An expanded mixed finite element method via a dual-dual formulation [1]

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Abstract

We apply an expanded mixed finite element method, which introduces the gradient as a third explicit unknown, to solve a linear second order elliptic equation in divergence form. Instead of using the standard dual form, we show that the corresponding variational formulation can be written as a *dual-dual* operator equation. We establish existence and uniqueness of solution for the continuous and discrete formulations, and provide the corresponding error analysis by using Raviart-Thomas elements. A generalization of the classical Babuška-Brezzi theory to this kind of formulations is the main tool of our analysis. In addition, we show that the corresponding dual-dual linear system can be efficiently solved by a preconditioned minimum residual method. Some numerical results, illustrating this fact and the rate of convergence of the mixed finite element method, are also provided.

Key words: Mixed finite elements, Raviart-Thomas spaces, inf-sup conditions, minimum residual method

[1] This research was partially supported by CONICYT-Chile through the Program A on Numerical Analysis of the FONDAP in Applied Mathematics, and the Fondecyt project No. 1980122.

1 Introduction

Let Ω be a simply connected and bounded domain in $\mathbf{R}^2$ with Lipschitz continuous boundary $\Gamma := \partial\Omega$. Then, given $f \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma)$ and a matrix valued continuous function $\boldsymbol{\kappa}$, we consider the non-homogeneous Dirichlet problem: *Find* $u \in H^1(\Omega)$ *such that*

$$\begin{aligned} -\operatorname{div}(\boldsymbol{\kappa}\nabla u) &= f \quad \text{in} \quad \Omega\,, \\ u &= g \quad \text{on} \quad \Gamma\,. \end{aligned} \tag{1}$$

Here, we assume that $\boldsymbol{\kappa}$ is symmetric and that there exists $C > 0$ such that

$$C\,||\xi||^2 \,\leq\, \sum_{i,j=1}^{2} \kappa_{ij}(x)\,\xi_i\,\xi_j \quad \forall\,\xi := (\xi_1, \xi_2) \in \mathbf{R}^2\,, \quad \forall\, x \in \bar{\Omega}\,, \tag{2}$$

with κ_{ij} being the entries of $\boldsymbol{\kappa}$.

The standard mixed finite element method for (1) requires first the definition of the flux $\boldsymbol{\sigma} := \boldsymbol{\kappa}\,\nabla u$ as an auxiliary unknown. Then, using that $\boldsymbol{\kappa}$ is invertible (because of (2)), the integration by parts procedure is applied to the relation $\nabla\, u = \boldsymbol{\kappa}^{-1}\,\boldsymbol{\sigma}$. Now, the idea of introducing $\boldsymbol{\theta} := \nabla u$ as an additional explicit unknown, was first employed in [13] and [14] in connection with the coupling of mixed finite element and boundary integral equation methods for solving nonlinear exterior transmission problems. This approach was also utilized in [7] and [8], where it was called an *expanded* mixed finite element method. Indeed, this formulation can be seen as an expansion of the usual method in the sense that three variables are explicitly treated as unknowns, namely the scalar u, the flux $\boldsymbol{\sigma}$ and the gradient $\boldsymbol{\theta}$. Nevertheless, it must be remarked that the idea of using an expanded mixed formulation had already been utilized in elasticity (see, e.g. [9]).

The purpose of this work is to develop the expanded mixed finite element method for the numerical solution of our model problem (1). However, instead of proceeding as in [7], we rewrite the variational formulation as a dual-dual operator equation so that an extension of the classical Babuška-Brezzi theory can be applied. In this way, our analysis becomes simpler than the one in [7] and provides an alternative numerical method for the mixed formulations of boundary value problems. In fact, as we will show in forthcoming papers, this approach can also be applied to linear and nonlinear problems arising in potential theory, heat conductivity and elastostatics. For simplicity, we have chosen here the model problem (1), which is sufficiently general to illustrate the main aspects of the expanded method via a dual-dual formulation. The rest of the present paper is organized as follows. In Section 2, we derive the dual-dual mixed formulation of (1). In Section 3, we present the main results concerning the above mentioned generalization of Babuška-Brezzi's theory. They constitute a particular case of the more general approach developed in [10]. Then, in Section 4 we apply the theorems from Section 3 and provide the existence and uniqueness of solution for the continuous and discrete dual-dual formulations, by using Raviart-Thomas elements of lowest order. Furthermore, we prove the Céa estimate and obtain, under usual regularity assumptions, an error bound of $O(h)$. An efficient iterative solution of the arising dual-dual linear systems is proposed in Section 5 by using the minimum residual method with preconditioning. Finally, several numerical results are given in Section 6.

2 The dual-dual formulation

According to the above analysis, the Dirichlet problem (1) can be reformulated as follows: *Find* $(\boldsymbol{\theta}, \boldsymbol{\sigma}, u) \in [L^2(\Omega)]^2 \times H(\mathrm{div};\Omega) \times L^2(\Omega)$ *such that*

$$
\begin{aligned}
u = g \quad \text{on} \quad \Gamma \quad , \quad \boldsymbol{\theta} = \nabla u \quad \text{in} \quad \Omega\,, \\
\boldsymbol{\sigma} = \boldsymbol{\kappa}\,\boldsymbol{\theta} \quad \text{in} \quad \Omega \quad \text{and} \quad \mathrm{div}\,\boldsymbol{\sigma} = -f \quad \text{in} \quad \Omega\,.
\end{aligned}
\tag{3}
$$

We recall here that $H(\mathrm{div};\Omega)$ is the space of functions $\boldsymbol{\tau} \in [L^2(\Omega)]^2$ such that $\mathrm{div}\,\boldsymbol{\tau} \in L^2(\Omega)$. It is well known that, provided with the inner product $< \boldsymbol{\tau}, \boldsymbol{\sigma} >_{H(\mathrm{div};\Omega)} := < \boldsymbol{\tau}, \boldsymbol{\sigma} >_{[L^2(\Omega)]^2} + < \mathrm{div}\,\boldsymbol{\tau}, \mathrm{div}\,\boldsymbol{\sigma} >_{L^2(\Omega)}$, $H(\mathrm{div};\Omega)$ is a Hilbert space. Moreover, for all $\boldsymbol{\tau} \in H(\mathrm{div};\Omega)$, $\boldsymbol{\tau} \cdot \nu \in H^{-1/2}(\Gamma)$ and $||\boldsymbol{\tau} \cdot \nu||_{H^{-1/2}(\Gamma)} \leq ||\boldsymbol{\tau}||_{H(\mathrm{div};\Omega)}$ (see [15] for the proof of these results), where ν is the unit outward normal to Γ.

Now, for the weak formulation, we first multiply the second equation in (3) by a function $\boldsymbol{\tau} \in H(\mathrm{div};\Omega)$, integrate by parts in Ω, and use that $u = g$ on Γ, to obtain

$$
-\int_\Omega \boldsymbol{\theta} \cdot \boldsymbol{\tau}\, dx - \int_\Omega u\, \mathrm{div}\,\boldsymbol{\tau}\, dx = - < g, \boldsymbol{\tau} \cdot \nu >\,, \tag{4}
$$

where $< \cdot, \cdot >$ stands for the duality pairing of $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ with respect to the $L^2(\Gamma)$-inner product.

Next, the third and fourth equations in (3) are tested against $\boldsymbol{\zeta} \in [L^2(\Omega)]^2$ and $v \in L^2(\Omega)$, respectively, which gives

$$\int_\Omega \boldsymbol{\kappa}\,\boldsymbol{\theta} \cdot \boldsymbol{\zeta}\, dx - \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\zeta}\, dx = 0 \tag{5}$$

and

$$- \int_\Omega v \operatorname{div} \boldsymbol{\sigma}\, dx = \int_\Omega f v\, dx\,. \tag{6}$$

Thus, collecting appropriately (4), (5) and (6), we arrive at the following variational formulation of (3): *Find* $(\boldsymbol{\theta}, \boldsymbol{\sigma}, u) \in [L^2(\Omega)]^2 \times H(\mathrm{div}; \Omega) \times L^2(\Omega)$ *such that*

$$\begin{array}{rcccl} \displaystyle\int_\Omega \boldsymbol{\kappa}\,\boldsymbol{\theta} \cdot \boldsymbol{\zeta}\, dx & \displaystyle - \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\zeta}\, dx & & = & 0\,, \\ \displaystyle - \int_\Omega \boldsymbol{\theta} \cdot \boldsymbol{\tau}\, dx & & \displaystyle - \int_\Omega u \operatorname{div} \boldsymbol{\tau}\, dx & = & - < g, \boldsymbol{\tau} \cdot \nu >\,, \\ & \displaystyle - \int_\Omega v \operatorname{div} \boldsymbol{\sigma}\, dx & & = & \displaystyle\int_\Omega f v\, dx\,, \end{array} \tag{7}$$

for all $(\boldsymbol{\zeta}, \boldsymbol{\tau}, v) \in [L^2(\Omega)]^2 \times H(\mathrm{div}; \Omega) \times L^2(\Omega)$.

We remark that the approach in [7] proceeds by adding the first and third equations from (7), and leaving the second one as it stands, so that (7) can be rewritten as the standard operator equation for constrained variational problems. We proceed differently, as shown next.

