Minimum Residual Iteration for a Dual-Dual Mixed Formulation of Exterior Transmission Problems *

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Abstract

We investigate the minimum residual method for symmetric, indefinite linear systems of a so-called dual-dual structure. These systems arise when using a combined dual-mixed finite element method with a Dirichlet-to-Neumann mapping to solve a class of exterior transmission problems. As a model problem we consider an elliptic equation of divergence form coupled with the Laplace equation in an unbounded region of the plane. We give abstract convergence results for the preconditioned minimum residual method for the solution of linear systems of the special dual-dual structure. For our model problem, we show that this iterative method provides an efficient solution procedure where standard preconditioners can directly be used.

Key words. Mixed finite elements, boundary elements, coupling, minimum residual method.

Mathematics subject classifications (1991). 65N30, 65N38, 65N22, 65F10

1 Introduction

The coupling of the boundary element and the finite element method is an efficient solution procedure for exterior transmission problems, see [6, 16, 19, 12]. Often, physical quantities like fluxes are more important to determine than potentials, which are approximated by standard finite elements. Therefore, for many problems, the coupling of boundary elements and mixed finite elements, where approximations to fluxes can be determined directly, is more adequate, see [4, 20, 14, 15, 11]. At the continuous level these methods are well understood for both linear and nonlinear problems. However, in the latter case the appearance of certain quotient spaces causes difficulties when one needs to define finite dimensional ansatz spaces. These difficulties can be avoided by using a so-called dual-dual mixed variational formulation that has

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been proposed in [10, 13]. This variational formulation allows for taking standard finite element subspaces. However, the linear systems that appear are not only indefinite but they also reflect the special dual-dual structure which gives rise to two-fold saddle point problems. It is important to remark that the application of primal mixed finite element and boundary element methods to solve problems in elastostatics also yields this kind of formulation, see [3].

The purpose of this paper is to provide an efficient iterative method for the solution of linear systems which possess this special structure. Since the appearing matrices are symmetric and indefinite, in which case the conjugate gradient method may fail, we propose the minimum residual method (MINRES) for their solution. We will show that a correct scaling of the basis functions being involved is important for the MINRES to converge at least moderately fast. Without employing a preconditioner the number of iterations, which are necessary to solve the system of our model problem up to a given accuracy, may grow like $O(h^{-3})$ (Theorem 7). Here, h is the mesh size for the finite element spaces. Moreover, we show that an efficient preconditioning method for this problem consists of three contributions, which are explained next for our model problem. First, one needs to take a preconditioner for the block of the stiffness matrix which belongs to the hypersingular operator (providing the Dirichlet-to-Neumann mapping). Secondly, the appearing divergence operator has to be taken into account by a preconditioner which is equivalent to the $H(\mathrm{div};\Omega)$-inner product on the respective ansatz space. Third, the basis functions of the remaining ansatz spaces must be scaled in the right manner. This will be demonstrated in §4. Provided the two individual preconditioners are optimal, the final preconditioned MINRES converges at a rate which is independent of the mesh size (Theorem 8).

The theory of the convergence of the MINRES is based on estimates of the spectrum of the stiffness matrix. Here, we give precise estimates for the spectrum of matrices with the two-fold saddle point structure. Related results, which deal with standard saddle point problems appearing when solving the Stokes problem by finite elements, are given, e.g., in [22, 31, 29]. Efficient preconditioned iterative methods for coupled finite element/boundary element systems of the same standard saddle point structure are given, e.g., in [23, 18, 17].

An outline of this paper is as follows. In §2 we introduce our model problem and its discretization which is of the aforementioned dual-dual structure. In §3 we investigate the preconditioned minimum residual method for the arising linear systems in an abstract form. We give the main results concerning the spectrum of matrices with dual-dual structure and the convergence of the MINRES (Theorems 5, 6). A simple numerical example underlining the sharpness of these results is discussed in §3.3. Eventually, in §4 we apply the abstract results to our model problem and show that, without preconditioner, the number of iterations of the MINRES grows at most like $O(h^{-3})$ and that it is bounded if an appropriate preconditioner is applied.

2 The exterior transmission problem

2.1 The continuous formulation

Let Ω_0 be a bounded and simply connected domain in $\mathbf{R}^2$ with Lipschitz continuous boundary Γ_0. Further, let Ω_1 be the annular domain bounded by Γ_0 and another Lipschitz continuous closed curve Γ_1 whose interior region contains Ω_0. Then, given a matrix function $\kappa(x) \in \mathbf{R}^{2\times 2}$ which is uniformly bounded and positive definite almost everywhere in Ω_1 with components $\in L^2(\Omega_1)$, and given a function $f_1 \in L^2(\Omega_1)$, the exterior transmission problem is as follows: *Find* $u_1 \in H^1(\Omega_1)$ *and* $u_2 \in H^1_{loc}(\mathbf{R}^2 \setminus \bar{\Omega}_0 \cup \bar{\Omega}_1)$ *such that*

$$\begin{array}{rcl} u_1 &=& 0 \quad \text{on } \Gamma_0, \\ -\operatorname{div}(\kappa \nabla u_1) &=& f_1 \quad \text{in } \Omega_1, \\ u_1 &=& u_2 \quad \text{on } \Gamma_1, \\ (\kappa \nabla u_1)\cdot \mathbf{n} &=& \frac{\partial u_2}{\partial \mathbf{n}} \quad \text{on } \Gamma_1, \\ -\Delta u_2 &=& 0 \quad \text{in } \mathbf{R}^2 \setminus \bar{\Omega}_0 \cup \bar{\Omega}_1, \\ u_2(x) &=& O(1) \quad \text{as } |x| \to \infty. \end{array} \tag{1}$$

Here, $\mathbf{n}$ denotes the unit outward normal to $\partial\Omega_1$.

Problem (1) is the linear case of a class of transmission problems that appear, e.g., in the computation of magnetic fields of electromagnetic devices, in some subsonic flow and fluid mechanics problems, and also in steady state heat conduction. In all these problems the fluxes themselves are of great interest and should be approximated directly. This can be realized by using mixed finite elements. In this paper we will concentrate on the linear case. The main objective is to investigate the minimum residual method as solver for the arising linear systems. This will be done in §3 in a more general framework. However, in the nonlinear case the efficient solution of linearized systems is even more important as they usually appear within an outer iteration.

