Approximation of Time-Dependent, Viscoelastic Fluid Flow: Crank–Nicolson, Finite Element Approximation*

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Abstract. In this article we analyze a fully discrete approximation to the time dependent viscoelasticity equations with an Oldroyd B constitutive equation in $\mathbb{R}^{\acute{d}}$, $\acute{d} = 2, 3$. We use a Crank–Nicolson discretization for the time derivatives. At each time level a linear system of equations is solved. To resolve the non-linearities we use a three step extrapolation for the prediction of the velocity and stress at the new time level. The approximation is stabilized by using a discontinuous Galerkin approximation for the constitutive equation. For the mesh parameter, h, and the temporal step size, Δt, sufficiently small and satisfying $\Delta t \leq C h^{\acute{d}/4}$, existence of the approximate solution is proven. A priori error estimates for the approximation in terms of Δt and h are also derived.

Key words. viscoelasticity, finite element method, fully discrete, discontinuous Galerkin

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1 Introduction

The accurate numerical simulations of time dependent viscoelastic flows are important in the ability to predict flow instabilities in non-Newtonian fluid mechanics. The underlying equations to be solved are the conservation of momentum and incompressibilty equations for fluid flow, coupled with a (hyperbolic) constitutive equation for the viscoelastic component of the stress. To avoid the introduction of spurious oscillations in the numerical approximation, some stabilization is needed in the discretization of the constitutive equation. This is commonly done via a discontinuous Galerkin (DG) approximation for the stress, or by using a Streamline Upwind Petrov Galerkin (SUPG) approximation for the constitutive equation.

In this paper we analyze a Crank-Nicolson, Finite Element Method (FEM) approximation scheme, and show that it is second order with respect to the time discretization (Δt). This work extends the results obtained in [3], and [7]. In [3], Baranger and Wardi studied an implicit Euler time discretization, with a DG approximation for the stress, and showed that in $\mathbb{R}^2$ the approximation of

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the velocity and the viscoelastic stress was first order in time, under the condition that $\Delta t \leq Ch^{3/2}$. In [7], Ervin and Miles analyzed an implicit Euler time discretization with a SUPG discretization of the constitutive equation and showed that, in $\mathbb{R}^{\acute{d}}$, the method was first order in time under the weaker condition of $\Delta t \leq Ch^{\acute{d}/2}$. No estimates for the approximation error for the pressure were given in [3],[7]. To obtain such an estimate one uses the discrete inf-sup condition together with the momentum equation. This requires a time differencing of the velocity approximation. For a first order temporal approximation for the velocity this would give an O(1) estimate for the error in the pressure. In this paper we are able to show that the approximation generates a first order temporal approximation for the pressure.

Heywood and Rannacher in [10] studied a Crank–Nicolson approximation for the non-stationary Navier-Stokes equations. The algorithm they analyzed required the solution of a non-linear system at each time level. The authors offered two suggestions to avoid having to solve a non-linear system while maintaining second order accuracy for the time discretization. These were: (i) linearize the non-linear system about the current approximation, and (ii) linearize the non-linear terms by using an extrapolation of the current and previous time level approximations (i.e. a two level scheme). This two level approach was implemented by Mu in [12] for the numerical simulation of the Ginzburg-Landau model of superconductivity.

In forming a Crank–Nicolson approximation for viscoelasticity our goal was to have the approximation determined at each time level by the solution of a single linear system. To do so we use the extrapolation approach. Linearizing the non-linear system would still have involved the complication of having the unknown velocity in the computation of the "edge jump contribution" arising from the DG discretization of the constitutive equation. We were not able to show second order accuracy in time using a two level discretization scheme. In the analysis the gradient of the velocity extrapolant is required to be bounded. We could not establish such a bound with a two level scheme. We therefore propose and analyze a three level scheme. However, the three level scheme analyzed can be considered a two level scheme for the *time averaged variables*. In deriving the error estimates we assume that the solution has the required regularity. For a discussion on the regularity issues associated with using the Crank-Nicolson discretization for the approximation of initial value problems we refer the reader to [10].

The paper is organized as follows. In section 2 we briefly describe the viscoelastic modeling equations. Herein we present the analysis for the Oldroyd B model, however the results can be readily extended to other differential models. Following the description of the model a variational formulation of the continuous problem is given. We then prove a perturbation result for the *distance* between the solution of the modeling equations and a *nearby* problem. The finite element approximation scheme is presented in section 3. The error analysis for the general scheme is then presented in section 4. Following in the appendix are several estimates used in the analysis of the general scheme, as well as an analysis of a suitable initialization procedure.

2 The Mathematical Model and the Approximating System

In this section we describe the modeling equations for viscoelastic fluid flow and the finite element approximation scheme.

2.1 The Mathematical Model

Consider a fluid flowing in a bounded, connected domain $\Omega \in \mathbb{R}^d$. The boundary of Ω, $\partial\Omega$, is assumed to be Lipschitz. The vector $\mathbf{n}$ represents the outward unit normal to $\partial\Omega$. The velocity vector is denoted by $\mathbf{u}$, pressure by p, total stress by $\mathbf{T}$, and extra stress by τ. The deformation tensor, $D(\mathbf{u})$, and the vorticity tensor, $W(\mathbf{u})$, are given by

$$D(\mathbf{u}) \;=\; \frac{1}{2}\left(\nabla\mathbf{u} \,+\, (\nabla\mathbf{u})^T\right)\,, \quad W(\mathbf{u}) \;=\; \frac{1}{2}\left(\nabla\mathbf{u} \,-\, (\nabla\mathbf{u})^T\right).$$

The Oldroyd model can be described using an *objective derivative* [2], denoted by $\hat{\partial}\sigma/\partial t$, where

$$\frac{\hat{\partial}\sigma}{\partial t} \;:=\; \frac{\partial\sigma}{\partial t} \,+\, \mathbf{u}\cdot\nabla\sigma \,+\, g_a(\sigma,\nabla\mathbf{u})\,, \qquad\qquad a \in [-1,1]$$

and

$$\begin{aligned} g_a(\sigma,\nabla\mathbf{u}) \;&:=\; \sigma W(\mathbf{u}) \,-\, W(\mathbf{u})\sigma \,-\, a(D(\mathbf{u})\sigma \,+\, \sigma D(\mathbf{u})) \\ &=\; \frac{1-a}{2}\left(\sigma\nabla\mathbf{u} \,+\, (\nabla\mathbf{u})^T\sigma\right) \,-\, \frac{1+a}{2}\left((\nabla\mathbf{u})\sigma \,+\, \sigma(\nabla\mathbf{u})^T\right). \end{aligned}$$

Oldroyd's model for stress employs a decomposition of the extra stress into two parts: a Newtonian part and a viscoelastic part. So $\tau \;=\; \tau_N + \tau_V$. The Newtonian part is given by $\tau_N \;=\; 2(1-\alpha)D(\mathbf{u})$. The $(1-\alpha)$ represents that part of the total viscosity which is considered Newtonian. Hence $\alpha \in (0,1)$ represents the proportion of the total viscosity that is considered to be viscoelastic in nature. For example, if a polymer is immersed within a Newtonian carrier fluid, α is related to the percentage of polymer in the mix. The constitutive law is [2]

$$\tau_V \,+\, \lambda\frac{\hat{\partial}\tau_V}{\partial t} \,-\, 2\alpha D(\mathbf{u}) \;=\; 0\,, \tag{2.1}$$

where λ is the Weissenberg number, which is a dimensionless constant defined as the product of the relaxation time and a characteristic strain rate [4]. For notational simplicity, the subscript, V, is dropped, and below τ will be used to denote the viscoelastic component of the extra stress.

The momentum balance for the fluid is given by

$$Re\left(\frac{d\mathbf{u}}{dt}\right) \;=\; -\nabla p \,+\, \nabla\cdot(2(1-\alpha)D(\mathbf{u}) \,+\, \tau) \,+\, \mathbf{f}\,, \tag{2.2}$$

where Re is the Reynolds number, $\mathbf{f}$ the body forces acting on the fluid, and $d\mathbf{u}/dt \;:=\; \partial\mathbf{u}/\partial t + \mathbf{u}\cdot\nabla\mathbf{u}$ denotes the material derivative.

In addition to (2.1) and (2.2) we also have the incompressibility condition:

$$\nabla\cdot\mathbf{u} \;=\; 0 \qquad \text{in } \Omega.$$

To fully specify the problem, appropriate boundary conditions must also be given. A condition for the velocity is required on each of the boundaries, and the stress specified on the inflow boundary. For simplicity, we consider homogeneous Dirichlet condition for velocity. In this case, there is no inflow boundary, and, thus, no boundary condition is required for stress. Summarizing, the modeling equations are:

$$
\begin{aligned}
Re\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + \nabla p - 2(1-\alpha)\nabla\cdot D(\mathbf{u}) - \nabla\cdot\tau &= \mathbf{f} && \text{in } \Omega, && (2.3)\\
\tau + \lambda\left(\frac{\partial \tau}{\partial t} + \mathbf{u}\cdot\nabla\tau + g_a(\tau,\nabla\mathbf{u})\right) - 2\alpha D(\mathbf{u}) &= 0 && \text{in } \Omega, && (2.4)\\
\nabla\cdot\mathbf{u} &= 0 && \text{in } \Omega, && (2.5)\\
\mathbf{u} &= 0 && \text{on } \partial\Omega, && (2.6)\\
\mathbf{u}(0,\mathbf{x}) &= \mathbf{u}_0(\mathbf{x}) && \text{in } \Omega, && (2.7)\\
\tau(0,\mathbf{x}) &= \tau_0(\mathbf{x}) && \text{in } \Omega. && (2.8)
\end{aligned}
$$

In [9], Guillope and Saut proved the following for the "slow-flow" model of (2.3)-(2.8) (i.e. $\mathbf{u}\cdot\nabla\mathbf{u}$ term in (2.3) is ignored):

1. local existence, in time, of a unique, regular solution, and
2. under a small data assumption on $\mathbf{f}, \mathbf{f}', \mathbf{u}_0, \tau_0$, the global existence (in time) of a unique solution for $\mathbf{u}$ and τ.

In contrast to the Navier–Stokes equations, well-posedness for general models in viscoelasticity is still not well understood. Results which are known fall into one of three types [14]:

1. for inital value problems, solutions have been shown to exist locally in time,
2. global existence (in time) of solutions if the initial conditions are small perturbations of the rest state, and
3. for steady-state problems, existence of solutions which are small perturbations of the analagous Newtonian case.

2.2 The Variational Formulation

In this section, we develop the variational formulation of (2.3)-(2.6). The following notation will be used. The $L^2(\Omega)$ norm and inner product will be denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. We use H^k to represent the Sobolev space W_2^k, and $\|\cdot\|_k$ denotes the norm in H^k. When $v(\mathbf{x},t)$ is defined on the entire time interval $(0,T)$, we define

$$
\|v\|_{\infty,k} := \sup_{0<t<T} \|v(\cdot,t)\|_k\ , \quad \|v\|_{0,k} := \left(\int_0^T \|v(\cdot,t)\|_k^2\, dt\right)^{1/2} .
$$

The following function spaces are used in the analysis:

$$
\begin{aligned}
\text{Velocity Space} \ &: \ X := H_0^1(\Omega) := \left\{\mathbf{u}\in H^1(\Omega) : \mathbf{u} = 0 \ \ on \ \ \partial\Omega\right\},\\
\text{Stress Space} \ &: \ S := \left\{\tau = (\tau_{ij}) : \tau_{ij} = \tau_{ji}; \tau_{ij}\in L^2(\Omega); 1 \le i,j \le \acute{d}\right\}
\end{aligned}
$$

$$\cap \Big\{ \tau = (\tau_{ij}) : \mathbf{u} \cdot \nabla \tau \in L^2(\Omega), \forall \mathbf{u} \in X \Big\},$$

$$\text{Pressure Space} \quad : \quad Q := L_0^2(\Omega) = \{ q \in L^2(\Omega) : \int_\Omega q \, dx = 0 \},$$

$$\text{Divergence} - \text{free Space} \quad : \quad Z := \{ v \in X : \int_\Omega q(\nabla \cdot v) \, dx = 0, \; \forall \, q \in Q \}.$$

The variational formulation of (2.3)-(2.6) proceeds in the usual manner. Taking the inner product of (2.3), (2.4), and (2.5) with a velocity test function, a stress test function, and a pressure test function respectively, we obtain

$$Re \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v} \right) - (p, \nabla \cdot \mathbf{v}) + (2(1-\alpha)D(\mathbf{u}) + \tau \; , \; D(\mathbf{v})) \;\; = \;\; (\mathbf{f}, \mathbf{v}), \qquad \forall \, \mathbf{v} \in X, \quad (2.9)$$

$$\left(\tau + \lambda \left(\frac{\partial \tau}{\partial t} + \mathbf{u} \cdot \nabla \tau + g_a(\tau, \nabla \mathbf{u}) \right) - 2\alpha D(\mathbf{u}) \; , \; \sigma \right) \;\; = \;\; 0, \qquad \forall \, \sigma \in S, \; (2.10)$$

$$(\nabla \cdot \mathbf{u}, q) \;\; = \;\; 0, \qquad \forall \, q \in Q. \; (2.11)$$

The space Z is the space of weakly divergence free functions. Note that the condition

$$(\nabla \cdot \mathbf{u}, q) = 0, \;\; \forall \, q \in Q, \;\; \mathbf{u} \in X,$$

is equivalent in a "distributional" sense to

$$(\mathbf{u}, \nabla q) = 0, \;\; \forall \, q \in Q, \;\; \mathbf{u} \in X, \qquad (2.12)$$

where in (2.12), $(\cdot, \cdot)$ denotes the duality pairing between H^{-1} and H_0^1 functions. In addition, note that the velocity and pressure spaces, X and Q, satisfy the *inf-sup* condition

$$\inf_{q \in Q} \sup_{\mathbf{v} \in X} \frac{(q, \nabla \cdot \mathbf{v})}{\|q\| \, \|\mathbf{v}\|_1} \geq \beta > 0. \qquad (2.13)$$

Since the inf-sup condition (2.13) holds, an equivalent variational formulation to (2.9)-(2.11) is: *Find* $\mathbf{u} \in Z, \tau \in S$ *satisfying*

$$Re \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v} \right) + (2(1-\alpha)D(\mathbf{u}) + \tau, D(\mathbf{v})) \;\; = \;\; (\mathbf{f}, \mathbf{v}), \;\; \forall \, \mathbf{v} \in Z, \qquad (2.14)$$

$$\left(\tau + \lambda \left(\frac{\partial \tau}{\partial t} + \mathbf{u} \cdot \nabla \tau + g_a(\tau, \nabla \mathbf{u}) \right) - 2\alpha D(\mathbf{u}), \sigma \right) \;\; = \;\; 0, \qquad \forall \, \sigma \in S. \qquad (2.15)$$

We assume that the fluid flow satisfies the following properties:

$$\|\mathbf{u}\|_\infty \, , \; \|\tau\|_\infty \, , \; \|\nabla \mathbf{u}\|_\infty \, , \; \|\nabla \tau\|_\infty \; \leq M, \quad \text{for all } t \in [0, T] \, . \qquad (2.16)$$

2.3 Perturbation Estimate

For the error analysis of the Crank-Nicolson time discretization of (2.14),(2.15), given below in (3.16)-(3.17), it is convenient to compare the approximation with the solution to a nearby problem. In this section we establish an error estimate between $(\mathbf{u}, \tau, p)$ satisfying (2.3)–(2.8) and $(\mathbf{w}, \eta, r)$ the solution of a nearby problem — assuming both solutions exist.

Let $(\mathbf{w}, \eta, r)$ denote the solution of

$$\begin{aligned}
Re\left(\frac{\partial \mathbf{w}}{\partial t} + \tilde{\mathbf{w}} \cdot \nabla \mathbf{w}\right) + \nabla r - 2(1-\alpha)\nabla \cdot D(\mathbf{w}) - \nabla \cdot \eta &= \mathbf{f} && \text{in } \Omega, && (2.17)\\
\eta + \lambda\left(\frac{\partial \eta}{\partial t} + \tilde{\mathbf{w}} \cdot \nabla \eta + g_a(\tilde{\eta}, \nabla \mathbf{w})\right) - 2\alpha D(\mathbf{w}) &= 0 && \text{in } \Omega, && (2.18)\\
\nabla \cdot \mathbf{w} &= 0 && \text{in } \Omega, && (2.19)\\
\mathbf{w} &= 0 && \text{on } \partial\Omega, && (2.20)\\
\mathbf{w}(0, \mathbf{x}) &= \mathbf{u}_0(\mathbf{x}) && \text{in } \Omega, && (2.21)\\
\eta(0, \mathbf{x}) &= \tau_0(\mathbf{x}) && \text{in } \Omega, && (2.22)
\end{aligned}$$

where $\tilde{\mathbf{w}} = \tilde{\mathbf{w}}(\mathbf{w}, \mathbf{x}, t) \in X$, and $\tilde{\eta} = \tilde{\eta}(\eta, \mathbf{x}, t) \in S$.

Analogous to (2.16) we assume that that

$$\|\mathbf{w}\|_\infty\,,\ \|\eta\|_\infty\,,\ \|\nabla \mathbf{w}\|_\infty\,,\ \|\nabla \eta\|_\infty \ \le M, \quad \text{for all } t \in [0, T]\,. \qquad (2.23)$$

Theorem 1 *Assume that there exist* $(\mathbf{u}, \tau, p) \in (X, S, Q)$ *satisfying (2.7)-(2.11), and* $(\mathbf{w}, \eta, r) \in (X, S, Q)$ *satisfying the analogous variational form of (2.17)-(2.22). Then, we have the following error estimate:*

$$\|\mathbf{u}-\mathbf{w}\|^2(T) + \|\tau - \eta\|^2(T) + \int_0^T \|D(\mathbf{u}-\mathbf{w})\|^2\, dt \ \le\ C \int_0^T \left(\|\mathbf{w} - \tilde{\mathbf{w}}\|^2 + \|\eta - \tilde{\eta}\|^2\right) dt\,. \quad (2.24)$$

Proof: Analogous to (2.14)-(2.15) we have that $\mathbf{w}, \eta$ satisfy

$$\begin{aligned}
Re\left(\frac{\partial \mathbf{w}}{\partial t} + \tilde{\mathbf{w}} \cdot \nabla \mathbf{w}, \mathbf{v}\right) + (2(1-\alpha)D(\mathbf{w}) + \eta\ ,\ D(\mathbf{v})) &= (\mathbf{f}, \mathbf{v}), \quad \forall\, \mathbf{v} \in Z, && (2.25)\\
\left(\eta + \lambda\left(\frac{\partial \eta}{\partial t} + \tilde{\mathbf{w}} \cdot \nabla \eta + g_a(\tilde{\eta}, \nabla \mathbf{w})\right) - 2\alpha D(\mathbf{w})\ ,\ \sigma\right) &= 0, \qquad \forall\, \sigma \in S. && (2.26)
\end{aligned}$$

Note that

$$\mathbf{u} \cdot \nabla \mathbf{u} - \tilde{\mathbf{w}} \cdot \nabla \mathbf{w} = (\mathbf{u} - \mathbf{w}) \cdot \nabla \mathbf{u} + \mathbf{w} \cdot \nabla(\mathbf{u} - \mathbf{w}) + (\mathbf{w} - \tilde{\mathbf{w}}) \cdot \nabla \mathbf{w}\,. \qquad (2.27)$$

Similarly,

$$\begin{aligned}
\mathbf{u} \cdot \nabla \tau - \tilde{\mathbf{w}} \cdot \nabla \eta &= (\mathbf{u} - \mathbf{w}) \cdot \nabla \tau + \mathbf{w} \cdot \nabla(\tau - \eta) + (\mathbf{w} - \tilde{\mathbf{w}}) \cdot \nabla \eta\,, && (2.28)\\
g_a(\tau, \nabla \mathbf{u}) - g_a(\tilde{\eta}, \nabla \mathbf{w}) &= g_a(\tau - \eta, \nabla \mathbf{u}) + g_a(\eta, \nabla(\mathbf{u} - \mathbf{w})) + g_a(\eta - \tilde{\eta}, \nabla \mathbf{w})\,. && (2.29)
\end{aligned}$$

Letting $\epsilon_u := \mathbf{u} - \mathbf{w}$, $\epsilon_\tau := \tau - \eta$, subtracting (2.25)-(2.26) from (2.14)-(2.15) and using (2.27)-(2.29) we have

$$\begin{aligned}
Re\left(\frac{\partial \epsilon_u}{\partial t}, \mathbf{v}\right) + Re\,(\epsilon_u \cdot \nabla \mathbf{u}, \mathbf{v}) + Re\,(\mathbf{w} \cdot \nabla \epsilon_u, \mathbf{v}) &+ (2(1-\alpha)D(\epsilon_u), D(\mathbf{v})) + (\epsilon_\tau, D(\mathbf{v}))\\
&= (-(\mathbf{w} - \tilde{\mathbf{w}}) \cdot \nabla \mathbf{w}, \mathbf{v}), \quad \forall\, \mathbf{v} \in Z, \quad (2.30)\\
(\epsilon_\tau, \sigma) + \lambda\left(\frac{\partial \epsilon_\tau}{\partial t}, \sigma\right) + \lambda\,(\epsilon_u \cdot \nabla \tau, \sigma) + \lambda\,(\mathbf{w} \cdot \nabla \epsilon_\tau, \sigma) &+ \lambda\,(g_a(\epsilon_\tau, \nabla \mathbf{u}), \sigma) + \lambda\,(g_a(\eta, \nabla \epsilon_u), \sigma)\\
-\,(2\alpha D(\epsilon_u), \sigma) = -((\mathbf{w} - \tilde{\mathbf{w}}) \cdot \nabla \eta, \sigma) &- \lambda\,(g_a(\eta - \tilde{\eta}, \nabla \mathbf{w}), \sigma)\,, \quad \forall\, \sigma \in S. \quad (2.31)
\end{aligned}$$

