Coupling of mixed finite element and stabilized boundary element methods for a fluid-solid interaction problem in 3D*

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Abstract

We introduce and analyze a Mixed-FEM and BEM coupling for a three-dimensional fluid-solid interaction problem. The media are governed by the acoustic and elastodynamic equations in time-harmonic regime coupled with adequate transmission conditions posed on the interface between the two media. We employ a dual-mixed variational formulation in the solid, in which the Cauchy stress tensor and the rotation are the only unknowns, and use the exterior acoustic problem to deduce a nonlocal boundary condition for this problem. The variational formulation in the solid is completed with boundary integral equations relating the Cauchy data of the acoustic problem on the coupling interface. Both the trace and the normal derivative of the pressure appear as boundary variables in the global FEM-BEM formulation and the pressure in the exterior domain may be recovered by means of an integral representation formula. A crucial point in our formulation is the stabilization technique introduced by Hiptmair and co-authors to avoid the well-known instability issue appearing in the BEM treatment of the exterior Helmholtz problem. The main novelty of this formulation, with respect to a previous approach, consists in reducing the computational domain to the solid media and providing a more accurate treatment of the far field effect. We show that a suitable decomposition of the space of stresses allows the application of the Babuška-Brezzi theory and the Fredholm alternative for concluding the solvability of the whole coupled problem. The unknowns of the solid are then approximated by the Arnold-Falk-Winther finite element of order 1, which yields a conforming Galerkin scheme. The stability and convergence of the discrete method relies on a stable decomposition of the finite element space used to approximate the stress and also on a classical result on conforming Galerkin approximations for Fredholm operators of index zero.

Key words: mixed finite elements, Helmholtz equation, elastodynamic equation

Mathematics subject classifications (1991): 65N30, 65N12, 65N15, 74F10, 74B05, 35J05

1 Introduction

In this paper we introduce a new numerical scheme to compute the scattered waves and the elastic vibrations that take place in the interaction between a bounded solid body and the compressible inviscid fluid surrounding it, when time–harmonic excitations of the system are imposed. Displacement-based

*This research was partially supported by BASAL project CMM, Universidad de Chile, by Centro de Investigación en Ingeniería Matemática (CI^{2}MA), Universidad de Concepción, by CONICYT-Chile through the Fondecyt project 1110324, and by the Ministery of Education of Spain through the project MTM2010-18427.

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formulations are generally used in the solid for this interaction problem (see for instance [4], [18], [19], [20], [21], [25], and [26]). Here we are rather interested in situations in which a direct finite element approximation of the stresses is needed. Our aim is to provide improvements of the approach using the dual-mixed formulation introduced in [11, 12, 13].

The interaction problem studied in [11] and [12] refers to a 2D model in which an elastic body is subject to a given incident wave that travels in the fluid surrounding it. The transmission conditions hold on the boundary of the solid and they are given by the equilibrium of forces and the equality of the normal displacements from both media. The original model is simplified a bit more in [11] by assuming that the fluid occupies an annular region, whence a Robin boundary condition imitating the behavior of the scattered field at infinity is imposed on its exterior boundary, which must be located sufficiently far from the obstacle. Then, a dual-mixed approach is used in the solid and the usual primal method is maintained in the fluid region. In addition, since the first transmission condition mentioned above becomes essential, it is enforced weakly by means of a Lagrange multiplier. In this way, the stress tensor in the solid and the pressure in the fluid constitute the main unknowns of the resulting formulation. A judicious decomposition of the space of stresses renders suitable the application of a Fredholm alternative for the analysis of the whole coupled problem. The associated discrete scheme is defined with PEERS [1] elements in the obstacle and the traditional first order Lagrange finite elements in the fluid domain. The stability and convergence of this Galerkin method also relies on a stable decomposition of the finite element space used to approximate the stress variable. In [12] the strategy from [11] is modified and, instead of considering a Robin condition on the exterior boundary, the far field behavior is imposed exactly through non-local absorbing boundary conditions based on boundary integral equations. In this case, the exterior boundary may be any parametrizable smooth closed curve containing the solid. In this way, the discretization procedure proposed in [12] couples the primal/dual-mixed finite element scheme from [11] with a suitable boundary element method arising from a combined double and single layer potential representation of the scattered wave (see [9]).

A new finite element method for the 3D version of the interaction problem studied in [11] is analyzed in [13]. The approach from [11] is simplified by incorporating the equilibrium of forces (see the first equation in (2.2) below) into the definition of the product space to which the stress $\boldsymbol{\sigma}$ of the solid and the pressure p of the fluid belong. This prevents from introducing further unknowns (Lagrange multipliers) on the boundary of the solid simplifying by the way the saddle point structure of the problem and reducing the number of unknowns. Moreover, the strategy involving a Lagrange multiplier on the transmission boundary requires the use of two finite element meshes satisfying a stability condition between their corresponding mesh sizes, which certainly constitutes a very cumbersome restriction in 3D computations. The discrete version of the problem is built upon the lowest order Arnold-Falk-Winther (AFW) element [3] in the solid and Lagrange finite element subspaces of order 1 for the pressure. It is worthwhile to notice here that, because of the coincidence between the polynomial shape functions approximating $\boldsymbol{\sigma}\boldsymbol{\nu}$ and $-p\boldsymbol{\nu}$ on the interface, this numerical scheme generates a conforming finite element subspace for the pair $(\boldsymbol{\sigma}, p)$. In other words, the essential transmission condition incorporated in the definition of the continuous space is also satisfied at the discrete level.

Now, the main purpose of the present paper is to incorporate more efficiently the far-field effects into the finite element discretization of the 3D problem presented in [13]. We will use the exterior Helmholtz problem to provide a nonlocal boundary condition for the interior elasticity problem through a Dirichlet-to-Neumann (**DtN**) operator associated with the acoustic problem and expressed in terms of boundary integral operators. It is well known that, if this strategy is not applied carefully, it leads to a variational formulation that suffers from serious drawbacks. Indeed, the well posedness of the resulting formulation (at the continuous level) requires regularity assumptions for the interface between the two media that may not be fulfilled in practice (cf. [4, 17, 27]). Moreover, the corresponding numerical method may exhibit an unstable behavior at the vicinity of a countable set of frequencies (cf. [4, 17, 25]). This is the reason for which the method presented in [12] (in the two dimensional case) imposes the absorbing boundary conditions on a smooth but arbitrary interface containing the obstacle in its interior. This procedure enlarges the domain for finite element computations, but the resulting method does not suffer from the limitations mentioned above.

Our analysis here (for the 3D case) is different, it relays upon the stabilization technique introduced in [8] and [16]. It permits to reduce the computational domain to the obstacle and gives rise to a convergent mixed-FEM and BEM scheme that is safe from the spurious modes that appear when the fluid frequency is related to the Dirichlet eigenvalues of Laplace problem in the interior domain. The rest of this work is organized as follows. In Sections 2 and 3 we describe the fluid-solid interaction problem and derive its continuous variational formulation. Then, in Section 4, we show that the resulting saddle point problem is well posed. Finally, the corresponding Galerkin scheme is analyzed in Section 5.

We end this section with further notations to be used below. Since in the sequel we deal with complex valued functions, we let $\mathbb{C}$ be the set of complex numbers, use the symbol $\imath$ for $\sqrt{-1}$, and denote by $\overline{z}$ and $|z|$ the conjugate and modulus, respectively, of each $z \in \mathbb{C}$. Also, we let $\mathbf{I}$ be the identity matrix of $\mathbb{C}^{3\times 3}$, and given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij}) \in \mathbb{C}^{3\times 3}$, we define the deviator tensor $\boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{3}\operatorname{tr}(\boldsymbol{\tau})\,\mathbf{I}$, the tensor product $\boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{3} \tau_{ij}\,\zeta_{ij}$, and the conjugate tensor $\overline{\boldsymbol{\tau}} := (\overline{\tau}_{ij})$. In turn, in what follows we utilize standard simplified terminology for Sobolev spaces and norms. In particular, if $\mathcal{O}$ is a domain, $\mathcal{S}$ is a closed Lipschitz curve, and $r \in \mathbb{R}$, we define

$$\mathbf{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^3\,, \quad \mathbb{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^{3\times 3}\,, \quad \text{and} \quad \mathbf{H}^r(\mathcal{S}) := [H^r(\mathcal{S})]^3\,.$$

However, when $r = 0$ we usually write $\mathbf{L}^2(\mathcal{O})$, $\mathbb{L}^2(\mathcal{O})$, and $\mathbf{L}^2(\mathcal{S})$ instead of $\mathbf{H}^0(\mathcal{O})$, $\mathbb{H}^0(\mathcal{O})$, and $\mathbf{H}^0(\mathcal{S})$, respectively. The corresponding norms are denoted by $\|\cdot\|_{r,\mathcal{O}}$ (for $H^r(\mathcal{O})$, $\mathbf{H}^r(\mathcal{O})$, and $\mathbb{H}^r(\mathcal{O})$) and $\|\cdot\|_{r,\mathcal{S}}$ (for $H^r(\mathcal{S})$ and $\mathbf{H}^r(\mathcal{S})$). In general, given any Hilbert space H, we use $\mathbf{H}$ and $\mathbb{H}$ to denote H^3 and $H^{3\times 3}$, respectively. Furthermore, the Hilbert space

$$\mathbf{H}(\mathrm{div};\mathcal{O}) := \left\{ \mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \quad \mathrm{div}\,\mathbf{w} \in L^2(\mathcal{O}) \right\},$$

is standard in the realm of mixed problems (see [6]). The space of matrix valued functions whose rows belong to $\mathbf{H}(\mathrm{div};\mathcal{O})$ will be denoted $\mathbb{H}(\mathbf{div};\mathcal{O})$. The Hilbert norms of $\mathbf{H}(\mathrm{div};\mathcal{O})$ and $\mathbb{H}(\mathbf{div};\mathcal{O})$ are denoted by $\|\cdot\|_{\mathrm{div};\mathcal{O}}$ and $\|\cdot\|_{\mathbf{div};\mathcal{O}}$, respectively. Note that if $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\mathcal{O})$, then $\mathbf{div}\,\boldsymbol{\tau} \in \mathbf{L}^2(\mathcal{O})$. Finally, we employ $\mathbf{0}$ to denote a generic null vector (including the null functional and operator), and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2 The fluid-solid interaction problem

We consider an incident acoustic wave upon a bounded elastic body (obstacle) in $\mathbb{R}^3$ that is fully surrounded by a fluid, and aim to determine both the response of the body and the scattered wave. We assume that the obstacle is represented by a polyhedron Ω whose boundary is denoted Γ. We also let $\Omega^+ := \mathbb{R}^3 \setminus \overline{\Omega}$. We assume that the incident wave and the volume force acting on the body exhibit a time-harmonic behaviour with frequency ω and amplitudes p_i and $\mathbf{f}$, respectively, so that p_i satisfies the Helmholtz equation in Ω^+. Hence, we may consider that this interaction problem is posed in the frequency domain. In this way, and since, following [11], we plan to employ a mixed variational formulation in the solid, our main unknowns become the amplitude $\boldsymbol{\sigma} : \Omega \to \mathbb{C}^{3\times 3}$ of the

Cauchy stress tensor, the amplitude $\mathbf{u} : \Omega \to \mathbb{C}^3$ of the displacement field, and the amplitude of the total (incident + scattered) pressure $p := p_i + p_s : \Omega^+ \to \mathbb{C}$.

The fluid is assumed to be perfect, compressible, and homogeneous, with mass density ρ_f and wave number $\kappa := \dfrac{\omega}{v_0}$, where v_0 is the speed of sound in the linearized fluid. In addition, the solid is supposed to be isotropic and linearly elastic with mass density ρ_s and Lamé constants μ and λ, which means, in particular, that the corresponding constitutive equation is given by

$$\boldsymbol{\sigma} = \mathcal{C}\,\boldsymbol{\varepsilon}(\mathbf{u}) \qquad \text{in} \quad \Omega\,,$$

where $\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathtt{t}}\right)$ is the strain tensor of small deformations, ∇ is the gradient tensor, and $\mathcal{C}$ is the elasticity operator given by Hooke's law, that is

$$\mathcal{C}\,\boldsymbol{\zeta} := \lambda\,\mathrm{tr}(\boldsymbol{\zeta})\,\mathbf{I} + 2\,\mu\,\boldsymbol{\zeta} \qquad \forall\,\boldsymbol{\zeta} \in \mathbb{L}^2(\Omega)\,. \tag{2.1}$$

Consequently, under the hypotheses of small oscillations, both in the solid and the fluid, the unknowns $\boldsymbol{\sigma}$, $\mathbf{u}$, and p satisfy the elastodynamic and acoustic equations in time-harmonic regime, that is:

$$\begin{aligned} \mathbf{div}\,\boldsymbol{\sigma} + \kappa_s^2\,\mathbf{u} &= -\,\mathbf{f} && \text{in} \quad \Omega\,,\\ \Delta p + \kappa^2\,p &= 0 && \text{in} \quad \Omega^+\,, \end{aligned}$$

where the wave number κ_s of the solid is defined by $\sqrt{\rho_s}\,\omega$, together with the transmission conditions:

$$\begin{aligned} \boldsymbol{\sigma}\,\boldsymbol{\nu} &= -\,p\,\boldsymbol{\nu} && \text{on} \quad \Gamma\,,\\ \rho_f\,\omega^2\,\mathbf{u}\cdot\boldsymbol{\nu} &= \frac{\partial p}{\partial \boldsymbol{\nu}} && \text{on} \quad \Gamma\,, \end{aligned} \tag{2.2}$$

and the behaviour at infinity given by

$$\frac{\partial(p - p_i)}{\partial r} - \imath\,\kappa\,(p - p_i) = o(r^{-1})\,, \tag{2.3}$$

as $r := \|\mathbf{x}\| \to +\infty$, uniformly for all directions $\dfrac{\mathbf{x}}{\|\mathbf{x}\|}$. Hereafter, $\mathbf{div}$ stands for the usual divergence operator div acting on each row of the tensor, $\|\mathbf{x}\|$ is the euclidean norm of a vector $\mathbf{x} := (x_1, x_2, x_3)^{\mathtt{t}} \in \mathbb{R}^3$, and $\boldsymbol{\nu}$ denotes the unit outward normal on Γ, that is pointing toward Ω^+. The transmission conditions given in (2.2) constitute the equilibrium of forces and the equality of the normal displacements of the solid and fluid, whereas the equation (2.3) is known as the Sommerfeld radiation condition. Notice that, as a consequence of (2.3), the outgoing waves are absorbed in the far field.

It is known that if $p_i = 0$ and $\mathbf{f} = \mathbf{0}$ then $p = 0$ and $\mathbf{u}$ is solution of (see [23])

$$\begin{aligned} \boldsymbol{\sigma} &= \mathcal{C}\,\boldsymbol{\varepsilon}(\mathbf{u}) && \text{in} \quad \Omega\,,\\ \mathbf{div}\,\boldsymbol{\sigma} + \kappa_s^2\,\mathbf{u} &= \mathbf{0} && \text{in} \quad \Omega\,,\\ \boldsymbol{\sigma}\boldsymbol{\nu} &= \mathbf{0} && \text{on} \quad \Gamma\,,\\ \mathbf{u}\cdot\boldsymbol{\nu} &= 0 && \text{on} \quad \Gamma\,. \end{aligned} \tag{2.4}$$

It turns out that for certain regions and some frequencies $\omega = \frac{\kappa_s}{\sqrt{\rho_s}}$, known as *Jones frequencies*, problem (2.4) has nontrivial solutions. This seems to be a rare eventuality but we will, in any case, assume that (2.4) only admits the trivial solution.

