Adaptive boundary element method for the exterior Stokes problem in three dimensions

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Abstract

We present an adaptive refinement strategy for the h-version of the boundary element method with weakly singular operators on surfaces. The model problem deals with the exterior Stokes problem, and thus considers vector functions. Our error indicators are computed by local projections onto one-dimensional subspaces defined by mesh refinement. These indicators measure the error separately for the vector components and allow for component independent adaption. Assuming a saturation condition the indicators give rise to an efficient and reliable error estimator. Also we describe how to deal with meshes containing quadrilaterals which are not shape regular. The theoretical results are underlined by numerical experiments. To justify the saturation assumption, in an appendix we prove optimal lower a priori error estimates for edge singularities on uniform and graded meshes.

Key words: Stokes problem, a posteriori error estimator, boundary element method, weakly singular operator
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1 Introduction and formulation of the problem

In this paper, we continue the investigation of adaptive strategies for Galerkin approximation based on indicators related to subspace decompositions. For initial ideas and the finite element method (FEM) see [12, 2]. For extensions to the boundary element method (BEM) with weakly singular and hypersingular operators in two and three dimensions, see [18, 20, 19, 17]. Here, we analyze this strategy for vector functions in three dimensions and the weakly singular operator. The Stokes problem serves as the model situation. Weakly singular operators act on the dual of the trace space of H^1-functions. This makes the analysis more technical than that for hypersingular operators (which act on the trace space of H^1-functions). This difference in

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the analysis applies to the three-dimensional situation since in two dimensions one may simply use differentiation and integration (on curves with respect to the arc length) to map between both energy spaces. For the three-dimensional case and weakly singular operators (and subspace decomposition based indicators) we only know of the references [18, 20]. In [18], however, no proofs for the weakly singular operator on surfaces are given, and in [20] only the scalar situation for rectangular quasi-uniform meshes is analyzed. Here, we give an analysis for vector functions on triangular/quadrilateral meshes. This analysis includes distorted quadrilateral elements if appropriate subspace decompositions are considered. We need to assume a saturation property. For a typical edge singularity, in an appendix we prove asymptotically optimal lower error estimates, analyzing the approximation on uniform and graded rectangular meshes. Using these lower bounds one can prove that the saturation property is satisfied for a sufficient refinement of the mesh. The error indicators we propose give local information on elements for different refinement directions and for the vector components separately. Our analysis of the stability of the subspace decompositions is based in part on [16] where the p-version for the weakly singular operator (of the Laplacian) is studied. There, the focus is on the p-version and rectangular quasi-uniform meshes. Here, we elaborate all the mesh dependent details and consider meshes consisting of quadrilateral and triangular elements.

In [6] several a posteriori error estimates for the BEM are studied. In particular, efficiency of a two-level estimator on curves is proved. The proof can be generalized to show efficiency of our estimator on surfaces (as has been indicated by one referee) for the case of the enrichment space $\mathbf{T}$ (see (2.4)) comprised of piecewise constants. In this paper we base our analysis on the additive Schwarz framework which is not restricted, a priori, to $\mathbf{T}$ being piecewise constant functions.

Let us describe our model problem. In what follows $\Gamma \subset \mathbf{R}^3$ can be an open or closed piecewise smooth obstacle. For ease of presentation, and since we will present numerical results only for a special situation, we restrict our presentation to an open plane polygonal screen. The homogeneous exterior Stokes problem reads as follows: *Find a velocity field* $\mathbf{u}$ *and a pressure field* p *such that*

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p &= 0 \quad \text{in } \Omega_\Gamma = \mathbf{R}^3 \setminus \bar{\Gamma}, \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in } \Omega_\Gamma \text{ and} \\ \mathbf{u} &= \mathbf{g} \quad \text{on } \Gamma. \end{aligned} \tag{1.1}$$

where ν is the given constant viscosity of the fluid. The pressure p is not unique. Extending Γ to a smooth closed surface $\partial\Omega$ with interior domain Ω one finds that p is determined within Ω only up to a constant. In the exterior domain $\Omega' := \mathbf{R}^3 \setminus \bar{\Omega}$, p is unique when requiring an appropriate decay condition. For uniqueness of $\mathbf{u}$ one also needs a decay condition. Following Wendland & Zhu [25] (see also, e.g., [15]) we incorporate such a condition by requiring $p \in L^2(\Omega_\Gamma)$ and assuming finite energy of the velocity in a weighted space, $\mathbf{u} \in \mathbf{W}^1(\Delta, \Omega_\Gamma) := (W^1(\Delta, \Omega_\Gamma))^3$, where

$$W^1(\Delta, \Omega_\Gamma) := \{u \in W^1(\Omega_\Gamma);\ \sqrt{1+r^2}\Delta u \in L^2(\Omega_\Gamma)\}$$

with

$$W^1(\Omega_\Gamma) := \{u;\ \frac{u}{\sqrt{1+r^2}} \in L^2(\Omega_\Gamma), \frac{\partial u}{\partial x_i} \in L^2(\Omega_\Gamma), r = |x|, i = 1, 2, 3\}.$$

The fundamental solution of (1.1) is given by (with identity $\mathbf{I}$)

$$\begin{array}{rcl} \mathbf{E}(x,y) & = & \frac{1}{8\pi\nu}\left(\frac{1}{|x-y|}\mathbf{I}+\frac{(x-y)(x-y)^T}{|x-y|^3}\right), \\ \mathbf{P}(x,y) & = & \frac{1}{4\pi}\frac{x-y}{|x-y|^3}. \end{array} \tag{1.2}$$

Let $\mathbf{v}\in\tilde{\mathbf{H}}_0^{-1/2}(\Gamma)$ be the solution of the following boundary integral equation with single layer potential operator $\mathbf{V}$:

$$\mathbf{V}\mathbf{v}(x) := \int_\Gamma \mathbf{E}(x,y)\mathbf{v}(y)\,dS_y = \mathbf{g}(x), \quad x\in\Gamma. \tag{1.3}$$

Then a variational solution $(\mathbf{u},p)$ of (1.1) with given $\mathbf{g}\in\mathbf{H}^{1/2}(\Gamma)$ is

$$\begin{array}{rcll} \mathbf{u}(x) & = & \int_\Gamma \mathbf{E}(x,y)\mathbf{v}(y)\,dS_y, & x\in\mathbf{R}^3, \\ p(x) & = & \int_\Gamma \mathbf{P}(x,y)\mathbf{v}(y)\,dS_y, & x\in\Omega_\Gamma, \end{array} \tag{1.4}$$

see [25, Theorem 2.1].

The operator $\mathbf{V}$ is positive definite on

$$\tilde{\mathbf{H}}_0^{-1/2}(\Gamma) := \{\mathbf{w}\in\tilde{\mathbf{H}}^{-1/2}(\Gamma);\ \langle\mathbf{w},\mathbf{n}\rangle = 0\},$$

see [25]. Here,

$$\tilde{\mathbf{H}}^{-1/2}(\Gamma) = \left(\tilde{H}^{-1/2}(\Gamma)\right)^3,$$

$\mathbf{n}$ is the normal vector on Γ (in a specified direction) and $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{L^2(\Gamma)}$ denotes the extension of the $(L^2(\Gamma))^3$-inner product by duality. Also, $\mathbf{H}^{1/2}(\Gamma)$ is the space of vector functions with components in $H^{1/2}(\Gamma)$. For a definition of $\tilde{H}^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ see §3. In the following, we use the symbol $\|\cdot\|_{\mathbf{V}}$ to denote the norm induced by the operator $\mathbf{V}$. This norm is equivalent to the $\tilde{\mathbf{H}}_0^{-1/2}(\Gamma)$-norm,

$$\langle\mathbf{V}\mathbf{w},\mathbf{w}\rangle \simeq \|\mathbf{w}\|^2_{\tilde{\mathbf{H}}^{-1/2}(\Gamma)} \quad \forall\mathbf{w}\in\tilde{\mathbf{H}}_0^{-1/2}(\Gamma), \tag{1.5}$$

see [25].

The remainder of the paper is organized as follows. In §2 we define the boundary element spaces for the approximate solution of (1.3). We define error indicators based on projections onto local subspaces defined by mesh refinement and state the main results, Theorems 2.1 and 2.2. Theorem 2.1 proves stability of the underlying subspace decomposition and Theorem 2.2 concludes efficiency and reliability of the resulting error estimator, assuming a saturation property and shape regular elements. In Theorem 2.2' we extend the results in Theorem 2.2 to include meshes containing distorted quadrilateral elements. All the technical details and the proof of Theorem 2.1 are given in §3. In §4 we present some numerical results for adaptive methods based on our error indicators. Moreover, the stability property stated by Theorem 2.1 is demonstrated for highly non-uniform rectangular and triangular meshes. In §5 we give asymptotically optimal error estimates for the approximation of an edge singularity which typically appears as part of the solution on open surfaces. Moreover, we comment on the saturation assumption.

2 Adaptive boundary element method

We solve (1.3) by the Galerkin method. To this end we introduce three sequences of shape regular meshes of triangles and/or quadrilaterals $\{\Gamma_{ij};\ j \in J_i\}$, $\bar{\Gamma} = \cup_{j\in J_i}\bar{\Gamma}_{ij}$ $(i = 1,2,3)$. Here J_i is the index set $\{1,2,\ldots,N_i\}$ with N_i being the number of elements of the corresponding mesh. By shape regular meshes we refer to meshes which need not be quasi-uniform (not even locally) but where the elements are shape regular, i.e. the smallest diameter of exterior circles can be uniformly bounded by a constant times the largest diameter of interior circles. For quadrilaterals one also bounds the interior angles away from π. The case of meshes which contain distorted quadrilateral elements (fulfilling the angle condition but not being shape regular) is considered at the end of this section. We do not need continuous basis functions. That means our basis functions (piecewise constants) are related with elements and not nodes or sides. Therefore, we do not use regular meshes. Our ansatz space consists of piecewise constant functions in each of the components. Formally we have three spaces of scalar functions

$$S_i := \{v \in L^2(\Gamma);\ v|_{\Gamma_{ij}} \text{ is constant } \forall j \in J_i\}, \quad i = 1,2,3. \tag{2.1}$$

In our numerical experiments we implement the constraint condition $\langle \mathbf{v}, \mathbf{n}\rangle = 0$ by a Lagrangian multiplier. The boundary element space then is

$$\mathbf{S} := \{\mathbf{v} = (v_1, v_2, v_3)^T;\ v_i \in S_i, i = 1,2,3,\ \langle \mathbf{v}, \mathbf{n}\rangle = 0\}. \tag{2.2}$$

Of course, the three meshes $\{\Gamma_{ij};\ j \in J_i\}$ $(i = 1,2,3)$ may coincide and then we have a space of vector functions being piecewise constant with respect to the same mesh.

The boundary element method then reads as follows: *Find* $\mathbf{v}_h \in \mathbf{S}$ *such that*

$$\langle \mathbf{V}\mathbf{v}_h, \mathbf{w}\rangle = \langle \mathbf{g}, \mathbf{w}\rangle \quad \forall \mathbf{w} \in \mathbf{S}. \tag{2.3}$$

In order to define an a posteriori error estimator we need a refined ansatz space that gives an improved approximation. To this end we divide all the triangles and quadrilaterals as in Figures 1, 2 and add the functions indicated on the right sides of the figures. The "+" and "–" signs mean positive and negative values which are constant on the respective elements. They are chosen such that the additional functions have integral mean zero. This is done for all components in the space $\mathbf{S}$. The enriched space is denoted by $\tilde{\mathbf{S}}$ and can be represented by the direct decomposition

$$\tilde{\mathbf{S}} = \mathbf{S} \oplus \mathbf{T}. \tag{2.4}$$

To obtain local error indicators we fully decompose $\mathbf{T}$, i.e. with respect to the vector components, with respect to the elements and with respect to the individual additional functions indicated on the right hand sides of Figures 1, 2. Formally we write this decomposition as

$$\tilde{\mathbf{S}} = \mathbf{S} \oplus \oplus_{i=1}^{3} \oplus_{j\in J_i} \oplus_{k\in\{\mathrm{a,b,c}\}} T_i^k(\Gamma_{ij}). \tag{2.5}$$

Here, $T_i^k(\Gamma_{ij})$ is the span of the vector function whose components different from i are zero and whose ith component has support $\bar{\Gamma}_{ij}$ and is piecewise constant of the type (k) (indicated by

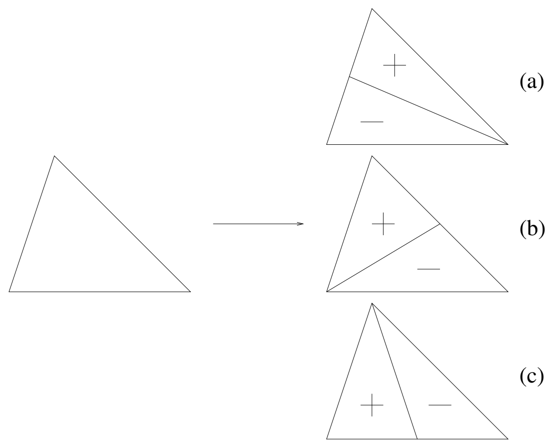


Figure 1: Division of triangles

(a), (b), or (c) in Figures 1, 2). By construction all the subspaces $T_i^k(\Gamma_{ij})$ consist of functions with integral mean zero. This is an important property that will be used in the analysis below.

Note that the construction of $\tilde{\mathbf{S}}$ is such that, on triangles, all the sides are halved independently and, on quadrilaterals, the sides are halved simultaneously. Thus, $\tilde{\mathbf{S}}$ contains at least all the piecewise constant functions on a mesh that comes from the previous mesh $\{\Gamma_{1j} \times \Gamma_{2,k} \times \Gamma_{3,l};\ j \in J_1, k \in J_2, l \in J_3\}$ by halving the longest sides. For a typical problem with singularities it is therefore likely that the following saturation assumption is satisfied (for more details see §5):

(A1) Let $\mathbf{v}_h \in \mathbf{S}$ be the Galerkin solution defined by (2.3) and let $\tilde{\mathbf{v}}_h \in \tilde{\mathbf{S}}$ be the improved Galerkin solution (by solving (2.3) within $\tilde{\mathbf{S}}$). Then there exists a constant $\sigma < 1$ being independent of h (a characteristic mesh size) such that

$$\|\mathbf{v} - \tilde{\mathbf{v}}_h\|_{\mathbf{V}} \leq \sigma \|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{V}}.$$

Here, $\mathbf{v}$ is the exact solution of (1.3).