Following [13] we reformulate the problem (1) such that mixed finite elements can be used to approximate the flux, whereas the exterior problem is dealt with by a Dirichlet-to-Neumann mapping provided by a hypersingular integral operator. To this end we first introduce a sufficiently large circle Γ with center at the origin such that its interior region covers $\bar{\Omega}_0 \cup \bar{\Omega}_1$. The annular region bounded by Γ_1 and Γ is denoted by Ω_2 and we define $\Omega := \Omega_1 \cup \Gamma_1 \cup \Omega_2$. Now let

$$u := \left\{ \begin{array}{lll} u_1 & \text{in} & \Omega_1 \\ u_2 & \text{in} & \Omega_2 \end{array} \right. , \quad \text{the flux variable } \boldsymbol{\sigma} := \left\{ \begin{array}{lll} \kappa\nabla u_1 & \text{in} & \Omega_1 \\ \nabla u_2 & \text{in} & \Omega_2 \end{array} \right. ,$$

the global data $f := \left\{ \begin{array}{lll} f_1 & \text{in} & \Omega_1 \\ 0 & \text{in} & \Omega_2 \end{array} \right.$, and let us introduce the auxiliary unknowns

$$\boldsymbol{\theta} := \nabla u \quad \text{in } \Omega \quad \text{and} \quad \xi := u|_\Gamma.$$

At this point we remark that in the standard mixed finite element method one just introduces the auxiliary unknown given by the flux, that is $\boldsymbol{\sigma} = \kappa\nabla u$, and then performs the integration by parts procedure to the relation $\nabla u = \kappa^{-1}\boldsymbol{\sigma}$. With the present approach, we avoid the inversion of the matrix κ by introducing the further unknown $\boldsymbol{\theta} = \nabla u$ (or $\boldsymbol{\theta}$ = strain tensor in elasticity), which has also been shown to be a very suitable way of dealing with nonlinear constitutive equations, say $\boldsymbol{\sigma} = \kappa(\nabla u)$, see, e.g., [14, 15, 13].

Now, to deal with the exterior problem we use the Dirichlet-to-Neumann mapping

$$\frac{\partial u}{\partial \boldsymbol{\nu}} = -2\mathbf{W}(u|_\Gamma) \quad \text{on } \Gamma, \text{ i.e.,} \quad \boldsymbol{\sigma}\cdot\boldsymbol{\nu} = -2\mathbf{W}\xi \quad \text{on } \Gamma.$$

Here, $\boldsymbol{\nu}$ is the unit outward normal to Γ and $\mathbf{W}$ is the hypersingular boundary integral operator associated with the Laplacian,

$$\mathbf{W}\phi(x) := \frac{1}{2\pi}\frac{\partial}{\partial \boldsymbol{\nu}_x}\int_\Gamma \phi(y)\frac{\partial}{\partial \boldsymbol{\nu}_y}\log|x-y|\,ds_y, \quad \text{for } x\in\Gamma, \phi\in H^{1/2}(\Gamma).$$

It is well-known that $\mathbf{W}: H^{1/2}(\Gamma)\to H^{-1/2}(\Gamma)$ is continuous and positive definite on $H_0^{1/2}(\Gamma)$, i.e., there exists $\delta_0>0$ such that

$$\langle \phi, \mathbf{W}\phi\rangle_{L^2(\Gamma)} \geq \delta_0\|\phi\|^2_{H^{1/2}(\Gamma)} \quad \text{for all } \phi\in H_0^{1/2}(\Gamma), \tag{2}$$

see [7], where

$$H_0^{1/2}(\Gamma) := \{\phi\in H^{1/2}(\Gamma):\ \langle\phi,1\rangle_{L^2(\Gamma)} = 0\}.$$

Since $\mathbf{W}1 = 0$ on Γ it suffices to look for the unknown ξ in the space $H_0^{1/2}(\Gamma)$. Moreover, by the decay condition of u at infinity, $\boldsymbol{\sigma}\in H_0(\mathrm{div};\Omega)$ where

$$H_0(\mathrm{div};\Omega) := \{\boldsymbol{\tau}\in H(\mathrm{div};\Omega):\ \langle 1, \boldsymbol{\tau}\cdot\boldsymbol{\nu}\rangle_{L^2(\Gamma)} = 0\}$$

and $H(\mathrm{div};\Omega)$ is the space of functions $\boldsymbol{\tau}\in L^2(\Omega)\times L^2(\Omega)$ such that $\mathrm{div}\,\boldsymbol{\tau}\in L^2(\Omega)$. $H(\mathrm{div};\Omega)$ is a Hilbert space with inner product $\langle\cdot,\cdot\rangle_{L^2(\Omega)\times L^2(\Omega)} + \langle \mathrm{div}\,\cdot, \mathrm{div}\,\cdot\rangle_{L^2(\Omega)}$.

Now the transmission problem (1) can be reformulated as the following boundary value problem in $\bar\Omega$: *Find* $(\boldsymbol{\theta},\xi,\boldsymbol{\sigma},u)\in(L^2(\Omega))^2\times H_0^{1/2}(\Gamma)\times H_0(\mathrm{div};\Omega)\times L^2(\Omega)$ *such that*

$$\begin{array}{rclcrcl}
u &=& 0 \quad \text{on } \Gamma_0, & & \boldsymbol{\theta} &=& \nabla u \quad \text{in } \Omega,\\
\boldsymbol{\sigma} &=& \left\{\begin{array}{ll}\kappa\boldsymbol{\theta} & \text{in } \Omega_1\\ \boldsymbol{\theta} & \text{in } \Omega_2\end{array}\right., & & \mathrm{div}\,\boldsymbol{\sigma} &=& -f \quad \text{in } \Omega,\\
\boldsymbol{\sigma}\cdot\boldsymbol{\nu} &=& -2\mathbf{W}\xi \quad \text{on } \Gamma, & & u &=& \xi \quad \text{on } \Gamma.
\end{array} \tag{3}$$