3 A variational formulation with non-local boundary conditions

Before dealing with the variational formulation of our problem, let us recall some basic properties of trace operators. First of all, we point out that we will use in the sequel Sobolev spaces $H^r(\Omega)$ and $H^r(\Gamma)$ of index $r \in \mathbb{R}$ whose definitions may be found in [24, 27]. We begin by denoting γ^- the interior trace on Γ. In other words, $\gamma^- : H^1(\Omega) \to H^{1/2}(\Gamma)$ is the unique bounded operator such that $\gamma^- v = v|_\Gamma$ for any smooth function $v \in C^\infty(\overline{\Omega})$. In the unbounded domain Ω^+, we consider the space $H^1_{\text{loc}}(\Omega^+)$ of distributions u in Ω^+ such that $\rho u \in H^1(\Omega^+)$ for any $\rho \in C_0^\infty(\Omega^+)$, and define similarly the exterior trace operator $\gamma^+ : H^1_{\text{loc}}(\Omega^+) \to H^{1/2}(\Gamma)$. It is well-known that both γ^- and γ^+ are surjective. We also recall that $H^{-1/2}(\Gamma)$ is the dual of $H^{1/2}(\Gamma)$ and that the bilinear form

$$\langle\, \xi, \varphi \,\rangle := \int_\Gamma \xi\varphi \quad \forall\, \xi, \varphi \in L^2(\Gamma)$$

can be continuously extended to a duality pairing on $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$, which is also denoted throughout the paper by $\langle \cdot, \cdot \rangle$. In turn, the inner product in $L^2(\Gamma)$ is given by $\langle\, \psi, \overline{\varphi} \,\rangle$. In addition, we let

$$H^1(\Delta, \Omega) := \Big\{ v \in H^1(\Omega) : \quad \Delta v \in L^2(\Omega) \Big\},$$

and define the interior normal derivative $\partial^-_{\boldsymbol{\nu}} : H^1(\Delta, \Omega) \to H^{-1/2}(\Gamma)$ by

$$\langle\, \partial^-_{\boldsymbol{\nu}} u, \varphi \,\rangle := \int_\Omega \nabla u \cdot \nabla v + \int_\Omega v \,\Delta u$$

with any $v \in H^1(\Omega)$ such that $\gamma^- v = \varphi \in H^{1/2}(\Gamma)$. Similarly, we let

$$H^1_{\text{loc}}(\Delta, \Omega^+) = \Big\{ v \in H^1_{\text{loc}}(\Omega^+) : \quad \Delta v \in L^2_{\text{loc}}(\Omega^+) \Big\},$$

and define the exterior normal derivative $\partial^+_{\boldsymbol{\nu}} : H^1_{\text{loc}}(\Delta, \Omega^+) \to H^{-1/2}(\Gamma)$ by

$$\langle\, \partial^+_{\boldsymbol{\nu}} u, \varphi \,\rangle := -\int_{\Omega^+} \nabla u \cdot \nabla v - \int_{\Omega^+} v \,\Delta u$$

with any $v \in H^1_{\text{loc}}(\Omega^+)$ of compact support such that $\gamma^+ v = \varphi \in H^{1/2}(\Gamma)$. The minus sign here is due to the fact that the normal vector is oriented towards the exterior of Ω. Also, we recall that there exists a unique bounded and onto application $\gamma_{\boldsymbol{\nu}} : \mathbb{H}(\mathbf{div}; \Omega) \to \mathbf{H}^{-1/2}(\Gamma)$ known as the normal trace operator and characterized by

$$\langle\, \gamma_{\boldsymbol{\nu}}\, \boldsymbol{\tau}, \boldsymbol{\xi} \,\rangle := \int_\Omega \mathbf{v} \cdot \mathbf{div}\, \boldsymbol{\tau} + \int_\Omega \boldsymbol{\tau} : \nabla \mathbf{v} \qquad \text{with any } \mathbf{v} \in \mathbf{H}^1(\Omega) \text{ such that } \gamma^- \mathbf{v} = \boldsymbol{\xi} \in \mathbf{H}^{1/2}(\Gamma)\,. \tag{3.1}$$

Hereafter, $\gamma^- \mathbf{v}$ is the usual trace operator γ^- acting componentwise. Finally, jumps and averages of the trace and the normal derivative operators across Γ are denoted

$$[\gamma u] := \gamma^+ u - \gamma^- u, \qquad [\partial_{\boldsymbol{\nu}} u] = \partial^+_{\boldsymbol{\nu}} u - \partial^-_{\boldsymbol{\nu}} u,$$

$$\{\gamma u\} = \tfrac{1}{2}(\gamma^+ u + \gamma^- u), \qquad \text{and} \qquad \{\partial_{\boldsymbol{\nu}} u\} = \tfrac{1}{2}(\partial^+_{\boldsymbol{\nu}} u + \partial^-_{\boldsymbol{\nu}} u)\,.$$

3.1 The variational formulation in the obstacle

In this section we describe the steps to obtain a mixed variational formulation in the solid Ω. We follow the usual procedure (see [1]) and introduce the rotation

$$\mathbf{r} := \tfrac{1}{2}(\nabla \mathbf{u} - (\nabla \mathbf{u})^{\mathtt{t}}) \in \mathbb{L}^2_{\mathtt{asym}}(\Omega)$$

as a further unknown, where

$$\mathbb{L}^2_{\mathtt{asym}}(\Omega) := \Big\{ \mathbf{s} \in \mathbb{L}^2(\Omega) : \quad \mathbf{s}^{\mathtt{t}} = -\mathbf{s} \Big\}.$$

The constitutive equation can then be rewritten in the form

$$\mathcal{C}^{-1}\boldsymbol{\sigma} = \boldsymbol{\varepsilon}(\mathbf{u}) = \nabla \mathbf{u} - \mathbf{r},$$

which, multiplying by a function $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega)$ and applying (3.1), yields

$$\int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} + \int_\Omega \mathbf{u} \cdot \mathbf{div}\,\boldsymbol{\tau} - \langle \gamma_{\boldsymbol{\nu}}\boldsymbol{\tau}, \gamma^- \mathbf{u} \rangle + \int_\Omega \boldsymbol{\tau} : \mathbf{r} = 0. \tag{3.2}$$

Then, replacing back

$$\mathbf{u} = -\frac{1}{\kappa_s^2}\big(\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}\big),$$

into (3.2), gives

$$\int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} - \frac{1}{\kappa_s^2}\int_\Omega \mathbf{div}\,\boldsymbol{\sigma} \cdot \mathbf{div}\,\boldsymbol{\tau} - \langle \gamma_{\boldsymbol{\nu}}\boldsymbol{\tau}, \gamma^- \mathbf{u} \rangle_\Gamma + \int_\Omega \boldsymbol{\tau} : \mathbf{r} = \frac{1}{\kappa_s^2}\int_\Omega \mathbf{f} \cdot \mathbf{div}\,\boldsymbol{\tau}. \tag{3.3}$$

¿From now on, the first transmission condition in (2.2), that is $\gamma_{\boldsymbol{\nu}}\boldsymbol{\sigma} = -(\gamma^+ p)\,\boldsymbol{\nu}$ on Γ will be imposed by tying the stress variable $\boldsymbol{\sigma}$ to an additional boundary unknown ψ representing $\gamma^+ p$. More precisely, we will look for the unknown $\widehat{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}, \psi)$ in the closed subspace $\mathbb{X}$ of $\mathbb{H}(\mathbf{div};\Omega) \times H^{1/2}(\Gamma)$ given by

$$\mathbb{X} := \Big\{ \widehat{\boldsymbol{\tau}} := (\boldsymbol{\tau}, \varphi) \in \mathbb{H}(\mathbf{div};\Omega) \times H^{1/2}(\Gamma) : \qquad \gamma_{\boldsymbol{\nu}}\boldsymbol{\tau} + \varphi\,\boldsymbol{\nu} = \mathbf{0} \quad \text{on} \quad \Gamma \Big\},$$

which is endowed with the norm

$$\|\widehat{\boldsymbol{\tau}}\|^2_{\mathbb{X}} := \|\boldsymbol{\tau}\|^2_{\mathbf{div};\Omega} + \|\varphi\|^2_{1/2,\Gamma}.$$

In addition, we also let from now on $\mathbb{Y} := \mathbb{L}^2_{\mathtt{asym}}(\Omega)$.

Now, taking the test function in (3.3) as the first component of the pair $\widehat{\boldsymbol{\tau}} := (\boldsymbol{\tau}, \varphi) \in \mathbb{X}$, we find that this equation becomes

$$\int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} - \frac{1}{\kappa_s^2}\int_\Omega \mathbf{div}\,\boldsymbol{\sigma} \cdot \mathbf{div}\,\boldsymbol{\tau} + \langle (\gamma^- \mathbf{u}) \cdot \boldsymbol{\nu}, \varphi \rangle_\Gamma + \int_\Omega \boldsymbol{\tau} : \mathbf{r} = \frac{1}{\kappa_s^2}\int_\Omega \mathbf{f} \cdot \mathbf{div}\,\boldsymbol{\tau} \tag{3.4}$$

for all $\widehat{\boldsymbol{\tau}} := (\boldsymbol{\tau}, \varphi) \in \mathbb{X}$. Next, according to the second transmission condition in (2.2), we replace $(\gamma^- \mathbf{u}) \cdot \boldsymbol{\nu}$ by $\frac{1}{\rho_f\,\omega^2}\,\partial^+_{\boldsymbol{\nu}}\,p$ and multiply by $-\rho_f\,\omega^2$ to obtain from (3.4)

$$\rho_f\,\omega^2 \left\{ \frac{1}{\kappa_s^2}\int_\Omega \mathbf{div}\,\boldsymbol{\sigma} \cdot \mathbf{div}\,\boldsymbol{\tau} - \int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} - \int_\Omega \boldsymbol{\tau} : \mathbf{r} \right\} - \langle \vartheta, \varphi \rangle = F_0(\widehat{\boldsymbol{\tau}})$$

for all $\widehat{\boldsymbol{\tau}} := (\boldsymbol{\tau}, \varphi) \in \mathbb{X}$, where

$$\vartheta := \partial_{\boldsymbol{\nu}}^{+} p_s \quad \text{and} \quad F_0(\widehat{\boldsymbol{\tau}}) := -\frac{\rho_f\,\omega^2}{\kappa_s^2} \int_\Omega \mathbf{f} \cdot \mathbf{div}\,\boldsymbol{\tau} + \langle\, \partial_{\boldsymbol{\nu}}^{+} p_i, \varphi \,\rangle.$$

Recall here that p_s is the scattered component of the total pressure p.

Imposing now weakly the symmetry of $\boldsymbol{\sigma}$, we arrive at the following variational formulation in the solid domain: Find $(\widehat{\boldsymbol{\sigma}}, \mathbf{r}) \in \mathbb{X} \times \mathbb{Y}$ such that

$$\begin{aligned} \rho_f\,\omega^2 \left\{ \frac{1}{\kappa_s^2} \int_\Omega \mathbf{div}\,\boldsymbol{\sigma} \cdot \mathbf{div}\,\boldsymbol{\tau} - \int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} \right\} - \rho_f\,\omega^2 \int_\Omega \boldsymbol{\tau} : \mathbf{r} - \langle\, \vartheta, \varphi \,\rangle &= F_0(\widehat{\boldsymbol{\tau}})\,, \\ - \rho_f\,\omega^2 \int_\Omega \boldsymbol{\sigma} : \mathbf{s} &= 0\,, \end{aligned} \tag{3.5}$$

for all $(\widehat{\boldsymbol{\tau}}, \mathbf{s}) \in \mathbb{X} \times \mathbb{Y}$.

3.2 Nonlocal boundary conditions

In order to deal with the problem in the unbounded region, we need to establish some notations and recall well-known properties of boundary integral operators. We begin with the integral representation formula

$$p_s = -\,\Psi_{SL}^{\kappa}(\vartheta) + \Psi_{DL}^{\kappa}(\gamma^{+} p_s) \qquad \text{in} \quad \Omega^{+}\,, \tag{3.6}$$

where κ is the wave number in the fluid (see definition in Section 2), and Ψ_{SL}^{κ} and Ψ_{DL}^{κ} are the single layer and double layer potentials, respectively, defined by

$$\Psi_{SL}^{\kappa}(\vartheta)(\mathbf{x}) = \int_\Gamma G_\kappa(\mathbf{x}, \mathbf{y})\,\vartheta(\mathbf{y})\,\mathrm{d}S(\mathbf{y}) \qquad \forall\,\mathbf{x} \in \Omega^{+}\,,$$

and

$$\Psi_{DL}^{\kappa}(\varphi)(\mathbf{x}) = \int_\Gamma \partial_{\boldsymbol{\nu}(\mathbf{y})}\, G_\kappa(\mathbf{x}, \mathbf{y})\,\varphi(\mathbf{y})\,\mathrm{d}S(\mathbf{y}) \qquad \forall\,\mathbf{x} \in \Omega^{+}\,,$$

with $G_\kappa(\mathbf{x}, \mathbf{y})$, given by

$$G_\kappa(\mathbf{x}, \mathbf{y}) := \frac{1}{4\pi} \frac{\exp(\imath\,\kappa\,\|\mathbf{x} - \mathbf{y}\|)}{\|\mathbf{x} - \mathbf{y}\|} \qquad \forall\,\mathbf{x},\,\mathbf{y} \in \mathbb{R}^3,\ \mathbf{x} \neq \mathbf{y}\,,$$

being the radial outgoing fundamental solution of the Helmholtz equation in $\mathbb{R}^3$.

It is well known that the boundary integral operators defined by the following averages

$$V_\kappa := \{\gamma \Psi_{SL}^{\kappa}\} :\ H^{s-1/2}(\Gamma) \to H^{s+1/2}(\Gamma), \qquad K_\kappa := \{\gamma \Psi_{DL}^{\kappa}\} :\ H^{s+1/2}(\Gamma) \to H^{s+1/2}(\Gamma),$$

$$K_\kappa^{\mathtt{t}} := \{\partial_{\boldsymbol{\nu}} \Psi_{SL}^{\kappa}\} :\ H^{s-1/2}(\Gamma) \to H^{s-1/2}(\Gamma), \qquad W_\kappa := -\{\partial_{\boldsymbol{\nu}} \Psi_{DL}^{\kappa}\} :\ H^{s+1/2}(\Gamma) \to H^{s-1/2}(\Gamma)$$

are bounded for any $|s| < 1/2$. Moreover, the classical jump relations

$$[\gamma \Psi_{SL}^{\kappa}(\vartheta)] = 0, \qquad [\partial_{\boldsymbol{\nu}} \Psi_{SL}^{\kappa}(\vartheta)] = -\vartheta, \qquad \forall\,\vartheta \in H^{-1/2}(\Gamma),$$

$$[\gamma \Psi_{DL}^{\kappa}(\psi)] = \psi, \qquad [\partial_{\boldsymbol{\nu}} \Psi_{DL}^{\kappa}(\psi)] = 0, \qquad \forall\,\psi \in H^{1/2}(\Gamma)$$

provide the identities

$$\begin{aligned} \gamma^{\pm} \Psi_{SL}^{\kappa} &= V_\kappa, & \partial_{\boldsymbol{\nu}}^{\pm} \Psi_{SL}^{\kappa} &= K_\kappa^{\mathtt{t}} \mp \tfrac{1}{2}\mathrm{id}, \\ \gamma^{\pm} \Psi_{DL}^{\kappa} &= K_\kappa \pm \tfrac{1}{2}\mathrm{id}, & \partial_{\boldsymbol{\nu}}^{\pm} \Psi_{DL}^{\kappa} &= -W_\kappa. \end{aligned} \tag{3.7}$$

Hereafter, id denotes the identity operator. Also, it can be shown that $K^{\mathtt{t}}_\kappa$ is the adjoint of K_κ, i.e.,

$$\langle\, \chi, K_\kappa \varphi \,\rangle = \langle\, K^{\mathtt{t}}_\kappa \chi, \varphi \,\rangle \qquad \forall\, \varphi \in H^{1/2}(\Gamma)\,, \quad \forall\, \chi \in H^{-1/2}(\Gamma)\,.$$

The following two lemmas will be essential in the sequel.

Lemma 3.1 *The following operators are compact:*

$$\begin{aligned} V_\kappa - V_0 :\ H^{-1/2}(\Gamma) \ &\to\ H^{1/2}(\Gamma), & K_\kappa - K_0 :\ H^{1/2}(\Gamma) \ &\to\ H^{1/2}(\Gamma), \\ K^{\mathtt{t}}_\kappa - K^{\mathtt{t}}_0 :\ H^{-1/2}(\Gamma) \ &\to\ H^{-1/2}(\Gamma), & W_\kappa - W_0 :\ H^{1/2}(\Gamma) \ &\to\ H^{-1/2}(\Gamma). \end{aligned}$$

Proof. See [27, Lemma 3.9.8].

□

Lemma 3.2 *There exists $C_0 > 0$ such that*

$$\langle\, \chi, V_0\, \overline{\chi} \,\rangle \ \geq\ C_0\, \|\chi\|^2_{-1/2,\Gamma} \quad \forall\, \chi \,\in\, H^{-1/2}(\Gamma) \tag{3.8}$$

and

$$\langle\, W_0\, \varphi, \overline{\varphi} \,\rangle \,+\, \left| \int_\Gamma \varphi \right|^2 \ \geq\ C_0\, \|\varphi\|^2_{1/2,\Gamma} \quad \forall\, \varphi \,\in\, H^{1/2}(\Gamma)\,. \tag{3.9}$$

Proof. See [27, Theorem 3.5.3].

□

Applying γ^+ and $\partial^+_{\boldsymbol{\nu}}$ to the integral representation formula (3.6) and recalling that $\vartheta = \partial^+_{\boldsymbol{\nu}} p_s$ and $\psi = \gamma^+ p = \gamma^+(p_s + p_i)$ we deduce that

$$\psi - \gamma^+ p_i \,=\, \left(\tfrac{1}{2}\,\mathrm{id} \,+\, K_\kappa\right)(\psi - \gamma^+ p_i) \,-\, V_\kappa \vartheta \tag{3.10}$$

and

$$\vartheta \,=\, -\,W_\kappa(\psi - \gamma^+ p_i) \,+\, \left(\tfrac{1}{2}\,\mathrm{id} \,-\, K^{\mathtt{t}}_\kappa\right) \vartheta\,. \tag{3.11}$$

Combining (3.5) with (3.10) and (3.11) leads to the classical symmetric BEM-FEM formulation of Costabel. However, for the exterior Helmholtz equation, this formulation suffers from spurious modes when the wave number is related to a Dirichlet eigenvalue of the Laplace operator $-\Delta$ in Ω. We remedy this situation by using the stabilization strategy presented in [16, 8]. To this end we need to introduce the operator $M :\ H^{-1}(\Gamma) \to H^1(\Gamma)$, which, given $\xi \in H^{-1}(\Gamma)$, is characterized by

$$\langle\, \nabla_\Gamma (M\xi), \nabla_\Gamma \varphi \,\rangle + \langle\, M\xi, \varphi \,\rangle = \langle\, \xi, \varphi \,\rangle \qquad \forall\, \varphi \,\in\, H^1(\Gamma)\,, \tag{3.12}$$

where ∇_Γ is the tangential gradient operator

$$\nabla_\Gamma \varphi := \boldsymbol{\nu} \times \left(\nabla(\gamma^{-1}\varphi) \times \boldsymbol{\nu}\right) \qquad \forall\, \varphi \in H^{1/2}(\Gamma)\,,$$

and $\gamma^{-1} :\ H^{1/2}(\Gamma) \to H^1(\Omega)$ is any continuous right inverse of γ^-. The following result is straightforward (see [8]).