Note that the decomposition (2.5) implicitly gives a component-wise decomposition, i.e.,

$$\tilde{S}_i := S_i \oplus \oplus_{j \in J_i} \oplus_{k \in \{\mathrm{a,b,c}\}} T_i^k(\Gamma_{ij}), \quad i = 1, 2, 3, \tag{2.6}$$

with $\tilde{\mathbf{S}} = (\oplus_{i=1}^3 \tilde{S}_i) \cap \{\mathbf{v};\ \langle \mathbf{v}, \mathbf{n} \rangle = 0\}$. In order to define error indicators and the error estimator we introduce for each of the subspaces a projection operator: $P_0 :\ \tilde{\mathbf{S}} \to \mathbf{S}$ by

$$\langle \mathbf{V} P_0 \mathbf{r}, \mathbf{w} \rangle = \langle \mathbf{V} \mathbf{r}, \mathbf{w} \rangle \quad \forall \mathbf{w} \in \mathbf{S}$$

and $P_{ij,k} :\ \tilde{\mathbf{S}} \to T_i^k(\Gamma_{ij})$ by

$$\langle \mathbf{V} P_{ij,k} \mathbf{r}, \mathbf{w} \rangle = \langle \mathbf{V} \mathbf{r}, \mathbf{w} \rangle \quad \forall \mathbf{w} \in T_i^k(\Gamma_{ij}). \tag{2.7}$$

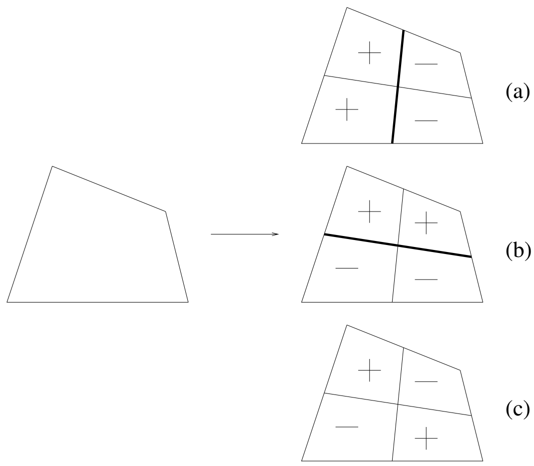


Figure 2: Division of quadrilaterals

The sum $P := P_0 + \sum_{i=1}^{3} \sum_{j\in J_i} \sum_{k\in\{\mathrm{a,b,c}\}} P_{ij,k}$ is known as the additive Schwarz operator and corresponds to a preconditioned stiffness matrix for $\mathbf{V}$. Now local error indicators are defined by using the projections $P_{ij,k}$ ($j \in J_i$, $i = 1,2,3$, $k \in \{\mathrm{a,b,c}\}$):

$$\theta_{ij,k}^2 := \langle \mathbf{V} P_{ij,k}(\mathbf{v}_h - \tilde{\mathbf{v}}_h), P_{ij,k}(\mathbf{v}_h - \tilde{\mathbf{v}}_h)\rangle.$$

The sum $\Theta^2 := \sum_{i=1}^{3} \sum_{j\in J_i} \sum_{k\in\{\mathrm{a,b,c}\}} \theta_{ij,k}^2$ is the square of our a posteriori error estimator. Note that the analogous indicator θ_0 for the space $\mathbf{S}$ for P_0 vanishes since $P_0(\mathbf{v}_h) = P_0(\tilde{\mathbf{v}}_h) = \mathbf{v}_h$. Also note that, since the subspaces $T_i^k(\Gamma_{ij})$ are one-dimensional, the error indicators can simply be calculated via

$$\theta_{ij,k}^2 = \langle \mathbf{V}\phi_{ij,k}, \phi_{ij,k}\rangle c_{ij,k}^2.$$

Here, $\phi_{ij,k}$ is the basis function spanning $T_i^k(\Gamma_{ij})$ and $P_{ij,k}(\mathbf{v}_h - \tilde{\mathbf{v}}_h) = c_{ij,k}\phi_{ij,k}$. For the calculation of $c_{ij,k}$ only a scalar equation must be inverted. In order to do so one does not need explicitly the improved approximation $\tilde{\mathbf{v}}_h$. By the Galerkin orthogonality of $\mathbf{v} - \tilde{\mathbf{v}}_h$ to $\tilde{\mathbf{S}}$, the right hand side in (2.7) for $\mathbf{r} = \tilde{\mathbf{v}}_h$ can be calculated by $\langle \mathbf{V}\tilde{\mathbf{v}}_h, \mathbf{w}\rangle = \langle \mathbf{g}, \mathbf{w}\rangle$ where $\mathbf{g}$ is the given function in (1.3).

It is well known that, depending on the parameter σ of the saturation assumption (A1), reliability and efficiency of the error estimator Θ can be estimated by using the minimum and maximum eigenvalues of the operator P (see, e.g., [2, 20]):

Proposition 2.1 *Let the assumption (A1) be satisfied and let* $\lambda_{\min}(P)$ *and* $\lambda_{\max}(P)$ *be the minimum and maximum eigenvalues of the additive Schwarz operator* P *implicitly defined by the decomposition (2.5). Then there holds*

$$\frac{\Theta^2}{\lambda_{\max}(P)} \le \|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{V}}^2 \le \frac{\Theta^2}{\lambda_{\min}(P)(1-\sigma^2)}.$$

The main results are stability of the decomposition (2.5) and, consequently, an estimate for the efficiency and reliability of our a posteriori error estimator.

Theorem 2.1 (stability) *Assume that the meshes* $\{\Gamma_{ij};\ j \in J_i\}$, $i = 1,2,3$, *are shape regular. Then there exist constants* c_1, $c_2 > 0$ *which are independent of the meshes (as long as they are shape regular) such that*

$$\lambda_{\min}(P) \geq c_1 \quad \textit{and} \quad \lambda_{\max}(P) \leq c_2.$$

Theorem 2.2 (efficiency and reliability) *Let the assumption (A1) be satisfied and assume that the meshes* $\{\Gamma_{ij};\ j \in J_i\}$, $i = 1,2,3$, *are shape regular. Then there exist constants* C_1, $C_2 > 0$ *which are independent of the meshes (as long as they are shape regular) such that*

$$C_1\Theta \leq \|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{V}} \leq C_2 \frac{\Theta}{\sqrt{1-\sigma^2}}.$$

Proof. Apply Proposition 2.1 selecting $C_1 = 1/\sqrt{c_2}$ and $C_2 = 1/\sqrt{c_1}$. □

Meshes with distorted quadrilateral elements

Theorem 2.2 assumes shape regularity of the elements which is used to prove stability of the underlying decomposition defining the local error indicators. However, to approximate efficiently edge singularities distorted elements (affine images of rectangles with high aspect ratio) are needed. Therefore, it is desirable to be able to deal with such elements in a posteriori error analysis. To this end we note that in our analysis shape regularity is needed just for the enriched ansatz space which is decomposed. Since no continuity of the basis functions is necessary one can easily ensure the shape regularity by subdividing the stretched quadrilaterals appropriately. In the following we describe which steps need to be changed.

First we use a different notation for the meshes used to define the ansatz space $\mathbf{S}$. The three meshes for the three components now are denoted by $\{\tilde{\Gamma}_{ij};\ j \in \tilde{J}_i\}$, $i = 1,2,3$. They may contain shape regular triangles and quadrilaterals which just satisfy the angle condition. Then the spaces S_i and $\mathbf{S}$ in (2.1), (2.2) are defined as before, using the elements $\tilde{\Gamma}_{ij}$.

In order to define the enriched ansatz space $\tilde{\mathbf{S}}$ we perform a previous mesh refinement. All the distorted quadrilateral elements are subdivided into shape regular quadrilaterals (see Figure 3), whereas the shape regular elements are not divided. After this refinement step the meshes are denoted by $\{\Gamma_{ij};\ j \in J_i\}$, $i = 1,2,3$, and they are shape regular. The new meshes define, as in (2.2) and using (2.1), a space of piecewise constant vector functions with integral mean zero. This space will be denoted by $\mathbf{T}_0$ and there holds $\mathbf{S} \subset \mathbf{T}_0$.

With the space $\mathbf{T}_0$ we proceed as previously described for $\mathbf{S}$. All the elements Γ_{ij} are divided as in Figures 1, 2, thus defining the spaces $T_i^k(\Gamma_{ij})$, $k \in \{\mathrm{a,b,c}\}$. Corresponding to (2.5), we then have the following decomposition of the enriched ansatz space $\tilde{\mathbf{S}}$:

$$\tilde{\mathbf{S}} = \mathbf{T}_0 \oplus \oplus_{i=1}^3 \oplus_{j \in J_i} \oplus_{k \in \{\mathrm{a,b,c}\}} T_i^k(\Gamma_{ij}). \tag{2.8}$$

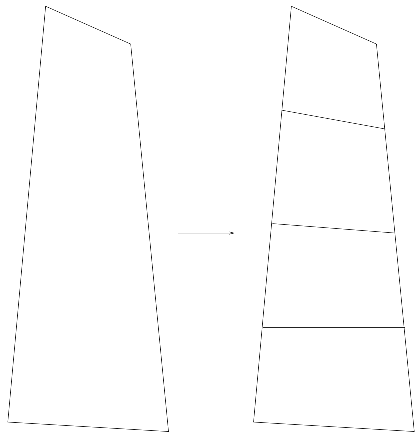


Figure 3: Division of distorted quadrilaterals

As in (2.6), this means decomposing component-wise as

$$\tilde{S}_i := T_{0i} \oplus \oplus_{j\in J_i} \oplus_{k\in\{\mathrm{a,b,c}\}} T_i^k(\Gamma_{ij}), \quad i = 1,2,3,$$

with $\tilde{\mathbf{S}} = (\oplus_{i=1}^3 \tilde{S}_i) \cap \{\mathbf{v};\ \langle \mathbf{v}, \mathbf{n}\rangle = 0\}$ and $\mathbf{T}_0 = (\oplus_{i=1}^3 T_{0i}) \cap \{\mathbf{v};\ \langle \mathbf{v}, \mathbf{n}\rangle = 0\}$.

The error indicators $\theta_{ij,k}^2$ are as before, and $\theta_0^2 := \langle \mathbf{V}P_0(\mathbf{v}_h - \tilde{\mathbf{v}}_h), P_0(\mathbf{v}_h - \tilde{\mathbf{v}}_h)\rangle$ with $P_0 : \tilde{\mathbf{S}} \to \mathbf{T}_0$ defined by

$$\langle \mathbf{V}P_0\mathbf{r}, \mathbf{w}\rangle = \langle \mathbf{V}\mathbf{r}, \mathbf{w}\rangle \quad \forall \mathbf{w} \in \mathbf{T}_0.$$

Note that we now have a global contribution θ_0 which usually does not vanish. The dimension of the system for the calculation of $P_0(\mathbf{v}_h - \tilde{\mathbf{v}}_h)$ depends on the number of distorted elements one needs to subdivide, and on their aspect ratio. Using this global contribution, the a posteriori error estimator now is $\Theta := (\theta_0^2 + \sum_{i=1}^3 \sum_{j\in J_i} \sum_{k\in\{\mathrm{a,b,c}\}} \theta_{ij,k}^2)^{1/2}$.

With these changes we obtain efficiency and reliability of Θ as before.

Theorem 2.2' *Let the assumption (A1) be satisfied and assume that the meshes $\{\tilde{\Gamma}_{ij};\ j \in \tilde{J}_i\}$, $i = 1,2,3$, consist of shape regular triangles and/or quadrilaterals which satisfy a maximum angle condition. Then there exist constants C_1, $C_2 > 0$ which are independent of the meshes such that*

$$C_1\Theta \le \|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{V}} \le C_2 \frac{\Theta}{\sqrt{1-\sigma^2}}.$$

Proof. Apply Proposition 2.1 using the decomposition (2.8) instead of (2.5). Bounds for the eigenvalues of the additive Schwarz operator P are given, as before, by Theorem 2.1. □

3 Technical details and proof of Theorem 2.1

First let us introduce the norms we will use. On Γ we take the standard L^2 and H^1-norms and define intermediate spaces by the K-method of interpolation (cf. [3]):

$$H^s(\Gamma) := [L^2(\Gamma), H^1(\Gamma)]_s \quad (0 < s < 1)$$

with norm

$$\|v\|_{H^s(\Gamma)} := \left(\int_0^\infty t^{-2s} K(t,v)^2 \frac{dt}{t} \right)^{1/2}.$$

Here, the K-functional is defined by

$$K(t,v) := \inf_{v=v_1+v_2} \left(\|v_1\|^2_{L^2(\Gamma)} + t^2 \|v_2\|^2_{H^1(\Gamma)} \right)^{1/2}.$$

The semi-norm in $H^s(\Gamma)$ is denoted by $|v|_{H^s(\Gamma)}$. It is defined by interpolation as before, using the semi-norm instead of the norm in $H^1(\Gamma)$. Also we need the spaces

$$\tilde{H}^s(\Gamma) := [L^2(\Gamma), H^1_0(\Gamma)]_s \quad (0 < s < 1)$$

where $H^1_0(\Gamma)$ is the completion of $C^\infty_0(\Gamma)$ in $H^1(\Gamma)$ and the norm $\|\cdot\|_{H^1_0(\Gamma)}$ is given by the semi-norm $|\cdot|_{H^1(\Gamma)}$ in $H^1(\Gamma)$. For negative s we define the Sobolev spaces by duality:

$$H^s(\Gamma) := (\tilde{H}^{-s}(\Gamma))', \quad \tilde{H}^s(\Gamma) := (H^{-s}(\Gamma))' \qquad (-1 \le s < 0).$$

For a subdomain $\gamma \subset \Gamma$ of diameter h, we analogously define the spaces $\tilde{H}^s(\gamma)$ for $0 < s < 1$, and $H^s(\gamma) := (\tilde{H}^{-s}(\gamma))'$ for $-1 \le s < 0$. Proceeding in the same way to define $\tilde{H}^s(\gamma)$ for negative s (as the dual space of $H^{-s}(\gamma)$) one does not get scalable norms. Therefore, we define

$$H^s_h(\gamma) := [L^2(\gamma), H^1_h(\gamma)]_s \quad (0 < s < 1) \quad \text{using} \quad \|\cdot\|_{H^1_h(\gamma)} := (h^{-2}\|\cdot\|^2_{L^2(\gamma)} + |\cdot|^2_{H^1(\gamma)})^{1/2}$$

with norm $\|\cdot\|_{H^s_h(\gamma)}$. For negative s we define, as before by duality,

$$\tilde{H}^s_h(\gamma) := (H^{-s}_h(\gamma))' \quad (-1 \le s < 0),$$

where the norm $\|\cdot\|_{H^{-s}_h(\gamma)}$ is taken in $H^{-s}_h(\gamma)$. Note that the index h in the notation of the norms always refers to the diameter of the subdomain under consideration.