We note that this formulation is very close to that proposed in [14, 15] and [11]. However, instead of using complicated quotient spaces as introduced in [14], we consider the following variational formulation proposed in [13]: *Find* $(\boldsymbol{\theta},\xi,\boldsymbol{\sigma},u)\in(L^2(\Omega))^2\times H_0^{1/2}(\Gamma)\times H_0(\mathrm{div};\Omega)\times L^2(\Omega)$ *such that*

$$\begin{array}{rcl}
\int_{\Omega_1}(\kappa\boldsymbol{\theta})\cdot\boldsymbol{\zeta}\,dx + \int_{\Omega_2}\boldsymbol{\theta}\cdot\boldsymbol{\zeta}\,dx + 2\langle\phi,\mathbf{W}\xi\rangle_{L^2(\Gamma)} - \int_\Omega\boldsymbol{\sigma}\cdot\boldsymbol{\zeta}\,dx + \langle\phi,\boldsymbol{\sigma}\cdot\boldsymbol{\nu}\rangle_{L^2(\Gamma)} &=& 0,\\
-\int_\Omega\boldsymbol{\theta}\cdot\boldsymbol{\tau}\,dx + \langle\xi,\boldsymbol{\tau}\cdot\boldsymbol{\nu}\rangle_{L^2(\Gamma)} - \int_\Omega u\,\mathrm{div}\,\boldsymbol{\tau}\,dx &=& 0,\\
-\int_\Omega v\,\mathrm{div}\,\boldsymbol{\sigma}\,dx &=& \int_\Omega fv\,dx,
\end{array} \tag{4}$$

for all $(\boldsymbol{\zeta},\phi,\boldsymbol{\tau},v)\in(L^2(\Omega))^2\times H_0^{1/2}(\Gamma)\times H_0(\mathrm{div};\Omega)\times L^2(\Omega)$.

Let us rewrite the variational form (4) as a system of operator equations. We use the notations $X_1 := (L^2(\Omega))^2 \times H_0^{1/2}(\Gamma)$, $M_1 := H_0(\mathrm{div};\Omega)$, $X := X_1 \times M_1$, $M := L^2(\Omega)$ and denote $\mathbf{t} := (\boldsymbol{\theta}, \xi)^T$, $\mathbf{s} := (\boldsymbol{\zeta}, \phi)^T \in X_1$. Moreover, we define the operators $\mathbf{A}_1 : X_1 \to X_1'$, $\mathbf{B}_1 : X_1 \to M_1'$, $\mathbf{A} : X \to X'$, $\mathbf{B} : M_1 \to M'$, and the functional $\mathbf{H} \in M'$ as follows:

$$[\mathbf{A}_1(\mathbf{t}), \mathbf{s}] := \int_{\Omega_1} (\kappa \boldsymbol{\theta}) \cdot \boldsymbol{\zeta} \, dx + \int_{\Omega_2} \boldsymbol{\theta} \cdot \boldsymbol{\zeta} \, dx + 2\langle \phi, \mathbf{W}\xi \rangle_{L^2(\Gamma)},$$

$$[\mathbf{B}_1(\mathbf{t}), \boldsymbol{\tau}] := -\int_{\Omega} \boldsymbol{\theta} \cdot \boldsymbol{\tau} \, dx + \langle \xi, \boldsymbol{\tau} \cdot \boldsymbol{\nu} \rangle_{L^2(\Gamma)},$$

$$[\mathbf{A}(\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{s}, \boldsymbol{\tau})] := [\mathbf{A}_1(\mathbf{t}), \mathbf{s}] + [\mathbf{B}_1(\mathbf{s}), \boldsymbol{\sigma}] + [\mathbf{B}_1(\mathbf{t}), \boldsymbol{\tau}],$$

$$[\mathbf{B}(\boldsymbol{\sigma}), v] := -\int_{\Omega} v \,\mathrm{div}\, \boldsymbol{\sigma} \, dx,$$

and

$$[\mathbf{H}, v] := \int_{\Omega} f v \, dx$$

for all $(\mathbf{t}, \boldsymbol{\sigma})$, $(\mathbf{s}, \boldsymbol{\tau}) \in X$ and for all $v \in M$. Here, $[\cdot,\cdot]$ stands for the duality pairing induced by the operators under consideration. Further, let $\mathbf{B}_1^* : M_1 \to X_1'$ and $\mathbf{B}^* : M \to M_1'$ denote the adjoints of $\mathbf{B}_1$ and $\mathbf{B}$, respectively. Then, the variational problem (4) can be formulated as the following system of operator equations: *Find* $(\mathbf{t}, \boldsymbol{\sigma}, u) \in X_1 \times M_1 \times M$ *such that*

$$\begin{pmatrix} \mathbf{A}_1 & \mathbf{B}_1^* & 0 \\ \mathbf{B}_1 & 0 & \mathbf{B}^* \\ 0 & \mathbf{B} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \boldsymbol{\sigma} \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \mathbf{H} \end{pmatrix}. \tag{5}$$

The problem (5) is called dual-dual mixed formulation of (3) since the operator

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{B}_1^* \\ \mathbf{B}_1 & 0 \end{pmatrix}$$

and the full system of operators

$$\begin{pmatrix} \mathbf{A} & (0\ \mathbf{B})^* \\ (0\ \mathbf{B}) & 0 \end{pmatrix}$$

both have a dual structure.

The problem (5) is uniquely solvable:

Theorem 1 [13, Theorem 5.5] *There exists a unique solution* $(\mathbf{t}, \boldsymbol{\sigma}, u) \in X_1 \times M_1 \times M$ *of* (5).

2.2 The Galerkin scheme

We now follow again [13] and introduce finite element spaces to approximately solve (5). The arising linear systems inherit their structure from the continuous problem and are indefinite and ill-conditioned. In §3 we will investigate the preconditioned minimum residual method for systems of this dual-dual type structure.

