An implicit-explicit residual error estimator for the coupling of dual-mixed finite elements and boundary elements in elastostatics

Gabriel N. Gatica * Norbert Heuer † Ernst P. Stephan ‡

Abstract

We consider the coupling of dual-mixed finite elements and boundary elements to solve a mixed Dirichlet-Neumann problem of plane elasticity. We derive an a-posteriori error estimate that is based on the solution of local Dirichlet problems and on a residual term defined on the coupling interface. The general error estimate does not make use of any special finite element or boundary element spaces. Here the residual term is given in a negative order Sobolev norm. In practical applications, where a certain boundary element subspace is used, this norm can be estimated by weighted local L^2-norms.

Key words. A-posteriori error estimate, mixed finite elements, boundary elements, coupling

Mathematics subject classifications (1991). 65N30, 65N38

1 Introduction

In this paper we propose an a-posteriori error estimate for the coupling of mixed finite elements and boundary elements to solve a mixed Dirichlet-Neumann problem of linear elasticity in $\mathbf{R}^2$. The coupled finite element/boundary element method (FEM/BEM) is often used for interface problems in unbounded domains where, e.g., nonlinearities are present in a bounded domain and the material in the exterior behaves linearly and isotropically, see, e.g., [14, 15, 22, 21]. The symmetric coupling of finite elements and boundary elements was proposed and analyzed by Costabel [14] where the displacements in the interior domain Ω_F (discretized by finite elements) are $H^1(\Omega_F)$-regular and the equilibrium of tractions across the interface is satisfied weakly. In applications, however, often the stresses are more important to determine than displacements.

*Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: ggatica@ing-mat.udec.cl The work of this author was supported by the German Academic Exchange Service (DAAD), and by Fondecyt-Chile through Project No. 1980122 and Program A on Numerical Analysis of the FONDAP in Applied Mathematics.

†Institut für Wissenschaftliche Datenverarbeitung, Universität Bremen, Postfach 33 04 40, 28334 Bremen, Germany, email: heuer@iwd.uni-bremen.de

‡Institut für Angewandte Mathematik, Universität Hannover, Welfengarten 1, 30167 Hannover, Germany, email: stephan@ifam.uni-hannover.de The work of this author was partially supported by the German Research Foundation (DFG) under Grant No. Ste 573/3-1.

Then, the coupling of boundary elements and mixed finite elements, where approximations to stresses can be determined directly, is more adequate, see [6, 21, 24, 16].

In this paper, for simplicity we deal with a model problem of plane linear elasticity in a bounded domain. For the finite element part Ω_F of the domain we use a dual-mixed formulation. That means the stresses $\boldsymbol{\sigma}$ are required to satisfy $\boldsymbol{\sigma} \in [L^2(\Omega_F)]^{2\times 2}$ and $\mathbf{div}\,\boldsymbol{\sigma} \in [L^2(\Omega_F)]^2$, whereas the displacements are only sought in $[L^2(\Omega_F)]^2$. Therefore, the approximate tractions are continuous across element sides and the interface boundary while the displacements are continuous only in a weak sense across element sides and the interface boundary.

We propose an a-posteriori error estimator that is based on the solution of local elliptic problems with Dirichlet boundary conditions and on a residual term defined on the coupling interface. The right hand side of the local problems involves residual terms and is therefore called an implicit residual estimator, cf. [4]. The advantage of this approach is that one can expect a good effectivity index, cf. [7]. In practice, the local problems are solved approximately by using a high order method that converges fast, cf. [1].

For an overview of a-posteriori error analysis in the finite element method we refer to [2, 28]. Explicit residual error indicators for boundary integral equations are considered, e.g., in [12]. Estimators based on hierarchical subspace decompositions for integral operators are given in [26]. For the coupling of FEM and BEM, explicit residual error indicators were derived in [11, 10]. Further, based on two-level additive Schwarz operators, hierarchical basis estimators for the coupling were introduced in [25]. Also, explicit residual error estimators for the coupling of mixed FEM and BEM for the Laplacian were provided in [19] and [9]. In [7] Brink and Stephan use a similar strategy as we do here to derive an implicit/explicit residual based error estimator. However, Brink and Stephan consider the symmetric coupling of standard finite elements and boundary elements whereas we use a dual-mixed variational formulation as in [6] (where no a-posteriori error estimate is given).

Our general a-posterior error estimate given by Theorem 2 below does not make use of any special knowledge about the approximate solution. It comprises a residual term on the coupling interface that has to be evaluated in a negative order Sobolev norm. We note that this general estimate can be derived in an analogous fashion for the elasticity problem in $\mathbf{R}^3$. For problems with nonlinearities, however, the inversion of the elasticity tensor $\mathcal{C}$ is not adequate. In that case a different so-called dual-dual formulation of the problem can be used [20, 18, 17]. This formulation avoids to directly invert $\mathcal{C}$. An a-posteriori error estimate for the pure FEM (no coupling with BEM) based on this dual-dual formulation is given in [5].

The remainder of this paper is as follows. In §2 we give the model problem of plane linear elasticity and rewrite it in a weak form. In §3 we propose and analyze our a-posteriori error estimator. Theorem 2 proves the reliability of this estimator. For practical applications, where one deals with a certain finite dimensional boundary element space, we show that the explicit residual term of the estimator can be bounded by a computable expression (Corollary 1). This bound is sharp on quasi-uniform boundary element meshes.

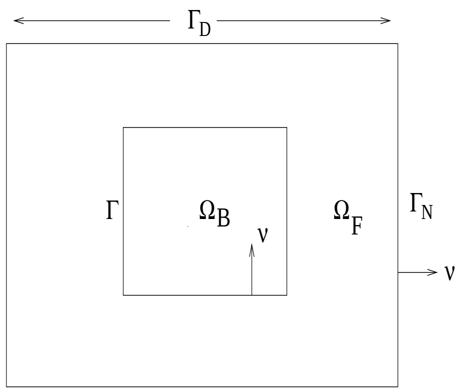

Figure 1: Geometry of the problem.

2 Linear elasticity problem

Let Ω be a polygonal domain in $\mathbf{R}^2$ with boundary $\partial\Omega = \bar{\Gamma}_N \cup \bar{\Gamma}_D$ where Γ_D is not empty. For a given body load on Ω and displacement $\mathbf{g}$ on Γ_D we consider the linear elasticity problem

$$\begin{array}{rcll} \mathbf{div}\,\boldsymbol{\sigma} &=& -\mathbf{f} & \text{in } \Omega \\ \boldsymbol{\sigma} &=& \mathcal{C}\mathbf{e}(\mathbf{u}) & \text{in } \Omega \\ \mathbf{u} &=& \mathbf{g} & \text{on } \Gamma_D \\ \boldsymbol{\sigma}\cdot\boldsymbol{\nu} &=& 0 & \text{on } \Gamma_N \end{array} \tag{1}$$

Here, $\mathbf{u}$ is the displacement field, $\mathbf{e}(\mathbf{u}) := \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$ is the strain tensor and $\boldsymbol{\nu}$ is the outward unit normal of Ω. The elasticity tensor $\mathcal{C}$ describes the stress-strain relationship. In the simplest case we have $\boldsymbol{\sigma} = \lambda\,\mathrm{tr}\,\mathbf{e}(\mathbf{u})\mathbf{I}_2 + 2\mu\mathbf{e}(\mathbf{u})$ where λ and μ are the Lamé coefficients, $\mathbf{I}_2$ denotes the identity matrix in $\mathbf{R}^{2\times 2}$ and $\mathrm{tr}\,(\boldsymbol{\tau}) := \sum_{i=1}^{2} \tau_{ii}$ for $\boldsymbol{\tau} := (\tau_{ij}) \in \mathbf{R}^{2\times 2}$.