The norms defined above on local subdomains are scalable under affine transformations onto a reference subdomain (or element). For ease of presentation we also use the notation $H^s_h(\gamma)$ for $s < 0$ and $\tilde{H}^s_h(\gamma)$ for $s > 0$ with norms $\|\cdot\|_{H^s_h(\gamma)}$ and $\|\cdot\|_{\tilde{H}^s_h(\gamma)}$ (where no L^2-terms occur which need a weight factor depending on h). The scaling properties of the norms are summarized by the following lemma.

Lemma 3.1 *Let γ and γ_h be two affine-equivalent open subsets of $\mathbf{R}^2$, $T_h(\gamma) = \gamma_h$ for an invertible affine mapping T_h. Assuming shape regularity of γ_h with* $\mathrm{diam}(\gamma_h) = h$ *and fixed γ with* $\mathrm{diam}(\gamma) = 1$*, there holds for $v \in H^s(\gamma)$ and $v_h := v \circ T_h^{-1}$ the equivalence of norms*

$$\|v_h\|^2_{H^s_h(\gamma_h)} \simeq h^{2-2s}\|v\|^2_{H^s(\gamma)}, \quad s \in [-1, 1]$$

uniformly for $h > 0$. Moreover, for $v \in \tilde{H}^s(\gamma)$ and with the above notation, there holds

$$\|v_h\|^2_{\tilde{H}^s_h(\gamma_h)} \simeq h^{2-2s}\|v\|^2_{\tilde{H}^s(\gamma)}, \quad s \in [-1, 1].$$

Again, the equivalence is uniform for $h > 0$.

Proof. Let the affine mapping be given by $T_h(x) = B_h x + b_h$ for $B_h \in \mathbf{R}^{2\times 2}$ and $b_h \in \mathbf{R}^2$. Standard estimates give, see, e.g., [8, Theorem 3.1.2],

$$|v|_{H^m(\gamma)} \le c\|B_h\|^m |\det(B_h)|^{-1/2}|v_h|_{H^m(\gamma_h)}, \quad m = 0, 1$$

and

$$|v_h|_{H^m(\gamma_h)} \le c\|B_h^{-1}\|^m |\det(B_h)|^{1/2}|v|_{H^m(\gamma)}, \quad m = 0, 1.$$

Both constants c are independent of γ_h. Also one finds

$$\|B_h\| \le \mathrm{diam}(\gamma_h)/\sup\{\mathrm{diam}(S);\ S \text{ is a ball contained in } \gamma\}$$

and

$$\|B_h^{-1}\| \le \mathrm{diam}(\gamma)/\sup\{\mathrm{diam}(S);\ S \text{ is a ball contained in } \gamma_h\},$$

see [8, Theorem 3.1.3]. Due to the relation $|\det(B_h)| = |\gamma_h|/|\gamma|$ and the shape regularity of γ_h, this gives, by definition of the norms,

$$\|v_h\|^2_{H^s_h(\gamma_h)} \simeq h^{2-2s}\|v\|^2_{H^s(\gamma)}, \quad s = 0, 1,$$

and

$$\|v_h\|^2_{\tilde{H}^s_h(\gamma_h)} \simeq h^{2-2s}\|v\|^2_{\tilde{H}^s(\gamma)}, \quad s = 0, 1.$$

Interpolation yields the analogous results for $s \in (0, 1)$. Using the definition of the norms for $s < 0$ by duality, these relations also hold for $s \in [-1, 0)$. □

Before dealing with domain decompositions for the scalable norms let us recall estimates for the standard norms from [1] (see also [22] where these estimate are given for the J-method of interpolation): Let Γ be partitioned into nonoverlapping Lipschitz subdomains Γ_j, $j = 1, \ldots, J$. Then, for $s \in [-1, 1]$, there hold

$$\sum_{j=1}^{J} \|v|_{\Gamma_j}\|^2_{H^s(\Gamma_j)} \le \|v\|^2_{H^s(\Gamma)} \qquad \forall v \in H^s(\Gamma) \tag{3.1}$$

and

$$\|v\|^2_{\tilde{H}^s(\Gamma)} \le \sum_{j=1}^{J} \|v|_{\Gamma_j}\|^2_{\tilde{H}^s(\Gamma_j)} \qquad \forall v \in \tilde{H}^s(\Gamma) \text{ with } v|_{\Gamma_j} \in \tilde{H}^s(\Gamma_j). \tag{3.2}$$

For the scalable norms introduced above one needs additional assumptions in order for these estimates to hold. We prove the following lemma.

Lemma 3.2 *Let Γ be partitioned into shape regular convex polygonal subdomains Γ_j, $j = 1,\ldots,J$, which are affine transformations of a fixed set of polygons. Then, for all $v \in \tilde{H}^s(\Gamma)$, $s \in [0,1]$, with $\int_{\Gamma_j} v\,dx = 0$, $j = 1,\ldots,J$, there holds*

$$\sum_{j=1}^{J} \|v|_{\Gamma_j}\|^2_{H^s_h(\Gamma_j)} \le c\|v\|^2_{H^s(\Gamma)} \le c\|v\|^2_{\tilde{H}^s(\Gamma)}. \tag{3.3}$$

The constant c is independent of v and the number of subdomains. Moreover, for $v \in \tilde{H}^s(\Gamma)$, $s \in [-1,0]$, with $v|_{\Gamma_j} \in \tilde{H}^s(\Gamma_j)$ and $\int_{\Gamma_j} v\,dx = 0$, $j = 1,\ldots,J$, there holds

$$\|v\|^2_{\tilde{H}^s(\Gamma)} \le c\sum_{j=1}^{J} \|v|_{\Gamma_j}\|^2_{\tilde{H}^s_h(\Gamma_j)}. \tag{3.4}$$

Again, the constant c is independent of v and J.

Proof. The second inequality in (3.3) is due to the definition of the norms. In order to prove the first estimate in (3.3) we show that, for $\bar{v}_j := 1/|\Gamma_j| \int_{\Gamma_j} v\,dx$, $j = 1,\ldots,J$, there holds

$$\sum_{j=1}^{J} \|v|_{\Gamma_j} - \bar{v}_j\|^2_{H^s_h(\Gamma_j)} \le c\|v\|^2_{H^s(\Gamma)} \quad \forall v \in H^s(\Gamma). \tag{3.5}$$

Note that $\bar{v}_j$ is well defined for $s \ge 0$ since $1|_{\Gamma_j} \in L^2(\Gamma_j) \subset \tilde{H}^{-s}_h(\Gamma_j) = (H^s_h(\Gamma_j))'$. On a fixed star shaped subdomain $\gamma \subset \Gamma$ there holds by the Poincaré-Friedrichs' inequality

$$\|v - \frac{1}{|\gamma|}\int_\gamma v\,dx\|_{H^1(\gamma)} \le C(\gamma)|v|_{H^1(\gamma)}, \tag{3.6}$$

see, e.g., [5, Lemma 4.3.14]. By affine transformations, this yields for $\gamma = \Gamma_j$ being one subdomain of Γ with diameter h the equivalence

$$\|v - \bar{v}\|^2_{H^1_h(\gamma)} = \frac{1}{h^2}\|v - \bar{v}\|^2_{L^2(\gamma)} + |v|^2_{H^1(\gamma)} \simeq |v|^2_{H^1(\gamma)}$$

with $\bar{v} := 1/|\gamma| \int_\gamma v\,dx$. This equivalence is uniform under affine transformations which ensure shape regularity. Noting that $\|v - \bar{v}\|_{L^2(\gamma)} \le \|v\|_{L^2(\gamma)}$ we obtain by interpolation

$$\|v - \bar{v}\|_{H^s_h(\gamma)} \le c|v|_{H^s(\gamma)} \le c\|v\|_{H^s(\gamma)} \quad \forall v \in H^s(\gamma), \quad s \in [0,1]. \tag{3.7}$$

The constant $c > 0$ is independent of the subdomain $\gamma = \Gamma_j$ of the partition. Then (3.5) follows by combining (3.7) and (3.1).

For $s \in [-1, 0]$ one obtains (3.4) by duality from (3.5) for $-s$ as follows. Let $v \in \tilde{H}^s(\Gamma)$ with $v|_{\Gamma_j} \in \tilde{H}^s(\Gamma_j)$ and $\int_{\Gamma_j} v\,dx = 0$ be given. Taking $\varphi \in H^{-s}(\Gamma)$ we find

$$\begin{aligned}
|\langle \varphi, v \rangle_{L^2(\Gamma)}|^2 &= \Big|\sum_{j=1}^{J} \langle \varphi|_{\Gamma_j}, v|_{\Gamma_j} \rangle_{L^2(\Gamma_j)}\Big|^2 = \Big|\sum_{j=1}^{J} \langle \varphi|_{\Gamma_j} - \bar{\varphi}_j, v|_{\Gamma_j} \rangle_{L^2(\Gamma_j)}\Big|^2 \\
&\leq \Big(\sum_{j=1}^{J} \|v|_{\Gamma_j}\|_{\tilde{H}^s_h(\Gamma_j)} \|\varphi|_{\Gamma_j} - \bar{\varphi}_j\|_{H_h^{-s}(\Gamma_j)}\Big)^2 \\
&\leq \sum_{j=1}^{J} \|v|_{\Gamma_j}\|^2_{\tilde{H}^s_h(\Gamma_j)} \sum_{j=1}^{J} \|\varphi|_{\Gamma_j} - \bar{\varphi}_j\|^2_{H_h^{-s}(\Gamma_j)} \\
&\overset{(3.5)}{\leq} \sum_{j=1}^{J} \|v|_{\Gamma_j}\|^2_{\tilde{H}^s_h(\Gamma_j)}\; c\|\varphi\|^2_{H^{-s}(\Gamma)},
\end{aligned}$$

hence

$$\|v\|^2_{\tilde{H}^s(\Gamma)} = \sup_{0 \neq \varphi \in H^{-s}(\Gamma)} \frac{|\langle \varphi, v \rangle_{L^2(\Gamma)}|^2}{\|\varphi\|^2_{H^{-s}(\Gamma)}} \leq c \sum_{j=1}^{J} \|v|_{\Gamma_j}\|^2_{\tilde{H}^s_h(\Gamma_j)}.$$

This proves (3.4). □

Remark 3.1 *Lemma* 3.2 *holds for more general partitions (than those described in* §2) *of* Γ*. Central point to its proof is establishing the Poincaré-Friedrichs' inequality on subdomains* (3.6)*. See* [13, 11] *for generalizations.*

Before proving Theorem 2.1 let us identify the Sobolev norm which is uniformly equivalent (under scalings) to the norm given by the integral operator $\mathbf{V}$.

Lemma 3.3 *Let* $\gamma \subset \mathbf{R}^2$ *be a Lipschitz domain with diameter* 1*, and for an invertible affine transformation* T_h *let* $\gamma_h := \{T_h(x);\ x \in \gamma\}$ *be the transformed domain with diameter* h*. Let us assume that* γ_h *is shape regular uniformly for* $h > 0$*. Then, for* $\mathbf{v} \in \tilde{\mathbf{H}}_0^{-1/2}(\gamma)$*, with integral mean zero normal component, and* $\mathbf{v}_h(x) := \mathbf{v}(T_h^{-1}(x))$ $(x \in \gamma_h)$ *there holds the equivalence of norms*

$$\langle \mathbf{V}\mathbf{v}_h, \mathbf{v}_h \rangle_{L^2(\gamma_h)} \simeq \|\mathbf{v}_h\|^2_{\tilde{\mathbf{H}}_h^{-1/2}(\gamma_h)}$$

uniformly for $0 < h \leq H$*. Here,* H *is a positive constant and the norm* $\|\cdot\|_{\tilde{\mathbf{H}}_h^{-1/2}(\gamma_h)}$ *is the product norm in* $\tilde{\mathbf{H}}_h^{-1/2}(\gamma_h)$ *using* $\|\cdot\|_{\tilde{H}_h^{-1/2}(\gamma_h)}$ *for the components. The bilinear form* $\langle \mathbf{V}\mathbf{v}_h, \mathbf{v}_h \rangle_{L^2(\gamma_h)}$ *is to be understood in the sense that* $\mathbf{V}$ *is defined for functions on* γ_h*.*

Proof. For a fixed domain γ, the equivalence of the norms is given by (1.5). Neglecting translations, any affine transformation T_h maintaining shape regularity can be written as a composition of a transformation T_0 (mapping γ to a shape regular domain γ_0 of diameter 1) and a scaling $x \mapsto hx$. We show the uniform equivalence of

$$\langle \mathbf{V}\mathbf{v}_0, \mathbf{v}_0 \rangle_{L^2(\gamma_0)} \simeq \|\mathbf{v}_0\|^2_{\tilde{\mathbf{H}}_h^{-1/2}(\gamma_0)} \tag{3.8}$$

under affine transformations T_0 which keep the diameter of γ_0 (= 1) and its shape regularity. Then, the assertion follows by proving the equivalence of

$$\langle \mathbf{V}\mathbf{v}_h, \mathbf{v}_h \rangle_{L^2(\gamma_h)} \simeq \|\mathbf{v}_h\|^2_{\tilde{\mathbf{H}}_h^{-1/2}(\gamma_h)} \tag{3.9}$$

under scalings $x \mapsto hx$ uniformly for $h > 0$.