First, we put $X_1 := [L^2(\Omega)]^2$, $M_1 := H(\mathrm{div}; \Omega)$, $X := X_1 \times M_1$, $M := L^2(\Omega)$, and define the bounded linear operators $\mathbf{A}_1 : X_1 \to X_1'$, $\mathbf{B}_1 : X_1 \to M_1'$, $\mathbf{A} : X \to X'$ and $\mathbf{B} : M_1 \to M'$, and the functionals $\mathbf{F}_1 \in X_1'$, $\mathbf{G}_1 \in M_1'$ and $\mathbf{G} \in M'$, as follows

$$[\mathbf{A}_1(\boldsymbol{\theta}), \boldsymbol{\zeta}] := \int_\Omega \boldsymbol{\kappa}\,\boldsymbol{\theta} \cdot \boldsymbol{\zeta}\, dx\,,$$

$$[\mathbf{B}_1(\boldsymbol{\theta}), \boldsymbol{\tau}] := - \int_\Omega \boldsymbol{\theta} \cdot \boldsymbol{\tau}\, dx\,,$$

$$[\mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\sigma}), (\boldsymbol{\zeta}, \boldsymbol{\tau})] := [\mathbf{A}_1(\boldsymbol{\theta}), \boldsymbol{\zeta}] + [\mathbf{B}_1(\boldsymbol{\zeta}), \boldsymbol{\sigma}] + [\mathbf{B}_1(\boldsymbol{\theta}), \boldsymbol{\tau}]\,,$$

$$[\mathbf{B}(\boldsymbol{\sigma}), v] := - \int_\Omega v \operatorname{div} \boldsymbol{\sigma}\, dx\,,$$

$$[\mathbf{F}_1, \boldsymbol{\zeta}] := 0\,, \quad [\mathbf{G}_1, \boldsymbol{\tau}] := - < g, \boldsymbol{\tau} \cdot \nu >$$

and

$$[\mathbf{G}, v] := \int_\Omega f v \, dx$$

for all $(\boldsymbol{\theta}, \boldsymbol{\sigma})$, $(\boldsymbol{\zeta}, \boldsymbol{\tau}) \in X$ and for all $v \in M$, where $[\,\cdot,\cdot\,]$ stands for the duality pairing induced by the operators appearing in each case.

Further, let $\mathbf{B}_1^* : M_1 \to X_1'$ and $\mathbf{B}^* : M \to M_1'$ be the transposes of $\mathbf{B}_1$ and $\mathbf{B}$, respectively, and let $\mathbf{O}$ denote the null operator. It follows that $\mathbf{A}$ can be defined, equivalently, as:

$$\mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\sigma}) := \begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1^* \\ \mathbf{B}_1 & \mathbf{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta} \\ \boldsymbol{\sigma} \end{bmatrix} \in X' := X_1' \times M_1' . \tag{8}$$

Therefore, the system (7) can be reformulated as the following operator equation: *Find* $((\boldsymbol{\theta}, \boldsymbol{\sigma}), u) \in X \times M$ *such that*

$$\begin{bmatrix} \mathbf{A} & \tilde{\mathbf{B}}^* \\ \tilde{\mathbf{B}} & \mathbf{O} \end{bmatrix} \begin{bmatrix} (\boldsymbol{\theta}, \boldsymbol{\sigma}) \\ u \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} , \tag{9}$$

where $\mathbf{F} := (\mathbf{F}_1, \mathbf{G}_1) \in X' := X_1' \times M_1'$ and $\tilde{\mathbf{B}} : X \to M'$ is given by $\tilde{\mathbf{B}} := (\mathbf{O} \quad \mathbf{B})$.

The equation (9) constitutes our so-called *dual-dual mixed formulation* of (7) since the operator $\mathbf{A}$ itself has the dual-type structure given by (8). We now need a generalization of the usual Babuška-Brezzi theory to this class of linear problems.

3 An extension of the Babuška-Brezzi theory

Let X_1, M_1, M be Hilbert spaces and define $X := X_1 \times M_1$. We consider bounded linear operators $\mathbf{B}_1 : X_1 \to M_1'$ and $\mathbf{B} : M_1 \to M'$, with transposes $\mathbf{B}_1^* : M_1 \to X_1'$ and $\mathbf{B}^* : M \to M_1'$, respectively. Also, let $\mathbf{A}_1 : X_1 \to X_1'$ be a bounded linear operator. With $\mathbf{A}_1$, $\mathbf{B}_1$ and $\mathbf{B}_1^*$ we define the operator $\mathbf{A} : X \to X'$ as in (8).

Then, given $(\mathbf{F}, \mathbf{G}) \in X' \times M'$, with $\mathbf{F} := (\mathbf{F}_1, \mathbf{G}_1) \in X_1' \times M_1'$, we are interested in the following variational problem: *Find* $((\boldsymbol{\theta},\boldsymbol{\sigma}),u) \in X \times M$ *such that*

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1^* & \mathbf{O} \\ \mathbf{B}_1 & \mathbf{O} & \mathbf{B}^* \\ \mathbf{O} & \mathbf{B} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta} \\ \boldsymbol{\sigma} \\ u \end{bmatrix} = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{G}_1 \\ \mathbf{G} \end{bmatrix}. \tag{10}$$

Sufficient conditions for the unique solvability of (10) are provided in the following theorem.

Theorem 1 *Let* $\tilde{M}_1 := Ker(\mathbf{B}) \subseteq M_1$ *and define* $V_1 := \{\boldsymbol{\zeta} \in X_1 : \; [\mathbf{B}_1(\boldsymbol{\zeta}),\boldsymbol{\tau}] = 0 \quad \forall\, \boldsymbol{\tau} \in \tilde{M}_1\}$. *Assume that*

i) *the linear operator* $\mathbf{A}_1 : X_1 \to X_1'$ *is bounded and* V_1*-elliptic, that is, there exist* γ, $\alpha > 0$ *such that*

$$||\mathbf{A}_1(\boldsymbol{\zeta})||_{X_1'} \;\leq\; \gamma\,||\boldsymbol{\zeta}||_{X_1} \quad \forall\,\boldsymbol{\zeta} \in X_1 \quad \textit{and} \quad [\mathbf{A}_1(\boldsymbol{\zeta}),\boldsymbol{\zeta}] \;\geq\; \alpha\,||\boldsymbol{\zeta}||_{X_1}^2 \quad \forall\,\boldsymbol{\zeta} \in V_1\,,$$

ii) *there exists* $\beta > 0$ *such that for all* $v \in M$:

$$\sup_{\substack{\boldsymbol{\tau} \in M_1 \\ \boldsymbol{\tau} \neq 0}} \frac{[\mathbf{B}(\boldsymbol{\tau}), v]}{||\boldsymbol{\tau}||_{M_1}} \;\geq\; \beta\,||v||_M\,,$$

iii) *there exists* $\beta_1 > 0$ *such that for all* $\boldsymbol{\tau} \in \tilde{M}_1$:

$$\sup_{\substack{\boldsymbol{\zeta} \in X_1 \\ \boldsymbol{\zeta} \neq 0}} \frac{[\mathbf{B}_1(\boldsymbol{\zeta}), \boldsymbol{\tau}]}{||\boldsymbol{\zeta}||_{X_1}} \;\geq\; \beta_1\,||\boldsymbol{\tau}||_{M_1}\,.$$

Then, for each $(\mathbf{F},\mathbf{G}) \in X' \times M'$ *there exists a unique solution* $((\boldsymbol{\theta},\boldsymbol{\sigma}),u) \in X \times M$ *of* (10)*, which depends continuously on the data.*

Proof. We adapt the analysis from [15] (Section 4, Chap. I) to the present situation. Hence, given $\mathbf{G} \in M'$, we set $V(\mathbf{G}) := X_1 \times \tilde{M}_1(\mathbf{G})$, where

$$\tilde{M}_1(\mathbf{G}) := \{\boldsymbol{\tau} \in M_1 : \quad [\mathbf{B}(\boldsymbol{\tau}), v] = [\mathbf{G}, v] \;\forall\, v \in M\}\,.$$

We observe that $V := V(\mathbf{O}) = X_1 \times \tilde{M}_1$.

Now, with (10) we associate the following problem: *Find* $(\boldsymbol{\theta},\boldsymbol{\sigma}) \in V(\mathbf{G})$ *such that*

$$[\mathbf{A}(\boldsymbol{\theta},\boldsymbol{\sigma}),(\boldsymbol{\zeta},\boldsymbol{\tau})] \;=\; [\mathbf{F},(\boldsymbol{\zeta},\boldsymbol{\tau})] \quad \forall\,(\boldsymbol{\zeta},\boldsymbol{\tau}) \in V\,. \tag{11}$$

Clearly, if $(\boldsymbol{\theta}, \boldsymbol{\sigma}) \in V(\mathbf{G})$ is a solution of (11), then, due to the inf-sup condition ii) and Lemma 4.1 in Chap. I of [15], there exists a unique $u \in M$ such that $((\boldsymbol{\theta}, \boldsymbol{\sigma}), u) \in X \times M$ is solution of (10).

Conversely, if $((\boldsymbol{\theta}, \boldsymbol{\sigma}), u) \in X \times M$ is a solution of (10), then $(\boldsymbol{\theta}, \boldsymbol{\sigma}) \in V(\mathbf{G})$ and $(\boldsymbol{\theta}, \boldsymbol{\sigma})$ is a solution of (11) since for all $\boldsymbol{\tau} \in \tilde{M}_1$, $[\mathbf{B}^*(u), \boldsymbol{\tau}] = [\mathbf{B}(\boldsymbol{\tau}), u] = 0$.