In the following we discretize the problem (1) by coupled dual-mixed finite elements and boundary elements. In order to do so let us partition the domain into $\Omega = \Omega_F \cup \Gamma \cup \Omega_B$ such that $\bar{\Omega}_B \cap (\bar{\Gamma}_N \cup \bar{\Gamma}_D) = \emptyset$, see Figure 1. We assume that this partition is given in such a way that the body load $\mathbf{f}$ vanishes on Ω_B and that the Lamé coefficients are constant on Ω_B. Below, this allows for representing $\mathbf{u}$ on Ω_B by pure boundary integral operators acting on Γ. Further, let $\boldsymbol{\nu}$ now denote the outward unit normal of Ω_F.

To derive an appropriate variational formulation let us define the tensor spaces

$$[L^2(\Omega_F)]^{2\times 2} := \left\{ \boldsymbol{\tau} = \begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{bmatrix};\ \tau_{ij} \in L^2(\Omega_F),\ i,j = 1,2 \right\}$$

and

$$\mathbf{H}(\mathbf{div};\Omega_F) := \left\{ \boldsymbol{\tau} \in [L^2(\Omega_F)]^{2\times 2};\ \mathbf{div}\,\boldsymbol{\tau} \in [L^2(\Omega_F)]^2 \right\}$$

with norms

$$\|\boldsymbol{\tau}\|_{[L^2(\Omega_F)]^{2\times 2}} := \left(\sum_{i,j=1}^{2} \|\tau_{ij}\|^2_{L^2(\Omega_F)}\right)^{1/2},$$

$$\|\boldsymbol{\tau}\|_{\mathbf{H}(\mathbf{div};\Omega_F)} := \left(\|\boldsymbol{\tau}\|^2_{[L^2(\Omega_F)]^{2\times 2}} + \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{[L^2(\Omega_F)]^2}\right)^{1/2}$$

and inner products

$$\int_{\Omega_F} \mathbf{s} : \boldsymbol{\tau}\, dx \quad \text{and} \quad \int_{\Omega_F} \mathbf{s} : \boldsymbol{\tau}\, dx + \int_{\Omega_F} \mathbf{div}\,\mathbf{s} \cdot \mathbf{div}\,\boldsymbol{\tau}\, dx.$$

Here, the symbol : denotes the product

$$\mathbf{s} : \boldsymbol{\tau} := \sum_{i,j=1}^{2} s_{ij}\tau_{ij} \quad \text{for } \mathbf{s}, \boldsymbol{\tau} \in [L^2(\Omega_F)]^{2\times 2}$$

and

$$\mathbf{div}\,\boldsymbol{\tau} := \begin{bmatrix} \operatorname{div}(\tau_{11}\ \tau_{12}) \\ \operatorname{div}(\tau_{21}\ \tau_{22}) \end{bmatrix} \quad \text{for } \boldsymbol{\tau} = \begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{bmatrix}.$$

We also define

$$\mathbf{H}_0(\mathbf{div};\Omega_F) := \{\boldsymbol{\tau} \in \mathbf{H}(\mathbf{div};\Omega_F);\ (\boldsymbol{\tau}\cdot\boldsymbol{\nu})|_{\Gamma_N} = 0\}.$$

Now we follow [6] in deriving a weak formulation of the problem (1). On Ω_F, we consider $\mathbf{u}$ and $\boldsymbol{\sigma}$ as independent unknowns. Testing (1.b) with $\boldsymbol{\tau} \in \mathbf{H}_0(\mathbf{div};\Omega_F)$ this gives

$$\int_{\Omega_F} \boldsymbol{\tau} : \mathcal{C}^{-1}\boldsymbol{\sigma}\, dx = \int_{\Omega_F} \boldsymbol{\tau} : \mathbf{e}(\mathbf{u})\, dx.$$

Integrating by parts and using $\mathbf{u} = \mathbf{g}$ on Γ_D we obtain

$$\begin{aligned} \int_{\Omega_F} \boldsymbol{\tau} : \mathcal{C}^{-1}\boldsymbol{\sigma}\, dx + \int_{\Omega_F} \mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau}\, dx + \int_{\Omega_F} \boldsymbol{\tau} : \gamma\, dx - \langle \boldsymbol{\phi}, \boldsymbol{\tau}\cdot\boldsymbol{\nu}\rangle_\Gamma & \\ = \langle \mathbf{g}, \boldsymbol{\tau}\cdot\boldsymbol{\nu}\rangle_{\Gamma_D} \quad \forall \boldsymbol{\tau} \in \mathbf{H}_0(\mathbf{div};\Omega_F) & \end{aligned} \tag{2}$$

where $\boldsymbol{\gamma} := \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^T)$ and

$$\boldsymbol{\phi} := \mathbf{u}|_\Gamma \in [H^{1/2}(\Gamma)]^2 \tag{3}$$

are introduced as further unknowns. Here, $\langle\cdot,\cdot\rangle_\Gamma$ stands for the duality pairing between $[H^{1/2}(\Gamma)]^2$ and $[H^{-1/2}(\Gamma)]^2$ whereas $\langle\cdot,\cdot\rangle_{\Gamma_D}$ denotes the duality between $[H^{1/2}(\Gamma_D)]^2$ and $[\tilde{H}^{-1/2}(\Gamma_D)]^2$. We note that $\boldsymbol{\gamma}$ represents rotations and lies in the space

$$H_0 := \left\{\boldsymbol{\tau} \in [L^2(\Omega_F)]^{2\times 2};\ \boldsymbol{\tau} + \boldsymbol{\tau}^T = 0\right\} = \left\{\begin{bmatrix} 0 & \tau \\ -\tau & 0 \end{bmatrix};\ \tau \in L^2(\Omega_F)\right\}.$$

Further, testing (1.a), we find that there holds

$$\int_{\Omega_F} \mathbf{v} \cdot \mathbf{div}\, \boldsymbol{\sigma}\, dx = -\int_{\Omega_F} \mathbf{f} \cdot \mathbf{v}\, dx \quad \forall \mathbf{v} \in [L^2(\Omega_F)]^2. \tag{4}$$