For a positive integer N, we let $0 = t_0 < t_1 < \cdots < t_N = 2\pi$ be a uniform partition of $[0, 2\pi]$ with $t_{j+1} - t_j = \tilde{h} = \frac{2\pi}{N}$ ($j \in \{0, 1, ..., N-1\}$). Let $\mathbf{z} : [0, 2\pi] \to \Gamma$ be the usual parameterization of the circle Γ given by $\mathbf{z}(t) := r\,(\cos(t), \sin(t))^T$ for all $t \in [0, 2\pi]$. By $\Omega_{\tilde{h}}$ we denote the annular domain bounded by Γ_0 and the polygonal line $\Gamma_{\tilde{h}}$ whose vertices are $\{\mathbf{z}(t_1), \mathbf{z}(t_2), ..., \mathbf{z}(t_N)\}$.

Let $\mathcal{T}_{\tilde{h}}$ be a quasi-uniform regular triangulation of $\Omega_{\tilde{h}}$ by triangles T of diameter h_T with mesh size $h := \sup_{T \in \mathcal{T}_{\tilde{h}}} h_T$. We assume that for each $T \in \mathcal{T}_{\tilde{h}}$, either $T \subset \bar{\Omega}_1$ or $T \subset \bar{\Omega}_2$. Then, we replace each triangle $T \in \mathcal{T}_{\tilde{h}}$ with one side along $\Gamma_{\tilde{h}}$ by the corresponding curved triangle with one side along Γ. In this way we obtain a triangulation $\mathcal{T}_h$ of $\bar{\Omega}$ consisting of straight and curved triangles.

Next, we consider the canonical triangle with vertices $\hat{P}_1 = (0,0)^T$, $\hat{P}_2 = (1,0)^T$, and $\hat{P}_3 = (0,1)^T$ as a reference triangle $\hat{T}$, and introduce a family of bijective mappings $\{F_T\}_{T \in \mathcal{T}_h}$, such that $F_T(\hat{T}) = T$. In particular, if T is a straight triangle of $\mathcal{T}_h$, then F_T is the well-known invertible affine mapping defined by $F_T(\hat{x}) = B_T\,\hat{x} + b_T$ where $B_T \in \mathbf{R}^{2\times 2}$ and $b_T \in \mathbf{R}^2$ depend on the vertices of T.

We now consider the lowest order Raviart-Thomas spaces. To this end let

$$\mathcal{RT}_0(\hat{T}) := \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} \right\}, \tag{6}$$

and for each triangle $T \in \mathcal{T}_h$ we define

$$\mathcal{RT}_0(T) := \{\boldsymbol{\tau} :\ \boldsymbol{\tau} = J(F_T)^{-1} D(F_T) \hat{\boldsymbol{\tau}} \circ F_T^{-1}, \hat{\boldsymbol{\tau}} \in \mathcal{RT}_0(\hat{T})\},$$

where $J(F_T)$ and $D(F_T)$ denote the Jacobian and the Frêchet differential, respectively, of the mapping F_T.

Then, the finite element subspaces for the unknowns $\boldsymbol{\theta}$ and $\boldsymbol{\sigma}$ are as follows:

$$X_{1,h}^{\boldsymbol{\theta}} := \left\{ \boldsymbol{\tau} \in (L(\Omega))^2 :\ \boldsymbol{\tau}|_T \in \mathcal{RT}_0(T) \quad \forall T \in \mathcal{T}_h \right\}$$

and

$$M_{1,h} := X_{1,h}^{\boldsymbol{\theta}} \cap H_0(\text{div}; \Omega).$$

Note that $X_{1,h}^{\boldsymbol{\theta}}$ does not require continuity of the normal components across the sides of the triangles while $M_{1,h}$ certainly does.

Next, we define

$$H_h(0, 2\pi) := \Big\{ \tilde{\lambda} :\ [0, 2\pi] \to \mathbf{R}, \quad \tilde{\lambda} \text{ is continuous and } 2\pi\text{-periodic},$$

$$\tilde{\lambda}|_{[t_{j-1}, t_j]} \in \mathbf{P}_1(t_{j-1}, t_j) \quad \forall j \in \{1, ..., N\}, \quad \int_0^{2\pi} \tilde{\lambda}(t)\, dt = 0 \Big\}$$

and take the following finite element subspace for the unknown ξ:

$$X^{\xi}_{1,h} := \{\lambda :\ \Gamma \to \mathbf{R}, \quad \lambda = \tilde{\lambda} \circ \mathbf{z}^{-1},\ \tilde{\lambda} \in H_h(0, 2\pi)\} \subset H^{1/2}_0(\Gamma).$$

Here and in the following, $\mathbf{P}_k(S)$ denotes the space of polynomials of degree $\leq k$ on a domain $S \subset \mathbf{R}$ or $\mathbf{R}^2$.

Finally, let

$$X_{1,h} := X^{\boldsymbol{\theta}}_{1,h} \times X^{\xi}_{1,h}$$

and consider piecewise constant functions as the finite element subspace for the unknown u, i.e.

$$M_h := \{v \in L^2(\Omega) :\ v|_T \in \mathbf{P}_0(T) \quad \forall T \in \mathcal{T}_h\}.$$

Then the Galerkin scheme for the continuous problem (5) reads as follows: *Find* $(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \in X_{1,h} \times M_{1,h} \times M_h$ *such that*

$$\begin{array}{lllll} [\mathbf{A}_1(\mathbf{t}_h), \mathbf{s}] & + [\mathbf{B}_1(\mathbf{s}), \boldsymbol{\sigma}_h] & & = & 0, \\ [\mathbf{B}_1(\mathbf{t}_h), \boldsymbol{\tau}] & & + [\mathbf{B}(\boldsymbol{\tau}), u_h] & = & 0, \\ & [\mathbf{B}(\boldsymbol{\sigma}_h), v] & & = & [\mathbf{H}, v] \end{array} \tag{7}$$

for all $(\mathbf{s}, \boldsymbol{\tau}, v) \in X_h \times M_h$.

The main results from [13] are collected by the following theorem.