Due to this equivalence, we now concentrate on problem (11). Again, by Lemma 4.1 in Chap. I of [15], there exists $\boldsymbol{\sigma}_0 \in M_1$ such that $\mathbf{B}(\boldsymbol{\sigma}_0) = \mathbf{G}$. Thus, problem (11) can be replaced by: *Find* $(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\sigma}}) \in V = X_1 \times \tilde{M}_1$ *such that*

$$\begin{aligned} [\mathbf{A}_1(\tilde{\boldsymbol{\theta}}), \boldsymbol{\zeta}] + [\mathbf{B}_1^*(\tilde{\boldsymbol{\sigma}}), \boldsymbol{\zeta}] &= [\mathbf{F}_1 - \mathbf{B}_1^*(\boldsymbol{\sigma}_0), \boldsymbol{\zeta}]\,, \\ [\mathbf{B}_1(\tilde{\boldsymbol{\theta}}), \boldsymbol{\tau}] &= [\mathbf{G}_1, \boldsymbol{\tau}]\,, \end{aligned} \tag{12}$$

for all $(\boldsymbol{\zeta}, \boldsymbol{\tau}) \in V$.

Next, we set

$$V_1(\mathbf{G}_1) := \{\boldsymbol{\zeta} \in X_1 : \quad [\mathbf{B}_1(\boldsymbol{\zeta}), \boldsymbol{\tau}] = [\mathbf{G}_1, \boldsymbol{\tau}] \ \forall \boldsymbol{\tau} \in \tilde{M}_1\}\,,$$

and observe that $V_1 = V_1(\mathbf{O})$.

Then, we associate with (12) the following problem: *Find* $\tilde{\boldsymbol{\theta}} \in V_1(\mathbf{G}_1)$ *such that*

$$[\mathbf{A}_1(\tilde{\boldsymbol{\theta}}), \boldsymbol{\zeta}] = [\mathbf{F}_1 - \mathbf{B}_1^*(\boldsymbol{\sigma}_0), \boldsymbol{\zeta}] \quad \forall \boldsymbol{\zeta} \in V_1\,. \tag{13}$$

At this point, we can apply the Lax-Milgram Theorem and conclude that (13) has a unique solution $\tilde{\boldsymbol{\theta}} \in V_1(\mathbf{G}_1)$. It follows, in virtue of iii) and Lemma 4.1 in Chap. I of [15], that there exists $\tilde{\boldsymbol{\sigma}} \in \tilde{M}_1$ such that $(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\sigma}}) \in V$ is the unique solution of (12). In this way, we deduce that $(\boldsymbol{\theta}, \boldsymbol{\sigma}) := (\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\sigma}} + \boldsymbol{\sigma}_0) \in V(\mathbf{G})$ is the unique solution of (11). Therefore, the equivalence between (10) and (11) completes the proof. $\square$

Now, for the Galerkin approximation of (10), we let $X_{1,h}$, $M_{1,h}$ and M_h be finite dimensional subspaces of X_1, M_1 and M, respectively, and let $X_h := X_{1,h} \times M_{1,h}$ be the corresponding subspace of X. Here, we assume that the index h is taken in a numerable family $\mathcal{I} := \{h_j\}_{j \in \mathbf{N}}$ such that $h_j \geq h_{j+1}$ for all $j \in \mathbf{N}$.

Then, the Galerkin scheme associated with (10) reads as follows: *Given* $(\mathbf{F}, \mathbf{G})$

$\in X' \times M'$, *find* $((\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h), u_h) \in X_h \times M_h$ *such that*

$$\begin{array}{lclcl}
[\mathbf{A}_1(\boldsymbol{\theta}_h), \boldsymbol{\zeta}_h] + [\mathbf{B}_1(\boldsymbol{\zeta}_h), \boldsymbol{\sigma}_h] & & & = & [\mathbf{F}_1, \boldsymbol{\zeta}_h] \\
[\mathbf{B}_1(\boldsymbol{\theta}_h), \boldsymbol{\tau}_h] & & + [\mathbf{B}(\boldsymbol{\tau}_h), u_h] & = & [\mathbf{G}_1, \boldsymbol{\tau}_h] \\
& [\mathbf{B}(\boldsymbol{\sigma}_h), v_h] & & = & [\mathbf{G}, v_h]
\end{array} \tag{14}$$

for all $((\boldsymbol{\zeta}_h, \boldsymbol{\tau}_h), v_h) \in X_h \times M_h$.

The discrete analogue of Theorem 1 is stated next.

Theorem 2 *Let* $\tilde{M}_{1,h} := \{\boldsymbol{\tau}_h \in M_{1,h} : \quad [\mathbf{B}(\boldsymbol{\tau}_h), v_h] = 0 \, \forall \, v_h \in M_h\}$ *and define* $V_{1,h} := \{\boldsymbol{\zeta}_h \in X_{1,h} : \quad [\mathbf{B}_1(\boldsymbol{\zeta}_h), \boldsymbol{\tau}_h] = 0 \quad \forall \, \boldsymbol{\tau}_h \in \tilde{M}_{1,h}\}$. *Assume that*

i) *the linear operator* $\mathbf{A}_1 : X_1 \to X_1'$ *is bounded and* $V_{1,h}$*-elliptic, that is, there exist* γ, $\alpha_h > 0$ *such that*

$$||\mathbf{A}_1(\boldsymbol{\zeta})||_{X_1'} \leq \gamma \, ||\boldsymbol{\zeta}||_{X_1} \quad \forall \, \boldsymbol{\zeta} \in X_1$$

and

$$[\mathbf{A}_1(\boldsymbol{\zeta}_h), \boldsymbol{\zeta}_h] \geq \alpha_h \, ||\boldsymbol{\zeta}_h||^2_{X_1} \quad \forall \, \boldsymbol{\zeta}_h \in V_{1,h} \, ,$$

ii) *there exists* $\beta^* > 0$*, independent of the subspaces involved, such that for all* $v_h \in M_h$:

$$\sup_{\substack{\boldsymbol{\tau}_h \in M_{1,h} \\ \boldsymbol{\tau}_h \neq 0}} \frac{[\mathbf{B}(\boldsymbol{\tau}_h), v_h]}{||\boldsymbol{\tau}_h||_{M_1}} \geq \beta^* \, ||v_h||_M \, ,$$

iii) *there exists* $\beta_1^* > 0$*, independent of the subspaces involved, such that for all* $\boldsymbol{\tau}_h \in \tilde{M}_{1,h}$:

$$\sup_{\substack{\boldsymbol{\zeta}_h \in X_{1,h} \\ \boldsymbol{\zeta}_h \neq 0}} \frac{[\mathbf{B}_1(\boldsymbol{\zeta}_h), \boldsymbol{\tau}_h]}{||\boldsymbol{\zeta}_h||_{X_1}} \geq \beta_1^* \, ||\boldsymbol{\tau}_h||_{M_1} \, .$$

Then, for each $(\mathbf{F}, \mathbf{G}) \in X' \times M'$ *there exists a unique solution* $((\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h), u_h) \in X_h \times M_h$ *of* (14).

Proof. It is similar to the proof of Theorem 1 and hence, we omit further details. $\square$

Finally, concerning the error analysis, we have the following result.

Theorem 3 *Assume that all the hypotheses of both Theorem* 1 *and Theorem* 2 *are satisfied, and let* $((\boldsymbol{\theta}, \boldsymbol{\sigma}), u) \in X \times M$ *and* $((\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h), u_h) \in X_h \times M_h$ *be* *the unique solutions of* (10) *and* (14)*, respectively. Let* $\mathbf{F} := (\mathbf{F}_1, \mathbf{G}_1) \in X'$*, with* $\mathbf{F}_1 \in X_1'$ *and* $\mathbf{G}_1 \in M_1'$*. In addition, suppose that there exists* $\tilde{\alpha} > 0$ *such that* α*,* $\alpha_h \geq \tilde{\alpha}$ *for all* $h \in \mathcal{I}$*. Then, there exists* $C > 0$*, depending only on* $\tilde{\alpha}$*,* γ*,* $||\mathbf{B}_1||$*,* β_1^**,* $||\mathbf{B}||$ *and* β^**, such that the following Strang-type error estimate holds for all* $h \in \mathcal{I}$*:*

$$\begin{aligned} & ||((\boldsymbol{\theta}, \boldsymbol{\sigma}), u) - ((\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h), u_h)|| \ \leq \\ C \Bigg\{ & \inf_{((\boldsymbol{\zeta}_h, \boldsymbol{\tau}_h), v_h) \in X_h \times M_h} ||((\boldsymbol{\theta}, \boldsymbol{\sigma}), u) - ((\boldsymbol{\zeta}_h, \boldsymbol{\tau}_h), v_h)|| \\ & + \sup_{\substack{\boldsymbol{\tau}_h \in \tilde{M}_{1,h} \\ \boldsymbol{\tau}_h \neq 0}} \left\{ \frac{[\mathbf{G}_1 - \mathbf{B}_1(\boldsymbol{\theta}), \boldsymbol{\tau}_h]}{||\boldsymbol{\tau}_h||} \right\} \Bigg\}. \end{aligned} \tag{15}$$

Proof. We just sketch the main ideas of the proof. For full details, we refer to Section 4 in [10].