The problem of linear elasticity within Ω_B is dealt with by boundary integral operators on Γ. Denoting by $\mathbf{V}$, $\mathbf{K}$, $\mathbf{K}'$ and $\mathbf{W}$ the integral operators of the single layer, double layer, adjoint of the double layer and hypersingular potentials, respectively, and using the well-known jump conditions (cf., e.g., [23]) we obtain

$$\boldsymbol{\phi} = (\frac{1}{2}\mathbf{I} + \mathbf{K})\boldsymbol{\phi} - \mathbf{V}(\boldsymbol{\sigma} \cdot \boldsymbol{\nu}) \quad \text{on } \Gamma \tag{5}$$

and

$$\mathbf{W}\boldsymbol{\phi} + (\frac{1}{2}\mathbf{I} + \mathbf{K}')(\boldsymbol{\sigma} \cdot \boldsymbol{\nu}) = 0 \quad \text{on } \Gamma. \tag{6}$$

Here, $\boldsymbol{\phi} = \mathbf{u}|_\Gamma$ (cf. (3)) and $\mathbf{I}$ is the identity operator on the corresponding spaces.

Eventually, we substitute (5) into (2), and we test (6) with functions $\boldsymbol{\psi} \in [H^{1/2}(\Gamma)]^2$. Then, collecting (2), (4) and (6) and requiring the symmetry of $\boldsymbol{\sigma}$ weakly by

$$\int_{\Omega_F} \boldsymbol{\sigma} : \boldsymbol{\delta}\, dx = 0 \quad \forall \boldsymbol{\delta} \in H_0,$$

we arrive at the following variational formulation of the boundary value problem (1): *Find* $(\boldsymbol{\sigma}, \boldsymbol{\phi}, \mathbf{u}, \boldsymbol{\gamma}) \in \mathbf{H}_0(\mathbf{div}; \Omega_F) \times [H^{1/2}(\Gamma)]^2 \times [L^2(\Omega_F)]^2 \times H_0$ *such that*

$$\begin{array}{rcl} a(\boldsymbol{\sigma}, \boldsymbol{\phi}; \boldsymbol{\tau}, \boldsymbol{\psi}) + b(\boldsymbol{\tau}; \mathbf{u}, \boldsymbol{\gamma}) & = & \langle \mathbf{g}, \boldsymbol{\tau} \cdot \boldsymbol{\nu} \rangle_{\Gamma_D} \\ b(\boldsymbol{\sigma}; \mathbf{v}, \boldsymbol{\delta}) & = & -\int_{\Omega_F} \mathbf{f} \cdot \mathbf{v}\, dx \end{array} \tag{7}$$

for all $(\boldsymbol{\tau}, \boldsymbol{\psi}, \mathbf{v}, \boldsymbol{\delta}) \in \mathbf{H}_0(\mathbf{div}; \Omega_F) \times [H^{1/2}(\Gamma)]^2 \times [L^2(\Omega_F)]^2 \times H_0$. Here,

$$a(\boldsymbol{\sigma}, \boldsymbol{\phi}; \boldsymbol{\tau}, \boldsymbol{\psi}) = a_F(\boldsymbol{\sigma}, \boldsymbol{\tau}) + a_B(\boldsymbol{\sigma}, \boldsymbol{\phi}; \boldsymbol{\tau}) - \langle \boldsymbol{\psi}, \mathbf{W}\boldsymbol{\phi} \rangle_\Gamma - \langle \boldsymbol{\psi}, (\frac{1}{2}\mathbf{I} + \mathbf{K}')(\boldsymbol{\sigma} \cdot \boldsymbol{\nu}) \rangle_\Gamma$$

with

$$\begin{array}{rcl} a_F(\boldsymbol{\sigma}, \boldsymbol{\tau}) & := & \displaystyle\int_{\Omega_F} \boldsymbol{\tau} : \mathcal{C}^{-1}\boldsymbol{\sigma}\, dx, \\ a_B(\boldsymbol{\sigma}, \boldsymbol{\phi}; \boldsymbol{\tau}) & := & \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, \mathbf{V}(\boldsymbol{\sigma} \cdot \boldsymbol{\nu}) \rangle_\Gamma - \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, (\frac{1}{2}\mathbf{I} + \mathbf{K})\boldsymbol{\phi} \rangle_\Gamma \end{array}$$

and

$$b(\boldsymbol{\sigma}; \mathbf{v}, \boldsymbol{\delta}) = \int_{\Omega_F} \mathbf{v} \cdot \mathbf{div}\, \boldsymbol{\sigma}\, dx + \int_{\Omega_F} \boldsymbol{\sigma} : \boldsymbol{\delta}\, dx.$$

Brink *et al.* [6] proved unique solvability of this weak formulation:

Theorem 1 [6] *For every* $\mathbf{f} \in [L^2(\Omega_F)]^2$ *and* $\mathbf{g} \in [H^{1/2}(\Gamma_D)]^2$ *the saddle point problem* (7) *has a unique solution satisfying*

$$\|\boldsymbol{\sigma}\|_{\mathbf{H}(\mathbf{div};\Omega_F)} + \|\boldsymbol{\phi}\|_{[H^{1/2}(\Gamma)]^2} + \|\mathbf{u}\|_{[L^2(\Omega_F)]^2} + \|\boldsymbol{\gamma}\|_{[L^2(\Omega_F)]^{2\times 2}} \le C \left\{ \|\mathbf{f}\|_{[L^2(\Omega_F)]^2} + \|\mathbf{g}\|_{[H^{1/2}(\Gamma_D)]^2} \right\}$$

with a positive constant C *that is independent of* $\mathbf{f}$ *and* $\mathbf{g}$.

3 Implicit residual error estimator

Let $\bar{\Omega}_F = \cup\{\bar{T};\ T \in \mathcal{T}_h\}$ be a partitioning of Ω_F. The elements T are triangles or quadrilaterals. For $T \neq T'$, $\bar{T} \cap \bar{T}'$ is either empty or a common vertex or edge.