Lemma 3.3 *The operator*

$$M :\ H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

is compact and

$$Re\{\langle\, \xi, M\overline{\xi} \,\rangle\} > 0 \qquad \forall\, \xi \,\in\, H^{-1/2}(\Gamma) \setminus \{0\}\,.$$

We now make redundant use of (3.11), rewritten as

$$(\tfrac{1}{2}\mathrm{id} + K_\kappa^{\mathtt{t}})\vartheta + W_\kappa(\psi - \gamma^+ p_i) \;=\; 0\,, \tag{3.13}$$

in such a way that, given $\eta \in \mathbb{R} \setminus \{0\}$, we substitute equation (3.10) by (3.10) $+\ \imath\,\eta\, M\big\{(3.13)\big\}$ to obtain the following relationships between the Cauchy data of the exterior Helmholtz problem:

$$\begin{aligned} (\tfrac{1}{2}\mathrm{id} - K_\kappa)\psi \;+\; V_\kappa\vartheta + \imath\eta M\big\{(\tfrac{1}{2}\mathrm{id} + K_\kappa^{\mathtt{t}})\vartheta \;+\; W_\kappa(\psi - \gamma^+ p_i)\big\} \;&=\; (\tfrac{1}{2}\mathrm{id} - K_\kappa)(\gamma^+ p_i)\,,\\ -W_\kappa\psi \;+\; (\tfrac{1}{2}\mathrm{id} - K_\kappa^{\mathtt{t}})\vartheta \;+\; W_\kappa(\gamma^+ p_i) \;&=\; \vartheta. \end{aligned} \tag{3.14}$$

In order to avoid dealing with a variational formulation involving the operator $M := (\mathrm{id} - \Delta_\Gamma)^{-1}$ we follow [16, 8] and introduce the boundary variable

$$z \;:=\; -\,M\big\{(\tfrac{1}{2}\mathrm{id} + K_\kappa^{\mathtt{t}})\vartheta + W_\kappa(\psi - \gamma^+ p_i)\big\} \;\in\; H^1(\Gamma)$$

to transform (3.14) into

$$\begin{aligned} (\tfrac{1}{2}\mathrm{id} - K_\kappa)\psi + V_\kappa\vartheta - \imath\eta z \;&=\; (\tfrac{1}{2}\mathrm{id} - K_\kappa)(\gamma^+ p_i)\,,\\ -W_\kappa\psi + (\tfrac{1}{2}\mathrm{id} - K_\kappa^{\mathtt{t}})\vartheta + W_\kappa\gamma^+ p_i \;&=\; \vartheta,\\ M^{-1}z + (\tfrac{1}{2}\mathrm{id} + K_\kappa^{\mathtt{t}})\vartheta + W_\kappa\psi \;&=\; W_\kappa(\gamma^+ p_i)\,. \end{aligned} \tag{3.15}$$

Now, replacing ϑ in (3.5) by the second equation from (3.15), and testing the remaining equations of (3.15) with $\chi \in H^{-1/2}(\Gamma)$ and $w \in H^1(\Gamma)$, respectively, we arrive at the following variational formulation of our problem: Find $\widehat{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}, \psi) \in \mathbb{X}$, $(\vartheta, z) \in H^{-1/2}(\Gamma) \times H^1(\Gamma)$ and $\mathbf{r} \in \mathbb{Y}$ such that

$$\begin{aligned} A(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\tau}}) \;-\; \rho_f\omega^2 \int_\Omega \boldsymbol{\tau} : \mathbf{r} \;-\; \langle\, (\tfrac{1}{2}\mathrm{id} - K_\kappa^{\mathtt{t}})\vartheta, \varphi \,\rangle \;&=\; F(\widehat{\boldsymbol{\tau}}) \quad \forall \widehat{\boldsymbol{\tau}} = (\boldsymbol{\tau}, \varphi) \in \mathbb{X}\,,\\ \langle\, \chi, V_\kappa\vartheta \,\rangle + \langle\, \chi, (\tfrac{1}{2}\mathrm{id} - K_\kappa)\psi \,\rangle - \imath\eta \,\langle\, \chi, z \,\rangle \;&=\; f(\chi) \quad \forall \chi \in H^{-1/2}(\Gamma),\\ \langle\, W_\kappa\psi, w \,\rangle + \langle\, (\tfrac{1}{2}\mathrm{id} + K_\kappa^{\mathtt{t}})\vartheta, w \,\rangle + b(z, w) \;&=\; g(w) \quad \forall w \in H^1(\Gamma),\\ -\,\rho_f\omega^2 \int_\Omega \boldsymbol{\sigma} : \mathbf{s} \;&=\; 0 \qquad\quad \forall \mathbf{s} \in \mathbb{Y}\,, \end{aligned} \tag{3.16}$$

where

$$A(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\tau}}) \;:=\; \frac{\rho_f\omega^2}{\kappa_s^2} \int_\Omega \mathbf{div}\,\boldsymbol{\sigma} \cdot \mathbf{div}\,\boldsymbol{\tau} \;-\; \rho_f\omega^2 \int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} \;+\; \langle\, W_\kappa\psi, \varphi \,\rangle, \tag{3.17}$$

$$F(\widehat{\boldsymbol{\tau}}) \;:=\; F_0(\widehat{\boldsymbol{\tau}}) \;+\; \langle\, W_\kappa(\gamma^+ p_i), \varphi \,\rangle, \qquad f(\chi) \;:=\; \langle\, \chi, (\tfrac{1}{2}\mathrm{id} - K_\kappa)(\gamma^+ p_i) \,\rangle, \qquad g(w) \;:=\; \langle\, W_\kappa(\gamma^+ p_i), w \,\rangle$$

and, according to (3.12),

$$b(z, w) \;:=\; \langle M^{-1}\, z, w \rangle \;=\; \int_\Gamma \nabla_\Gamma z \cdot \nabla_\Gamma w \;+\; \int_\Gamma z w. \tag{3.18}$$

For economy of notation we introduce the space $\widehat{\mathbb{X}} := \mathbb{X} \times H^{-1/2}(\Gamma) \times H^1(\Gamma)$, and denote its norm by

$$\|(\widehat{\boldsymbol{\tau}}, \chi, w)\|_{\widehat{\mathbb{X}}}^2 \;:=\; \|\widehat{\boldsymbol{\tau}}\|_{\mathbb{X}}^2 \;+\; \|\chi\|_{-1/2,\Gamma}^2 \;+\; \|w\|_{1,\Gamma}^2 \qquad \forall\, (\widehat{\boldsymbol{\tau}}, \chi, w) \;\in\; \widehat{\mathbb{X}}\,.$$

Let us also define the bilinear forms $\mathbb{A} : \widehat{\mathbb{X}} \times \widehat{\mathbb{X}} \to \mathbb{C}$ and $\mathbb{B} : \widehat{\mathbb{X}} \times \mathbb{Y} \to \mathbb{C}$, and the linear functional $\mathbb{F} : \widehat{\mathbb{X}} \to \mathbb{C}$, by

$$\begin{aligned} \mathbb{A}((\widehat{\boldsymbol{\sigma}}, \vartheta, z), (\widehat{\boldsymbol{\tau}}, \chi, w)) \;:=\;& A(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\tau}}) + \langle\, \chi, V_\kappa\vartheta \,\rangle + b(z, w) - \langle\, (\tfrac{1}{2}\mathrm{id} - K_\kappa^{\mathtt{t}})\vartheta, \varphi \,\rangle\\ +\;& \langle\, \chi, (\tfrac{1}{2}\mathrm{id} - K_\kappa)\psi \,\rangle - \imath\eta \,\langle\, \chi, z \,\rangle + \langle\, W_\kappa\psi, w \,\rangle + \langle\, (\tfrac{1}{2}\mathrm{id} + K_\kappa^{\mathtt{t}})\vartheta, w \,\rangle\,, \end{aligned} \tag{3.19}$$

$$\mathbb{B}((\widehat{\boldsymbol{\tau}},\chi,w),\mathbf{s}) \,:=\, -\,\rho_f\,\omega^2\int_\Omega \mathbf{s}:\boldsymbol{\tau}\,, \tag{3.20}$$

and

$$\mathbb{F}((\widehat{\boldsymbol{\tau}},\chi,w)) \,:=\, F(\widehat{\boldsymbol{\tau}}) \,+\, f(\chi) \,+\, g(w)\,.$$

It is now clear that problem (3.16) has the following saddle point structure: Find $((\widehat{\boldsymbol{\sigma}},\vartheta,z),\mathbf{r}) \in \widehat{\mathbb{X}}\times\mathbb{Y}$ such that

$$\begin{aligned} \mathbb{A}((\widehat{\boldsymbol{\sigma}},\vartheta,z),(\widehat{\boldsymbol{\tau}},\chi,w)) \,+\, \mathbb{B}((\widehat{\boldsymbol{\tau}},\chi,w),\mathbf{r}) \,&=\, \mathbb{F}((\widehat{\boldsymbol{\tau}},\chi,w)) \quad \forall\,(\widehat{\boldsymbol{\tau}},\chi,w)\,\in\,\widehat{\mathbb{X}}\,,\\ \mathbb{B}((\widehat{\boldsymbol{\sigma}},\vartheta,z),\mathbf{s}) \,&=\, 0 \qquad\qquad\qquad \forall\,\mathbf{s}\,\in\,\mathbb{Y}\,. \end{aligned} \tag{3.21}$$

It is easy to see that $\mathbb{F}$, $\mathbb{A}$ and $\mathbb{B}$ are all bounded with constants depending on ω, ρ_f, ρ_s, κ, and κ_s. Concerning the form A, we also observe from (2.1) that the inverse operator $\mathcal{C}^{-1}$ reduces to

$$\mathcal{C}^{-1}\,\boldsymbol{\zeta} \,:=\, \frac{1}{2\,\mu}\,\boldsymbol{\zeta} \,-\, \frac{\lambda}{3\,\mu\,(3\lambda+2\mu)}\,\mathrm{tr}(\boldsymbol{\zeta})\,\mathbf{I} \qquad \forall\,\boldsymbol{\zeta}\,\in\,\mathbb{L}^2(\Omega)\,,$$

which implies that

$$\int_\Omega \mathcal{C}^{-1}\,\boldsymbol{\zeta}:\boldsymbol{\tau} \,=\, \frac{1}{2\,\mu}\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}}:\boldsymbol{\tau}^{\mathrm{d}} \,+\, \frac{1}{3\,(3\lambda+2\mu)}\int_\Omega \mathrm{tr}(\boldsymbol{\zeta})\,\mathrm{tr}(\boldsymbol{\tau}) \quad \forall\,\boldsymbol{\zeta},\,\boldsymbol{\tau}\,\in\,\mathbb{L}^2(\Omega)\,,$$

and hence

$$\int_\Omega \mathcal{C}^{-1}\,\boldsymbol{\zeta}:\overline{\boldsymbol{\zeta}} \,\geq\, \frac{1}{2\mu}\,\|\boldsymbol{\zeta}^{\mathrm{d}}\|^2_{[L^2(\Omega)]^{3\times 3}} \qquad \forall\,\boldsymbol{\zeta}\,\in\,\mathbb{L}^2(\Omega)\,. \tag{3.22}$$

This estimate will be useful for our analysis below.

4 Analysis of the continuous variational formulation

In this section we proceed analogously to [11] and employ a suitable decomposition of $\mathbb{X}$ to show that (3.21) becomes a compact perturbation of a well-posed problem. For this purpose, we now need to introduce a projector defined in terms of an auxiliary Neumann boundary value problem in Ω.

4.1 The associated projector

Let $\mathbb{RM}(\Omega)$ be the space of rigid body motions in Ω, that is

$$\mathbb{RM}(\Omega) := \Big\{\mathbf{v}:\Omega\to\mathbb{C}^3 \,:\, \mathbf{v}(\mathbf{x}) = \mathbf{a}+\mathbf{b}\times\mathbf{x} \quad \forall\,\mathbf{x}\in\Omega\,,\ \mathbf{a},\,\mathbf{b}\,\in\mathbb{C}^3\Big\}\,,$$

and let $\mathbf{M}:\mathbf{L}^2(\Omega)\to\mathbb{RM}(\Omega)$ be the associated orthogonal projector. Then, given $\widehat{\boldsymbol{\tau}} = (\boldsymbol{\tau},\varphi)\in\mathbb{X}$, we consider the boundary value problem

$$\begin{aligned} \tilde{\boldsymbol{\sigma}} \,=\, \mathcal{C}\,\boldsymbol{\varepsilon}(\tilde{\mathbf{u}}) \quad \text{in} \quad \Omega\,, \quad \mathbf{div}\,\tilde{\boldsymbol{\sigma}} \,=\, \mathbf{div}\,\boldsymbol{\tau} + \mathbf{v}(\widehat{\boldsymbol{\tau}}) \quad \text{in} \quad \Omega\,,\\ \gamma_{\boldsymbol{\nu}}\,\tilde{\boldsymbol{\sigma}} \,=\, -\,\varphi\,\boldsymbol{\nu} \quad \text{on} \quad \Gamma\,, \quad \tilde{\mathbf{u}}\in(\mathrm{id}-\mathbf{M})(\mathbf{L}^2(\Omega))\,, \end{aligned} \tag{4.1}$$

where $\mathcal{C}\,\boldsymbol{\varepsilon}(\tilde{\mathbf{u}})$ is defined according to (2.1) and $\mathbf{v}(\widehat{\boldsymbol{\tau}})\in\mathbb{RM}(\Omega)$ is characterized by

$$\int_\Omega \mathbf{v}(\widehat{\boldsymbol{\tau}})\cdot\mathbf{w} \,=\, -\,\langle\,\varphi\,\boldsymbol{\nu},\gamma^-\mathbf{w}\,\rangle - \int_\Omega \mathbf{w}\cdot\mathbf{div}\,\boldsymbol{\tau} \qquad \forall\,\mathbf{w}\,\in\,\mathbb{RM}(\Omega)\,.$$

Note that $\mathbf{v}(\widehat{\boldsymbol{\tau}})$ is just an auxiliary rigid motion that is needed to guarantee the usual compatibility condition required for the Neumann problem (4.1) (cf. [5, Theorem 9.2.30]) and that the orthogonality condition on $\tilde{\mathbf{u}}$ is required for uniqueness. The well-posedness of (4.1) is already well known (see, e.g. [2, Section 11.7, Theorem 11.7] or [12, Section 3, Theorem 3.1]). In addition, owing to the regularity result for the elasticity problem with Neumann boundary conditions (see [10]), we know that $(\tilde{\boldsymbol{\sigma}}, \tilde{\mathbf{u}}) \in \mathbb{H}^{\epsilon}(\Omega) \times \mathbf{H}^{1+\epsilon}(\Omega)$, for some $\epsilon > 0$, and there holds

$$\|\tilde{\boldsymbol{\sigma}}\|_{\epsilon,\Omega} + \|\tilde{\mathbf{u}}\|_{1+\epsilon,\Omega} \le C \left\{ \|\mathbf{div}\, \boldsymbol{\tau}\|_{0,\Omega} + \|\varphi\|_{1/2,\Gamma} \right\}. \tag{4.2}$$

Note that the embedding $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$ is used here to bound $\|\varphi\, \boldsymbol{\nu}\|_{0,\Gamma}$ by $C\,\|\varphi\|_{1/2,\Gamma}$.

