A DUAL-DUAL FORMULATION FOR THE COUPLING OF MIXED-FEM AND BEM IN HYPERELASTICITY*

GABRIEL N. GATICA† AND NORBERT HEUER‡

Abstract. We combine a dual-mixed finite element method with the boundary integral equation method, to study the solvability and Galerkin approximations of a linear-nonlinear transmission problem arising in plane elastostatics. Our approach is based on the introduction of the strain tensor as an auxiliary unknown in the finite element domain, and it leads to what we call a *dual-dual* mixed formulation since the corresponding operator equation becomes of a two-fold saddle point structure. We derive existence and uniqueness of solution for the continuous and discrete formulations, and provide the associated error analysis. In particular, we extend the finite element space for plane elasticity given by the PEERS element and define an explicit finite element/boundary element subspace satisfying all the required compatibility conditions. Most of our analysis makes use of an extension of the classical Babuška-Brezzi theory to a class of nonlinear saddle-point problems.

Key words. Mixed finite elements, boundary elements, saddle-point problems, hyperelastic material, PEERS

AMS subject classifications. 65N30, 65N38, 35J25, 35J65

1. Introduction. The coupling of the boundary element method (BEM) and the finite element method (FEM) is an efficient solution procedure for linear and nonlinear transmission problems, see [4, 12, 16, 17]. Often, physical quantities like stresses (or fluxes) are more important to determine than displacements (or potentials), which are approximated by standard finite elements. Therefore, for many problems, the coupling of boundary elements and mixed finite elements, where approximations to stresses and fluxes can be determined directly, is more adequate, see [3, 7, 14, 15, 19]. At the continuous level these methods are well understood for both linear and nonlinear problems. However, in the latter case the appearance of certain quotient spaces causes difficulties when one needs to define finite dimensional ansatz spaces. This drawback has motivated the alternative use of primal-mixed methods (see, e.g. [2]) and also the derivation of new tools for studying the operator equations representing the dual-mixed formulations. For example, by simplifying the boundary integral equation method to a Dirichlet-to-Neumann mapping, one can develop a new dual-dual approach for a transmission problem in potential theory, which provides satisfactory results for both the continuous and Galerkin schemes (see [13]). In particular, explicit finite dimensional subspaces of Raviart-Thomas type, satisfying the corresponding compatibility conditions, are given there. Nevertheless, there is no analogue results available neither for the utilization of full BEM instead of Dirichlet-to-Neumann mappings or for other nonlinear transmission problems, such as the one from hyperelasticity originally studied in [15].

The purpose of this paper is, precisely, to extend the dual-dual approach from [13] to the mixed-FEM/BEM formulation of the hyperelastic interface problem in [15]. Our results are mainly based on a slight modification of the extended Babuška-Brezzi theory introduced in [8]. Up to the authors's knowledge, the present work is the first

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†Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, e-mail: ggatica@ing-mat.udec.cl

‡Institut für Wissenschaftliche Datenverarbeitung, Universität Bremen, Postfach 330440, 28334 Bremen, Germany, e-mail: heuer@iwd.uni-bremen.de

one providing complete continuous and discrete analyses for the coupling of mixed-FEM and BEM, as applied to a nonlinear transmission problem. We also remark that the linear systems that appear in the dual-dual formulations are not only indefinite but they also reflect this special structure which gives rise to two-fold saddle point problems. Efficient algorithms for solving this kind of systems, which are based on minimum residual and conjugate gradient methods, can be seen in [9, 10].

The rest of the paper is organized as follows. In Section 2 we introduce our linear-nonlinear transmission problem. The mixed variational formulation is derived in Section 3. Then, in Section 4 we rewrite the variational formulation as an operator equation of dual-dual structure and prove its unique solvability. The Galerkin method is analyzed in Section 5 and a Cea's type error estimate, which includes a consistency term, is provided. Finally, a specific BEM/FEM subspace satisfying the discrete compatibility conditions, which is an extension of the well known PEERS element, is described in Section 6. In particular, it is shown that the usual Cea estimate holds.

2. The boundary value problem. Let $\Omega_0 \subset \mathbf{R}^2$ be a bounded and simply connected domain with Lipschitz continuous boundary Γ. Let Ω be the annular region bounded by Γ and another closed, Lipschitz continuous curve $\Gamma_1 \subset \mathbf{R}^2 \setminus \bar{\Omega}_0$. Further, let Γ_D and Γ_N be two disjoint subsets of Γ_1 such that $\Gamma_1 = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ and $\Gamma_D \neq \emptyset$. Our goal is to determine the displacements and stresses of a material occupying the region $\bar{\Omega}_0 \cup \bar{\Omega}$ which is linear elastic in Ω_0 and hyperelastic in the surrounding annular domain Ω.

In what follows, $\mathbf{R}^{2\times 2}$ is the space of square matrices of order 2 with real entries, $\mathbf{I}_2$ is the identity matrix of $\mathbf{R}^{2\times 2}$, and given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\sigma} := (\sigma_{ij}) \in \mathbf{R}^{2\times 2}$, we use the notations

$$\operatorname{tr}\,(\boldsymbol{\tau}) := \sum_{i=1}^{2} \tau_{ii} \quad \text{and} \quad \boldsymbol{\sigma} : \boldsymbol{\tau} := \sum_{i,j=1}^{2} \sigma_{ij}\,\tau_{ij}\,.$$

Now, we assume that the nonlinear behavior of the hyperelastic material in Ω can be modeled by the Hencky–von Mises stress-strain relation. Therefore, given displacement fields $\mathbf{u}_0$ and $\mathbf{u}$ in Ω_0 and Ω, respectively, strain fields $\mathbf{e}(\mathbf{u}_0)$, $\mathbf{e}(\mathbf{u})$, and stress fields $\boldsymbol{\sigma}_0$, $\boldsymbol{\sigma}$, there hold the following constitutive equations:

$$\begin{aligned} &\boldsymbol{\sigma}_0 = \lambda_0 \operatorname{tr}\mathbf{e}(\mathbf{u}_0)\mathbf{I}_2 + 2\mu_0\mathbf{e}(\mathbf{u}_0) && \text{in } \Omega_0\,, \\ &\boldsymbol{\sigma} = [\kappa(\cdot) - \mu(\cdot,\rho(\mathbf{u}))] \operatorname{tr}\mathbf{e}(\mathbf{u})\mathbf{I}_2 + 2\mu(\cdot,\rho(\mathbf{u}))\mathbf{e}(\mathbf{u}) && \text{in } \Omega\,, \end{aligned} \tag{2.1}$$

and the equilibrium equations

$$\mathbf{div}\,\boldsymbol{\sigma}_0 = 0 \quad \text{in} \quad \Omega_0, \qquad \mathbf{div}\,\boldsymbol{\sigma} = -\mathbf{f} \quad \text{in} \quad \Omega. \tag{2.2}$$

Here, λ_0 and μ_0 are constants such that $\mu_0 > 0$ and $\lambda_0 + \mu_0 > 0$, $\mathbf{f}$ is a given volume force and

$$\mathbf{div}\,\boldsymbol{\tau} := \begin{bmatrix} \operatorname{div}(\tau_{11}\ \tau_{12}) \\ \operatorname{div}(\tau_{21}\ \tau_{22}) \end{bmatrix} \quad \text{for all } \boldsymbol{\tau} = \begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{bmatrix} \in \mathbf{R}^{2\times 2}.$$

Moreover, $\kappa : \ \Omega \to \mathbf{R}$ is the bulk modulus, $\mu : \ \Omega \times \mathbf{R}^+ \to \mathbf{R}$ is a nonlinear Lamé function, and for any displacement $\mathbf{v} := (v_1, v_2)^T$ we define

$$\rho(\mathbf{v}) := \left[\mathbf{e}(\mathbf{v}) - \frac{1}{2}\operatorname{tr}\mathbf{e}(\mathbf{v})\mathbf{I}_2\right] : \left[\mathbf{e}(\mathbf{v}) - \frac{1}{2}\operatorname{tr}\mathbf{e}(\mathbf{v})\mathbf{I}_2\right]\,.$$

We assume that the functions κ and μ are continuous and that $\mu(x,\cdot) \in C^1(\mathbf{R}^+)$ for any $x \in \Omega$. Furthermore, we assume that there exist constants k_0, k_1, μ_1 and μ_2 such that

$$0 < k_0 \leq \kappa(x) \leq k_1, \quad 0 < \mu_1 \leq \mu(x,\rho) \leq \kappa(x) \quad \text{for all } (x,\rho) \in \bar{\Omega} \times \mathbf{R}^+ \tag{2.3}$$

and

$$0 < \mu_1 \leq \mu(x,\rho) + 2\,\rho\,\frac{\partial}{\partial\rho}\,\mu(x,\rho) \leq \mu_2 \quad \text{for all } (x,\rho) \in \Omega \times \mathbf{R}^+. \tag{2.4}$$

According to (2.1) and (2.2), and given $\mathbf{f} \in [L^2(\Omega)]^2$, $\mathbf{g} \in [H^{-1/2}(\Gamma_N)]^2$, our linear-nonlinear transmission problem reads as: *Find* $\mathbf{u}_0 \in [H^1(\Omega_0)]^2$ *and* $\mathbf{u} \in [H^1(\Omega)]^2$ *such that*

$$\mu_0\Delta\mathbf{u}_0 + (\lambda_0+\mu_0)\nabla(\operatorname{div}\mathbf{u}_0) = 0 \quad \text{in } \Omega_0, \tag{2.5}$$
$$\mathbf{u}_0 - \mathbf{u} = 0 \quad \text{on } \Gamma, \qquad \boldsymbol{\sigma}_0\mathbf{n} - \boldsymbol{\sigma}\mathbf{n} = 0 \quad \text{on } \Gamma, \tag{2.6}$$
$$\boldsymbol{\sigma} = [\kappa(\cdot) - \mu(\cdot,\rho(\mathbf{u}))]\ \operatorname{tr}\mathbf{e}(\mathbf{u})\mathbf{I}_2 + 2\mu(\cdot,\rho(\mathbf{u}))\,\mathbf{e}(\mathbf{u}) \quad \text{in } \Omega, \tag{2.7}$$
$$\mathbf{div}\,\boldsymbol{\sigma} = -\mathbf{f} \quad \text{in } \Omega, \tag{2.8}$$
$$\mathbf{u} = 0 \quad \text{on } \Gamma_D, \qquad \boldsymbol{\sigma}\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_N, \tag{2.9}$$

where $\mathbf{n}$ denotes the unit outward normal to $\partial\Omega := \Gamma \cup \Gamma_1$.

3. The weak formulation. We need to define the tensor spaces

$$\mathcal{M}(L^2(\Omega)) := \left\{ \boldsymbol{\tau} = \begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{bmatrix} :\ \tau_{ij} \in L^2(\Omega),\ i,j \in \{1,2\} \right\} \tag{3.1}$$

and

$$\mathbf{H}(\mathbf{div};\Omega) := \left\{ \boldsymbol{\tau} \in \mathcal{M}(L^2(\Omega)) :\ \mathbf{div}\,\boldsymbol{\tau} \in [L^2(\Omega)]^2 \right\} \tag{3.2}$$

with inner products

$$\langle \mathbf{s}, \boldsymbol{\tau} \rangle_{\mathcal{M}(L^2(\Omega))} := \int_\Omega \mathbf{s} : \boldsymbol{\tau}\, dx\,, \quad \langle \mathbf{s}, \boldsymbol{\tau} \rangle_{\mathbf{H}(\mathbf{div};\Omega)} := \int_\Omega \mathbf{s} : \boldsymbol{\tau}\, dx + \int_\Omega \mathbf{div}\,\mathbf{s} \cdot \mathbf{div}\,\boldsymbol{\tau}\, dx,$$

and induced norms

$$\|\boldsymbol{\tau}\|_{\mathcal{M}(L^2(\Omega))} := \left(\sum_{i,j=1}^{2} \|\tau_{ij}\|^2_{L^2(\Omega)} \right)^{1/2},$$

$$\|\boldsymbol{\tau}\|_{\mathbf{H}(\mathbf{div};\Omega)} := \left(\|\boldsymbol{\tau}\|^2_{\mathcal{M}(L^2(\Omega))} + \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{[L^2(\Omega)]^2} \right)^{1/2}.$$

We also use the space of symmetric tensors

$$\mathcal{M}_s(L^2(\Omega)) := \{ \boldsymbol{\tau} \in \mathcal{M}(L^2(\Omega)) :\ \boldsymbol{\tau} = \boldsymbol{\tau}^T \} \tag{3.3}$$

and

$$\mathbf{H}_0(\mathbf{div};\Omega) := \{ \boldsymbol{\tau} \in \mathbf{H}(\mathbf{div};\Omega) :\ \langle 1, \boldsymbol{\tau}\mathbf{n} \rangle_\Gamma \ = \ 0 \}\,.$$

Hereafter, $\langle\cdot,\cdot\rangle_\Gamma$ and $\langle\cdot,\cdot\rangle_{\Gamma_N}$ stand, respectively, for the duality pairings between $[H^{1/2}(\Gamma)]^2$ and $[H^{-1/2}(\Gamma)]^2$, and between $[\tilde{H}^{1/2}(\Gamma_N)]^2$ and $[H^{-1/2}(\Gamma_N)]^2$, where

$$[\tilde{H}^{1/2}(\Gamma_N)]^2 := \{\mathbf{z}|_{\Gamma_N} : \quad \mathbf{z} \in [H^1(\Omega)]^2 \quad \text{and} \quad \mathbf{z} = 0 \text{ on } \Gamma_D\}$$

and $[H^{-1/2}(\Gamma_N)]^2$ is its dual.

Now we follow [15] in deriving a weak formulation of the problem (2.5)-(2.9). However, here we use weaker Sobolev spaces which allow for simpler structured finite element ansatz spaces later on, cf. Section 6.

We first introduce the auxiliary unknown

$$\mathbf{t} := \mathbf{e}(\mathbf{u}) \quad \text{in } \Omega, \tag{3.4}$$

which must belong to $\mathcal{M}_s(L^2(\Omega))$. Hence, (2.7) becomes

$$\boldsymbol{\sigma} = \hat{\lambda}(\mathbf{t})\,\mathrm{tr}\,(\mathbf{t})\mathbf{I}_2 + 2\hat{\mu}(\mathbf{t})\mathbf{t} \quad \text{in } \Omega, \tag{3.5}$$

where $\hat{\lambda}(\mathbf{t}) := \kappa(\cdot) - \hat{\mu}(\mathbf{t})$ and $\hat{\mu}(\mathbf{t}) := \mu(\cdot, \hat{\rho}(\mathbf{t}))$ with

$$\hat{\rho}(\mathbf{t}) := \left[\mathbf{t} - \frac{1}{2}\,\mathrm{tr}\,(\mathbf{t})\mathbf{I}_2\right] : \left[\mathbf{t} - \frac{1}{2}\,\mathrm{tr}\,(\mathbf{t})\mathbf{I}_2\right].$$

According to the definition of $\mathbf{e}(\mathbf{u})$ we can rewrite (3.4) as

$$\mathbf{t} = \nabla\mathbf{u} - \boldsymbol{\gamma}, \tag{3.6}$$

where $\boldsymbol{\gamma} := \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^T)$ represents rotations and lies in the space

$$H_0 := \left\{\boldsymbol{\delta} \in \mathcal{M}(L^2(\Omega)) :\ \boldsymbol{\delta} + \boldsymbol{\delta}^T = 0\right\} = \left\{\begin{bmatrix} 0 & \delta \\ -\delta & 0 \end{bmatrix} :\ \delta \in L^2(\Omega)\right\}. \tag{3.7}$$

Note that $\langle 1, \boldsymbol{\sigma}\mathbf{n}\rangle_\Gamma = \langle 1, \boldsymbol{\sigma}_0\mathbf{n}\rangle_\Gamma$ by (2.6), and thus $\langle 1, \boldsymbol{\sigma}\mathbf{n}\rangle_\Gamma = -\int_{\Omega_0} \mathbf{div}\boldsymbol{\sigma}_0\,dx = 0$ by (2.2), i.e., it suffices to look for $\boldsymbol{\sigma} \in \mathbf{H}_0(\mathbf{div};\Omega)$.