To define our error estimator we need an elliptic, continuous, symmetric bilinear form $\hat{a}(\cdot,\cdot)$ on $\mathbf{H}_0(\mathbf{div};\Omega_F) \times [H^{1/2}(\Gamma)]_0^2$. Here, $[H^{1/2}(\Gamma)]_0^2$ is the quotient space $[H^{1/2}(\Gamma)]^2_{\ker(\mathbf{e}|_\Gamma)}$ that identifies the rigid body motions $\ker(\mathbf{e}|_\Gamma)$. For simplicity we take the bilinear forms

$$\hat{a}_F(\boldsymbol{\sigma},\boldsymbol{\tau}) := \int_{\Omega_F} \boldsymbol{\sigma} : \boldsymbol{\tau}\, dx + \int_{\Omega_F} \mathbf{div}\,\boldsymbol{\sigma} \cdot \mathbf{div}\,\boldsymbol{\tau}\, dx \quad \text{on} \quad \mathbf{H}_0(\mathbf{div};\Omega_F) \times \mathbf{H}_0(\mathbf{div};\Omega_F)$$

and

$$\hat{a}_B(\boldsymbol{\phi},\boldsymbol{\psi}) := \langle \boldsymbol{\psi}, \mathbf{W}\boldsymbol{\phi}\rangle_\Gamma \quad \text{on} \quad [H^{1/2}(\Gamma)]^2 \times [H^{1/2}(\Gamma)]^2$$

and define

$$\hat{a}(\boldsymbol{\sigma},\boldsymbol{\phi};\boldsymbol{\tau},\boldsymbol{\psi}) := \hat{a}_F(\boldsymbol{\sigma},\boldsymbol{\tau}) + \hat{a}_B(\boldsymbol{\phi},\boldsymbol{\psi}). \tag{8}$$

On $\mathbf{H}_0(\mathbf{div};\Omega_F) \times [H^{1/2}(\Gamma)]_0^2$, $\hat{a}$ is an inner product. The restriction of $\hat{a}_F$ to an element $T \in \mathcal{T}_h$ is denoted by $\hat{a}_{F,T}$, that is

$$\hat{a}_F(\boldsymbol{\sigma},\boldsymbol{\tau}) = \sum_{T\in\mathcal{T}_h} \hat{a}_{F,T}(\boldsymbol{\sigma}|_T, \boldsymbol{\tau}|_T) \quad \text{for all} \quad \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathbf{H}_0(\mathbf{div};\Omega_F).$$

Now let us assume that we have an approximate solution $(\boldsymbol{\sigma}_h, \boldsymbol{\phi}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h) \in \mathbf{H}_0(\mathbf{div};\Omega_F) \times [H^{1/2}(\Gamma)]^2 \times [L^2(\Omega_F)]^2 \times H_0$ to (7). In practice this will be a coupled finite element/boundary element solution obtained by restricting (7) to some discrete subspaces.

For the error estimator the following local problems are defined: *For $T \in \mathcal{T}_h$ find*

$$\boldsymbol{\sigma}_T \in \mathbf{H}_0(\mathbf{div};T) := \{\boldsymbol{\tau} \in [L^2(T)]^{2\times 2};\ \mathbf{div}\,\boldsymbol{\tau} \in [L^2(T)]^2, (\boldsymbol{\tau}\cdot\boldsymbol{\nu})|_{\partial T \cap \Gamma_N} = 0\}$$

such that

$$\hat{a}_{F,T}(\boldsymbol{\sigma}_T,\boldsymbol{\tau}) = -F_T(\boldsymbol{\tau}) \quad \textit{for all} \quad \boldsymbol{\tau} \in \mathbf{H}_0(\mathbf{div};T) \tag{9}$$

where

$$F_T(\boldsymbol{\tau}) := a_{F,T}(\boldsymbol{\sigma}_h,\boldsymbol{\tau}) + a_{B,T}(\boldsymbol{\sigma}_h,\boldsymbol{\phi}_h;\boldsymbol{\tau}) + b_T(\boldsymbol{\tau};\mathbf{u}_h,\boldsymbol{\gamma}_h) - \int_{\partial T\setminus\Gamma} \boldsymbol{\lambda}\cdot(\boldsymbol{\tau}\cdot\boldsymbol{n})\, ds$$

($\boldsymbol{n}$ is the outward unit normal of T) with

$$\begin{aligned}
a_{F,T}(\boldsymbol{\sigma}_h,\boldsymbol{\tau}) &:= \int_T \boldsymbol{\tau} : \mathcal{C}^{-1}\boldsymbol{\sigma}_h\, dx,\\
a_{B,T}(\boldsymbol{\sigma}_h,\boldsymbol{\phi}_h;\boldsymbol{\tau}) &:= \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, \mathbf{V}(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu})\rangle_{\partial T\cap\Gamma} - \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, (\tfrac{1}{2}\mathbf{I}+\mathbf{K})\boldsymbol{\phi}_h\rangle_{\partial T\cap\Gamma},\\
b_T(\boldsymbol{\tau};\mathbf{u}_h,\boldsymbol{\gamma}_h) &:= \int_T \mathbf{u}_h\cdot\mathbf{div}\,\boldsymbol{\tau}\, dx + \int_T \boldsymbol{\tau} : \boldsymbol{\gamma}_h\, dx.
\end{aligned}$$

Here, $\boldsymbol{\lambda} \in [H^{1/2}(\cup_{T\in\mathcal{T}_h}\partial T \setminus \Gamma)]^2$ is arbitrary on the element boundaries interior to Ω_F and on Γ_N. We require that $\boldsymbol{\lambda} = \mathbf{g}$ on Γ_D. Note that, if $\boldsymbol{\lambda}|_{\partial T\setminus\{\Gamma_N\cup\Gamma\}} = \mathbf{u}|_{\partial T\setminus\{\Gamma_N\cup\Gamma\}}$ for $T \in \mathcal{T}_h$ (which in particular implies that $\boldsymbol{\lambda}|_{\Gamma_D} = \mathbf{g}$) then the solution $\boldsymbol{\sigma}_T$ of (9) converges to 0 in $\mathbf{H}_0(\mathbf{div}; T)$ if $(\boldsymbol{\sigma}_h, \boldsymbol{\phi}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h)$ converges to $(\boldsymbol{\sigma}, \boldsymbol{\phi}, \mathbf{u}, \boldsymbol{\gamma})$. The local solution $\boldsymbol{\sigma}_T$ can be considered as a projection of the error $(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\phi} - \boldsymbol{\phi}_h, \mathbf{u} - \mathbf{u}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h)$ onto the local space $\mathbf{H}_0(\mathbf{div}; T)$. We expect that a good approximation of $\boldsymbol{\lambda}$ to $\mathbf{u}$ on interior element edges improves the efficiency of our error estimator.

The next theorem is our main result. It gives an a-posteriori error estimate based on the local problems (9) and proves its reliability.

Theorem 2 *Let $\boldsymbol{\lambda} \in [H^{1/2}(\cup_{T\in\mathcal{T}_h}\partial T \setminus \Gamma)]^2$ with $\boldsymbol{\lambda}|_{\Gamma_D} = \mathbf{g}$. Further define for any $T \in \mathcal{T}_h$ the function $\boldsymbol{\sigma}_T \in \mathbf{H}_0(\mathbf{div}; T)$ by the local problem (9). Then there holds the a-posteriori error estimate*

$$\begin{aligned}
&\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{H}(\mathbf{div};\Omega_F)} + \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{[H^{1/2}(\Gamma)]^2} + \|\mathbf{u} - \mathbf{u}_h\|_{[L^2(\Omega_F)]^2} + \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{[L^2(\Omega_F)]^{2\times 2}} \\
\leq\ & C \left\{ [\sum_{T\in\mathcal{T}_h} \hat{a}_{F,T}(\boldsymbol{\sigma}_T, \boldsymbol{\sigma}_T) + R_\Gamma^2]^{1/2} + \|\mathbf{f} + \mathbf{div}\, \boldsymbol{\sigma}_h\|_{[L^2(\Omega_F)]^2} + \|\frac{1}{2}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^T)\|_{[L^2(\Omega_F)]^{2\times 2}} \right\}
\end{aligned}$$