To prove (3.8) we assume without loss of generality that $\gamma \subset \Gamma$ and $\gamma_0 \subset \Gamma$. We consider $\mathbf{v} \in \tilde{\mathbf{H}}_0^{-1/2}(\gamma)$ with transformed function $\mathbf{v}_0(x) := \mathbf{v}(T_0^{-1}(x))$ on γ_0. We denote by $\mathbf{v}_0^*$ the extension by 0 of $\mathbf{v}_0$ onto Γ. By the equivalence (1.5) for any $\mathbf{w} \in \tilde{\mathbf{H}}_0^{-1/2}(\Gamma)$ there holds

$$\langle \mathbf{V}\mathbf{v}_0, \mathbf{v}_0 \rangle_{L^2(\gamma_0)} = \langle \mathbf{V}\mathbf{v}_0^*, \mathbf{v}_0^* \rangle_{L^2(\Gamma)} \simeq \|\mathbf{v}_0^*\|^2_{\tilde{\mathbf{H}}^{-1/2}(\Gamma)}.$$

Therefore, in order to prove (3.8), we only have to show the uniform equivalence of the norms

$$\|\mathbf{v}_0^*\|_{\tilde{\mathbf{H}}^{-1/2}(\Gamma)} \simeq \|\mathbf{v}_0\|_{\tilde{\mathbf{H}}^{-1/2}(\gamma_0)}.$$

Writing $\mathbf{v}_0^* = (v_{0,1}^*, v_{0,2}^*, v_{0,3}^*)$ there holds

$$\|\mathbf{v}_0^*\|^2_{\tilde{\mathbf{H}}^{-1/2}(\Gamma)} = \sum_{i=1}^{3} \|v_{0,i}^*\|^2_{\tilde{H}^{-1/2}(\Gamma)}$$

and

$$\|v_{0,i}^*\|_{\tilde{H}^{-1/2}(\Gamma)} = \sup_{0 \neq w \in H^{1/2}(\Gamma)} \frac{\langle w, v_{0,i}^* \rangle_{L^2(\Gamma)}}{\|w\|_{H^{1/2}(\Gamma)}} = \sup_{0 \neq w \in H^{1/2}(\Gamma)} \frac{\langle w, v_{0,i} \rangle_{L^2(\gamma_0)}}{\|w\|_{H^{1/2}(\Gamma)}}.$$

Since $\|w\|_{H^{1/2}(\Gamma)} \geq \|w\|_{H^{1/2}(\gamma_0)}$ we directly obtain

$$\|v_{0,i}^*\|_{\tilde{H}^{-1/2}(\Gamma)} \leq \sup_{0 \neq w \in H^{1/2}(\gamma_0)} \frac{\langle w, v_{0,i} \rangle_{L^2(\gamma_0)}}{\|w\|_{H^{1/2}(\gamma_0)}} = \|v_{0,i}\|_{\tilde{H}^{-1/2}(\gamma_0)}.$$

On the other hand, for the Lipschitz domain γ_0 there exists an extension operator $E: H^{1/2}(\gamma_0) \to H^{1/2}(\Gamma)$ which is bounded and whose bound depends only on the number of Lipschitz mappings used to describe the boundary of γ_0 and their Lipschitz constants, see, e.g., [21]. More precisely, by Theorem 5 in [21, Chapter VI, Section 3] there exists a bounded operator $E : H^k(\gamma_0) \to H^k(\mathbf{R}^2)$ $(k = 0, 1)$ such that, by interpolation, $E : H^{1/2}(\gamma_0) \to H^{1/2}(\mathbf{R}^2)$. Then one obtains, by taking the restriction $H^{1/2}(\mathbf{R}^2) \to H^{1/2}(\Gamma)$, an operator $E:\ H^{1/2}(\gamma_0) \to H^{1/2}(\Gamma)$ with bound $\|E\|$. Therefore,

$$\begin{aligned}
\|v_{0,i}^*\|_{\tilde H^{-1/2}(\Gamma)} &= \sup_{0\neq w\in H^{1/2}(\Gamma)} \frac{\langle w, v_{0,i}^*\rangle_{L^2(\Gamma)}}{\|w\|_{H^{1/2}(\Gamma)}} \geq \sup_{0\neq w_0\in H^{1/2}(\gamma_0)} \frac{\langle Ew_0, v_{0,i}^*\rangle_{L^2(\Gamma)}}{\|Ew_0\|_{H^{1/2}(\Gamma)}}\\
&\geq \|E\|^{-1} \sup_{0\neq w_0\in H^{1/2}(\gamma_0)} \frac{\langle w_0, v_{0,i}\rangle_{L^2(\gamma)}}{\|w_0\|_{H^{1/2}(\gamma_0)}} = \|E\|^{-1}\|v_{0,i}\|_{\tilde H^{-1/2}(\gamma_0)}.
\end{aligned}$$

For affine transformations to shape regular domains γ_0, $\|E\|^{-1}$ is uniformly bounded from below by a positive constant. This finishes the proof of (3.8).

It remains to prove (3.9) for scalings $x \mapsto hx$. Transforming γ_0 to $\gamma_h := h\gamma_0$ and using the homogeneity of the kernel $\mathbf{E}(hx, hy) = h^{-1}\mathbf{E}(x, y)$, we compute

$$\begin{aligned}
\langle \mathbf{V}\mathbf{v}_h, \mathbf{v}_h\rangle_{L^2(\gamma_h)} &= \int_{\gamma_h}\int_{\gamma_h} \mathbf{E}(x,y)\mathbf{v}_h(x)\mathbf{v}_h(y)\,dy\,dx\\
&= h^4 \int_{\gamma_0}\int_{\gamma_0} \mathbf{E}(hx,hy)\mathbf{v}_0(x)\mathbf{v}_0(y)\,dy\,dx\\
&= h^3 \int_{\gamma_0}\int_{\gamma_0} \mathbf{E}(x,y)\mathbf{v}_0(x)\mathbf{v}_0(y)\,dy\,dx = h^3\ \langle \mathbf{V}\mathbf{v}_0, \mathbf{v}_0\rangle_{L^2(\gamma_0)}.
\end{aligned}$$

On the other hand, due to Lemma 3.1, there holds for a single component v_0 of $\mathbf{v}_0$ on γ_0 (and with v_h being the transformed component)

$$\|v_h\|^2_{\tilde H_h^{-1/2}(\gamma_h)} \simeq h^3\|v_0\|^2_{\tilde H^{-1/2}(\gamma_0)}$$

which, together with (3.8), proves (3.9). □

Proof of Theorem 2.1. By standard results of the additive Schwarz theory (see, e.g., [7]), and since the decomposition (2.5) is direct, one has to show that there exist constants c_1, $c_2 > 0$ such that

$$c_1\left(\langle \mathbf{V}\mathbf{w}_0, \mathbf{w}_0\rangle + \sum_{i,j,k}\langle \mathbf{V}\mathbf{w}_{ij,k}, \mathbf{w}_{ij,k}\rangle\right) \leq \left\langle \mathbf{V}(\mathbf{w}_0 + \sum_{i,j,k}\mathbf{w}_{ij,k}), (\mathbf{w}_0 + \sum_{i,j,k}\mathbf{w}_{ij,k})\right\rangle \tag{3.10}$$

and

$$\left\langle \mathbf{V}(\mathbf{w}_0 + \sum_{i,j,k}\mathbf{w}_{ij,k}), (\mathbf{w}_0 + \sum_{i,j,k}\mathbf{w}_{ij,k})\right\rangle \leq c_2\left(\langle \mathbf{V}\mathbf{w}_0, \mathbf{w}_0\rangle + \sum_{i,j,k}\langle \mathbf{V}\mathbf{w}_{ij,k}, \mathbf{w}_{ij,k}\rangle\right) \tag{3.11}$$

for any $\mathbf{w}_0 \in \mathbf{S}$ and any $\mathbf{w}_{ij,k} \in T_i^k(\Gamma_{ij})$, $j \in J_i$, $i = 1, 2, 3$, $k \in \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$.

Let us denote the ith component of $\mathbf{w}_{ij,k}$ by $w_{ij,k}$ (the other components of $\mathbf{w}_{ij,k}$ vanish by construction), the ith component of $\mathbf{w}_0$ by w_{i0} and $w_i := w_{i0} + \sum_{j\in J_i}\sum_{k\in\{\mathrm{a,b,c}\}} w_{ij,k}$, $i = 1, 2, 3$.

Then, using the equivalence of $\langle \mathbf{V}\mathbf{w}_0, \mathbf{w}_0\rangle$ and $\|\mathbf{w}_0\|^2_{\tilde{\mathbf{H}}^{-1/2}(\Gamma)}$ due to (1.5), the equivalence of the norm in $\tilde{\mathbf{H}}_h^{-1/2}$ and the norm given by $\mathbf{V}$ for functions with integral mean zero (by Lemma 3.3), and recalling the product structure of the space $\tilde{\mathbf{H}}_0^{-1/2}(\Gamma)$, one finds that (3.10), (3.11) are equivalent to the existence of positive constants c_1, c_2 such that

$$c_1 \left(\sum_{i=1}^{3} \|w_{i0}\|^2_{\tilde{H}^{-1/2}(\Gamma)} + \sum_{i,j,k} \|w_{ij,k}\|^2_{\tilde{H}_h^{-1/2}(\Gamma_{ij})} \right) \le \sum_{i=1}^{3} \|w_i\|^2_{\tilde{H}^{-1/2}(\Gamma)} \tag{3.12}$$

and

$$\sum_{i=1}^{3} \|w_i\|^2_{\tilde{H}^{-1/2}(\Gamma)} \le c_2 \left(\sum_{i=1}^{3} \|w_{i0}\|^2_{\tilde{H}^{-1/2}(\Gamma)} + \sum_{i,j,k} \|w_{ij,k}\|^2_{\tilde{H}_h^{-1/2}(\Gamma_{ij})} \right). \tag{3.13}$$

It therefore suffices to estimate

$$c_{i1} \left(\|w_{i0}\|^2_{\tilde{H}^{-1/2}(\Gamma)} + \sum_{j,k} \|w_{ij,k}\|^2_{\tilde{H}_h^{-1/2}(\Gamma_{ij})} \right) \le \|w_i\|^2_{\tilde{H}^{-1/2}(\Gamma)} \tag{3.14}$$

and

$$\|w_i\|^2_{\tilde{H}^{-1/2}(\Gamma)} \le c_{i2} \left(\|w_{i0}\|^2_{\tilde{H}^{-1/2}(\Gamma)} + \sum_{j,k} \|w_{ij,k}\|^2_{\tilde{H}_h^{-1/2}(\Gamma_{ij})} \right) \tag{3.15}$$

for any $w_{i0} \in S_i$ and any $w_{ij,k} \in T_i^k(\Gamma_{ij})$, $j \in J_i$, $i = 1, 2, 3$, $k \in \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$. Then setting $c_1 := \min\{c_{11}, c_{21}, c_{31}\}$ and $c_2 := \max\{c_{12}, c_{22}, c_{32}\}$ we obtain (3.12), (3.13) and the theorem is proved.

Note that the numbers c_{i1}, c_{i2} in (3.14) and (3.15) are bounds for the extremum eigenvalues of the additive Schwarz operator which belongs to the decomposition (2.6). The set of functions $\{w_{ij,k};\ j \in J_i,\ k \in \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}\}$ consists of triples of functions which are non-zero on the same element. These functions span the three-dimensional spaces $T_i(\Gamma_{ij}) := \oplus_{k\in\{\mathrm{a,b,c}\}} T_i^k(\Gamma_{ij})$, $j \in J_i$, where $T_i^k(\Gamma_{ij})$ is the span of the function (over Γ_{ij}) denoted by (k) on the right side of Figure 1 (if Γ_{ij} is a triangle) or of Figure 2 (if Γ_{ij} is a quadrilateral).

We prove stability of the splitting of w_i with respect to the elements, i.e., the existence of constants c_{i1}, $c_{i2} > 0$ such that

$$c_{i1} \left(\|w_{i0}\|^2_{\tilde{H}^{-1/2}(\Gamma)} + \sum_{j\in J_i} \|w_{ij}\|^2_{\tilde{H}_h^{-1/2}(\Gamma_{ij})} \right) \le \|w_i\|^2_{\tilde{H}^{-1/2}(\Gamma)} \tag{3.16}$$

and

$$\|w_i\|^2_{\tilde{H}^{-1/2}(\Gamma)} \le c_{i2} \left(\|w_{i0}\|^2_{\tilde{H}^{-1/2}(\Gamma)} + \sum_{j\in J_i} \|w_{ij}\|^2_{\tilde{H}_h^{-1/2}(\Gamma_{ij})} \right) \tag{3.17}$$

where $w_{ij} = \sum_{k\in\{\mathrm{a,b,c}\}} w_{ij,k} \in T_i(\Gamma_{ij})$ in the representation (3.14), (3.15). Since the functions $w_{ij,k}$, $k = \mathrm{a,b,c}$, are linearly independent the expressions

$$\|w_{ij}\|_{\tilde H_h^{-1/2}(\Gamma_{ij})} \quad \text{and} \quad \left(\|w_{ij,\mathrm{a}}\|^2_{\tilde H_h^{-1/2}(\Gamma_{ij})} + \|w_{ij,\mathrm{b}}\|^2_{\tilde H_h^{-1/2}(\Gamma_{ij})} + \|w_{ij,\mathrm{c}}\|^2_{\tilde H_h^{-1/2}(\Gamma_{ij})}\right)^{1/2}$$

are equivalent norms. The uniformity of this equivalence with respect to the elements Γ_{ij} can be seen by considering affine transformations of a fixed element to shape regular elements using Lemma 3.1. It therefore suffices to prove (3.16) and (3.17). This is the scalar situation presented in [20] (without giving a proof) for the situation of quasi-uniform rectangular meshes (see also [16, Corollary 1] for an improved estimate under the same restrictions).