Multiplying (2.30) by 2α and adding to (2.31) we obtain for the choice $\mathbf{v} = \epsilon_u$, $\sigma = \epsilon_\tau$

$$\begin{aligned}
\alpha\, Re\, \|\epsilon_u\|_t^2 \;&+\; 2\alpha\, Re\, (\epsilon_u \cdot \nabla \mathbf{u}, \epsilon_u) + 2\alpha\, Re\, (\mathbf{w} \cdot \nabla \epsilon_u, \epsilon_u) \;+\; 4\alpha\,(1-\alpha)\,\|D(\epsilon_u)\|^2 \;+\; \|\epsilon_\tau\|^2 \\
+\frac{1}{2}\lambda\, \|\epsilon_\tau\|_t^2 \;&+\; \lambda\,(\epsilon_u \cdot \nabla \tau, \epsilon_\tau) \;+\; \lambda\,(\mathbf{w} \cdot \nabla \epsilon_\tau, \epsilon_\tau) \;+\; \lambda\,(g_a(\epsilon_\tau, \nabla \mathbf{u}), \epsilon_\tau) \;+\; \lambda\,(g_a(\eta, \nabla \epsilon_u), \epsilon_\tau) \\
&=\; -2\alpha((\mathbf{w} - \tilde{\mathbf{w}}) \cdot \nabla \mathbf{w}, \epsilon_u) \;-\; ((\mathbf{w} - \tilde{\mathbf{w}}) \cdot \nabla \eta, \epsilon_\tau) \;-\; \lambda\,(g_a(\eta - \tilde{\eta}, \nabla \mathbf{w}), \epsilon_\tau\;)\;. \qquad (2.32)
\end{aligned}$$

Note that, using (2.19), we have

$$(\mathbf{w} \cdot \nabla \epsilon_u, \epsilon_u) \;=\; -\,(\nabla \cdot \mathbf{w}\, \epsilon_u, \epsilon_u) \;-\; (\mathbf{w} \cdot \nabla \epsilon_u, \epsilon_u) \;=\; -\,(\mathbf{w} \cdot \nabla \epsilon_u, \epsilon_u)\;.$$

Thus,

$$(\mathbf{w} \cdot \nabla \epsilon_u, \epsilon_u) \;=\; 0\,, \qquad (2.33)$$

and similarly,

$$(\mathbf{w} \cdot \nabla \epsilon_\tau, \epsilon_\tau) \;=\; 0\,. \qquad (2.34)$$

Using (2.33),(2.34), equation (2.32) may be rewritten as

$$\begin{aligned}
\alpha\, Re\, \|\epsilon_u\|_t^2 \;&+\; \frac{1}{2}\lambda\, \|\epsilon_\tau\|_t^2 \;+\; 4\alpha\,(1-\alpha)\,\|D(\epsilon_u)\|^2 \;+\; \|\epsilon_\tau\|^2 \\
&=\; -2\alpha\, Re\, (\epsilon_u \cdot \nabla \mathbf{u}, \epsilon_u) \;-\; \lambda\,(\epsilon_u \cdot \nabla \tau, \epsilon_\tau) \;-\; \lambda\,(g_a(\epsilon_\tau, \nabla \mathbf{u}), \epsilon_\tau) \;-\; \lambda\,(g_a(\eta, \nabla \epsilon_u), \epsilon_\tau) \\
&\quad -2\alpha((\mathbf{w} - \tilde{\mathbf{w}}) \cdot \nabla \mathbf{w}, \epsilon_u) \;-\; ((\mathbf{w} - \tilde{\mathbf{w}}) \cdot \nabla \eta, \epsilon_\tau) \;-\; \lambda\,(g_a(\eta - \tilde{\eta}, \nabla \mathbf{w}), \epsilon_\tau)\;. \qquad (2.35)
\end{aligned}$$

We now bound each of the terms on the right hand side of (2.35).

$$-2\alpha\, Re\, (\epsilon_u \cdot \nabla \mathbf{u}, \epsilon_u) \;\le\; 2\alpha\, Re\, \|\epsilon_u \cdot \nabla \mathbf{u}\|\, \|\epsilon_u\| \;\le\; 2\alpha\, \acute{d}\, Re\, \|\nabla \mathbf{u}\|_\infty\, \|\epsilon_u\|^2\;. \qquad (2.36)$$

Similarly,

$$\begin{aligned}
-\lambda\,(\epsilon_u \cdot \nabla \tau, \epsilon_\tau) \;&\le\; \lambda\, \|\epsilon_u \cdot \nabla \tau\|\, \|\epsilon_\tau\| \;\le\; \lambda\, \acute{d}\, \|\nabla \tau\|_\infty\, \|\epsilon_u\|\, \|\epsilon_\tau\| \\
&\le\; \frac{\lambda\, \acute{d}}{2}\, \|\nabla \tau\|_\infty\, \|\epsilon_u\|^2 \;+\; \frac{\lambda\, \acute{d}}{2}\, \|\nabla \tau\|_\infty\, \|\epsilon_\tau\|^2\;, \qquad (2.37)
\end{aligned}$$

$$-\lambda\,(g_a(\epsilon_\tau, \nabla \mathbf{u}), \epsilon_\tau) \;\le\; \lambda\, \|g_a(\epsilon_\tau, \nabla \mathbf{u})\|\, \|\epsilon_\tau\| \;\le\; 4\lambda\, \acute{d}\, \|\nabla \mathbf{u}\|_\infty\, \|\epsilon_\tau\|^2\;, \qquad (2.38)$$

$$-\lambda\,(g_a(\eta, \nabla \epsilon_u), \epsilon_\tau) \;\le\; 4\lambda\, \acute{d}\, \|\eta\|_\infty\, \|\nabla \epsilon_u\|\, \|\epsilon_\tau\| \;\le\; \epsilon_1\, \|D(\epsilon_u)\|^2 \;+\; \frac{4\lambda^2\, \acute{d}^2\, C_K^2\, \|\eta\|_\infty^2}{\epsilon_1}\, \|\epsilon_\tau\|^2\;, \qquad (2.39)$$

(using Korn's lemma)

$$-2\alpha((\mathbf{w} - \tilde{\mathbf{w}}) \cdot \nabla \mathbf{w}, \epsilon_u) \;\le\; 2\alpha\, \acute{d}\, \|\nabla \mathbf{w}\|_\infty\, \|\mathbf{w} - \tilde{\mathbf{w}}\|\, \|\epsilon_u\| \;\le\; \|\mathbf{w} - \tilde{\mathbf{w}}\|^2 \;+\; \alpha^2\, \acute{d}^2\, \|\nabla \mathbf{w}\|_\infty^2\, \|\epsilon_u\|^2\;, \qquad (2.40)$$

$$-((\mathbf{w}-\tilde{\mathbf{w}})\cdot\nabla\eta,\epsilon_\tau) \le \|\mathbf{w}-\tilde{\mathbf{w}}\|^2 \,+\, \frac{\acute{d}^2}{4}\,\|\nabla\eta\|_\infty^2\,\|\epsilon_\tau\|^2\,, \tag{2.41}$$

$$-\lambda\,(g_a(\eta-\tilde{\eta},\nabla\mathbf{w}),\epsilon_\tau) \le \lambda\,4\acute{d}\,\|\eta-\tilde{\eta}\|\,\|\nabla\mathbf{w}\|_\infty\,\|\epsilon_\tau\| \le \|\eta-\tilde{\eta}\|^2 \,+\, \lambda^2\,4\acute{d}^2\,\|\nabla\mathbf{w}\|_\infty^2\,\|\epsilon_\tau\|^2\,. \tag{2.42}$$

Substituting (2.36)-(2.42) into (2.35) we obtain

$$\begin{aligned}
\alpha\,Re\,\|\epsilon_u\|_t^2 \;+\; & \frac{1}{2}\lambda\,\|\epsilon_\tau\|_t^2 \,+\, (4\alpha\,(1-\alpha)-\epsilon_1)\,\|D(\epsilon_u)\|^2 \,+\, \|\epsilon_\tau\|^2 \\
\le\; & \|\epsilon_u\|^2\left(2\alpha\,\acute{d}\,Re\,\|\nabla\mathbf{u}\|_\infty \,+\, \lambda\frac{\acute{d}}{2}\,\|\nabla\tau\|_\infty \,+\, \alpha^2\,\acute{d}^2\,\|\nabla\mathbf{w}\|_\infty^2\right) \\
& + \|\epsilon_\tau\|^2\left(4\lambda\,\acute{d}\,\|\nabla\mathbf{u}\|_\infty \,+\, \lambda\frac{\acute{d}}{2}\,\|\nabla\tau\|_\infty \,+\, 4\lambda^2\,\acute{d}^2\,\|\nabla\mathbf{w}\|_\infty^2\right. \\
& \qquad\qquad \left. + \frac{4}{\epsilon_1}\lambda^2\,C_K^2\,\acute{d}^2\,\|\eta\|_\infty^2 \,+\, \frac{\acute{d}^2}{4}\,\|\nabla\eta\|_\infty^2\right) \\
& + 2\,\|\mathbf{w}-\tilde{\mathbf{w}}\|^2 \,+\, \|\eta-\tilde{\eta}\|^2\,.
\end{aligned} \tag{2.43}$$

Applying Gronwall's lemma, we obtain (2.24).

■

Of particular interest in what follows is the case corresponding to $\tilde{\mathbf{w}}(\mathbf{x},t)$, $\tilde{\eta}(\mathbf{x},t)$ given by

$$\tilde{\mathbf{w}}(\cdot,t) \;:=\; \mathbf{w}(\cdot,t-\frac{\Delta t}{2}) \,+\, \frac{1}{2}\mathbf{w}(\cdot,t-\frac{3\Delta t}{2}) \,-\, \frac{1}{2}\mathbf{w}(\cdot,t-\frac{5\Delta t}{2})\,, \tag{2.44}$$

$$\text{and}\;\; \tilde{\eta}(\cdot,t) \;:=\; \eta(\cdot,t-\frac{\Delta t}{2}) \,+\, \frac{1}{2}\eta(\cdot,t-\frac{3\Delta t}{2}) \,-\, \frac{1}{2}\eta(\cdot,t-\frac{5\Delta t}{2})\,. \tag{2.45}$$

Corollary 1 *For $\tilde{\mathbf{w}}$ and $\tilde{\eta}$ defined in (2.44),(2.45) we have that $(\mathbf{u},\tau)$ and $(\mathbf{w},\eta)$ given respectively by (2.14),(2.15), and (2.25),(2.26) satisfy*

$$\|\mathbf{u}-\mathbf{w}\|^2(T) \;+\; \|\tau-\eta\|^2(T) \;+\int_0^T \|D(\mathbf{u}-\mathbf{w})\|^2\,dt \;\le\; C\,(\Delta t)^4\int_{-5\Delta t/2}^{T}\left(\|\mathbf{w}_{tt}\|^2 \,+\, \|\eta_{tt}\|^2\right)dt\,. \tag{2.46}$$

Proof: In view of (2.24), from (A.13) (in the appendix) we have

$$\begin{aligned}
\int_0^T \left(\|\mathbf{w}-\tilde{\mathbf{w}}\|^2 \,+\, \|\eta-\tilde{\eta}\|^2\right)dt \;&\le\; \frac{39}{8}(\Delta t)^3\int_0^T\left(\int_{t-5\Delta t/2}^{t}\left(\|\mathbf{w}_{tt}\|^2 \,+\, \|\eta_{tt}\|^2\right)d\mu\right)dt \\
&\le\; \frac{39}{8}(\Delta t)^3\,\frac{5}{2}\Delta t\int_{-5\Delta t/2}^{T}\left(\|\mathbf{w}_{tt}\|^2 \,+\, \|\eta_{tt}\|^2\right)dt\,.
\end{aligned}$$

■

3 Finite Element Approximation

In this section we formulate a fully discrete finite element method for solving the viscoelastic fluid flow equations, and prove the solvability of the approximation at each step (for sufficiently small $\Delta t,\ h$). To avoid having a non-linear algebraic system for the Crank-Nicolson discretization, the approximation is a *three–level* scheme, involving computed approximations at the three previous time levels.

We begin by describing the finite element approximation framework and listing the approximating properties and inverse estimates used in the analysis. We assume throughout that the viscoelastic stress tensors, τ, η, are continuous. This assumption is consistent with that used in [3] of $\tau \in H^2(\Omega)$ for $\Omega \subset \mathbb{R}^2$.

Let $\Omega \subset \mathbb{R}^{\acute{d}} (\acute{d} = 2, 3)$ be a polygonal domain and let T_h be a triangulation of Ω made of triangles (in $\mathbb{R}^2$) or tetrahedrons (in $\mathbb{R}^3$). Thus, the computational domain is defined by

$$\Omega = \bigcup K; \quad K \in T_h.$$

We assume that there exist constants c_1, c_2 such that

$$c_1 h \leq h_K \leq c_2 \rho_K \,,$$

where h_K is the diameter of triangle (tetrahedral) K, ρ_K is the diameter of the greatest ball (sphere) included in K, and $h = \max_{K \in T_h} h_K$. Let $P_k(A)$ denote the space of polynomials on A of degree no greater than k. Then we define the finite element spaces as follows.

$$\begin{aligned}
X_h &:= \left\{ \mathbf{v} \in X \cap C(\bar{\Omega})^2 : \mathbf{v}|_K \in P_k(K),\ \forall K \in T_h \right\}, \\
S_h &:= \left\{ \sigma \in S : \sigma|_K \in P_m(K),\ \forall K \in T_h \right\}, \\
Q_h &:= \left\{ q \in Q \cap C(\bar{\Omega}) : q|_K \in P_q(K),\ \forall K \in T_h \right\}, \\
Z_h &:= \left\{ \mathbf{v} \in X_h : (q, \nabla\cdot\mathbf{v}) = 0,\ \forall q \in Q_h \right\}.
\end{aligned}$$

Analogous to the continuous spaces, we assume that X_h and Q_h satisfy the discrete *inf-sup* condition

$$\inf_{q \in Q_h} \sup_{\mathbf{v} \in X_h} \frac{(q, \nabla \cdot \mathbf{v})}{\|q\| \, \|\mathbf{v}\|_1} \geq \beta > 0\,. \tag{3.1}$$

We summarize several properties of finite element spaces and Sobolev's spaces which we will use in our subsequent analysis. For $(\mathbf{w}, r) \in H^{k+1}(\Omega)^{\acute{d}} \times H^{q+1}(\Omega)$ we have (see [8]) that there exists $(\mathcal{U}, \mathcal{P}) \in Z_h \times Q_h$ such that

$$\|\mathbf{w} - \mathcal{U}\| \ + \ h\|\nabla(\mathbf{w} - \mathcal{U})\| \ \leq \ C_I h^{k+1} \|\mathbf{w}\|_{k+1} \,, \tag{3.2}$$

$$\|r - \mathcal{P}\| \ \leq \ C_I h^{q+1} \|r\|_{q+1} \,. \tag{3.3}$$

Let $\mathcal{T} \in S_h$ be a P_m continuous interpolant of η. For $\eta \in H^{m+1}(\Omega)^{\acute{d}\times\acute{d}}$ we have that

$$\|\eta - \mathcal{T}\| \ + \ h\|\nabla(\eta - \mathcal{T})\| \ \leq \ C_I h^{m+1} \|\eta\|_{m+1} \,. \tag{3.4}$$

Note that for $\mathbf{w}$, η given by (2.25),(2.26) and $\mathcal{U}$, $\mathcal{T}$ by (3.2),(3.4), it follows from (2.23) and inverse estimates, [5], that

$$\|\mathcal{U}^n\|_\infty , \|\nabla\mathcal{U}^n\|_\infty , \|\mathcal{T}^n\|_\infty , \|\nabla\mathcal{T}^n\|_\infty \ \le\ \tilde{M} \approx M \ . \tag{3.5}$$

Below, for simplicity, we take $\tilde{M} = M$.

Let Δt denote the step size for t so that $t_n = n\Delta t$, $n = 0, 1, 2, \ldots, N$. For notational convenience, we denote $v^n := v(\cdot, t_n)$. Also, let

$$d_t f \ := \ \frac{f(t_n) - f(t_{n-1})}{\Delta t} \tag{3.6}$$

$$\bar{f}^n \ := \ \frac{f^n \,+\, f^{n-1}}{2} \tag{3.7}$$

$$\tilde{f}^n \ := \ f^{n-1} \,+\, \frac{1}{2} f^{n-2} \,-\, \frac{1}{2} f^{n-3} \ . \tag{3.8}$$

The following norms are also used in the analysis:

$$\|\!|v|\!\|_{\infty,k} \ := \ \max_{1\le n\le N} \|v^n\|_k$$

$$\|\!|v|\!\|_{0,k} \ := \ \left[\sum_{n=1}^{N} \|v^n\|_k^2 \, \Delta t \right]^{\frac{1}{2}} \ .$$

In order to describe the approximation of the constitutive equation by the method of discontinuous finite elements, following [2], we introduce $\partial K^-(\mathbf{u}) := \{x \in \partial K, \mathbf{u}\cdot\mathbf{n} < 0\}$, where ∂K is the boundary of K and $\mathbf{n}$ is the outward unit normal and $\tau^\pm(\mathbf{u})(x) := \lim_{\epsilon\to 0^\pm} \tau(x + \epsilon\mathbf{u})$.

We define

$$(\tau, \sigma)_h \ := \sum_{K\in T_h} (\tau, \sigma)_K \ , \quad \langle \tau^\pm, \sigma^\pm \rangle_{h,\mathbf{u}} \ := \sum_{K\in T_h} \int_{\partial K^-(\mathbf{u})} (\tau^\pm(\mathbf{u}) : \sigma^\pm(\mathbf{u})) |\mathbf{u}\cdot\mathbf{n}| \, ds \ , \tag{3.9}$$

and

$$\langle\langle \tau^\pm \rangle\rangle_{h,\mathbf{u}} \ := \ \langle \tau^\pm, \tau^\pm \rangle_{h,\mathbf{u}}^{1/2} \ . \tag{3.10}$$

The operator B on $X_h \times S_h \times S_h$ is defined by

$$B(\mathbf{u}, \tau, \sigma) := (\mathbf{u}\cdot\nabla\tau, \sigma)_h \,+\, \frac{1}{2}(\nabla\cdot\mathbf{u}\,\tau, \sigma) \,+\, \langle \tau^+ - \tau^-, \sigma^+ \rangle_{h,\mathbf{u}} \ . \tag{3.11}$$

We have on applying Green's Theorem to (3.11) that

$$B(\mathbf{u}, \tau, \sigma) := -(\mathbf{u}\cdot\nabla\sigma, \tau)_h \,-\, \frac{1}{2}(\nabla\cdot\mathbf{u}\,\sigma, \tau) \,+\, \langle \tau^-, \sigma^- - \sigma^+ \rangle_{h,\mathbf{u}} \ , \tag{3.12}$$

which on combining with (3.11) yields some "coercivity" for B

$$B(\mathbf{u}, \tau, \tau) \ = \ \frac{1}{2} \langle\langle \tau^+ - \tau^- \rangle\rangle_{h,\mathbf{u}}^2 \ . \tag{3.13}$$

Also used in the analysis, for notation convenience, is the operator c, defined on $X_h \times X_h \times X_h$, by

$$c(\mathbf{w}, \mathbf{u}, \mathbf{v}) := (\mathbf{w}\cdot\nabla\mathbf{u}, \mathbf{v}) \,, \tag{3.14}$$

and $\hat{\lambda} := 2\alpha/\lambda$.

As we are assuming "slow flow", i.e. $Re \equiv O(1)$, we use a conforming finite element method to discretize the momentum equation.