The discrete inf-sup condition satisfied by $\mathbf{B}$ guarantees the existence of $\boldsymbol{\sigma}_{0,h} \in M_{1,h}$ such that $[\mathbf{B}(\boldsymbol{\sigma}_{0,h}), v_h] = [\mathbf{G}, v_h]$ for all $v_h \in M_h$. Then, by using the properties of the operators $\mathbf{A}_1$ and $\mathbf{B}_1$, one proves that for all $h \in \mathcal{I}$:

$$\begin{aligned} ||\boldsymbol{\theta} - \boldsymbol{\theta}_h|| \ \leq & \left(1 + \frac{\gamma}{\tilde{\alpha}}\right) \inf_{\boldsymbol{\zeta}_h \in V_{1,h}(\mathbf{G}_1)} ||\boldsymbol{\theta} - \boldsymbol{\zeta}_h|| \\ & + \frac{||\mathbf{B}_1||}{\tilde{\alpha}} \inf_{\boldsymbol{\tau}_h \in \tilde{M}_{1,h}} ||\boldsymbol{\sigma} - (\boldsymbol{\tau}_h + \boldsymbol{\sigma}_{0,h})||\,, \end{aligned} \tag{16}$$

where

$$V_{1,h}(\mathbf{G}_1) := \{\boldsymbol{\zeta}_h \in X_{1,h} : \quad [\mathbf{B}_1(\boldsymbol{\zeta}_h), \boldsymbol{\tau}_h] = [\mathbf{G}_1, \boldsymbol{\tau}_h] \ \forall \, \boldsymbol{\tau}_h \in \tilde{M}_{1,h}\}\,.$$

Now, the discrete inf-sup condition for $\mathbf{B}_1$ allows to improve the bound provided by the first term on the right hand side of (16). Indeed, one shows that the following estimate holds for all $h \in \mathcal{I}$:

$$\begin{aligned} \inf_{\boldsymbol{\zeta}_h \in V_{1,h}(\mathbf{G}_1)} ||\boldsymbol{\theta} - \boldsymbol{\zeta}_h|| \ \leq & \left(1 + \frac{||\mathbf{B}_1||}{\beta_1^*}\right) \inf_{\boldsymbol{\zeta}_h \in X_{1,h}} ||\boldsymbol{\theta} - \boldsymbol{\zeta}_h|| \\ & + \frac{1}{\beta_1^*} \sup_{\substack{\boldsymbol{\tau}_h \in \tilde{M}_{1,h} \\ \boldsymbol{\tau}_h \neq 0}} \left\{ \frac{[\mathbf{G}_1 - \mathbf{B}_1(\boldsymbol{\theta}), \boldsymbol{\tau}_h]}{||\boldsymbol{\tau}_h||} \right\}. \end{aligned} \tag{17}$$

On the other hand, by applying again the properties of the operators $\mathbf{A}_1$ and

$\mathbf{B}_1$, we obtain that:

$$\begin{aligned} ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h|| \;&\le\; \frac{\gamma}{\beta_1^*}\, ||\boldsymbol{\theta} - \boldsymbol{\theta}_h|| \\ &+ \left(1 + \frac{||\mathbf{B}_1||}{\beta_1^*}\right) \inf_{\boldsymbol{\tau}_h \in \tilde{M}_{1,h}} ||\boldsymbol{\sigma} - (\boldsymbol{\tau}_h + \boldsymbol{\sigma}_{0,h})|| \,. \end{aligned} \tag{18}$$

Hence, as a consequence of (16), (17) and (18), we deduce that there exists $\tilde{C} > 0$, depending only on $\tilde{\alpha}$, γ, $||\mathbf{B}_1||$, β_1^*, $||\mathbf{B}||$ and β^*, such that for all $h \in \mathcal{I}$:

$$\begin{aligned} ||(\boldsymbol{\theta}, \boldsymbol{\sigma}) - (\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h)|| \;\le\; \tilde{C} \Bigg\{ & \inf_{(\boldsymbol{\zeta}_h, \boldsymbol{\tau}_h) \in X_h} ||(\boldsymbol{\theta}, \boldsymbol{\sigma}) - (\boldsymbol{\zeta}_h, \boldsymbol{\tau}_h)|| \\ &+ \sup_{\substack{\boldsymbol{\tau}_h \in \tilde{M}_{1,h} \\ \boldsymbol{\tau}_h \neq 0}} \left\{ \frac{[\mathbf{G}_1 - \mathbf{B}_1(\boldsymbol{\theta}), \boldsymbol{\tau}_h]}{||\boldsymbol{\tau}_h||} \right\} \Bigg\} . \end{aligned} \tag{19}$$

Finally, following the usual approach from [15], and applying now the properties of the operators $\mathbf{A}$ and $\mathbf{B}$, one can prove that there exists $\bar{C} > 0$, depending only on γ, $||\mathbf{B}_1||$, $||\mathbf{B}||$ and β^*, such that for all $h \in \mathcal{I}$:

$$||u - u_h|| \;\le\; \bar{C} \left\{ ||(\boldsymbol{\theta}, \boldsymbol{\sigma}) - (\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h)|| + \inf_{v_h \in M_h} ||u - v_h|| \right\} . \tag{20}$$

Consequently, (19) and (20) yield (15). $\square$

It is important to observe that if $\tilde{M}_{1,h} \subseteq \tilde{M}_1$, then from the second row in (12) (see also (10)) we deduce that

$$\sup_{\substack{\boldsymbol{\tau}_h \in \tilde{M}_{1,h} \\ \boldsymbol{\tau}_h \neq 0}} \left\{ \frac{[\mathbf{G}_1 - \mathbf{B}_1(\boldsymbol{\theta}), \boldsymbol{\tau}_h]}{||\boldsymbol{\tau}_h||} \right\} \;=\; 0 \,,$$

and hence, (15) becomes the usual Céa estimate for the Galerkin error. In other words, the second term on the right hand side of (15) constitutes the consistency error for the case in which $\tilde{M}_{1,h}$ is not a subspace of $\tilde{M}_1$.

4 Solvability and approximation results

We now go back to the continuous problem (9). Our main result concerning its solvability is stated first.

Theorem 4 *There exists a unique solution* $((\boldsymbol{\theta}, \boldsymbol{\sigma}), u) \in X \times M$ *of the dual-dual mixed formulation* (9).

Proof. The assumption (2) guarantees that the bounded linear operator $\mathbf{A}_1 : X_1 \to X_1'$ is X_1-elliptic. Now, the proof of the continuous inf-sup condition for $\mathbf{B}$ is quite standard and we refer the interested reader to [15], [13] or [18].

On the other hand, it is straigthforward to see that

$$\tilde{M}_1 := \operatorname{Ker}(\mathbf{B}) = \{\, \boldsymbol{\tau} \in M_1 : \quad \operatorname{div} \boldsymbol{\tau} = 0 \quad \text{in} \quad \Omega \,\} \,.$$

Thus, since $\tilde{M}_1 \subseteq M_1 \subseteq X_1$, we deduce that

$$\sup_{\substack{\boldsymbol{\zeta} \in X_1 \\ \boldsymbol{\zeta} \neq 0}} \frac{[\mathbf{B}_1(\boldsymbol{\zeta}), \boldsymbol{\tau}]}{||\boldsymbol{\zeta}||_{X_1}} \;=\; \sup_{\substack{\boldsymbol{\zeta} \in X_1 \\ \boldsymbol{\zeta} \neq 0}} \frac{-\int_\Omega \boldsymbol{\zeta} \cdot \boldsymbol{\tau} \, dx}{||\boldsymbol{\zeta}||_{[L^2(\Omega)]^2}} \;=\; ||\boldsymbol{\tau}||_{[L^2(\Omega)]^2} \;=\; ||\boldsymbol{\tau}||_{H(\mathrm{div};\Omega)}$$

for all $\boldsymbol{\tau} \in \tilde{M}_1$, which establishes the continuous inf-sup condition for $\mathbf{B}_1$.

Therefore, these results and a direct application of the abstract Theorem 1 complete the proof. $\Box$

At this point we remark that, using classical regularity results, one can show (cf. Lemma 4.4 in [18]) that there exists $\beta > 0$ such that

$$\sup_{\substack{\boldsymbol{\tau} \in [H^1(\Omega)]^2 \\ \boldsymbol{\tau} \neq 0}} \frac{-\int_\Omega v \operatorname{div} \boldsymbol{\tau} \, dx}{||\boldsymbol{\tau}||_{[H^1(\Omega)]^2}} \;\geq\; \beta \, ||v||_{L^2(\Omega)} \quad \forall\, v \in L^2(\Omega) \,. \tag{21}$$

This inf-sup condition will be needed below to prove the discrete inf-sup condition for $\mathbf{B}$.

In what follows, we introduce specific finite element subspaces and define the associated Galerkin scheme. For simplicity, from now on we assume that the boundary Γ of Ω is a polygonal curve.

First, let $\mathcal{T}_h$ be a regular triangulation of Ω made up of triangles T of diameter h_T such that $h := \sup\limits_{T \in \mathcal{T}_h} h_T$ and $\bar{\Omega} = \cup \{\, T : \quad T \in \mathcal{T}_h \,\}$. Next, we consider the canonical triangle with vertices $\hat{P}_1 = (0,0)^T$, $\hat{P}_2 = (1,0)^T$ and $\hat{P}_3 = (0,1)^T$ as a reference triangle $\hat{T}$, and introduce the family of bijective affine mappings $\{F_T\}_{T \in \mathcal{T}_h}$, such that $F_T(\hat{T}) = T$. It is well known that $F_T(\hat{x}) = B_T \,\hat{x} + b_T$ for all $\hat{x} \in \hat{T}$, where B_T, a square matrix of order 2, and $b_T \in \mathbf{R}^2$, depend only on the vertices of T.

We now consider the lowest order Raviart-Thomas spaces. For each triangle $T \in \mathcal{T}_h$, we put

$$\mathcal{R}T_0(T) := \{ \boldsymbol{\tau} : \quad \boldsymbol{\tau} = |\det(B_T)|^{-1} B_T \, \hat{\boldsymbol{\tau}} \circ F_T^{-1} \,, \ \hat{\boldsymbol{\tau}} \in \mathcal{R}T_0(\hat{T}) \} \,,$$

where

$$\mathcal{R}T_0(\hat{T}) := \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} \right\} .$$

Then, we define the finite element subspaces for the unknowns $\boldsymbol{\theta}$ and $\boldsymbol{\sigma}$, respectively, as follows:

$$X_{1,h} := \left\{ \boldsymbol{\zeta}_h \in [L^2(\Omega)]^2 : \quad \boldsymbol{\zeta}_h|_T \in \mathcal{R}T_0(T) \quad \forall \, T \in \mathcal{T}_h \right\} \tag{22}$$

and

$$M_{1,h} \; := \; \{ \boldsymbol{\tau}_h \in H(\text{div};\Omega) : \quad \boldsymbol{\tau}_h|_T \in \mathcal{R}T_0(T) \quad \forall \, T \in \mathcal{T}_h \} \,. \tag{23}$$

We remark that $X_{1,h}$ does not require continuity of the normal components through the sides of each triangle T, while $M_{1,h}$ certainly does.