We now introduce the linear operators $P : \mathbb{X} \to \mathbb{H}(\mathbf{div};\Omega)$ and $\mathbf{P} : \mathbb{X} \to \mathbb{X}$ defined by

$$P(\widehat{\boldsymbol{\tau}}) := \tilde{\boldsymbol{\sigma}} \qquad \text{and} \qquad \mathbf{P}(\widehat{\boldsymbol{\tau}}) := (P(\widehat{\boldsymbol{\tau}}), \varphi) \qquad \forall\, \widehat{\boldsymbol{\tau}} = (\boldsymbol{\tau}, \varphi) \in \mathbb{X}, \tag{4.3}$$

where $\tilde{\boldsymbol{\sigma}} := \mathcal{C}\, \boldsymbol{\varepsilon}(\tilde{\mathbf{u}})$ and $\tilde{\mathbf{u}}$ is the unique solution of (4.1). It is clear from (4.1) that

$$P(\widehat{\boldsymbol{\tau}})^{\mathtt{t}} = P(\widehat{\boldsymbol{\tau}}) \quad \text{in} \quad \Omega, \quad \mathbf{div}\,(P(\widehat{\boldsymbol{\tau}})) = \mathbf{div}\, \boldsymbol{\tau} + \mathbf{v}(\widehat{\boldsymbol{\tau}}) \quad \text{in} \quad \Omega \tag{4.4}$$

and

$$\gamma_{\boldsymbol{\nu}}\,(P(\widehat{\boldsymbol{\tau}})) = -\,\varphi\, \boldsymbol{\nu} \quad \text{on} \quad \Gamma, \tag{4.5}$$

which confirms that $\mathbf{P}(\widehat{\boldsymbol{\tau}})$ belongs to $\mathbb{X}$. Then, thanks to the continuous dependence result for (4.1), we find that

$$\|P(\widehat{\boldsymbol{\tau}})\|_{\mathbf{div};\Omega} \le C \left\{ \|\mathbf{div}\, \boldsymbol{\tau}\|_{0,\Omega} + \|\varphi\|_{1/2,\Gamma} \right\} \qquad \forall\, \widehat{\boldsymbol{\tau}} = (\boldsymbol{\tau}, \varphi) \in \mathbb{X},$$

which shows that $\mathbf{P}$ is bounded. Moreover, it is easy to see from (4.1), (4.3), (4.4), and (4.5) that $\mathbf{P}$ is actually a projector, and hence there holds

$$\mathbb{X} = \mathbf{P}(\mathbb{X}) \oplus (\mathrm{id} - \mathbf{P})(\mathbb{X}). \tag{4.6}$$

Finally, it is clear from (4.2) that $P(\widehat{\boldsymbol{\tau}}) \in \mathbb{H}^{\epsilon}(\Omega)$ and

$$\|P(\widehat{\boldsymbol{\tau}})\|_{\epsilon,\Omega} \le C \left\{ \|\mathbf{div}\, \boldsymbol{\tau}\|_{0,\Omega} + \|\varphi\|_{1/2,\Gamma} \right\} \qquad \forall\, \widehat{\boldsymbol{\tau}} = (\boldsymbol{\tau}, \varphi) \in \mathbb{X}. \tag{4.7}$$

4.2 Well-posedness of the continuous formulation

In order to show that our coupled problem (3.21) is well posed, we now employ the stable decomposition (4.6) to reformulate it in a more suitable form. We begin by observing, according to (4.4), (4.5), the symmetry of $P(\widehat{\boldsymbol{\tau}})$, and the fact that $\nabla \mathbf{v} \in \mathbb{L}^2_{\mathtt{asym}}(\Omega)$ $\forall\, \mathbf{v} \in \mathbb{RM}(\Omega)$, that for all $\widehat{\boldsymbol{\sigma}} = (\boldsymbol{\sigma}, \psi)$, $\widehat{\boldsymbol{\tau}} = (\boldsymbol{\tau}, \varphi) \in \mathbb{X}$ there holds

$$\begin{aligned} \int_\Omega \left\{ \mathbf{div}\, \boldsymbol{\sigma} - \mathbf{div}\, P(\widehat{\boldsymbol{\sigma}}) \right\} \cdot \mathbf{div}\, P(\widehat{\boldsymbol{\tau}}) &= -\int_\Omega \mathbf{v}(\widehat{\boldsymbol{\sigma}}) \cdot \mathbf{div}\, P(\widehat{\boldsymbol{\tau}}) \\ = \int_\Omega \nabla \mathbf{v}(\widehat{\boldsymbol{\sigma}}) : P(\widehat{\boldsymbol{\tau}}) - \langle \gamma_{\boldsymbol{\nu}}(P(\widehat{\boldsymbol{\tau}})), \mathbf{v}(\widehat{\boldsymbol{\sigma}}) \rangle &= \int_\Gamma (\mathbf{v}(\widehat{\boldsymbol{\sigma}}) \cdot \boldsymbol{\nu})\, \varphi. \end{aligned} \tag{4.8}$$

Then, writing $\widehat{\boldsymbol{\sigma}} = \mathbf{P}(\widehat{\boldsymbol{\sigma}}) + (\mathrm{id} - \mathbf{P})(\widehat{\boldsymbol{\sigma}})$ and $\widehat{\boldsymbol{\tau}} = \mathbf{P}(\widehat{\boldsymbol{\tau}}) + (\mathrm{id} - \mathbf{P})(\widehat{\boldsymbol{\tau}})$ in (3.17), similarly as we did in [11], using the identity (4.8), and adding and substracting suitable terms, we find that A (cf. (3.17)) can be decomposed as

$$A(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\tau}}) = A_0(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\tau}}) + K(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\tau}}) \qquad \forall\, \widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\tau}} \in \mathbb{X},$$

where $A_0 : \mathbb{X} \times \mathbb{X} \to \mathbb{C}$ and $K : \mathbb{X} \times \mathbb{X} \to \mathbb{C}$ are the bounded and symmetric bilinear forms given by

$$A_0(\widehat{\boldsymbol{\sigma}},\widehat{\boldsymbol{\tau}}) \,:=\, \mathcal{A}(\mathbf{P}(\widehat{\boldsymbol{\sigma}}),\mathbf{P}(\widehat{\boldsymbol{\tau}})) \,-\, \mathcal{A}((\mathrm{id}-\mathbf{P})(\widehat{\boldsymbol{\sigma}}),(\mathrm{id}-\mathbf{P})(\widehat{\boldsymbol{\tau}})) \tag{4.9}$$

with

$$\mathcal{A}(\widehat{\boldsymbol{\sigma}},\widehat{\boldsymbol{\tau}}) \,:=\, \rho_f\,\omega^2 \int_\Omega \mathcal{C}^{-1}\,\boldsymbol{\sigma} : \boldsymbol{\tau} \,+\, \frac{\rho_f\,\omega^2}{\kappa_s^2} \int_\Omega \mathbf{div}\,\boldsymbol{\sigma}\cdot\mathbf{div}\,\boldsymbol{\tau} \,+\, \langle\, W_0\psi,\varphi\,\rangle \,+\, \left\{\int_\Gamma \psi\right\}\left\{\int_\Gamma \varphi\right\}, \tag{4.10}$$

and

$$\begin{aligned} K(\widehat{\boldsymbol{\sigma}},\widehat{\boldsymbol{\tau}}) \,:=\, & -2\,\rho_f\,\omega^2 \int_\Omega \mathcal{C}^{-1}\,P(\widehat{\boldsymbol{\sigma}}) : P(\widehat{\boldsymbol{\tau}}) \,-\, \rho_f\,\omega^2 \int_\Omega \mathcal{C}^{-1}\,(\boldsymbol{\sigma} - P(\widehat{\boldsymbol{\sigma}})) : P(\widehat{\boldsymbol{\tau}}) \\ & -\, \rho_f\,\omega^2 \int_\Omega \mathcal{C}^{-1}\,P(\widehat{\boldsymbol{\sigma}}) : (\boldsymbol{\tau} - \mathbf{P}(\widehat{\boldsymbol{\tau}})) \,+\, \frac{2\,\rho_f\,\omega^2}{\kappa_s^2}\int_\Omega \mathbf{v}(\widehat{\boldsymbol{\sigma}})\cdot\mathbf{v}(\widehat{\boldsymbol{\tau}}) \,+\, \frac{\rho_f\,\omega^2}{\kappa_s^2}\int_\Gamma \big(\mathbf{v}(\widehat{\boldsymbol{\tau}})\cdot\boldsymbol{\nu}\big)\,\psi \\ & +\, \frac{\rho_f\,\omega^2}{\kappa_s^2}\int_\Gamma \big(\mathbf{v}(\widehat{\boldsymbol{\sigma}})\cdot\boldsymbol{\nu}\big)\,\varphi \,-\, \left\{\int_\Gamma \psi\right\}\left\{\int_\Gamma \varphi\right\} \,+\, \langle\,(W_\kappa - W_0)\psi,\varphi\,\rangle, \end{aligned} \tag{4.11}$$

for all $\widehat{\boldsymbol{\sigma}} = (\boldsymbol{\sigma},\psi)$, $\widehat{\boldsymbol{\tau}} = (\boldsymbol{\tau},\varphi) \in \mathbb{X}$. The above suggests to decompose $\mathbb{A}$ (cf. (3.19)) as

$$\mathbb{A} \,=\, \mathbb{A}_0 \,+\, \mathbb{K}, \tag{4.12}$$

where, given $(\widehat{\boldsymbol{\sigma}},\vartheta,z)$, $(\widehat{\boldsymbol{\tau}},\chi,w) \in \widehat{\mathbb{X}}$,

$$\mathbb{A}_0((\widehat{\boldsymbol{\sigma}},\vartheta,z),(\widehat{\boldsymbol{\tau}},\chi,w)) := A_0(\widehat{\boldsymbol{\sigma}},\widehat{\boldsymbol{\tau}}) + \langle\chi,V_0\,\vartheta\rangle + b(z,w) - \langle(\tfrac{1}{2}\mathrm{id} - K_0^{\mathtt{t}})\vartheta,\varphi\rangle + \langle\chi,(\tfrac{1}{2}\mathrm{id} - K_0)\psi\rangle, \tag{4.13}$$

and

$$\begin{aligned} \mathbb{K}((\widehat{\boldsymbol{\sigma}},\vartheta,z),(\widehat{\boldsymbol{\tau}},\chi,w)) \,:=\, & K(\widehat{\boldsymbol{\sigma}},\widehat{\boldsymbol{\tau}}) \,+\, \langle\,\chi,(V_\kappa - V_0)\vartheta\,\rangle \,+\, \langle\,(K_\kappa^{\mathtt{t}} - K_0^{\mathtt{t}})\vartheta,\varphi\,\rangle \\ + \, & \langle\,\chi,(K_0 - K_\kappa)\psi\,\rangle \,-\, \imath\,\eta\,\langle\,\chi,z\,\rangle \,+\, \langle\,W_\kappa\psi,w\,\rangle \,+\, \langle\,(\tfrac{1}{2}\mathrm{id} + K_\kappa^{\mathtt{t}})\vartheta,w\,\rangle\,. \end{aligned} \tag{4.14}$$

Next, we let $\mathbf{A}_0 : \widehat{\mathbb{X}} \to \widehat{\mathbb{X}}$, $\mathbf{K} : \widehat{\mathbb{X}} \to \widehat{\mathbb{X}}$, and $\mathbf{B} : \widehat{\mathbb{X}} \to \mathbb{Y}$ be the linear and bounded operators induced by the bilinear forms $\mathbb{A}_0$, $\mathbb{K}$, and $\mathbb{B}$, respectively. In addition, we let $\mathbf{F} \in \widehat{\mathbb{X}}$ be the Riesz representant of $\mathbb{F}$. Hence, using these notations and taking into account the decomposition (4.12), the variational formulation (3.21) can be rewritten as the following operator equation: Find $((\widehat{\boldsymbol{\sigma}},\vartheta,z),\mathbf{r}) \in \widehat{\mathbb{X}} \times \mathbb{Y}$ such that

$$\begin{pmatrix} \mathbf{A}_0 & \mathbf{B}^* \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} (\widehat{\boldsymbol{\sigma}},\vartheta,z) \\ \mathbf{r} \end{pmatrix} \,+\, \begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} (\widehat{\boldsymbol{\sigma}},\vartheta,z) \\ \mathbf{r} \end{pmatrix} \,=\, \begin{pmatrix} \mathbf{F} \\ \mathbf{0} \end{pmatrix}. \tag{4.15}$$

Throughout the rest of this section we prove that the matrix operators on the left hand side of (4.15) are invertible and compact, respectively.

Because of the saddle point structure of the matrix operator involving $\mathbf{A}_0$ and $\mathbf{B}$, and according to the classical Babuška-Brezzi theory, we begin the analysis with the inf-sup condition for $\mathbb{B}$.

Lemma 4.1 *There exists $\beta > 0$ such that*

$$\sup_{\substack{(\widehat{\boldsymbol{\tau}},\chi,w) \in \widehat{\mathbb{X}} \\ (\widehat{\boldsymbol{\tau}},\chi,w) \neq \mathbf{0}}} \frac{|\,\mathbb{B}((\widehat{\boldsymbol{\tau}},\chi,w),\mathbf{s})\,|}{\|(\widehat{\boldsymbol{\tau}},\chi,w)\|_{\widehat{\mathbb{X}}}} \,\geq\, \beta\,\|\mathbf{s}\|_{0,\Omega} \qquad \forall\,\mathbf{s} \,\in\, \mathbb{Y}\,.$$

Proof. See [3, Lemma 4.1]. $\square$

Our next goal is to prove that $\mathbf{A}_0$ is an isomorphism on the kernel of $\mathbf{B}$. For this purpose, we now recall the decomposition

$$\mathbb{H}(\mathbf{div};\,\Omega) \,=\, \mathbb{H}_0(\mathbf{div};\,\Omega) \,\oplus\, \mathbb{C}\,\mathbf{I}\,,$$

where

$$\mathbb{H}_0(\mathbf{div};\,\Omega) \,:=\, \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\,\Omega) : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) \,=\, 0 \right\}.$$

This means that for any $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\,\Omega)$ there exist unique $\boldsymbol{\tau}_0 \in \mathbb{H}_0(\mathbf{div};\,\Omega)$ and $d \in \mathbb{C}$ given by $d := \dfrac{1}{3\,|\Omega|}\displaystyle\int_\Omega \mathrm{tr}(\boldsymbol{\tau})$, where $|\Omega|$ denotes the measure of Ω, such that $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + d\,\mathbf{I}$.

Our subsequent analysis will strongly depend on the inequalities provided by the following lemmas.

Lemma 4.2 *There exists $c_1 > 0$, depending only on Ω, such that*

$$c_1\,\|\boldsymbol{\tau}_0\|^2_{0,\Omega} \;\le\; \|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,\Omega} \,+\, \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{0,\Omega} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\,\Omega)\,. \tag{4.16}$$

Proof. See [6, Proposition 3.1, Chapter IV]. $\square$

Lemma 4.3 *There exists $c_2 > 0$, depending on c_1, κ_s, ρ_f, and ω^2, such that*

$$\mathrm{Re}\Big\{\mathcal{A}(\widehat{\boldsymbol{\tau}},\overline{\widehat{\boldsymbol{\tau}}})\Big\} \;\ge\; c_2\,\|\widehat{\boldsymbol{\tau}}\|^2_{\mathbb{X}} \qquad \forall\,\widehat{\boldsymbol{\tau}} \,\in\, \mathbb{X}\,. \tag{4.17}$$

Proof. Let $\widehat{\boldsymbol{\tau}} \,=\, (\boldsymbol{\tau},\varphi) \,\in\, \mathbb{X}$ with $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + d\,\mathbf{I}$. We first notice, from the definition of $\mathcal{A}$ (cf. (4.10)) and the inequalities (3.22) and (3.9) (cf. Lemma 3.2), that

$$\mathrm{Re}\Big\{\mathcal{A}(\widehat{\boldsymbol{\tau}},\overline{\widehat{\boldsymbol{\tau}}})\Big\} \;\ge\; C_0\,\Big\{\|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,\Omega} \,+\, \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{0,\Omega} \,+\, \|\varphi\|^2_{1/2,\Gamma}\Big\}\,. \tag{4.18}$$

On the other hand, since $\gamma_{\boldsymbol{\nu}}\,\boldsymbol{\tau} \;=\; -\,\varphi\,\boldsymbol{\nu}$ on Γ, we see that $-\varphi\,\boldsymbol{\nu} \;=\; \gamma_{\boldsymbol{\nu}}\,\boldsymbol{\tau}_0 \,+\, d\,\boldsymbol{\nu}$ in $\mathbf{H}^{-1/2}(\Gamma)$, from which, applying the trace theorem in $\mathbb{H}(\mathbf{div};\,\Omega)$ together with the continuity of the canonical embeddings $\mathbf{L}^2(\Gamma) \hookrightarrow \mathbf{H}^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$, we deduce that

$$\begin{aligned} |d|\,\|\boldsymbol{\nu}\|_{-1/2,\Gamma} \;&\le\; \|\gamma_{\boldsymbol{\nu}}\,\boldsymbol{\tau}_0\|_{-1/2,\Gamma} \,+\, \|\varphi\,\boldsymbol{\nu}\|_{-1/2,\Gamma} \\ &\le\quad C_1\,\Big\{\|\boldsymbol{\tau}_0\|_{\mathbf{div};\Omega} \,+\, \|\varphi\|_{1/2,\Gamma}\Big\}\,. \end{aligned}$$

It follows that

$$\begin{aligned} \|\boldsymbol{\tau}\|^2_{\mathbf{div};\Omega} \,+\, \|\varphi\|^2_{1/2,\Gamma} \;&=\; \|\boldsymbol{\tau}_0\|^2_{\mathbf{div};\Omega} \,+\, 3\,d^2\,|\Omega| \,+\, \|\varphi\|^2_{1/2,\Gamma} \\ &\le\quad C_2\,\Big\{\|\boldsymbol{\tau}_0\|^2_{\mathbf{div};\Omega} \,+\, \|\varphi\|^2_{1/2,\Gamma}\Big\}\,, \end{aligned}$$

which, thanks to (4.16), yields

$$\|\boldsymbol{\tau}\|^2_{\mathbf{div}} \,+\, \|\varphi\|^2_{1/2,\Gamma} \;\le\; C_3\,\Big\{\|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,\Omega} \,+\, \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{0,\Omega} \,+\, \|\varphi\|^2_{1/2,\Gamma}\Big\}$$

for all $\widehat{\boldsymbol{\tau}} \,=\, (\boldsymbol{\tau},\varphi) \,\in\, \mathbb{X}$. The above estimate and (4.18) imply (4.17) and finish the proof. $\square$