where C depends on the norm of

$$\mathbf{W}^{-1} :\ [H^{-1/2}(\Gamma)]^2 \to [H^{1/2}(\Gamma)]^2/\mathrm{ker}\mathbf{W},$$

and on the inf-sup constant β of the bilinear form of the saddle point problem (7), but is independent of h. Here, R_Γ is the norm of the residual for the boundary integral equation (6), that is

$$R_\Gamma := \|\mathbf{W}\boldsymbol{\phi}_h + (\frac{1}{2}\mathbf{I} + \mathbf{K}')(\boldsymbol{\sigma}_h \cdot \boldsymbol{\nu})\|_{[H^{-1/2}(\Gamma)]^2},$$

and the kernel of $\mathbf{W}$ consists of the rigid body motions, i.e., $[H^{1/2}(\Gamma)]^2/\mathrm{ker}\mathbf{W} = [H^{1/2}(\Gamma)]_0^2$.

Proof. Define $(\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\phi}}) \in \mathbf{H}_0(\mathbf{div}; \Omega_F) \times [H^{1/2}(\Gamma)]_0^2$ to be the unique solution of

$$\hat{a}(\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\phi}}; \boldsymbol{\tau}, \boldsymbol{\psi}) = a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\phi} - \boldsymbol{\phi}_h; \boldsymbol{\tau}, \boldsymbol{\psi}) + b(\boldsymbol{\tau}; \mathbf{u} - \mathbf{u}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h)$$

for all $(\boldsymbol{\tau}, \boldsymbol{\psi}) \in \mathbf{H}_0(\mathbf{div}; \Omega_F) \times [H^{1/2}(\Gamma)]_0^2$. Due to (7.a) there holds

$$\hat{a}(\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\phi}}; \boldsymbol{\tau}, \boldsymbol{\psi}) = -a(\boldsymbol{\sigma}_h, \boldsymbol{\phi}_h; \boldsymbol{\tau}, \boldsymbol{\psi}) - b(\boldsymbol{\tau}; \mathbf{u}_h, \boldsymbol{\gamma}_h) + \langle \mathbf{g}, \boldsymbol{\tau} \cdot \boldsymbol{\nu} \rangle_{\Gamma_D}. \tag{10}$$

Let us define

$$\mathbf{H}_0(\mathbf{div}; \Omega_F)^{br} := \{\boldsymbol{\tau} \in [L^2(\Omega_F)]^{2\times 2};\ \mathbf{div}\, \boldsymbol{\tau}|_T \in [L^2(T)]^2 \text{ and } (\boldsymbol{\tau} \cdot \boldsymbol{\nu})|_{\partial T\cap\Gamma_N} = 0\ \forall T \in \mathcal{T}_h\}.$$

For $\boldsymbol{\tau} \in \mathbf{H}_0(\mathbf{div}; \Omega_F)^{br}$ there holds $\boldsymbol{\tau} \in \mathbf{H}_0(\mathbf{div}; \Omega_F)$ if and only if

$$\sum_{T\in\mathcal{T}_h} \int_{\partial T\setminus\{\Gamma_D\cup\Gamma\}} \mathbf{v} \cdot (\boldsymbol{\tau} \cdot \boldsymbol{n})\, ds = 0 \quad \text{for all} \quad \mathbf{v} \in [H^1(\Omega_F)]^2.$$

Therefore, by using (10) and the property $\boldsymbol{\lambda}|_{\Gamma_D} = \mathbf{g}$, we obtain

$$
\begin{aligned}
-\frac{1}{2}\hat{a}(\hat{\boldsymbol{\sigma}},\hat{\boldsymbol{\phi}};\hat{\boldsymbol{\sigma}},\hat{\boldsymbol{\phi}}) &= \inf_{(\boldsymbol{\tau},\boldsymbol{\psi})\in\mathbf{H}_0(\mathbf{div};\Omega_F)\times[H^{1/2}(\Gamma)]^2}\{\frac{1}{2}\hat{a}(\boldsymbol{\tau},\boldsymbol{\psi};\boldsymbol{\tau},\boldsymbol{\psi}) - \hat{a}(\hat{\boldsymbol{\sigma}},\hat{\boldsymbol{\phi}};\boldsymbol{\tau},\boldsymbol{\psi})\} \\
&= \inf_{(\boldsymbol{\tau},\boldsymbol{\psi})\in\mathbf{H}_0(\mathbf{div};\Omega_F)\times[H^{1/2}(\Gamma)]^2}\{\frac{1}{2}\hat{a}(\boldsymbol{\tau},\boldsymbol{\psi};\boldsymbol{\tau},\boldsymbol{\psi}) + a(\boldsymbol{\sigma}_h,\boldsymbol{\phi}_h;\boldsymbol{\tau},\boldsymbol{\psi}) + b(\boldsymbol{\tau};\mathbf{u}_h,\gamma_h) - \langle\mathbf{g},\boldsymbol{\tau}\cdot\boldsymbol{\nu}\rangle_{\Gamma_D}\} \\
&= \inf_{(\boldsymbol{\tau},\boldsymbol{\psi})\in\mathbf{H}_0(\mathbf{div};\Omega_F)\times[H^{1/2}(\Gamma)]^2}\{\frac{1}{2}\sum_{T\in\mathcal{T}_h}\hat{a}_{F,T}(\boldsymbol{\tau},\boldsymbol{\tau}) + \frac{1}{2}\hat{a}_B(\boldsymbol{\psi},\boldsymbol{\psi}) \\
&\qquad + \sum_{T\in\mathcal{T}_h} a_{F,T}(\boldsymbol{\sigma}_h,\boldsymbol{\tau}) + \sum_{T\in\mathcal{T}_h} a_{B,T}(\boldsymbol{\sigma}_h,\boldsymbol{\phi}_h;\boldsymbol{\tau}) + \sum_{T\in\mathcal{T}_h} b_T(\boldsymbol{\tau};\mathbf{u}_h,\gamma_h) \\
&\qquad - \langle\boldsymbol{\psi},\mathbf{W}\boldsymbol{\phi}_h\rangle_\Gamma - \langle\boldsymbol{\psi},(\frac{1}{2}\mathbf{I}+\mathbf{K}')(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu})\rangle_\Gamma \\
&\qquad - \langle\mathbf{g},\boldsymbol{\tau}\cdot\boldsymbol{\nu}\rangle_{\Gamma_D} - \sum_{T\in\mathcal{T}_h}\int_{\partial T\setminus\{\Gamma_D\cup\Gamma\}}\boldsymbol{\lambda}\cdot(\boldsymbol{\tau}\cdot\boldsymbol{n})\,ds\} \\
&\geq \inf_{\boldsymbol{\tau}\in\mathbf{H}_0(\mathbf{div};\Omega_F)^{br}}\{\frac{1}{2}\sum_{T\in\mathcal{T}_h}\hat{a}_{F,T}(\boldsymbol{\tau},\boldsymbol{\tau}) + \sum_{T\in\mathcal{T}_h} F_T(\boldsymbol{\tau})\} \\
&\quad + \inf_{\boldsymbol{\psi}\in[H^{1/2}(\Gamma)]^2}\{\frac{1}{2}\langle\boldsymbol{\psi},\mathbf{W}\boldsymbol{\psi}\rangle_\Gamma - \langle\boldsymbol{\psi},\mathbf{W}\boldsymbol{\phi}_h\rangle_\Gamma - \langle\boldsymbol{\psi},(\frac{1}{2}\mathbf{I}+\mathbf{K}')(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu})\rangle_\Gamma\} \\
&\geq \sum_{T\in\mathcal{T}_h}\inf_{\boldsymbol{\tau}\in\mathbf{H}_0(\mathbf{div};T)}\{\frac{1}{2}\hat{a}_{F,T}(\boldsymbol{\tau},\boldsymbol{\tau}) + F_T(\boldsymbol{\tau})\} \\
&\quad + \inf_{\boldsymbol{\psi}\in[H^{1/2}(\Gamma)]^2}\{\frac{1}{2}\langle\boldsymbol{\psi},\mathbf{W}\boldsymbol{\psi}\rangle_\Gamma - \langle\boldsymbol{\psi},\mathbf{W}\boldsymbol{\phi}_h\rangle_\Gamma - \langle\boldsymbol{\psi},(\frac{1}{2}\mathbf{I}+\mathbf{K}')(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu})\rangle_\Gamma\}.
\end{aligned}
\tag{11}
$$