Initialization of the Approximation Scheme

The approximation scheme described, and analyzed below, is a three level scheme. To initialize the procedure suitable approximates are required for $\mathbf{u}_h^n$, and τ_h^n for $n = 0, 1, 2$. Here we state our assumptions on these initial approximates. (An initialization procedure is presented in the appendix.)

$$\begin{aligned}\|\mathbf{u}_h^n - \mathbf{w}(n\Delta t)\|^2 &+ \|\tau_h^n - \eta(n\Delta t)\|^2 + \Delta t \, \|D(\mathbf{u}_h^n - \mathbf{w}(n\Delta t))\|^2 \\ &\leq C\,(\Delta t)^4 + C\left(h^{2k} + h^{2m} + h^{2q+2}\right) \\ &= \frac{1}{3} G_I(\Delta t, h)\,, \text{ for } n = 0, 1, 2\,. \end{aligned} \tag{3.15}$$

Approximating System

For $n = 3, 4, \ldots, N$, find $\mathbf{u}_h^n \in Z_h, \tau_h^n \in S_h$ such that

$$Re(d_t\mathbf{u}_h^n, \mathbf{v}) + Re\, c(\tilde{\mathbf{u}}_h^n, \bar{\mathbf{u}}_h^n, \mathbf{v}) + 2(1-\alpha)(D(\bar{\mathbf{u}}_h^n), D(\mathbf{v})) + (\bar{\tau}_h^n, D(\mathbf{v})) = (\bar{\mathbf{f}}^n, \mathbf{v})\,, \ \mathbf{v} \in Z_h \tag{3.16}$$

$$\frac{1}{\lambda}(\bar{\tau}_h^n, \sigma) + (d_t\tau_h^n, \sigma) + B(\tilde{\mathbf{u}}_h^n, \bar{\tau}_h^n, \sigma) - \hat{\lambda}(D(\bar{\mathbf{u}}_h^n), \sigma) + (g_a(\tilde{\tau}_h^n, \nabla\bar{\mathbf{u}}_h^n), \sigma) = 0\,, \ \sigma \in S_h\,. \tag{3.17}$$

To ensure computability of the algorithm, we begin by showing that (3.16)-(3.17) is uniquely solvable for $\mathbf{u}_h$ and τ_h at each time step n. We use the following induction hypothesis.

$$(IH1) \qquad \left\|\mathbf{u}_h^{n-1}\right\|_\infty, \left\|\tau_h^{n-1}\right\|_\infty \leq K.$$

Lemma 1 *Assume (IH1) is true. For a sufficiently small step size Δt, there exists a unique solution $(\mathbf{u}_h^n, \tau_h^n) \in Z_h \times S_h$ satisfying (3.16)-(3.17).*

Proof: For notational simplicity, in this proof we drop the subscript h from the variables. Choosing $\mathbf{v} = \mathbf{u}_h^n, \sigma = \tau_h^n$, multiplying (3.16) by $\hat{\lambda}$ and adding to (3.17) we obtain

$$\begin{aligned} a(\mathbf{u}^n, \tau^n; \mathbf{u}^n, \tau^n) &= \hat{\lambda}(\bar{\mathbf{f}}^n, \mathbf{u}^n) + \hat{\lambda}\frac{Re}{\Delta t}(\mathbf{u}^{n-1}, \mathbf{u}^n) - \hat{\lambda}\frac{Re}{2}\, c(\tilde{\mathbf{u}}^n, \mathbf{u}^{n-1}, \mathbf{u}^n) - \hat{\lambda}(1-\alpha)(D(\mathbf{u}^{n-1}), D(\mathbf{u}^n)) \\ &- \frac{1}{2}\hat{\lambda}(\tau^{n-1}, D(\mathbf{u}^n)) - \frac{1}{2\lambda}(\tau^{n-1}, \tau^n) + \frac{1}{\Delta t}(\tau^{n-1}, \tau^n) - \frac{1}{2}B(\tilde{\mathbf{u}}^n, \tau^{n-1}, \tau^n) \\ &+ \frac{1}{2}\hat{\lambda}(D(\mathbf{u}^{n-1}), \tau^n) - \frac{1}{2}(g_a(\tilde{\tau}^n, \nabla\mathbf{u}^{n-1}), \tau^n) \end{aligned} \tag{3.18}$$

where the bilinear form $a(\mathbf{u}, \tau; \mathbf{v}, \sigma)$ is defined as:

$$\begin{aligned} a(\mathbf{u}, \tau; \mathbf{v}, \sigma) &:= \hat{\lambda}\frac{Re}{\Delta t}(\mathbf{u}, \mathbf{v}) + \hat{\lambda}\frac{Re}{2}\, c(\tilde{\mathbf{u}}^n, \mathbf{u}, \mathbf{v}) + \hat{\lambda}(1-\alpha)(D(\mathbf{u}), D(\mathbf{v})) + \frac{1}{2\lambda}(\tau, \sigma) \\ &+ \frac{1}{\Delta t}(\tau, \sigma) + \frac{1}{2}B(\tilde{\mathbf{u}}^n, \tau, \sigma) + \frac{1}{2}(g_a(\tilde{\tau}^n, \nabla\mathbf{u}), \sigma)\,. \end{aligned} \tag{3.19}$$

We now estimate the terms in $a(\mathbf{u}^n, \tau^n; \mathbf{u}^n, \tau^n)$. We have

$$
\begin{aligned}
|c(\tilde{\mathbf{u}}^n, \mathbf{u}, \mathbf{u})| = |(\tilde{\mathbf{u}}^n \cdot \nabla \mathbf{u}, \mathbf{u})| &\leq \acute{d}^{1/2} \, \|\tilde{\mathbf{u}}^n\|_\infty \, \|\nabla \mathbf{u}\| \, \|\mathbf{u}\| \\
&\leq \acute{d}^{1/2} \, \|\tilde{\mathbf{u}}^n\|_\infty \, C_k \, \|D(\mathbf{u})\| \, \|\mathbf{u}\| \ , \ \text{(using Korn's lemma)} \\
&\leq \epsilon_1 \, \|D(\mathbf{u})\|^2 \, + \, \frac{\acute{d} K^2 C_k^2}{4\epsilon_1} \, \|\mathbf{u}\|^2 \ , \\
B(\tilde{\mathbf{u}}^n, \tau, \tau) &= \frac{1}{2} \langle\langle \tau^+ - \tau^- \rangle\rangle_{h, \tilde{\mathbf{u}}^n}^2 , \\
|(g_a(\tilde{\tau}^n, \nabla \mathbf{u}), \tau^n)| &\leq 4 \, \|\tilde{\tau}^n \nabla \mathbf{u}\| \, \|\tau^n\| \\
&\leq 4 \acute{d}^{1/2} \, \|\tilde{\tau}^n\|_\infty \, \|\nabla \mathbf{u}\| \, \|\tau^n\| \\
&\leq 4 \acute{d}^{1/2} C_k 2K \, \|\tau^n\| \, \|D(\mathbf{u})\| \\
&\leq \epsilon_2 \, \|D(\mathbf{u})\|^2 \, + \, \frac{16 \acute{d} C_k^2 K^2}{\epsilon_2} \, \|\tau^n\|^2 \ .
\end{aligned}
$$

Applying these inequalities to the bilinear form $a(\cdot, \cdot; \cdot, \cdot)$ yields

$$
\begin{aligned}
a(\mathbf{u}^n, \tau^n; \mathbf{u}^n, \tau^n) \ \geq \ & \hat{\lambda} \, Re \left(\frac{1}{\Delta t} - \frac{\acute{d} K^2 C_k^2}{8\epsilon_1} \right) \|\mathbf{u}^n\|^2 \, + \, \left(\hat{\lambda}[(1-\alpha) - \frac{Re \, \epsilon_1}{2}] - \frac{\epsilon_2}{2} \right) \|D(\mathbf{u}^n)\|^2 \\
& + \left(\frac{1}{2\lambda} + \frac{1}{\Delta t} - \frac{8 \acute{d} C_k^2 K^2}{\epsilon_2} \right) \|\tau^n\|^2 \, + \, \frac{1}{4} \langle\langle \tau^+ - \tau^- \rangle\rangle_{h, \tilde{\mathbf{u}}^n}^2 .
\end{aligned}
$$

Choosing $\epsilon_1 = \frac{(1-\alpha)}{2Re}, \epsilon_2 = \frac{\hat{\lambda}(1-\alpha)}{2}$, and $\Delta t \leq \frac{(1-\alpha)}{\acute{d} C_k^2 K^2} \min\left\{ \frac{4}{Re} , \frac{\hat{\lambda}}{16} \right\}$, it follows that the bilinear form $a(\cdot, \cdot; \cdot, \cdot)$ is positive. Hence, (3.18) has at most one solution. Since (3.18) is a finite dimensional linear system, the uniqueness of the solution implies the existence of the solution.

■

The discrete Gronwall's lemma plays an important role in the following analysis.

Lemma 2 (Discrete Gronwall's Lemma) *[10] Let Δt, H, and a_n, b_n, c_n, γ_n , (for integers $n \geq 0$), be nonnegative numbers such that*

$$
a_l \, + \, \Delta t \sum_{n=0}^{l} b_n \ \leq \ \Delta t \sum_{n=0}^{l} \gamma_n \, a_n \, + \, \Delta t \sum_{n=0}^{l} c_n \, + \, H \quad \text{for } l \geq 0 \, .
$$

Suppose that $\Delta t \, \gamma_n < 1$, for all n, and set $\sigma_n = (1 - \Delta t \, \gamma_n)^{-1}$. Then,

$$
a_l \, + \, \Delta t \sum_{n=0}^{l} b_n \ \leq \ \exp\left(\Delta t \sum_{n=0}^{l} \sigma_n \, \gamma_n \right) \left\{ \Delta t \sum_{n=0}^{l} c_n \, + \, H \right\} \quad \text{for } l \geq 0 \, . \tag{3.20}
$$

■

4 A Priori Error Estimate

In this section we analyze the error between the finite element approximation given by (3.16)-(3.17) and the true solution. A priori error estimates for the approximation are given in Theorem 2.

Theorem 2 *There exists a constant $c_1 > 0$ such that for $\Delta t < c_1 h^{d/4}$, the finite element approximation (3.16)-(3.17) is convergent to the solution $(\mathbf{u}, \tau)$ of (2.14)-(2.15) on the interval $(0,T)$ as $\Delta t,\, h \to 0$. In addition, the approximation $(\mathbf{u}_h, \tau_h)$ satisfies the following error estimates:*

$$
\begin{aligned}
|||\mathbf{u}_h - \mathbf{u}|||_{\infty,0} + |||\tau_h - \tau|||_{\infty,0} &\leq \mathbf{F}_1(\Delta t, h)\,, & (4.1)\\
|||\bar{\mathbf{u}}_h - \bar{\mathbf{u}}|||_{0,1} + |||\bar{\tau}_h - \bar{\tau}|||_{0,0} &\leq C\,(1 + T^{1/2})\;\mathbf{F}_1(\Delta t, h) + C\,\mathbf{F}_2(\Delta t)\,, & (4.2)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{F}_1(\Delta t, h) \;:=\;& C\,\left(h^k\,|||\mathbf{w}|||_{0,k+1} + h^m\,|||\eta|||_{0,m+1} + h^{q+1}\,|||r|||_{0,q+1}\right)\\
&+ C\,\left(h^{k+1}\,\|\mathbf{w}_t\|_{0,k+1} + h^{m+1}\,\|\eta_t\|_{0,m+1}\right)\\
&+ C\,(\Delta t)^2\left(\|\mathbf{w}_{tt}\|_{0,1} + \|\mathbf{w}_{ttt}\|_{0,0} + \|\eta_{tt}\|_{0,1} + \|\eta_{ttt}\|_{0,0} + \|r_{tt}\|_{0,0}\right)\\
&+ C\,(G_I(\Delta t, h))^{1/2} + C\,(\Delta t)^2\left(\int_{-5\Delta t/2}^{T}\left(\|\mathbf{w}_{tt}\|^2 + \|\eta_{tt}\|^2\right)\,dt\right)^{1/2},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{F}_2(\Delta t) \;:=\;& (\Delta t)^2\left(\|\nabla\mathbf{u}\|_{0,2} + \|\nabla\mathbf{w}\|_{0,2} + \|\nabla\mathbf{u}_t\|_{0,2} + \|\nabla\mathbf{w}_t\|_{0,2} + \|\nabla\mathbf{u}_{tt}\|_{0,2}\right.\\
&\left.+ \|\nabla\mathbf{w}_{tt}\|_{0,2} + \|\nabla\mathbf{u}_{ttt}\|_{0,2} + \|\nabla\mathbf{w}_{ttt}\|_{0,2} + \|\nabla\mathbf{u}_{tttt}\|_{0,2} + \|\nabla\mathbf{w}_{tttt}\|_{0,2}\right)\,, \quad (4.3)
\end{aligned}
$$

and $\mathbf{w}$, η, r satisfy (2.17)-(2.22),(2.44),(2.45), and $G_I(\Delta t, h)$, defined in (3.15), represents the initialization error.

The structure of the proof of Theorem 2 is as follows.

Let $\mathcal{U}^n$, $\mathcal{T}^n$ denote elements in Z_h, S_h, satisfying (3.2) and (3.4), respectively, and define $\mathbf{\Lambda}^n, \mathbf{E}^n, \mathbf{\Gamma}^n, \mathbf{F}^n$, $\epsilon_w, \epsilon_\eta$ as

$$
\begin{aligned}
\mathbf{\Lambda}^n &= \mathbf{w}^n - \mathcal{U}^n, & \mathbf{E}^n &= \mathcal{U}^n - \mathbf{u}_h^n, & (4.4)\\
\mathbf{\Gamma}^n &= \eta^n - \mathcal{T}^n, & \mathbf{F}^n &= \mathcal{T}^n - \tau_h^n, & (4.5)\\
\epsilon_w &= \mathbf{w}^n - \mathbf{u}_h^n, & \epsilon_\eta &= \eta^n - \tau_h^n. & (4.6)
\end{aligned}
$$

As introduced above, we use a bar to denote average between levels n and $n-1$ and a tilde to denote extrapolation from levels $n-1$, $n-2$, and $n-3$, i.e.,

$$
\begin{aligned}
\bar{\mathbf{\Lambda}}^n &= \frac{1}{2}\left(\mathbf{\Lambda}^n + \mathbf{\Lambda}^{n-1}\right)\,,\\
\tilde{\mathbf{\Lambda}}^n &= \mathbf{\Lambda}^{n-1} + \frac{1}{2}\Lambda^{n-2} - \frac{1}{2}\Lambda^{n-3} = 2\bar{\Lambda}^{n-1} - \bar{\Lambda}^{n-2}\,.
\end{aligned}
$$

Step 1. We prove the following lemma.

Lemma 3 *Under the induction hypothesis (IH) we have that for $l = 3, 4, \ldots, N$,*

$$
\left\|\mathbf{E}^l\right\|^2 + \left\|\mathbf{F}^l\right\|^2 + \sum_{n=3}^{l}\Delta t\,\left\|D(\bar{\mathbf{E}}^n)\right\|^2 \quad\leq\quad G(\Delta t, h) + C\,G_I(\Delta t, h)\,, \qquad (4.7)
$$

where

$$\begin{aligned} G(\Delta t,h) &= C\,(\Delta t)^4\left(\|\mathbf{w}_{tt}\|_{0,1}^2 + \|\mathbf{w}_{ttt}\|_{0,0}^2 + \|\eta_{tt}\|_{0,1}^2 + \|\eta_{ttt}\|_{0,0}^2 + \|r_{tt}\|_{0,0}^2\right) \\ &\quad + C\left(h^{2k}\,\|\mathbf{w}\|_{0,k+1}^2 + h^{2m}\,\|\eta\|_{0,m+1}^2 + h^{2q+2}\,\|k\|_{0,q+1}^2\right) \\ &\quad + C\left(h^{2k+2}\,\|\mathbf{w}_t\|_{0,k+1}^2 + h^{2m+2}\,\|\eta_t\|_{0,m+1}^2\right)\,. \end{aligned}$$

Step 2. We show that the induction hypothesis, $(IH1)$, is true.

Step 3. We derive the error estimates in (4.1) and (4.2).

Step 1. **Proof of Lemma 3:** From (2.17)-(2.18), it is clear that the true solution $(\mathbf{w},\eta,r)$ satisfies

$$\begin{aligned} Re\,(d_t\mathbf{w}^n,\mathbf{v}) \;&+\; Re\,c(\tilde{\mathbf{w}}^n,\bar{\mathbf{w}}^n,\mathbf{v}) + 2(1-\alpha)(D(\bar{\mathbf{w}}^n),D(\mathbf{v})) + (\bar{\eta}^n,D(\mathbf{v})) \\ &= (\bar{\mathbf{f}}^n,\mathbf{v}) + (r^{n-1/2},\nabla\cdot\mathbf{v}) + R_1(\mathbf{v})\,,\ \forall\,\mathbf{v}\in Z_h, \qquad (4.8) \\ \frac{1}{\lambda}(\bar{\eta}^n,\sigma) \;&+\; (d_t\eta^n,\sigma) + B(\tilde{\mathbf{w}}^n,\bar{\eta}^n,\sigma) - \hat{\lambda}(D(\bar{\mathbf{w}}^n),\sigma) + (g_a(\tilde{\eta}^n,\nabla\bar{\mathbf{w}}^n),\sigma) \\ &= R_2(\sigma)\,,\ \forall\,\sigma\in S_h, \qquad (4.9) \end{aligned}$$

where

$$\begin{aligned} R_1(\mathbf{v}) \;:=\; & Re\,(d_t\mathbf{w}^n - \mathbf{w}_t^{n-1/2},\mathbf{v}) + Re\,c(\tilde{\mathbf{w}}^n,\bar{\mathbf{w}}^n - \mathbf{w}^{n-1/2},\mathbf{v}) \\ & + 2(1-\alpha)(D(\bar{\mathbf{w}}^n - \mathbf{w}^{n-1/2}),D(\mathbf{v})) + (\bar{\eta}^n - \eta^{n-1/2},D(\mathbf{v}))\,, \qquad (4.10) \end{aligned}$$

and

$$\begin{aligned} R_2(\sigma) \;:=\; & \frac{1}{\lambda}(\bar{\eta}^n - \eta^{n-1/2},\sigma) + (d_t\eta^n - \eta_t^{n-1/2},\sigma) + B(\tilde{\mathbf{w}}^n,\bar{\eta}^n - \eta^{n-1/2},\sigma) \qquad (4.11) \\ & - \hat{\lambda}(D(\bar{\mathbf{w}}^n - \mathbf{w}^{n-1/2}),\sigma) + (g_a(\tilde{\eta}^n,\nabla(\bar{\mathbf{w}}^n - \mathbf{w}^{n-1/2}),\sigma)\,. \qquad (4.12) \end{aligned}$$

Subtracting (3.16)-(3.17) from (4.8)-(4.9) we obtain the following equations for ϵ_w and ϵ_η:

$$\begin{aligned} Re\,(d_t\epsilon_w,\mathbf{v}) \;&+\; Re\,c(\tilde{\mathbf{u}}_h^n,\bar{\epsilon}_w,\mathbf{v}) + 2(1-\alpha)(D(\bar{\epsilon}_w),D(\mathbf{v})) + (\bar{\epsilon}_\eta,D(\mathbf{v})) \\ &= (r^{n-1/2},\nabla\cdot\mathbf{v}) + R_1(\mathbf{v}) + Re\,c(\tilde{\mathbf{u}}_h^n - \tilde{\mathbf{w}}^n,\bar{\mathbf{w}}^n,\mathbf{v})\,,\ \forall\,\mathbf{v}\in Z_h, \qquad (4.13) \\ \frac{1}{\lambda}(\bar{\epsilon}_\eta,\sigma) \;&+\; (d_t\epsilon_\eta,\sigma) + B(\tilde{\mathbf{u}}_h^n,\bar{\epsilon}_\eta,\sigma) - \hat{\lambda}(D(\bar{\epsilon}_w),\sigma) + (g_a(\tilde{\tau}_h^n,\nabla\bar{\epsilon}_w),\sigma) \\ &= R_2(\sigma) + B(\tilde{\mathbf{u}}_h^n - \tilde{\mathbf{w}}^n,\bar{\eta}^n,\sigma) + (g_a(\tilde{\tau}_h^n - \tilde{\eta}^n,\nabla\bar{\mathbf{w}}^n),\sigma)\,,\ \forall\,\sigma\in S_h. \qquad (4.14) \end{aligned}$$