Theorem 2 [13, Theorems 5.9, 5.10]
There exists a unique solution $((\boldsymbol{\theta}_h, \xi_h), \boldsymbol{\sigma}_h, u_h) := (\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \in X_{1,h} \times M_{1,h} \times M_h$ *of* (7). *Let* $(\boldsymbol{\theta}, \xi, \boldsymbol{\sigma}, u)$ *be the unique solution of* (4) *and assume that* $\boldsymbol{\theta}|_T \in (H^1(T))^2\ \forall T \in \mathcal{T}_h$, $\xi \in H^{3/2}(\Gamma)$, $\boldsymbol{\sigma} \in (H^1(\Omega))^2$, $\operatorname{div} \boldsymbol{\sigma} \in H^1(\Omega)$, *and* $u \in H^1(\Omega)$. *Then, there exists a positive constant* C *which is independent of* h *such that*

$$\begin{aligned} &\|(\boldsymbol{\theta}, \xi, \boldsymbol{\sigma}, u) - (\boldsymbol{\theta}_h, \xi_h, \boldsymbol{\sigma}_h, u_h)\| \\ &\quad \leq Ch \left\{ \sum_{T \in \mathcal{T}_h} \|\boldsymbol{\theta}\|^2_{(H^1(T))^2} + \|\xi\|^2_{H^{3/2}(\Gamma)} + \|\boldsymbol{\sigma}\|^2_{(H^1(\Omega))^2} + \|\operatorname{div} \boldsymbol{\sigma}\|^2_{H^1(\Omega)} + \|u\|^2_{H^1(\Omega)} \right\}^{1/2}. \end{aligned}$$

3 The preconditioned minimum residual method

In this section we deal with the solution of the discrete problem (7), in short form $\mathcal{A}\vec{x} = \vec{b}$. The stiffness matrix is symmetric and indefinite and, therefore, we use the preconditioned minimum residual method (MINRES), see [2, 29]. We do that in a slightly more general abstract framework than given in the preceding section. This framework is given in §3.1 where all the essential assumptions are collected. These assumptions ensure unique solvability of the continuous and the discrete problems on one hand and, on the other hand, they also guarantee rapid convergence of the preconditioned MINRES. Precise estimates for the convergence of the MINRES are given in §3.2.

First let us recall important properties of the MINRES. The minimum residual method belongs to the family of Krylov subspace methods. Stable formulations of this method for symmetric and indefinite systems are given in [21, 5]. If we denote the iterates by $\vec{x}_k$, $k = 0, 1, \ldots$, with corresponding residual verctors $\vec{r}_k$, there holds

$$\|\vec{r}_k\|_{l^2} = \|\vec{b} - \mathcal{A}\vec{x}_k\|_{l^2} = \min_{\vec{x} \in \vec{x}_0 + \mathcal{K}_k(\mathcal{A}, \vec{r}_0)} \|\vec{b} - \mathcal{A}\vec{x}\|_{l^2}$$

where $\mathcal{K}_k(\mathcal{A}, \vec{r}_0) = \operatorname{span}\{\vec{r}_0, \mathcal{A}\vec{r}_0, \ldots, \mathcal{A}^{k-1}\vec{r}_0\}$ denotes the Krylov subspace and $\vec{r}_0$ is the initial residual $\vec{r}_0 = \vec{b} - \mathcal{A}\vec{x}_0$. For a symmetric, positive definite preconditioner $\mathcal{P}$ this relation becomes

$$\|b - \mathcal{A}\vec{x}_k\|_{\mathcal{P}^{-1}} = \min_{\vec{x} \in \vec{x}_0 + \mathcal{K}_k(\mathcal{P}^{-1}\mathcal{A}, \mathcal{P}^{-1}\vec{r}_0)} \|\vec{b} - \mathcal{A}\vec{x}\|_{\mathcal{P}^{-1}}$$

where $\|\vec{z}\|^2_{\mathcal{P}^{-1}} = \vec{z}^T \mathcal{P}^{-1} \vec{z}$, see [31, 29].

The following theorem gives an error estimate for the MINRES which is based on bounds for the spectrum of the iteration matrix. Therefore, it is clear that, in order to study the convergence of the MINRES for our model problem, we need to investigate the spectrum of matrices with the special dual-dual structure. This will be done in §3.2.

Theorem 3 [1] *Let the set of eigenvalues Λ of $\mathcal{P}^{-1}\mathcal{A}$ be such that $\Lambda \subset [-a, -b] \cup [c, d]$ with $-a < -b < 0 < c < d$ and $b - a = d - c$. Then there holds*

$$\left(\frac{\|\vec{r}_k\|_{l^2}}{\|\vec{r}_0\|_{l^2}} \right)^{\frac{1}{k}} \leq 2^{\frac{1}{2k}} \left(\frac{1 - \sqrt{bc/ad}}{1 + \sqrt{bc/ad}} \right)^{\frac{1}{2}}.$$

The number of iterations of the preconditioned MINRES which are required to solve $\mathcal{A}\vec{x} = \vec{b}$ up to a given accuracy is bounded by $O(\sqrt{bc/ad})^{-1}$.

Remark. *We note that the convergence of the minimum residual method has been studied also for more general inclusion sets Λ. For instance, results for the case where the interior bounds $-b$ and c tend to zero (if $h \to 0$) at different rates are given in* [29, 30].

3.1 Abstract setting of the problem

The abstract setting of the continuous problem we deal with is as follows. Let X_1, M_1, M be Hilbert spaces. We consider linear bounded operators $\mathbf{A}_1 : X_1 \to X_1'$ and $\mathbf{B}_1 : X_1 \to M_1'$, $\mathbf{B} : M_1 \to M'$ with adjoints $\mathbf{B}_1^* : M_1 \to X_1'$ and $\mathbf{B}^* : M \to M_1'$, respectively. Moreover, $\mathbf{A}_1$ is assumed to be symmetric positive definite. Therefore, there exist positive constants α_1, α_0, γ_1, and γ such that there holds

$$\|\mathbf{A}_1(\mathbf{s})\|_{X_1'} \leq \alpha_1 \|\mathbf{s}\|_{X_1} \quad \text{for any } \mathbf{s} \in X_1, \tag{8}$$

$$[\mathbf{A}_1(\mathbf{s}), \mathbf{s}] \geq \alpha_0 \|\mathbf{s}\|^2_{X_1} \quad \text{for any } \mathbf{s} \in X_1, \tag{9}$$

$$[\mathbf{B}_1(\mathbf{s}), \boldsymbol{\tau}] \leq \gamma_1 \|\mathbf{s}\|_{X_1} \|\boldsymbol{\tau}\|_{M_1} \quad \text{for any } \mathbf{s} \in X_1, \boldsymbol{\tau} \in M_1, \tag{10}$$

and

$$[\mathbf{B}(\boldsymbol{\tau}), v] \leq \gamma \|\boldsymbol{\tau}\|_{M_1} \|v\|_M \quad \text{for any } \boldsymbol{\tau} \in M_1, v \in M. \tag{11}$$

Here, $[\cdot, \cdot]$ denotes the duality pairing.