The minimizers to the element-wise quadratic forms $\frac{1}{2}\hat{a}_{F,T}(\boldsymbol{\tau},\boldsymbol{\tau}) + F_T(\boldsymbol{\tau})$ are the solutions to the local problems (9), i.e.,

$$\frac{1}{2}\hat{a}_{F,T}(\boldsymbol{\tau},\boldsymbol{\tau}) + F_T(\boldsymbol{\tau}) \geq \frac{1}{2}\hat{a}_{F,T}(\boldsymbol{\sigma}_T,\boldsymbol{\sigma}_T) + F_T(\boldsymbol{\sigma}_T) = -\frac{1}{2}\hat{a}_{F,T}(\boldsymbol{\sigma}_T,\boldsymbol{\sigma}_T). \tag{12}$$

On the other hand, the minimizer to the quadratic form

$$\frac{1}{2}\langle\boldsymbol{\psi},\mathbf{W}\boldsymbol{\psi}\rangle_\Gamma - \langle\boldsymbol{\psi},\mathbf{W}\boldsymbol{\phi}_h\rangle_\Gamma - \langle\boldsymbol{\psi},(\frac{1}{2}\mathbf{I}+\mathbf{K}')(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu})\rangle_\Gamma$$

is the solution $\tilde{\boldsymbol{\psi}} \in [H^{1/2}(\Gamma)]^2$ to the equation

$$\langle\boldsymbol{\psi},\mathbf{W}\tilde{\boldsymbol{\psi}}\rangle_\Gamma = \langle\boldsymbol{\psi},\mathbf{W}\boldsymbol{\phi}_h\rangle_\Gamma + \langle\boldsymbol{\psi},(\frac{1}{2}\mathbf{I}+\mathbf{K}')(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu})\rangle_\Gamma \quad \text{for all} \quad \boldsymbol{\psi}\in[H^{1/2}(\Gamma)]^2.$$

Since $\mathbf{W}: [H^{1/2}(\Gamma)]^2 \to [H^{-1/2}(\Gamma)]^2$ is positive semi-definite with $\ker\mathbf{W} = \ker(\mathbf{e}|_\Gamma)$, $\tilde{\boldsymbol{\psi}}$ is unique up to rigid body motions. Furthermore, there holds

$$\|\tilde{\boldsymbol{\psi}}\|_{[H^{1/2}(\Gamma)]^2} \leq \|\mathbf{W}^{-1}\|_{[H^{-1/2}(\Gamma)]^2\to[H^{1/2}(\Gamma)]^2_0}\|\mathbf{W}\boldsymbol{\phi}_h + (\frac{1}{2}\mathbf{I}+\mathbf{K}')(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu})\|_{[H^{-1/2}(\Gamma)]^2}.$$

For proofs of the mapping properties of the integral operators we refer to [13, 15]. Therefore, we obtain

$$\begin{aligned}
& \inf_{\boldsymbol{\psi}\in[H^{1/2}(\Gamma)]^2} \{\frac{1}{2}\langle\boldsymbol{\psi},\mathbf{W}\boldsymbol{\psi}\rangle_\Gamma - \langle\boldsymbol{\psi},\mathbf{W}\boldsymbol{\phi}_h\rangle_\Gamma - \langle\boldsymbol{\psi},(\frac{1}{2}\mathbf{I}+\mathbf{K}')(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu})\rangle_\Gamma\} \\
= & -\frac{1}{2}\langle\tilde{\boldsymbol{\psi}},\mathbf{W}\boldsymbol{\phi}_h + (\frac{1}{2}\mathbf{I}+\mathbf{K}')(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu})\rangle_\Gamma \\
\geq & -\frac{1}{2}\|\mathbf{W}^{-1}\|\|\mathbf{W}\boldsymbol{\phi}_h + (\frac{1}{2}\mathbf{I}+\mathbf{K}')(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu})\|^2_{[H^{-1/2}(\Gamma)]^2}.
\end{aligned} \tag{13}$$

Now, first using the definition (8) of the bilinear form $\hat{a}$ and noting that the norm in $[H^{1/2}(\Gamma)]_0^2$ is equivalent to the norm induced by $\mathbf{W}$, and then collecting (11), (12), (13), this gives

$$\begin{aligned}
\|\hat{\boldsymbol{\sigma}}\|^2_{\mathbf{H}(\mathbf{div};\Omega_F)} + \|\hat{\boldsymbol{\phi}}\|^2_{[H^{1/2}(\Gamma)]^2} & \leq c\hat{a}(\hat{\boldsymbol{\sigma}},\hat{\boldsymbol{\phi}};\hat{\boldsymbol{\sigma}},\hat{\boldsymbol{\phi}}) \\
& \leq c\sum_{T\in\mathcal{T}_h}\hat{a}_{F,T}(\boldsymbol{\sigma}_T,\boldsymbol{\sigma}_T) + c\|\mathbf{W}^{-1}\|R_\Gamma^2
\end{aligned} \tag{14}$$

with

$$R_\Gamma = \|\mathbf{W}\boldsymbol{\phi}_h + (\frac{1}{2}\mathbf{I}+\mathbf{K}')(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu})\|_{[H^{-1/2}(\Gamma)]^2}.$$