Substituting $\epsilon_w = \mathbf{E}^n + \mathbf{\Lambda}^n,\ \epsilon_\eta = \mathbf{F}^n + \mathbf{\Gamma}^n,\ \mathbf{v} = \bar{\mathbf{E}}^n,\ \sigma = \bar{\mathbf{F}}^n$ into (4.13)-(4.14), we obtain

$$\begin{aligned} Re\,(d_t\mathbf{E}^n,\bar{\mathbf{E}}^n) + Re\,c(\tilde{\mathbf{u}}_h^n,\bar{\mathbf{E}}^n,\bar{\mathbf{E}}^n) + 2(1-\alpha)(D(\bar{\mathbf{E}}^n),D(\bar{\mathbf{E}}^n)) + (\bar{\mathbf{F}}^n,D(\bar{\mathbf{E}}^n)) &= \mathcal{F}_1(\bar{\mathbf{E}}^n) \qquad (4.15) \\ \frac{1}{\lambda}(\bar{\mathbf{F}}^n,\bar{\mathbf{F}}^n) + (d_t\mathbf{F}^n,\bar{\mathbf{F}}^n) + B(\tilde{\mathbf{u}}_h^n,\bar{\mathbf{F}}^n,\bar{\mathbf{F}}^n) - \hat{\lambda}(D(\bar{\mathbf{E}}^n),\bar{\mathbf{F}}^n) + (g_a(\tilde{\tau}_h^n,\nabla\bar{\mathbf{E}}^n),\bar{\mathbf{F}}^n) &= \mathcal{F}_2(\bar{\mathbf{F}}^n) \qquad (4.16) \end{aligned}$$

where,

$$\begin{aligned} \mathcal{F}_1(\bar{\mathbf{E}}^n) &= (r^{n-1/2},\nabla\cdot\bar{\mathbf{E}}^n) + R_1(\bar{\mathbf{E}}^n) + Re\,c(\tilde{\mathbf{u}}_h^n - \tilde{\mathbf{w}}^n,\bar{\mathbf{w}}^n,\bar{\mathbf{E}}^n) - Re\,(d_t\mathbf{\Lambda}^n,\bar{\mathbf{E}}^n) \\ &\quad - Re\,c(\tilde{\mathbf{u}}_h^n,\bar{\mathbf{\Lambda}}^n,\bar{\mathbf{E}}^n) - 2(1-\alpha)(D(\bar{\mathbf{\Lambda}}^n),D(\bar{\mathbf{E}}^n)) - (\bar{\mathbf{\Gamma}}^n,D(\bar{\mathbf{E}}^n))\,, \\ \mathcal{F}_2(\bar{\mathbf{F}}^n) &= R_2(\bar{\mathbf{F}}^n) + B(\tilde{\mathbf{u}}_h^n - \tilde{\mathbf{w}}^n,\bar{\eta}^n,\bar{\mathbf{F}}^n) + (g_a(\tilde{\tau}_h^n - \tilde{\eta}^n,\nabla\bar{\mathbf{w}}^n),\bar{\mathbf{F}}^n) - \frac{1}{\lambda}(\bar{\mathbf{\Gamma}}^n,\bar{\mathbf{F}}^n) - (d_t\mathbf{\Gamma}^n,\bar{\mathbf{F}}^n) \\ &\quad - B(\tilde{\mathbf{u}}_h^n,\bar{\mathbf{\Gamma}}^n,\bar{\mathbf{F}}^n) + \hat{\lambda}(D(\bar{\mathbf{\Lambda}}^n),\bar{\mathbf{F}}^n) - (g_a(\tilde{\tau}_h^n,\nabla\bar{\mathbf{\Lambda}}^n),\bar{\mathbf{F}}^n). \end{aligned}$$

Multiplying (4.15) by $\hat{\lambda}$ and adding to (4.16) yields the single equation

$$\begin{aligned} Re\hat{\lambda}\,(d_t\mathbf{E}^n,\bar{\mathbf{E}}^n) &+ Re\hat{\lambda}\,c(\tilde{\mathbf{u}}_h^n,\bar{\mathbf{E}}^n,\bar{\mathbf{E}}^n)+2\hat{\lambda}(1-\alpha)(D(\bar{\mathbf{E}}^n),D(\bar{\mathbf{E}}^n))+\frac{1}{\lambda}(\bar{\mathbf{F}}^n,\bar{\mathbf{F}}^n)+(d_t\mathbf{F}^n,\bar{\mathbf{F}}^n) \\ &+ B(\tilde{\mathbf{u}}_h^n,\bar{\mathbf{F}}^n,\bar{\mathbf{F}}^n)+(g_a(\tilde{\tau}_h^n,\nabla\bar{\mathbf{E}}^n),\bar{\mathbf{F}}^n) \;=\; \hat{\lambda}\mathcal{F}_1(\bar{\mathbf{E}}^n)\,+\,\mathcal{F}_2(\bar{\mathbf{F}}^n)\,. \end{aligned}$$

Note that

$$(d_t\mathbf{E}^n,\bar{\mathbf{E}}^n) \;=\; \left(\frac{\mathbf{E}^n-\mathbf{E}^{n-1}}{\Delta t},\frac{\mathbf{E}^n+\mathbf{E}^{n-1}}{2}\right) \;=\; \frac{1}{2\Delta t}\left(\|\mathbf{E}^n\|^2-\left\|\mathbf{E}^{n-1}\right\|^2\right)\,,$$

and similarly,

$$(d_t\mathbf{F}^n,\bar{\mathbf{F}}^n) \;=\; \frac{1}{2\Delta t}\left(\|\mathbf{F}^n\|^2-\left\|\mathbf{F}^{n-1}\right\|^2\right)\,.$$

Thus we have

$$\begin{aligned} \frac{Re\hat{\lambda}}{2\Delta t}\left(\|\mathbf{E}^n\|^2-\left\|\mathbf{E}^{n-1}\right\|^2\right) &+ \frac{1}{2\Delta t}\left(\|\mathbf{F}^n\|^2-\left\|\mathbf{F}^{n-1}\right\|^2\right)+2\hat{\lambda}(1-\alpha)\left\|D(\bar{\mathbf{E}}^n)\right\|^2+\frac{1}{\lambda}\left\|\bar{\mathbf{F}}^n\right\|^2 \\ +\frac{1}{2}\langle\langle\bar{\mathbf{F}}^{n+}-\bar{\mathbf{F}}^{n-}\rangle\rangle^2_{h,\tilde{\mathbf{u}}_h^n} &= -Re\hat{\lambda}\,c(\tilde{\mathbf{u}}_h^n,\bar{\mathbf{E}}^n,\bar{\mathbf{E}}^n)-(g_a(\tilde{\tau}_h^n,\nabla\bar{\mathbf{E}}^n),\bar{\mathbf{F}}^n)+\hat{\lambda}\mathcal{F}_1(\bar{\mathbf{E}}^n) \\ &\quad +\mathcal{F}_2(\bar{\mathbf{F}}^n)\,. \end{aligned} \tag{4.17}$$

Multiplying (4.17) by $2\Delta t$ and summing from $n=3,\ldots,l$ we have

$$\begin{aligned} Re\hat{\lambda}\left(\left\|\mathbf{E}^l\right\|^2-\left\|\mathbf{E}^2\right\|^2\right) &+ \left(\left\|\mathbf{F}^l\right\|^2-\left\|\mathbf{F}^2\right\|^2\right)+4\hat{\lambda}(1-\alpha)\sum_{n=3}^{l}\Delta t\,\left\|D(\bar{\mathbf{E}}^n)\right\|^2+\frac{2}{\lambda}\sum_{n=3}^{l}\Delta t\,\left\|\bar{\mathbf{F}}^n\right\|^2 \\ &\le 2\Delta t\sum_{n=3}^{l}\left[-Re\hat{\lambda}\,c(\tilde{\mathbf{u}}_h^n,\bar{\mathbf{E}}^n,\bar{\mathbf{E}}^n)-(g_a(\tilde{\tau}_h^n,\nabla\bar{\mathbf{E}}^n),\bar{\mathbf{F}}^n)\right]+2\hat{\lambda}\Delta t\sum_{n=3}^{l}\mathcal{F}_1(\bar{\mathbf{E}}^n) \\ &\quad +2\Delta t\sum_{n=3}^{l}\mathcal{F}_2(\bar{\mathbf{F}}^n)\,. \end{aligned} \tag{4.18}$$

We now estimate each term on the right hand side of (4.18):

$$\begin{aligned} |c(\tilde{\mathbf{u}}_h^n,\bar{\mathbf{E}}^n,\bar{\mathbf{E}}^n)| &\le |(\tilde{\mathbf{u}}_h^n\cdot\nabla\bar{\mathbf{E}}^n,\bar{\mathbf{E}}^n)| \\ &\le \left\|\tilde{\mathbf{u}}_h^n\cdot\nabla\bar{\mathbf{E}}^n\right\|\,\left\|\bar{\mathbf{E}}^n\right\| \\ &\le \left\|\tilde{\mathbf{u}}_h^n\right\|_\infty \acute{d}^{1/2}\left\|\nabla\bar{\mathbf{E}}^n\right\|\,\left\|\bar{\mathbf{E}}^n\right\| \\ &\le C_k\left\|D(\bar{\mathbf{E}}^n)\right\|\,2K\acute{d}^{1/2}\left\|\bar{\mathbf{E}}^n\right\| \\ &\le \epsilon_1\left\|D(\bar{\mathbf{E}}^n)\right\|^2+\frac{K^2\acute{d}C_k^2}{\epsilon_1}\left\|\bar{\mathbf{E}}^n\right\|^2\,. \end{aligned}$$

$$\begin{aligned} |(g_a(\tilde{\tau}_h^n,\nabla\bar{\mathbf{E}}^n),\bar{\mathbf{F}}^n)| &\le \left\|g_a(\tilde{\tau}_h^n,\nabla\bar{\mathbf{E}}^n)\right\|\left\|\bar{\mathbf{F}}^n\right\| \\ &\le 4\left\|\tilde{\tau}_h^n\right\|_\infty\acute{d}\left\|\nabla\bar{\mathbf{E}}^n\right\|\,\left\|\bar{\mathbf{F}}^n\right\| \\ &\le C_k\left\|D(\bar{\mathbf{E}}^n)\right\|8K\acute{d}\left\|\bar{\mathbf{F}}^n\right\| \\ &\le \epsilon_2\left\|D(\bar{\mathbf{E}}^n)\right\|^2+\frac{16K^2\acute{d}^2C_k^2}{\epsilon_2}\left\|\bar{\mathbf{F}}^n\right\|^2\,. \end{aligned}$$

Thus for the first summation on the right hand side of (4.18) we have

$$2\Delta t \sum_{n=3}^{l} [-Re\hat{\lambda}\, c(\tilde{\mathbf{u}}_h^n, \bar{\mathbf{E}}^n, \bar{\mathbf{E}}^n) \quad - \quad (g_a(\tilde{\tau}_h^n, \nabla \bar{\mathbf{E}}^n), \bar{\mathbf{F}}^n)] \;\leq\; 2\Delta t \sum_{n=3}^{l} \left(Re\hat{\lambda}\epsilon_1 + \epsilon_2 \right) \|D(\bar{\mathbf{E}}^n)\|^2$$
$$+2\Delta t \sum_{n=3}^{l} \frac{Re\hat{\lambda}\acute{d}K^2C_k^2}{\epsilon_1} \|\bar{\mathbf{E}}^n\|^2 + 2\Delta t \sum_{n=3}^{l} \frac{16\acute{d}^2K^2C_k^2}{\epsilon_2} \|\bar{\mathbf{F}}^n\|^2 . \quad (4.19)$$

Next we consider $\mathcal{F}_1(\bar{\mathbf{E}}^n)$. For $\mathcal{P}^n$ and $\mathcal{P}^{n-1}$ elements in Q_h satisfying (3.3),

$$\begin{aligned}
|(r^{n-1/2}, \nabla\cdot \bar{\mathbf{E}}^n)| &= |(r^{n-1/2} - \bar{\mathcal{P}}^n, \nabla\cdot \bar{\mathbf{E}}^n)| \\
&\leq |(r^{n-1/2} - \bar{r}^n, \nabla\cdot \bar{\mathbf{E}}^n)| + |(\bar{r}^n - \bar{\mathcal{P}}^n, \nabla\cdot \bar{\mathbf{E}}^n)| \\
&\leq \left\| r^{n-1/2} - \bar{r}^n \right\| \|\nabla\cdot \bar{\mathbf{E}}^n\| + \|\bar{r}^n - \bar{\mathcal{P}}^n\| \|\nabla\cdot \bar{\mathbf{E}}^n\| \\
&\leq \left\| r^{n-1/2} - \bar{r}^n \right\| \acute{d}^{1/2} \|\nabla \bar{\mathbf{E}}^n\| + \|\bar{r}^n - \bar{\mathcal{P}}^n\| \acute{d}^{1/2} \|\nabla \bar{\mathbf{E}}^n\| \\
&\leq C_k \|D(\bar{\mathbf{E}}^n)\| \acute{d}^{1/2} \left\| r^{n-1/2} - \bar{r}^n \right\| + C_k \|D(\bar{\mathbf{E}}^n)\| \acute{d}^{1/2} \|\bar{r}^n - \bar{\mathcal{P}}^n\| \\
&\leq (\epsilon_3 + \epsilon_4) \|D(\bar{\mathbf{E}}^n)\|^2 + \frac{C_k^2 \acute{d}}{4\epsilon_3} \left\| r^{n-1/2} - \bar{r}^n \right\|^2 + \frac{C_k^2 \acute{d}}{4\epsilon_4} \|\bar{r}^n - \bar{\mathcal{P}}^n\|^2 \ . \quad (4.20)
\end{aligned}$$

$$\begin{aligned}
|c(\tilde{\mathbf{u}}_h^n - \tilde{\mathbf{w}}^n, \bar{\mathbf{w}}^n, \bar{\mathbf{E}}^n)| &\leq \|(\tilde{\mathbf{u}}_h^n - \tilde{\mathbf{w}}^n) \cdot \nabla \bar{\mathbf{w}}^n\| \, \|\bar{\mathbf{E}}^n\| \\
&\leq \acute{d} \|\nabla \bar{\mathbf{w}}^n\|_\infty \|\tilde{\mathbf{u}}_h^n - \tilde{\mathbf{w}}^n\| \, \|\bar{\mathbf{E}}^n\| \\
&\leq \frac{1}{2} \acute{d}M \, \|\tilde{\mathbf{u}}_h^n - \tilde{\mathbf{w}}^n\|^2 + \frac{1}{2} \acute{d}M \, \|\bar{\mathbf{E}}^n\|^2 \\
&\leq \frac{1}{2} \acute{d}M \, \|\bar{\mathbf{E}}^n\|^2 + \acute{d}M \left\| \tilde{\mathbf{E}}^n \right\|^2 + \acute{d}M \left\| \tilde{\mathbf{\Lambda}}^n \right\|^2 . \quad (4.21)
\end{aligned}$$

$$\begin{aligned}
|(d_t \mathbf{\Lambda}^n, \bar{\mathbf{E}}^n)| &\leq \|\bar{\mathbf{E}}^n\| \, \|d_t \mathbf{\Lambda}^n\| \\
&\leq \|\bar{\mathbf{E}}^n\|^2 + \frac{1}{4} \|d_t \mathbf{\Lambda}^n\|^2 \ . \quad (4.22)
\end{aligned}$$

$$\begin{aligned}
|c(\tilde{\mathbf{u}}_h^n, \bar{\mathbf{\Lambda}}^n, \bar{\mathbf{E}}^n)| &\leq \|\bar{\mathbf{E}}^n\| \, \|\tilde{\mathbf{u}}_h^n \cdot \nabla \bar{\mathbf{\Lambda}}^n\| \\
&\leq \|\bar{\mathbf{E}}^n\| \, \|\tilde{\mathbf{u}}_h^n\|_\infty \acute{d}^{1/2} \|\nabla \bar{\mathbf{\Lambda}}^n\| \\
&\leq \|\bar{\mathbf{E}}^n\|^2 + K^2 \acute{d} \|\nabla \bar{\mathbf{\Lambda}}^n\|^2 \ . \quad (4.23)
\end{aligned}$$

$$\begin{aligned}
|(D(\bar{\mathbf{\Lambda}}^n), D(\bar{\mathbf{E}}^n))| &\leq \|D(\bar{\mathbf{\Lambda}}^n)\| \, \|D(\bar{\mathbf{E}}^n)\| \\
&\leq \epsilon_5 \|D(\bar{\mathbf{E}}^n)\|^2 + \frac{1}{4\epsilon_5} \|\nabla \bar{\mathbf{\Lambda}}^n\|^2 \ . \quad (4.24)
\end{aligned}$$

$$
\begin{aligned}
|(\bar{\mathbf{\Gamma}}^n, D(\bar{\mathbf{E}}^n))| &\leq \|\bar{\mathbf{\Gamma}}^n\| \, \|D(\bar{\mathbf{E}}^n)\| \\
&\leq \epsilon_6 \|D(\bar{\mathbf{E}}^n)\|^2 + \frac{1}{4\epsilon_6} \|\bar{\mathbf{\Gamma}}^n\|^2 \ .
\end{aligned} \tag{4.25}
$$

For the $R_1(\bar{\mathbf{E}}^n)$ terms we have:

$$
|(d_t\mathbf{w}^n - \mathbf{w}_t^{n-1/2}, \bar{\mathbf{E}}^n)| \leq \|\bar{\mathbf{E}}^n\|^2 + \frac{1}{4} \left\| d_t\mathbf{w}^n - \mathbf{w}_t^{n-1/2} \right\|^2 \ . \tag{4.26}
$$

$$
\begin{aligned}
|c(\tilde{\mathbf{w}}^n, \bar{\mathbf{w}}^n - \mathbf{w}^{n-1/2}, \bar{\mathbf{E}}^n)| &= |(\tilde{\mathbf{w}}^n \cdot \nabla(\bar{\mathbf{w}}^n - \mathbf{w}^{n-1/2}), \bar{\mathbf{E}}^n)| \\
&\leq \left\| \tilde{\mathbf{w}}^n \cdot \nabla(\bar{\mathbf{w}}^n - \mathbf{w}^{n-1/2}) \right\| \, \|\bar{\mathbf{E}}^n\| \\
&\leq \|\tilde{\mathbf{w}}^n\|_\infty \, \acute{d}^{1/2} \left\| \nabla(\bar{\mathbf{w}}^n - \mathbf{w}^{n-1/2}) \right\| \, \|\bar{\mathbf{E}}^n\| \\
&\leq M \left\| \nabla(\bar{\mathbf{w}}^n - \mathbf{w}^{n-1/2}) \right\|^2 + \, M\acute{d} \, \|\bar{\mathbf{E}}^n\|^2 \ .
\end{aligned} \tag{4.27}
$$

$$
\begin{aligned}
|(D(\bar{\mathbf{w}}^n - \mathbf{w}^{n-1/2}), D(\bar{\mathbf{E}}^n))| &\leq \left\| D(\bar{\mathbf{w}}^n - \mathbf{w}^{n-1/2}) \right\| \, \|D(\bar{\mathbf{E}}^n)\| \\
&\leq \epsilon_7 \|D(\bar{\mathbf{E}}^n)\|^2 + \frac{1}{4\epsilon_7} \left\| \nabla(\bar{\mathbf{w}}^n - \mathbf{w}^{n-1/2}) \right\|^2 \ .
\end{aligned} \tag{4.28}
$$

$$
|(\bar{\eta}^n - \eta^{n-1/2}, D(\bar{\mathbf{E}}^n))| \leq \epsilon_8 \|D(\bar{\mathbf{E}}^n)\|^2 + \frac{1}{4\epsilon_8} \left\| \bar{\eta}^n - \eta^{n-1/2} \right\|^2 \ . \tag{4.29}
$$

Combining (4.20)-(4.29) we have the following estimate for $\mathcal{F}_1(\bar{\mathbf{E}}^n)$.

$$
\begin{aligned}
\hat{\lambda}\mathcal{F}_1(\bar{\mathbf{E}}^n) \leq\ & \|D(\bar{\mathbf{E}}^n)\|^2 \left(\hat{\lambda}(\epsilon_3 + \epsilon_4 + \epsilon_6 + \epsilon_8) \, + \, \hat{\lambda}2(1-\alpha)(\epsilon_5 + \epsilon_7) \right) \\
+\ & \|\bar{\mathbf{E}}^n\|^2 \left(3\hat{\lambda}Re + \hat{\lambda}Re\, \acute{d}M + \frac{1}{2}\hat{\lambda}Re\, \acute{d}M \right) \, + \, \left\| \tilde{\mathbf{E}}^n \right\|^2 \left(\hat{\lambda}Re\, \acute{d}M \right) \\
+\ & \|\nabla\bar{\mathbf{\Lambda}}^n\|^2 \left(\hat{\lambda}Re\, \acute{d}K^2 + \hat{\lambda}2(1-\alpha)\frac{1}{4\epsilon_5} \right) + \|d_t\mathbf{\Lambda}^n\|^2 \left(\hat{\lambda}\frac{Re}{4} \right) + \left\| \tilde{\mathbf{\Lambda}}^n \right\|^2 \left(\hat{\lambda}Re\, \acute{d}M \right) \\
+\ & \left\| \nabla(\bar{\mathbf{w}}^n - \mathbf{w}^{n-1/2}) \right\|^2 \left(\hat{\lambda}Re\, M + \hat{\lambda}2(1-\alpha)\frac{1}{4\epsilon_7} \right) + \left\| d_t\mathbf{w}^n - \mathbf{w}_t^{n-1/2} \right\|^2 \left(\hat{\lambda}Re\frac{1}{4} \right) \\
+\ & \|\bar{\mathbf{\Gamma}}^n\|^2 \left(\hat{\lambda}\frac{1}{4\epsilon_6} \right) + \left\| \bar{\eta}^n - \eta^{n-1/2} \right\|^2 \left(\hat{\lambda}\frac{1}{4\epsilon_8} \right) \\
+\ & \|\bar{r}^n - \bar{\mathcal{P}}^n\|^2 \left(\hat{\lambda}\frac{C_k^2\acute{d}}{4\epsilon_4} \right) + \left\| r^{n-1/2} - \bar{r}^n \right\|^2 \left(\hat{\lambda}\frac{C_k^2\acute{d}}{4\epsilon_3} \right) \ .
\end{aligned} \tag{4.30}
$$

Next we consider the terms in $\mathcal{F}_2(\bar{\mathbf{F}}^n)$.