Then we multiply (3.6) by a test function $\boldsymbol{\tau} \in \mathbf{H}_0(\mathbf{div};\Omega)$, integrate by parts on Ω and make use of $\mathbf{u} = 0$ on Γ_D. This gives

$$\int_\Omega \mathbf{t} : \boldsymbol{\tau}\,dx + \int_\Omega \mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau}\,dx + \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau}\,dx + \langle\boldsymbol{\eta}, \boldsymbol{\tau}\mathbf{n}\rangle_{\Gamma_N} - \langle\boldsymbol{\phi}, \boldsymbol{\tau}\mathbf{n}\rangle_\Gamma = 0 \tag{3.8}$$

for all $\boldsymbol{\tau} \in \mathbf{H}_0(\mathbf{div};\Omega)$, where

$$\boldsymbol{\phi} := \mathbf{u}|_\Gamma \in [H^{1/2}(\Gamma)]^2 \quad \text{and} \quad \boldsymbol{\eta} := -\mathbf{u}|_{\Gamma_N} \in [\tilde{H}^{1/2}(\Gamma_N)]^2 \tag{3.9}$$

are introduced as further unknowns.

Note that $\langle 1, \boldsymbol{\tau}\mathbf{n}\rangle_\Gamma = 0$ for any $\boldsymbol{\tau} \in \mathbf{H}_0(\mathbf{div};\Omega)$ and, therefore, it suffices to look for

$$\boldsymbol{\phi} \in [H_0^{1/2}(\Gamma)]^2 := \{\boldsymbol{\psi} \in [H^{1/2}(\Gamma)]^2 :\ \langle\boldsymbol{\psi}, 1\rangle_\Gamma = 0\}.$$

Now, testing (3.5) with a function $\mathbf{s} \in \mathcal{M}_s(L^2(\Omega))$, we obtain

$$\int_\Omega \left[\hat{\lambda}(\mathbf{t})\,\mathrm{tr}\,(\mathbf{t})\,\mathrm{tr}\,(\mathbf{s}) + 2\hat{\mu}(\mathbf{t})\mathbf{t} : \mathbf{s}\right] dx - \int_\Omega \boldsymbol{\sigma} : \mathbf{s}\,dx = 0 \quad \forall \mathbf{s} \in \mathcal{M}_s(L^2(\Omega)), \tag{3.10}$$

and, testing (2.8),

$$\int_\Omega \mathbf{v} \cdot \mathbf{div}\, \boldsymbol{\sigma}\, dx = -\int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx \quad \forall \mathbf{v} \in [L^2(\Omega)]^2. \tag{3.11}$$

The weak formulation of the Neumann condition is obtained by testing the second equation in (2.9) by a function $\boldsymbol{\xi} \in [\tilde{H}^{1/2}(\Gamma_N)]^2$:

$$\langle \boldsymbol{\xi}, \boldsymbol{\sigma}\mathbf{n} \rangle_{\Gamma_N} = \langle \boldsymbol{\xi}, \mathbf{g} \rangle_{\Gamma_N} \quad \forall \boldsymbol{\xi} \in [\tilde{H}^{1/2}(\Gamma_N)]^2. \tag{3.12}$$

The problem of linear elasticity within Ω_0 is dealt with by boundary integral operators on Γ. Denoting by $\mathbf{V}$, $\mathbf{K}$, $\mathbf{K}'$ and $\mathbf{W}$ the integral operators of the single, double, adjoint of the double and hypersingular layer potentials, respectively, and using the well-known jump conditions (cf., e.g., [18]) and the transmission conditions (2.6), we obtain

$$\boldsymbol{\phi} = (\frac{1}{2}\mathbf{I} + \mathbf{K})\boldsymbol{\phi} - \mathbf{V}(\boldsymbol{\sigma}\mathbf{n}) \quad \text{on } \Gamma \tag{3.13}$$

and

$$\mathbf{W}\boldsymbol{\phi} + (\frac{1}{2}\mathbf{I} + \mathbf{K}')(\boldsymbol{\sigma}\mathbf{n}) = 0 \quad \text{on } \Gamma. \tag{3.14}$$

Here, $\mathbf{I}$ is the identity operator on the corresponding spaces.

Eventually, we substitute (3.13) into (3.8), and we test (3.14) with functions $\boldsymbol{\psi} \in [H_0^{1/2}(\Gamma)]^2$.

Then, collecting (3.8), (3.10), (3.11), (3.12) and (3.14), and requiring the symmetry of $\boldsymbol{\sigma}$ weakly by

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\delta}\, dx = 0 \quad \forall \boldsymbol{\delta} \in H_0,$$

we arrive at the following variational formulation of the boundary value problem (2.5)-(2.9): *Find* $(\mathbf{t}, \boldsymbol{\sigma}, \boldsymbol{\phi}, \mathbf{u}, \gamma, \boldsymbol{\eta}) \in \mathcal{M}_s(L^2(\Omega)) \times \mathbf{H}_0(\mathbf{div};\Omega) \times [H_0^{1/2}(\Gamma)]^2 \times [L^2(\Omega)]^2 \times H_0 \times [\tilde{H}^{1/2}(\Gamma_N)]^2$ *such that*

$$\int_\Omega [\hat{\lambda}(\mathbf{t})\ \mathrm{tr}\,(\mathbf{t})\ \mathrm{tr}\,(\mathbf{s}) + 2\hat{\mu}(\mathbf{t})\mathbf{t} : \mathbf{s}]\, dx - \int_\Omega \boldsymbol{\sigma} : \mathbf{s}\, dx = 0, \tag{3.15}$$

$$-\int_\Omega \mathbf{t} : \boldsymbol{\tau}\, dx - \langle \mathbf{V}(\boldsymbol{\sigma}\mathbf{n}), \boldsymbol{\tau}\mathbf{n} \rangle_\Gamma + \langle (\frac{1}{2}\mathbf{I} + \mathbf{K})\boldsymbol{\phi}, \boldsymbol{\tau}\mathbf{n} \rangle_\Gamma$$

$$-\int_\Omega \mathbf{u} \cdot \mathbf{div}\, \boldsymbol{\tau}\, dx - \int_\Omega \gamma : \boldsymbol{\tau}\, dx - \langle \boldsymbol{\eta}, \boldsymbol{\tau}\mathbf{n} \rangle_{\Gamma_N} = 0, \tag{3.16}$$

$$\langle \boldsymbol{\psi}, \mathbf{W}\boldsymbol{\phi} \rangle_\Gamma + \langle \boldsymbol{\psi}, (\frac{1}{2}\mathbf{I} + \mathbf{K}')(\boldsymbol{\sigma}\mathbf{n}) \rangle_\Gamma = 0, \tag{3.17}$$

$$-\int_\Omega \mathbf{v} \cdot \mathbf{div}\, \boldsymbol{\sigma}\, dx = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx, \tag{3.18}$$

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\delta}\, dx = 0, \tag{3.19}$$

$$\langle \boldsymbol{\xi}, \boldsymbol{\sigma}\mathbf{n} \rangle_{\Gamma_N} = \langle \boldsymbol{\xi}, \mathbf{g} \rangle_{\Gamma_N} \tag{3.20}$$

for all $(\mathbf{s}, \boldsymbol{\tau}, \boldsymbol{\psi}, \mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\xi}) \in \mathcal{M}_s(L^2(\Omega)) \times \mathbf{H}_0(\mathbf{div};\Omega) \times [H_0^{1/2}(\Gamma)]^2 \times [L^2(\Omega)]^2 \times H_0 \times [\tilde{H}^{1/2}(\Gamma_N)]^2$.

4. Unique solvability of the problem. As in Section 4 of [15] we rewrite the variational problem (3.15)-(3.20) as an operator equation of an appropriate form that simplifies the analysis. However, different from [15], we do not invert the linear operator of the corresponding formulation but deal with the non-linear contribution. In this way, we do not need the strong ellipticity of the single layer potential operator $\mathbf{V}$, which was the key point of the analysis in [15]. Due to this fact we can deal with the weak formulation (3.15)-(3.20) in relatively simple spaces. More precisely, we deal with the spaces $\mathcal{M}_s(L^2(\Omega))$ and $\mathbf{H}_0(\mathbf{div};\Omega)$ instead of the quotient spaces $\mathcal{M}_s(L^2(\Omega))/\mathcal{K}$ and $\mathbf{H}(\mathbf{div};\Omega)/\mathcal{K}$, where

$$\mathcal{K} = \{\boldsymbol{\tau} \in \mathbf{H}(\mathbf{div};\Omega) : \quad \mathbf{div}\,\boldsymbol{\tau} = 0 \text{ in } \Omega, \quad \boldsymbol{\tau}\mathbf{n} = 0 \text{ on } \Gamma \cup \Gamma_N\} \tag{4.1}$$

as in [15].

Now let us introduce the spaces

$$\mathbf{X} := \mathcal{M}_s(L^2(\Omega)) \times [H_0^{1/2}(\Gamma)]^2,\ \mathbf{Y} := \mathbf{H}_0(\mathbf{div};\Omega),\ \mathbf{Z} := [L^2(\Omega)]^2 \times H_0 \times [\tilde{H}^{1/2}(\Gamma_N)]^2,$$

and the operators

$$\begin{aligned} \mathbf{A}_1 &: \mathbf{X} \to \mathbf{X}', \quad \mathbf{B}_1 : \mathbf{X} \to \mathbf{Y}', \\ \mathbf{B} &: \mathbf{Y} \to \mathbf{Z}', \quad \tilde{\mathbf{V}} : \mathbf{Y} \to \mathbf{Y}' \end{aligned}$$

as follows:

$$\begin{aligned} [\mathbf{A}_1(\mathbf{t},\boldsymbol{\phi}),(\mathbf{s},\boldsymbol{\psi})]_{\mathbf{X}'\times\mathbf{X}} &:= \int_\Omega [\hat{\lambda}(\mathbf{t})\ \mathrm{tr}\,(\mathbf{t})\ \mathrm{tr}\,(\mathbf{s}) + 2\hat{\mu}(\mathbf{t})\mathbf{t} : \mathbf{s}]\,dx + \langle \boldsymbol{\psi}, \mathbf{W}\boldsymbol{\phi}\rangle_\Gamma, \\ [\mathbf{B}_1(\mathbf{t},\boldsymbol{\phi}),\boldsymbol{\tau}]_{\mathbf{Y}'\times\mathbf{Y}} &:= -\int_\Omega \mathbf{t} : \boldsymbol{\tau}\,dx + \langle (\frac{1}{2}\mathbf{I} + \mathbf{K})\boldsymbol{\phi}, \boldsymbol{\tau}\mathbf{n}\rangle_\Gamma, \\ [\mathbf{B}(\boldsymbol{\sigma}),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\xi})]_{\mathbf{Z}'\times\mathbf{Z}} &:= -\int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\sigma}\,dx - \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\delta}\,dx - \langle \boldsymbol{\xi}, \boldsymbol{\sigma}\mathbf{n}\rangle_{\Gamma_N}, \\ [\tilde{\mathbf{V}}(\boldsymbol{\sigma}),\boldsymbol{\tau}]_{\mathbf{Y}'\times\mathbf{Y}} &:= \langle \mathbf{V}(\boldsymbol{\sigma}\mathbf{n}), \boldsymbol{\tau}\mathbf{n}\rangle_\Gamma \end{aligned}$$

where the brackets $[\cdot,\cdot]$ denote the duality pairings indicated by the subscripts. Let us also introduce a generic null operator/functional $\mathbf{O}$, and the functional $\mathbf{F} \in \mathbf{Z}'$ where

$$\mathbf{F}(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\xi}) := \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx - \langle \boldsymbol{\xi}, \mathbf{g}\rangle_{\Gamma_N}.$$

Using these operators and functionals the variational problem (3.15)-(3.20) can be equivalently formulated as: *Find* $((\mathbf{t},\boldsymbol{\phi}),\boldsymbol{\sigma},(\mathbf{u},\gamma,\boldsymbol{\eta})) \in \mathbf{X}\times\mathbf{Y}\times\mathbf{Z}$ *such that*

$$\mathcal{A}\begin{bmatrix} (\mathbf{t},\boldsymbol{\phi}) \\ \boldsymbol{\sigma} \\ (\mathbf{u},\gamma,\boldsymbol{\eta}) \end{bmatrix} = \begin{bmatrix} \mathbf{O} \\ \mathbf{O} \\ \mathbf{F} \end{bmatrix}, \tag{4.2}$$

where

$$\mathcal{A} := \begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1' & \mathbf{O} \\ \mathbf{B}_1 & -\tilde{\mathbf{V}} & \mathbf{B}' \\ \mathbf{O} & \mathbf{B} & \mathbf{O} \end{bmatrix}. \tag{4.3}$$

Note that $\mathbf{A}_1$ is a nonlinear operator and that $\mathbf{B}_1$, $\mathbf{B}$ and $\tilde{\mathbf{V}}$ are linear and bounded. The operators $\mathbf{B}_1' : \mathbf{Y} \to \mathbf{X}'$ and $\mathbf{B}' : \mathbf{Z} \to \mathbf{Y}'$ are the adjoints of $\mathbf{B}_1$ and $\mathbf{B}$, respectively.

At this point we recall that the unique solvability of (4.2) was proved in [15] by using the invertibility of the operator given by

$$\begin{bmatrix} \mathbf{W} & \frac{1}{2}\tilde{\mathbf{I}} + \tilde{\mathbf{K}}' & \mathbf{O} \\ \frac{1}{2}\mathbf{I} + \mathbf{K} & -\tilde{\mathbf{V}} & \mathbf{B}' \\ \mathbf{O} & \mathbf{B} & \mathbf{O} \end{bmatrix}$$

and, in particular, the invertibility of $\tilde{\mathbf{V}}$ on the quotient space $\mathbf{H}(\mathbf{div};\Omega)/\mathcal{K}$. Here, $\tilde{\mathbf{I}}$ and $\tilde{\mathbf{K}}'$ denote, respectively, the operators $\mathbf{I}$ and $\mathbf{K}'$ acting on the traces of the normal components of $\mathbf{H}_0(\mathbf{div};\Omega)$.

In contrast, in this section we prove the unique solvability of (4.2) by using that $\tilde{\mathbf{V}}$ is only positive semi-definite and by inverting the nonlinear operator $\mathbf{A}_1$. In fact, we have the following result.