We want to solve the following variational problem: *Given* $(\mathbf{F}, \mathbf{G}, \mathbf{H}) \in X_1' \times M_1' \times M'$ *find* $(\mathbf{t}, \boldsymbol{\sigma}, u) \in X_1 \times M_1 \times M$ *such that*

$$\begin{pmatrix} \mathbf{A}_1 & \mathbf{B}_1^* & 0 \\ \mathbf{B}_1 & 0 & \mathbf{B}^* \\ 0 & \mathbf{B} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \boldsymbol{\sigma} \\ u \end{pmatrix} = \begin{pmatrix} \mathbf{F} \\ \mathbf{G} \\ \mathbf{H} \end{pmatrix}. \tag{12}$$

We assume that this continuous problem is uniquely solvable. This can be guaranteed by inf-sup conditions for $\mathbf{B}_1$ and $\mathbf{B}$, see [10].

The Galerkin method for approximately solving (12) is as follows. *Given finite dimensional subspaces* $X_{1,h} \times M_{1,h} \times M_h \subset X_1 \times M_1 \times M$ *find* $(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \in X_{1,h} \times M_{1,h} \times M_h$ *such that*

$$\left[\begin{pmatrix} \mathbf{A}_1 & \mathbf{B}_1^* & 0 \\ \mathbf{B}_1 & 0 & \mathbf{B}^* \\ 0 & \mathbf{B} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t}_h \\ \boldsymbol{\sigma}_h \\ u_h \end{pmatrix}, \begin{pmatrix} \mathbf{s} \\ \boldsymbol{\tau} \\ v \end{pmatrix} \right] = \left[\begin{pmatrix} \mathbf{F} \\ \mathbf{G} \\ \mathbf{H} \end{pmatrix}, \begin{pmatrix} \mathbf{s} \\ \boldsymbol{\tau} \\ v \end{pmatrix} \right] \tag{13}$$

for any $(\mathbf{s}, \boldsymbol{\tau}, v) \in X_{1,h} \times M_{1,h} \times M_h$.

The main properties ensuring unique solvability of the discrete Galerkin systems are discrete inf-sup conditions for $\mathbf{B}$ and $\mathbf{B}_1$. Let

$$\tilde{M}_{1,h} := \{\boldsymbol{\tau} \in M_{1,h} : \ [\mathbf{B}(\boldsymbol{\tau}), v] = 0 \ \forall v \in M_h\}.$$

We assume that there exist positive constants β and β_1, not depending on h, such that

$$\sup_{\boldsymbol{\tau} \in M_{1,h} \setminus \{0\}} \frac{[\mathbf{B}(\boldsymbol{\tau}), v]}{\|\boldsymbol{\tau}\|_X} \geq \beta \|v\|_M \quad \text{for all } v \in M_h, \tag{14}$$

and

$$\sup_{\mathbf{s} \in X_{1,h} \setminus \{0\}} \frac{[\mathbf{B}_1(\mathbf{s}), \boldsymbol{\tau}]}{\|\mathbf{s}\|_{X_1}} \geq \beta_1 \|\boldsymbol{\tau}\|_{M_1} \quad \text{for all } \boldsymbol{\tau} \in \tilde{M}_{1,h}. \tag{15}$$

Moreover, we require that $\mathbf{B}_1$ satisfies an inf-sup condition on the whole space $M_{1,h}$ where the inf-sup constant may depend on h:

$$\sup_{\mathbf{s} \in X_{1,h} \setminus \{0\}} \frac{[\mathbf{B}_1(\mathbf{s}), \boldsymbol{\tau}]}{\|\mathbf{s}\|_{X_1}} \geq \beta_0 \|\boldsymbol{\tau}\|_{M_1} \quad \text{for all } \boldsymbol{\tau} \in M_{1,h} \tag{16}$$

where $\beta_0 = \beta_0(h) > 0$.

Theorem 4 [10, Theorem 3.2] *Let the assumptions* (8) *through* (11) *and* (14), (15) *hold. Then, for each* $(\mathbf{F}, \mathbf{G}, \mathbf{H}) \in X_1' \times M_1' \times M'$ *there exists a unique solution* $(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \in X_{1,h} \times M_{1,h} \times M_h$ *of* (13).

3.2 MINRES for linear systems of dual-dual structure

In the following we use plain symbols for matrices which correspond to operators with the analogous bold symbol. For a coefficient vector which belongs to a certain function we use the same symbol as for the function and add an arrow for clarity.

The structure of the linear system we have to deal with looks like

$$\mathcal{A}\begin{pmatrix} \vec{\mathbf{t}}_h \\ \vec{\boldsymbol{\sigma}}_h \\ \vec{u}_h \end{pmatrix} := \begin{pmatrix} A_1 & B_1^T & 0 \\ B_1 & 0 & B^T \\ 0 & B & 0 \end{pmatrix}\begin{pmatrix} \vec{\mathbf{t}}_h \\ \vec{\boldsymbol{\sigma}}_h \\ \vec{u}_h \end{pmatrix} = \begin{pmatrix} F \\ G \\ H \end{pmatrix}. \tag{17}$$

Here, $A_1 \in \mathbf{R}^{L\times L}$ is symmetric positive definite and the matrices $B_1 \in \mathbf{R}^{M\times L}$ and $B \in \mathbf{R}^{N\times M}$ have maximum rank. As usual, B_1^T and B^T denote the transposes of B_1 and B, respectively. The solution vector $(\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{u}_h)$ corresponds to a solution $(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h)$ within the Hilbert space $X_{1,h} \times M_{1,h} \times M_h$.