$$
\begin{aligned}
|B(\tilde{\mathbf{u}}_h^n - \tilde{\mathbf{w}}^n, \bar{\eta}^n, \bar{\mathbf{F}}^n)| \leq\ & |((\tilde{\mathbf{u}}_h^n - \tilde{\mathbf{w}}^n) \cdot \nabla\bar{\eta}^n, \bar{\mathbf{F}}^n)_h| + \frac{1}{2}|(\nabla \cdot (\tilde{\mathbf{u}}_h^n - \tilde{\mathbf{w}}^n) \ \bar{\eta}^n, \bar{\mathbf{F}}^n)| \\
& + |\langle \bar{\eta}^{n+} - \bar{\eta}^{n-}, \bar{\mathbf{F}}^{n+} \rangle_{h,(\tilde{\mathbf{u}}_h^n - \tilde{\mathbf{w}}^n)}| \, .
\end{aligned} \tag{4.31}
$$

For the first term in (4.31)

$$\begin{aligned}
|((\tilde{\mathbf{u}}_h^n - \tilde{\mathbf{w}}^n)\cdot\nabla\bar{\eta}^n, \bar{\mathbf{F}}^n)_h| &\le \|(\tilde{\mathbf{u}}_h^n - \tilde{\mathbf{w}}^n)\cdot\nabla\bar{\eta}^n\|\,\|\bar{\mathbf{F}}^n\| \\
&\le \acute{d}^{3/2}\,\|\nabla\bar{\eta}^n\|_\infty\,\|(\tilde{\mathbf{u}}_h^n - \tilde{\mathbf{w}}^n)\|\,\|\bar{\mathbf{F}}^n\| \\
&\le \frac{1}{2}\acute{d}^{3/2}M\,\|(\tilde{\mathbf{u}}_h^n - \tilde{\mathbf{w}}^n)\|^2 + \frac{1}{2}\acute{d}^{3/2}M\,\|\bar{\mathbf{F}}^n\|^2 \\
&\le \acute{d}^{3/2}M\left\|\tilde{\mathbf{E}}^n\right\|^2 + \acute{d}^{3/2}M\left\|\tilde{\mathbf{\Lambda}}^n\right\|^2 + \frac{1}{2}\acute{d}^{3/2}M\,\|\bar{\mathbf{F}}^n\|^2 . \qquad (4.32)
\end{aligned}$$

The second term is handled via

$$\begin{aligned}
|(\nabla\cdot(\tilde{\mathbf{u}}_h^n - \tilde{\mathbf{w}}^n)\,\bar{\eta}^n, \bar{\mathbf{F}}^n)| &\le \|\nabla\cdot(\tilde{\mathbf{u}}_h^n - \tilde{\mathbf{w}}^n)\,\bar{\eta}^n\|\,\|\bar{\mathbf{F}}^n\| \\
&\le \acute{d}\,\|\bar{\eta}^n\|_\infty\,\|\nabla\cdot(\tilde{\mathbf{u}}_h^n - \tilde{\mathbf{w}}^n)\|\,\|\bar{\mathbf{F}}^n\| \\
&\le \acute{d}^{3/2}M\,\|\nabla(\tilde{\mathbf{u}}_h^n - \tilde{\mathbf{w}}^n)\|\,\|\bar{\mathbf{F}}^n\| \\
&\le \acute{d}^{3/2}M\left\|\nabla\tilde{\mathbf{E}}^n\right\|\|\bar{\mathbf{F}}^n\| + \acute{d}^{3/2}M\left\|\nabla\tilde{\mathbf{\Lambda}}^n\right\|\|\bar{\mathbf{F}}^n\| \\
&\le \acute{d}^{3/2}M\,C_K\left\|D(\tilde{\mathbf{E}}^n)\right\|\|\bar{\mathbf{F}}^n\| + \acute{d}^{3/2}M\left\|\nabla\tilde{\mathbf{\Lambda}}^n\right\|\|\bar{\mathbf{F}}^n\| \\
&\le \epsilon_9\left\|D(\tilde{\mathbf{E}}^n)\right\|^2 + \frac{1}{4}\acute{d}^3M^2\left(1+\frac{C_K^2}{\epsilon_9}\right)\|\bar{\mathbf{F}}^n\|^2 + \left\|\nabla\tilde{\mathbf{\Lambda}}^n\right\|^2 . \qquad (4.33)
\end{aligned}$$

For the third term

$$|\langle\bar{\eta}^{n+} - \bar{\eta}^{n-}, \bar{\mathbf{F}}^{n+}\rangle_{h,(\tilde{\mathbf{u}}_h^n - \tilde{\mathbf{w}}^n)}| = 0\,, \qquad (4.34)$$

by the continuity of $\bar{\eta}^n$.

$$\begin{aligned}
|(g_a(\tilde{\tau}_h^n - \tilde{\eta}^n, \nabla\bar{\mathbf{w}}^n), \bar{\mathbf{F}}^n)| &\le \|g_a(\tilde{\tau}_h^n - \tilde{\eta}^n, \nabla\bar{\mathbf{w}}^n)\|\,\|\bar{\mathbf{F}}^n\| \\
&\le 4\acute{d}\,\|\nabla\bar{\mathbf{w}}^n\|_\infty\,\|\tilde{\tau}_h^n - \tilde{\eta}^n\|\,\|\bar{\mathbf{F}}^n\| \\
&\le 2\acute{d}M\,\|\tilde{\tau}_h^n - \tilde{\eta}^n\|^2 + 2\acute{d}M\,\|\bar{\mathbf{F}}^n\| \\
&\le 2\acute{d}M\,\|\bar{\mathbf{F}}^n\| + 4\acute{d}M\left\|\tilde{\mathbf{F}}^n\right\|^2 + 4\acute{d}M\left\|\tilde{\mathbf{\Gamma}}^n\right\|^2 . \qquad (4.35)
\end{aligned}$$

$$|(\bar{\mathbf{\Gamma}}^n, \bar{\mathbf{F}}^n)| \le \|\bar{\mathbf{F}}^n\|^2 + \frac{1}{4}\|\bar{\mathbf{\Gamma}}^n\|^2 . \qquad (4.36)$$

$$|(d_t\mathbf{\Gamma}^n, \bar{\mathbf{F}}^n)| \le \|\bar{\mathbf{F}}^n\|^2 + \frac{1}{4}\|d_t\mathbf{\Gamma}^n\|^2 . \qquad (4.37)$$

$$|B(\tilde{\mathbf{u}}_h^n, \bar{\mathbf{\Gamma}}^n, \bar{\mathbf{F}}^n)| \le |(\tilde{\mathbf{u}}_h^n\cdot\nabla\bar{\mathbf{\Gamma}}^n, \bar{\mathbf{F}}^n)_h| + \frac{1}{2}|(\nabla\cdot\tilde{\mathbf{u}}_h^n\,\bar{\mathbf{\Gamma}}^n, \bar{\mathbf{F}}^n)| + |\langle\bar{\mathbf{\Gamma}}^{n+} - \bar{\mathbf{\Gamma}}^{n-}, \bar{\mathbf{F}}^{n+}\rangle_{h,\tilde{\mathbf{u}}^n}|\,.$$

Each of these terms may be bounded via:

$$\begin{aligned}
|(\tilde{\mathbf{u}}_h^n\cdot\nabla\bar{\mathbf{\Gamma}}^n, \bar{\mathbf{F}}^n)_h| &\le \|\tilde{\mathbf{u}}_h^n\cdot\nabla\bar{\mathbf{\Gamma}}^n\|\,\|\bar{\mathbf{F}}^n\| \\
&\le \acute{d}^{1/2}\,\|\tilde{\mathbf{u}}_h^n\|_\infty\,\|\nabla\bar{\mathbf{\Gamma}}^n\|\,\|\bar{\mathbf{F}}^n\| \\
&\le \|\bar{\mathbf{F}}^n\|^2 + \acute{d}K^2\,\|\nabla\bar{\mathbf{\Gamma}}^n\|^2 . \qquad (4.38)
\end{aligned}$$

$$
\begin{aligned}
\frac{1}{2}|(\nabla\cdot\tilde{\mathbf{u}}_h^n\ \bar{\mathbf{\Gamma}}^n,\bar{\mathbf{F}}^n)| &\leq \frac{1}{2}\|\nabla\cdot\tilde{\mathbf{u}}_h^n\ \bar{\mathbf{\Gamma}}^n\|\,\|\bar{\mathbf{F}}^n\| \\
&\leq \frac{1}{2}\acute{d}\,\|\nabla\tilde{\mathbf{u}}_h^n\|_\infty\,\|\bar{\mathbf{\Gamma}}^n\|\,\|\bar{\mathbf{F}}^n\| \\
&\leq \frac{1}{2}\,\acute{d}\,C_i\,h^{-1}\,\|\tilde{\mathbf{u}}_h^n\|_\infty\,\|\bar{\mathbf{\Gamma}}^n\|\,\|\bar{\mathbf{F}}^n\| \\
&\leq \|\bar{\mathbf{F}}^n\|^2\ +\ \frac{1}{4}\acute{d}^2\,C_i^2\,K^2\,h^{-2}\,\|\bar{\mathbf{\Gamma}}^n\|^2\ .
\end{aligned}
\tag{4.39}
$$

$$
|\langle\bar{\mathbf{\Gamma}}^{n+}-\bar{\mathbf{\Gamma}}^{n-},\bar{\mathbf{F}}^{n+}\rangle_{h,\tilde{\mathbf{u}}^n}|\ =\ 0\ ,\ (\text{by the continuity of } \bar{\mathbf{\Gamma}}^n)\ .
\tag{4.40}
$$

$$
\begin{aligned}
|(D(\bar{\mathbf{\Lambda}}^n),\bar{\mathbf{F}}^n)| &\leq \|\bar{\mathbf{F}}^n\|\,\|D(\bar{\mathbf{\Lambda}}^n)\| \\
&\leq \|\bar{\mathbf{F}}^n\|^2+\frac{1}{4}\,\|\nabla\bar{\mathbf{\Lambda}}^n\|^2\ .
\end{aligned}
\tag{4.41}
$$

$$
\begin{aligned}
|(g_a(\tilde{\tau}_h^n,\nabla\bar{\mathbf{\Lambda}}^n),\bar{\mathbf{F}}^n)| &\leq \|g_a(\tilde{\tau}_h^n,\nabla\bar{\mathbf{\Lambda}}^n)\|\,\|\bar{\mathbf{F}}^n\| \\
&\leq 4\acute{d}\,\|\tilde{\tau}_h^n\|_\infty\,\|\nabla\bar{\mathbf{\Lambda}}^n\|\,\|\bar{\mathbf{F}}^n\| \\
&\leq \|\bar{\mathbf{F}}^n\|^2+16\acute{d}^2K^2\,\|\nabla\bar{\mathbf{\Lambda}}^n\|^2\ .
\end{aligned}
\tag{4.42}
$$

Now for the $R_2(\bar{\mathbf{F}}^n)$ terms.

$$
\begin{aligned}
|(\bar{\eta}^n-\eta^{n-1/2},\bar{\mathbf{F}}^n)| &\leq \left\|\bar{\eta}^n-\eta^{n-1/2}\right\|\,\|\bar{\mathbf{F}}^n\| \\
&\leq \|\bar{\mathbf{F}}^n\|^2+\frac{1}{4}\left\|\bar{\eta}^n-\eta^{n-1/2}\right\|^2\ .
\end{aligned}
\tag{4.43}
$$

$$
|(d_t\eta^n-\eta_t^{n-1/2},\bar{\mathbf{F}}^n)|\leq\|\bar{\mathbf{F}}^n\|^2+\frac{1}{4}\left\|d_t\eta^n-\eta_t^{n-1/2}\right\|^2\ .
\tag{4.44}
$$

$$
|(D(\bar{\mathbf{w}}^n-\mathbf{w}^{n-1/2}),\bar{\mathbf{F}}^n)|\leq\|\bar{\mathbf{F}}^n\|^2+\frac{1}{4}\left\|\nabla(\bar{\mathbf{w}}^n-\mathbf{w}^{n-1/2})\right\|^2\ .
\tag{4.45}
$$

Next,

$$
\begin{aligned}
|(g_a(\tilde{\eta}^n,\nabla(\bar{\mathbf{w}}^n-\mathbf{w}^{n-1/2}),\bar{\mathbf{F}}^n)| &\leq \left\|g_a(\tilde{\eta}^n,\nabla(\bar{\mathbf{w}}^n-\mathbf{w}^{n-1/2}))\right\|\ \|\bar{\mathbf{F}}^n\| \\
&\leq 4\acute{d}\,\|\tilde{\eta}^n\|_\infty\,\left\|\nabla(\bar{\mathbf{w}}^n-\mathbf{w}^{n-1/2})\right\|\ \|\bar{\mathbf{F}}^n\| \\
&\leq \|\bar{\mathbf{F}}^n\|^2\ +\ 16\acute{d}^2M^2\left\|\nabla(\bar{\mathbf{w}}^n-\mathbf{w}^{n-1/2})\right\|^2\ .
\end{aligned}
\tag{4.46}
$$

Now for $B(\tilde{\mathbf{w}}^n,\bar{\eta}^n-\eta^{n-1/2},\bar{\mathbf{F}}^n)$ we have

$$
\begin{aligned}
|B(\tilde{\mathbf{w}}^n,\bar{\eta}^n-\eta^{n-1/2},\bar{\mathbf{F}}^n)| &\leq |(\tilde{\mathbf{w}}^n\cdot\nabla(\bar{\eta}^n-\eta^{n-1/2}),\bar{\mathbf{F}}^n)_h|+\frac{1}{2}|(\nabla\cdot\tilde{\mathbf{w}}^n\ (\bar{\eta}^n-\eta^{n-1/2}),\bar{\mathbf{F}}^n)| \\
&\quad+|\langle(\bar{\eta}^n-\eta^{n-1/2})^+-(\bar{\eta}^n-\eta^{n-1/2})^-,\bar{\mathbf{F}}^{n+}\rangle_{h,\tilde{\mathbf{w}}^n}|\,.
\end{aligned}
\tag{4.47}
$$

For the first term in (4.47)

$$
\begin{aligned}
|(\tilde{\mathbf{w}}^n \cdot \nabla(\bar{\eta}^n - \eta^{n-1/2}), \bar{\mathbf{F}}^n)_h| &\leq \|\tilde{\mathbf{w}}^n\|_\infty \, \acute{d}^{1/2} \left\|\nabla(\bar{\eta}^n - \eta^{n-1/2})\right\| \, \|\bar{\mathbf{F}}^n\| \\
&\leq \|\bar{\mathbf{F}}^n\|^2 + \acute{d} M^2 \left\|\nabla(\bar{\eta}^n - \eta^{n-1/2})\right\|^2 . \qquad (4.48)
\end{aligned}
$$

The second term in (4.47) is bounded via

$$
\begin{aligned}
\frac{1}{2}|(\nabla \cdot \tilde{\mathbf{w}}^n \, (\bar{\eta}^n - \eta^{n-1/2}), \bar{\mathbf{F}}^n)| &\leq \frac{1}{2}\|\nabla \cdot \tilde{\mathbf{w}}^n\|_\infty \left\|\bar{\eta}^n - \eta^{n-1/2}\right\| \, \|\bar{\mathbf{F}}^n\| \\
&\leq \frac{1}{2}\acute{d}\, \|\nabla \tilde{\mathbf{w}}^n\|_\infty \left\|\bar{\eta}^n - \eta^{n-1/2}\right\| \, \|\bar{\mathbf{F}}^n\| \\
&\leq \|\bar{\mathbf{F}}^n\|^2 + \frac{1}{4}\acute{d}^2 \, M^2 \left\|\bar{\eta}^n - \eta^{n-1/2}\right\|^2 . \qquad (4.49)
\end{aligned}
$$

For the third term in (4.47) we have

$$
|\langle(\bar{\eta}^n - \eta^{n-1/2})^+ - (\bar{\eta}^n - \eta^{n-1/2})^-, \bar{\mathbf{F}}^{n+}\rangle_{h,\tilde{\mathbf{w}}^n}| \;=\; 0 \, , \qquad (4.50)
$$

by the continuity of η.

Combining the estimates in (4.31)-(4.50) we obtain the following estimate for $\mathcal{F}_2(\bar{\mathbf{F}}^n)$.

$$
\begin{aligned}
|\mathcal{F}_2(\bar{\mathbf{F}}^n)| \;\leq\; & \left\|\tilde{\mathbf{E}}^n\right\|^2 \left(\acute{d}^{3/2} M\right) + \left\|D(\tilde{\mathbf{E}}^n)\right\|^2 (\epsilon_9) + \left\|\tilde{\mathbf{\Lambda}}^n\right\|^2 \left(\acute{d}^{3/2} M\right) + \left\|\nabla \tilde{\mathbf{\Lambda}}^n\right\|^2 \\
& + \|\nabla \bar{\mathbf{\Lambda}}^n\|^2 \left(\frac{\hat{\lambda}}{4} + 16\acute{d}^2 K^2\right) + \left\|\nabla(\bar{\mathbf{w}}^n - \mathbf{w}^{n-1/2})\right\|^2 \left(\frac{\hat{\lambda}}{4} + 16\acute{d}^2 M^2\right) \\
& + \|\bar{\mathbf{F}}^n\|^2 \left(\frac{1}{2}\acute{d}^{3/2} M + \frac{1}{4}\acute{d}^3 M^2 \left(1 + \frac{C_K^2}{\epsilon_9}\right) + 8 + 2\acute{d}\, M + 2\hat{\lambda} + \frac{2}{\lambda}\right) + \left\|\tilde{\mathbf{F}}^n\right\|^2 \left(4\acute{d}\, M\right) \\
& + \|\bar{\mathbf{\Gamma}}^n\|^2 \left(\frac{1}{4\lambda} + \frac{1}{4}\acute{d}^2 \, C_i^2 \, K^2 \, h^{-2}\right) + \left\|\tilde{\mathbf{\Gamma}}^n\right\|^2 \left(4\acute{d}\, M\right) + \|\nabla \bar{\mathbf{\Gamma}}^n\|^2 \left(\acute{d} K^2\right) \\
& + \|d_t \mathbf{\Gamma}^n\|^2 \left(\frac{1}{4}\right) + \left\|\bar{\eta}^n - \eta^{n-1/2}\right\|^2 \left(\frac{1}{4\lambda} + \frac{\acute{d}^2 \, M^2}{4}\right) \\
& + \left\|\nabla(\bar{\eta}^n - \eta^{n-1/2})\right\|^2 \left(\acute{d}\, M^2\right) + \left\|d_t \eta^n - \eta_t^{n-1/2}\right\|^2 \left(\frac{1}{4}\right) . \qquad (4.51)
\end{aligned}
$$

Note that

$$
\begin{aligned}
\|\bar{\mathbf{F}}^n\|^2 &\leq \frac{1}{2}\|\mathbf{F}^n\|^2 + \frac{1}{2}\left\|\mathbf{F}^{n-1}\right\|^2 , \qquad (4.52) \\
\left\|\tilde{\mathbf{F}}^n\right\|^2 &\leq \frac{3}{2}\left\|\mathbf{F}^{n-1}\right\|^2 + \frac{3}{2}\left\|\mathbf{F}^{n-2}\right\|^2 + \frac{3}{2}\left\|\mathbf{F}^{n-3}\right\|^2 . \qquad (4.53)
\end{aligned}
$$

with analogous estimates also holding for $\bar{\mathbf{E}}^n$.