Next, we put

$$X_h \; := \; X_{1,h} \, \times \, M_{1,h} \,,$$

and consider the piecewise constant functions as the finite element subspace for the unknown u, that is

$$M_h := \{ v_h \in L^2(\Omega) : \quad v_h|_T \quad \text{is constant} \quad \forall \, T \in \mathcal{T}_h \} \,. \tag{24}$$

In this way, the Galerkin scheme associated with the continuous problem (9) (or (7)) reads as follows: *Find* $((\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h), u_h) \in X_h \times M_h$ *such that*

$$\begin{array}{llcl} [\mathbf{A}_1(\boldsymbol{\theta}_h), \boldsymbol{\zeta}_h] + [\mathbf{B}_1(\boldsymbol{\zeta}_h), \boldsymbol{\sigma}_h] & & = & 0 \\ [\mathbf{B}_1(\boldsymbol{\theta}_h), \boldsymbol{\tau}_h] & + [\mathbf{B}(\boldsymbol{\tau}_h), u_h] & = & [\mathbf{G}_1, \boldsymbol{\tau}_h] \\ \qquad\qquad [\mathbf{B}(\boldsymbol{\sigma}_h), v_h] & & = & [\mathbf{G}, v_h] \end{array} \tag{25}$$

for all $((\boldsymbol{\zeta}_h, \boldsymbol{\tau}_h), v_h) \in X_h \times M_h$.

The unique solvability of the Galerkin scheme (25) and the corresponding error estimate can be established now.

Theorem 5 *There exists a unique solution* $((\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h), u_h) \in X_h \times M_h$ *of the Galerkin scheme* (25)*. In addition, there exists* $C > 0$*, independent of* h*, such that the following Céa estimate holds*

$$\begin{aligned} & ||((\boldsymbol{\theta}, \boldsymbol{\sigma}), u) \, - \, ((\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h), u_h)|| \\ \leq \; C & \inf_{((\boldsymbol{\zeta}_h, \boldsymbol{\tau}_h), v_h) \in X_h \times M_h} ||((\boldsymbol{\theta}, \boldsymbol{\sigma}), u) \, - \, ((\boldsymbol{\zeta}_h, \boldsymbol{\tau}_h), v_h)|| \, . \end{aligned}$$

Proof. We first prove that there exists $\beta^* > 0$, independent of the subspaces involved, such that for all $v_h \in M_h$:

$$\sup_{\substack{\tau_h \in M_{1,h} \\ \tau_h \neq 0}} \frac{[\mathbf{B}(\tau_h), v_h]}{||\tau_h||_{M_1}} \;\geq\; \beta^* \, ||v_h||_M \, . \tag{26}$$

To this end, we proceed as in Lemma 5.6 of [12] and introduce the equilibrium interpolation operator (cf. [5], [20]) $E_h : [H^1(\Omega)]^2 \to M_{1,h}$, which satisfies the commuting diagram property

$$\operatorname{div}(E_h \boldsymbol{\tau}) \;=\; \mathcal{P}_h \, (\operatorname{div} \boldsymbol{\tau}) \quad \forall \boldsymbol{\tau} \in [H^1(\Omega)]^2 \, , \tag{27}$$

where $\mathcal{P}_h$ is the orthogonal projection from $L^2(\Omega)$ onto the subspace M_h. Moreover, using (27) and the approximation properties of E_h (cf. [5] or [20]), we conclude that the family of operators $E_h : [H^1(\Omega)]^2 \to M_{1,h}$ is uniformly bounded. This means that there exists $\tilde{C} > 0$, independent of $h \in \mathcal{I}$, such that

$$||E_h \, \boldsymbol{\tau}||_{H(\mathrm{div};\Omega)} \;\leq\; \tilde{C} \, ||\boldsymbol{\tau}||_{[H^1(\Omega)]^2} \quad \forall \, \boldsymbol{\tau} \in [H^1(\Omega)]^2 \, . \tag{28}$$

Then, using the orthogonality relation satisfied by $\mathcal{P}_h$, the uniform bound (28) and the stronger continuous inf-sup condition (21), we obtain

$$\begin{aligned} \sup_{\substack{\boldsymbol{\tau}_h \in M_{1,h} \\ \boldsymbol{\tau}_h \neq 0}} \frac{-\int_\Omega v_h \operatorname{div} \boldsymbol{\tau}_h \, dx}{||\boldsymbol{\tau}_h||_{H(\mathrm{div};\Omega)}} \;&\geq\; \sup_{\substack{\boldsymbol{\tau} \in [H^1(\Omega)]^2 \\ \boldsymbol{\tau} \neq 0}} \frac{\int_\Omega v_h \operatorname{div}(E_h \boldsymbol{\tau}) \, dx}{||E_h \boldsymbol{\tau}||_{H(\mathrm{div};\Omega)}} \\ &\geq\; \frac{1}{\tilde{C}} \sup_{\substack{\boldsymbol{\tau} \in [H^1(\Omega)]^2 \\ \boldsymbol{\tau} \neq 0}} \frac{\int_\Omega v_h \operatorname{div} \boldsymbol{\tau} \, dx}{||\boldsymbol{\tau}||_{[H^1(\Omega)]^2}} \;\geq\; \beta^* \, ||v_h||_M \, , \end{aligned}$$

which proves the discrete inf-sup condition (26).

On the other hand, according to the definition of $\mathbf{B}$, we have

$$\tilde{M}_{1,h} = \{ \, \boldsymbol{\tau}_h \in M_{1,h} : \quad \int_\Omega v_h \operatorname{div} \boldsymbol{\tau}_h \, dx = 0 \; \forall \, v_h \in M_h \} \, .$$

Then, given $\boldsymbol{\tau}_h \in \tilde{M}_{1,h}$ and $T \in \mathcal{T}_h$, we may choose $v_h \in M_h$ such that $v_h|_T \equiv 1$ and $v_h \equiv 0$ in $\Omega - T$. It follows that $\int\limits_T \operatorname{div} \boldsymbol{\tau}_h \, dx = 0$ for all $T \in \mathcal{T}_h$, which, using that $\operatorname{div}(\boldsymbol{\tau}_h|_T)$ is constant, implies that $\operatorname{div}(\boldsymbol{\tau}_h|_T) = 0$. This shows that

$$\tilde{M}_{1,h} \;=\; \{\, \boldsymbol{\tau}_h \in M_{1,h} : \quad \operatorname{div} \boldsymbol{\tau}_h = 0 \quad \text{in} \quad \Omega \,\}\,.$$

Now, since $\tilde{M}_{1,h} \subseteq M_{1,h} \subseteq X_{1,h}$, we have for all $\boldsymbol{\tau}_h \in \tilde{M}_{1,h}$

$$\sup_{\substack{\boldsymbol{\zeta}_h \in X_{1,h} \\ \boldsymbol{\zeta}_h \neq 0}} \frac{[\mathbf{B}_1(\boldsymbol{\zeta}_h), \boldsymbol{\tau}_h]}{||\boldsymbol{\zeta}_h||_{X_1}} \;=\; \sup_{\substack{\boldsymbol{\zeta}_h \in X_{1,h} \\ \boldsymbol{\zeta}_h \neq 0}} \frac{-\int_\Omega \boldsymbol{\tau}_h \cdot \boldsymbol{\zeta}_h \, dx}{||\boldsymbol{\zeta}_h||_{[L^2(\Omega)]^2}}$$

$$=\; ||\boldsymbol{\tau}_h||_{[L^2(\Omega)]^2} \;=\; ||\boldsymbol{\tau}_h||_{H(\operatorname{div};\Omega)}\,,$$

which yields the discrete inf-sup condition for $\mathbf{B}_1$.

Consequently, noting that $\tilde{M}_{1,h} \subseteq \tilde{M}_1$, a direct application of the abstract Theorems 2 and 3 finishes the proof. $\square$

As a consequence of the Céa estimate given by Theorem 5, we obtain the following error bound.