Here we show that (3.16) and (3.17) can be proved under the only assumption of shape regularity of the elements. Estimate in (3.17) is a combination of the triangle inequality and Lemma 3.2:

$$\begin{aligned}\|w_i\|^2_{\tilde H^{-1/2}(\Gamma)} &\le 2\left(\|w_{i0}\|^2_{\tilde H^{-1/2}(\Gamma)} + \|\sum_{j\in J_i} w_{ij}\|^2_{\tilde H^{-1/2}(\Gamma)}\right)\\ &\le c\left(\|w_{i0}\|^2_{\tilde H^{-1/2}(\Gamma)} + \sum_{j\in J_i}\|w_{ij}\|^2_{\tilde H_h^{-1/2}(\Gamma_{ij})}\right).\end{aligned}$$

To prove (3.16) we show that

$$\sum_{j\in J_i}\|w_{ij}\|^2_{\tilde H_h^{-1/2}(\Gamma_{ij})} \le c\|w_i\|^2_{\tilde H^{-1/2}(\Gamma)}. \tag{3.18}$$

The assertion then follows by the triangle inequality and (3.4). Denoting by Q_{ij} the $L^2(\Gamma_{ij})$-projection operator onto the constants we can write $w_{ij} = w_i - Q_{ij}(w_i)$ on Γ_{ij}. Using the orthogonality $\langle Q_{ij}(w_i), v - Q_{ij}(v)\rangle = 0$ for arbitrary functions $v \in L^2(\Gamma_{ij})$, this gives

$$\langle w_{ij}, v\rangle_{L^2(\Gamma_{ij})} = \langle w_i, v - Q_{ij}(v)\rangle_{L^2(\Gamma_{ij})} \quad \forall\, v \in L^2(\Gamma_{ij}).$$

Therefore, we can bound the dual norm of w_{ij} by

$$\begin{aligned}\|w_{ij}\|_{\tilde H_h^{-1/2+\epsilon}(\Gamma_{ij})} &= \sup_{v\in H^{1/2-\epsilon}(\Gamma_{ij})}\frac{\langle w_{ij}, v\rangle}{\|v\|_{H_h^{1/2-\epsilon}(\Gamma_{ij})}}\\ &\le \|w_i\|_{H_h^{-1/2+\epsilon}(\Gamma_{ij})}\sup_{v\in H^{1/2-\epsilon}(\Gamma_{ij})}\frac{\|v - Q_{ij}(v)\|_{\tilde H_h^{1/2-\epsilon}(\Gamma_{ij})}}{\|v\|_{H_h^{1/2-\epsilon}(\Gamma_{ij})}}.\end{aligned} \tag{3.19}$$

Here, $\epsilon > 0$ is arbitrary but sufficiently small and we used that $H_h^{-1/2+\epsilon}(\Gamma_{ij})$ is the dual space of $\tilde H_h^{1/2-\epsilon}(\Gamma_{ij})$ with uniformly equivalent norms (under scalings).

It is well known that the norms in $\tilde{H}^s$ and H^s on Lipschitz domains are equivalent for $|s| < 1/2$, see [14]. More precisely, by [16, Lemma 5] there holds

$$\|v - Q_{ij}(v)\|_{\tilde{H}_h^{1/2-\epsilon}(\Gamma_{ij})} \le \frac{c}{\epsilon}\|v - Q_{ij}(v)\|_{H_h^{1/2-\epsilon}(\Gamma_{ij})}.$$

Here, the constant c does not depend on ϵ nor the diameter of Γ_{ij} since the norms in $\tilde{H}_h^{1/2-\epsilon}(\Gamma_{ij})$ and $H_h^{1/2-\epsilon}(\Gamma_{ij})$ both scale the same way due to Lemma 3.1. This, of course, hings on the shape regularity of the elements Γ_{ij}. The operator Q_{ij} is continuous from $L^2(\Gamma_{ij}) \to L^2(\Gamma_{ij})$ and from $H^1(\Gamma_{ij}) \to H^1(\Gamma_{ij})$, see [16, (24)]. The latter continuity simply follows from $|Q_{ij}(v)|_{H^1(\Gamma_{ij})} = 0$. Therefore, one also obtains $Q_{ij}: H_h^1(\Gamma_{ij}) \to H_h^1(\Gamma_{ij})$, such that, by interpolation, $Q_{ij}: H_h^{1/2-\epsilon}(\Gamma_{ij}) \to H_h^{1/2-\epsilon}(\Gamma_{ij})$. Then we obtain from (3.19) and the previous observation the estimate

$$\|w_{ij}\|_{\tilde{H}_h^{-1/2+\epsilon}(\Gamma_{ij})} \le \frac{c}{\epsilon}\|w_i\|_{H_h^{-1/2+\epsilon}(\Gamma_{ij})}.$$

Using this estimate one finds

$$\|w_{ij}\|_{\tilde{H}_h^{-1/2}(\Gamma_{ij})} \le c\,\mathrm{diam}\,(\Gamma_{ij})^\epsilon \|w_{ij}\|_{\tilde{H}_h^{-1/2+\epsilon}(\Gamma_{ij})} \le \frac{c}{\epsilon}\,\mathrm{diam}\,(\Gamma_{ij})^\epsilon \|w_i\|_{H_h^{-1/2+\epsilon}(\Gamma_{ij})}. \tag{3.20}$$

Here, for the first estimate, one simply transforms back and forth to a reference elements and uses the scaling properties of the norms in $\tilde{H}_h^{-1/2}$ and $\tilde{H}_h^{-1/2+\epsilon}$ on shape regular elements. Using again Lemma 3.1, now for the norms in $H_h^{-1/2+\epsilon}$ and $H_h^{-1/2}(\Gamma_{ij})$, one finds by the same argument

$$\|w_i\|_{H_h^{-1/2+\epsilon}(\Gamma_{ij})} \le \frac{c}{\mathrm{diam}\,(\Gamma_{ij})^\epsilon}\|w_i\|_{H_h^{-1/2}(\Gamma_{ij})} \tag{3.21}$$

which is an inverse property for piecewise constant functions. Combining (3.20) and (3.21) we obtain

$$\|w_{ij}\|_{\tilde{H}_h^{-1/2}(\Gamma_{ij})} \le \frac{c}{\epsilon}\|w_i\|_{H_h^{-1/2}(\Gamma_{ij})}.$$

Taking a fixed, sufficiently small $\epsilon > 0$ and summing the squares of the last estimation this yields

$$\sum_{j\in J_i}\|w_{ij}\|^2_{\tilde{H}_h^{-1/2}(\Gamma_{ij})} \le c\sum_{j\in J_i}\|w_i\|^2_{H_h^{-1/2}(\Gamma_{ij})} \le c\|w_i\|^2_{\tilde{H}^{-1/2}(\Gamma)}.$$

The last upper bound is due to (3.1) by noting that $\|\cdot\|_{H_h^{-1/2}(\Gamma_{ij})} = \|\cdot\|_{H^{-1/2}(\Gamma_{ij})}$ by definition of $H^s(\gamma)$ for $s \in [-1, 0]$ at the beginning of Section 3. This proves (3.18) and the proof of the theorem is finished. □

4 Numerical results

4.1 Example 1: polygonal surface

We solve the integral equation (1.3) on the plane surface piece Γ indicated by Figure 4 with right hand side function $\mathbf{g} = (g_1, g_2, g_3)$ where $g_1 = g_2 = 1$ and $g_3(x_1, x_2) = x_1 + x_2$, $x = (x_1, x_2) \in \Gamma$.

We did not take $g_3 = 1$ since testing with functions with mean zero normal component, here the third component, would give the trivial solution for the third component of $\mathbf{v}$. The horizontal and vertical sides of Γ, as subdomain of $\mathbf{R}^2$, have length 1. Since Γ is an open surface piece the solution $\mathbf{v}$ of (1.3) usually exhibits strong singularities at the edges and corners. Due to the chosen shape of Γ it is not obvious what an optimal mesh for the boundary element method looks like (which, for a rectangle, just needs rectangular elements). We start with the initial mesh indicated in Figure 4, consisting of quadrilaterals and triangles, and use piecewise constant trial functions. The scheme (2.3) then gives an approximation for the true solution $\mathbf{v}$ and we use our indicators to refine the mesh thus giving improved approximations. For simplicity we do not subdivide distorted rectangles into shape regular sub-elements (as is necessary for the proof of Theorem 2.2'). The efficiency of the estimator in our numerical experiments is still convincing, see Figure 6.

Our adaptive algorithm is as follows. We determine the maximum of all the elemental indicators

$$\theta_{\max} = \max_{i,j} \theta_{ij}$$

where

$$\theta_{ij} := \theta_{ij,\mathrm{a}}^2 + \theta_{ij,\mathrm{b}}^2 + \theta_{ij,\mathrm{c}}^2$$

according to the full decomposition (2.5), and refine the element Γ_{ij} (the jth element for the ith component) whenever $\theta_{ij} \geq \frac{1}{2}\theta_{\max}$. The refinement of Γ_{ij} is done anisotropically as follows. For a **triangle** Γ_{ij}: if $0.7\,\theta_{ij,k}$ is larger than the sum of the other two indicators then the element is halved as referenced by (k) in Figure 1. If none of the indicators fulfills this condition then the element is divided into four triangles by connecting the midpoints of the sides. Additionally we restrict halving elements by a minimum angle condition. For a **quadrilateral**: if $0.7\,\theta_{ij,\mathrm{a}}$ is larger than both the other indicators then the element is halved along the bold line in Figure 2(a), and analogously in the case (b). Otherwise the element is divided into four quadrilaterals by connecting opposite midpoints of the sides. This is the standard strategy and indicated by "**indicators w.r.t. directions**" in the figures. To underline its efficiency we also study pure elemental indicators by just using the terms θ_{ij} and performing isotropic refinement, i.e. subdivision into four elements where $\theta_{ij} \geq \frac{1}{2}\theta_{\max}$. This strategy is referred to by "**indicators w.r.t. elements**". Finally, to demonstrate the influence of the individual adaption of the components, we also perform an adaption which is uniform with respect to the three vector components, but still uses the anisotropic refinement from the first strategy. This is realized by joining the indicators for the components,

$$\theta_{j,k}^2 := \theta_{1j,k}^2 + \theta_{2j,k}^2 + \theta_{3j,k}^2, \quad k \in \{\mathrm{a}, \mathrm{b}, \mathrm{c}\},$$

defining

$$\theta_j^2 := \theta_{j,\mathrm{a}}^2 + \theta_{j,\mathrm{b}}^2 + \theta_{j,\mathrm{c}}^2, \quad \theta_{\max}^* := \max_{j \in J} \theta_j,$$

(the index sets are equal, $J := J_1 = J_2 = J_3$) and performing as in the standard situation: refine Γ_{ij} according to the directional strategy by using the direction indicators $\theta_{j,k}$, k =a,b,c, whenever $\theta_j \geq \frac{1}{2}\theta_{\max}^*$. This adaption is referred to by "**indicators w.r.t. directions (uniform)**".

In order to present the relative errors in energy norm for the different methods we approximate the norm of the true solution $\mathbf{v}$ by extrapolation and use the symmetry of the operator $\mathbf{V}$ to obtain

$$\|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{V}}^2 = \|\mathbf{v}\|_{\mathbf{V}}^2 - \|\mathbf{v}_h\|_{\mathbf{V}}^2.$$

Of course, substituting $\|\mathbf{v}\|_{\mathbf{V}}$ by an extrapolated value, we only obtain approximations for the errors. Our results underline the expected behavior of the strategies. Figure 5 shows the errors for the uniform h-version (initial mesh as in Figure 4 and subsequent uniform element divisions), and for the adaptive versions. Obviously, the adaptive versions converge faster than the uniform version and, moreover, anisotropic refinements based on indicators for directions lead to better convergence than isotropic refinements. This is reasonable since our problem exhibits edge singularities which can be best approximated by anisotropic meshes. It also becomes clear that component independent adaption further reduces the number of unknowns and leads to better convergence. Of course, we cannot expect a better convergence rate for this method since we save asymptotically at most two thirds of the unknowns (e.g., when two of the components of the true solution are smooth).

Figure 6 plots the error estimator Θ (belonging to the full decomposition) divided by the error in energy norm (its approximation by extrapolation), for sequences of meshes obtained by the uniform h-version and the three adaptive strategies. The results are almost constant and reflect good efficiency of the estimator, as stated by Theorem 2.2 for shape regular meshes. The statement of Theorem 2.2, however, depends on assumption (A1), i.e. on the saturation parameter σ. Table 1 lists numerical approximations for this parameter, for the uniform method and the adaptive versions based on elemental and directional indicators. Here, N denotes the dimension of the actual ansatz space and $\tilde{N}$ is the dimension of the enriched ansatz space. Except the second value for the uniform method (which needs the norm of the boundary element solution on a quite fine mesh) all the values are around 0.7. This is the value which one expects in the presence of edge singularities of the type $\mathrm{dist}(x, \partial\Gamma)^{-0.5}$. In fact, the a priori error estimate $O(h^{1/2})$ in this case indicates for mesh halving an asymptotic saturation parameter $\sigma \approx (1/2)^{1/2} \approx 0.7$. We refer to §5 for more details. In Figures 7, 8, 9 we present the meshes (always one for each of the three components of $\mathbf{v}_h$) obtained by the three adaptive strategies for the step when the error in the approximation is approximately 10% in the energy norm. All the meshes exhibit refinement towards edges and corners. Note that the incoming corner does not exhibit a strong singularity since we are dealing with the problem exterior to Γ. The mesh refinements work well for rectangular and triangular elements. Also note that the meshes for the three components are different in Figures 7 and 9. We do not have a regularity theory for the individual components of the solution at hand. But the different meshes (with different refinement priorities) indicate at least different dominant terms in the singularity expansions of the components of $\mathbf{v}$.