With the following choices: $\epsilon_1 = 3(1-\alpha)/(18Re)$, $\epsilon_2 = 3\hat{\lambda}(1-\alpha)/18$, $\epsilon_3 = \epsilon_4 = \epsilon_6 = \epsilon_8 = 3(1-\alpha)/18$, $\epsilon_5 = \epsilon_7 = 3/36$, $\epsilon_9 = \hat{\lambda}(1-\alpha)/60$, substituting (4.19),(4.30),(4.51), and (4.52),(4.53) into (4.18) we obtain

$$
\begin{aligned}
Re\hat{\lambda}\left(\left\|\mathbf{E}^l\right\|^2-\left\|\mathbf{E}^2\right\|^2\right) \quad &+\quad \left(\left\|\mathbf{F}^l\right\|^2-\left\|\mathbf{F}^2\right\|^2\right)+\hat{\lambda}(1-\alpha)\sum_{n=3}^{l}\Delta t\ \|D(\bar{\mathbf{E}}^n)\|^2+\frac{2}{\lambda}\sum_{n=3}^{l}\Delta t\ \|\bar{\mathbf{F}}^n\|^2 \\
&\leq\ C_1\sum_{n=3}^{l}\Delta t\ \|\mathbf{E}^n\|^2\ +\ C_2\sum_{n=3}^{l}\Delta t\ \|\mathbf{F}^n\|^2 \\
&\quad +C_3\sum_{n=0}^{l}\Delta t\ \|\mathbf{\Lambda}^n\|^2\ +\ C_4\sum_{n=0}^{l}\Delta t\ \|\nabla\mathbf{\Lambda}^n\|^2\ +\ C_5\sum_{n=3}^{l}\Delta t\ \|d_t\mathbf{\Lambda}^n\|^2 \\
&\quad +\ \left(C_6+C_7h^{-2}\right)\sum_{n=0}^{l}\Delta t\ \|\mathbf{\Gamma}^n\|^2\ +\ C_8\sum_{n=2}^{l}\Delta t\ \|\nabla\mathbf{\Gamma}^n\|^2\ +\ C_9\sum_{n=3}^{l}\Delta t\ \|d_t\mathbf{\Gamma}^n\|^2 \\
&\quad +C_{10}\sum_{n=3}^{l}\Delta t\ \left\|\nabla(\bar{\mathbf{w}}^n-\mathbf{w}^{n-1/2})\right\|^2\ +\ C_{11}\sum_{n=3}^{l}\Delta t\ \left\|d_t\mathbf{w}^n-\mathbf{w}_t^{n-1/2}\right\|^2 \\
&\quad +C_{12}\sum_{n=3}^{l}\Delta t\ \left\|\bar{\eta}^n-\eta^{n-1/2}\right\|^2\ +\ C_{13}\sum_{n=3}^{l}\Delta t\ \left\|\nabla(\bar{\eta}^n-\eta^{n-1/2})\right\|^2 \\
&\quad +C_{14}\sum_{n=3}^{l}\Delta t\ \left\|d_t\eta^n-\eta_t^{n-1/2}\right\|^2 \\
&\quad +C_{15}\sum_{n=3}^{l}\Delta t\ \left\|\bar{r}^n-r^{n-1/2}\right\|^2\ +\ C_{16}\sum_{n=2}^{l}\Delta t\ \|r^n-\mathcal{P}^n\|^2 \\
&\quad +C_{17}\sum_{i=0}^{2}\Delta t\ \left(\left\|\mathbf{E}^i\right\|^2\ +\ \left\|\mathbf{F}^i\right\|^2\ +\ \left\|D(\bar{\mathbf{E}}^i)\right\|^2\right)\ . \qquad (4.54)
\end{aligned}
$$

We now apply the interpolation properties of the approximating spaces to estimate the terms on the right hand side of (4.54). Using elements of order k for velocity, elements of order m for stress, and elements of order q for pressure, we have

$$
\begin{aligned}
\sum_{n=0}^{l}\Delta t\ \|\nabla\mathbf{\Lambda}^n\|^2\ +\ \sum_{n=2}^{l}\Delta t\ \|\nabla\mathbf{\Gamma}^n\|^2\ &\leq\ C\left(h^{2k}\sum_{n=0}^{l}\Delta t\ \|\mathbf{w}^n\|_{k+1}^2\ +\ h^{2m}\sum_{n=2}^{l}\Delta t\ \|\eta^n\|_{m+1}^2\right) \\
&\leq\ C\left(h^{2k}\ \|\!|\mathbf{w}|\!\|_{0,k+1}^2\ +\ h^{2m}\ \|\!|\eta|\!\|_{0,m+1}^2\right)\ . \qquad (4.55)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n=0}^{l}\Delta t\ \|\mathbf{\Lambda}^n\|^2\ +\ \sum_{n=0}^{l}\Delta t\ \|\mathbf{\Gamma}^n\|^2\ +\ \sum_{n=3}^{l}\Delta t\ \|r^n-\mathcal{P}^n\|^2 \\
&\qquad \leq\ C\left(h^{2k+2}\sum_{n=0}^{l}\Delta t\ \|\mathbf{w}^n\|_{k+1}^2\ +\ h^{2m+2}\sum_{n=0}^{l}\Delta t\ \|\eta^n\|_{m+1}^2\ +\ h^{2q+2}\sum_{n=3}^{l}\Delta t\ \|r^n\|_{q+1}^2\right) \\
&\qquad \leq\ C\left(h^{2k+2}\ \|\!|\mathbf{w}|\!\|_{0,k+1}^2\ +\ h^{2m+2}\ \|\!|\eta|\!\|_{0,m+1}^2\ +\ h^{2q+2}\ \|\!|r|\!\|_{0,q+1}^2\right)\ . \qquad (4.56)
\end{aligned}
$$

$$
\sum_{n=3}^{l}\Delta t\ \|d_t\mathbf{\Lambda}^n\|^2\ =\ \sum_{n=3}^{l}\Delta t\ \left\|\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}1\frac{\partial\Lambda}{\partial t}\,dt\right\|^2
$$

$$\leq \sum_{n=3}^{l} \Delta t \left(\frac{1}{\Delta t}\right)^2 \int_\Omega \left(\int_{t_{n-1}}^{t_n} 1\, dt\right) \left(\int_{t_{n-1}}^{t_n} \left(\frac{\partial \Lambda}{\partial t}\right)^2 dt\right) d\mathbf{x}$$

$$\leq Ch^{2k+2} \|\mathbf{w}_t\|^2_{0,k+1} \, , \tag{4.57}$$

and similarly,

$$\sum_{n=3}^{l} \Delta t \, \|d_t \mathbf{\Gamma}^n\|^2 \leq Ch^{2m+2} \|\eta_t\|^2_{0,m+1} \, . \tag{4.58}$$

Using (A.10),(A.11) and (A.12) we have

$$\sum_{n=3}^{l} \Delta t \left\|\nabla(\bar{\mathbf{w}}^n - \mathbf{w}^{n-1/2})\right\|^2 \leq C(\Delta t)^4 \|\mathbf{w}_{tt}\|^2_{0,1} \, , \tag{4.59}$$

$$\sum_{n=3}^{l} \Delta t \left\|d_t \mathbf{w}^n - \mathbf{w}_t^{n-1/2}\right\|^2 \leq C(\Delta t)^4 \|\mathbf{w}_{ttt}\|^2_{0,0} \, , \tag{4.60}$$

$$\sum_{n=3}^{l} \Delta t \left\|\bar{\eta}^n - \eta^{n-1/2}\right\|^2 \leq C(\Delta t)^4 \|\eta_{tt}\|^2_{0,0} \, , \tag{4.61}$$

$$\sum_{n=3}^{l} \Delta t \left\|\nabla(\bar{\eta}^n - \eta^{n-1/2})\right\|^2 \leq C(\Delta t)^4 \|\eta_{tt}\|^2_{0,1} \, , \tag{4.62}$$

$$\sum_{n=3}^{l} \Delta t \left\|d_t \eta^n - \eta_t^{n-1/2}\right\|^2 \leq C(\Delta t)^4 \|\eta_{ttt}\|^2_{0,0} \, , \tag{4.63}$$

$$\sum_{n=3}^{l} \Delta t \left\|\bar{r}^n - r^{n-1/2}\right\|^2 \leq C(\Delta t)^4 \|r_{tt}\|^2_{0,0} \, . \tag{4.64}$$

Substituting (4.55)-(4.64) into (4.54) gives the estimate

$$\begin{aligned} Re\hat{\lambda}\left(\left\|\mathbf{E}^l\right\|^2 - \left\|\mathbf{E}^2\right\|^2\right) &+ \left(\left\|\mathbf{F}^l\right\|^2 - \left\|\mathbf{F}^2\right\|^2\right) + \hat{\lambda}(1-\alpha)\sum_{n=3}^{l} \Delta t \, \|D(\bar{\mathbf{E}}^n)\|^2 + \frac{2}{\lambda}\sum_{n=3}^{l} \Delta t \, \|\bar{\mathbf{F}}^n\|^2 \\ &\leq C\sum_{n=3}^{l} \Delta t \, \|\mathbf{E}^n\|^2 + C\sum_{n=3}^{l} \Delta t \, \|\mathbf{F}^n\|^2 \\ &\quad + C\,(\Delta t)^4 \left(\|\mathbf{w}_{tt}\|^2_{0,1} + \|\mathbf{w}_{ttt}\|^2_{0,0} + \|\eta_{tt}\|^2_{0,1} + \|\eta_{ttt}\|^2_{0,0} + \|r_{tt}\|^2_{0,0}\right) \\ &\quad + C\left(h^{2k} \, \|\mathbf{w}\|^2_{0,k+1} + h^{2m} \, \|\eta\|^2_{0,m+1} + h^{2q+2} \, \|r\|^2_{0,q+1}\right) \\ &\quad + C\left(h^{2k+2} \, \|\mathbf{w}_t\|^2_{0,k+1} + h^{2m+2} \, \|\eta_t\|^2_{0,m+1}\right) \\ &\quad + C\sum_{i=0}^{2} \Delta t \left(\left\|\mathbf{E}^i\right\|^2 + \left\|\mathbf{F}^i\right\|^2 + \left\|D(\bar{\mathbf{E}}^i)\right\|^2\right) \, , \end{aligned} \tag{4.65}$$

where C denotes a constant independent of l, Δt, h. Thus, combining (4.65) with (3.15) and, for Δt sufficiently small, applying Gronwall's lemma to (4.65), estimate (4.7) follows.

■

Next we verify that the induction hypothesis (IH) holds.

Step 2. Verification of (IH1)
Assume that (IH) holds true for $n = 1, 2, \ldots, l-1$. By interpolation properties, inverse estimates and (4.7), we have that

$$\begin{aligned}
\left\|\mathbf{u}_h^l\right\|_\infty &\le \left\|\mathbf{u}_h^l - \mathbf{w}^l\right\|_\infty + \left\|\mathbf{w}^l\right\|_\infty \\
&\le \left\|\mathbf{E}^l\right\|_\infty + \left\|\Lambda^l\right\|_\infty + M \\
&\le C\, h^{-\frac{\acute{d}}{2}} \left\|\mathbf{E}^l\right\| + C\, h^{-\frac{\acute{d}}{2}} \left\|\Lambda^l\right\| + M \\
&\le C \left((\Delta t)^2\, h^{-\frac{\acute{d}}{2}} + h^{k-\acute{d}/2} + h^{m-\acute{d}/2} + h^{q+1-\acute{d}/2} \right) + M. \qquad (4.66)
\end{aligned}$$

Note that the expression $C \left((\Delta t)^2\, h^{-\frac{\acute{d}}{2}} + h^{k-\acute{d}/2} + h^{m-\acute{d}/2} + h^{q+1-\acute{d}/2} \right)$ is independent of l. Hence, if we set $k, m \ge \acute{d}/2$, $q \ge \acute{d}/2 - 1$, and choose $h, \Delta t$ such that

$$h^{k-\acute{d}/2}, h^{m-\acute{d}/2}, h^{q+1-\acute{d}/2} \le \frac{1}{C}, \quad \Delta t \le \frac{h^{\acute{d}/4}}{C^{1/2}}, \qquad (4.67)$$

then from (4.66)

$$\left\|\mathbf{u}_h^l\right\|_\infty \le M + 4\,.$$

Similarly it follows that $\left\|\tau_h^l\right\|_\infty \le M + 4$.

■

Step 3. Proof of the Theorem 2.
We have that

$$\begin{aligned}
\left\|\mathbf{u}^l - \mathbf{u}_h^l\right\|^2 + \left\|\tau^l - \tau_h^l\right\|^2 \;\le\; & 3\left\|\mathbf{u}^l - \mathbf{w}^l\right\|^2 + 3\left\|\mathbf{E}^l\right\|^2 + 3\left\|\Lambda^l\right\|^2 \\
& + 3\left\|\tau^l - \eta^l\right\|^2 + 3\left\|\mathbf{F}^l\right\|^2 + 3\left\|\Gamma^l\right\|^2 . \qquad (4.68)
\end{aligned}$$

Now, (4.1) follows from (4.68) using Corollary 1, Lemma 3, the approximation properties, and taking the maximum over l.

To establish (4.2), using (4.1)and (3.15), we have that

$$\sum_{n=1}^{N} \Delta t\, \|\bar{\mathbf{u}}_h^n - \bar{\mathbf{u}}^n\|^2 \;\le\; \sum_{n=1}^{N} \Delta t\, \|\mathbf{u}_h^n - \mathbf{u}^n\|^2 \;\le\; T\, \mathbf{F}_1^2(\Delta t, h)\,, \qquad (4.69)$$

with the same estimate also valid for $\sum_{n=1}^{N} \Delta t\, \|\bar{\tau}_h^n - \bar{\tau}^n\|^2$.

To estimate $\sum_{n=1}^{N} \Delta t\, \|\nabla(\bar{\mathbf{u}}_h^n - \bar{\mathbf{u}}^n)\|^2 \equiv \sum_{n=1}^{N} \Delta t\, \|D(\bar{\mathbf{u}}_h^n - \bar{\mathbf{u}}^n)\|^2$, note that

$$\begin{aligned}
\sum_{n=1}^{N} \Delta t\, \|D(\bar{\mathbf{u}}_h^n - \bar{\mathbf{u}}^n)\|^2 \;\le\; & 3\sum_{n=3}^{N} \Delta t\, \|D(\bar{\mathbf{u}}^n - \bar{\mathbf{w}}^n)\|^2 + 3\sum_{n=3}^{N} \Delta t\, \|D(\bar{\mathbf{\Lambda}}^n)\|^2 + 3\sum_{n=3}^{N} \Delta t\, \|D(\bar{\mathbf{E}}^n)\|^2 \\
& + \sum_{n=1}^{2} \Delta t\, \|D(\bar{\mathbf{u}}_h^n - \bar{\mathbf{u}}^n)\|^2 \,. \qquad (4.70)
\end{aligned}$$

The second, third, and fourth terms on the right hand side of (4.70) can be bounded using the interpolation properties, Lemma 3, and the initialization assumption (3.15), respectively. To bound the first term we proceed as follows. For simplicity, we assume N is even.

$$\begin{aligned}
\sum_{n=3}^{N} \Delta t \, \|D(\bar{\mathbf{u}}^n - \bar{\mathbf{w}}^n)\|^2 \;&\leq\; \sum_{n=3}^{N} \frac{\Delta t}{2} \left(\|D(\mathbf{u}^n - \mathbf{w}^n)\|^2 \,+\, \left\|D(\mathbf{u}^{n-1} - \mathbf{w}^{n-1})\right\|^2 \right) \\
&\leq\; \frac{3}{2} \sum_{i=1}^{(N-2)/2} \frac{2\Delta t}{6} \left(\left\|D(\mathbf{u}^{2i} - \mathbf{w}^{2i})\right\|^2 \,+\, 4 \left\|D(\mathbf{u}^{2i+1} - \mathbf{w}^{2i+1})\right\|^2 \right. \\
&\qquad\qquad \left. +\, \left\|D(\mathbf{u}^{2i+2} - \mathbf{w}^{2i+2})\right\|^2 \right) \\
&\quad \text{using Simpson's rule,} \\
&\leq\; \frac{3}{2} \left(\int_{t_2}^{t_N} \|D(\mathbf{u} - \mathbf{w})\|^2 \, dt \,+\, \frac{7}{72} (\Delta t)^4 \int_{t_2}^{t_N} |(\|D(\mathbf{u} - \mathbf{w})\|^2)_{tttt}| \, dt \right) \\
&\quad \text{using (2.46),} \\
&\leq\; C\,(\Delta t)^4 \int_{-5\Delta t/2}^{T} \left(\|\mathbf{w}_{tt}\|^2 \,+\, \|\eta_{tt}\|^2 \right) dt \,+\, \frac{7}{48} (\Delta t)^4 \int_{t_2}^{t_N} \left(\|\nabla(\mathbf{u} - \mathbf{w})\|^2 \right. \\
+\,4\,\|\nabla(\mathbf{u} - \mathbf{w})_t\|^2 \;&+\; 6\,\|\nabla(\mathbf{u} - \mathbf{w})_{tt}\|^2 \,+\, 4\,\|\nabla(\mathbf{u} - \mathbf{w})_{ttt}\|^2 \,+\, \left. \|\nabla(\mathbf{u} - \mathbf{w})_{tttt}\|^2 \right) dt \\
&\leq\; C\,(\Delta t)^4 \left(\int_{-5\Delta t/2}^{T} \left(\|\mathbf{w}_{tt}\|^2 \,+\, \|\eta_{tt}\|^2 \right) dt \right) \,+\, C\, \mathbf{F}_2^2(\Delta t)\,, \qquad (4.71)
\end{aligned}$$

where $\mathbf{F}_2(\Delta t)$ is defined in (4.3).

The stated result, (4.2), now follows.

■

We are now in a position to consider the error estimate for the pressure. Similar to (2.9) the approximation for the pressure p_h^n satisfies the equation

$$\begin{aligned}
Re(d_t \mathbf{u}_h^n, \mathbf{v}) \,+\, Re\, c(\tilde{\mathbf{u}}_h^n, \bar{\mathbf{u}}_h^n, \mathbf{v}) \,+\, 2(1-\alpha)(D(\bar{\mathbf{u}}_h^n), D(\mathbf{v})) \;&-\; (\bar{p}_h^n, \nabla \cdot \mathbf{v}) \,+\, (\bar{\tau}_h^n, D(\mathbf{v})) \\
&= (\bar{\mathbf{f}}^n, \mathbf{v})\,, \text{ for all } \mathbf{v} \in X_h \quad (4.72)
\end{aligned}$$

Corollary 2 *With the hypotheses of Theorem 2, we have that*

$$\begin{aligned}
\left(\sum_{n=1}^{N} \left(\left\|\mathbf{u}^{n-1/2} \,-\, \bar{\mathbf{u}}_h^n\right\|^2 \;+\; \left\|D(\mathbf{u}^{n-1/2} \,-\, \bar{\mathbf{u}}_h^n)\right\|^2 \right) \Delta t \right)^{1/2} &+ \left(\sum_{n=1}^{N} \left\|\tau^{n-1/2} \,-\, \bar{\tau}_h^n\right\|^2 \Delta t \right)^{1/2} \\
&\leq\; C\,(1 \,+\, T^{1/2})\; \mathbf{F}_1(\Delta t, h) \,+\, C\, \mathbf{F}_2(\Delta t) \\
&\quad +C\,(\Delta t)^2 \left(\|\mathbf{u}_{tt}\|_{0,0} \,+\, \|\nabla \mathbf{u}_{tt}\|_{0,0} \,+\, \|\tau_{tt}\|_{0,0} \right), \qquad (4.73) \\
\left(\sum_{n=1}^{N} \left\|p^{n-1/2} \,-\, \bar{p}_h^n\right\|^2 \Delta t \right)^{1/2} \;&\leq\; C\,(\Delta t)^{-1} T^{1/2}\; \mathbf{F}_1(\Delta t, h) \,+\, C\,(1 \,+\, T^{1/2})\; \mathbf{F}_1(\Delta t, h) \,+\, C\, \mathbf{F}_2(\Delta t) \\
&\quad +C\,(\Delta t)^2 \left(\|\mathbf{u}_{tt}\|_{0,0} \,+\, \|\nabla \mathbf{u}_{tt}\|_{0,0} \,+\, \|\tau_{tt}\|_{0,0} \right) \\
&\quad +C\,(\Delta t)^2 \left(\|\mathbf{u}_{ttt}\|_{0,0} \,+\, \|p_{tt}\|_{0,0} \,+\, \|\mathbf{f}_{tt}\|_{0,0} \right)
\end{aligned}$$

$$+C\left(h^{q+1}\,|||p|||_{0,q+1}\;+\;h^{q+1}(\Delta t)^{1/2}\left\|p^0\right\|_{q+1}\right)$$
$$+C\,(\Delta t)^2\left(\int_{-2\Delta t}^{T}\|\mathbf{u}_{tt}\|^2\;dt\right)^{1/2}. \tag{4.74}$$

Proof of Corollary 2:
The estimate (4.73) follows from (4.2), the triangle inequality and (A.10),(A.12).