In what follows we make frequent use of the linear and bounded operator

$$\Xi := (2\,\mathbf{P} - \mathrm{id}) : \mathbb{X} \to \mathbb{X}.$$

Lemma 4.4 *There exists $C > 0$, depending on c_2, such that*

$$\mathrm{Re}\Big\{ A_0(\widehat{\boldsymbol{\tau}}, \Xi(\overline{\widehat{\boldsymbol{\tau}}})) \Big\} \,\geq\, C\, \|\widehat{\boldsymbol{\tau}}\|_{\mathbb{X}}^2 \qquad \forall\, \widehat{\boldsymbol{\tau}} \,:=\, (\boldsymbol{\tau}, \varphi) \,\in\, \mathbb{X}\,. \tag{4.19}$$

Proof. Since $\mathbf{P}$ is a projector we easily observe that

$$\mathbf{P}\,\Xi(\widehat{\boldsymbol{\tau}}) \,=\, \mathbf{P}(\widehat{\boldsymbol{\tau}}) \quad \text{and} \quad (\mathrm{id}-\mathbf{P})\,\Xi(\widehat{\boldsymbol{\tau}}) \,=\, -\,(\mathrm{id}-\mathbf{P})(\widehat{\boldsymbol{\tau}}) \qquad \forall\, \widehat{\boldsymbol{\tau}} \,\in\, \mathbb{X}\,,$$

which, according to the definition of A_0 (cf. (4.9)), gives

$$A_0(\widehat{\boldsymbol{\tau}}, \Xi(\overline{\widehat{\boldsymbol{\tau}}})) \;=\; \mathcal{A}(\mathbf{P}(\widehat{\boldsymbol{\tau}}), \mathbf{P}(\overline{\widehat{\boldsymbol{\tau}}})) \;+\; \mathcal{A}((\mathrm{id}-\mathbf{P})(\widehat{\boldsymbol{\tau}}), (\mathrm{id}-\mathbf{P})(\overline{\widehat{\boldsymbol{\tau}}}))\,. \tag{4.20}$$

Hence, the inequality (4.19) follows directly from (4.20), Lemma 4.3, the fact that both $\mathbf{P}(\widehat{\boldsymbol{\tau}})$ and $(\mathrm{id}-\mathbf{P})(\widehat{\boldsymbol{\tau}})$ belong to $\mathbb{X}$, and the stability of the decomposition (4.6).

□

We now let $\widehat{\mathbb{V}}$ be the kernel of $\mathbf{B}$, that is

$$\widehat{\mathbb{V}} \,:=\, \Big\{ (\widehat{\boldsymbol{\tau}}, \chi, w) \,\in\, \widehat{\mathbb{X}} : \quad \mathbf{B}(\widehat{\boldsymbol{\tau}}, \chi, w) \,=\, 0 \Big\},$$

which, recalling that $\mathbf{B}(\widehat{\boldsymbol{\tau}}, \chi, w) \,:=\, -\,\frac{1}{2}\,\rho_f\,\omega^2\left(\boldsymbol{\tau} - \boldsymbol{\tau}^{\mathtt{t}}\right)$ $\forall\,(\widehat{\boldsymbol{\tau}}, \chi, w) \,:=\, ((\boldsymbol{\tau}, \varphi), \chi, w) \,\in\, \widehat{\mathbb{X}}$, becomes

$$\widehat{\mathbb{V}} \,=\, \Big\{ (\widehat{\boldsymbol{\tau}}, \chi, w) \,:=\, ((\boldsymbol{\tau}, \varphi), \chi, w) \,\in\, \widehat{\mathbb{X}} : \quad \boldsymbol{\tau}^{\mathtt{t}} \,=\, \boldsymbol{\tau} \Big\}.$$

Hence, we are now in a position to establish the following lemma, which includes, in particular, the weak coercivity of $\mathbb{A}_0$ on $\widehat{\mathbb{V}}$.

Lemma 4.5 *There exist $\alpha,\, C \,>\, 0$ such that*

$$\mathrm{Re}\Big\{\mathbb{A}_0((\widehat{\boldsymbol{\sigma}}, \vartheta, z), (\Xi(\overline{\widehat{\boldsymbol{\sigma}}}), \overline{\vartheta}, \overline{z}))\Big\} \,\geq\, \alpha\, \|(\widehat{\boldsymbol{\sigma}}, \vartheta, z)\|_{\widehat{\mathbb{X}}}^2 \qquad \forall\,(\widehat{\boldsymbol{\sigma}}, \vartheta, z) \in \widehat{\mathbb{X}}\,, \tag{4.21}$$

and

$$\sup_{\substack{(\widehat{\boldsymbol{\tau}},\chi,w)\in\widehat{\mathbb{V}} \\ (\widehat{\boldsymbol{\tau}},\chi,w)\neq\mathbf{0}}} \frac{|\,\mathbb{A}_0((\widehat{\boldsymbol{\sigma}}, \vartheta, z), (\widehat{\boldsymbol{\tau}}, \chi, w))\,|}{\|(\widehat{\boldsymbol{\tau}}, \chi, w)\|_{\widehat{\mathbb{X}}}} \;\geq\; C\,\|(\widehat{\boldsymbol{\sigma}}, \vartheta, z)\|_{\widehat{\mathbb{X}}} \qquad \forall\,(\widehat{\boldsymbol{\sigma}}, \vartheta, z) \in \widehat{\mathbb{V}}\,. \tag{4.22}$$

In addition, there holds

$$\sup_{(\widehat{\boldsymbol{\sigma}},\vartheta,z)\in\widehat{\mathbb{V}}} |\,\mathbb{A}_0((\widehat{\boldsymbol{\sigma}}, \vartheta, z), (\widehat{\boldsymbol{\tau}}, \chi, w))\,| \;>\; 0 \qquad \forall\,(\widehat{\boldsymbol{\tau}}, \chi, w) \in \widehat{\mathbb{V}}\,,\; (\widehat{\boldsymbol{\tau}}, \chi, w) \neq \mathbf{0}\,. \tag{4.23}$$

Proof. Having in mind the definition of $\mathbb{A}_0$ (cf. (4.13)), recalling that $K_0^{\mathtt{t}}$ is the adjoint of K_0, and applying (4.19), the inequality (3.8) (cf. Lemma 3.2), and the fact that $b(z,z) \,\geq\, c\,\|z\|_{1,\Gamma}^2\;\forall\, z \in H^1(\Gamma)$ (cf. (3.18)), we easily deduce the ellipticity-type estimate given by (4.21). Next, since $P(\widehat{\boldsymbol{\tau}})^{\mathtt{t}} \,=\, P(\widehat{\boldsymbol{\tau}})$ $\forall\,\widehat{\boldsymbol{\tau}} \in \mathbb{X}$ (cf. (4.4)), we find that $(\mathbf{P}(\widehat{\boldsymbol{\tau}}), \chi, w)$, and hence $(\Xi(\widehat{\boldsymbol{\tau}}), \chi, w)$, belong to $\widehat{\mathbb{V}}$ for any $(\widehat{\boldsymbol{\tau}}, \chi, w) \in \widehat{\mathbb{V}}$. According to the above, for any $(\widehat{\boldsymbol{\sigma}}, \vartheta, z) \,\in\, \widehat{\mathbb{V}}$, $(\widehat{\boldsymbol{\sigma}}, \vartheta, z) \neq \mathbf{0}$, we can write

$$\sup_{\substack{(\widehat{\boldsymbol{\tau}},\chi,w)\in\widehat{\mathbb{V}} \\ (\widehat{\boldsymbol{\tau}},\chi,w)\neq\mathbf{0}}} \frac{|\,\mathbb{A}_0((\widehat{\boldsymbol{\sigma}}, \vartheta, z), (\widehat{\boldsymbol{\tau}}, \chi, w))\,|}{\|(\widehat{\boldsymbol{\tau}}, \chi, w)\|_{\widehat{\mathbb{X}}}} \;\geq\; \frac{|\,\mathbb{A}_0((\widehat{\boldsymbol{\sigma}}, \vartheta, z), (\Xi(\overline{\widehat{\boldsymbol{\sigma}}}), \overline{\vartheta}, \overline{z}))\,|}{\|(\Xi(\overline{\widehat{\boldsymbol{\sigma}}}), \overline{\vartheta}, \overline{z})\|_{\widehat{\mathbb{X}}}}$$
$$\geq\; \frac{\Big|\,\mathrm{Re}\Big\{\mathbb{A}_0((\widehat{\boldsymbol{\sigma}}, \vartheta, z), (\Xi(\overline{\widehat{\boldsymbol{\sigma}}}), \overline{\vartheta}, \overline{z}))\Big\}\,\Big|}{\|(\Xi(\overline{\widehat{\boldsymbol{\sigma}}}), \overline{\vartheta}, \overline{z})\|_{\widehat{\mathbb{X}}}}\,,$$

which, thanks to (4.21) and the boundedness of Ξ, yields straightforwardly the inf-sup condition (4.22). Finally, the fact that $\mathbb{A}_0((\Xi(\overline{\widehat{\boldsymbol{\sigma}}}), \overline{\vartheta}, \overline{z}), (\widehat{\boldsymbol{\sigma}}, \vartheta, z)) \;=\; \mathbb{A}_0((\widehat{\boldsymbol{\sigma}}, \vartheta, z), (\Xi(\overline{\widehat{\boldsymbol{\sigma}}}), \overline{\vartheta}, \overline{z}))$ and (4.22) imply (4.23) and complete the proof. □

Lemma 4.6 *The operator* $\mathbf{K} : \widehat{\mathbb{X}} \to \widehat{\mathbb{X}}$ *is compact.*

Proof. We begin by recalling (cf. (4.7)) that there exists $\epsilon > 0$ such that $P(\widehat{\boldsymbol{\tau}}) \in \mathbb{H}^\epsilon(\Omega)$ for all $\widehat{\boldsymbol{\tau}} \in \mathbb{X}$, which, according to the compact imbeddings $H^\epsilon(\Omega) \stackrel{c}{\hookrightarrow} L^2(\Omega)$ and $H^{1/2}(\Gamma) \stackrel{c}{\hookrightarrow} L^2(\Gamma)$, yields the compactness of $\mathbf{P} : \mathbb{X} \to \mathbb{L}^2(\Omega) \times L^2(\Gamma)$. It follows that $\mathbf{P}^* : \mathbb{L}^2(\Omega) \times L^2(\Gamma) \to \mathbb{X}$, $\mathbf{P}^* \mathcal{C}^{-1} \mathbf{P}$, $\mathbf{P}^* \mathcal{C}^{-1} (\mathrm{id} - \mathbf{P})$, and $(\mathrm{id} - \mathbf{P})^* \mathcal{C}^{-1} \mathbf{P}$ are all compact. This shows that the operator induced by the first three terms defining K (cf. (4.11)) becomes compact, and the four following terms are finite rank operators. Finally, we deduce from Lemma 3.1 and the compactness of the embedding $H^1(\Gamma) \hookrightarrow H^{1/2}(\Gamma)$ that the last term defining K, and all the remaning terms appearing in the definition of $\mathbb{K}$ (cf. (4.14)) are compact.

□

We are now able to establish the main result of this section. Some aspects of the proof, mainly those involving the boundary integral operators, follow very closely the arguments from [16].

Theorem 4.1 *Assume that* $\frac{\kappa_s}{\sqrt{\rho_s}}$ *is not a Jones frequency. Then, given* $\mathbf{f} \in \mathbf{L}^2(\Omega)$ *and a smooth incident wave* p_i*, there exists a unique solution* $((\widehat{\boldsymbol{\sigma}}, \vartheta, z), \mathbf{r}) \in \widehat{\mathbb{X}} \times \mathbb{Y}$ *to* (3.21) (*equivalently* (3.16) *or* (4.15))*. In addition, there exists* $C > 0$ *such that*

$$\|((\widehat{\boldsymbol{\sigma}}, \vartheta, z), \mathbf{r})\|_{\widehat{\mathbb{X}} \times \mathbb{Y}} \;\le\; C \left\{ \|\mathbf{f}\|_{0,\Omega} + \|\gamma^+ p_i\|_{1/2,\Gamma} + \|\partial^+_{\boldsymbol{\nu}} p_i\|_{-1/2,\Gamma} \right\}.$$

Proof. We notice that the left hand side of (4.15) constitutes a Fredholm operator of index zero. Actually, Lemmata 4.1 and 4.5 imply that $\begin{pmatrix} \mathbf{A}_0 & \mathbf{B}^* \\ \mathbf{B} & \mathbf{0} \end{pmatrix}$ is an isomorphism, and Lemma 4.6 yields the compactness of $\begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$. Hence, we just have to show that problem (3.21) (equivalently (3.16) or (4.15)) admits a unique solution. Let us then assume that $((\widehat{\boldsymbol{\sigma}}, \vartheta, z), \mathbf{r}) \in \widehat{\mathbb{X}} \times \mathbb{Y}$ solves (3.16) with $\mathbf{f} = \mathbf{0}$ and $p_i = 0$. We first introduce the variable

$$\mathbf{u} \;=\; -\frac{1}{\kappa_s^2}\, \mathbf{div}\, \boldsymbol{\sigma} \;\in\; \mathbf{L}^2(\Omega) \tag{4.24}$$

in the first equation of (3.16), and then, given $\boldsymbol{\tau} \in \mathbb{C}_0^\infty(\Omega)$, test this equation with $(\boldsymbol{\tau}, 0) \in \mathbb{X}$, to obtain

$$\int_\Omega \nabla \mathbf{u} : \boldsymbol{\tau} \;=\; \int_\Omega \mathcal{C}^{-1} \boldsymbol{\sigma} : \boldsymbol{\tau} \;+\; \int_\Omega \mathbf{r} : \boldsymbol{\tau}\,,$$

where the first integral is in the distributional sense. This shows that

$$\nabla \mathbf{u} \;=\; \mathcal{C}^{-1} \boldsymbol{\sigma} \;+\; \mathbf{r} \;\in\; \mathbb{L}^2(\Omega)\,, \tag{4.25}$$

and hence $\mathbf{u} \in \mathbf{H}^1(\Omega)$. In addition, using from the last equation of (3.16) that $\boldsymbol{\sigma}$, and hence $\mathcal{C}^{-1}\boldsymbol{\sigma}$, are symmetric, and recalling that $\mathbf{r} \in \mathbb{L}^2_{\mathtt{asym}}(\Omega)$, the identity (4.25) gives inmediately that $\mathcal{C}^{-1}\boldsymbol{\sigma}$ and $\mathbf{r}$ constitute, respectively, the symmetric and asymmetric components of $\nabla \mathbf{u}$, that is

$$\mathcal{C}^{-1} \boldsymbol{\sigma} \;=\; \boldsymbol{\varepsilon}(\mathbf{u}) \qquad \text{and} \qquad \mathbf{r} \;=\; \tfrac{1}{2}\,(\nabla \mathbf{u} - (\nabla \mathbf{u})^{\mathtt{t}}) \quad \text{in} \quad \Omega\,. \tag{4.26}$$

Now, testing the first equation of (3.16) with $\widehat{\boldsymbol{\tau}} := (\boldsymbol{\tau}, \varphi) \in \mathbb{X}$ and integrating by parts, we arrive at

$$-\rho_f\, \omega^2 \,\langle\, \gamma_{\boldsymbol{\nu}} \boldsymbol{\tau}, \gamma^- \mathbf{u} \,\rangle \;+\; \langle\, W_\kappa \psi, \varphi \,\rangle \;-\; \langle\, (\tfrac{1}{2}\mathrm{id} - K_\kappa^{\mathtt{t}}) \vartheta, \varphi \,\rangle \;=\; 0\,,$$

that is, recalling that $\gamma_{\boldsymbol{\nu}}\, \boldsymbol{\tau} \;=\; -\,\varphi\, \boldsymbol{\nu}$,

$$\rho_f\, \omega^2\, \langle \gamma^- \mathbf{u} \cdot \boldsymbol{\nu}, \varphi \rangle \;=\; \langle\, (\tfrac{1}{2}\mathrm{id} - K_\kappa^{\mathtt{t}}) \vartheta \;-\; W_\kappa \psi, \varphi \,\rangle\,,$$

and since the surjectivity of $\gamma_{\boldsymbol{\nu}}$ guarantees that this holds for each $\varphi \in H^{1/2}(\Gamma)$, we obtain the identity

$$\rho_f\, \omega^2\, \gamma^- \mathbf{u} \cdot \boldsymbol{\nu} \,=\, (\tfrac{1}{2}\,\mathrm{id} \,-\, K_\kappa^{\mathtt{t}})\vartheta - W_\kappa \psi\,. \tag{4.27}$$

Finally, from the second and third equations of (3.16) we have

$$\psi \,=\, (\tfrac{1}{2}\mathrm{id} \,+\, K_\kappa)\psi - V_\kappa \vartheta \,+\, \imath\, \eta\, z \tag{4.28}$$

with

$$z \,=\, -\, M\big\{ (\tfrac{1}{2}\mathrm{id} + K_\kappa^{\mathtt{t}})\vartheta + W_\kappa \psi \big\}\,. \tag{4.29}$$