LEMMA 4.1. *The operator* $\mathbf{A}_1 : \mathbf{X} \to \mathbf{X}'$ *is an isomorphism. More precisely,* $\mathbf{A}_1$ *is Lipschitz continuous and strongly monotone, i.e., there exist positive constants* M_1 *and* μ_1 *such that*

$$\|\mathbf{A}_1(\mathbf{t},\boldsymbol{\phi}) - \mathbf{A}_1(\mathbf{s},\boldsymbol{\psi})\|_{\mathbf{X}'} \leq M_1 \|(\mathbf{t},\boldsymbol{\phi}) - (\mathbf{s},\boldsymbol{\psi})\|_{\mathbf{X}} \quad \forall (\mathbf{t},\boldsymbol{\phi}),(\mathbf{s},\boldsymbol{\psi}) \in \mathbf{X}$$

and

$$[\mathbf{A}_1(\mathbf{t},\boldsymbol{\phi}) - \mathbf{A}_1(\mathbf{s},\boldsymbol{\psi}), (\mathbf{t},\boldsymbol{\phi}) - (\mathbf{s},\boldsymbol{\psi})]_{\mathbf{X}'\times\mathbf{X}} \geq \mu_1 \|(\mathbf{t},\boldsymbol{\phi}) - (\mathbf{s},\boldsymbol{\psi})\|_{\mathbf{X}}^2 \quad \forall (\mathbf{t},\boldsymbol{\phi}),(\mathbf{s},\boldsymbol{\psi}) \in \mathbf{X}.$$

Proof. First we note that $\mathbf{W} : [H^{1/2}(\Gamma)]^2 \to [H^{-1/2}(\Gamma)]^2$ is linear, continuous and positive definite on $[H_0^{1/2}(\Gamma)]^2$, see [5]. Moreover, according to the assumptions given by (2.3) and (2.4), one can prove that the nonlinear part $\mathbf{S}$ of $\mathbf{A}_1$ defined by

$$[\mathbf{S}(\mathbf{t}),\mathbf{s}]_{\mathcal{M}_s(L^2(\Omega))\times\mathcal{M}_s(L^2(\Omega))} = \int_\Omega [\hat{\lambda}(\mathbf{t}) \text{ tr}(\mathbf{t}) \text{ tr}(\mathbf{s}) + 2\hat{\mu}(\mathbf{t})\mathbf{t} : \mathbf{s}]\, dx \quad \forall \mathbf{s} \in \mathcal{M}_s(L^2(\Omega))$$

is strongly monotone and Lipschitz continuous, see [15, Theorem 5.2]. Therefore,

$$\mathbf{A}_1 = \begin{bmatrix} \mathbf{S} & \mathbf{O} \\ \mathbf{O} & \mathbf{W} \end{bmatrix}$$

is also strongly monotone and Lipschitz continuous. Thus, the invertibility of $\mathbf{A}_1$ follows from a classical result on nonlinear functional analysis, see, e.g., [20, Theorem 3.3.23]. □

Now, given a closed subspace $\mathcal{V}$ of $\mathbf{Y}$, we consider a problem of the form: *Find* $((\mathbf{t},\boldsymbol{\phi}),\tilde{\boldsymbol{\sigma}}) \in \mathbf{X} \times \mathcal{V}$ *such that*

$$\mathcal{P}\begin{bmatrix} (\mathbf{t},\boldsymbol{\phi}) \\ \tilde{\boldsymbol{\sigma}} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{bmatrix} \tag{4.4}$$

where $\mathbf{F}_1 \in \mathbf{X}'$, $\mathbf{F}_2 \in \mathcal{V}'$, and $\mathcal{P} : \mathbf{X} \times \mathcal{V} \to \mathbf{X}' \times \mathcal{V}'$ is given by

$$\mathcal{P} := \begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1' \\ \mathbf{B}_1 & -\tilde{\mathbf{V}} \end{bmatrix}$$

The unique solvability of this kind of problems is shown next.

LEMMA 4.2. *Assume that* $\mathbf{B}_1$ *satisfies an inf-sup condition:*

$$\exists \beta_1 > 0 \qquad \sup_{(\mathbf{s},\boldsymbol{\psi})\in\mathbf{X}\setminus\{0\}} \frac{[\mathbf{B}_1(\mathbf{s},\boldsymbol{\psi}),\tilde{\boldsymbol{\tau}}]_{\mathbf{Y}'\times\mathbf{Y}}}{\|(\mathbf{s},\boldsymbol{\psi})\|_{\mathbf{X}}} \geq \beta_1 \|\tilde{\boldsymbol{\tau}}\|_{\mathbf{Y}} \quad \forall \tilde{\boldsymbol{\tau}} \in \mathcal{V}.$$

Then there exists a unique solution $((\mathbf{t},\boldsymbol{\phi}),\tilde{\boldsymbol{\sigma}}) \in \mathbf{X}\times\mathcal{V}$ *of* (4.4).

Proof. Block eliminations show that $\mathcal{P}$ is invertible if and only if the nonlinear operator

$$\mathcal{T} := \tilde{\mathbf{V}} + \mathbf{B}_1\mathbf{A}_1^{-1}\mathbf{B}_1' : \mathcal{V} \to \mathcal{V}'$$

is invertible, see [6, Theorem 2.1]. It therefore suffices to show that $\mathcal{T}$ is strongly monotone and Lipschitz continuous. The integral operator $\mathbf{V} : [H^{-1/2}(\Gamma)]^2 \to [H^{1/2}(\Gamma)]^2$ is continuous and positive semi-definite, see [5]. Therefore, $\tilde{\mathbf{V}}$ is linear, bounded due to the continuity of the trace operator $\mathbf{H}(\mathbf{div};\Omega) \to [H^{-1/2}(\Gamma)]^2$, and positive semi-definite. Thus, we only need to show that $\mathbf{B}_1\mathbf{A}_1^{-1}\mathbf{B}_1'$ is strongly monotone and Lipschitz continuous.

Let M_1 and μ_1 denote the Lipschitz and monotonicity constants, respectively, of $\mathbf{A}_1$ (cf. Lemma 4.1). For arbitrary $\tilde{\boldsymbol{\tau}}_1$, $\tilde{\boldsymbol{\tau}}_2 \in \mathcal{V}$ define $(\mathbf{s},\boldsymbol{\psi})_i' := \mathbf{B}_1'\tilde{\boldsymbol{\tau}}_i \in \mathbf{X}'$ and $(\mathbf{s},\boldsymbol{\psi})_i := \mathbf{A}_1^{-1}(\mathbf{s},\boldsymbol{\psi})_i' \in \mathbf{X}$, $i = 1,2$. Then there holds

$$\begin{aligned}
&[\mathbf{B}_1\mathbf{A}_1^{-1}\mathbf{B}_1'(\tilde{\boldsymbol{\tau}}_1) - \mathbf{B}_1\mathbf{A}_1^{-1}\mathbf{B}_1'(\tilde{\boldsymbol{\tau}}_2), \tilde{\boldsymbol{\tau}}_1 - \tilde{\boldsymbol{\tau}}_2]_{\mathbf{Y}'\times\mathbf{Y}} = \\
&\quad [\mathbf{A}_1^{-1}((\mathbf{s},\boldsymbol{\psi})_1') - \mathbf{A}_1^{-1}((\mathbf{s},\boldsymbol{\psi})_2'), (\mathbf{s},\boldsymbol{\psi})_1' - (\mathbf{s},\boldsymbol{\psi})_2']_{\mathbf{X}\times\mathbf{X}'} \\
&= [(\mathbf{s},\boldsymbol{\psi})_1 - (\mathbf{s},\boldsymbol{\psi})_2, \mathbf{A}_1((\mathbf{s},\boldsymbol{\psi})_1) - \mathbf{A}_1((\mathbf{s},\boldsymbol{\psi})_2)]_{\mathbf{X}\times\mathbf{X}'} \\
&\geq \mu_1 \|(\mathbf{s},\boldsymbol{\psi})_1 - (\mathbf{s},\boldsymbol{\psi})_2\|_{\mathbf{X}}^2 \geq \frac{\mu_1}{M_1^2}\|\mathbf{A}_1((\mathbf{s},\boldsymbol{\psi})_1) - \mathbf{A}_1((\mathbf{s},\boldsymbol{\psi})_2)\|_{\mathbf{X}'}^2 \\
&= \frac{\mu_1}{M_1^2}\|(\mathbf{s},\boldsymbol{\psi})_1' - (\mathbf{s},\boldsymbol{\psi})_2'\|_{\mathbf{X}'}^2 = \frac{\mu_1}{M_1^2}\|\mathbf{B}_1'\tilde{\boldsymbol{\tau}}_1 - \mathbf{B}_1'\tilde{\boldsymbol{\tau}}_2\|_{\mathbf{X}'}^2 \\
&\geq \beta_1^2 \frac{\mu_1}{M_1^2}\|\tilde{\boldsymbol{\tau}}_1 - \tilde{\boldsymbol{\tau}}_2\|_{\mathbf{Y}}^2.
\end{aligned}$$

For the last estimate we used the inf-sup condition for $\mathbf{B}_1$ which means that $\|\mathbf{B}_1'\|_{\mathcal{V}\to\mathbf{X}'} \geq \beta_1$. This proves the strong monotonicity of $\mathbf{B}_1\mathbf{A}_1^{-1}\mathbf{B}_1'$.

Now, we show that the operator $\mathbf{A}_1^{-1} : \mathbf{X}' \to \mathbf{X}$ is Lipschitz continuous. Indeed, let $(\mathbf{s},\boldsymbol{\psi})_1'$, $(\mathbf{s},\boldsymbol{\psi})_2' \in \mathbf{X}'$ be given and define $(\mathbf{s},\boldsymbol{\psi})_i := \mathbf{A}_1^{-1}(\mathbf{s},\boldsymbol{\psi})_i' \in \mathbf{X}$, $i = 1,2$. Then

$$\begin{aligned}
&\|\mathbf{A}_1^{-1}((\mathbf{s},\boldsymbol{\psi})_1') - \mathbf{A}_1^{-1}((\mathbf{s},\boldsymbol{\psi})_2')\|_{\mathbf{X}}^2 = \|(\mathbf{s},\boldsymbol{\psi})_1 - (\mathbf{s},\boldsymbol{\psi})_2\|_{\mathbf{X}}^2 \\
&\leq \frac{1}{\mu_1}[\mathbf{A}_1((\mathbf{s},\boldsymbol{\psi})_1) - \mathbf{A}_1((\mathbf{s},\boldsymbol{\psi})_2), (\mathbf{s},\boldsymbol{\psi})_1 - (\mathbf{s},\boldsymbol{\psi})_2]_{\mathbf{X}'\times\mathbf{X}} \\
&\leq \frac{1}{\mu_1}\|\mathbf{A}_1((\mathbf{s},\boldsymbol{\psi})_1) - \mathbf{A}_1((\mathbf{s},\boldsymbol{\psi})_2)\|_{\mathbf{X}'}\|(\mathbf{s},\boldsymbol{\psi})_1 - (\mathbf{s},\boldsymbol{\psi})_2\|_{\mathbf{X}} \\
&= \frac{1}{\mu_1}\|(\mathbf{s},\boldsymbol{\psi})_1' - (\mathbf{s},\boldsymbol{\psi})_2'\|_{\mathbf{X}'}\|\mathbf{A}_1^{-1}((\mathbf{s},\boldsymbol{\psi})_1') - \mathbf{A}_1^{-1}((\mathbf{s},\boldsymbol{\psi})_2')\|_{\mathbf{X}},
\end{aligned}$$

whence

$$\|\mathbf{A}_1^{-1}((\mathbf{s},\boldsymbol{\psi})_1') - \mathbf{A}_1^{-1}((\mathbf{s},\boldsymbol{\psi})_2')\|_{\mathbf{X}} \leq \frac{1}{\mu_1}\|(\mathbf{s},\boldsymbol{\psi})_1' - (\mathbf{s},\boldsymbol{\psi})_2'\|_{\mathbf{X}'}.$$

Finally, the Lipschitz continuity of $\mathbf{B}_1\mathbf{A}_1^{-1}\mathbf{B}_1'$ follows from the linearity and continuity of $\mathbf{B}_1$ and from the Lipschitz continuity of $\mathbf{A}_1^{-1}$. $\square$

It is important to remark that a problem of the type (4.4) was also considered in [15], see also [6]. However, in those references the key point is the injectivity of $\tilde{\mathbf{V}}$ which is not valid in our situation. Instead, here we need an inf-sup condition for the operator $\mathbf{B}_1$.

We are now in a position to state the main result of this section.

THEOREM 4.3. *There exists a unique solution* $((\mathbf{t},\boldsymbol{\phi}),\boldsymbol{\sigma},(\mathbf{u},\gamma,\boldsymbol{\eta})) \in \mathbf{X}\times\mathbf{Y}\times\mathbf{Z}$ *of* (4.2).

Proof. First, for any $\mathbf{F}\in\mathbf{Z}'$, we define

$$\mathcal{V}(\mathbf{F}) := \{\,\boldsymbol{\tau}\in\mathbf{Y} : \quad \mathbf{B}(\boldsymbol{\tau}) = \mathbf{F}\}\,,$$

and, in particular, we put $\mathcal{V} := \mathcal{V}(\mathbf{O})$.

Then, with (4.2) we associate the following problem: *Find* $((\mathbf{t},\boldsymbol{\phi}),\boldsymbol{\sigma})\in\mathbf{X}\times\mathcal{V}(\mathbf{F})$ *such that*

$$\left[\begin{pmatrix}\mathbf{A}_1 & \mathbf{B}_1' \\ \mathbf{B}_1 & -\tilde{\mathbf{V}}\end{pmatrix}\begin{pmatrix}(\mathbf{t},\boldsymbol{\phi}) \\ \boldsymbol{\sigma}\end{pmatrix},\begin{pmatrix}(\mathbf{s},\boldsymbol{\psi}) \\ \boldsymbol{\tau}\end{pmatrix}\right] = 0 \tag{4.5}$$

for all $((\mathbf{s},\boldsymbol{\psi}),\boldsymbol{\tau})\in\mathbf{X}\times\mathcal{V}$. By [8, Theorem 2.1] we know that there exists a unique solution to (4.2) if and only if (4.5) is uniquely solvable and the operator $\mathbf{B}$ satisfies an inf-sup condition:

$$\exists\beta>0 \qquad \sup_{\boldsymbol{\tau}\in\mathbf{Y}\setminus\{0\}} \frac{[\mathbf{B}(\boldsymbol{\tau}),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\xi})]_{\mathbf{Z}'\times\mathbf{Z}}}{\|\boldsymbol{\tau}\|_{\mathbf{Y}}} \geq \beta\|(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\xi})\|_{\mathbf{Z}} \quad \forall(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\xi})\in\mathbf{Z}. \tag{4.6}$$

In fact, due to [15, Lemma 4.5] (cf. the definition of $\mathbf{B}$ [15, (46)]) there exists $\beta>0$ such that

$$\sup_{\boldsymbol{\tau}\in(\mathbf{H}(\mathbf{div};\Omega)/\mathcal{K})\setminus\{0\}} \frac{[\mathbf{B}(\boldsymbol{\tau}),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\xi})]_{\mathbf{Z}'\times\mathbf{Z}}}{\|\boldsymbol{\tau}\|_{\mathbf{H}(\mathbf{div};\Omega)}} \geq \beta\|(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\xi})\|_{\mathbf{Z}} \quad \forall(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\xi})\in\mathbf{Z}.$$

For the definition of $\mathcal{K}$ see (4.1). However, analyzing the proof of that lemma it actually has been shown the stronger inf-sup condion

$$\sup_{\boldsymbol{\tau}\in(\mathbf{H}_0(\mathbf{div};\Omega)/\mathcal{K})\setminus\{0\}} \frac{[\mathbf{B}(\boldsymbol{\tau}),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\xi})]_{\mathbf{Z}'\times\mathbf{Z}}}{\|\boldsymbol{\tau}\|_{\mathbf{H}(\mathbf{div};\Omega)}} \geq \beta\|(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\xi})\|_{\mathbf{Z}} \quad \forall(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\xi})\in\mathbf{Z}$$

since all $\boldsymbol{\tau}$ used in that proof satisfy the condition $\langle 1,\boldsymbol{\tau}\mathbf{n}\rangle_\Gamma = 0$. Therefore, the inf-sup condition (4.6) is satisfied since $\mathbf{H}_0(\mathbf{div};\Omega)/\mathcal{K}\simeq\mathcal{K}^\perp$ in $\mathbf{H}_0(\mathbf{div};\Omega)$ and $\mathcal{K}^\perp\subset\mathbf{H}_0(\mathbf{div};\Omega)=\mathbf{Y}$. It is left to prove the unique solvability of (4.5).