To estimate the error in the pressure, let $\mathcal{P} \in Q_h$ be such that

$$\|p-\mathcal{P}\|\;\leq\;C_I h^{q+1}\|p\|_{q+1}\,. \tag{4.75}$$

From the discrete *inf-sup* condition (3.1), and using (A.10), we have

$$\begin{aligned}
\|\bar{\mathcal{P}}^n-\bar{p}_h^n\| \;&\leq\; \frac{1}{\beta}\sup_{\mathbf{v}\in X_h}\frac{|(\bar{\mathcal{P}}^n-\bar{p}_h^n,\nabla\cdot\mathbf{v})|}{\|\mathbf{v}\|_1}\\
&\leq\; \frac{1}{\beta}\sup_{\mathbf{v}\in X_h}\frac{|(p^{n-1/2}-\bar{\mathcal{P}}^n,\nabla\cdot\mathbf{v})|}{\|\mathbf{v}\|_1}\;+\;\frac{1}{\beta}\sup_{\mathbf{v}\in X_h}\frac{|(p^{n-1/2}-\bar{p}_h^n,\nabla\cdot\mathbf{v})|}{\|\mathbf{v}\|_1}\\
&\leq\; \frac{1}{\beta}\acute{d}^{1/2}\left(\left\|p^{n-1/2}-\bar{p}^n\right\|\;+\;\|\bar{p}^n-\bar{\mathcal{P}}^n\|\right)\;+\;\frac{1}{\beta}\sup_{\mathbf{v}\in X_h}\frac{|(p^{n-1/2}-\bar{p}_h^n,\nabla\cdot\mathbf{v})|}{\|\mathbf{v}\|_1}\\
&\leq\; \frac{C}{\beta}\acute{d}^{1/2}\left\{(\Delta t)^{3/2}\left(\int_{t_{n-1}}^{t_n}\|p_{tt}\|^2\;dt\right)^{1/2}\;+\;h^{q+1}\left(\|p^n\|_{q+1}\;+\;\left\|p^{n-1}\right\|_{q+1}\right)\right\}\\
&\quad+\frac{1}{\beta}\sup_{\mathbf{v}\in X_h}\frac{|(p^{n-1/2}-\bar{p}_h^n,\nabla\cdot\mathbf{v})|}{\|\mathbf{v}\|_1}\,.
\end{aligned} \tag{4.76}$$

Subtracting (4.72) from (2.9)implies

$$\begin{aligned}
(\bar{p}_h^n-p^{n-1/2},\nabla\cdot\mathbf{v}) \;&=\; (\mathbf{f}^{n-1/2}-\bar{\mathbf{f}}^n,\mathbf{v})\;-\;Re\,(\mathbf{u}_t^{n-1/2}-d_t\mathbf{u}^n,\mathbf{v})\;-\;Re\,(d_t(\mathbf{u}^n-\mathbf{u}_h^n),\mathbf{v})\\
&\quad-Re\,c(\mathbf{u}^{n-1/2}-\tilde{\mathbf{u}}^n,\mathbf{u}^{n-1/2},\mathbf{v})\;-\;Re\,c(\tilde{\mathbf{u}}^n-\tilde{\mathbf{u}}_h^n,\mathbf{u}^{n-1/2},\mathbf{v})\;-\;Re\,c(\tilde{\mathbf{u}}_h^n,\mathbf{u}^{n-1/2}-\bar{\mathbf{u}}_h^n,\mathbf{v})\\
&\quad-2(1-\alpha)(D(\mathbf{u}^{n-1/2}-\bar{\mathbf{u}}_h^n),D(\mathbf{v}))\;-\;(\tau^{n-1/2}-\bar{\tau}_h^n,D(\mathbf{v}))\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{|(p^{n-1/2}-\bar{p}_h^n,\nabla\cdot\mathbf{v})|}{\|\mathbf{v}\|_1}\;&\leq\;\left\|\mathbf{f}^{n-1/2}-\bar{\mathbf{f}}^n\right\|\;+\;Re\left\|\mathbf{u}_t^{n-1/2}-d_t\mathbf{u}^n\right\|\;+\;Re\,\|d_t(\mathbf{u}^n-\mathbf{u}_h^n)\|\\
&\quad+Re\,\acute{d}^{3/2}\,M\left\|\mathbf{u}^{n-1/2}-\tilde{\mathbf{u}}^n\right\|\;+\;Re\,\acute{d}^{3/2}\,M\;\|\tilde{\mathbf{u}}^n-\tilde{\mathbf{u}}_h^n\|\\
&\quad+\left(2K\acute{d}^{1/2}\,C_k\;+\;2(1-\alpha)\right)\left\|D(\mathbf{u}^{n-1/2}-\bar{\mathbf{u}}_h^n)\right\|\;+\;\left\|\tau^{n-1/2}-\bar{\tau}_h^n\right\|.
\end{aligned} \tag{4.77}$$

All the terms on the right hand side of (4.77) may be bounded in a similar manner as:

$$\left\|\mathbf{f}^{n-1/2}-\bar{\mathbf{f}}^n\right\|\;\leq\;C\,(\Delta t)^{3/2}\left(\int_{t_{n-1}}^{t_n}\|\mathbf{f}_{tt}\|^2\;dt\right)^{1/2},\;\text{using }(A.10), \tag{4.78}$$

$$\left\| \mathbf{u}_t^{n-1/2} - d_t \mathbf{u}^n \right\| \;\leq\; C\,(\Delta t)^{3/2} \left(\int_{t_{n-1}}^{t_n} \|\mathbf{u}_{ttt}\|^2 \, dt \right)^{1/2} , \text{ using } (A.11), \tag{4.79}$$

$$\left\| \mathbf{u}^{n-1/2} - \tilde{\mathbf{u}}^n \right\| \;\leq\; C\,(\Delta t)^{3/2} \left(\int_{t_{n-3}}^{t_{n-1/2}} \|\mathbf{u}_{tt}\|^2 \, dt \right)^{1/2} , \text{ using } (A.13), \tag{4.80}$$

$$\|\tilde{\mathbf{u}}^n - \tilde{\mathbf{u}}_h^n\| \;\leq\; 2\,\mathbf{F}_1(\Delta t, h)\,, \text{ using } (4.1), \tag{4.81}$$

$$\|d_t(\mathbf{u}^n - \mathbf{u}_h^n)\| \;=\; \frac{\left\| (\mathbf{u}^n - \mathbf{u}_h^n) \;-\; (\mathbf{u}^{n-1} - \mathbf{u}_h^{n-1}) \right\|}{\Delta t} \;\leq\; 2\,(\Delta t)^{-1}\,\mathbf{F}_1(\Delta t, h)\,, \text{ using } (4.1)(4.82)$$

As

$$\left\| p^{n-1/2} - \bar{p}_h^n \right\|^2 \;\leq\; 3\, \left\| p^{n-1/2} - \bar{p}^n \right\|^2 \;+\; 3\, \|\bar{p}^n - \bar{\mathcal{P}}^n\|^2 \;+\; 3\, \|\bar{\mathcal{P}}^n - \bar{p}_h^n\|^2 \,,$$

combining the estimates from (4.76)-(4.82), and using (4.73), the stated result (4.74) now follows. ■

Note that the estimate for the pressure is only first order with respect to the time discretization.

In concluding we again remark that the estimates in Theorem 2 and Corollary 2 are derived under the assumption that the solution to the continuous problem has the necessary regularity. For a discussion of the regularity assumption for the Navier-Stokes equations see [10].

A Appendix

On repeated integration by parts we have the following representations:

$$\mathbf{u}^n \;=\; \mathbf{u}(t_n) \;=\; \mathbf{u}^{n-1/2} + \int_{t_{n-1/2}}^{t_n} \mathbf{u}_t(\cdot, t)\, dt$$

$$= \mathbf{u}^{n-1/2} + \frac{\Delta t}{2}\mathbf{u}_t^{n-1/2} + \int_{t_{n-1/2}}^{t_n} \mathbf{u}_{tt}(\cdot, t)\,(t_n - t)\, dt \tag{A.1}$$

$$= \mathbf{u}^{n-1/2} + \frac{\Delta t}{2}\mathbf{u}_t^{n-1/2} + \frac{1}{2}\frac{(\Delta t)^2}{4}\mathbf{u}_{tt}^{n-1/2} + \frac{1}{2}\int_{t_{n-1/2}}^{t_n} \mathbf{u}_{ttt}(\cdot, t)\,(t_n - t)^2\, dt\,. \tag{A.2}$$

Also

$$\mathbf{u}^n \;=\; \mathbf{u}^{n-1} + \Delta t\, \mathbf{u}_t^{n-1} + \int_{t_{n-1}}^{t_n} \mathbf{u}_{tt}(\cdot, t)\,(t_n - t)\, dt \tag{A.3}$$

For $\mathbf{u}^{n-1}$,$\mathbf{u}^{n-2}$ and $\mathbf{u}^{n-3}$ we have

$$\mathbf{u}^{n-1} \;=\; \mathbf{u}(t_{n-1}) \;=\; \mathbf{u}^{n-1/2} - \frac{\Delta t}{2}\mathbf{u}_t^{n-1/2} + \int_{t_{n-1}}^{t_{n-1/2}} \mathbf{u}_{tt}(\cdot, t)\,(t - t_{n-1})\, dt \tag{A.4}$$

$$= \mathbf{u}^{n-1/2} \;-\; \frac{\Delta t}{2}\mathbf{u}_t^{n-1/2} + \frac{1}{2}\frac{(\Delta t)^2}{4}\mathbf{u}_{tt}^{n-1/2} - \frac{1}{2}\int_{t_{n-1}}^{t_{n-1/2}} \mathbf{u}_{ttt}(\cdot, t)\,(t - t_{n-1})^2\, dt\;, \tag{A.5}$$

$$\mathbf{u}^{n-2} \;=\; \mathbf{u}(t_{n-2}) \;=\; \mathbf{u}^{n-1/2} - \frac{3\,\Delta t}{2}\mathbf{u}_t^{n-1/2} + \int_{t_{n-2}}^{t_{n-1/2}} \mathbf{u}_{tt}(\cdot, t)\,(t - t_{n-2})\, dt \tag{A.6}$$

$$\mathbf{u}^{n-2} \;=\; \mathbf{u}^{n-1} - \Delta t\, \mathbf{u}_t^{n-1} + \int_{t_{n-2}}^{t_{n-1}} \mathbf{u}_{tt}(\cdot, t)\,(t - t_{n-2})\, dt\;, \tag{A.7}$$

$$\mathbf{u}^{n-3} \;=\; \mathbf{u}(t_{n-3}) \;=\; \mathbf{u}^{n-1/2} - \frac{5\,\Delta t}{2}\mathbf{u}_t^{n-1/2} + \int_{t_{n-3}}^{t_{n-1/2}} \mathbf{u}_{tt}(\cdot,t)\,(t-t_{n-3})\,dt \tag{A.8}$$

$$\mathbf{u}^{n-3} \;=\; \mathbf{u}^{n-1} - 2\Delta t\,\mathbf{u}_t^{n-1} + \int_{t_{n-3}}^{t_{n-1}} \mathbf{u}_{tt}(\cdot,t)\,(t-t_{n-3})\,dt\ . \tag{A.9}$$

Lemma 4

$$\left\|\bar{\mathbf{u}}^n - \mathbf{u}^{n-1/2}\right\|^2 \;\le\; \frac{1}{48}(\Delta t)^3 \int_{t_{n-1}}^{t_n} \|\mathbf{u}_{tt}\|^2\;dt\,. \tag{A.10}$$

Proof of Lemma 4:

$$\begin{aligned}
\left\|\bar{\mathbf{u}}^n - \mathbf{u}^{n-1/2}\right\|^2 &= \left\|\frac{1}{2}(\mathbf{u}^n + \mathbf{u}^{n-1}) - \mathbf{u}^{n-1/2}\right\|^2 \\
&= \frac{1}{4}\int_\Omega \left[\int_{t_{n-1/2}}^{t_n} \mathbf{u}_{tt}(\cdot,t)\,(t_n-t)\,dt \;+\; \int_{t_{n-1}}^{t_{n-1/2}} \mathbf{u}_{tt}(\cdot,t)\,(t-t_{n-1})\,dt\right]^2\;d\mathbf{x} \\
&\le \frac{1}{4}\int_\Omega 2\left[\left(\int_{t_{n-1/2}}^{t_n} \mathbf{u}_{tt}(\cdot,t)\,(t_n-t)\,dt\right)^2 \;+\; \left(\int_{t_{n-1}}^{t_{n-1/2}} \mathbf{u}_{tt}(\cdot,t)\,(t-t_{n-1})\,dt\right)^2\right]\;d\mathbf{x} \\
&\le \frac{1}{2}\int_\Omega \left[\int_{t_{n-1/2}}^{t_n} (\mathbf{u}_{tt}(\cdot,t))^2\;dt\;\int_{t_{n-1/2}}^{t_n} (t_n-t)^2\,dt\right. \\
&\qquad \left. + \int_{t_{n-1}}^{t_{n-1/2}} (\mathbf{u}_{tt}(\cdot,t))^2\;dt\;\int_{t_{n-1}}^{t_{n-1/2}} (t-t_{n-1})^2\,dt\right]\;d\mathbf{x} \\
&= \frac{1}{2}\int_\Omega \left[\frac{1}{3}\left(\frac{\Delta t}{2}\right)^3 \int_{t_{n-1/2}}^{t_n} (\mathbf{u}_{tt}(\cdot,t))^2\;dt \;+\; \frac{1}{3}\left(\frac{\Delta t}{2}\right)^3 \int_{t_{n-1}}^{t_{n-1/2}} (\mathbf{u}_{tt}(\cdot,t))^2\;dt\right]\;d\mathbf{x} \\
&= \frac{1}{48}(\Delta t)^3 \int_\Omega \int_{t_{n-1}}^{t_n} (\mathbf{u}_{tt}(\cdot,t))^2\;dt\,d\mathbf{x} \\
&= \frac{1}{48}(\Delta t)^3 \int_{t_{n-1}}^{t_n} \|\mathbf{u}_{tt}\|^2\;dt\,.
\end{aligned}$$

■

Lemma 5

$$\left\|d_t\mathbf{u}^n - \mathbf{u}_t^{n-1/2}\right\|^2 \;\le\; \frac{1}{1280}(\Delta t)^3 \int_{t_{n-1}}^{t_n} \|\mathbf{u}_{ttt}\|^2\;dt\,. \tag{A.11}$$

Proof of Lemma 5:

$$\begin{aligned}
\left\|d_t\mathbf{u}^n - \mathbf{u}_t^{n-1/2}\right\|^2 &= \left\|\frac{1}{\Delta t}(\mathbf{u}^n - \mathbf{u}^{n-1}) - \mathbf{u}_t^{n-1/2}\right\|^2 \\
&= \left(\frac{1}{4\,\Delta t}\right)^2 \int_\Omega \left[\int_{t_{n-1/2}}^{t_n} \mathbf{u}_{ttt}(\cdot,t)\,(t_n-t)^2\,dt \;+\; \int_{t_{n-1}}^{t_{n-1/2}} \mathbf{u}_{ttt}(\cdot,t)\,(t-t_{n-1})^2\,dt\right]^2\;d\mathbf{x} \\
&\le \left(\frac{1}{4\,\Delta t}\right)^2 \int_\Omega 2\left[\left(\int_{t_{n-1/2}}^{t_n} \mathbf{u}_{ttt}(\cdot,t)\,(t_n-t)^2\,dt\right)^2 \;+\; \left(\int_{t_{n-1}}^{t_{n-1/2}} \mathbf{u}_{ttt}(\cdot,t)\,(t-t_{n-1})^2\,dt\right)^2\right]\;d\mathbf{x}
\end{aligned}$$

$$
\begin{aligned}
&\leq 2\left(\frac{1}{4\,\Delta t}\right)^2 \int_\Omega \left[\int_{t_{n-1/2}}^{t_n} (\mathbf{u}_{ttt}(\cdot,t))^2 \; dt \; \int_{t_{n-1/2}}^{t_n} (t_n - t)^4 \, dt \right. \\
&\quad \left. + \int_{t_{n-1}}^{t_{n-1/2}} (\mathbf{u}_{ttt}(\cdot,t))^2 \; dt \; \int_{t_{n-1}}^{t_{n-1/2}} (t - t_{n-1})^4 \, dt \right] d\mathbf{x} \\
&= 2\left(\frac{1}{4\,\Delta t}\right)^2 \int_\Omega \left[\frac{1}{5}\left(\frac{\Delta t}{2}\right)^5 \int_{t_{n-1/2}}^{t_n} (\mathbf{u}_{ttt}(\cdot,t))^2 \; dt \; + \; \frac{1}{5}\left(\frac{\Delta t}{2}\right)^5 \int_{t_{n-1}}^{t_{n-1/2}} (\mathbf{u}_{ttt}(\cdot,t))^2 \; dt\right] d\mathbf{x} \\
&= \frac{1}{1280}(\Delta t)^3 \int_\Omega \int_{t_{n-1}}^{t_n} (\mathbf{u}_{ttt}(\cdot,t))^2 \; dt \, d\mathbf{x} \\
&= \frac{1}{1280}(\Delta t)^3 \int_{t_{n-1}}^{t_n} \|\mathbf{u}_{ttt}\|^2 \; dt \, .
\end{aligned}
$$

■

For the vector $\mathbf{u}$, $\mathbf{u}^{(i)}$, $i = 1, \ldots \acute{d}$, denotes the ith component of the vector.

Lemma 6

$$
\left\|\nabla(\bar{\mathbf{u}}^n - \mathbf{u}^{n-1/2})\right\|^2 \; \leq \; \frac{(\Delta t)^3}{48} \int_{t_{n-1}}^{t_n} \|\nabla \mathbf{u}_{tt}\|^2 \; dt \, . \qquad \text{(A.12)}
$$

Proof of Lemma 6:

$$
\begin{aligned}
\left\|\nabla(\bar{\mathbf{u}}^n - \mathbf{u}^{n-1/2})\right\|^2 &= \frac{1}{4}\int_\Omega \nabla\left\{\int_{t_{n-1/2}}^{t_n} \mathbf{u}_{tt}(\cdot,t)\,(t_n - t)\,dt \; + \; \int_{t_{n-1}}^{t_{n-1/2}} \mathbf{u}_{tt}(\cdot,t)\,(t - t_{n-1})\,dt\right\} \\
&\quad : \; \nabla\left\{\int_{t_{n-1/2}}^{t_n} \mathbf{u}_{tt}(\cdot,t)\,(t_n - t)\,dt \; + \; \int_{t_{n-1}}^{t_{n-1/2}} \mathbf{u}_{tt}(\cdot,t)\,(t - t_{n-1})\,dt\right\} \; d\mathbf{x} \\
&\quad \text{interchanging differentiation and integration} \\
&= \frac{1}{4}\int_\Omega \left\{\int_{t_{n-1/2}}^{t_n} \nabla\mathbf{u}_{tt}(\cdot,t)\,(t_n - t)\,dt \; + \; \int_{t_{n-1}}^{t_{n-1/2}} \nabla\mathbf{u}_{tt}(\cdot,t)\,(t - t_{n-1})\,dt\right\} \\
&\quad : \; \left\{\int_{t_{n-1/2}}^{t_n} \nabla\mathbf{u}_{tt}(\cdot,t)\,(t_n - t)\,dt \; + \; \int_{t_{n-1}}^{t_{n-1/2}} \nabla\mathbf{u}_{tt}(\cdot,t)\,(t - t_{n-1})\,dt\right\} \; d\mathbf{x} \\
&= \sum_{i,j=1}^{\acute{d}} \; \frac{1}{4}\int_\Omega \left(\int_{t_{n-1/2}}^{t_n} \mathbf{u}^i_{ttx_j}(\cdot,t)\,(t_n - t)\,dt \; + \; \int_{t_{n-1}}^{t_{n-1/2}} \mathbf{u}^i_{ttx_j}(\cdot,t)\,(t - t_{n-1})\,dt\right)^2 \; d\mathbf{x} \\
&\leq \sum_{i,j=1}^{\acute{d}} \; \frac{1}{4}\int_\Omega 2\left[\left(\int_{t_{n-1/2}}^{t_n} \mathbf{u}^i_{ttx_j}(\cdot,t)\,(t_n - t)\,dt\right)^2 \; + \; \left(\int_{t_{n-1}}^{t_{n-1/2}} \mathbf{u}^i_{ttx_j}(\cdot,t)\,(t - t_{n-1})\,dt\right)^2\right] \; d\mathbf{x} \\
&\leq \sum_{i,j=1}^{\acute{d}} \; \frac{1}{4}\int_\Omega 2\left[\int_{t_{n-1/2}}^{t_n} \left(\mathbf{u}^i_{ttx_j}(\cdot,t)\right)^2 \; dt \; \int_{t_{n-1/2}}^{t_n} (t_n - t)^2 \, dt \right. \\
&\quad \left. + \; \int_{t_{n-1}}^{t_{n-1/2}} \left(\mathbf{u}^i_{ttx_j}(\cdot,t)\right)^2 \; dt \; \int_{t_{n-1}}^{t_{n-1/2}} (t - t_{n-1})^2 \, dt\right] \; d\mathbf{x} \\
&= \sum_{i,j=1}^{\acute{d}} \; \frac{1}{4}\int_\Omega 2\,\frac{1}{3}\left(\frac{\Delta t}{2}\right)^3 \int_{t_{n-1}}^{t_n} \left(\mathbf{u}^i_{ttx_j}(\cdot,t)\right)^2 \; dt \; d\mathbf{x}
\end{aligned}
$$

$$= \frac{(\Delta t)^3}{48} \int_{t_{n-1}}^{t_n} \|\nabla \mathbf{u}_{tt}\|^2 \, dt \, .$$

■

Lemma 7

$$\left\|\tilde{\mathbf{u}}^n - \mathbf{u}^{n-1/2}\right\|^2 \;\le\; \frac{39}{8}(\Delta t)^3 \int_{t_{n-3}}^{t_{n-1/2}} \|\mathbf{u}_{tt}\|^2 \, dt \, . \tag{A.13}$$