The matrix $\mathcal{A}$ is symmetric and indefinite as can be seen by the following calculation. Defining the non-singular matrices

$$T_1 := \begin{pmatrix} I & 0 \\ -B_1A_1^{-1} & I \end{pmatrix} \quad \text{and} \quad T := \begin{pmatrix} I & 0 \\ -(0\ B)(T_1AT_1^T)^{-1} & I \end{pmatrix}$$

with

$$A := \begin{pmatrix} A_1 & B_1^T \\ B_1 & 0 \end{pmatrix}$$

we obtain

$$T\begin{pmatrix} T_1 & 0 \\ 0 & I \end{pmatrix}\mathcal{A}\begin{pmatrix} T_1^T & 0 \\ 0 & I \end{pmatrix}T^T = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & -B_1A_1^{-1}B_1^T & 0 \\ 0 & 0 & B(B_1A_1^{-1}B_1^T)^{-1}B^T \end{pmatrix}.$$

Here we note that, since A_1 is positive definite and since $\mathbf{B}_1$ satisfies the inf-sup condition (16), the matrix $B_1A_1^{-1}B_1^T$ is positive definite. Moreover, by (14), also $B(B_1A_1^{-1}B_1^T)^{-1}B^T$ is positive definite. Therefore, by the above transformation, Sylvester's law of inertia yields the existence of $L+N$ positive and M negative eigenvalues for $\mathcal{A}$.

In this paper we consider a symmetric positive definite preconditioner of the general block form

$$\mathcal{P} = \begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{pmatrix} \tag{18}$$

where P_1 is a preconditioner for A_1 and P_2 and P_3 are equivalent to the inner products in $M_{1,h}$ and M_h, respectively. More precisely, we require that there exist positive numbers c_0, c_1, C_0, and C_1 such that

$$c_0\vec{\boldsymbol{\tau}}^TP_2\vec{\boldsymbol{\tau}} \le \|\boldsymbol{\tau}\|_{M_1}^2 \le c_1\vec{\boldsymbol{\tau}}^TP_2\vec{\boldsymbol{\tau}} \quad \text{for any } \boldsymbol{\tau} \in M_{1,h}, \tag{19}$$

$$C_0\vec{v}^T P_3 \vec{v} \le \|v\|_M^2 \le C_1 \vec{v}^T P_3 \vec{v} \quad \text{for any } v \in M_h. \tag{20}$$

Here, $\vec{\boldsymbol{\tau}}$ and $\vec{v}$ are the vectors of coefficients of $\boldsymbol{\tau}$ and v, respectively, for the chosen bases in $M_{1,h}$ and M_h. In the following we will assume for simplicity that the numbers c_0, c_1, C_0, and C_1 are constants which are independent of h. However, one certainly can use diagonal matrices for P_2 and P_3 to scale the basis functions in $M_{1,h}$ and M_h. Depending on the norms in M_1 and M then some of the constants may depend on h.

Now let us take a closer look at the stiffness matrix. In the symmetric form the preconditioned stiffness matrix looks like

$$\begin{aligned}
\tilde{\mathcal{A}} := \mathcal{P}^{-1/2}\mathcal{A}\mathcal{P}^{-1/2} &= \begin{pmatrix} P_1^{-1/2}A_1P_1^{-1/2} & P_1^{-1/2}B_1^T P_2^{-1/2} & 0 \\ P_2^{-1/2}B_1P_1^{-1/2} & 0 & P_2^{-1/2}B^T P_3^{-1/2} \\ 0 & P_3^{-1/2}BP_2^{-1/2} & 0 \end{pmatrix} \\
&=: \begin{pmatrix} \tilde{A}_1 & \tilde{B}_1^T & 0 \\ \tilde{B}_1 & 0 & \tilde{B}^T \\ 0 & \tilde{B} & 0 \end{pmatrix}.
\end{aligned}$$

By the above transformation we can assume that the spectrum Λ of $\tilde{\mathcal{A}}$ is contained in the set

$$\Lambda \subset [-a, -b] \cup [c, d]$$

where a, b, c, and d are positive numbers which might depend on discretization parameters like the mesh size h or the polynomial degree p used for the construction of the ansatz spaces. More precisely, we order the eigenvalues of $\tilde{\mathcal{A}}$ as

$$\mu_{-M} \le \mu_{-M+1} \le \cdots \le \mu_{-1} < 0 < \mu_1 \le \mu_2 \le \cdots \le \mu_{L+N}.$$

We also need the eigenvalues of $\tilde{A}_1$,

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_L,$$

the singular values of $\tilde{B}$,

$$0 < \eta_1 \le \eta_2 \le \cdots \le \eta_N$$

and those of $\tilde{B}_1$,

$$0 < \sigma_1 \le \sigma_2 \le \cdots \le \sigma_M.$$

Indeed, $\eta_1 > 0$ by the inf-sup condition (14), see Lemma 1 below. There also holds $\sigma_1 > 0$. However, some singular values of $\tilde{B}_1$ can tend to zero quite rapidly. This is due to the fact that the inf-sup constant β_0 in (16) might depend on h. Therefore, we define the following singular value of $\tilde{B}_1$ by only considering the space $\ker(\tilde{B})$ on which $\tilde{B}_1$ satisfies an optimal inf-sup condition (cf. (15)):

$$\sigma_m := \min_{\boldsymbol{\tau} \in \ker(\tilde{B}) \setminus \{0\}} \max_{\mathbf{s} \in X_{1,h} \setminus \{0\}} \frac{\vec{\boldsymbol{\tau}}^T \tilde{B}_1 \vec{\mathbf{s}}}{\sqrt{\vec{\boldsymbol{\tau}}^T \vec{\boldsymbol{\tau}}}\sqrt{\vec{\mathbf{s}}^T \vec{\mathbf{s}}}}. \tag{21}$$

Here and in the following we identify matrices with their corresponding linear mappings. At the end of this section we will prove that, if λ_L is bounded, the eigenvalues of $\tilde{\mathcal{A}}$ are asymptotically covered by the set

$$[-O(1), -O(\lambda_1)] \cup [O(\lambda_1), O(1)] \quad (h \to 0),$$

see Theorem 6. The boundedness of λ_L can be guaranteed by a simple scaling and it is not a severe restriction.