Since $\mathbf{B}$ satisfies the inf-sup condition the mapping $\mathbf{B}:\ \mathbf{Y}\to\mathbf{Z}'$ is surjective, and hence there exists $\boldsymbol{\sigma}_0\in\mathbf{Y}$ such that $\mathbf{B}(\boldsymbol{\sigma}_0)=\mathbf{F}$. In this way, (4.5) can be rewritten as: Find $((\mathbf{t},\boldsymbol{\phi}),\tilde{\boldsymbol{\sigma}})\in\mathbf{X}\times\mathcal{V}$ such that

$$\left[\begin{pmatrix}\mathbf{A}_1 & \mathbf{B}_1' \\ \mathbf{B}_1 & -\tilde{\mathbf{V}}\end{pmatrix}\begin{pmatrix}(\mathbf{t},\boldsymbol{\phi}) \\ \tilde{\boldsymbol{\sigma}}\end{pmatrix},\begin{pmatrix}(\mathbf{s},\boldsymbol{\psi}) \\ \boldsymbol{\tau}\end{pmatrix}\right] = \left[\begin{pmatrix}-\mathbf{B}_1'(\boldsymbol{\sigma}_0) \\ \tilde{\mathbf{V}}(\boldsymbol{\sigma}_0)\end{pmatrix},\begin{pmatrix}(\mathbf{s},\boldsymbol{\psi}) \\ \boldsymbol{\tau}\end{pmatrix}\right] \tag{4.7}$$

for all $((\mathbf{s},\boldsymbol{\psi}),\boldsymbol{\tau})\in\mathbf{X}\times\mathcal{V}$, which is in the form (4.4).

We now observe that $\mathbf{B}_1$ satisfies the inf-sup condition required by Lemma 4.2. Indeed, let $\boldsymbol{\tau}\in\mathcal{V}$, i.e., in particular $\operatorname{div}\boldsymbol{\tau}=0$ in Ω. Then

$$\sup_{(\mathbf{s},\boldsymbol{\psi})\in\mathbf{X}\setminus\{0\}} \frac{[\mathbf{B}_1(\mathbf{s},\boldsymbol{\psi}),\boldsymbol{\tau}]_{\mathbf{Y}'\times\mathbf{Y}}}{\|(\mathbf{s},\boldsymbol{\psi})\|_{\mathcal{M}_s(L^2(\Omega))\times[H_0^{1/2}(\Gamma)]^2}} \geq \sup_{\mathbf{s}\in\mathcal{M}_s(L^2(\Omega))\setminus\{0\}} \frac{[\mathbf{B}_1(\mathbf{s},0),\boldsymbol{\tau}]_{\mathbf{Y}'\times\mathbf{Y}}}{\|\mathbf{s}\|_{\mathcal{M}_s(L^2(\Omega))}}$$

$$= \sup_{\mathbf{s} \in \mathcal{M}_s(L^2(\Omega)) \setminus \{0\}} \frac{-\int_\Omega \mathbf{s} : \boldsymbol{\tau} \, dx}{\|\mathbf{s}\|_{\mathcal{M}_s(L^2(\Omega))}} = \|\boldsymbol{\tau}\|_{\mathcal{M}_s(L^2(\Omega))} = \|\boldsymbol{\tau}\|_{\mathbf{H}(\mathbf{div};\Omega)}.$$

Therefore, (4.7) is uniquely solvable and $((\mathbf{t}, \boldsymbol{\phi}), \tilde{\boldsymbol{\sigma}} + \boldsymbol{\sigma}_0) \in \mathbf{X} \times \mathcal{V}(\mathbf{F})$ becomes the unique solution of (4.5). This completes the proof of the theorem. □

5. Galerkin approximation. In this section we study the Galerkin method to obtain an approximate solution to the continuous problem (4.2). Here we deal with that problem in an abstract way, as suggested by (4.2). Later on, in Section 6, we will present a specific approximation for the problem (3.15)–(3.20).

5.1. Main estimates. Recall that the operator $\mathcal{A}$ given in (4.3) maps the space $\mathbf{X} \times \mathbf{Y} \times \mathbf{Z}$ onto its dual $\mathbf{X}' \times \mathbf{Y}' \times \mathbf{Z}'$. For its approximate inversion we consider finite dimensional subspaces $\mathbf{X}_h \subset \mathbf{X}$, $\mathbf{Y}_h \subset \mathbf{Y}$, $\mathbf{Z}_h \subset \mathbf{Z}$ where h denotes, as usual, a discretization parameter. The Galerkin method then looks like: *Find* $(\mathbf{t}_h, \boldsymbol{\phi}_h) \in \mathbf{X}_h$, $\boldsymbol{\sigma}_h \in \mathbf{Y}_h$, $(\mathbf{u}_h, \gamma_h, \boldsymbol{\eta}_h) \in \mathbf{Z}_h$ *such that*

$$\left[\mathcal{A} \begin{pmatrix} (\mathbf{t}_h, \boldsymbol{\phi}_h) \\ \boldsymbol{\sigma}_h \\ (\mathbf{u}_h, \gamma_h, \boldsymbol{\eta}_h) \end{pmatrix}, \begin{pmatrix} (\mathbf{s}_h, \boldsymbol{\psi}_h) \\ \boldsymbol{\tau}_h \\ (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\xi}_h) \end{pmatrix} \right] = [\mathbf{F}, (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\xi}_h)] \tag{5.1}$$

for all $(\mathbf{s}_h, \boldsymbol{\psi}_h) \in \mathbf{X}_h$, $\boldsymbol{\tau}_h \in \mathbf{Y}_h$, $(\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\xi}_h) \in \mathbf{Z}_h$. Here and in the following the generic notation $[\cdot, \cdot]$ is used to denote the duality for the corresponding discrete and continuous spaces.

Like (4.2), (5.1) can also be written in operator form. In order to do so we introduce the canonical injections $\mathbf{i}_h : \mathbf{X}_h \to \mathbf{X}$, $\mathbf{j}_h : \mathbf{Y}_h \to \mathbf{Y}$, $\mathbf{k}_h : \mathbf{Z}_h \to \mathbf{Z}$ with adjoints $\mathbf{i}_h'$, $\mathbf{j}_h'$, $\mathbf{k}_h'$. Then we define the operators

$$\mathbf{A}_{1,h} := \mathbf{i}_h' \mathbf{A}_1 \mathbf{i}_h : \mathbf{X}_h \to \mathbf{X}_h', \quad \mathbf{B}_{1,h} := \mathbf{j}_h' \mathbf{B}_1 \mathbf{i}_h : \mathbf{X}_h \to \mathbf{Y}_h',$$
$$\tilde{\mathbf{V}}_h := \mathbf{j}_h' \tilde{\mathbf{V}} \mathbf{j}_h : \mathbf{Y}_h \to \mathbf{Y}_h', \quad \mathbf{B}_h := \mathbf{k}_h' \mathbf{B} \mathbf{j}_h : \mathbf{Y}_h \to \mathbf{Z}_h',$$

and the functional

$$\mathbf{F}_h := \mathbf{F} \mathbf{k}_h \in \mathbf{Z}_h'.$$

Using these operators, (5.1) can be written in the form

$$\mathcal{A}_h \begin{bmatrix} (\mathbf{t}_h, \boldsymbol{\phi}_h) \\ \boldsymbol{\sigma}_h \\ (\mathbf{u}_h, \gamma_h, \boldsymbol{\eta}_h) \end{bmatrix} = \begin{bmatrix} \mathbf{O} \\ \mathbf{O} \\ \mathbf{F}_h \end{bmatrix}, \tag{5.2}$$

where

$$\mathcal{A}_h := \begin{bmatrix} \mathbf{A}_{1,h} & \mathbf{B}_{1,h}' & \mathbf{O} \\ \mathbf{B}_{1,h} & -\tilde{\mathbf{V}}_h & \mathbf{B}_h' \\ \mathbf{O} & \mathbf{B}_h & \mathbf{O} \end{bmatrix}.$$

We also need the following spaces:

$$\mathcal{V}_h(\mathbf{F}_h) := \{\boldsymbol{\tau}_h \in \mathbf{Y}_h : \quad \mathbf{B}_h(\boldsymbol{\tau}_h) = \mathbf{F}_h\} \quad \text{and} \quad \mathcal{V}_h := \mathcal{V}_h(\mathbf{O}).$$

As in the continuous case, $\mathcal{V}_h(\mathbf{F}_h)$ becomes non empty if $\mathbf{B}_h$ satisfies an inf-sup condition. Thus, given $\boldsymbol{\sigma}_{0,h} \in \mathcal{V}_h(\mathbf{F}_h)$, the problem (5.2) is equivalent to: Find $(\mathbf{t}_h, \boldsymbol{\phi}_h) \in \mathbf{X}_h$, $\boldsymbol{\sigma}_h := \tilde{\boldsymbol{\sigma}}_h + \boldsymbol{\sigma}_{0,h} \in \mathcal{V}_h \oplus \mathcal{V}_h(\mathbf{F}_h)$ such that

$$\left[\begin{pmatrix} \mathbf{A}_{1,h} & \mathbf{B}_{1,h}' \\ \mathbf{B}_{1,h} & -\tilde{\mathbf{V}}_h \end{pmatrix} \begin{pmatrix} (\mathbf{t}_h, \boldsymbol{\phi}_h) \\ \tilde{\boldsymbol{\sigma}}_h + \boldsymbol{\sigma}_{0,h} \end{pmatrix}, \begin{pmatrix} (\mathbf{s}_h, \boldsymbol{\psi}_h) \\ \tilde{\boldsymbol{\tau}}_h \end{pmatrix} \right] = 0 \tag{5.3}$$

for all $(\mathbf{s}_h, \boldsymbol{\psi}_h) \in \mathbf{X}_h$, $\tilde{\boldsymbol{\tau}}_h \in \mathcal{V}_h$.

We then obtain the following main result.

THEOREM 5.1. *Let the subspaces $\mathbf{X}_h$, $\mathbf{Y}_h$ and $\mathbf{Z}_h$ be chosen such that $\mathbf{B}_h$ and $\mathbf{B}_{1,h}$ satisfy the following inf-sup conditions:*

$$\exists \beta^* > 0 : \quad \sup_{\boldsymbol{\tau}_h \in \mathbf{Y}_h \setminus \{0\}} \frac{[\mathbf{B}_h(\boldsymbol{\tau}_h), (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\xi}_h)]}{\|\boldsymbol{\tau}_h\|_{\mathbf{Y}}} \geq \beta^* \|(\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\xi}_h)\|_{\mathbf{Z}} \quad \forall\, (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\xi}_h) \in \mathbf{Z}_h,$$

$$\exists \beta_1^* > 0 : \quad \sup_{(\mathbf{s}_h, \boldsymbol{\psi}_h) \in \mathbf{X}_h \setminus \{0\}} \frac{[\mathbf{B}_{1,h}(\mathbf{s}_h, \boldsymbol{\psi}_h), \boldsymbol{\tau}_h]}{\|(\mathbf{s}_h, \boldsymbol{\psi}_h)\|_{\mathbf{X}}} \geq \beta_1^* \|\boldsymbol{\tau}_h\|_{\mathbf{Y}} \quad \forall\, \boldsymbol{\tau}_h \in \mathcal{V}_h\,.$$

Then the discrete problem (5.2) *is uniquely solvable and the following error estimate holds:*

$$\begin{aligned}
&\|(\mathbf{t}, \boldsymbol{\phi}, \boldsymbol{\sigma}, \mathbf{u}, \gamma, \boldsymbol{\eta}) - (\mathbf{t}_h, \boldsymbol{\phi}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h, \gamma_h, \boldsymbol{\eta}_h)\|_{\mathbf{X}\times\mathbf{Y}\times\mathbf{Z}} \\
&\leq C \Bigg(\inf_{(\mathbf{s}_h, \boldsymbol{\psi}_h, \boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\xi}_h) \in \mathbf{X}_h \times \mathbf{Y}_h \times \mathbf{Z}_h} \|(\mathbf{t}, \boldsymbol{\phi}, \boldsymbol{\sigma}, \mathbf{u}, \gamma, \boldsymbol{\eta}) - (\mathbf{s}_h, \boldsymbol{\psi}_h, \boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\xi}_h)\| \\
&\qquad + \sup_{\tilde{\boldsymbol{\tau}}_h \in \mathcal{V}_h \setminus \{0\}} \frac{[\tilde{\mathbf{V}}(\boldsymbol{\sigma}) - \mathbf{B}_1(\mathbf{t}, \boldsymbol{\phi}), \tilde{\boldsymbol{\tau}}_h]}{\|\tilde{\boldsymbol{\tau}}_h\|_{\mathbf{Y}}} \Bigg)
\end{aligned}$$

where the constant C is independent of h and of the solution $(\mathbf{t}, \boldsymbol{\phi}, \boldsymbol{\sigma}, \mathbf{u}, \gamma, \boldsymbol{\eta})$ to (4.2).

For the proof of the error estimate provided by Theorem 5.1 we exploit techniques presented in [15] and [8] and adapt them to our specific problem. The required intermediate results are presented in the following lemmata. ¿From now on we assume that the assumptions of Theorem 5.1 are satisfied. Hence, for the time being we assume that (5.2) is uniquely solvable, which will be shown later in Section 5.2.

The next lemma bounds the error $\|(\mathbf{t}, \boldsymbol{\phi}) - (\mathbf{t}_h, \boldsymbol{\phi}_h)\|_{\mathbf{X}}$ and Lemma 5.3 below is used to further bound the obtained estimate.

LEMMA 5.2. *There holds*

$$\|(\mathbf{t}, \boldsymbol{\phi}) - (\mathbf{t}_h, \boldsymbol{\phi}_h)\|_{\mathbf{X}} \leq (1 + \frac{M_1}{\mu_1}) \inf_{(\mathbf{s}_h, \boldsymbol{\psi}_h) \in \mathcal{V}_{1,h}(\tilde{\mathbf{V}}_h(\boldsymbol{\sigma}_h))} \|(\mathbf{t}, \boldsymbol{\phi}) - (\mathbf{s}_h, \boldsymbol{\psi}_h)\|_{\mathbf{X}}$$

$$+ \quad \frac{\|\mathbf{B}_1\|}{\mu_1} \inf_{\tilde{\boldsymbol{\tau}}_h \in \mathcal{V}_h} \|\boldsymbol{\sigma} - (\tilde{\boldsymbol{\tau}}_h + \boldsymbol{\sigma}_{0,h})\|_{\mathbf{Y}}$$

where

$$\mathcal{V}_{1,h}(\tilde{\mathbf{V}}_h(\boldsymbol{\sigma}_h)) := \{(\mathbf{s}_h, \boldsymbol{\psi}_h) \in \mathbf{X}_h : \quad [\mathbf{B}_{1,h}(\mathbf{s}_h, \boldsymbol{\psi}_h), \tilde{\boldsymbol{\tau}}_h] = [\tilde{\mathbf{V}}_h(\boldsymbol{\sigma}_h), \tilde{\boldsymbol{\tau}}_h] \quad \forall \tilde{\boldsymbol{\tau}}_h \in \mathcal{V}_h\}.$$

Here, M_1 and $\mu_1 > 0$ are the Lipschitz and monotonicity constants, respectively, of $\mathbf{A}_{1,h}$.