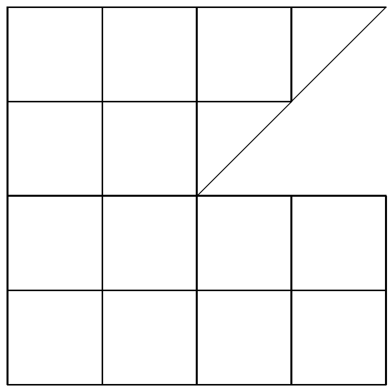


Figure 4: The surface with initial mesh of rectangles and triangles.

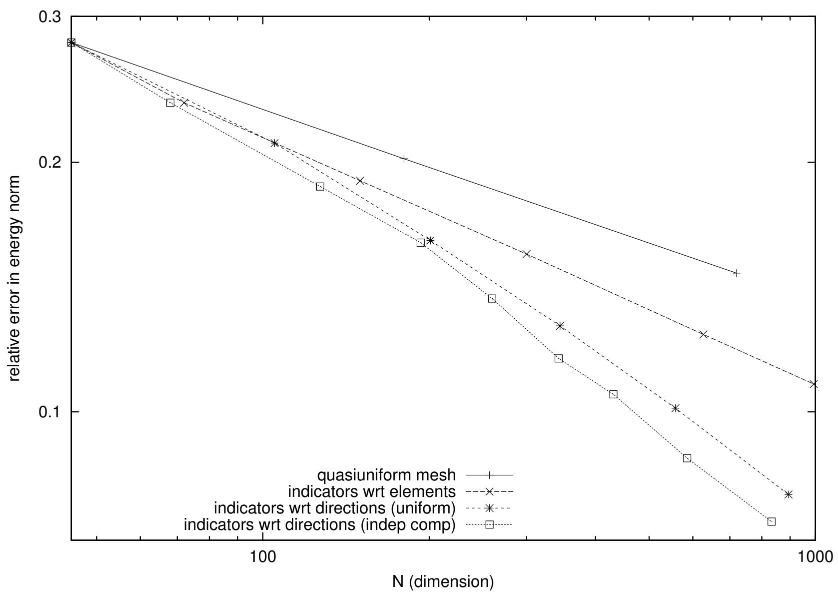


Figure 5: Relative error in energy norm: uniform and adaptive methods.

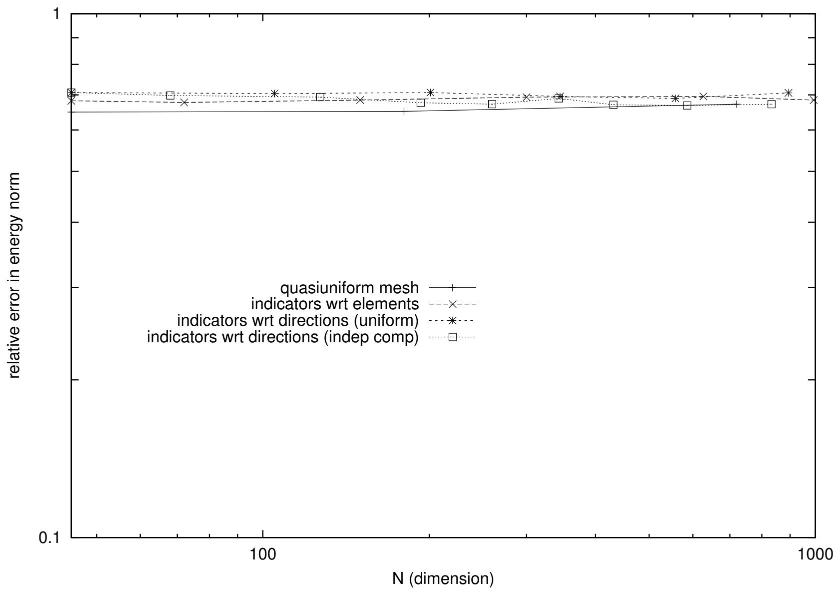
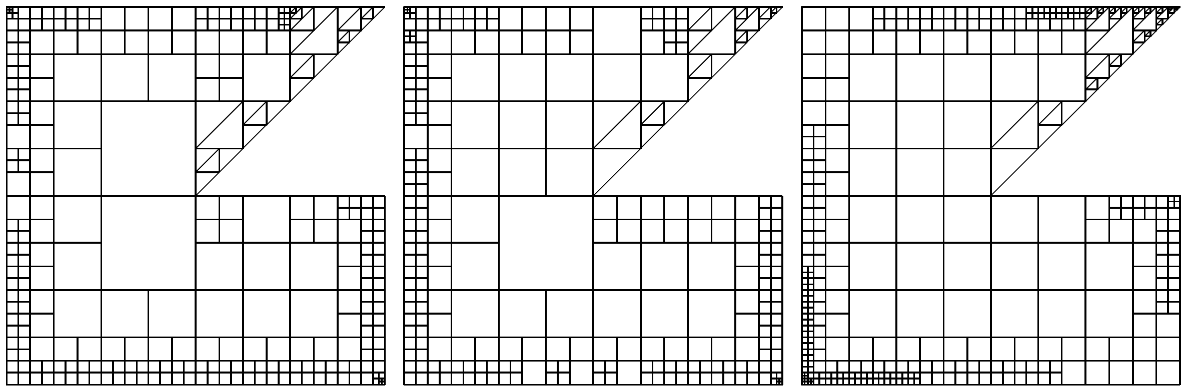


Figure 6: Testing efficiency/reliability: error estimator divided by error in energy norm.

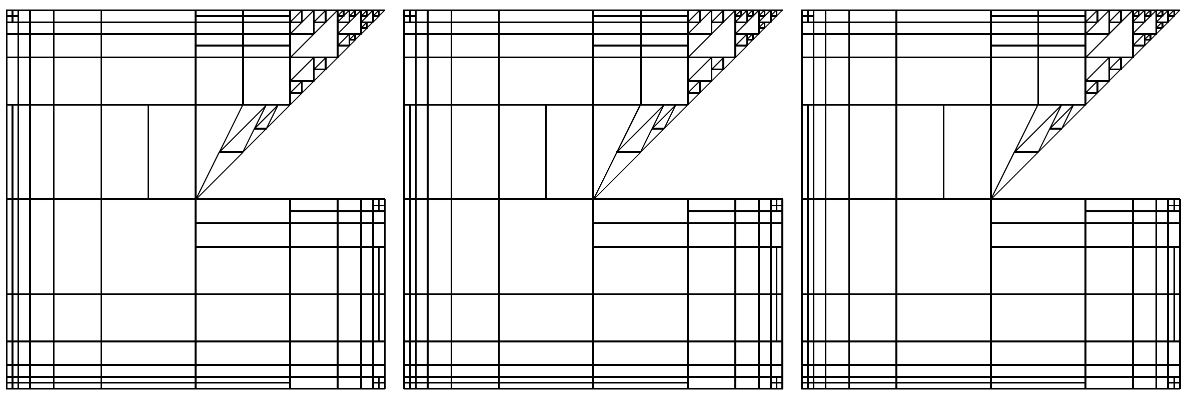
N	$\tilde{N}$	(1)	(2)	(3)
45	180	0.759	0.760	0.760
68	272			0.725
72	288		0.726	
127	508			0.722
150	600		0.723	
180	720	0.857		
193	772			0.712
260	1040			0.698
300	1200		0.707	
343	1372			0.675
627	2508		0.659	

Table 1: The saturation parameter σ for uniform refinement (1), shape regular refinement using indicators w.r.t. elements (2), and refinement using indicators w.r.t. directions (3).



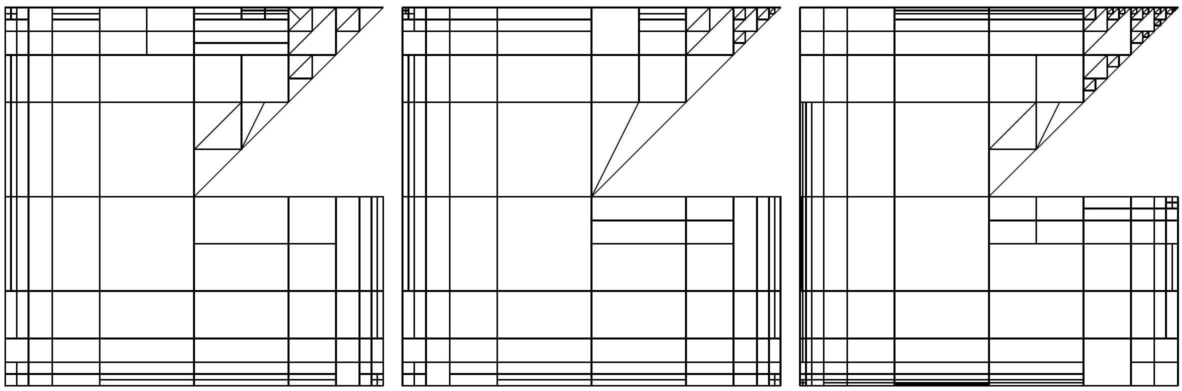
306 elements 291 elements 396 elements

Figure 7: Adaptively refined meshes (for three components) using indicators w.r.t. elements, 10.8% error for 993 unknowns.



186 elements 186 elements 186 elements

Figure 8: Adaptively refined mesh (same mesh for the three components) using indicators w.r.t. directions (uniform), 10.1% error for 558 unknowns.



122 elements 123 elements 186 elements

Figure 9: Adaptively refined meshes (for three components) using indicators w.r.t. directions, 10.5% error for 431 unknowns.

4.2 Example 2: testing stability.

The numerical results presented in the previous section underline the statement of Theorem 2.2 for the reliability and efficiency of the error estimator Θ on shape regular meshes. In this section we numerically investigate the theoretical basis of Theorem 2.2. This is the stability of decomposition (2.5) on shape regular meshes, stated by Theorem 2.1. Here, we do not approximate a real problem but artificially create highly non-uniform meshes. Then, assembling the corresponding stiffness matrices for the integral operator $\mathbf{V}$, we calculate the minimum and maximum eigenvalues of the corresponding additive Schwarz operator P and those of the stiffness matrix A. Figure 10 shows the types of meshes used to generate a sequence of non-uniform rectangular meshes. The left, middle and right meshes together are an example in this sequence of meshes used for the first, second and third components, respectively, of the unknown function. Table 2 lists the corresponding results, depending on the maximum mesh ratio (being the same for the three components): maximum side length divided by minimum side length. As before, N denotes the dimension of the ansatz space formed by the three non-uniform meshes. Analogously, we study a sequence of highly refined triangular meshes as in Figure 11. The corresponding results are given in Table 3. All the results demonstrate independence of the extremum eigenvalues of P on the mesh ratio, as described in Theorem 2.1. We have also performed numerical experiments (not presented here) which indicate that the extremum eigenvalues of P do indeed depend on the aspect ratio of the elements.

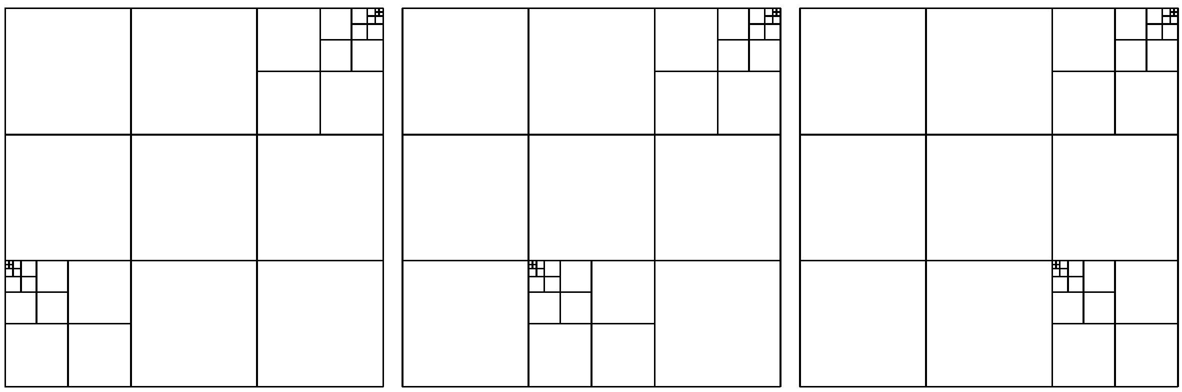


Figure 10: Test meshes: shape regular rectangular elements.

N	$h_{\max}/h_{\min}$	$\lambda_{\min}(A)$	$\lambda_{\max}(A)$	$\lambda_{\min}(P)$	$\lambda_{\max}(P)$
27	1	0.474E-01	0.501	0.447	1.83
45	2	0.692E-02	0.451	0.370	1.89
63	4	0.865E-03	0.449	0.329	1.88
81	8	0.108E-03	0.449	0.326	1.89
99	16	0.135E-04	0.449	0.332	1.88
117	32	0.169E-05	0.449	0.338	1.88
135	64	0.211E-06	0.449	0.340	1.88
153	128	0.264E-07	0.449	0.341	1.87
171	256	0.330E-08	0.449	0.342	1.87
189	512	0.413E-09	0.449	0.342	1.87

Table 2: Test meshes like in Fig. 10: extremum eigenvalues of the stiffness matrix and of the additive Schwarz operator for enriched spaces.

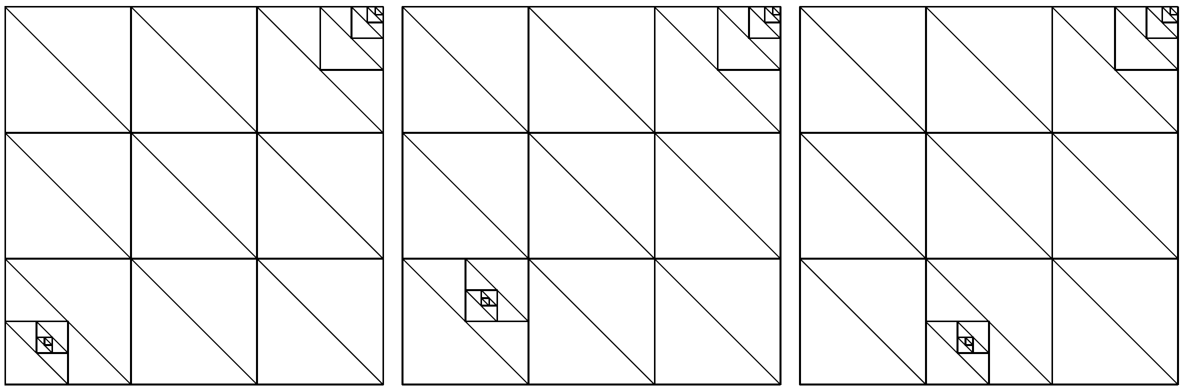


Figure 11: Test meshes: shape regular triangular elements.