Note at this point that (4.28) and (4.27) can be rewritten as

$$\begin{pmatrix} \mathrm{id} & \imath\,\eta\, M \\ 0 & \mathrm{id} \end{pmatrix} \begin{pmatrix} \frac{1}{2}\mathrm{id} - K_\kappa & V_\kappa \\ W_\kappa & \frac{1}{2}\mathrm{id} + K_\kappa^{\mathtt{t}} \end{pmatrix} \begin{pmatrix} \psi \\ \vartheta \end{pmatrix} = \begin{pmatrix} 0 \\ \vartheta \,-\, \rho_f\, \omega^2\, \gamma^- \mathbf{u} \cdot \boldsymbol{\nu} \end{pmatrix}. \tag{4.30}$$

Let us now introduce the function $q:\ \Omega \to \mathbb{C}$ defined by

$$q(\mathbf{x}) := (\Psi_{SL}^{\kappa} \vartheta)(\mathbf{x}) \,-\, (\Psi_{DL}^{\kappa} \psi)(\mathbf{x}) \qquad \forall\, \mathbf{x} \,\in\, \Omega\,.$$

Then, using the trace properties (3.7), we find that

$$\begin{pmatrix} \frac{1}{2}\mathrm{id} - K_\kappa & V_\kappa \\ W_\kappa & \frac{1}{2}\mathrm{id} + K_\kappa^{\mathtt{t}} \end{pmatrix} \begin{pmatrix} \psi \\ \vartheta \end{pmatrix} = \begin{pmatrix} \gamma^- q \\ \partial_{\boldsymbol{\nu}}^- q \end{pmatrix},$$

which, after applying the operator $\begin{pmatrix} \mathrm{id} & \imath\,\eta\, M \\ 0 & \mathrm{id} \end{pmatrix}$ and comparing with (4.30), implies that

$$\begin{pmatrix} \mathrm{id} & \imath\,\eta\, M \\ 0 & \mathrm{id} \end{pmatrix} \begin{pmatrix} \gamma^- q \\ \partial_{\boldsymbol{\nu}}^- q \end{pmatrix} = \begin{pmatrix} 0 \\ \vartheta \,-\, \rho_f\, \omega^2\, \gamma^- \mathbf{u} \cdot \boldsymbol{\nu} \end{pmatrix}. \tag{4.31}$$

It follows that q satisfies

$$\begin{aligned} -\Delta q \,-\, \kappa^2\, q \quad &= \quad 0 \quad \text{in} \quad \Omega\,, \\ \gamma^- q \,+\, \imath\,\eta\, M\big\{ \partial_{\boldsymbol{\nu}}^- q \big\} \quad &= \quad 0 \quad \text{on} \quad \Gamma\,, \end{aligned} \tag{4.32}$$

where the boundary condition is a consequence of the first equation of (4.31). Testing this Hemholtz equation with $\overline{q}$ and integrating by parts yields

$$\|\nabla q\|_{0,\Omega}^2 \,-\, \kappa^2\, \|q\|_{0,\Omega}^2 \,=\, \langle\, \partial_{\boldsymbol{\nu}}^- q, \gamma^- \overline{q} \,\rangle\,,$$

which, using the boundary condition from (4.32), becomes

$$\|\nabla q\|_{0,\Omega}^2 \,-\, \kappa^2\, \|q\|_{0,\Omega}^2 \,=\, \imath\,\eta\, \langle\, \partial_{\boldsymbol{\nu}}^- q, M\big(\overline{\partial_{\boldsymbol{\nu}}^- q}\big) \,\rangle\,.$$

It follows that $\mathrm{Re}\{\langle\, \partial_{\boldsymbol{\nu}}^- q, M\big(\overline{\partial_{\boldsymbol{\nu}}^- q}\big)\,\rangle\} \,=\, 0$, which, according to Lemma 3.3, yields $\partial_{\boldsymbol{\nu}}^- q \,=\, 0$. This identity and (4.31) imply that

$$\gamma^- q \,=\, 0 \quad \text{and} \quad \vartheta \,=\, \rho_f\, \omega^2\, \gamma^- \mathbf{u} \cdot \boldsymbol{\nu} \quad \text{on} \quad \Gamma\,, \tag{4.33}$$

whence, in particular, (4.27) becomes $(\frac{1}{2}\,\mathrm{id} \,+\, K_\kappa^{\mathtt{t}})\vartheta \,+\, W_\kappa \psi \,=\, 0$, and therefore z (cf. (4.29)) vanishes identically. As a consequence, (4.28) and (4.27) are simplified to

$$\begin{pmatrix} \frac{1}{2}\mathrm{id} + K_\kappa & -V_\kappa \\ -W_\kappa & \frac{1}{2}\mathrm{id} - K_\kappa^{\mathtt{t}} \end{pmatrix} \begin{pmatrix} \psi \\ \vartheta \end{pmatrix} \,=\, \begin{pmatrix} \psi \\ \vartheta \end{pmatrix}. \tag{4.34}$$

Introducing now the function $p:\ \Omega^+ \to \mathbb{C}$ defined by

$$p(\mathbf{x}) \,:=\, (\Psi^{\kappa}_{DL}\psi)(\mathbf{x}) \,-\, (\Psi^{\kappa}_{SL}\vartheta)(\mathbf{x}) \qquad \forall\, \mathbf{x} \,\in\, \Omega^+\,,$$

and applying again the trace properties (3.7), we obtain

$$\begin{pmatrix} \frac{1}{2}\mathrm{id} + K_\kappa & -V_\kappa \\ -W_\kappa & \frac{1}{2}\mathrm{id} - K^{\mathtt{t}}_\kappa \end{pmatrix} \begin{pmatrix} \psi \\ \vartheta \end{pmatrix} \,=\, \begin{pmatrix} \gamma^+ p \\ \partial^+_{\boldsymbol{\nu}}\, p \end{pmatrix}\,,$$

which, compared with (4.34), and using (4.33), gives

$$\gamma^+ p \,=\, \psi \quad \text{and} \quad \partial^+_{\boldsymbol{\nu}}\, p \,=\, \vartheta \,=\, \rho_f\, \omega^2\, \gamma^- \mathbf{u}\cdot\boldsymbol{\nu} \quad \text{on} \quad \Gamma\,. \tag{4.35}$$

Then, recalling that $\widehat{\boldsymbol{\sigma}} \,:=\, (\boldsymbol{\sigma}, \psi) \,\in\, \mathbb{X}$, and integrating by parts in Ω, we find that

$$\begin{aligned} \frac{1}{\rho_f\,\omega^2}\,\langle \partial^+_{\boldsymbol{\nu}}\,\bar{p}, \gamma^+ p\rangle \,&=\, \langle \psi\,\boldsymbol{\nu}, \gamma^-\bar{\mathbf{u}}\rangle \,=\, -\,\langle \gamma_{\boldsymbol{\nu}}\,\boldsymbol{\sigma}, \gamma^-\bar{\mathbf{u}}\rangle \\ &=\, -\int_\Omega \bar{\mathbf{u}}\cdot\mathbf{div}\,\boldsymbol{\sigma} \,-\, \int_\Omega \boldsymbol{\sigma} : \nabla\bar{\mathbf{u}} \,=\, \kappa_s^2\,\|\mathbf{u}\|^2_{0,\Omega} \,-\, \int_\Omega \mathcal{C}\,\boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\bar{\mathbf{u}})\,, \end{aligned}$$

which yields

$$\mathrm{Im}\left\{\langle \partial^+_{\boldsymbol{\nu}}\,\bar{p}, \gamma^+ p\rangle\right\} \,=\, 0\,.$$

This identity and the fact that p satisfies the Helmholtz equation $\Delta p \,+\, \kappa^2\, p \,=\, 0$ in Ω^+ and the Sommerfeld radiation condition (2.3) with $p_i \,=\, 0$, imply, thanks to the Rellich theorem (cf. [9, Theorem 2.12] or [24, Lemma 9.9]), that p vanishes identically in Ω^+. Thus, it is clear from (4.35) that $\psi \,=\, 0$ and $\vartheta \,=\, 0$ on Γ, whence,

$$\gamma_{\boldsymbol{\nu}}\,\boldsymbol{\sigma} \,=\, 0 \quad \text{and} \quad \gamma^-\mathbf{u}\cdot\boldsymbol{\nu} \,=\, 0 \quad \text{on} \quad \Gamma\,.$$

The above together with (4.24) and the first equation in (4.26) show that $\mathbf{u}$ solves (2.4), and therefore, because of our hypothesis on κ_s, $\mathbf{u}$ must also vanish identically in Ω, which completes the proof.

□

At this point we remark that analogue arguments to those employed in the previous theorem allow to show that z also vanishes in the original non-homogeneous coupled problem (3.16) (equivalently (3.21)). In other words, z is an artificial unknown that is introduced in the formulation only to stabilize the boundary integral equations. A similar situation holds in [16].

5 Analysis of the Galerkin scheme

In this section we introduce a Galerkin approximation of (3.5) and prove its well-posedness.

5.1 Preliminaries

We first let $\{\mathcal{T}_h\}_{h>0}$ be a shape-regular family of triangulations of the polyhedral region $\bar{\Omega}$ by tetrahedra T of diameter h_T with mesh sizes $h := \max\{\, h_T : \quad T \in \mathcal{T}_h\,\}$. We denote by $\{\mathcal{T}_h(\Gamma)\}_{h>0}$ the family of triangulations induced by $\{\mathcal{T}_h\}_{h>0}$ on Γ. In what follows, $C(\Gamma)$ is the space of continuous functions on Γ, and given an integer $\ell \geq 0$ and a subset S of $\mathbb{R}^3$, $P_\ell(S)$ denotes the space of polynomials defined in S of total degree $\leq\, \ell$. In addition, following the same terminology described at the end of the introduction, we denote $\mathbf{P}_\ell(S) \,:=\, [P_\ell(S)]^3$ and $\mathbb{P}_\ell(S) \,:=\, [P_\ell(S)]^{3\times 3}$. Then, we define

$$\mathbb{H}_h \,:=\, \left\{\boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\,\Omega) : \quad \boldsymbol{\tau}_h|_T \,\in\, \mathbb{P}_1(T) \quad \forall\, T \,\in\, \mathcal{T}_h\right\},$$

$$\Phi_h := \Big\{ \varphi_h \in C(\Gamma): \quad \varphi_h|_F \in P_1(F) \quad \forall\, F \in \mathcal{T}_h(\Gamma) \Big\},$$

$$\Theta_h^r := \Big\{ \chi_h \in L^2(\Gamma): \quad \chi_h|_F \in P_r(F) \quad \forall\, F \in \mathcal{T}_h(\Gamma) \Big\}, \quad r \in \{0,1\},$$

and introduce the finite element subspaces of $\mathbb{X}$ and $\mathbb{Y}$, given, respectively, by

$$\mathbb{X}_h := \Big\{ \widehat{\boldsymbol{\tau}}_h = (\boldsymbol{\tau}_h, \varphi_h) \in \mathbb{H}_h \times \Phi_h : \quad \boldsymbol{\tau}_h \boldsymbol{\nu} = -\varphi_h \boldsymbol{\nu} \quad \text{on} \quad \Gamma \Big\}, \tag{5.1}$$

and

$$\mathbb{Y}_h := \Big\{ \mathbf{s}_h \in \mathbb{Y}: \quad \mathbf{s}_h|_T \in \mathbb{P}_0(T) \quad \forall\, T \in \mathcal{T}_h \Big\}.$$

In addition, throughout the analysis below we will also need the space

$$\mathbf{U}_h := \Big\{ \mathbf{v}_h \in \mathbf{L}^2(\Omega): \quad \mathbf{v}_h|_T \in \mathbf{P}_0(T) \quad \forall\, T \in \mathcal{T}_h \Big\}.$$

Note that $\mathbb{H}_h \times \mathbf{U}_h \times \mathbb{Y}_h$ constitutes the lowest order mixed finite element approximation of the linear elasticity problem introduced recently by Arnold, Falk and Winther (see [3], [2], [7]).

Hence, the finite element scheme associated to our coupled problem (3.5) is defined as: Find $(\widehat{\boldsymbol{\sigma}}_h, \vartheta_h, z_h) \in \widehat{\mathbb{X}}_h := \mathbb{X}_h \times \Theta_h^0 \times \Phi_h$ and $\mathbf{r}_h \in \mathbb{Y}_h$ such that

$$\begin{aligned} \mathbb{A}((\widehat{\boldsymbol{\sigma}}_h, \vartheta_h, z_h), (\widehat{\boldsymbol{\tau}}, \chi, w)) + \mathbb{B}((\widehat{\boldsymbol{\tau}}, \chi, w), \mathbf{r}_h) &= \mathbb{F}((\widehat{\boldsymbol{\tau}}, \chi, w)) \quad \forall (\widehat{\boldsymbol{\tau}}, \chi, w) \in \widehat{\mathbb{X}}_h, \\ \mathbb{B}((\widehat{\boldsymbol{\sigma}}_h, \vartheta_h, z_h), \mathbf{s}) &= 0 \qquad \forall\, \mathbf{s} \in \mathbb{Y}_h. \end{aligned} \tag{5.2}$$

The well-posedness of (5.2) will be proved below in Section 5.3. We previously collect in what remains of this section the approximation properties of the subspaces involved, and then in Section 5.2 we introduce a mixed finite element approximation of the operator $\mathbf{P}|_{\mathbb{X}_h}$ (cf. (4.3)).

Given $\delta \in (0,1]$, we let $\mathcal{E}_h : \mathbb{H}^\delta(\Omega) \cap \mathbb{H}(\mathbf{div};\Omega) \to \mathbb{H}_h$ be the usual BDM interpolation operator (see [6]), which, given $\boldsymbol{\tau} \in \mathbb{H}^\delta(\Omega) \cap \mathbb{H}(\mathbf{div};\Omega)$ is characterized by the identities

$$\int_F \mathcal{E}_h(\boldsymbol{\tau})\boldsymbol{\nu}\cdot\mathbf{p} = \int_F \gamma_{\boldsymbol{\nu}}\boldsymbol{\tau}\cdot\mathbf{p} \quad \forall\, \mathbf{p} \in \mathbf{P}_1(F), \quad \forall F \in \mathcal{T}_h(\Gamma). \tag{5.3}$$

Moreover, the commuting diagram property yields

$$\mathbf{div}(\mathcal{E}_h(\boldsymbol{\tau})) = \mathcal{P}_h(\mathbf{div}\,\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}^\delta(\Omega) \cap \mathbb{H}(\mathbf{div};\Omega), \tag{5.4}$$

where $\mathcal{P}_h : \mathbf{L}^2(\Omega) \to \mathbf{U}_h$ is the $\mathbf{L}^2(\Omega)$-orthogonal projector. In addition, it is easy to show, using the Bramble-Hilbert Lemma and the boundedness of the local interpolation operators on the reference element $\widehat{T}$ (see, e.g. [15, equation (3.39)]), that there exists $C > 0$, independent of h, such that for any $\boldsymbol{\tau} \in \mathbb{H}^\delta(\Omega) \cap \mathbb{H}(\mathbf{div};\Omega)$ there holds

$$\|\boldsymbol{\tau} - \mathcal{E}_h(\boldsymbol{\tau})\|_{0,T} \le C\, h_T^\delta \Big\{ |\boldsymbol{\tau}|_{\delta,T} + \|\mathbf{div}\,\boldsymbol{\tau}\|_{0,T} \Big\} \qquad \forall\, T \in \mathcal{T}_h. \tag{5.5}$$

We now let $\Pi_h : \mathbb{H}(\mathbf{div};\Omega) \to \mathbb{H}_h$, $\pi_h : H^{1/2}(\Gamma) \to \Phi_h$, and $\mathcal{R}_h : \mathbb{L}^2(\Omega) \to \mathbb{Y}_h$ be the corresponding orthogonal projectors. Then we have (see [6]):

($\mathrm{AP}_h^{\boldsymbol{\sigma}}$) For any $\delta \in (0,1]$ and for any $\boldsymbol{\tau} \in \mathbb{H}^\delta(\Omega)$, with $\mathbf{div}\,\boldsymbol{\tau} \in \mathbf{H}^\delta(\Omega)$, there holds

$$\|\boldsymbol{\tau} - \Pi_h(\boldsymbol{\tau})\|_{\mathbf{div};\Omega} \le C\, h^\delta \Big\{ \|\boldsymbol{\tau}\|_{\delta,\Omega} + \|\mathbf{div}\,\boldsymbol{\tau}\|_{\delta,\Omega} \Big\}.$$

(AP_h^{ψ}) For any $s \in (1,2]$ and for any $\varphi \in H^s(\Gamma)$ there holds (cf. [27, Theorem 4.3.22])

$$\|\varphi - \pi_h(\varphi)\|_{1/2,\Gamma} \,\leq\, C\, h^{s-1/2}\, \|\varphi\|_{s,\Gamma}\,.$$

($\mathrm{AP}_h^{\vartheta}$) For any $0 \leq t \leq s \leq 1$ and for any $\chi \in H^s(\Gamma)$, there holds (cf. [27, Theorem 4.3.20])

$$\inf_{\chi_h \in \Theta_h^0} \|\chi - \chi_h\|_{-t,\Gamma} \,\leq\, C\, h^{s+t}\, \|\chi\|_{s,\Gamma}\,.$$

($\mathrm{AP}_h^{\mathbf{r}}$) For any $\epsilon \in (0,1]$ and for any $\mathbf{s} \in \mathbb{H}^{\epsilon}(\Omega) \cap \mathbb{L}^2_{\mathtt{asym}}(\Omega)$, there holds

$$\|\mathbf{s} - \mathcal{R}_h(\mathbf{s})\|_{0,\Omega} \,\leq\, C\, h^{\epsilon}\, \|\mathbf{s}\|_{\epsilon,\Omega}\,.$$

($\mathrm{AP}_h^{\mathbf{u}}$) For any $t \in (0,1]$ and for any $\mathbf{v} \in \mathbf{H}^t(\Omega)$, there holds

$$\|\mathbf{v} - \mathcal{P}_h(\mathbf{v})\|_{0,\Omega} \,\leq\, C\, h^{t}\, \|\mathbf{v}\|_{t,\Omega}\,.$$

Note here that ($\mathrm{AP}_h^{\boldsymbol{\sigma}}$) is a consequence of the inequality $\|\boldsymbol{\tau} - \Pi_h(\boldsymbol{\tau})\|_{\mathbf{div};\Omega} \,\leq\, \|\boldsymbol{\tau} - \mathcal{E}_h(\boldsymbol{\tau})\|_{\mathbf{div};\Omega}$ together with (5.4), (5.5), and ($\mathrm{AP}_h^{\mathbf{u}}$).