Theorem 6 *Let* $((\boldsymbol{\theta}, \boldsymbol{\sigma}), u)$ *and* $((\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h), u_h)$ *be the unique solutions of* (9) *and* (25). *Assume that* $\boldsymbol{\theta}|_T \in [H^1(T)]^2 \; \forall T \in \mathcal{T}_h$, $\boldsymbol{\sigma} \in [H^1(\Omega)]^2$, $\operatorname{div} \boldsymbol{\sigma} \in H^1(\Omega)$ *and* $u \in H^1(\Omega)$. *Then, there exists* $\tilde{C} > 0$, *independent of* h, *such that the following estimate holds*

$$||((\boldsymbol{\theta}, \boldsymbol{\sigma}), u) - ((\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h), u_h)||$$

$$\leq \tilde{C}\, h \left\{ \sum_{T \in \mathcal{T}_h} ||\boldsymbol{\theta}||^2_{[H^1(T)]^2} + ||\boldsymbol{\sigma}||^2_{[H^1(\Omega)]^2} + ||\operatorname{div} \boldsymbol{\sigma}||^2_{H^1(\Omega)} \;+\; ||u||^2_{H^1(\Omega)} \right\}^{1/2}.$$

Proof. Let $E_{h,T}$ be the restriction of the equilibrium operator E_h to a given triangle T of $\mathcal{T}_h$. Then, from Theorem 5, and using again the commuting diagram property (27), we deduce that

$$||((\boldsymbol{\theta}, \boldsymbol{\sigma}), u) - ((\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h), u_h)||^2$$

$$\leq\; C \left\{ \sum_{T \in \mathcal{T}_h} ||\boldsymbol{\theta} - E_{h,T}\, \boldsymbol{\theta}||^2_{[L^2(T)]^2} \;+\; ||\boldsymbol{\sigma} - E_h\, \boldsymbol{\sigma}||^2_{[L^2(\Omega)]^2} \right.$$

$$\left. + \, ||\operatorname{div} \boldsymbol{\sigma} - \mathcal{P}_h\,(\operatorname{div} \boldsymbol{\sigma})||^2_{L^2(\Omega)} + ||u - \mathcal{P}_h u||^2_{L^2(\Omega)} \right\}.$$

Hence, the result follows from classical error estimates for interpolation and projection operators in the corresponding Sobolev spaces. We omit further details. $\square$

5 Iterative solution of the linear system

We now demonstrate that the arising linear systems of the dual-dual mixed formulation can be efficiently solved. The stiffness matrix of our system (25) is symmetric but indefinite. Indeed, as it has been shown in [11] for general systems of this structure, the stiffness matrix has n negative eigenvalues and $k+m$ positive eigenvalues where $n := \dim M_{1,h}$, $k := \dim X_{1,h}$, $m := \dim M_h$. Therefore, the conjugate gradient method may fail and we take the minimum residual method (MINRES) as iterative solver, see, e.g., [2,23]. Related results for coupled finite element/boundary element systems of the standard saddle point structure are given in [16,17,21]. The MINRES belongs to the family of Krylov subspace methods. Stable formulations of this method for symmetric and indefinite systems are given in [19,6].

If we denote the iterates by $\vec{x}_k$, $k = 0, 1, \ldots$, with corresponding residual vectors $\vec{r}_k$, there holds

$$||\vec{r}_k||_{l^2} = ||\vec{b} - \mathcal{A}\vec{x}_k||_{l^2} = \min_{\vec{x} \in \vec{x}_0 + \mathcal{K}_k(\mathcal{A}, \vec{r}_0)} ||\vec{b} - \mathcal{A}\vec{x}||_{l^2}$$

where $\mathcal{K}_k(\mathcal{A}, \vec{r}_0) = \operatorname{span}\{\vec{r}_0, \mathcal{A}\vec{r}_0, \ldots, \mathcal{A}^{k-1}\vec{r}_0\}$ denotes the Krylov subspace and $\vec{r}_0$ is the initial residual $\vec{r}_0 = \vec{b} - \mathcal{A}\vec{x}_0$. Here we use the notations $\mathcal{A}$, $\vec{b}$, and $\vec{x}$ for the stiffness matrix, the right hand side vector and the solution vector, respectively. For a symmetric, positive definite preconditioner $\mathcal{P}$ this relation becomes

$$||b - \mathcal{A}\vec{x}_k||_{\mathcal{P}^{-1}} = \min_{\vec{x} \in \vec{x}_0 + \mathcal{K}_k(\mathcal{P}^{-1}\mathcal{A}, \mathcal{P}^{-1}\vec{r}_0)} ||\vec{b} - \mathcal{A}\vec{x}||_{\mathcal{P}^{-1}}$$

where $||\vec{z}||^2_{\mathcal{P}^{-1}} = \vec{z}^T \mathcal{P}^{-1} \vec{z}$, see [24,23].

The following theorem gives an error estimate for the MINRES which is based on bounds for the spectrum of the iteration matrix.

Theorem 7 [1,23] *Let the set of eigenvalues Λ of $\mathcal{P}^{-1}\mathcal{A}$ be such that $\Lambda \subset [-a, -b] \cup [c, d]$ with $-a < -b < 0 < c < d$ and $b - a = d - c$. Then there holds*

$$\left(\frac{||\vec{r}_k||_{l^2}}{||\vec{r}_0||_{l^2}} \right)^{\frac{1}{k}} \leq 2^{\frac{1}{2k}} \left(\frac{1 - \sqrt{bc/ad}}{1 + \sqrt{bc/ad}} \right)^{\frac{1}{2}} .$$

The number of iterations of the preconditioned MINRES which are required to solve $\mathcal{A}\vec{x} = \vec{b}$ up to a given accuracy is bounded by $O(\sqrt{bc/ad})^{-1}$.

The same estimate holds for spectra of the form

$$\Lambda \subset [-a, -bh^{2\delta}] \cup [ch^{\delta}, d],$$

giving a bound like $O(h^{-3\delta/2})$ for the number of iterations.

In the following we consider a symmetric positive definite preconditioner (or just a scaling) of the general block form

$$\mathcal{P} = \begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{pmatrix} \tag{29}$$

where $P_1 \in \mathbf{R}^{k\times k}$, $P_2 \in \mathbf{R}^{n\times n}$, and $P_3 \in \mathbf{R}^{m\times m}$. Let d_0, d_1, c_0, c_1, C_0 and C_1 be positive numbers such that there holds

$$d_0 \vec{\boldsymbol{\zeta}}^T P_1 \vec{\boldsymbol{\zeta}} \le \|\boldsymbol{\zeta}\|^2_{X_1} \le d_1 \vec{\boldsymbol{\zeta}}^T P_1 \vec{\boldsymbol{\zeta}} \quad \text{for any } \boldsymbol{\zeta} \in X_{1,h}, \tag{30}$$

$$c_0 \vec{\boldsymbol{\tau}}^T P_2 \vec{\boldsymbol{\tau}} \le \|\boldsymbol{\tau}\|^2_{M_1} \le c_1 \vec{\boldsymbol{\tau}}^T P_2 \vec{\boldsymbol{\tau}} \quad \text{for any } \boldsymbol{\tau} \in M_{1,h}, \tag{31}$$

$$C_0 \vec{v}^T P_3 \vec{v} \le \|v\|^2_{M} \le C_1 \vec{v}^T P_3 \vec{v} \quad \text{for any } v \in M_h. \tag{32}$$

Here, $\vec{\boldsymbol{\zeta}}$, $\vec{\boldsymbol{\tau}}$ and $\vec{v}$ are the vectors of coefficients of $\boldsymbol{\zeta}$, $\boldsymbol{\tau}$ and v, respectively, for the chosen bases in $X_{1,h}$, $M_{1,h}$ and M_h. Depending on the norms and the scaling of basis functions some of the numbers d_0, d_1, c_0, c_1, C_0 and C_1 may depend on h.

Now let us take a closer look at the stiffness matrix. In the symmetric form the preconditioned stiffness matrix looks like

$$\begin{aligned} \tilde{\mathcal{A}} := \mathcal{P}^{-1/2}\mathcal{A}\mathcal{P}^{-1/2} &= \begin{bmatrix} P_1^{-1/2} A_1 P_1^{-1/2} & P_1^{-1/2} B_1^T P_2^{-1/2} & 0 \\ P_2^{-1/2} B_1 P_1^{-1/2} & 0 & P_2^{-1/2} B^T P_3^{-1/2} \\ 0 & P_3^{-1/2} B P_2^{-1/2} & 0 \end{bmatrix} \\ &=: \begin{bmatrix} \tilde{A}_1 & \tilde{B}_1^T & 0 \\ \tilde{B}_1 & 0 & \tilde{B}^T \\ 0 & \tilde{B} & 0 \end{bmatrix}. \end{aligned}$$

Let us denote the eigenvalues of $\tilde{\mathcal{A}}$ by

$$\mu_{-n} \leq \mu_{-n+1} \leq \cdots \leq \mu_{-1} < 0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{k+m}.$$

We also need the singular values of $\tilde{B}$,

$$0 < \eta_1 \leq \eta_2 \leq \cdots \leq \eta_m$$

and those of $\tilde{B}_1$,

$$0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_n.$$

¿From [11] we cite the following general result giving bounds on the spectrum of $\tilde{\mathcal{A}}$.

Lemma 8 [11, Lemma 2] *There holds*

$$-\frac{\sigma_n^2 + \eta_m^2}{2d_0} - \sqrt{\frac{(\sigma_n^2 + \eta_m^2)^2}{4d_0^2} + \eta_m^2} \leq \mu_{-n},$$

$$\mu_{-1} \leq \max\left\{\frac{d_1}{2} - \sqrt{\frac{d_1^2}{4} + \sigma_1^2}, -\eta_1\right\},$$

$$\min\left\{d_0, -\frac{\sigma_n^2 + \eta_1^2}{2d_0} + \sqrt{\frac{(\sigma_n^2 + \eta_1^2)^2}{4d_0^2} + \eta_1^2}\right\} \leq \mu_1,$$

$$\mu_{k+m} \leq \frac{d_1}{2} + \sqrt{\frac{d_1^2}{4} + \sigma_n^2 + \eta_m^2}$$

Our first method uses just simple scalings. That means we can neglect the preconditioner $\mathcal{P}$ by scaling the basis functions appropriately. We obtain the following result.

Theorem 9 *Let the basis functions of $X_{1,h}$, $M_{1,h}$ and M_h be scaled such that their L^∞-norms are $O(h^{-1})$, $O(1)$, and $O(h^{-1})$, respectively. Then the spectrum of $\tilde{\mathcal{A}} = \mathcal{A}$ is asymptotically covered by the two intervals*

$$[-a, -bh^2] \cup [ch, d]$$

where the positive numbers a, b, c, and d do not depend on h. Moreover, the number of iterations of the MINRES is bounded by $O(h^{-3/2})$.