N	$h_{\max}/h_{\min}$	$\lambda_{\min}(A)$	$\lambda_{\max}(A)$	$\lambda_{\min}(P)$	$\lambda_{\max}(P)$
54	1	0.135E-01	0.252	0.409	2.15
72	2	0.207E-02	0.243	0.396	2.16
90	4	0.254E-03	0.242	0.378	2.16
108	8	0.318E-04	0.242	0.366	2.15
126	16	0.397E-05	0.242	0.360	2.15
144	32	0.496E-06	0.242	0.355	2.15
162	64	0.621E-07	0.242	0.352	2.15
180	128	0.776E-08	0.242	0.350	2.15
198	256	0.970E-09	0.242	0.349	2.15

Table 3: Test meshes like in Fig. 11: extremum eigenvalues of the stiffness matrix and of the additive Schwarz operator for enriched spaces.

5 Appendix

In this section we analyze the saturation assumption (A1) in the presence of strong singularities and for uniform and graded meshes. The solution $\mathbf{v}$ of (1.3) is, up to a normal vector of constant length, the jump of the stress vector across Γ, see [25, (2.15)]. Even for smooth data, this stress vector in general exhibits corner and corner edge singularities, see [25, 10, 9] and the references cited there. For representations of these singularities in tensor product form see [23].

We consider $\Gamma = \{(x_1, x_2, 0);\ 0 < x_1, x_2 < 1\}$ and assume that the solution $\mathbf{v}$ of (1.3) is, close to the lower edge and the corner $(0, 0, 0)$, of the form

$$\mathbf{v}(x_1, x_2) = \mathbf{w}(x_1, x_2) x_1^{\lambda-1/2} x_2^{-1/2} \tag{5.1}$$

with $\lambda > 0$ and a smooth vector function $\mathbf{w}$ whose components are equal to 1 close to the edge $x_2 = 0$ (for small angles in polar coordinates with center $(0, 0)$). We assume that the singular behaviour of $\mathbf{v}$ at the other corners/edges is analogous. Of course, there are more singularities of different types, but the one assumed above is the strongest for an open surface and smooth data [10, 25, 4]. This singularity therefore dominates the convergence of the boundary element approximation, which is studied now.

We consider sequences of rectangular meshes which are graded towards the edges. For a precise definition we divide Γ into four squares and map each square to $Q = (0, 1) \times (0, 1) \times \{0\}$ such that the part of the boundary of Γ is mapped to the edges $x_1 = 0$ and $x_2 = 0$ of Q. For a grading parameter $\beta \geq 1$, integer $N > 0$ and $h = 1/N$ we introduce the graded mesh generated by the lines

$$x_1 = \left(\frac{i}{N}\right)^{\beta},\ x_2 = \left(\frac{j}{N}\right)^{\beta},\ i, j = 0, \ldots, N.$$

This mesh on Q defines a mesh on Γ which is graded towards the edges for $\beta > 1$. For $\beta = 1$

the mesh is uniform. Considering vector functions with piecewise constant components on the given mesh, this defines the boundary element space $\mathbf{S}$, cf. (2.2).

Then we have the following asymptotically optimal error estimate.

Theorem 5.1 *Let* $\Gamma = (0,1)\times(0,1)\times\{0\}$ *and assume that the singular behaviour of the solution* $\mathbf{v}$ *of* (1.3) *is as described in* (5.1)*. Define the Galerkin approximation* $\mathbf{v}_h$ *by* (2.3) *with ansatz space* $\mathbf{S}$ *as defined before. Then there exist constants* c_0*,* $c_1 > 0$ *such that*

$$c_0 h^\alpha \le \|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{V}} \le c_1 h^\alpha$$

with $\alpha = 1/2$ *for* $\beta = 1$ *and* $\alpha = 3/2$ *for* $\beta > 3$*.*

Proof. By the quasi-optimal convergence of the Galerkin method, the a priori error estimate is, up to constant factors, a result of best approximation in the Sobolev space $\tilde{\mathbf{H}}^{-1/2}(\Gamma)$. The upper bound of the approximation error for the assumed singularity is given by [24, Lemma 3.1] with detailed proofs cited from [22]. (The appearing $\epsilon > 0$ in that result is for technical reasons due to another type of singularity.) It remains to prove the lower bound. Let $\gamma := (a,b)\times(0,d)\times\{0\} \subset \Gamma$ with $0 < a < b, d \le 1/2$ denote a surface piece which touches the boundary of Γ at $(a,b)\times\{0\}\times\{0\}$. We assume that γ is so small that the function $\mathbf{w}$ from (5.1) satisfies $\mathbf{w}|_\gamma = \mathbf{1}$ (the constant vector with components 1). It follows from (1.5) and the definition of $\tilde{H}^s(\Gamma)$ that

$$\|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{V}} \ge c\|v_i - v_{h,i}\|_{H^{-1/2}(\gamma)}.$$

Here, for $i \in \{1,2,3\}$, v_i and $v_{h,i}$ are the ith components of $\mathbf{v}$ and $\mathbf{v}_h$. We choose an arbitrary i. It suffices to prove that

$$\|x_1^{\lambda-1/2} x_2^{-1/2} - \phi\|_{H^{-1/2}(\gamma)} \ge ch^\alpha \tag{5.2}$$

for any piecewise constant function ϕ (with respect to a given mesh) and a constant $c > 0$.

Without loss of generality assume that, for a given mesh grading parameter $\beta \ge 1$, the subdomain is $\gamma = ((1/3)^\beta, (1/2)^\beta) \times (0, (1/3)^\beta) \times \{0\}$ and that the mesh on γ (which is induced from the one on Γ) is given by the lines

$$x_1 = s_i,\ x_2 = s_j, \quad i = 2N, \ldots, 3N,\ j = 0, \ldots, 2N$$

with

$$s_j := \left(\frac{j}{6N}\right)^\beta.$$

The elements of the mesh are $\gamma_{ij} := (s_{i-1}, s_i)\times(s_{j-1}, s_j)\times\{0\}$, $i = 2N{+}1, \ldots, 3N$, $j = 1, \ldots, 2N$. Their lengths in x_1- and x_2-directions are h_i and h_j, respectively, with $h_j := s_j - s_{j-1}$. By the selection of γ (depending on β) their holds $h_i > h_j$ for any element γ_{ij} on γ.

In order to establish (5.2), we begin by showing that

$$\|\psi\|_{H^{-1/2}(\gamma_{ij})} \succeq h_i^{1/2} h_j \|\tilde{\psi}\|_{H^{-1/2}(Q)} \tag{5.3}$$

where " $\tilde{}$ " denotes the affine transformation of a function from γ_{ij} to $Q = (0,1) \times (0,1) \times \{0\}$: $\tilde{\psi}(x_1, x_2) := \psi\Big((1-x_1)s_{i-1} + x_1 s_i, (1-x_2)s_{j-1} + x_2 s_j\Big)$. (Here and in the following, $a(h) \succeq b(h)$ means that there exists a constant $c > 0$ which is independent of h such that $a(h) \geq c\, b(h)$ for all h, and analogously for "$\preceq$". For expressions $a(h)$, $b(h)$ involving general functions the constant c does not depend on them. The relation $a(h) \simeq b(h)$ means $a(h) \succeq b(h)$ and $a(h) \preceq b(h)$.)

Since $h_i > h_j$, we have

$$\|\phi\|_{L^2(\gamma_{ij})} \simeq h_i^{1/2} h_j^{1/2} \|\tilde{\phi}\|_{L^2(Q)}, \qquad |\phi|_{H^1(\gamma_{ij})} \preceq \left(\frac{h_i}{h_j}\right)^{1/2} |\tilde{\phi}|_{H^1(Q)},$$

and by interpolation,

$$\|\phi\|_{\tilde{H}^{1/2}(\gamma_{ij})} \preceq h_i^{1/2} \|\tilde{\phi}\|_{\tilde{H}^{1/2}(Q)}.$$

By duality this gives

$$\begin{aligned} \|\psi\|_{H^{-1/2}(\gamma_{ij})} &= \sup_{\phi \in \tilde{H}^{1/2}(\gamma_{ij}) \setminus \{0\}} \frac{\langle \psi, \phi \rangle_{L^2(\gamma_{ij})}}{\|\phi\|_{\tilde{H}^{1/2}(\gamma_{ij})}} \\ &\succeq \sup_{\tilde{\phi} \in \tilde{H}^{1/2}(Q) \setminus \{0\}} \frac{h_i h_j \langle \tilde{\psi}, \tilde{\phi} \rangle_{L^2(Q)}}{h_i^{1/2} \|\tilde{\phi}\|_{\tilde{H}^{1/2}(Q)}} = h_i^{1/2} h_j \|\tilde{\psi}\|_{H^{-1/2}(Q)} \end{aligned}$$

which is (5.3).

Let us estimate the error on the elements γ_{i1}, $i = 2N+1, \ldots, 3N$, which are adjacent to the edge $x_2 = 0$. By (5.3) we obtain for any $c_{i1} \in \mathbf{R}$

$$\begin{aligned} \|x_1^{\lambda-1/2} x_2^{-1/2} - c_{i1}\|_{H^{-1/2}(\gamma_{i1})} &\succeq h_i^{1/2} h_1 \|(s_{i-1} + h_i x_1)^{\lambda-1/2} (s_1 x_2)^{-1/2} - c_{i1}\|_{H^{-1/2}(Q)} \\ &= h_i^{1/2} h_1 s_1^{-1/2} s_{i-1}^{\lambda-1/2} \|(1 + \frac{h_i}{s_{i-1}} x_1)^{\lambda-1/2} x_2^{-1/2} - \tilde{c}_{i1}\|_{H^{-1/2}(Q)} \end{aligned} \tag{5.4}$$

with constant $\tilde{c}_{i1} = s_1^{1/2} s_{i-1}^{1/2-\lambda} c_{i1}$. There holds

$$\inf_{\tilde{c}_{i1} \in \mathrm{R}} \|(1 + \frac{h_i}{s_{i-1}} x_1)^{\lambda-1/2} x_2^{-1/2} - \tilde{c}_{i1}\|_{H^{-1/2}(Q)} =: d_{N,i1} \geq c > 0 \tag{5.5}$$

for a constant c which is independent of i and N for $i = 2N+1, \ldots, 3N$ (we write $d_{N,i1}$ to emphasize the dependence on N). In fact, for any i, the function $(1 + \frac{h_i}{s_{i-1}} x_1)^{\lambda-1/2} x_2^{-1/2}$ is not constant such that $d_{N,i1} > 0$ for any N and $i \in \{2N+1, \ldots, 3N\}$. If (5.5) did not hold then there must exist subsequences N_n and $i_n \to \infty$ such that $d_{N_n,i_n 1} \to 0$ for $n \to \infty$ (note that $N_n \to \infty$ if and only if $i_n \to \infty$). Since $h_i/s_{i-1} \to 0$ for $i \to \infty$ we then obtain the existence of a constant c such that $x_2^{-1/2} - c = 0$ in $H^{-1/2}(Q)$ which is a contradiction. Therefore, (5.5) holds.

Next, $h_1 = s_1 = h^\beta$, $s_{i-1} \geq (1/3)^\beta$ for $i \geq 2N+1$, and we conclude from (5.4) that

$$\|x_1^{\lambda-1/2} x_2^{-1/2} - c_{i1}\|_{H^{-1/2}(\gamma_{i1})} \succeq h_i^{1/2} h^{\beta/2} \tag{5.6}$$

uniformly for $i = 2N+1, \dots, 6N$.