Here, c is a positive constant that is independent of the mesh $\mathcal{T}_h$ and the approximate solution $(\boldsymbol{\sigma}_h,\boldsymbol{\phi}_h,\mathbf{u}_h,\boldsymbol{\gamma}_h)$ defining $(\hat{\boldsymbol{\sigma}},\hat{\boldsymbol{\phi}})$, see (10). Since the bilinear form of the saddle point problem (7) satisfies an inf-sup condition there holds

$$\begin{aligned}
& \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{H}(\mathbf{div};\Omega_F)} + \|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{[H^{1/2}(\Gamma)]^2} + \|\mathbf{u}-\mathbf{u}_h\|_{[L^2(\Omega_F)]^2} + \|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_{[L^2(\Omega_F)]^{2\times2}} \\
\leq & \frac{1}{\beta}\sup_{(\boldsymbol{\tau},\boldsymbol{\psi},\mathbf{v},\boldsymbol{\delta})} \frac{a(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{\phi}-\boldsymbol{\phi}_h;\boldsymbol{\tau},\boldsymbol{\psi}) + b(\boldsymbol{\tau};\mathbf{u}-\mathbf{u}_h,\boldsymbol{\gamma}-\boldsymbol{\gamma}_h) + b(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h;\mathbf{v},\boldsymbol{\delta})}{\|\boldsymbol{\tau}\|_{\mathbf{H}(\mathbf{div};\Omega_F)} + \|\boldsymbol{\psi}\|_{[H^{1/2}(\Gamma)]^2} + \|\mathbf{v}\|_{[L^2(\Omega_F)]^2} + \|\boldsymbol{\delta}\|_{[L^2(\Omega_F)]^{2\times2}}} \\
= & \frac{1}{\beta}\sup_{(\boldsymbol{\tau},\boldsymbol{\psi},\mathbf{v},\boldsymbol{\delta})} \frac{\hat{a}(\hat{\boldsymbol{\sigma}},\hat{\boldsymbol{\phi}};\boldsymbol{\tau},\boldsymbol{\psi}) + \int_{\Omega_F}\mathbf{v}\cdot\mathbf{div}\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\,dx + \int_{\Omega_F}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h):\boldsymbol{\delta}\,dx}{\|\boldsymbol{\tau}\|_{\mathbf{H}(\mathbf{div};\Omega_F)} + \|\boldsymbol{\psi}\|_{[H^{1/2}(\Gamma)]^2} + \|\mathbf{v}\|_{[L^2(\Omega_F)]^2} + \|\boldsymbol{\delta}\|_{[L^2(\Omega_F)]^{2\times2}}} \\
\leq & \frac{1}{\beta}\sup_{(\boldsymbol{\tau},\boldsymbol{\psi},\mathbf{v},\boldsymbol{\delta})} \frac{\left\{\begin{array}{c}\|\hat{\boldsymbol{\sigma}}\|_{\mathbf{H}(\mathbf{div};\Omega_F)}\|\boldsymbol{\tau}\|_{\mathbf{H}(\mathbf{div};\Omega_F)} + \|\hat{\boldsymbol{\phi}}\|_{[H^{1/2}(\Gamma)]^2}\|\boldsymbol{\psi}\|_{[H^{1/2}(\Gamma)]^2} \\ +\|\mathbf{f}+\mathbf{div}\,\boldsymbol{\sigma}_h\|_{[L^2(\Omega_F)]^2}\|\mathbf{v}\|_{[L^2(\Omega_F)]^2} \\ +\|\frac{1}{2}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_h^T)\|_{[L^2(\Omega_F)]^{2\times2}}\|\boldsymbol{\delta}\|_{[L^2(\Omega_F)]^{2\times2}}\end{array}\right\}}{\|\boldsymbol{\tau}\|_{\mathbf{H}(\mathbf{div};\Omega_F)} + \|\boldsymbol{\psi}\|_{[H^{1/2}(\Gamma)]^2} + \|\mathbf{v}\|_{[L^2(\Omega_F)]^2} + \|\boldsymbol{\delta}\|_{[L^2(\Omega_F)]^{2\times2}}} \\
\leq & \frac{1}{\beta}\left(\|\hat{\boldsymbol{\sigma}}\|_{\mathbf{H}(\mathbf{div};\Omega_F)} + \|\hat{\boldsymbol{\phi}}\|_{[H^{1/2}(\Gamma)]^2} + \|\mathbf{f}+\mathbf{div}\,\boldsymbol{\sigma}_h\|_{[L^2(\Omega_F)]^2} + \|\frac{1}{2}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_h^T)\|_{[L^2(\Omega_F)]^{2\times2}}\right)
\end{aligned}$$

where the suprema have to be taken over all $(\boldsymbol{\tau},\boldsymbol{\psi},\mathbf{v},\boldsymbol{\delta}) \in \mathbf{H}(\mathbf{div};\Omega_F)\times[H^{1/2}(\Gamma)]^2\times[L^2(\Omega_F)]^2\times H_0$. Here we used the Cauchy-Schwarz inequality, the relation $\mathbf{div}\,\boldsymbol{\sigma} = -\mathbf{f}$ and the symmetry of $\boldsymbol{\sigma}$. Making use of the estimate (14) this yields

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{[L^2(\Omega_F)]^{2\times2}} + \|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{[H^{1/2}(\Gamma)]^2} + \|\mathbf{u}-\mathbf{u}_h\|_{[L^2(\Omega_F)]^2} + \|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_{[L^2(\Omega_F)]^{2\times2}}$$

$$\leq \frac{c}{\beta}\left(\sum_{T\in\mathcal{T}_h}\hat{a}_{F,T}(\boldsymbol{\sigma}_T,\boldsymbol{\sigma}_T)+\|\mathbf{W}^{-1}\|R_\Gamma^2\right)^{1/2}$$
$$+\frac{c}{\beta}\|\mathbf{f}+\mathbf{div}\,\boldsymbol{\sigma}_h\|_{[L^2(\Omega_F)]^2}+\frac{c}{\beta}\|\frac{1}{2}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_h^T)\|_{[L^2(\Omega_F)]^{2\times 2}}.$$

□

In this paper we do not consider approximation and convergence of a discrete solution to (7). This has been done by Brink *et al.* [6] where only an a-priori error estimate is given whereas, in this paper, an a-posteriori error estimate is analyzed (cf. Theorem 2).

In order to bound the explicit residual contribution R_Γ to our a-posteriori error estimate by a computable expression we need some more information about the approximate solution $(\boldsymbol{\sigma}_h,\boldsymbol{\phi}_h,\mathbf{u}_h,\boldsymbol{\gamma}_h)$. More precisely, we need an orthogonal property of the residual $\mathbf{W}\boldsymbol{\phi}_h+(\frac{1}{2}\mathbf{I}+\mathbf{K}')(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu})$.