Proof. Since the norms in $X_{1,h}$ and M_h are $[L^2(\Omega)]^2$ and $L^2(\Omega)$, respectively, the scaling of the basis functions in these spaces to $O(h^{-1})$ directly give bounded constants d_0, d_1, C_0, and C_1 in (30) and (32). Now, by the inverse property of the basis functions in $M_{1,h}$,

$$\|\boldsymbol{\tau}\|^2_{[L^2(\Omega)]^2} \leq \|\boldsymbol{\tau}\|_{H(\mathrm{div};\Omega)} = \|\boldsymbol{\tau}\|^2_{M_1} \leq c(1+h^{-2})\|\boldsymbol{\tau}\|^2_{[L^2(\Omega)]^2}$$

for any $\boldsymbol{\tau} \in M_{1,h}$, for a constant $c > 0$ that is independent of h. Therefore, by the scaling to $O(1)$ in $M_{1,h}$, the constants in (31) behave like $c_0 = O(h^2)$ and $c_1 = O(1)$. Here, in (30)–(32), P_1, P_2 and P_3 are simply the identities.

To estimate the singular values of B we use [11, Lemma 1] giving

$$\beta\sqrt{c_0 C_0} \leq \eta_1 \leq \eta_m \leq \|\mathbf{B}\|\sqrt{c_1 C_1},$$

where β is the discrete inf-sup constant of $\mathbf{B}$. Therefore, since $c_0 = O(h^2)$, there holds

$$\eta_1 = O(h), \quad \eta_m = O(1).$$

Furthermore, for the singular values of B_1 we obtain $\sigma_1 = O(h)$ and $\sigma_n = O(h)$. This can be seen by noting that B_1 is nothing but the Gram matrix between $X_{1,h}$ and $M_{1,h}$ and using the scalings $O(h^{-1})$ and $O(1)$, respectively, in these spaces. Therefore, the bounds for the spectrum of $\tilde{\mathcal{A}} = \mathcal{A}$ follow by making use of Lemma 8. The bound for the number of iterations of the MINRES holds by Theorem 7. $\square$

By the previous theorem we expect an increase of the iteration numbers of the MINRES when the mesh size h is reduced to improve the approximation given by the mixed finite element solution. However, a quite simple preconditioner can be used to bound the number of iterations independently of h. More precisely, we retain the scalings of the basis functions in $X_{1,h}$ and M_h, and use, instead of a simple scaling within $M_{1,h}$, a preconditioner P_2. We require that P_2 is equivalent to the $H(\mathrm{div};\Omega)$ inner product in $M_{1,h}$, i.e.

$$c_0 \vec{\boldsymbol{\tau}}^T P_2 \vec{\boldsymbol{\tau}} \leq \|\boldsymbol{\tau}\|^2_{M_1} \leq c_1 \vec{\boldsymbol{\tau}}^T P_2 \vec{\boldsymbol{\tau}} \quad \text{for any } \boldsymbol{\tau} \in M_{1,h},$$

where c_0 and c_1 do not depend on h. Then, analogously to Theorem 9, we obtain the following result.

Theorem 10 *Let the basis functions of $X_{1,h}$ and M_h be scaled such that their L^∞-norms are $O(h^{-1})$. Moreover, assume that P_2 is given such that* (31) *holds* *for constants c_0 and c_1 that are independent of h. Then the spectrum of $\tilde{\mathcal{A}}$ is bounded (away from 0 and $\pm\infty$) and the number of iterations of the preconditioned MINRES is bounded as well.*

6 Numerical results

For the computational implementation of (25) we choose the finite element subspaces according to (22), (23) and (24).

Let n and m be the number of edges and the number of triangles, respectively, of the triangulation $\mathcal{T}_h$, and let $k = 3m$. Then, we let $\{\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, ..., \boldsymbol{\theta}_k\}$, $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, ..., \boldsymbol{\sigma}_n\}$ and $\{u_1, u_2, ..., u_m\}$ be bases of $X_{1,h}$, $M_{1,h}$ and M_h, respectively. In particular, if $\{e_1, e_2, ..., e_n\}$ denote the edges of $\mathcal{T}_h$, the functions $\boldsymbol{\sigma}_j$ can be characterized by the relation

$$\boldsymbol{\sigma}_j \in M_{1,h} \quad \text{and} \quad \boldsymbol{\sigma}_j|_{e_i} \cdot \nu_i = c_j\, \delta_{ij} \quad \forall\, i, j \in \{1, 2, ..., n\}\,,$$

where the c_j are scaling constants and ν_i denotes the unit normal on the edge e_i (in a previously chosen direction). In addition, if $\{T_1, T_2, ..., T_m\}$ denote the triangles of $\mathcal{T}_h$, we can take u_i such that $u_i|_{T_j} = \hat{c}_i\, \delta_{ij}$ for all $i, j \in \{1, 2, ..., m\}$, where the $\hat{c}_i$ are also scaling constants.

It follows that there exist unknown vectors $\vec{\boldsymbol{\theta}} \in \mathbf{R}^k$, $\vec{\boldsymbol{\sigma}} \in \mathbf{R}^n$ and $\vec{u} \in \mathbf{R}^m$ such that

$$\boldsymbol{\theta}_h = \vec{\boldsymbol{\theta}} \cdot (\boldsymbol{\theta}_1, ..., \boldsymbol{\theta}_k), \quad \boldsymbol{\sigma}_h = \vec{\boldsymbol{\sigma}} \cdot (\boldsymbol{\sigma}_1, ..., \boldsymbol{\sigma}_n)$$

and

$$u_h = \vec{u} \cdot (u_1, ..., u_m)\,.$$

Thus, it is not difficult to see that (25) reduces to: *Find $\vec{\boldsymbol{\theta}} \in \mathbf{R}^k$, $\vec{\boldsymbol{\sigma}} \in \mathbf{R}^n$ and $\vec{u} \in \mathbf{R}^m$ such that*

$$\begin{bmatrix} A_1 & B_1^T & O \\ B_1 & O & B^T \\ 0 & B & O \end{bmatrix} \begin{bmatrix} \vec{\boldsymbol{\theta}} \\ \vec{\boldsymbol{\sigma}} \\ \vec{u} \end{bmatrix} = \begin{bmatrix} O \\ G_1 \\ G \end{bmatrix}, \tag{33}$$

where the matrices A_1, B_1 and B, and the vectors G_1 and G, are defined in terms of the corresponding operators and functionals given in Section 2.

More precisely, we find that

$$A_1 := (a_{ij})_{k\times k} \quad \text{with} \quad a_{ij} := \int_\Omega \kappa\, \boldsymbol{\theta}_i \cdot \boldsymbol{\theta}_j\, dx\,,$$

$$B_1 := (b_{ij}^{(1)})_{n\times k} \quad \text{with} \quad b_{ij}^{(1)} := - \int\limits_{\Omega} \boldsymbol{\sigma}_i \cdot \boldsymbol{\theta}_j \, dx \,,$$

$$B := (b_{ij})_{m\times n} \quad \text{with} \quad b_{ij} := - \int\limits_{T_i} \operatorname{div} \boldsymbol{\sigma}_j \, dx \;=\; \begin{cases} -\operatorname{div}(\boldsymbol{\sigma}_j)\,|T_i|, & \text{if } e_j \subseteq \overline{T}_i\,, \\ 0, & \text{otherwise}\,, \end{cases}$$

$$G_1 := (g_i^{(1)})_{n\times 1} \text{ with } g_i^{(1)} := - \int\limits_{\Gamma} g\,\boldsymbol{\sigma}_i \cdot \nu \, ds \;=\; \begin{cases} -\,(\boldsymbol{\sigma}_i \cdot \nu_i) \int_{e_i} g\,ds, & \text{if } e_i \subseteq \Gamma\,, \\ 0, & \text{otherwise}\,, \end{cases}$$

and

$$G := (g_i)_{m\times 1} \quad \text{with} \quad g_i := \hat{c}_i \int\limits_{T_i} f \, dx \,.$$

For our numerical example we choose $\Omega = (0,1)\times(0,1)$ and take the right hand side functions f and g in (1) such that $u(x_1,x_2) = 1/(x_1+x_2+1)$ and $\boldsymbol{\kappa} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Of course, then the regularity assumptions of Theorem 6 are satisfied. Considering a sequence of uniform triangular meshes we expect an approximation error which decays like $O(h)$. In Figure 1 the approximation errors for the individual unknowns $\boldsymbol{\theta}$, $\boldsymbol{\sigma}$, and u are given. Obviously all three errors individually decay like h, and this confirms the theoretical error bound given by Theorem 6.

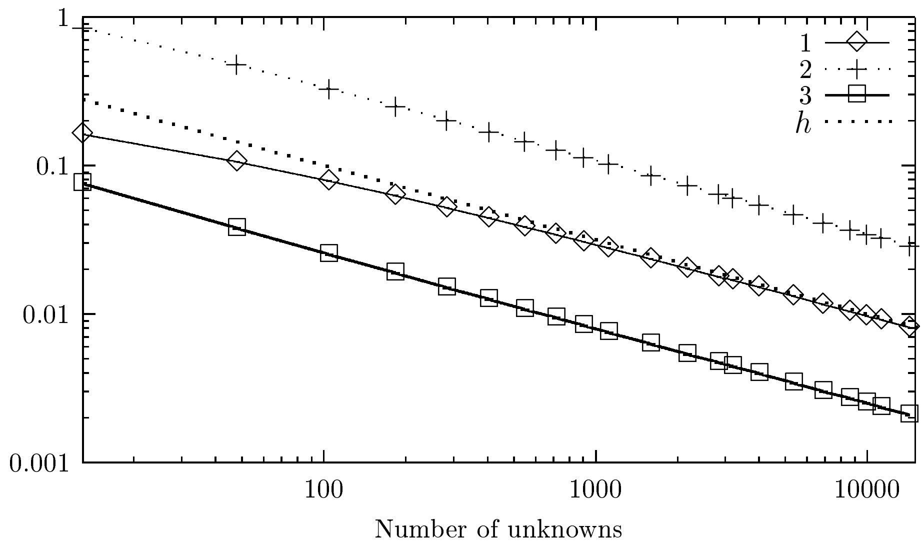

Fig. 1. Approximation errors for the individual unknowns (1: $\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_{[L^2(\Omega)]^2}$, 2: $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{H(\mathrm{div};\Omega)}$, 3: $\|u-u_h\|_{L^2(\Omega)}$).