We end this section with an approximation property of our finite element subspace $\mathbb{X}_h$ (cf. (5.1)). For this purpose, we assume from now on that $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform around Γ. This means that there exists an open neighborhood of Γ, say Ω_Γ, with Lipschitz boundary, and such that the elements of $\mathcal{T}_h$ intersecting that region are more or less of the same size. In other words, we define

$$\mathcal{T}_{\Gamma,h} \,:=\, \Big\{ T \in \mathcal{T}_h : \quad T \,\cap\, \Omega_\Gamma \neq \emptyset \Big\}\,,$$

and assume that there exists $c > 0$, independent of h, such that

$$\max_{T \in \mathcal{T}_{\Gamma,h}} h_T \;\leq\; c \min_{T \in \mathcal{T}_{\Gamma,h}} h_T \qquad \forall\, h > 0\,. \tag{5.6}$$

Note that this assumption and the shape-regularity property of the meshes imply that $\mathcal{T}_h(\Gamma)$, the partition on Γ inherited from $\mathcal{T}_h$, is also quasi-uniform, which means that there exists $C > 0$, independent of h, such that

$$h_\Gamma \,:=\, \max\Big\{ \mathrm{diam}\,\{F\} : \quad F \in \mathcal{T}_h(\Gamma) \Big\} \,\leq\, C\, \min\Big\{ \mathrm{diam}\,\{F\} : \quad F \in \mathcal{T}_h(\Gamma) \Big\}\,.$$

In addition, the quasi-uniformity of $\mathcal{T}_h(\Gamma)$ guarantees the inverse inequality on Θ_h^1, the subspace of $L^2(\Gamma)$ given by the piecewise polynomials of degree ≤ 1, that is, in particular,

$$\|\chi_h\|_{0,\Gamma} \,\leq\, C\, h_\Gamma^{-1/2}\, \|\chi_h\|_{-1/2,\Gamma} \qquad \forall\, \chi_h \in \Theta_h^1\,. \tag{5.7}$$

We are now in a position to establish the following lemma.

Lemma 5.1 *There exists $C > 0$, independent of h, such that for any $\widehat{\boldsymbol{\tau}} = (\boldsymbol{\tau}, \varphi) \in \mathbb{X}$ there holds*

$$\inf_{\widehat{\boldsymbol{\tau}}_h \in \mathbb{X}_h} \|\widehat{\boldsymbol{\tau}} \,-\, \widehat{\boldsymbol{\tau}}_h\|_{\mathbb{X}} \,\leq\, C\,\Big\{ \|\boldsymbol{\tau} - \Pi_h(\boldsymbol{\tau})\|_{\mathbf{div};\Omega} \,+\, \|\varphi - \pi_h(\varphi)\|_{1/2,\Gamma} \Big\}\,. \tag{5.8}$$

Proof. Given $\widehat{\boldsymbol{\tau}} := (\boldsymbol{\tau}, \varphi) \in \mathbb{X}$, we let $\mathbf{v} \in \mathbf{H}^1(\Omega)$ be the unique solution (guaranteed by the Lax-Milgram Lemma) of the vectorial Laplace problem

$$\begin{aligned} \Delta \mathbf{v} \,&=\, \frac{1}{|\Omega|} \int_\Gamma \Big\{ \Pi_h(\boldsymbol{\tau})\, \boldsymbol{\nu} \,+\, \pi_h(\varphi)\, \boldsymbol{\nu} \Big\} \quad \text{in} \quad \Omega \\ \frac{\partial \mathbf{v}}{\partial \boldsymbol{\nu}} \,&=\, \Pi_h(\boldsymbol{\tau})\, \boldsymbol{\nu} \,+\, \pi_h(\varphi)\, \boldsymbol{\nu} \quad \text{on} \quad \Gamma\,, \qquad \int_\Omega \mathbf{v} \,=\, \mathbf{0}\,, \end{aligned} \tag{5.9}$$

whose corresponding continuous dependence result states that

$$\|\mathbf{v}\|_{1,\Omega} \,\le\, C\, \|\Pi_h(\boldsymbol{\tau})\, \boldsymbol{\nu} \,+\, \pi_h(\varphi)\, \boldsymbol{\nu}\|_{-1/2,\Gamma}\,. \tag{5.10}$$

Since the Neumann datum $\Pi_h(\boldsymbol{\tau})\, \boldsymbol{\nu} \,+\, \pi_h(\varphi)\, \boldsymbol{\nu}$, being a piecewise polynomial of degree ≤ 1, belongs to $\mathbf{H}^\delta(\Gamma)$ for any $\delta \in [0, 1/2)$, we deduce that we have at least $\mathbf{H}^{3/2}(\Omega)$-regularity for the solution $\mathbf{v}$ and (see [14])

$$\|\mathbf{v}\|_{3/2,\Omega} \,\le\, C\, \|\Pi_h(\boldsymbol{\tau})\, \boldsymbol{\nu} \,+\, \pi_h(\varphi)\, \boldsymbol{\nu}\|_{0,\Gamma}\,. \tag{5.11}$$

Moreover, since $\Omega^{\mathbf{int}} := \Omega \backslash \Omega_\Gamma$ is an interior region of Ω, the interior elliptic regularity estimate (see, e.g. [24, Theorem 4.16]) says that

$$\|\mathbf{v}\|_{2,\Omega^{\mathbf{int}}} \,\le\, C\, \|\Pi_h(\boldsymbol{\tau})\, \boldsymbol{\nu} \,+\, \pi_h(\varphi)\, \boldsymbol{\nu}\|_{-1/2,\Gamma}\,. \tag{5.12}$$

Next, we define $\boldsymbol{\zeta} := \nabla \mathbf{v}$ in Ω, whence $\boldsymbol{\zeta} \in \mathbb{H}^{1/2+\delta}(\Omega)$, and observe from (5.9) that

$$\mathbf{div}\, \boldsymbol{\zeta} \,=\, \frac{1}{|\Omega|} \int_\Gamma \Big\{ \Pi_h(\boldsymbol{\tau})\, \boldsymbol{\nu} \,+\, \pi_h(\varphi)\, \boldsymbol{\nu} \Big\} \quad \text{in} \quad \Omega\,, \quad \text{and} \quad \gamma_{\boldsymbol{\nu}}\, \boldsymbol{\zeta} \,=\, \Pi_h(\boldsymbol{\tau})\, \boldsymbol{\nu} \,+\, \pi_h(\varphi)\, \boldsymbol{\nu} \quad \text{on} \quad \Gamma\,, \tag{5.13}$$

which, in particular, implies that $\boldsymbol{\zeta} \in \mathbb{H}(\mathbf{div};\, \Omega)$. Hence, we now set

$$\widehat{\boldsymbol{\tau}}_h \,:=\, \big(\Pi_h(\boldsymbol{\tau}) - \mathcal{E}_h(\boldsymbol{\zeta}), \pi_h(\varphi) \big) \,\in\, \mathbb{H}_h \times \varPhi_h\,,$$

and show that $\widehat{\boldsymbol{\tau}}_h \in \mathbb{X}_h$. In fact, employing the characterization (5.3) and the second identity in (5.13), we find that for any $F \in \mathcal{T}_h(\Gamma)$ and for any $\mathbf{p} \in \mathbf{P}_1(F)$, there holds

$$\int_F \mathcal{E}_h(\boldsymbol{\zeta})\, \boldsymbol{\nu} \cdot \mathbf{p} \,=\, \int_F \gamma_{\boldsymbol{\nu}}\, \boldsymbol{\zeta} \cdot \mathbf{p} \,=\, \int_F \Big\{ \Pi_h(\boldsymbol{\tau})\, \boldsymbol{\nu} \,+\, \pi_h(\varphi)\, \boldsymbol{\nu} \Big\} \cdot \mathbf{p}$$

which, noting that $\Big\{ \big(\Pi_h(\boldsymbol{\tau}) - \mathcal{E}_h(\boldsymbol{\zeta})\big)\, \boldsymbol{\nu} + \pi_h(\varphi)\, \boldsymbol{\nu} \Big\}\Big|_F \in \mathbf{P}_1(F)$, yields $\big(\Pi_h(\boldsymbol{\tau}) - \mathcal{E}_h(\boldsymbol{\zeta})\big)\, \boldsymbol{\nu} \,=\, -\,\pi_h(\varphi)\, \boldsymbol{\nu}$ on Γ.

We now aim to prove (5.8). We first observe, applying the triangle inequality, that

$$\|\widehat{\boldsymbol{\tau}} - \widehat{\boldsymbol{\tau}}_h\|_{\mathbb{X}}^2 \,\le\, 2\, \|\boldsymbol{\tau} - \Pi_h(\boldsymbol{\tau})\|_{\mathbf{div};\Omega}^2 \,+\, 2\, \|\mathcal{E}_h(\boldsymbol{\zeta})\|_{\mathbf{div};\Omega}^2 \,+\, \|\varphi - \pi_h(\varphi)\|_{1/2,\Gamma}^2\,. \tag{5.14}$$

Then, using the first identity in (5.13), which says that $\mathbf{div}\, \boldsymbol{\zeta} \in \mathbf{U}_h$, and (5.4), we deduce that

$$\begin{aligned} \|\mathcal{E}_h(\boldsymbol{\zeta})\|_{\mathbf{div};\Omega}^2 \,&=\, \|\mathcal{E}_h(\boldsymbol{\zeta})\|_{0,\Omega}^2 \,+\, \|\mathbf{div}\, \boldsymbol{\zeta}\|_{0,\Omega}^2 \\ &\le\, C_0 \Big\{ \|\mathcal{E}_h(\boldsymbol{\zeta})\|_{0,\Omega}^2 \,+\, \|\Pi_h(\boldsymbol{\tau})\, \boldsymbol{\nu} \,+\, \pi_h(\varphi)\, \boldsymbol{\nu}\|_{-1/2,\Gamma}^2 \Big\}\,. \end{aligned} \tag{5.15}$$

Now, adding and substracting $\gamma_{\boldsymbol{\nu}}\, \boldsymbol{\tau} \,=\, -\,\varphi\, \boldsymbol{\nu}$ on Γ, and applying the trace theorem in $\mathbb{H}(\mathbf{div};\, \Omega)$, we find that

$$\begin{aligned} \|\Pi_h(\boldsymbol{\tau})\, \boldsymbol{\nu} \,+\, \pi_h(\varphi)\, \boldsymbol{\nu}\|_{-1/2,\Gamma} \,&\le\, \|\gamma_{\boldsymbol{\nu}}\big(\boldsymbol{\tau} - \Pi_h(\boldsymbol{\tau})\big)\|_{-1/2,\Gamma} \,+\, \|\big(\varphi \,-\, \pi_h(\varphi)\big)\, \boldsymbol{\nu}\|_{-1/2,\Gamma} \\ &\le\, C_1 \Big\{ \|\boldsymbol{\tau} - \Pi_h(\boldsymbol{\tau})\|_{\mathbf{div};\Omega} \,+\, \|\varphi \,-\, \pi_h(\varphi)\|_{0,\Gamma} \Big\} \\ &\le\, C_1 \Big\{ \|\boldsymbol{\tau} - \Pi_h(\boldsymbol{\tau})\|_{\mathbf{div};\Omega} \,+\, \|\varphi - \pi_h(\varphi)\|_{1/2,\Gamma} \Big\}\,. \end{aligned} \tag{5.16}$$

It remains to estimate $\|\mathcal{E}_h(\boldsymbol{\zeta})\|_{0,\Omega}$. In fact, defining the sets

$$\Omega_{\Gamma,h} := \cup\Big\{ T : \quad T \in \mathcal{T}_{\Gamma,h} \Big\} \quad \text{and} \quad \Omega_h^{\mathtt{int}} := \Omega\backslash\Omega_{\Gamma,h} \subseteq \Omega^{\mathtt{int}} ,$$

and using the stability of $\mathcal{E}_h$ when applied to $\mathbf{H}^1(\Omega_h^{\mathtt{int}})$, and the estimate (5.12), we find that

$$\begin{aligned} \|\mathcal{E}_h(\boldsymbol{\zeta})\|_{0,\Omega} \,&\leq\, \|\mathcal{E}_h(\boldsymbol{\zeta})\|_{0,\Omega_h^{\mathtt{int}}} \,+\, \|\mathcal{E}_h(\boldsymbol{\zeta})\|_{0,\Omega_{\Gamma,h}} \\ &\leq\, C_2\,\|\mathbf{v}\|_{2,\Omega^{\mathtt{int}}} \,+\, \|\mathcal{E}_h(\boldsymbol{\zeta})\|_{0,\Omega_{\Gamma,h}} \\ &\leq\, C_3\,\|\Pi_h(\boldsymbol{\tau})\,\boldsymbol{\nu} \,+\, \pi_h(\varphi)\,\boldsymbol{\nu}\|_{-1/2,\Gamma} \,+\, \|\mathcal{E}_h(\boldsymbol{\zeta})\|_{0,\Omega_{\Gamma,h}}\,. \end{aligned} \tag{5.17}$$

In turn, adding and substracting $\boldsymbol{\zeta} = \nabla\mathbf{v}$, and utilizing the upper bound (5.10), the estimates (5.5) (with $\delta = 1/2$) and (5.11), the first identity in (5.13), the quasi-uniformity bound (5.6), and the inverse inequality (5.7), we arrive at

$$\begin{aligned} \|\mathcal{E}_h(\boldsymbol{\zeta})\|^2_{0,\Omega_{\Gamma,h}} \,&\leq\, C_4\,\Big\{\|\boldsymbol{\zeta} - \mathcal{E}_h(\boldsymbol{\zeta})\|^2_{0,\Omega_{\Gamma,h}} \,+\, \|\Pi_h(\boldsymbol{\tau})\,\boldsymbol{\nu} \,+\, \pi_h(\varphi)\,\boldsymbol{\nu}\|^2_{-1/2,\Gamma}\Big\} \\ &\leq\, C_5 \sum_{T\in\mathcal{T}_{\Gamma,h}} h_T\,\|\mathbf{v}\|^2_{3/2,T} \,+\, C_5\,\|\mathbf{div}\,\boldsymbol{\zeta}\|^2_{0,\Omega} \,+\, C_4\,\|\Pi_h(\boldsymbol{\tau})\,\boldsymbol{\nu} \,+\, \pi_h(\varphi)\,\boldsymbol{\nu}\|^2_{-1/2,\Gamma} \\ &\leq\, C_6\,h_\Gamma\,\|\Pi_h(\boldsymbol{\tau})\,\boldsymbol{\nu} \,+\, \pi_h(\varphi)\,\boldsymbol{\nu}\|^2_{0,\Gamma} \,+\, \tilde{C}_4\,\|\Pi_h(\boldsymbol{\tau})\,\boldsymbol{\nu} \,+\, \pi_h(\varphi)\,\boldsymbol{\nu}\|^2_{-1/2,\Gamma} \\ &\leq\, C_7\,\|\Pi_h(\boldsymbol{\tau})\,\boldsymbol{\nu} \,+\, \pi_h(\varphi)\,\boldsymbol{\nu}\|^2_{-1/2,\Gamma}\,. \end{aligned} \tag{5.18}$$

Finally, (5.14), (5.15), (5.16), (5.17) and (5.18) finish the proof.

□

5.2 A mixed finite element approximation of $\mathbf{P}|_{\mathbb{X}_h}$

In what follows we introduce uniformly bounded linear operators $\mathbf{P}_h : \mathbb{X}_h \to \mathbb{X}_h$ so that $\mathbf{P}_h(\widehat{\boldsymbol{\tau}}_h)$ becomes a suitable discrete approximation of $\mathbf{P}(\widehat{\boldsymbol{\tau}}_h)$ for any $\widehat{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_h, \varphi_h) \in \mathbb{X}_h$, and then provide the corresponding estimate for the error $\|\mathbf{P}(\widehat{\boldsymbol{\tau}}_h) - \mathbf{P}_h(\widehat{\boldsymbol{\tau}}_h)\|_{\mathbf{div};\Omega}$.