Proof. We first observe that the discrete inf-sup condition satisfied by $\mathbf{B}_{1,h}$ guarantees that $\mathcal{V}_{1,h}(\tilde{\mathbf{V}}_h(\boldsymbol{\sigma}_h))$ is not empty. In particular, it follows from (5.3) that $(\mathbf{t}_h, \boldsymbol{\phi}_h) \in \mathcal{V}_{1,h}(\tilde{\mathbf{V}}_h(\boldsymbol{\sigma}_h))$. Now, by the strong monotonicity of $\mathbf{A}_{1,h}$ (which follows from the strong monotonicity of $\mathbf{A}_1$) there holds for any $(\mathbf{s}_h, \boldsymbol{\psi}_h) \in \mathcal{V}_{1,h}(\tilde{\mathbf{V}}_h(\boldsymbol{\sigma}_h))$

$$\mu_1 \|(\mathbf{t}_h, \boldsymbol{\phi}_h) - (\mathbf{s}_h, \boldsymbol{\psi}_h)\|_{\mathbf{X}}^2$$

$$\leq [\mathbf{A}_1(\mathbf{t}_h,\boldsymbol{\phi}_h),(\mathbf{t}_h,\boldsymbol{\phi}_h)-(\mathbf{s}_h,\boldsymbol{\psi}_h)] - [\mathbf{A}_1(\mathbf{s}_h,\boldsymbol{\psi}_h),(\mathbf{t}_h,\boldsymbol{\phi}_h)-(\mathbf{s}_h,\boldsymbol{\psi}_h)]\,.$$

Since $(\mathbf{t}_h,\boldsymbol{\phi}_h)-(\mathbf{s}_h,\boldsymbol{\psi}_h)\in\mathcal{V}_{1,h}:=\mathcal{V}_{1,h}(\mathbf{O})$ and $\tilde{\boldsymbol{\sigma}}_h\in\mathcal{V}_h$, we obtain from (5.3)

$$\begin{aligned}[\mathbf{A}_1(\mathbf{t}_h,\boldsymbol{\phi}_h),(\mathbf{t}_h,\boldsymbol{\phi}_h)-(\mathbf{s}_h,\boldsymbol{\psi}_h)] &= -[\mathbf{B}'_{1,h}(\boldsymbol{\sigma}_{0,h}),(\mathbf{t}_h,\boldsymbol{\phi}_h)-(\mathbf{s}_h,\boldsymbol{\psi}_h)]\\ &= -[\boldsymbol{\sigma}_{0,h},\mathbf{B}_1\left((\mathbf{t}_h,\boldsymbol{\phi}_h)-(\mathbf{s}_h,\boldsymbol{\psi}_h)\right)].\end{aligned}$$

Therefore, by adding $[\mathbf{B}_1\left((\mathbf{t}_h,\boldsymbol{\phi}_h)-(\mathbf{s}_h,\boldsymbol{\psi}_h)\right),\boldsymbol{\sigma}] + [\mathbf{A}_1(\mathbf{t},\boldsymbol{\phi}),(\mathbf{t}_h,\boldsymbol{\phi}_h)-(\mathbf{s}_h,\boldsymbol{\psi}_h)]$ (which vanishes by (4.2) because $(\mathbf{t}_h,\boldsymbol{\phi}_h)-(\mathbf{s}_h,\boldsymbol{\psi}_h)\in\mathbf{X}_h\subseteq\mathbf{X}$), we get

$$\begin{aligned}&\mu_1\|(\mathbf{t}_h,\boldsymbol{\phi}_h)-(\mathbf{s}_h,\boldsymbol{\psi}_h)\|_{\mathbf{X}}^2\leq\\ &\quad -[\mathbf{B}_1\left((\mathbf{t}_h,\boldsymbol{\phi}_h)-(\mathbf{s}_h,\boldsymbol{\psi}_h)\right),\boldsymbol{\sigma}_{0,h}] - [\mathbf{A}_1(\mathbf{s}_h,\boldsymbol{\psi}_h),(\mathbf{t}_h,\boldsymbol{\phi}_h)-(\mathbf{s}_h,\boldsymbol{\psi}_h)]\\ &\quad +[\mathbf{B}_1\left((\mathbf{t}_h,\boldsymbol{\phi}_h)-(\mathbf{s}_h,\boldsymbol{\psi}_h)\right),\boldsymbol{\sigma}] + [\mathbf{A}_1(\mathbf{t},\boldsymbol{\phi}),(\mathbf{t}_h,\boldsymbol{\phi}_h)-(\mathbf{s}_h,\boldsymbol{\psi}_h)]\\ &= [\mathbf{A}_1(\mathbf{t},\boldsymbol{\phi})-\mathbf{A}_1(\mathbf{s}_h,\boldsymbol{\psi}_h),(\mathbf{t}_h,\boldsymbol{\phi}_h)-(\mathbf{s}_h,\boldsymbol{\psi}_h)]\\ &\quad +[\mathbf{B}_1\left((\mathbf{t}_h,\boldsymbol{\phi}_h)-(\mathbf{s}_h,\boldsymbol{\psi}_h)\right),\boldsymbol{\sigma}-(\tilde{\boldsymbol{\tau}}_h+\boldsymbol{\sigma}_{0,h})]\quad\forall\tilde{\boldsymbol{\tau}}_h\in\mathcal{V}_h.\end{aligned}$$

Here we used that for $(\mathbf{t}_h,\boldsymbol{\phi}_h)-(\mathbf{s}_h,\boldsymbol{\psi}_h)\in\mathcal{V}_{1,h}$

$$[\mathbf{B}_1\left((\mathbf{t}_h,\boldsymbol{\phi}_h)-(\mathbf{s}_h,\boldsymbol{\psi}_h)\right),\tilde{\boldsymbol{\tau}}_h]=0\quad\forall\tilde{\boldsymbol{\tau}}_h\in\mathcal{V}_h.$$

By the Lipschitz continuity of $\mathbf{A}_1$ and the boundedness of $\mathbf{B}_1$ we conclude that there holds

$$\|(\mathbf{t}_h,\boldsymbol{\phi}_h)-(\mathbf{s}_h,\boldsymbol{\psi}_h)\|_{\mathbf{X}}\leq\frac{M_1}{\mu_1}\|(\mathbf{t},\boldsymbol{\phi})-(\mathbf{s}_h,\boldsymbol{\psi}_h)\|_{\mathbf{X}}+\frac{\|\mathbf{B}_1\|}{\mu_1}\|\boldsymbol{\sigma}-(\tilde{\boldsymbol{\tau}}_h+\boldsymbol{\sigma}_{0,h})\|_{\mathbf{Y}}.$$

Due to the triangle inequality this gives

$$\begin{aligned}\|(\mathbf{t},\boldsymbol{\phi})-(\mathbf{t}_h,\boldsymbol{\phi}_h)\|_{\mathbf{X}}&\leq\|(\mathbf{t},\boldsymbol{\phi})-(\mathbf{s}_h,\boldsymbol{\psi}_h)\|_{\mathbf{X}}+\|(\mathbf{t}_h,\boldsymbol{\phi}_h)-(\mathbf{s}_h,\boldsymbol{\psi}_h)\|_{\mathbf{X}}\\ &\leq\ (1+\frac{M_1}{\mu_1})\inf_{(\mathbf{s}_h,\boldsymbol{\psi}_h)\in\mathcal{V}_{1,h}(\tilde{\mathbf{V}}_h(\boldsymbol{\sigma}_h))}\|(\mathbf{t},\boldsymbol{\phi})-(\mathbf{s}_h,\boldsymbol{\psi}_h)\|_{\mathbf{X}}\\ &\quad+\frac{\|\mathbf{B}_1\|}{\mu_1}\inf_{\tilde{\boldsymbol{\tau}}_h\in\mathcal{V}_h}\|\boldsymbol{\sigma}-(\tilde{\boldsymbol{\tau}}_h+\boldsymbol{\sigma}_{0,h})\|_{\mathbf{Y}}\end{aligned}$$

which is the assertion. $\square$

Now we bound the first term on the right hand side of the estimate given by Lemma 5.2.

LEMMA 5.3. *There holds*

$$\begin{aligned}&\inf_{(\tilde{\mathbf{s}}_h,\tilde{\boldsymbol{\psi}}_h)\in\mathcal{V}_{1,h}(\tilde{\mathbf{V}}_h(\boldsymbol{\sigma}_h))}\|(\mathbf{t},\boldsymbol{\phi})-(\tilde{\mathbf{s}}_h,\tilde{\boldsymbol{\psi}}_h)\|_{\mathbf{X}}\\ &\leq\left(1+\frac{\|\mathbf{B}_1\|}{\beta_1^*}\right)\inf_{(\mathbf{s}_h,\boldsymbol{\psi}_h)\in\mathbf{X}_h}\|(\mathbf{t},\boldsymbol{\phi})-(\mathbf{s}_h,\boldsymbol{\psi}_h)\|_{\mathbf{X}}\\ &\quad+\frac{1}{\beta_1^*}\sup_{\tilde{\boldsymbol{\tau}}_h\in\mathcal{V}_h\setminus\{0\}}\frac{[\tilde{\mathbf{V}}(\boldsymbol{\sigma})-\mathbf{B}_1(\mathbf{t},\boldsymbol{\phi}),\tilde{\boldsymbol{\tau}}_h]}{\|\tilde{\boldsymbol{\tau}}_h\|_{\mathbf{Y}}}+\frac{\|\tilde{\mathbf{V}}\|}{\beta_1^*}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{Y}}\,.\end{aligned}$$

Proof. Let $(\mathbf{s}_h, \boldsymbol{\psi}_h) \in \mathbf{X}_h$ be given. Since $\mathbf{B}_{1,h}$ satisfies an inf-sup condition on $\mathbf{X}_h \times \mathcal{V}_h$ there exists a unique $(\bar{\mathbf{s}}_h, \bar{\boldsymbol{\psi}}_h) \in \mathcal{V}_{1,h}^{\perp} \cap \mathbf{X}_h$ such that

$$[\mathbf{B}_{1,h}(\bar{\mathbf{s}}_h, \bar{\boldsymbol{\psi}}_h), \tilde{\boldsymbol{\tau}}_h] = [\mathbf{B}_1\left((\mathbf{t}, \boldsymbol{\phi}) - (\mathbf{s}_h, \boldsymbol{\psi}_h)\right), \tilde{\boldsymbol{\tau}}_h] + [\tilde{\mathbf{V}}(\boldsymbol{\sigma}_h) - \mathbf{B}_1(\mathbf{t}, \boldsymbol{\phi}), \tilde{\boldsymbol{\tau}}_h] \quad \forall \tilde{\boldsymbol{\tau}}_h \in \mathcal{V}_h.$$

Then, by the inf-sup condition for $\mathbf{B}_{1,h}$ and by the boundedness of $\mathbf{B}_1$,

$$\|(\bar{\mathbf{s}}_h, \bar{\boldsymbol{\psi}}_h)\|_{\mathbf{X}} \leq \frac{\|\mathbf{B}_1\|}{\beta_1^*}\|(\mathbf{t}, \boldsymbol{\phi}) - (\mathbf{s}_h, \boldsymbol{\psi}_h)\|_{\mathbf{X}} + \frac{1}{\beta_1^*} \sup_{\tilde{\boldsymbol{\tau}}_h \in \mathcal{V}_h \setminus \{0\}} \frac{[\tilde{\mathbf{V}}(\boldsymbol{\sigma}_h) - \mathbf{B}_1(\mathbf{t}, \boldsymbol{\phi}), \tilde{\boldsymbol{\tau}}_h]}{\|\tilde{\boldsymbol{\tau}}_h\|_{\mathbf{Y}}}.$$

Therefore we obtain, by using the fact that $(\bar{\mathbf{s}}_h, \bar{\boldsymbol{\psi}}_h) + (\mathbf{s}_h, \boldsymbol{\psi}_h) \in \mathcal{V}_{1,h}(\tilde{\mathbf{V}}_h(\boldsymbol{\sigma}_h))$ due to the definition of $(\bar{\mathbf{s}}_h, \bar{\boldsymbol{\psi}}_h)$,

$$\begin{aligned}
\inf_{(\tilde{\mathbf{s}}_h, \tilde{\boldsymbol{\psi}}_h) \in \mathcal{V}_{1,h}(\tilde{\mathbf{V}}_h(\boldsymbol{\sigma}_h))} \|(\mathbf{t}, \boldsymbol{\phi}) - (\tilde{\mathbf{s}}_h, \tilde{\boldsymbol{\psi}}_h)\|_{\mathbf{X}} &\leq \|(\mathbf{t}, \boldsymbol{\phi}) - \left((\bar{\mathbf{s}}_h, \bar{\boldsymbol{\psi}}_h) + (\mathbf{s}_h, \boldsymbol{\psi}_h)\right)\|_{\mathbf{X}} \\
&\leq \|(\mathbf{t}, \boldsymbol{\phi}) - (\mathbf{s}_h, \boldsymbol{\psi}_h)\|_{\mathbf{X}} + \|(\bar{\mathbf{s}}_h, \bar{\boldsymbol{\psi}}_h)\|_{\mathbf{X}} \\
&\leq (1 + \frac{\|\mathbf{B}_1\|}{\beta_1^*})\|(\mathbf{t}, \boldsymbol{\phi}) - (\mathbf{s}_h, \boldsymbol{\psi}_h)\|_{\mathbf{X}} + \frac{1}{\beta_1^*} \sup_{\tilde{\boldsymbol{\tau}}_h \in \mathcal{V}_h \setminus \{0\}} \frac{[\tilde{\mathbf{V}}(\boldsymbol{\sigma}_h) - \mathbf{B}_1(\mathbf{t}, \boldsymbol{\phi}), \tilde{\boldsymbol{\tau}}_h]}{\|\tilde{\boldsymbol{\tau}}_h\|_{\mathbf{Y}}}
\end{aligned}$$

for all $(\mathbf{s}_h, \boldsymbol{\psi}_h) \in \mathbf{X}_h$. Making use of the triangle inequality and the continuity of $\tilde{\mathbf{V}}$, we obtain

$$\sup_{\tilde{\boldsymbol{\tau}}_h \in \mathcal{V}_h \setminus \{0\}} \frac{[\tilde{\mathbf{V}}(\boldsymbol{\sigma}_h) - \mathbf{B}_1(\mathbf{t}, \boldsymbol{\phi}), \tilde{\boldsymbol{\tau}}_h]}{\|\tilde{\boldsymbol{\tau}}_h\|_{\mathbf{Y}}} \leq \sup_{\tilde{\boldsymbol{\tau}}_h \in \mathcal{V}_h \setminus \{0\}} \frac{[\tilde{\mathbf{V}}(\boldsymbol{\sigma}) - \mathbf{B}_1(\mathbf{t}, \boldsymbol{\phi}), \tilde{\boldsymbol{\tau}}_h]}{\|\tilde{\boldsymbol{\tau}}_h\|_{\mathbf{Y}}} + \|\tilde{\mathbf{V}}\|\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|$$

and this, together with the previous estimate, finishes the proof. ◻

An upper bound for the error $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|$ is given by the following lemma.