Proof of Lemma 7:

$$\begin{aligned}
\left\|\tilde{\mathbf{u}}^n - \mathbf{u}^{n-1/2}\right\|^2 &= \left\|(\mathbf{u}^{n-1} + \frac{1}{2}\mathbf{u}^{n-2} - \frac{1}{2}\mathbf{u}^{n-3}) - \mathbf{u}^{n-1/2}\right\|^2 \\
&= \int_\Omega \left[\int_{t_{n-1}}^{t_{n-1/2}} \mathbf{u}_{tt}(\cdot,t)\,(t - t_{n-1})\,dt \;+\; \frac{1}{2}\int_{t_{n-2}}^{t_{n-1/2}} \mathbf{u}_{tt}(\cdot,t)\,(t - t_{n-2})\,dt \right. \\
&\quad \left. -\,\frac{1}{2}\int_{t_{n-3}}^{t_{n-1/2}} \mathbf{u}_{tt}(\cdot,t)\,(t - t_{n-3})\,dt\right]^2 \, d\mathbf{x} \\
&\le \int_\Omega 3\left[\left(\int_{t_{n-1}}^{t_{n-1/2}} \mathbf{u}_{tt}(\cdot,t)\,(t - t_{n-1})\,dt\right)^2 \;+\; \frac{1}{4}\left(\int_{t_{n-2}}^{t_{n-1/2}} \mathbf{u}_{tt}(\cdot,t)\,(t - t_{n-2})\,dt\right)^2 \right. \\
&\quad \left. +\,\frac{1}{4}\left(\int_{t_{n-3}}^{t_{n-1/2}} \mathbf{u}_{tt}(\cdot,t)\,(t - t_{n-3})\,dt\right)^2\right] \, d\mathbf{x} \\
&\le 3\int_\Omega \left[\int_{t_{n-1}}^{t_{n-1/2}} (\mathbf{u}_{tt}(\cdot,t))^2 \, dt \; \int_{t_{n-1}}^{t_{n-1/2}} (t - t_{n-1})^2\,dt \right. \\
&\quad +\,\frac{1}{4}\int_{t_{n-2}}^{t_{n-1/2}} (\mathbf{u}_{tt}(\cdot,t))^2 \, dt \; \int_{t_{n-2}}^{t_{n-1/2}} (t - t_{n-2})^2\,dt \\
&\quad \left. +\,\frac{1}{4}\int_{t_{n-3}}^{t_{n-1/2}} (\mathbf{u}_{tt}(\cdot,t))^2 \, dt \; \int_{t_{n-3}}^{t_{n-1/2}} (t - t_{n-3})^2\,dt\right] \, d\mathbf{x} \\
&= 3\int_\Omega \left[\frac{1}{3}\left(\frac{\Delta t}{2}\right)^3 \int_{t_{n-1}}^{t_{n-1/2}} (\mathbf{u}_{tt}(\cdot,t))^2 \, dt \;+\; \frac{1}{12}\left(\frac{3\Delta t}{2}\right)^3 \int_{t_{n-2}}^{t_{n-1/2}} (\mathbf{u}_{tt}(\cdot,t))^2 \, dt \right. \\
&\quad \left. +\;\frac{1}{12}\left(\frac{5\Delta t}{2}\right)^3 \int_{t_{n-3}}^{t_{n-1/2}} (\mathbf{u}_{tt}(\cdot,t))^2 \, dt\right] \, d\mathbf{x} \\
&\le \frac{39}{8}(\Delta t)^3 \int_\Omega \int_{t_{n-3}}^{t_{n-1/2}} (\mathbf{u}_{tt}(\cdot,t))^2 \, dt\, d\mathbf{x} \\
&= \frac{39}{8}(\Delta t)^3 \int_{t_{n-3}}^{t_{n-1/2}} \|\mathbf{u}_{tt}\|^2 \, dt \, .
\end{aligned}$$

■

Lemma 8

$$\left\|\tilde{\mathbf{u}}^n - \mathbf{u}^{n-1}\right\|^2 \;\le\; \frac{1}{4}\Delta t \int_{t_{n-3}}^{t_{n-2}} \|\mathbf{u}_t\|^2 \, dt \, . \tag{A.14}$$

Proof of Lemma 8:

$$
\begin{aligned}
\left\|\tilde{\mathbf{u}}^n - \mathbf{u}^{n-1}\right\|^2 &= \left\|\left(\mathbf{u}^{n-1} + \frac{1}{2}\mathbf{u}^{n-2} - \frac{1}{2}\mathbf{u}^{n-3}\right) - \mathbf{u}^{n-1}\right\|^2 = \left\|\frac{1}{2}(\mathbf{u}^{n-2} - \mathbf{u}^{n-3})\right\|^2 \\
&= \frac{1}{4}\int_\Omega \left[\int_{t_{n-3}}^{t_{n-2}} 1\, \mathbf{u}_t(\cdot,t)\, dt\right]^2 d\mathbf{x} \\
&\le \frac{1}{4}\int_\Omega \left[\int_{t_{n-3}}^{t_{n-2}} 1\, dt \int_{t_{n-3}}^{t_{n-2}} (\mathbf{u}_t(\cdot,t))^2\, dt\right] d\mathbf{x} \\
&= \frac{1}{4}\Delta t \int_{t_{n-3}}^{t_{n-2}} \|\mathbf{u}_t(\cdot,t)\|^2\, dt\,.
\end{aligned}
$$

■

B Appendix B

In this section we give an example of an initialization procedure for the Crank-Nicolson approximation described in (3.16),(3.17). There are three steps to the procedure:
Step 1. Apply a single step implicit Euler approximation to approximate $\mathbf{u}_h^2$, τ_h^2, (i.e. $\mathbf{z}_h^2$, θ_h^2).
Step 2. Apply a single step Crank-Nicolson approximation to compute $\mathbf{u}_h^2$, τ_h^2 .
Step3. Interpolate to determine $\mathbf{u}_h^1$, τ_h^1 .

We introduce the following notation:

$$
\begin{aligned}
d_{2t}f &:= \frac{f(t_n) - f(t_{n-2})}{2\Delta t} && \text{(B.1)} \\
\breve{f}^n &:= \frac{f^n + f^{n-2}}{2} && \text{(B.2)} \\
B_2(\mathbf{u},\tau,\sigma) &:= (\mathbf{u}\cdot\nabla\tau,\sigma)_h + \langle\tau^+ - \tau^-, \sigma^+\rangle_{h,\mathbf{u}}\,. && \text{(B.3)}
\end{aligned}
$$

Similar to (3.13) for $B(\mathbf{u},\tau,\tau)$, we have

$$
B_2(\mathbf{u},\tau,\tau) = -\frac{1}{2}(\nabla\cdot\mathbf{u}\ \tau,\tau)_h + \frac{1}{2}\langle\langle\tau^+ - \tau^-\rangle\rangle_{h,\mathbf{u}}^2\,. \tag{B.4}
$$

For the initial data we take $\mathbf{u}_h^0 = \mathcal{U}^0$, $\tau_h^0 = \mathcal{T}^0$ which implies that $\|\mathbf{E}^0\| = \|\mathbf{F}^0\| = 0$.

<u>**Initialization**</u>
Step 1. *For* $\mathbf{z}_h^0 = \mathbf{u}_h^0$, $\theta_h^0 = \tau_h^0$, *find* $\mathbf{z}_h^2 \in Z_h$, $\theta_h^2 \in S_h$ *such that*

$$
\begin{aligned}
Re(d_{2t}\mathbf{z}_h^2,\mathbf{v}) &+ Re\, c(\mathbf{u}_h^0,\mathbf{z}_h^2,\mathbf{v}) + 2(1-\alpha)(D(\mathbf{z}_h^2),D(\mathbf{v})) + (\theta_h^2,D(\mathbf{v})) = (\mathbf{f}^2,\mathbf{v})\,,\ \mathbf{v}\in Z_h && \text{(B.5)} \\
\frac{1}{\lambda}(\theta_h^2,\sigma) &+ (d_{2t}\theta_h^2,\sigma) + B(\mathbf{u}_h^0,\theta_h^2,\sigma) - \hat{\lambda}(D(\mathbf{z}_h^2),\sigma) + (g_a(\tau_h^0,\nabla\mathbf{z}_h^2),\sigma) = 0\,,\ \sigma\in S_h\,. && \text{(B.6)}
\end{aligned}
$$

Step 2. *For* $\mathbf{z}_h^1 = (\mathbf{z}_h^2 + \mathbf{z}_h^0)/2$, $\theta_h^1 = (\theta_h^2 + \theta_h^0)/2$, *find* $\mathbf{u}_h^2 \in Z_h$, $\tau_h^2 \in S_h$ *such that*

$$
\begin{aligned}
Re(d_{2t}\mathbf{u}_h^2,\mathbf{v}) &+ Re\, c(\mathbf{z}_h^1,\breve{\mathbf{u}}_h^2,\mathbf{v}) + 2(1-\alpha)(D(\breve{\mathbf{u}}_h^2),D(\mathbf{v})) + (\breve{\tau}_h^2,D(\mathbf{v})) = (\mathbf{f}^1,\mathbf{v})\,,\ \mathbf{v}\in Z_h && \text{(B.7)} \\
\frac{1}{\lambda}(\breve{\tau}_h^2,\sigma) &+ (d_{2t}\tau_h^2,\sigma) + B_2(\mathbf{z}_h^1,\breve{\tau}_h^2,\sigma) - \hat{\lambda}(D(\breve{\mathbf{u}}_h^2),\sigma) + (g_a(\theta_h^1,\nabla\mathbf{z}_h^2),\sigma) = 0\,,\ \sigma\in S_h && \text{(B.8)}
\end{aligned}
$$

Step 3. $\mathbf{u}_h^1 \,=\, (\mathbf{u}_h^2 + \mathbf{u}_h^0)/2$, $\theta_h^1 \,=\, (\theta_h^2 + \theta_h^0)/2$.

For the error in **Step 1** of the initialization we have:

Lemma 9 *For $\mathbf{z}_h^2$, θ_h^2 given by (B.5),(B.6)and $\mathbf{w}$, η satisfying (2.25),(2.26) we have that*

$$\left\|\mathbf{z}_h^2 \,-\, \mathbf{w}(2\Delta t)\right\|^2 \,+\, \left\|\theta_h^2 \,-\, \eta(2\Delta t)\right\|^2 \;\le\; \mathcal{E}_2^1(\Delta t, h) \ , \tag{B.9}$$

where,

$$\begin{aligned}
\mathcal{E}_2^1(\Delta t, h) \;&=\; C\,\Delta t\left(h^{2k}\left\|\mathbf{w}^0\right\|_{k+1}^2 \,+\, h^{2m}\left\|\eta^0\right\|_{m+1}^2 \,+\, h^{2q+2}\left\|r^0\right\|_{q+1}^2\right) \\
&\quad + C\,(\Delta t)^3\left(\left\|\mathbf{w}_t^0\right\|^2 \,+\, \left\|\nabla\mathbf{w}_t^0\right\|^2 \,+\, \left\|\mathbf{w}_{tt}^0\right\|^2 \,+\, \left\|\eta_t^0\right\|^2 \,+\, \left\|\eta_{tt}^0\right\|^2\right) \\
&\quad + C\,\Delta t\left(h^{2k+2}\left\|w_t^0\right\|_{k+1}^2 \,+\, h^{2m+2}\left\|\eta_t^0\right\|_{m+1}^2\right) .
\end{aligned} \tag{B.10}$$

Proof: Proceeding in a analogous fashion to above (see also [7]) with

$$\mathbf{E}_z^n := \mathcal{U}^n - \mathbf{z}_h^n, \qquad\qquad \mathbf{F}_z^n := \mathcal{T}^n - \theta_h^n,$$

and noting that

$$\mathbf{z}_h^0 \,=\, \mathbf{u}_h^0 \,=\, \mathcal{U}^0, \quad \theta_h^0 \,=\, \tau_h^0 \,=\, \mathcal{T}^0, \quad \text{and} \;\; D(\mathbf{E}_z^0) \,=\, \mathbf{0}\,,$$

we obtain

$$\begin{aligned}
Re\hat{\lambda}\left\|\mathbf{E}_z^2\right\|^2 \;&+\; \left\|\mathbf{F}_z^2\right\|^2 \,+\, \Delta t\,\hat{\lambda}(1-\alpha)\left\|D(\mathbf{E}_z^2)\right\|^2 \,+\, \Delta t\frac{2}{\lambda}\left\|\mathbf{F}_z^2\right\|^2 \\
\le&\;\; C_1\Delta t\left\|\mathbf{E}_z^2\right\|^2 \,+\, C_2\Delta t\left\|\mathbf{F}_z^2\right\|^2 \\
&+ C_3\Delta t\left(\left\|\Lambda^0\right\|^2 \,+\, \left\|\nabla\Lambda^0\right\|^2 \,+\, \left\|\nabla\Lambda^2\right\|^2\right) \,+\, C_4\Delta t\left\|d_{2t}\Lambda^2\right\|^2 \\
&+ C_5\Delta t\left(\left\|\Gamma^0\right\|^2 \,+\, \left\|\Gamma^2\right\|^2 \,+\, \left\|\nabla\Gamma^2\right\|^2\right) \,+\, C_6\Delta t\left\|d_{2t}\Gamma^2\right\|^2 \\
&+ C_7\,\Delta t\left(\left\|\mathbf{w}^0 - \tilde{\mathbf{w}}^2\right\|^2 \,+\, \left\|\nabla(\mathbf{w}^0 - \tilde{\mathbf{w}}^2)\right\|^2 \,+\, \left\|\eta^0 - \tilde{\eta}^2\right\|^2\right) \\
&+ C_8\,\Delta t\left(\left\|d_{2t}\mathbf{w}^2 - \mathbf{w}_t^2\right\|^2 \,+\, \left\|d_{2t}\eta^2 - \eta_t^2\right\|^2\right) \\
&+ C_9\,\Delta t\left\|r^2 - \mathcal{P}^2\right\|^2 .
\end{aligned} \tag{B.11}$$

Using $\mathbf{z}_h^2 \,-\, \mathbf{w}(2\Delta t) \,=\, \mathbf{E}_z^2 \,+\, \Lambda^2$, $\theta_h^2 \,-\, \eta(2\Delta t) \,=\, \mathbf{F}_z^2 \,+\, \Gamma^2$, and the approximation properties, the stated result follows.

■

Next we estimate the error in u_h^2, τ_h^2 generated in **Step 2**.

Note in this step a modified B operator, B_2 is used. We have that

$$|(\nabla\cdot\mathbf{z}_h^1\,\tau,\tau)| \;\;\le\;\; \left\|\nabla\cdot\mathbf{z}_h^1\,\tau\right\|\,\|\tau\|$$

$$
\begin{aligned}
&\le \acute{d}\, \left\|\nabla \mathbf{z}_h^1\right\|_\infty \|\tau\|^2 \\
&\le \acute{d}\, C_I h^{-1} \left\|\mathbf{z}_h^1\right\|_\infty \|\tau\|^2 \\
&\le \acute{d}\, C_I\, K\, h^{-1} \|\tau\|^2 .
\end{aligned}
$$

Therefore, from (B.4),

$$
B_2(\mathbf{z}_h^1, \tau, \tau) \;\ge\; -\frac{1}{2}\acute{d}\, C_I\, K\, h^{-1} \|\tau\|^2 + \frac{1}{2}\langle\langle \tau^+ - \tau^- \rangle\rangle_{h,\mathbf{u}}^2 . \tag{B.12}
$$

Lemma 10 *For $\mathbf{u}_h^2$, τ_h^2 given by (B.7),(B.8), $\mathbf{w}$, η satisfying (2.25),(2.26), and $\mathbf{E}^2$, $\mathbf{F}^2$ defined in (4.4),(4.5), we have that*

$$
\left\|\mathbf{E}^2\right\|^2 + \left\|\mathbf{F}^2\right\|^2 + \Delta t \left\|D(\mathbf{E}^2)\right\|^2 \;\le\; \mathcal{E}_2(\Delta t, h) , \tag{B.13}
$$

where,

$$
\begin{aligned}
\mathcal{E}_2(\Delta t, h) \;=\;& C\,\Delta t \left(h^{2k} \left\|\mathbf{w}^0\right\|_{k+1}^2 + h^{2m} \left\|\eta^0\right\|_{m+1}^2 + h^{2q+2} \left\|r^0\right\|_{q+1}^2 \right) \\
&+ C\,\Delta t \left(h^{2k+2} \left\|\mathbf{w}_t^0\right\|_{k+1}^2 + h^{2m+2} \left\|\eta_t^0\right\|_{m+1}^2 \right) \\
&+ C\,(\Delta t)^4 \left(\left\|\mathbf{w}_t^0\right\|^2 + \left\|\nabla \mathbf{w}_t^0\right\|^2 + \left\|\mathbf{w}_{tt}^0\right\|^2 + \left\|\eta_t^0\right\|^2 + \left\|\eta_{tt}^0\right\|^2 \right) \\
&+ C\,(\Delta t)^5 \left(\left\|\nabla \mathbf{w}_{tt}^0\right\|^2 + \left\|\eta_{tt}^0\right\|^2 + \left\|\nabla \eta_{tt}^0\right\|^2 + \left\|\nabla \mathbf{w}_{ttt}^0\right\|^2 + \left\|\eta_{ttt}^0\right\|^2 \right) .
\end{aligned} \tag{B.14}
$$

Proof: Proceeding in an analogous fashion to the general case with the $B_2(\mathbf{z}_h^1, \breve{\mathbf{F}}^2, \breve{\mathbf{F}}^2)$ term on the left hand side of the equation bounded using (B.12), we obtain

$$
\begin{aligned}
Re\,\hat{\lambda} \left\|\mathbf{E}^2\right\|^2 &+ \left(1 + \frac{4\Delta t}{\lambda} - \frac{1}{2}\acute{d}\, C_I K h^{-1} \Delta t\right) \left\|\mathbf{F}^2\right\|^2 + \hat{\lambda}(1-\alpha)\Delta t \left\|D(\mathbf{E}^2)\right\|^2 \\
\le\;& C_1 \Delta t \left\|\mathbf{E}^2\right\|^2 + C_2 \Delta t \left\|\mathbf{F}^2\right\|^2 \\
&+ C_3 \Delta t \left(\left\|\Lambda^0\right\|^2 + \left\|\nabla \Lambda^0\right\|^2 + \left\|\Lambda^2\right\|^2 + \left\|\nabla \Lambda^2\right\|^2 \right) + C_4 \Delta t \left\|d_{2t}\Lambda^2\right\|^2 \\
&+ C_5 \Delta t \left(\left\|\Gamma^0\right\|^2 + \left\|\nabla \Gamma^0\right\|^2 + \left\|\Gamma^2\right\|^2 + \left\|\nabla \Gamma^2\right\|^2 \right) + C_6 \Delta t \left\|d_{2t}\Gamma^2\right\|^2 \\
&+ C_7\, \Delta t \left(\left\|\mathbf{z}_h^1 - \tilde{\mathbf{w}}^1\right\|^2 + \left\|\theta_h^1 - \tilde{\eta}^1\right\|^2 \right) \\
&+ C_8\, \Delta t \left(\left\|\nabla(\breve{\mathbf{w}}^2 - \mathbf{w}^1)\right\|^2 + \left\|\breve{\eta}^2 - \eta^1\right\|^2 + \left\|\nabla(\breve{\eta}^2 - \eta^1)\right\|^2 \right) \\
&+ C_9\, \Delta t \left(\left\|d_{2t}\mathbf{w}^2 - \mathbf{w}_t^1\right\|^2 + \left\|d_{2t}\eta^2 - \eta_t^1\right\|^2 \right) \\
&+ C_{10}\, \Delta t \left\|r^1 - \mathcal{P}^1\right\|^2 .
\end{aligned} \tag{B.15}
$$

To estimate $\left\|\mathbf{z}_h^1 - \tilde{\mathbf{w}}^1\right\|^2$ (analogously, $\left\|\theta_h^1 - \tilde{\eta}^1\right\|^2$), we have

$$
\left\|\mathbf{z}_h^1 - \tilde{\mathbf{w}}^1\right\|^2 \;\le\; 2 \left\|\mathbf{z}_h^1 - \frac{1}{2}(\mathbf{w}^0 + \mathbf{w}^2)\right\|^2 + 2 \left\|\frac{1}{2}(\mathbf{w}^0 + \mathbf{w}^2) - \tilde{\mathbf{w}}^1\right\|^2 .
$$

Now,

$$2\left\|\mathbf{z}_h^1 - \frac{1}{2}(\mathbf{w}^0 + \mathbf{w}^2)\right\|^2 \;\leq\; \left\|\mathbf{z}_h^0 - \mathbf{w}^0\right\|^2 + \left\|\mathbf{z}_h^2 - \mathbf{w}^2\right\|^2 \tag{B.16}$$

$$\leq\; \left\|\Lambda^0\right\|^2 + \mathcal{E}_2^1(\Delta t, h)\,. \tag{B.17}$$

For $\left\|\frac{1}{2}(\mathbf{w}^0 + \mathbf{w}^2) - \tilde{\mathbf{w}}^1\right\|^2$, using $\tilde{\mathbf{w}}^1 = \mathbf{w}(\cdot, \Delta t/2) + \mathbf{w}(\cdot, -\Delta t/2)/2 - \mathbf{w}(\cdot, -3\Delta t/2)/2$ and expanding about $t = \Delta t$ we have that

$$\left\|\frac{1}{2}(\mathbf{w}^0 + \mathbf{w}^2) - \tilde{\mathbf{w}}^1\right\|^2 \;\leq\; C\,(\Delta t)^3 \int_{-5\Delta t/2}^{2\Delta t} \|\mathbf{w}_{tt}\|^2 \, dt \;\leq\; C\,(\Delta t)^4 \left\|\mathbf{w}_{tt}^0\right\|^2 \;. \tag{B.18}$$

Estimates for $\|\nabla(\breve{\mathbf{w}}^2 - \mathbf{w}^1)\|^2$, $\|\breve{\eta}^2 - \eta^1\|^2$, $\|\nabla(\breve{\eta}^2 - \eta^1)\|^2$ follow as in (A.10),(A.12), and $\|d_{2t}\mathbf{w}^2 - \mathbf{w}_t^1\|^2$, $\|d_{2t}\eta^2 - \eta_t^1\|^2$ from (A.11).

The terms $\|d_{2t}\Lambda^2\|^2$, $\|d_{2t}\Gamma^2\|^2$ are estimated as in (4.57).

On combining the above with the approximation properties, estimate (B.13) follows.

■

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