Given $\widehat{\boldsymbol{\tau}}_h = (\boldsymbol{\tau}_h, \varphi_h) \in \mathbb{X}_h \subseteq \mathbb{X}$, we first recall from (4.1) and (4.3) that

$$P(\widehat{\boldsymbol{\tau}}_h) \,=\, \tilde{\boldsymbol{\sigma}} \qquad \text{and} \qquad \mathbf{P}(\widehat{\boldsymbol{\tau}}_h) \,=\, (P(\widehat{\boldsymbol{\tau}}_h), \varphi_h)\,, \tag{5.19}$$

where $\tilde{\boldsymbol{\sigma}} := \mathcal{C}\,\boldsymbol{\varepsilon}(\tilde{\mathbf{u}})$ and $\tilde{\mathbf{u}}$ is the unique solution of the problem

$$\begin{aligned} \tilde{\boldsymbol{\sigma}} \,=\, \mathcal{C}\,\boldsymbol{\varepsilon}(\tilde{\mathbf{u}}) \quad \text{in} \quad \Omega\,, \quad \mathbf{div}\,\tilde{\boldsymbol{\sigma}} \,=\, \mathbf{div}\,\boldsymbol{\tau}_h + \mathbf{v}(\widehat{\boldsymbol{\tau}}_h) \quad \text{in} \quad \Omega\,, \\ \gamma_{\boldsymbol{\nu}}\,\tilde{\boldsymbol{\sigma}} \,=\, -\,\varphi_h\,\boldsymbol{\nu} \quad \text{on} \quad \Gamma\,, \quad \tilde{\mathbf{u}} \in (\mathrm{id} - \mathbf{M})(\mathbf{L}^2(\Omega))\,. \end{aligned} \tag{5.20}$$

Then, proceeding analogously to [13, Section 5.2], we let $(\tilde{\boldsymbol{\sigma}}_h, \tilde{\mathbf{u}}_h, \tilde{\mathbf{r}}_h) \in \mathbb{H}_h \times (\mathrm{id} - \mathbf{M})(\mathbf{U}_h) \times \mathbb{Y}_h$ be the usual mixed finite element approximation of the solution of (5.20) and define

$$P_h(\widehat{\boldsymbol{\tau}}_h) \,:=\, \tilde{\boldsymbol{\sigma}}_h \qquad \text{and} \qquad \mathbf{P}_h(\widehat{\boldsymbol{\tau}}_h) \,:=\, (P_h(\widehat{\boldsymbol{\tau}}_h), \varphi_h)\,.$$

It is not difficult to show, as in [13, Section 5.2, Theorem 5.1 and eq. (5.27)], that there hold

$$\|P_h(\widehat{\boldsymbol{\tau}}_h)\|_{\mathbf{div};\Omega} \,\leq\, C\,\Big\{\|\boldsymbol{\tau}_h\|_{\mathbf{div};\Omega} \,+\, \|\varphi_h\|_{1/2,\Gamma}\Big\}\,, \tag{5.21}$$

$$P_h(\widehat{\boldsymbol{\tau}}_h)\,\boldsymbol{\nu} \,=\, -\,\varphi_h\,\boldsymbol{\nu} \quad \text{on} \quad \Gamma \qquad \text{and} \qquad \int_\Omega P_h(\widehat{\boldsymbol{\tau}}_h) : \tilde{\mathbf{s}}_h \,=\, 0 \quad \forall\, \tilde{\mathbf{s}}_h \,\in\, \mathbb{Y}_h\,. \tag{5.22}$$

It is clear that (5.21) yields the uniform boundedness of $\mathbf{P}_h$, while the first equation of (5.22) guarantees that $\mathbf{P}_h(\widehat{\boldsymbol{\tau}}_h)$ belongs to $\mathbb{X}_h$. In addition, following the same arguments of the proof of [13, Lemma 5.4], that is using the definition (5.19), the commuting diagram identity (5.4), the approximation properties (5.5), ($\mathrm{AP}_h^{\mathbf{u}}$), and ($\mathrm{AP}_h^{\mathbf{r}}$), and the regularity estimate for (5.20) (cf. (4.2), (4.7)), we can prove the following error estimate.

Lemma 5.2 *Let $\epsilon > 0$ be the parameter defining the regularity of the solution of* (5.20)*. Then, there exists $C > 0$, independent of h, such that for any $\widehat{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_h, \varphi_h) \in \mathbb{X}_h$ there holds*

$$\|\mathbf{P}(\widehat{\boldsymbol{\tau}}_h) \,-\, \mathbf{P}_h(\widehat{\boldsymbol{\tau}}_h)\|_{\mathbf{div};\Omega} \,\leq\, C\,h^\epsilon \Big\{ \|\mathbf{div}\,\boldsymbol{\tau}_h\|_{0,\Omega} \,+\, \|\varphi_h\|_{1/2,\Gamma} \Big\}\,. \tag{5.23}$$

5.3 Well-posedness of the discrete formulation

In this section we prove the well-posedness of our mixed finite element scheme (5.2). To this end, as established by a classical result on projection methods for Fredholm operators of index zero (see, e.g. [22, Theorem 13.7]), it suffices to show that the Galerkin scheme associated to the isomorphism $\begin{pmatrix} \mathbf{A}_0 & \mathbf{B}^* \\ \mathbf{B} & \mathbf{0} \end{pmatrix}$ is well-posed. Therefore, in what follows we prove that $\mathbb{B}$ (cf. (3.20)) and $\mathbb{A}_0$ (cf. (4.13)) satisfy the corresponding inf-sup conditions on the finite element subspace $\mathbb{X}_h \times \mathbb{Y}_h$, thus providing the discrete analogues of Lemmata 4.1 and 4.5.

We begin with the discrete inf-sup condition for $\mathbb{B}$.

Lemma 5.3 *There exists $\beta > 0$, independent of h, such that for any $\mathbf{s}_h \in \mathbb{Y}_h$, there holds*

$$\sup_{\substack{(\widehat{\boldsymbol{\tau}}_h,\chi_h,w_h)\,\in\,\widehat{\mathbb{X}}_h \\ (\widehat{\boldsymbol{\tau}}_h,\chi_h,w_h)\neq\mathbf{0}}} \frac{|\,\mathbb{B}((\widehat{\boldsymbol{\tau}}_h,\chi_h,w_h),\mathbf{s}_h)\,|}{\|(\widehat{\boldsymbol{\tau}}_h,\chi_h,w_h)\|_{\widehat{\mathbb{X}}}} \,\geq\, \beta\,\|\mathbf{s}_h\|_{0,\Omega}\,.$$

Proof. It suffices to note that $\mathbb{B}((\widehat{\boldsymbol{\tau}}_h,\chi_h,w_h),\mathbf{s}_h) \,=\, -\,\rho_f\,\omega^2 \displaystyle\int_\Omega \boldsymbol{\tau}_h : \mathbf{s}_h$ and then apply the inf-sup condition given by [13, Lemma 5.5]. □

We now let $\widehat{\mathbb{V}}_h$ be the discrete kernel of $\mathbb{B}$, that is

$$\begin{aligned} \widehat{\mathbb{V}}_h \,:=\, & \Big\{ (\widehat{\boldsymbol{\tau}}_h,\chi_h,w_h) \,\in\, \widehat{\mathbb{X}}_h : \quad B((\widehat{\boldsymbol{\tau}}_h,\chi_h,w_h),\mathbf{s}_h) \,=\, 0 \quad \forall\, \mathbf{s}_h \,\in\, \mathbb{Y}_h \Big\} \\ =\, & \Big\{ (\widehat{\boldsymbol{\tau}}_h,\chi_h,w_h) \,\in\, \widehat{\mathbb{X}}_h : \quad \int_\Omega \boldsymbol{\tau}_h : \mathbf{s}_h \,=\, 0 \quad \forall\, \mathbf{s}_h \in \mathbb{Y}_h \Big\}\,. \end{aligned}$$

Then, the discrete weak coercivity of $\mathbb{A}_0$ is established as follows.

Lemma 5.4 *There exist C, $h_1 > 0$, independent of h, such that for any $h \leq h_1$ there holds*

$$\sup_{\substack{(\widehat{\boldsymbol{\sigma}}_h,\vartheta_h,z_h)\in\widehat{\mathbb{V}}_h \\ (\widehat{\boldsymbol{\sigma}}_h,\vartheta_h,z_h)\neq\mathbf{0}}} \frac{|\,\mathbb{A}_0((\widehat{\boldsymbol{\tau}}_h,\chi_h,w_h),(\widehat{\boldsymbol{\sigma}}_h,\vartheta_h,z_h))\,|}{\|(\widehat{\boldsymbol{\sigma}}_h,\vartheta_h,z_h)\|_{\widehat{\mathbb{X}}}} \,\geq\, C\,\|(\widehat{\boldsymbol{\tau}}_h,\chi_h,w_h)\|_{\widehat{\mathbb{X}}} \qquad \forall\,(\widehat{\boldsymbol{\tau}}_h,\chi_h,w_h) \in \widehat{\mathbb{V}}_h\,. \tag{5.24}$$

Proof. Let us introduce the linear and bounded operator $\Xi_h := (2\,\mathbf{P}_h - \mathrm{id}) : \mathbb{X}_h \to \mathbb{X}_h$, which constitutes a natural discrete approximation of the operator $\Xi := (2\,\mathbf{P} - \mathrm{id}) : \mathbb{X} \to \mathbb{X}$ (cf. Section 4.2). It follows from (5.23) (cf. Lemma 5.2) that for any $\widehat{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_h, \varphi_h) \in \mathbb{X}_h$ there holds

$$\|\Xi(\widehat{\boldsymbol{\tau}}_h) - \Xi_h(\widehat{\boldsymbol{\tau}}_h)\|_{\mathbf{div};\Omega} \le C\,h^\epsilon \Big\{\|\mathbf{div}\,\boldsymbol{\tau}_h\|_{0,\Omega} + \|\varphi_h\|_{1/2,\Gamma}\Big\} \le C\,h^\epsilon\,\|\widehat{\boldsymbol{\tau}}_h\|_{\mathbb{X}}\,.$$

Then, adding and substracting $(\Xi(\overline{\widehat{\boldsymbol{\tau}}}_h), \overline{\chi}_h, \overline{w}_h))$, using the boundedness of $\mathbb{A}_0$, and applying the inequality (4.21) (cf. Lemma 4.5), we find that for any $(\widehat{\boldsymbol{\tau}}_h, \chi_h, w_h) \in \widehat{\mathbb{X}}_h$ there holds

$$\begin{aligned}
&\Big|\mathrm{Re}\Big\{\,\mathbb{A}_0((\widehat{\boldsymbol{\tau}}_h, \chi_h, w_h), (\Xi_h(\overline{\widehat{\boldsymbol{\tau}}}_h), \overline{\chi}_h, \overline{w}_h))\Big\}\Big| \\
&\quad\ge\ \Big|\mathrm{Re}\Big\{\,\mathbb{A}_0((\widehat{\boldsymbol{\tau}}_h, \chi_h, w_h), (\Xi(\overline{\widehat{\boldsymbol{\tau}}}_h), \overline{\chi}_h, \overline{w}_h))\Big\}\Big| - C\,h^\epsilon \|\widehat{\boldsymbol{\tau}}_h\|^2_{\mathbb{X}} \\
&\quad\ge\ \Big\{\alpha - C\,h^\epsilon\Big\}\,\|(\widehat{\boldsymbol{\tau}}_h, \chi_h, w_h)\|^2_{\widehat{\mathbb{X}}}\,,
\end{aligned}$$

from which we deduce the existence of c, $h_1 > 0$ such that for any $h \le h_1$ there holds

$$\Big|\,\mathrm{Re}\Big\{\,\mathbb{A}_0((\widehat{\boldsymbol{\tau}}_h, \chi_h, w_h), (\Xi_h(\overline{\widehat{\boldsymbol{\tau}}}_h), \overline{\chi}_h, \overline{w}_h))\,\Big\}\,\Big| \ge c\,\|(\widehat{\boldsymbol{\tau}}_h, \chi_h, w_h)\|^2_{\widehat{\mathbb{X}}} \qquad \forall\,(\widehat{\boldsymbol{\tau}}_h, \chi_h, w_h) \in \widehat{\mathbb{X}}_h\,. \tag{5.25}$$

Note, thanks to this inequality, that $(\Xi_h(\overline{\widehat{\boldsymbol{\tau}}}_h), \overline{\chi}_h, \overline{w}_h)) \neq \mathbf{0}$ for each $(\widehat{\boldsymbol{\tau}}_h, \chi_h, w_h) \neq \mathbf{0}$. Furthermore, the second equation of (5.22) and the above characterization of $\widehat{\mathbb{V}}_h$ imply that $(\Xi_h(\widehat{\boldsymbol{\tau}}_h), \chi_h, w_h) \in \widehat{\mathbb{V}}_h$ for any $(\widehat{\boldsymbol{\tau}}_h, \chi_h, w_h) \in \widehat{\mathbb{V}}_h$. In this way, the discrete inf-sup condition (5.24) follows straightforwardly from (5.25) and the uniform boundedness of Ξ_h. □

The well-posedness and convergence of the discrete scheme (5.2) can now be established.

Theorem 5.1 *Assume that problem (2.4) has only the trivial solution. Let $h_1 > 0$ be the constant provided by Lemma 5.4. Then, there exists $h_0 \in (0, h_1]$ such that for any $h \le h_0$, the mixed finite element scheme (5.2) has a unique solution $((\widehat{\boldsymbol{\sigma}}_h, \vartheta_h, z_h), \mathbf{r}_h) \in \widehat{\mathbb{X}}_h \times \mathbb{Y}_h$. In addition, there exist C_1, $C_2 > 0$, independent of h, such that*

$$\|((\widehat{\boldsymbol{\sigma}}_h, \vartheta_h, z_h), \mathbf{r}_h)\|_{\widehat{\mathbb{X}}\times\mathbb{Y}} \le C\,\|\mathbb{F}|_{\widehat{\mathbb{X}}_h}\| \le C_1\Big\{\,\|\mathbf{f}\|_{0,\Omega} + \|\partial^+_{\boldsymbol{\nu}}\,p_i\|_{-1/2,\Gamma} + \|\gamma^+ p_i\|_{1/2,\Gamma}\,\Big\}$$

and

$$\begin{aligned}
&\|((\widehat{\boldsymbol{\sigma}}, \vartheta, z), \mathbf{r}) - (\widehat{\boldsymbol{\sigma}}_h, \vartheta_h, z_h), \mathbf{r}_h)\|_{\widehat{\mathbb{X}}\times\mathbb{Y}} \\
&\quad\le\ C_2 \inf_{((\widehat{\boldsymbol{\tau}}_h, \chi_h, w_h), \mathbf{s}_h) \in \widehat{\mathbb{X}}_h\times\mathbb{Y}_h} \|((\widehat{\boldsymbol{\sigma}}, \vartheta, z), \mathbf{r}) - ((\widehat{\boldsymbol{\tau}}_h, \chi_h, w_h), \mathbf{s}_h)\|_{\widehat{\mathbb{X}}\times\mathbb{Y}}\,.
\end{aligned} \tag{5.26}$$

Furthermore, if there exists $\delta \in (0, 1]$ such that $\boldsymbol{\sigma} \in \mathbb{H}^\delta(\Omega)$, $\mathbf{div}\,\boldsymbol{\sigma} \in \mathbf{H}^\delta(\Omega)$, $\psi \in H^{1/2+\delta}(\Gamma)$, $\vartheta \in H^{-1/2+\delta}(\Gamma)$, and $\mathbf{r} \in \mathbb{H}^\delta(\Omega)$, then there holds

$$\begin{aligned}
&\|((\widehat{\boldsymbol{\sigma}}, \vartheta, z), \mathbf{r}) - ((\widehat{\boldsymbol{\sigma}}_h, \vartheta_h, z_h), \mathbf{r}_h)\|_{\widehat{\mathbb{X}}\times\mathbb{Y}} \\
&\quad\le\ C_3\,h^\delta\Big\{\,\|\boldsymbol{\sigma}\|_{\delta,\Omega} + \|\mathbf{div}\,\boldsymbol{\sigma}\|_{\delta,\Omega} + \|\psi\|_{1/2+\delta,\Gamma} + \|\vartheta\|_{-1/2+\delta,\Gamma} + \|\mathbf{r}\|_{\delta,\Omega}\,\Big\}\,,
\end{aligned} \tag{5.27}$$

with a constant $C_3 > 0$, independent of h.

Proof. Thanks to Theorem 4.1 and Lemmata 5.3 and 5.4, the proof of the first part is a direct application of [22, Theorem 13.7], whereas the rate of convergence (5.27) follows directly from the Céa estimate (5.26), the fact that $z = 0$, and the approximation properties ($\mathrm{AP}^{\boldsymbol{\sigma}}_h$), ($\mathrm{AP}^{\psi}_h$), ($\mathrm{AP}^{\vartheta}_h$), ($\mathrm{AP}^{\mathbf{r}}_h$), and the one for $\mathbb{X}_h$ given by Lemma 5.1 (cf. (5.8)). □

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