LEMMA 5.4. *There exists a constant C, independent of h, such that there holds*

$$\begin{aligned}
\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{Y}} \leq C \Bigg\{ & \inf_{(\mathbf{s}_h, \boldsymbol{\psi}_h) \in \mathbf{X}_h} \|(\mathbf{t}, \boldsymbol{\phi}) - (\mathbf{s}_h, \boldsymbol{\psi}_h)\|_{\mathbf{X}} \\
& + \inf_{\tilde{\boldsymbol{\tau}}_h \in \mathcal{V}_h} \|\boldsymbol{\sigma} - (\tilde{\boldsymbol{\tau}}_h + \boldsymbol{\sigma}_{0,h})\|_{\mathbf{Y}} + \sup_{\tilde{\boldsymbol{\tau}}_h \in \mathcal{V}_h \setminus \{0\}} \frac{[\tilde{\mathbf{V}}(\boldsymbol{\sigma}) - \mathbf{B}_1(\mathbf{t}, \boldsymbol{\phi}), \tilde{\boldsymbol{\tau}}_h]}{\|\tilde{\boldsymbol{\tau}}_h\|_{\mathbf{Y}}} \Bigg\}.
\end{aligned}$$

Proof. For given $\boldsymbol{\sigma}_{0,h}$, as defined in (5.3), let us define $\boldsymbol{\sigma}_h^* \in \mathcal{V}_h$ by the solution of

$$[\mathcal{T}(\boldsymbol{\sigma}_h^* + \boldsymbol{\sigma}_{0,h}), \tilde{\boldsymbol{\tau}}_h] = 0 \quad \forall \tilde{\boldsymbol{\tau}}_h \in \mathcal{V}_h \tag{5.4}$$

where $\mathcal{T} := \tilde{\mathbf{V}} + \mathbf{B}_1 \mathbf{A}_1^{-1} \mathbf{B}_1'$ is considered now acting from $\mathbf{Y}$ into $\mathbf{Y}'$. Since $\mathbf{A}_1 : \mathbf{X} \to \mathbf{X}'$ is Lipschitz continuous (cf. Lemma 4.2), we deduce that $\mathcal{T}$ is also Lipschitz continuous (with constant M_2). In addition, similarly as in the proof of Lemma 4.2, one can show that $\mathcal{T}(\cdot + \boldsymbol{\sigma}_{0,h})$ is strongly monotone on $\mathcal{V}_h$ with a constant μ_2 that does not depend on h. Therefore, $\boldsymbol{\sigma}_h^*$ is uniquely defined and we claim that there holds

$$\begin{aligned}
\|\boldsymbol{\sigma} - (\boldsymbol{\sigma}_h^* + \boldsymbol{\sigma}_{0,h})\|_{\mathbf{Y}} &\leq (1 + \frac{M_2}{\mu_2}) \inf_{\tilde{\boldsymbol{\tau}}_h \in \mathcal{V}_h} \|\boldsymbol{\sigma} - (\tilde{\boldsymbol{\tau}}_h + \boldsymbol{\sigma}_{0,h})\|_{\mathbf{Y}} \\
&\quad + \frac{1}{\mu_2} \sup_{\hat{\boldsymbol{\tau}}_h \in \mathcal{V}_h \setminus \{0\}} \frac{[\mathcal{T}(\boldsymbol{\sigma}), \hat{\boldsymbol{\tau}}_h]}{\|\hat{\boldsymbol{\tau}}_h\|_{\mathbf{Y}}}.
\end{aligned} \tag{5.5}$$

Note that we are approximating the solution $\boldsymbol{\sigma} = \tilde{\boldsymbol{\sigma}} + \boldsymbol{\sigma}_0 \in \mathcal{V} \oplus \mathcal{V}(\mathbf{F})$ of $\mathcal{T}(\boldsymbol{\sigma}) = 0$ in $\mathcal{V}'$ by the solution $\boldsymbol{\sigma}_h^* + \boldsymbol{\sigma}_{0,h} \in \mathcal{V}_h \oplus \mathcal{V}_h(\mathbf{F}_h)$ of (5.4). That means, in the continuous case we invert the operator $\mathcal{T}(\cdot + \boldsymbol{\sigma}_0) : \mathcal{V} \to \mathcal{V}'$ and in the discrete case we invert $\mathcal{T}(\cdot + \boldsymbol{\sigma}_{0,h}) : \mathcal{V}_h \to \mathcal{V}_h'$ which is an operator depending on h and different from the restriction of $\mathcal{T}(\cdot + \boldsymbol{\sigma}_0)$ onto $\mathcal{V}_h$. This type of situations has been studied in Theorem 6.4 of [11] and here we follow that reference to prove (5.5).

Let $\tilde{\boldsymbol{\tau}}_h \in \mathcal{V}_h$ be arbitrary. By the strong monotonicity of $\mathcal{T}(\cdot + \boldsymbol{\sigma}_{0,h})$ and by the relation (5.4) we obtain

$$\mu_2 \|\tilde{\boldsymbol{\tau}}_h - \boldsymbol{\sigma}_h^*\|_{\mathbf{Y}}^2 \le [\mathcal{T}(\tilde{\boldsymbol{\tau}}_h + \boldsymbol{\sigma}_{0,h}) - \mathcal{T}(\boldsymbol{\sigma}_h^* + \boldsymbol{\sigma}_{0,h}), \tilde{\boldsymbol{\tau}}_h - \boldsymbol{\sigma}_h^*] = [\mathcal{T}(\tilde{\boldsymbol{\tau}}_h + \boldsymbol{\sigma}_{0,h}), \tilde{\boldsymbol{\tau}}_h - \boldsymbol{\sigma}_h^*].$$

Now, adding and subtracting an appropriate term and using the Lipschitz continuity of $\mathcal{T}$, we obtain

$$\begin{aligned}\mu_2 \|\tilde{\boldsymbol{\tau}}_h - \boldsymbol{\sigma}_h^*\|_{\mathbf{Y}}^2 &\le [\mathcal{T}(\tilde{\boldsymbol{\tau}}_h + \boldsymbol{\sigma}_{0,h}) - \mathcal{T}(\boldsymbol{\sigma}), \tilde{\boldsymbol{\tau}}_h - \boldsymbol{\sigma}_h^*] + [\mathcal{T}(\boldsymbol{\sigma}), \tilde{\boldsymbol{\tau}}_h - \boldsymbol{\sigma}_h^*] \\ &\le M_2 \|\boldsymbol{\sigma} - (\tilde{\boldsymbol{\tau}}_h + \boldsymbol{\sigma}_{0,h})\|_{\mathbf{Y}} \|\tilde{\boldsymbol{\tau}}_h - \boldsymbol{\sigma}_h^*\|_{\mathbf{Y}} + [\mathcal{T}(\boldsymbol{\sigma}), \tilde{\boldsymbol{\tau}}_h - \boldsymbol{\sigma}_h^*],\end{aligned}$$

i.e.,

$$\|\tilde{\boldsymbol{\tau}}_h - \boldsymbol{\sigma}_h^*\|_{\mathbf{Y}} \le \frac{M_2}{\mu_2} \|\boldsymbol{\sigma} - (\tilde{\boldsymbol{\tau}}_h + \boldsymbol{\sigma}_{0,h})\|_{\mathbf{Y}} + \frac{1}{\mu_2} \sup_{\hat{\boldsymbol{\tau}}_h \in \mathcal{V}_h \setminus \{0\}} \frac{[\mathcal{T}(\boldsymbol{\sigma}), \hat{\boldsymbol{\tau}}_h]}{\|\hat{\boldsymbol{\tau}}_h\|_{\mathbf{Y}}}.$$

The triangle inequality applied to

$$\|\boldsymbol{\sigma} - (\boldsymbol{\sigma}_h^* + \boldsymbol{\sigma}_{0,h})\|_{\mathbf{Y}} = \|\boldsymbol{\sigma} - (\tilde{\boldsymbol{\tau}}_h + \boldsymbol{\sigma}_{0,h}) + (\tilde{\boldsymbol{\tau}}_h - \boldsymbol{\sigma}_h^*)\|_{\mathbf{Y}}$$

finishes the proof of (5.5).

Now we proceed with the proof of the lemma. Analogously to the continuous case we define the discrete operator $\mathcal{T}_h(\cdot + \boldsymbol{\sigma}_{0,h}) := \tilde{\mathbf{V}}_h + \mathbf{B}_{1,h} \mathbf{A}_{1,h}^{-1} \mathbf{B}_{1,h}' : \mathcal{V}_h \to \mathcal{V}_h'$ which is also Lipschitz continuous and strongly monotone (with constants M_2 and μ_2, respectively). By the strong monotonicity of $\mathcal{T}_h(\cdot + \boldsymbol{\sigma}_{0,h})$ and by using that $[\mathcal{T}_h(\boldsymbol{\sigma}_h), \tilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h^*] = 0$ since $\tilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h^* \in \mathcal{V}_h$ (cf. (5.3)), and using also the relation defining $\boldsymbol{\sigma}_h^*$ (cf. (5.4)), we obtain

$$\begin{aligned}&\mu_2 \|\tilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h^*\|_{\mathbf{Y}}^2 \le [\mathcal{T}_h(\tilde{\boldsymbol{\sigma}}_h + \boldsymbol{\sigma}_{0,h}) - \mathcal{T}_h(\boldsymbol{\sigma}_h^* + \boldsymbol{\sigma}_{0,h}), \tilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h^*] \\ &= -[\mathcal{T}_h(\boldsymbol{\sigma}_h^* + \boldsymbol{\sigma}_{0,h}), \tilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h^*] \\ &= [\mathcal{T}(\boldsymbol{\sigma}_h^* + \boldsymbol{\sigma}_{0,h}) - \mathcal{T}_h(\boldsymbol{\sigma}_h^* + \boldsymbol{\sigma}_{0,h}), \tilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h^*] \\ &= [(\tilde{\mathbf{V}} + \mathbf{B}_1 \mathbf{A}_1^{-1} \mathbf{B}_1')(\boldsymbol{\sigma}_h^* + \boldsymbol{\sigma}_{0,h}) - (\tilde{\mathbf{V}}_h + \mathbf{B}_{1,h} \mathbf{A}_{1,h}^{-1} \mathbf{B}_{1,h}')(\boldsymbol{\sigma}_h^* + \boldsymbol{\sigma}_{0,h}), \tilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h^*] \\ &= [\mathbf{B}_1 \mathbf{A}_1^{-1} \mathbf{B}_1'(\boldsymbol{\sigma}_h^* + \boldsymbol{\sigma}_{0,h}) - \mathbf{B}_{1,h} \mathbf{A}_{1,h}^{-1} \mathbf{B}_{1,h}'(\boldsymbol{\sigma}_h^* + \boldsymbol{\sigma}_{0,h}), \tilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h^*] \\ &= [\mathbf{A}_1^{-1} \mathbf{B}_1'(\boldsymbol{\sigma}_h^* + \boldsymbol{\sigma}_{0,h}) - \mathbf{A}_{1,h}^{-1} \mathbf{B}_{1,h}'(\boldsymbol{\sigma}_h^* + \boldsymbol{\sigma}_{0,h}), \mathbf{B}_1'(\tilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h^*)] \\ &= [(\mathbf{A}_1^{-1} - \mathbf{A}_{1,h}^{-1} \mathbf{i}_h') \mathbf{B}_1'(\boldsymbol{\sigma}_h^* + \boldsymbol{\sigma}_{0,h}), \mathbf{B}_1'(\tilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h^*)] \,.\end{aligned}$$

By adding and subtracting an appropriate term we get

$$\begin{aligned}&\mu_2 \|\tilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h^*\|_{\mathbf{Y}}^2 \le [(\mathbf{A}_1^{-1} - \mathbf{A}_{1,h}^{-1} \mathbf{i}_h') \mathbf{B}_1'(\boldsymbol{\sigma}_h^* + \boldsymbol{\sigma}_{0,h}), \mathbf{B}_1'(\tilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h^*)] \\ &\quad + [(\mathbf{A}_1^{-1} - \mathbf{A}_{1,h}^{-1} \mathbf{i}_h') \mathbf{B}_1' \boldsymbol{\sigma}, \mathbf{B}_1'(\tilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h^*)] - [(\mathbf{A}_1^{-1} - \mathbf{A}_{1,h}^{-1} \mathbf{i}_h') \mathbf{B}_1' \boldsymbol{\sigma}, \mathbf{B}_1'(\tilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h^*)] \\ &= [(\mathbf{A}_1^{-1} - \mathbf{A}_{1,h}^{-1} \mathbf{i}_h') \mathbf{B}_1'(\boldsymbol{\sigma}_h^* + \boldsymbol{\sigma}_{0,h}) - (\mathbf{A}_1^{-1} - \mathbf{A}_{1,h}^{-1} \mathbf{i}_h') \mathbf{B}_1' \boldsymbol{\sigma}, \mathbf{B}_1'(\tilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h^*)] \\ &\quad + [(\mathbf{A}_1^{-1} - \mathbf{A}_{1,h}^{-1} \mathbf{i}_h') \mathbf{B}_1' \boldsymbol{\sigma}, \mathbf{B}_1'(\tilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h^*)].\end{aligned}$$

Now, making use of the Lipschitz continuity of $\mathbf{A}_1^{-1} - \mathbf{A}_{1,h}^{-1}\mathbf{i}_h'$, with a positive constant M, independent of h, and using also the continuity of $\mathbf{B}_1'$, we obtain

$$(5.6)\quad \|\tilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h^*\|_{\mathbf{Y}} \leq M\,\frac{\|\mathbf{B}_1\|^2}{\mu_2}\,\|\boldsymbol{\sigma} - (\boldsymbol{\sigma}_h^* + \boldsymbol{\sigma}_{0,h})\|_{\mathbf{Y}} + \frac{\|\mathbf{B}_1\|}{\mu_2}\|(\mathbf{A}_1^{-1} - \mathbf{A}_{1,h}^{-1}\mathbf{i}_h')\mathbf{B}_1'\boldsymbol{\sigma}\|_{\mathbf{X}}.$$

Since $\mathbf{A}_{1,h}^{-1}\mathbf{i}_h' : \mathbf{X}' \to \mathbf{X}_h$ is just the Galerkin projection of $\mathbf{A}_1^{-1}$ on $\mathbf{X}_h$, the second term on the right hand side can be bounded by the Céa estimate for the strongly monotone and Lipschitz-continuous operator $\mathbf{A}_1$, that is

$$(5.7)\quad \begin{aligned}\|(\mathbf{A}_1^{-1} - \mathbf{A}_{1,h}^{-1}\mathbf{i}_h')\mathbf{B}_1'\boldsymbol{\sigma}\|_{\mathbf{X}} &\leq C \inf_{(\mathbf{s}_h,\boldsymbol{\psi}_h)\in\mathbf{X}_h} \|\mathbf{A}_1^{-1}\mathbf{B}_1'\boldsymbol{\sigma} - (\mathbf{s}_h,\boldsymbol{\psi}_h)\|_{\mathbf{X}}\\ &= C \inf_{(\mathbf{s}_h,\boldsymbol{\psi}_h)\in\mathbf{X}_h} \|(\mathbf{t},\boldsymbol{\phi}) - (\mathbf{s}_h,\boldsymbol{\psi}_h)\|_{\mathbf{X}}\end{aligned}$$

with a constant C, independent of h. Here we used that $\mathbf{B}_1'\boldsymbol{\sigma} = -\mathbf{A}_1(\mathbf{t},\boldsymbol{\phi})$ by (4.2).

The triangle inequality together with (5.6) and (5.7) now gives

$$\begin{aligned}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{Y}} &\leq \|\boldsymbol{\sigma} - (\boldsymbol{\sigma}_h^* + \boldsymbol{\sigma}_{0,h})\|_{\mathbf{Y}} + \|\tilde{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h^*\|_{\mathbf{Y}}\\ &\leq C\left(\|\boldsymbol{\sigma} - (\boldsymbol{\sigma}_h^* + \boldsymbol{\sigma}_{0,h})\|_{\mathbf{Y}} + \inf_{(\mathbf{s}_h,\boldsymbol{\psi}_h)\in\mathbf{X}_h} \|(\mathbf{t},\boldsymbol{\phi}) - (\mathbf{s}_h,\boldsymbol{\psi}_h)\|_{\mathbf{X}}\right).\end{aligned}$$

Finally, noting that $\mathcal{T}(\boldsymbol{\sigma}) = \tilde{\mathbf{V}}(\boldsymbol{\sigma}) - \mathbf{B}_1(\mathbf{t},\boldsymbol{\phi})$ by definition of $\mathcal{T}$ and by (4.2), the error bound (5.5) and the above estimate complete the proof. □

Now, we present a bound for the error $\|(\mathbf{u},\gamma,\boldsymbol{\eta}) - (\mathbf{u}_h,\gamma_h,\boldsymbol{\eta}_h)\|_{\mathbf{Z}}$.