Now we estimate the norms on the remaining elements γ_{ij}, $i \geq 2N+1$, $j > 1$. As before we obtain by transformation

$$\begin{aligned} \|x_1^{\lambda-1/2} x_2^{-1/2} - c_{ij}\|_{H^{-1/2}(\gamma_{ij})} &\succeq h_i^{1/2} h_j \|(s_{i-1}+h_i x_1)^{\lambda-1/2}(s_{j-1}+h_j x_2)^{-1/2} - c_{ij}\|_{H^{-1/2}(Q)} \\ &= h_i^{1/2} h_j (s_{j-1}^{-1/2} - s_j^{-1/2}) s_{i-1}^{\lambda-1/2} \|(1+\frac{h_i}{s_{i-1}}x_1)^{\lambda-1/2} \frac{(s_{j-1}+h_j x_2)^{-1/2}}{s_{j-1}^{-1/2} - s_j^{-1/2}} - \tilde{c}_{ij}\|_{H^{-1/2}(Q)} \\ &= h_i^{1/2} h_j (s_{j-1}^{-1/2} - s_j^{-1/2}) s_{i-1}^{\lambda-1/2} \|\phi_i(x_1)\psi_j(x_2) - \tilde{c}_{ij}\|_{H^{-1/2}(Q)} \end{aligned} \tag{5.7}$$

with constant $\tilde{c}_{ij} = c_{ij} s_{i-1}^{1/2-\lambda}/(s_{j-1}^{-1/2} - s_j^{-1/2})$ and functions $\phi_i(x_1) = (1+\frac{h_i}{s_{i-1}}x_1)^{\lambda-1/2}$, $\psi_j(x_2) = (s_{j-1}+h_j x_2)^{-1/2}/(s_{j-1}^{-1/2} - s_j^{-1/2})$. We prove that there holds

$$\inf_{\tilde{c}_{ij}\in \mathrm{R}} \|\phi_i(x_1)\psi_j(x_2) - \tilde{c}_{ij}\|_{H^{-1/2}(Q)} =: d_{N,ij} \geq \tilde{c} > 0 \tag{5.8}$$

for a constant $\tilde{c}$ which does not depend on i, j and N ($i \in \{2N+1, \dots, 3N\}$, $j \in \{2, \dots, 2N\}$). As for (5.5) we note that $d_{N,ij} > 0$ for any i and j in the above mentioned ranges. If (5.8) did not hold then there must exist subsequences N_n, (i_n, j_n), $n = 1, 2, \dots$, such that $d_{N_n, i_n j_n} \to 0$ as $n \to \infty$. As above, since as $n \to \infty$ $N_n \to \infty$, then $i_n \to \infty$ also. Hence

$$\phi_{i_n}(x_1) = (1+\frac{h_{i_n}}{s_{i_n-1}}x_1)^{\lambda-1/2} \to 1 \qquad (n\to\infty).$$

If the sequence j_n is bounded then there must exist a j^* such that a subsequence of j_n, $j_{n_m} = j^*$. Then $\phi_{i_{n_m}}(x_1)\psi_{j_{n_m}}(x_2) = \phi_{i_{n_m}}(x_1)\psi_{j^*}(x_2) \to \psi_{j^*}(x_2)$ in $H^{-1/2}(Q)$ $(m \to \infty)$, but

$$\inf_{\tilde{c}_{ij^*}\in \mathrm{R}} \|\psi_{j^*}(x_2) - \tilde{c}_{ij^*}\|_{H^{-1/2}(Q)} = \inf_{\tilde{c}_{ij^*}\in \mathrm{R}} \|\frac{(s_{j^*-1}+h_{j^*}x_2)^{-1/2}}{s_{j^*-1}^{-1/2} - s_{j^*}^{-1/2}} - \tilde{c}_{ij^*}\|_{H^{-1/2}(Q)} > 0.$$

Thus, if (5.8) fails the sequence j_n must be unbounded. Without loss of generality we may assume that the sequence j_n is strictly increasing. We estimate

$$\begin{aligned} &\|\phi_{i_n}\psi_{j_n} - \tilde{c}_{i_n j_n}\|_{H^{-1/2}(Q)} \geq \\ &\quad\Big| \|\phi_{i_n}\psi_{j_n} - \psi_{j_n} - \frac{\phi_{i_n}-1}{1-\left(\frac{j_n}{j_n-1}\right)^{-\beta/2}}\|_{H^{-1/2}(Q)} - \|\psi_{j_n} + \frac{\phi_{i_n}-1}{1-\left(\frac{j_n}{j_n-1}\right)^{-\beta/2}} - \tilde{c}_{i_n j_n}\|_{H^{-1/2}(Q)} \Big|. \end{aligned} \tag{5.9}$$

Note that

$$\psi_j(x_2) = \frac{(s_{j-1}+h_j x_2)^{-1/2}}{s_{j-1}^{-1/2} - s_j^{-1/2}} = \frac{(1+[(\frac{j}{j-1})^{\beta}-1]x_2)^{-1/2}}{1-(\frac{j}{j-1})^{-\beta/2}},$$

thus

$$\psi_{j_n}(x_2) - \frac{1}{1-(\frac{j_n}{j_n-1})^{-\beta/2}} \to -x_2 \quad (n\to\infty) \tag{5.10}$$

and

$$\phi_{i_n}(x_1)\psi_{j_n}(x_2) - \psi_{j_n}(x_2) - \frac{\phi_{i_n}(x_1) - 1}{1 - \left(\frac{j_n}{j_n - 1}\right)^{-\beta/2}} =$$

$$\underbrace{(\phi_{i_n}(x_1) - 1)}_{\to 0 \ (n\to\infty)} \underbrace{\left(\psi_{j_n}(x_2) - \frac{1}{1 - \left(\frac{j_n}{j_n - 1}\right)^{-\beta/2}}\right)}_{\to -x_2 \ (n\to\infty)} \to 0 \ (n \to \infty).$$

Therefore, continuing with (5.9),

$$\begin{aligned} &\liminf_{n\to\infty} \inf_{\tilde{c}_{i_n j_n}\in \mathrm{R}} \|\phi_{i_n}\psi_{j_n} - \tilde{c}_{i_n j_n}\|_{H^{-1/2}(Q)} \\ &\geq \liminf_{n\to\infty} \inf_{\tilde{c}_{i_n j_n}\in \mathrm{R}} \|\psi_{j_n} + \frac{\phi_{i_n} - 1}{1 - \left(\frac{j_n}{j_n-1}\right)^{-\beta/2}} - \tilde{c}_{i_n j_n}\|_{H^{-1/2}(Q)} \\ &= \liminf_{n\to\infty} \inf_{\tilde{c}_{i_n j_n}\in \mathrm{R}} \|\psi_{j_n} + \frac{\phi_{i_n}}{1 - \left(\frac{j_n}{j_n-1}\right)^{-\beta/2}} - \tilde{c}_{i_n j_n}\|_{H^{-1/2}(Q)} \end{aligned} \tag{5.11}$$

Now, a direct calculation shows that for $i_n \in \{2N_n + 1, \ldots, 3N_n\}$ and $j_n \in \{2, \ldots, 2N_n\}$ there exists a sequence of constants $\{c_n\}$ such that

$$\frac{\phi_{i_n}(x_1)}{1 - \left(\frac{j_n}{j_n-1}\right)^{-\beta/2}} - c_n - (2\lambda - 1)\frac{j_n}{i_n}x_1 \to 0 \quad (n \to \infty).$$

Thus, again using (5.10), we obtain

$$\begin{aligned} &\liminf_{n\to\infty} \inf_{\tilde{c}_{i_n j_n}\in \mathrm{R}} \|\psi_{j_n} + \frac{\phi_{i_n}}{1 - \left(\frac{j_n}{j_n-1}\right)^{-\beta/2}} - \tilde{c}_{i_n j_n}\|_{H^{-1/2}(Q)} \\ &= \liminf_{n\to\infty} \inf_{\tilde{c}_{i_n j_n}\in \mathrm{R}} \| - x_2 + (2\lambda - 1)\frac{j_n}{i_n}x_1 - \tilde{c}_{i_n j_n}\|_{H^{-1/2}(Q)}. \end{aligned} \tag{5.12}$$

As $i_n \in \{2N_n + 1, \ldots, 3N_n\}$ and $j_n \in \{2, \ldots, 2N_n\}$, $j_n/i_n \in (0, 1)$. Hence,

$$\begin{aligned} &\liminf_{n\to\infty} \inf_{\tilde{c}_{i_n j_n}\in \mathrm{R}} \| - x_2 + (2\lambda - 1)\frac{j_n}{i_n}x_1 - \tilde{c}_{i_n j_n}\|_{H^{-1/2}(Q)} \geq \\ &\inf_{c\in(0,1)} \inf_{\tilde{c}_{i_n j_n}\in \mathrm{R}} \| - x_2 + (2\lambda - 1)cx_1 - \tilde{c}_{i_n j_n}\|_{H^{-1/2}(Q)} > 0. \end{aligned} \tag{5.13}$$

Combining (5.11)–(5.13) yields a contradiction to the assumption $d_{N_n, i_n j_n} \to 0$. Hence (5.8) holds.

Noting that $s_{i-1} \ge (1/3)^\beta$ for $i \ge 2N+1$, then from (5.7) we obtain, for another constant $\tilde{c} > 0$,

$$\begin{aligned} \|x_1^{\lambda-1/2} x_2^{-1/2} - c_{ij}\|_{H^{-1/2}(\gamma_{ij})} &\succeq \tilde{c} h_i^{1/2} h_j (s_{j-1}^{-1/2} - s_j^{-1/2}) && (5.14) \\ &= \tilde{c} h_i^{1/2} \left(\frac{j}{6N}\right)^\beta \left(1 - (\frac{j-1}{j})^\beta\right) \left(\frac{j}{6N}\right)^{-\beta/2} \left((\frac{j-1}{j})^{-\beta/2} - 1\right) \\ &\succeq h_i^{1/2} j^{\frac{\beta}{2}-2} N^{-\frac{\beta}{2}} && (5.15) \end{aligned}$$

for $i = 2N+1, \dots, 3N$, $j = 2, \dots, 2N$. Using (3.1) and combining (5.6) with (5.15) we obtain

$$\begin{aligned} \|x_1^{\lambda-1/2} x_2^{-1/2} - \phi\|^2_{H^{-1/2}(\gamma)} &\succeq \sum_{i=2N+1}^{3N} \sum_{j=1}^{2N} \|x_1^{\lambda-1/2} x_2^{-1/2} - \phi\|^2_{H^{-1/2}(\gamma_{ij})} \\ &\succeq \sum_{i=2N+1}^{3N} h_i \left(h^\beta + N^{-\beta} \sum_{j=2}^{2N} j^{\beta-4} \right) \succeq h^\beta + N^{-\beta} N^{\beta-3} = h^\beta + h^3. \end{aligned}$$

Choosing $\beta = 1$ and $\beta > 3$ we obtain the smallest lower bounds for the uniform mesh and the optimally graded mesh, respectively, as stated in the theorem. □

Remark 5.1 *Using the optimal a priori error estimate of the theorem one obtains*

$$\|\mathbf{v} - \mathbf{v}_{h_2}\|_{\mathbf{V}} \le c \left(\frac{h_2}{h_1}\right)^\alpha \|\mathbf{v} - \mathbf{v}_{h_1}\|_{\mathbf{V}}$$

for the solution $\mathbf{v}$ *and Galerkin approximations* $\mathbf{v}_{h_i}$ *of* (1.3) *in the presence of the strong edge singularities where* $\alpha \in [1/2, 3/2]$ *depends on the mesh grading. Therefore, choosing a sufficiently fine mesh for the definition of the enriched ansatz space* $\tilde{\mathbf{S}}$ *in* (2.4) *one obtains a constant parameter* $\sigma < 1$ *in the saturation assumption* (A1)*, in the case of graded or uniform meshes.*

Remark 5.2 *A posteriori error estimation and adaptive mesh refinement are different tasks. In order to guarantee the saturation property asymptotically (in the presence of singularities) for adaptively refined meshes additional refinement criteria seem to be necessary.*

For instance, convergence for problems with solutions that behave like (5.1) *requires that every element will be refined eventually. When starting with rectangular elements, our adaptive algorithm refines them by halving in one or both directions, thus giving meshes where quotients of widths (in a certain direction) of neighboring elements (longer length divided by shorter or equal length) are integer powers of* 2*.*

Let us consider an edge defined by $x = 0$ *and let us exclude corners such that the dominant behaviour of the unknown solution is* $x_2^{-1/2}$*. Our numerical results show that, in the preasymptotic range and close to the edge (e.g. on the subdomain* $\gamma = (0,1) \times (0,1) \times \{0\}$*) the algorithm produces a mesh that is geometrically graded towards the edge. It is of the type* $\bar{\gamma} = \cup_{j=1}^{J+1} \bar{\gamma}_j$ *for a* *certain integer J with $\gamma_j = (0,1)\times(s_{j-1},s_j)\times\{0\}$ and $s_0 = 0$, $s_j = 1/2^{J-j+1}$ ($j = 1,\dots,J+1$). Designing the algorithm such that the width of the largest elements is bounded from above (here only in direction x_2), e.g. $h_j := s_j - s_{j-1} \le 1/2^l$ for an integer $l \le J$ (where $l \to \infty$ is necessary to guarantee convergence), we obtain a mixed graded-uniform mesh. The x_2-coordinates of the element edges are*

s_0,	s_1,	s_2,	$\cdots$	s_{J-l},	$s_{J-l+1} = \tilde s_{J-l+1}$,	$\tilde s_{J-l+2}$,	$\cdots$	$\tilde s_{J-l+2^l}$
0,	$\frac{1}{2^J}$,	$\frac{1}{2^{J-1}}$,	$\cdots$	$\frac{1}{2^{l+1}}$,	$\frac{1}{2^l}$,	$\frac{2}{2^l}$,	$\cdots$	$\frac{2^l}{2^l} = 1$

That means that $\{\gamma_j = (0,1)\times(s_{j-1},s_j)\times\{0\};\ j = 1,\dots,J-l+1\}$ constitutes a geometrically graded mesh whereas $\{\gamma_j = (0,1)\times(\tilde s_{j-1},\tilde s_j)\times\{0\};\ j = J-l+2,\dots,J-l+2^l\}$ form a (in x_2-direction) uniform mesh.

Direct application of the estimates of this section (substituting the old mesh data s_j, h_j by the new data s_j, $\tilde s_j$ and h_j), and considering the particular case of the edge singularity $x_2^{-1/2}$, yields (cf. (5.14))

$$\|x_2^{-1/2} - c_j\|_{H^{-1/2}(\gamma_j)} \succeq \frac{1}{2^{(J-j)/2}} \qquad (j = 1,\dots,J-l+1)$$

and

$$\|x_2^{-1/2} - c_j\|_{H^{-1/2}(\gamma_{J-l+j})} \succeq \frac{1}{2^{l/2}}(j-1)^{-1/2}(1-(1-\frac{1}{j})^{1/2}) \qquad (j = 2,\dots,2^l).$$

This gives

$$\|x_2^{-1/2} - c_j\|^2_{H^{-1/2}(\gamma)} \succeq \sum_{j=1}^{J-l+1}\frac{1}{2^{J-j}} + \frac{1}{2^l}\sum_{j=2}^{2^l}(j-1)^{-1}(1-(1-\frac{1}{j})^{1/2})^2 \succeq \frac{1}{2^l} = \max_j h_j.$$

Therefore, the uniform part of the mesh dominates the convergence which is linear in $h := \max_j h_j$. Since the mixed graded-uniform mesh is a refinement of the uniform mesh of width h (in x_2-direction), the approximation error for the mixed mesh can only be less than or equal to the error for the uniform mesh. For uniform meshes linear convergence is obtained such that, together with the lower bound above, the convergence for the mixed mesh is linear in h. Therefore, we conjecture that the analysis of this section applies to the meshes generated by our algorithm (if one incorporates an upper bound for the maximum elements' width) if singularities of the mentioned types (5.1) *are present. In particular, the saturation assumption then follows for sufficiently fine meshes from Remark* 5.1 *with $\alpha = 1$.*

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