ_1 (below, $\Gamma_j\in\Delta_h$ is an arbitrary element):

$$\phi_1|_{\Gamma_j}:=\begin{cases}\psi_{1,j}+\frac{1}{|\Gamma_j|}\int_{\Gamma_j}(\varphi_1-\psi_{1,j})\,dx & \text{if } \Gamma_j\subset A_v,\\ 0 & \text{if } \Gamma_j\subset(\Gamma\backslash A_v).\end{cases}$$

Then $\phi_1\in V^{h,p}(\Gamma)$ and $\int_{\Gamma_j}(\varphi_1-\phi_1)\,dx=0$ for any $\Gamma_j\in\Delta_h$. Therefore, recalling that the number ν_v of elements in A_v is bounded independently of h and making use of Lemmas 3.3, 3.4 we obtain by (5.27)

$$\|\varphi_1-\phi_1\|_{\tilde{H}^s(\Gamma)}\le C\,h^{\lambda-s}\,(p+1)^{-2(\lambda-s)}\left(1+\log\tfrac{p+1}{h}\right)^{\beta},\quad -1\le s<\min\{0,\lambda\}. \tag{5.28}$$

The smooth function $\varphi_2\in H^m(\Gamma)$ (see (5.26)) is approximated by using Theorem 4.1: there exists $\phi_2\in V^{h,p}(\Gamma)$ such that for $s\in[-1,0]$

$$\|\varphi_2-\phi_2\|_{\tilde{H}^s(\Gamma)}\le Ch^{\mu-s}(p+1)^{s-k}\|\varphi_2\|_{H^k(\Gamma)}, \tag{5.29}$$

where $k\in[0,m]$ and $\mu=\min\{k,p+1\}$. Furthermore, recalling the definition of χ^v (see (2.6)), we find by simple calculations (cf. estimate (6.10) in [6] with λ replaced by $\lambda-1$)

$$\|\varphi_2\|^2_{H^k(\Gamma)}\le C(\log(1/h))^{2\beta}\int\limits_{\tau_v h_0}^{2\tau_v} r^{2(\lambda-1-k)}\,rdr,\quad 0\le k\le m. \tag{5.30}$$

Thus, for any integer k satisfying $\max\{0,\lambda\}\le k\le m$, estimates (5.29) and (5.30) yield

$$\|\varphi_2-\phi_2\|_{\tilde{H}^s(\Gamma)}\le Ch^{\mu-s+\lambda-k}(p+1)^{s-k}(\log(1/h))^{\beta+\nu},\quad s\in[-1,0], \tag{5.31}$$

where $\nu=\frac{1}{2}$ if $k=\lambda$ and $\nu=0$ if $k>\lambda$.

If $p\ge\lambda-1$, then similarly as in the proof of Theorem 5.1 we select an integer k such that $\mu=k$ in (5.31) and $(p+1)^{s-k}\le C(\lambda)\,(p+1)^{-2(\lambda-s)}$ for any $s\in[-1,0]$. Then combination of (5.28) and (5.31) gives (5.24) with $u_{hp}:=\phi_1+\phi_2\in V^{h,p}(\Gamma)$.

The proof of estimate (5.25) is analogous to the proof of the corresponding results in Theorems 5.1 and 5.2. □

6 The general approximation result

Combining the approximation results for smooth and singular functions from Sections 4 and 5, we estimate the approximation error for the function u given by (2.4)–(2.10).

Theorem 6.1 *Let the function u be given by* (2.4)–(2.10) *on Γ with $\gamma_1^e > 0$ and $\lambda_1^v > -\frac{1}{2}$. Also, let $v_0 \in V$, $e_0 \in E(v_0)$ be such that $\min\{\lambda_1^{v_0} + 1/2, \gamma_1^{e_0}\} = \min_{v\in V, e\in E(v)} \min\{\lambda_1^v + 1/2, \gamma_1^e\}$, with λ_1^v and γ_1^e being as in* (2.5)–(2.8)*. Then, for any $h > 0$ and every $p \geq \min\{\lambda_1^{v_0} - 1, \gamma_1^{e_0} - 3/2\}$, there exists a function $u_{hp} \in V^{h,p}$ such that for $-1 \leq s < \min\{0, \lambda_1^{v_0}, \gamma_1^{e_0} - 1/2\}$*

$$\|u - u_{hp}\|_{\tilde{H}^s(\Gamma)} \leq C \max\Big\{h^{\min\{k,p+1\}-s}(p+1)^{s-k}, \\ h^{\min\{\lambda_1^{v_0}, \gamma_1^{e_0}-1/2\}-s}(p+1)^{-2(\min\{\lambda_1^{v_0}, \gamma_1^{e_0}-1/2\}-s)}\Big(1 + \log\frac{p+1}{h}\Big)^{\beta+\nu}\Big\}, \tag{6.1}$$

where β and ν are defined by (2.12) *and* (2.13)*, respectively.*

If $0 \leq p < \min\{\lambda_1^{v_0} - 1, \gamma_1^{e_0} - 3/2\}$, then for any $h > 0$ there exists $u_{hp} \in V^{h,p}$ such that for $s \in [-1, 0]$

$$\|u - u_{hp}\|_{\tilde{H}^s(\Gamma)} \leq C\, h^{\min\{k,p+1\}-s}. \tag{6.2}$$

Proof. To approximate the smooth part $u_{\text{reg}} \in H^k(\Gamma)$ of decomposition (2.4) we use Theorem 4.1, and applying Theorems 5.1, 5.2, and 5.3 we find piecewise polynomial approximations for the singularities u^{ev} and u^v on Γ. We also observe that the proof of Theorem 5.2 applies to the edge singularity terms given by (2.5). In fact, each component of u^e can be written in the more general form (5.18) and the statement of Theorem 5.2 remains valid for $u = u^e$. Thus combining the corresponding error estimates from the mentioned theorems we obtain (6.1) and (6.2). □

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