LEMMA 5.5. *There holds*

$$\begin{aligned}&\|(\mathbf{u},\gamma,\boldsymbol{\eta}) - (\mathbf{u}_h,\gamma_h,\boldsymbol{\eta}_h)\|_{\mathbf{Z}}\\ &\leq \frac{1}{\beta^*}\left(M_1 + 2\|\mathbf{B}_1\| + \|\tilde{\mathbf{V}}\|\right)\|((\mathbf{t},\boldsymbol{\phi}),\boldsymbol{\sigma}) - ((\mathbf{t}_h,\boldsymbol{\phi}_h),\boldsymbol{\sigma}_h)\|_{\mathbf{X}\times\mathbf{Y}}\\ &+\frac{1}{\beta^*}(1 + \|\mathbf{B}\|)\inf_{(\mathbf{v}_h,\boldsymbol{\delta}_h,\boldsymbol{\xi}_h)\in\mathbf{Z}_h\setminus\{0\}} \|(\mathbf{u},\gamma,\boldsymbol{\eta}) - (\mathbf{v}_h,\boldsymbol{\delta}_h,\boldsymbol{\xi}_h)\|_{\mathbf{Z}}\end{aligned}$$

where M_1 is the Lipschitz constant of $\mathbf{A}_1$.

Proof. The assertion follows directly from [8, Theorem 4.6] by using the Lipschitz continuity of the operator

$$\mathbf{A} := \begin{bmatrix}\mathbf{A}_1 & \mathbf{B}_1'\\ \mathbf{B}_1 & -\tilde{\mathbf{V}}\end{bmatrix} : \mathbf{X}\times\mathbf{Y} \to \mathbf{X}'\times\mathbf{Y}'.$$

Note that [8] deals with operators of the form $\mathbf{A}$ with $\tilde{\mathbf{V}} = \mathbf{O}$. □

Now we only need to collect the intermediate results to complete the proof of Theorem 5.1.

5.2. Proof of Theorem 5.1. As in the continuous case, (5.2) is uniquely solvable if and only if (5.3) is uniquely solvable and $\mathbf{B}_h$ satifies an inf-sup condition (which holds by assumption). Referring to Lemma 4.2 and its proof, the problem (5.3) is uniquely solvable since $\tilde{\mathbf{V}}_h$ and $\mathbf{B}_{1,h}$ are linear and bounded, since $\mathbf{B}_{1,h}$ satisfies an inf-sup condition, since $\tilde{\mathbf{V}}_h$ is positive semi-definite (which follows from the same property of $\tilde{\mathbf{V}}$), and since $\mathbf{A}_{1,h}$ is Lipschitz continuous and strongly monotone. The latter two properties directly follow from the continuous case, cf. Lemma 4.1.

The error estimate follows by combination of Lemmata 5.2–5.5. More precisely, by Lemmata 5.2, 5.3 and 5.4, we know that there exists a constant C such that

$$\|(\mathbf{t},\boldsymbol{\phi},\boldsymbol{\sigma})-(\mathbf{t}_h,\boldsymbol{\phi}_h,\boldsymbol{\sigma}_h)\|_{\mathbf{X}\times\mathbf{Y}} \le C\left\{\inf_{(\mathbf{s}_h,\boldsymbol{\psi}_h)\in\mathbf{X}_h}\|(\mathbf{t},\boldsymbol{\phi})-(\mathbf{s}_h,\boldsymbol{\psi}_h)\|_{\mathbf{X}}\right.$$

$$\left. +\inf_{\tilde{\boldsymbol{\tau}}_h\in\mathcal{V}_h}\|\boldsymbol{\sigma}-(\tilde{\boldsymbol{\tau}}_h+\boldsymbol{\sigma}_{0,h})\|_{\mathbf{Y}}+\sup_{\tilde{\boldsymbol{\tau}}_h\in\mathcal{V}_h\setminus\{0\}}\frac{[\tilde{\mathbf{V}}(\boldsymbol{\sigma})-\mathbf{B}_1(\mathbf{t},\boldsymbol{\phi}),\tilde{\boldsymbol{\tau}}_h]}{\|\tilde{\boldsymbol{\tau}}_h\|_{\mathbf{Y}}}\right\}. \tag{5.8}$$

Now we proceed as in the proof of Lemma 5.4 to bound the second term on the right hand side of (5.8).

Let $\boldsymbol{\tau}_h\in\mathbf{Y}_h$ be arbitrary but fixed for the moment. By the inf-sup condition for $\mathbf{B}_h$ there exists a unique $\bar{\boldsymbol{\tau}}_h\in\mathcal{V}_h^\perp\cap\mathbf{Y}_h$ such that

$$[\mathbf{B}_h(\bar{\boldsymbol{\tau}}_h),(\mathbf{v}_h,\boldsymbol{\delta}_h,\boldsymbol{\xi}_h)]=[\mathbf{B}(\boldsymbol{\sigma}-\boldsymbol{\tau}_h),(\mathbf{v}_h,\boldsymbol{\delta}_h,\boldsymbol{\xi}_h)]\quad\forall(\mathbf{v}_h,\boldsymbol{\delta}_h,\boldsymbol{\xi}_h)\in\mathbf{Z}_h,$$

and there holds

$$\|\bar{\boldsymbol{\tau}}_h\|_{\mathbf{Y}}\le\frac{\|\mathbf{B}\|}{\beta^*}\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|_{\mathbf{Y}}.$$

Then $\mathbf{B}_h(\bar{\boldsymbol{\tau}}_h+\boldsymbol{\tau}_h)=\mathbf{B}(\boldsymbol{\sigma})=\mathbf{F}_h$ in $\mathbf{Z}_h'$ by definition of $\bar{\boldsymbol{\tau}}_h$ and by (5.2), i.e., $\bar{\boldsymbol{\tau}}_h+\boldsymbol{\tau}_h\in\mathcal{V}_h(\mathbf{F})$. Therefore, we conclude that there holds

$$\begin{aligned}\inf_{\tilde{\boldsymbol{\tau}}_h\in\mathcal{V}_h}\|\boldsymbol{\sigma}-(\tilde{\boldsymbol{\tau}}_h+\boldsymbol{\sigma}_{0,h})\|_{\mathbf{Y}} &= \inf_{\tilde{\boldsymbol{\tau}}_h\in\mathcal{V}_h(\mathbf{F})}\|\boldsymbol{\sigma}-\tilde{\boldsymbol{\tau}}_h\|_{\mathbf{Y}}\le\|\boldsymbol{\sigma}-(\bar{\boldsymbol{\tau}}_h+\boldsymbol{\tau}_h)\|_{\mathbf{Y}}\\ &\le\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|_{\mathbf{Y}}+\|\bar{\boldsymbol{\tau}}_h\|_{\mathbf{Y}}\le(1+\frac{\|\mathbf{B}\|}{\beta^*})\,\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|_{\mathbf{Y}}\end{aligned}$$

for any $\boldsymbol{\tau}_h\in\mathbf{Y}_h$, and hence

$$\inf_{\tilde{\boldsymbol{\tau}}_h\in\mathcal{V}_h}\|\boldsymbol{\sigma}-(\tilde{\boldsymbol{\tau}}_h+\boldsymbol{\sigma}_{0,h})\|_{\mathbf{Y}}\le(1+\frac{\|\mathbf{B}\|}{\beta^*})\inf_{\boldsymbol{\tau}_h\in\mathbf{Y}_h}\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|_{\mathbf{Y}}.$$

This estimate together with (5.8) and Lemma 5.5 finally proves the error estimate of Theorem 5.1.

6. Finite element spaces. Now we present finite element spaces $\mathbf{X}_h$, $\mathbf{Y}_h$, $\mathbf{Z}_h$ that satisfy the required properties of Section 5, i.e., they lead to a convergent coupled mixed finite element and boundary element scheme. For simplicity we only consider the boundary value problem (2.5)–(2.9) with pure Dirichlet boundary condition, i.e., $\Gamma_N=\emptyset$. Then we can use a finite element/boundary element space that is an extension of the finite element space for plane elasticity given by the PEERS element of Arnold, Brezzi and Douglas [1]. This extended space is described in the following.

Let $\mathcal{T}_h$ be a quasi-uniform regular triangulation of Ω by triangles T of diameter h_T with mesh size $h:=\sup_{T\in\mathcal{T}_h}h_T$. By $\mathbf{P}_l$ we denote the space of polynomials in two variables of total degree at most l and we let

$$RT:=\operatorname{span}\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}x_1\\x_2\end{pmatrix}\right\}\subset\mathbf{P}_1\times\mathbf{P}_1\,.$$

We define

$$
\begin{aligned}
M^l &:= \{v \in L^2(\Omega) : \quad v|_T \in \mathbf{P}_l \ \forall T \in \mathcal{T}_h\},\\
\mathcal{M}^l &:= \{\mathbf{v} = (v_1, v_2) : \quad v_i \in M^l,\ i = 1,2\},\\
\mathcal{RT} &:= \{\tau = (\tau_1, \tau_2) \in [L^2(\Omega)]^2 : \quad \tau|_T \in RT \ \forall T \in \mathcal{T}_h\},\\
\mathcal{RT}^2 &:= \left\{ \boldsymbol{\tau} = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} : \quad (\tau_{i1}, \tau_{i2}) \in \mathcal{RT},\ i = 1,2 \right\},\\
M_0^l &:= M^l \cap H^1(\Omega),\\
\mathcal{RT}_0 &:= \mathcal{RT} \cap \mathbf{H}(\mathbf{div};\Omega),\\
\mathcal{RT}_0^2 &:= \left\{ \boldsymbol{\tau} = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} : \quad (\tau_{i1}, \tau_{i2}) \in \mathcal{RT}_0,\ i = 1,2 \right\}.
\end{aligned}
$$

The space $\mathcal{RT}_0$ is the $H(\mathrm{div};\Omega)$-part of the well-known lowest order Raviart-Thomas space. In order to define the space of PEERS elements we need to enrich the space $\mathcal{RT}_0^2$ by rotations of cubic triangular bubble functions. For $T \in \mathcal{T}_h$ we take the unique cubic polynomial b_T on T that vanishes on ∂T and is normalized by $\int_T b_T\,dx = 1$. This polynomial is considered to be extended by 0 onto $\Omega \setminus \bar{T}$ and therefore $b_T \in M_0^3$. Let us define

$$
\begin{aligned}
\mathcal{B} &:= \operatorname{span}\{\operatorname{rot} b_T,\ T \in \mathcal{T}_h\},\\
\mathcal{B}^2 &:= \left\{ \boldsymbol{\tau} = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} : \quad (\tau_{i1}, \tau_{i2}) \in \mathcal{B},\ i = 1,2 \right\},
\end{aligned}
$$

where $\operatorname{rot} v := (\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1})$.

The PEERS space given by Arnold, Brezzi and Douglas in [1] for the approximation of the unknowns $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma})$ is

$$
\left(\mathcal{RT}_0^2 + \mathcal{B}^2\right) \times \mathcal{M}^0 \times M_0^1\, \boldsymbol{\chi} \ \subset\ \mathbf{H}(\mathbf{div};\Omega) \times [L^2(\Omega)]^2 \times H^1(\Omega)\, \boldsymbol{\chi}
$$

where

$$
\boldsymbol{\chi} := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
$$

Note that $H^1(\Omega)\,\boldsymbol{\chi} \subset H_0$ (cf. (3.7)). In order to define a finite element space for an approximation of the full solution $(\mathbf{t}, \boldsymbol{\phi}, \boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma})$ to the problem (3.15)–(3.20) (note that the unknown $\boldsymbol{\eta}$ does not appear for $\Gamma_N = \emptyset$) we need to ensure $\int_\Gamma \boldsymbol{\sigma}\mathbf{n}\,ds = 0$ and we need approximation spaces for $\mathbf{t}$ and $\boldsymbol{\phi}$. For $\mathbf{t}$ we simply take the part of the PEERS element that approximates $\boldsymbol{\sigma}$ without requiring $\mathbf{H}(\mathbf{div};\Omega)$-conformity and both components of $\boldsymbol{\phi}$ will be approximated by continuous piecewise polynomials of degree 1 on the mesh $\mathcal{T}_h|_\Gamma$. Therefore, using the notation of Section 5, we define the ansatz spaces

$$
\mathbf{X}_h := \left[\, (\mathcal{RT}^2 + \mathcal{B}^2) \cap \mathcal{M}_s(L^2(\Omega)) \,\right] \times \left[M_0^1|_\Gamma \cap \{\varphi : \quad \int_\Gamma \varphi\, ds = 0\} \right]^2
$$

$$
\subset \mathcal{M}_s(L^2(\Omega)) \times [H_0^{1/2}(\Gamma)]^2 = \mathbf{X}
$$

for the approximation of $(\mathbf{t}, \boldsymbol{\phi})$,

$$
\mathbf{Y}_h := (\mathcal{RT}_0^2 + \mathcal{B}^2) \cap \{\boldsymbol{\tau} : \quad \int_\Gamma \boldsymbol{\tau}\mathbf{n}\, ds = 0\} \subset \mathbf{H}_0(\mathbf{div};\Omega) = \mathbf{Y}
$$

for the approximation of $\boldsymbol{\sigma}$ and

$$\mathbf{Z}_h := \mathcal{M}^0 \times M_0^1\, \boldsymbol{\chi} \subset [L^2(\Omega)]^2 \times H_0 = \mathbf{Z}$$

for the approximation of $(\mathbf{u}, \boldsymbol{\gamma})$.

Then the Galerkin procedure for approximately solving our model problem (2.5)–(2.9) with $\Gamma_N = \emptyset$ is: *Find* $(\mathbf{t}_h, \boldsymbol{\phi}_h) \in \mathbf{X}_h$, $\boldsymbol{\sigma}_h \in \mathbf{Y}_h$, $(\mathbf{u}_h, \boldsymbol{\gamma}_h) \in \mathbf{Z}_h$ *such that*

$$\left[\begin{pmatrix} \mathbf{A}_1 & \mathbf{B}_1' & \mathbf{O} \\ \mathbf{B}_1 & -\tilde{\mathbf{V}} & \mathbf{B}' \\ \mathbf{O} & \mathbf{B} & \mathbf{O} \end{pmatrix} \begin{pmatrix} (\mathbf{t}_h, \boldsymbol{\phi}_h) \\ \boldsymbol{\sigma}_h \\ (\mathbf{u}_h, \boldsymbol{\gamma}_h) \end{pmatrix}, \begin{pmatrix} (\mathbf{s}_h, \boldsymbol{\psi}_h) \\ \boldsymbol{\tau}_h \\ (\mathbf{v}_h, \boldsymbol{\delta}_h) \end{pmatrix} \right] = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \, dx \tag{6.1}$$

for all $(\mathbf{s}_h, \boldsymbol{\psi}_h) \in \mathbf{X}_h$, $\boldsymbol{\tau}_h \in \mathbf{Y}_h$, $(\mathbf{v}_h, \boldsymbol{\delta}_h) \in \mathbf{Z}_h$. Here, $\mathbf{A}_1$, $\mathbf{B}_1$ and $\tilde{\mathbf{V}}$ are defined as in Section 4 and $\mathbf{B}$ has to be simplified to

$$[\mathbf{B}(\boldsymbol{\sigma}), (\mathbf{v}, \boldsymbol{\delta})] := -\int_\Omega \mathbf{v} \cdot \mathbf{div}\, \boldsymbol{\sigma} \, dx - \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\delta} \, dx \,.$$

We obtain the following quasi-optimal error estimate.

THEOREM 6.1. *The discrete problem* (6.1) *is uniquely solvable and the following error estimate holds:*

$$\begin{aligned} &\|(\mathbf{t}, \boldsymbol{\phi}, \boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}) - (\mathbf{t}_h, \boldsymbol{\phi}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h)\|_{\mathbf{X}\times\mathbf{Y}\times\mathbf{Z}} \\ &\le C \inf_{(\mathbf{s}_h, \boldsymbol{\psi}_h, \boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\delta}_h) \in \mathbf{X}_h \times \mathbf{Y}_h \times \mathbf{Z}_h} \|(\mathbf{t}, \boldsymbol{\phi}, \boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}) - (\mathbf{s}_h, \boldsymbol{\psi}_h, \boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\delta}_h)\|_{\mathbf{X}\times\mathbf{Y}\times\mathbf{Z}} \end{aligned}$$

where the constant C *is independent of* h.