To this end let us briefly introduce finite element and boundary element spaces. For simplicity we take the mesh $\mathcal{T}_h$ of our error estimator as the mesh to define finite element subspaces $\mathbf{H}_0(\mathbf{div};\Omega_F)_h\subset\mathbf{H}_0(\mathbf{div};\Omega_F)$, $[L^2(\Omega_F)]_h^2\subset[L^2(\Omega_F)]^2$, and $H_{0,h}\subset H_0$. For example, these spaces can be defined by the PEERS element of Arnold *et al.* [3] (see also [17]), or by the improved elements of Stenberg [27] as in [6].

Now, for the boundary element subspace we consider a mesh $\Gamma=\cup\{\bar\gamma;\ \gamma\in\mathcal{G}_h\}$ that is independent of $\mathcal{T}_h$ and define

$$[H^{1/2}(\Gamma)]_h^2:=\{\boldsymbol{\psi}\in[C^0(\Gamma)]^2;\ \boldsymbol{\psi}|_\gamma\in[P_1(\gamma)]^2\ \forall\gamma\in\mathcal{G}_h\}\subset[H^{1/2}(\Gamma)]^2.$$

Here, $P_1(\gamma)$ denotes the set of polynomials of degree 1 on γ.

The coupled finite element/boundary element procedure is:

Find $(\boldsymbol{\sigma}_h,\boldsymbol{\phi}_h,\mathbf{u}_h,\boldsymbol{\gamma}_h)\in\mathbf{H}_0(\mathbf{div};\Omega_F)_h\times[H^{1/2}(\Gamma)]_h^2\times[L^2(\Omega_F)]_h^2\times H_{0,h}$ *such that*

$$\begin{array}{lcl} a(\boldsymbol{\sigma}_h,\boldsymbol{\phi}_h;\boldsymbol{\tau},\boldsymbol{\psi})+b(\boldsymbol{\tau};\mathbf{u}_h,\boldsymbol{\gamma}_h) & = & \langle\mathbf{g},\boldsymbol{\tau}\cdot\boldsymbol{\nu}\rangle_{\Gamma_D}\\ b(\boldsymbol{\sigma}_h;\mathbf{v},\boldsymbol{\delta}) & = & -\int_{\Omega_F}\mathbf{f}\cdot\mathbf{v}\,dx \end{array}\tag{15}$$

for all $(\boldsymbol{\tau},\boldsymbol{\psi},\mathbf{v},\boldsymbol{\delta})\in\mathbf{H}_0(\mathbf{div};\Omega_F)_h\times[H^{1/2}(\Gamma)]_h^2\times[L^2(\Omega_F)]_h^2\times H_{0,h}$.

We assume unique solvability of (15) and refer for the analysis to [6]. By the above choice of the boundary element space $[H^{1/2}(\Gamma)]_h^2$ and for the examples of finite element spaces mentioned above we have the properties

a) $\boldsymbol{\phi}_h\in[H^1(\Gamma)]^2$ and

b) $(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu})|_\Gamma\in[L^2(\Gamma)]^2$

and this regularity is assumed in the following.

Corollary 1 *Let the assumptions of Theorem* 2 *and* a), b) *hold. Then there holds the a-posteriori error estimate*

$$\begin{aligned}&\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{H}(\mathbf{div};\Omega_F)}+\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{[H^{1/2}(\Gamma)]^2}+\|\mathbf{u}-\mathbf{u}_h\|_{[L^2(\Omega_F)]^2}+\|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_{[L^2(\Omega_F)]^{2\times 2}}\\ \le\ & C\{[\sum_{T\in\mathcal{T}_h}\hat{a}_{F,T}(\boldsymbol{\sigma}_T,\boldsymbol{\sigma}_T)+c(\mathcal{G}_h)\sum_{\gamma\in\mathcal{G}_h}h_\gamma\|\mathbf{W}\boldsymbol{\phi}_h+(\frac{1}{2}\mathbf{I}+\mathbf{K}')(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu})\|^2_{[L^2(\gamma)]^2}]^{1/2}\\ &+\|\mathbf{f}+\mathbf{div}\,\boldsymbol{\sigma}_h\|_{[L^2(\Omega_F)]^2}+\|\frac{1}{2}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_h^T)\|_{[L^2(\Omega_F)]^{2\times 2}}\}\end{aligned}$$

where C is as in Theorem 2*. Here, h_γ is the length of the boundary element γ and*

$$c(\mathcal{G}_h)=c\log(1+\kappa),\quad \kappa=\max\{\frac{h_\gamma}{h_{\gamma'}};\ \gamma,\gamma'\in\mathcal{G}_h,\ \gamma \textit{ is neighbor of } \gamma'\}$$

with a constant c that is independent of h.

Proof. By Theorem 2 we only have to prove the bound

$$\begin{aligned}R_\Gamma^2 &= \|\mathbf{W}\boldsymbol{\phi}_h+(\frac{1}{2}\mathbf{I}+\mathbf{K}')(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu})\|^2_{[H^{-1/2}(\Gamma)]^2}\\ &\le c(\mathcal{G}_h)\sum_{\gamma\in\mathcal{G}_h}h_\gamma\|\mathbf{W}\boldsymbol{\phi}_h+(\frac{1}{2}\mathbf{I}+\mathbf{K}')(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu})\|^2_{[L^2(\gamma)]^2}.\end{aligned}$$

But this estimate precisely is the assertion of [8, Theorem 2], provided
(i) $\mathbf{W}\boldsymbol{\phi}_h+(\frac{1}{2}\mathbf{I}+\mathbf{K}')(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu})\in[L^2(\Gamma)]^2$ and
(ii) $\mathbf{W}\boldsymbol{\phi}_h+(\frac{1}{2}\mathbf{I}+\mathbf{K}')(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu})$ is $[L^2(\Gamma)]^2$-orthogonal to $[H^{1/2}(\Gamma)]^2_h$.

The condition (i) holds by the mapping properties of the integral operators $\mathbf{W}$ and $\mathbf{K}'$, see [13, 15]. The condition (ii) is satisfied by the definition of the coupled finite element/boundary element problem (15). Indeed, since only the terms $\mathbf{W}\boldsymbol{\phi}_h$ and $(\frac{1}{2}\mathbf{I}+\mathbf{K}')(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu})$ are tested against $\boldsymbol{\psi}\in[H^{1/2}(\Gamma)]^2_h$ (cf. the definition of the bilinear forms below (7)), there holds

$$\langle\boldsymbol{\psi},\mathbf{W}\boldsymbol{\phi}_h\rangle_\Gamma+\langle\boldsymbol{\psi},(\frac{1}{2}\mathbf{I}+\mathbf{K}')(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu})\rangle_\Gamma=0\quad\forall\boldsymbol{\psi}\in[H^{1/2}(\Gamma)]^2_h.$$

This is the required $[L^2(\Gamma)]^2$-orthogonality and the proof of the corollary is finished. □

We remark that the upper bound for the residual term R_Γ given by Corollary 1 is also a lower bound to R_Γ (with a different constant) as long as $\mathcal{G}_h$ is quasi-uniform, see [8, Theorem 3].

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