Here we compute the entries g_i by using the quadrature rule for a triangle determined by the middle points of the three edges. In addition, the computation of $g_i^{(1)}$ is performed via the 8 point Gaussian quadrature formula on any edge of Γ. On the other hand, the approximation errors shown in Figure 1 are

$k+n+m$	h	μ_{-n}	μ_{-1}	μ_1	μ_{k+m}
13	1.00000	-2.475016	-0.132258	0.328510	2.493110
48	0.50000	-2.670952	-0.043429	0.331933	2.674731
105	0.33333	-2.741588	-0.020459	0.332542	2.743098
285	0.20000	-2.790894	-0.007541	0.332515	2.791384
553	0.14286	-2.807653	-0.003872	0.326753	2.807892
909	0.11111	-2.815268	-0.002348	0.271471	2.815409
1353	0.09091	-2.819353	-0.001574	0.226370	2.819446
1885	0.07692	-2.821795	-0.001128	0.193918	2.821861
2505	0.06667	-2.823369	-0.000848	0.169556	2.823419
3213	0.05882	-2.824443	-0.000660	0.150613	2.824481

Table 1
The extreme eigenvalues of the un-preconditioned matrix $\mathcal{A}$.

calculated on each triangle by the 7 knot quadrature rule from [22] (p. 314) (see also [3] (p. 171)).

Now we check the theoretical results of Section 5. For the un-preconditioned MINRES we scale the basis functions in $X_{1,h}$, $M_{1,h}$ and M_h to $O(h^{-1})$, $O(1)$ and $O(h^{-1})$, respectively. Due to Theorem 9 we then expect that the eigenvalues of the stiffness matrix $\mathcal{A}$ are contained in the set

$$[-a, -bh^2] \cup [ch, d]$$

where a, b, c, and d are positive numbers being independent of h. Table 1 presents the eigenvalues μ_{-n}, μ_{-1}, μ_1 and μ_{k+m} for our numerical example. Obviously, μ_{-n} and μ_{k+m} are bounded and μ_{-1} and μ_1 tend to zero if $h \to 0$. Indeed, $-\mu_{-1} = O(h^2)$ and $\mu_1 = O(h)$ as it can be seen from Figure 2 where a double logarithmic scale is used. Also from Theorem 9 we expect that the number of iterations of the MINRES is bounded by $O(h^{-3/2})$ in this case. However, Figure 3 demonstrates that, for this example, the theoretical bound on the iteration numbers is not sharp. The numerically obtained iteration numbers behave like $O(h^{-1})$. Here we stopped the iteration when the discrete l^2-norm of the initial residual was reduced by the factor 10^{-6}. It appears that the actual numbers of iterations are very large for this example. Therefore, one is forced to use a preconditioner.

For our example we simply take the $H(\mathrm{div};\Omega)$-inner product to construct the preconditioner P_2 in (29) and take the identities for P_1 and P_3 (and retain the scalings for $X_{1,h}$ and M_h from above). Of course, this preconditioner is not expensive since an application of P_2^{-1} to a vector requires only the solution

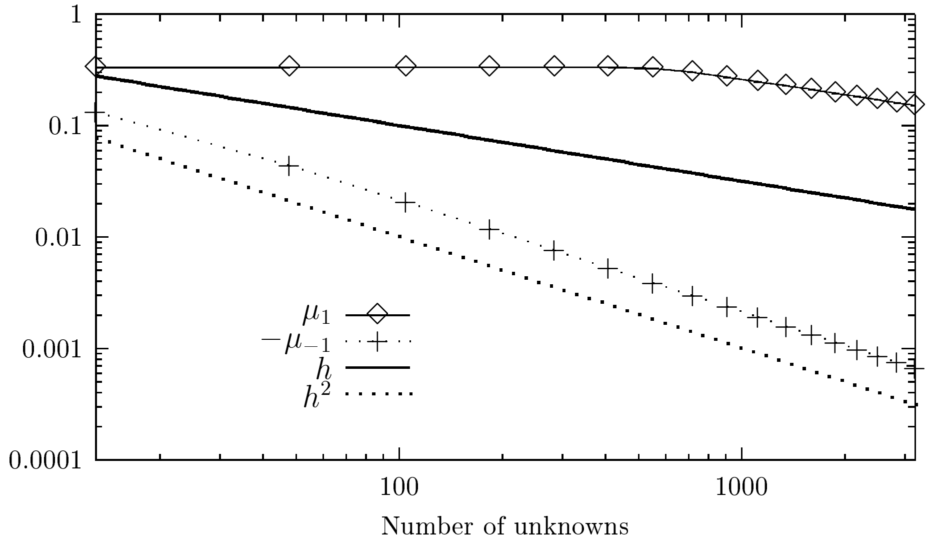

Fig. 2. Asymptotic behavior of the extreme eigenvalues μ_{-1} and μ_1 of the un-preconditioned matrix $\mathcal{A}$.

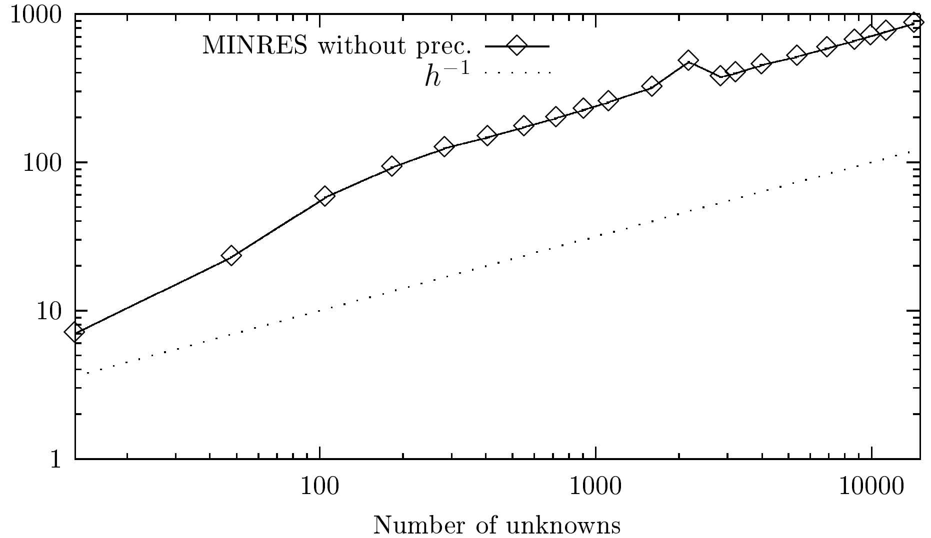

Fig. 3. Numbers of iterations of the MINRES for reducing the initial residual by 10^{-6}.

of a sparse linear system. Due to Theorem 10 we expect a bounded spectrum for the preconditioned stiffness matrix $\tilde{\mathcal{A}}$ and also bounded iteration numbers of the preconditioned MINRES. Indeed, Table 2 confirms the boundedness of the extreme eigenvalues μ_{-n}, μ_{-1}, μ_1 and μ_{k+m}. Moreover, as presented by Table 3, the iteration numbers are bounded when using the preconditioner and are very large when only using scaled basis functions (see also Figure 3).

$k+n+m$	h	μ_{-n}	μ_{-1}	μ_1	μ_{k+m}
13	1.00000	-0.700085	-0.312822	0.327477	1.366025
48	0.50000	-0.705323	-0.312840	0.331764	1.366025
105	0.33333	-0.706312	-0.312843	0.332608	1.366025
285	0.20000	-0.706820	-0.312843	0.333063	1.366025
553	0.14286	-0.706960	-0.312842	0.333193	1.366025
909	0.11111	-0.707018	-0.312840	0.333248	1.366025
1353	0.09091	-0.707048	-0.312838	0.333276	1.366025
1885	0.07692	-0.707064	-0.312835	0.333292	1.366025
2505	0.06667	-0.707075	-0.312832	0.333302	1.366025
3213	0.05882	-0.707082	-0.312828	0.333309	1.366025

Table 2
The extreme eigenvalues of the preconditioned matrix $\tilde{\mathcal{A}}$.

$k+n+m$	h	without prec.	with prec.
13	1.00000	7	7
48	0.50000	23	17
105	0.33333	58	23
285	0.20000	125	27
553	0.14286	172	27
909	0.11111	226	27
1353	0.09091	292	27
1885	0.07692	337	27
2505	0.06667	356	27
3213	0.05882	399	27
4009	0.05263	450	27
4893	0.04762	495	27
5865	0.04348	538	27
6925	0.04000	586	27
8680	0.03571	659	27
10633	0.03226	734	27
11328	0.03125	757	27
14328	0.02778	859	27

Table 3
Numbers of iterations of the MINRES for reducing the initial residual by 10^{-6}.

Acknowledgement. This research was done while N. Heuer was visiting the Departamento de Ingeniería Matemática of the Universidad de Concepción, Chile, during the period April - September, 1999. This author expresses his gratitude to the colleagues of that Department for their great hospitality.

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