Proof. We apply Theorem 5.1 and therefore have to prove the inf-sup conditions for $\mathbf{B}_h$ and $\mathbf{B}_{1,h}$. In a second step we then bound the consistency term appearing in the error estimate of Theorem 5.1.

First let us check the inf-sup condition for $\mathbf{B}_h$. ¿From [1, Lemma 4.4] we know that, for given $(\mathbf{v}_h, \boldsymbol{\delta}_h) \in \mathbf{Z}_h$, there exists $\boldsymbol{\tau}_h \in (\mathcal{RT}_0^2 + \mathcal{B}^2) \setminus \{0\}$ such that

$$[\mathbf{B}(\boldsymbol{\tau}_h), (\mathbf{v}_h, \boldsymbol{\delta}_h)] \ge c\, \|\boldsymbol{\tau}_h\|_{\mathbf{Y}}\ \|(\mathbf{v}_h, \boldsymbol{\delta}_h)\|_{\mathbf{Z}}.$$

Here, c is a positive constant that does not depend on h and $(\mathbf{v}_h, \boldsymbol{\delta}_h)$. Moreover, taking a closer look at the proof of that lemma in [1], it appears that also the condition $\int_\Gamma \boldsymbol{\tau}_h \cdot \mathbf{n}\, ds = 0$ can be satisfied, i.e., $\boldsymbol{\tau}_h \in \mathbf{Y}_h$. This proves the inf-sup condition for $\mathbf{B}_h$:

$$\sup_{\boldsymbol{\tau}_h \in \mathbf{Y}_h \setminus \{0\}} \frac{[\mathbf{B}(\boldsymbol{\tau}_h), (\mathbf{v}_h, \boldsymbol{\delta}_h)]}{\|\boldsymbol{\tau}_h\|_{\mathbf{Y}}} \ge c\, \|(\mathbf{v}_h, \boldsymbol{\delta}_h)\|_{\mathbf{Z}} \quad \forall (\mathbf{v}_h, \boldsymbol{\delta}_h) \in \mathbf{Z}_h.$$

To show the inf-sup condition for $\mathbf{B}_{1,h}$ on $\mathbf{X}_h \times \mathcal{V}_h$ we first need some information about the space $\mathcal{V}_h$ which is the discrete kernel of $\mathbf{B}_h$:

$$\mathcal{V}_h = \{\boldsymbol{\tau}_h \in \mathbf{Y}_h : \quad [\mathbf{B}(\boldsymbol{\tau}_h), (\mathbf{v}_h, \boldsymbol{\delta}_h)] = 0 \quad \forall (\mathbf{v}_h, \boldsymbol{\delta}_h) \in \mathbf{Z}_h\}.$$

Obviously, $\mathcal{V}_h$ is included in

$$\mathcal{V}_{h,0} := \{\boldsymbol{\tau}_h \in \mathbf{Y}_h : \quad [\mathbf{B}(\boldsymbol{\tau}_h), (\mathbf{v}_h, 0)] = 0 \quad \forall \mathbf{v}_h \in \mathcal{M}^0\}\,,$$

and hence, it suffices to prove an inf-sup condition for $\mathbf{B}_{1,h}$ on $\mathbf{X}_h \times \mathcal{V}_{h,0}$. Since $\mathbf{div}\,\mathcal{B}^2 = 0$ by construction it follows that $\mathbf{div}\,\boldsymbol{\tau}_h$ is constant on the elements for $\boldsymbol{\tau}_h \in (\mathcal{RT}_0^2 + \mathcal{B}^2)$. Therefore, since $\mathbf{v}_h \in \mathcal{M}^0$ is piecewise constant as well, there holds

$$\mathcal{V}_{h,0} = \{\boldsymbol{\tau}_h \in \mathbf{Y}_h : \quad \mathbf{div}\,\boldsymbol{\tau}_h = 0\}.$$

Now the inf-sup condition for $\mathbf{B}_{1,h}$ follows easily by using the relation $\mathcal{V}_{h,0} \subset (\mathcal{RT}^2 + \mathcal{B}^2)$ (cf. also [9, 13]):

$$\begin{aligned}\sup_{(\mathbf{s}_h,\boldsymbol{\psi}_h)\in\mathbf{X}_h\setminus\{0\}} \frac{[\mathbf{B}_1(\mathbf{s}_h,\boldsymbol{\psi}_h),\boldsymbol{\tau}_h]}{\|(\mathbf{s}_h,\boldsymbol{\psi}_h)\|_{\mathbf{X}}} &\geq \sup_{\mathbf{s}_h\in(\mathcal{RT}^2+\mathcal{B}^2)\setminus\{0\}} \frac{-\int_\Omega \mathbf{s}_h : \boldsymbol{\tau}_h\,dx}{\|\mathbf{s}_h\|_{[L^2(\Omega)]^2}} \\ &= \|\boldsymbol{\tau}_h\|_{[L^2(\Omega)]^2} = \|\boldsymbol{\tau}_h\|_{\mathbf{H}(\mathbf{div};\Omega)} \quad \forall\, \boldsymbol{\tau}_h \in \mathcal{V}_{h,0} \supset \mathcal{V}_h.\end{aligned}$$

In this way, a direct application of Theorem 5.1 gives

$$\begin{aligned}&\|(\mathbf{t},\boldsymbol{\phi},\boldsymbol{\sigma},\mathbf{u},\gamma) - (\mathbf{t}_h,\boldsymbol{\phi}_h,\boldsymbol{\sigma}_h,\mathbf{u}_h,\gamma_h)\|_{\mathbf{X}\times\mathbf{Y}\times\mathbf{Z}} \\ &\leq C\left(\inf_{(\mathbf{s}_h,\boldsymbol{\psi}_h,\boldsymbol{\tau}_h,\mathbf{v}_h,\boldsymbol{\delta}_h)\in\mathbf{X}_h\times\mathbf{Y}_h\times\mathbf{Z}_h} \|(\mathbf{t},\boldsymbol{\phi},\boldsymbol{\sigma},\mathbf{u},\gamma) - (\mathbf{s}_h,\boldsymbol{\psi}_h,\boldsymbol{\tau}_h,\mathbf{v}_h,\boldsymbol{\delta}_h)\|_{\mathbf{X}\times\mathbf{Y}\times\mathbf{Z}}\right. \\ &\qquad \left. + \sup_{\tilde{\boldsymbol{\tau}}_h\in\mathcal{V}_h\setminus\{0\}} \frac{[\tilde{\mathbf{V}}(\boldsymbol{\sigma}) - \mathbf{B}_1(\mathbf{t},\boldsymbol{\phi}),\tilde{\boldsymbol{\tau}}_h]}{\|\tilde{\boldsymbol{\tau}}_h\|_{\mathbf{Y}}}\right).\end{aligned}$$

It remains to estimate the consistency term. We first observe from (4.2) that $\tilde{\mathbf{V}}(\boldsymbol{\sigma}) - \mathbf{B}_1(\mathbf{t},\boldsymbol{\phi}) = \mathbf{B}'(\mathbf{u},\gamma)$. Then, given $\tilde{\boldsymbol{\tau}}_h \in \mathcal{V}_h$, which satisfies in particular that $\mathbf{div}\,\tilde{\boldsymbol{\tau}}_h = 0$, we get for any $\boldsymbol{\delta}_h \in \mathcal{M}_0^1\,\boldsymbol{\chi}$

$$\begin{aligned}[\tilde{\mathbf{V}}(\boldsymbol{\sigma}) - \mathbf{B}_1(\mathbf{t},\boldsymbol{\phi}),\tilde{\boldsymbol{\tau}}_h] = [\mathbf{B}'(\mathbf{u},\gamma),\tilde{\boldsymbol{\tau}}_h] &= -\int_\Omega \mathbf{u}\cdot\mathbf{div}\tilde{\boldsymbol{\tau}}_h\,dx - \int_\Omega \tilde{\boldsymbol{\tau}}_h : \gamma\,dx \\ &= \int_\Omega \tilde{\boldsymbol{\tau}}_h : (\boldsymbol{\delta}_h - \gamma)\,dx - \int_\Omega \tilde{\boldsymbol{\tau}}_h : \boldsymbol{\delta}_h\,dx = \int_\Omega \tilde{\boldsymbol{\tau}}_h : (\boldsymbol{\delta}_h - \gamma)\,dx.\end{aligned}$$

Here we used the orthogonality

$$\int_\Omega \tilde{\boldsymbol{\tau}}_h : \boldsymbol{\delta}_h\,dx = 0 \quad \forall\tilde{\boldsymbol{\tau}}_h \in \mathcal{V}_h\ \forall\boldsymbol{\delta}_h \in \mathcal{M}_0^1\,\boldsymbol{\chi}.$$

Therefore, since $\|\tilde{\boldsymbol{\tau}}_h\|_{\mathbf{H}(\mathbf{div};\Omega)} = \|\tilde{\boldsymbol{\tau}}_h\|_{[L^2(\Omega)]^2}$ for $\tilde{\boldsymbol{\tau}}_h \in \mathcal{V}_h$,

$$\begin{aligned}\sup_{\tilde{\boldsymbol{\tau}}_h\in\mathcal{V}_h\setminus\{0\}} \frac{[\tilde{\mathbf{V}}(\boldsymbol{\sigma}) - \mathbf{B}_1(\mathbf{t},\boldsymbol{\phi}),\tilde{\boldsymbol{\tau}}_h]}{\|\tilde{\boldsymbol{\tau}}_h\|_{\mathbf{Y}}} &= \sup_{\tilde{\boldsymbol{\tau}}_h\in\mathcal{V}_h\setminus\{0\}} \frac{\int_\Omega \tilde{\boldsymbol{\tau}}_h : (\boldsymbol{\delta}_h - \boldsymbol{\gamma})\,dx}{\|\tilde{\boldsymbol{\tau}}_h\|_{\mathbf{Y}}} \\ &\leq \|\boldsymbol{\gamma} - \boldsymbol{\delta}_h\|_{[L^2(\Omega)]^2} \sup_{\tilde{\boldsymbol{\tau}}_h\in\mathcal{V}_h\setminus\{0\}} \frac{\|\tilde{\boldsymbol{\tau}}_h\|_{[L^2(\Omega)]^2}}{\|\tilde{\boldsymbol{\tau}}_h\|_{\mathbf{H}(\mathbf{div};\Omega)}} \\ &= \|\boldsymbol{\gamma} - \boldsymbol{\delta}_h\|_{[L^2(\Omega)]^2} \quad \forall\boldsymbol{\delta}_h \in \mathcal{M}_0^1\,\boldsymbol{\chi},\end{aligned}$$

that is

$$\sup_{\tilde{\boldsymbol{\tau}}_h\in\mathcal{V}_h\setminus\{0\}} \frac{[\tilde{\mathbf{V}}(\boldsymbol{\sigma}) - \mathbf{B}_1(\mathbf{t},\boldsymbol{\phi}),\tilde{\boldsymbol{\tau}}_h]}{\|\tilde{\boldsymbol{\tau}}_h\|_{\mathbf{Y}}} \leq \inf_{\boldsymbol{\delta}_h\in\mathcal{M}_0^1\boldsymbol{\chi}} \|\boldsymbol{\gamma} - \boldsymbol{\delta}_h\|_{[L^2(\Omega)]^2}.$$

This finishes the proof of the theorem. $\square$

We end this paper by remarking that, under usual regularity assumptions, the Cea estimate from Theorem 6.1 together with classical error estimates for interpolation and projection operators, yield an asymptotic rate of convergence of $O(h)$.

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REFERENCES

[1] D. N. Arnold, F. Brezzi, and J. Douglas, Jr., *PEERS: a new mixed finite element for plane elasticity*, Japan J. Appl. Math., 1 (1984), pp. 347–367.
[2] G. R. Barrenechea, G. N. Gatica, and J.-M. Thomas, *Primal mixed formulations for the coupling of FEM and BEM. part I: Linear problems*, Numer. Funct. Anal. Optim., 19 (1998), pp. 7–32.
[3] U. Brink, C. Carstensen, and E. Stein, *Symmetric coupling of boundary elements and Raviart-Thomas-type mixed finite elements in elastostatics*, Numer. Math., 75 (1996), pp. 153–174.
[4] M. Costabel, *Symmetric methods for the coupling of finite elements and boundary elements*, in Boundary Elements IX, C. A. B. et al., ed., vol. 1, Berlin, 1987, Springer–Verlag, pp. 411–420.
[5] M. Costabel, *Boundary integral operators on Lipschitz domains: Elementary results*, SIAM J. Math. Anal., 19 (1988), pp. 613–626.
[6] M. Costabel and E. P. Stephan, *Coupling of finite element and boundary element methods for an elasto-plastic interface problem*, SIAM J. Numer. Anal., 27 (1990), pp. 1212–1226.
[7] G. N. Gatica, *Combination of mixed finite element and Dirichlet-to-Neumann methods in nonlinear plane elasticity*, Appl. Math. Lett., 10 (1997), pp. 29–35.
[8] G. N. Gatica, *Solvability and Galerkin approximations of a class of nonlinear operator equations.* Technical Report 99-03, Departamento de Ingeniería Matemática, Universidad de Concepción, Chile. `http://www.ing-mat.udec.cl/inf-loc-dim.html`
[9] G. N. Gatica and N. Heuer, *Minimum residual iteration for a dual-dual mixed formulation of exterior transmission problems*, Technical Report 99-07, Departamento de Ingeniería Matemática, Universidad de Concepción, Chile. `http://www.ing-mat.udec.cl/inf-loc-dim.html`
[10] G.N. Gatica and N. Heuer, *Conjugate gradient method for dual-dual mixed formulations*, Technical Report 99-16, Departamento de Ingeniería Matemática, Universidad de Concepción. `http://www.ing-mat.udec.cl/inf-loc-dim.html`
[11] G. N. Gatica and G. C. Hsiao, *On the coupled BEM and FEM for a nonlinear exterior Dirichlet problem in* $\mathbb{R}^2$, Numer. Math., 61 (1992), pp. 171–214.
[12] G. N. Gatica and G. C. Hsiao, *Boundary-Field Equation Methods for a Class of Nonlinear Problems*, vol. 331 of Pitman Research Notes in Mathematics Series, Longman, 1995.
[13] G. N. Gatica and S. Meddahi, *A dual-dual mixed formulation for nonlinear exterior transmission problems.* Technical Report 99-04, Departamento de Ingeniería Matemática, Universidad de Concepción, Chile. `http://www.ing-mat.udec.cl/inf-loc-dim.html`
[14] G. N. Gatica and W. L. Wendland, *Coupling of mixed finite elements and boundary elements for linear and nonlinear elliptic problems*, Appl. Anal., 63 (1996), pp. 39–75.
[15] G. N. Gatica and W. L. Wendland, *Coupling of mixed finite elements and boundary elements for a hyperelastic interface problem*, SIAM J. Numer. Anal., 34 (1997), pp. 2335–2356.
[16] H. Han, *A new class of variational formulations for the coupling of finite and boundary element methods*, J. Comput. Math., 8 (1990), pp. 223–232.
[17] G. C. Hsiao, *The coupling of boundary element and finite element methods*, Z. Angew. Math. Mech., 70 (1990), pp. 493–503. A plenary lecture at the GAMM-Tagung 1989.
[18] G. C. Hsiao, E. P. Stephan, and W. L. Wendland, *On the Dirichlet problem in elasticity for a domain exterior to an arc*, J. Comput. Appl. Math., 34 (1991), pp. 1–19.
[19] S. Meddahi, J. Valdés, O. Menéndez, and P. Pérez, *On the coupling of boundary integral and mixed finite element methods*, J. Comput. Appl. Math., 69 (1996), pp. 113–124.
[20] J. Nečas, *Introduction to the Theory of Nonlinear Elliptic Equations*, Wiley, New York, 1986.