Before giving the main results of this section (Theorems 5 and 6) we need two technical lemmata. First we bound the singular values of $\tilde{B}$ and $\tilde{B}_1$.

Lemma 1 *There holds*

$$\beta_1 \sqrt{\frac{c_0 \lambda_1}{\alpha_1}} \leq \sigma_m \leq \sigma_M \leq \gamma_1 \sqrt{\frac{c_1 \lambda_L}{\alpha_0}}, \tag{22}$$

$$\beta \sqrt{c_0 C_0} \leq \eta_1 \leq \eta_N \leq \gamma \sqrt{c_1 C_1}. \tag{23}$$

Proof. The proof for the lower bound of σ_m is as follows. First we note that there holds

$$\ker(\tilde{B}) = \ker(P_3^{-1/2} B P_2^{-1/2}) = P_2^{1/2} \ker(B) = P_2^{1/2} \tilde{M}_{1,h}.$$

Therefore, using the definition of $\tilde{B}_1$, the inf-sup condition (15) for $\mathbf{B}_1$ and (8), (19) we obtain

$$\begin{aligned}
\sigma_m &= \min_{\boldsymbol{\tau} \in \ker(\tilde{B}) \setminus \{0\}} \max_{\mathbf{s} \in X_{1,h} \setminus \{0\}} \frac{\vec{\boldsymbol{\tau}}^T P_2^{-1/2} B_1 P_1^{-1/2} \vec{\mathbf{s}}}{\sqrt{\vec{\boldsymbol{\tau}}^T \vec{\boldsymbol{\tau}}} \sqrt{\vec{\mathbf{s}}^T \vec{\mathbf{s}}}} = \min_{\boldsymbol{\tau} \in \tilde{M}_{1,h} \setminus \{0\}} \max_{\mathbf{s} \in X_{1,h} \setminus \{0\}} \frac{\vec{\boldsymbol{\tau}}^T B_1 \vec{\mathbf{s}}}{\sqrt{\vec{\boldsymbol{\tau}}^T P_2 \vec{\boldsymbol{\tau}}} \sqrt{\vec{\mathbf{s}}^T P_1 \vec{\mathbf{s}}}} \\
&= \min_{\boldsymbol{\tau} \in \tilde{M}_{1,h} \setminus \{0\}} \max_{\mathbf{s} \in X_{1,h} \setminus \{0\}} \frac{[\mathbf{B}_1(\mathbf{s}), \boldsymbol{\tau}]}{\|\boldsymbol{\tau}\|_{M_1} \|\mathbf{s}\|_{X_1}} \frac{\|\boldsymbol{\tau}\|_{M_1}}{\sqrt{\vec{\boldsymbol{\tau}}^T P_2 \vec{\boldsymbol{\tau}}}} \frac{\|\mathbf{s}\|_{X_1}}{\sqrt{\vec{\mathbf{s}}^T A_1 \vec{\mathbf{s}}}} \frac{\sqrt{\vec{\mathbf{s}}^T A_1 \vec{\mathbf{s}}}}{\sqrt{\vec{\mathbf{s}}^T P_1 \vec{\mathbf{s}}}} \geq \beta_1 \sqrt{\frac{c_0 \lambda_1}{\alpha_1}}.
\end{aligned}$$

Similarly, we estimate σ_M by

$$\sigma_M = \max_{\boldsymbol{\tau} \in M_{1,h} \setminus \{0\}} \max_{\mathbf{s} \in X_{1,h} \setminus \{0\}} \frac{[\mathbf{B}_1(\mathbf{s}), \boldsymbol{\tau}]}{\|\boldsymbol{\tau}\|_{M_1} \|\mathbf{s}\|_{X_1}} \frac{\|\boldsymbol{\tau}\|_{M_1}}{\sqrt{\vec{\boldsymbol{\tau}}^T P_2 \vec{\boldsymbol{\tau}}}} \frac{\|\mathbf{s}\|_{X_1}}{\sqrt{\vec{\mathbf{s}}^T A_1 \vec{\mathbf{s}}}} \frac{\sqrt{\vec{\mathbf{s}}^T A_1 \vec{\mathbf{s}}}}{\sqrt{\vec{\mathbf{s}}^T P_1 \vec{\mathbf{s}}}} \leq \gamma_1 \sqrt{\frac{c_1 \lambda_L}{\alpha_0}}.$$

To derive a lower bound for η_1 we need the inf-sup condition (14) for $\mathbf{B}$. By the properties (20) and (19), this gives

$$\begin{aligned}
\eta_1 &= \min_{v \in M_h \setminus \{0\}} \max_{\boldsymbol{\tau} \in M_{1,h} \setminus \{0\}} \frac{\vec{v}^T P_3^{-1/2} B P_2^{-1/2} \vec{\boldsymbol{\tau}}}{\sqrt{\vec{v}^T \vec{v}} \sqrt{\vec{\boldsymbol{\tau}}^T \vec{\boldsymbol{\tau}}}} \\
&= \min_{v \in M_h \setminus \{0\}} \max_{\boldsymbol{\tau} \in M_{1,h} \setminus \{0\}} \frac{[\mathbf{B}(\boldsymbol{\tau}), v]}{\|v\|_M \|\boldsymbol{\tau}\|_{M_1}} \frac{\|v\|_M \|\boldsymbol{\tau}\|_{M_1}}{\sqrt{\vec{v}^T P_3 \vec{v}} \sqrt{\vec{\boldsymbol{\tau}}^T P_2 \vec{\boldsymbol{\tau}}}} \geq \beta \sqrt{c_0 C_0}.
\end{aligned}$$

For an upper bound of η_N we need the continuity (11) of $\mathbf{B}$ and obtain by (19) and (20)

$$\eta_N = \max_{v \in M_h \setminus \{0\}} \max_{\boldsymbol{\tau} \in M_{1,h} \setminus \{0\}} \frac{[\mathbf{B}(\boldsymbol{\tau}), v]}{\|v\|_M \|\boldsymbol{\tau}\|_{M_1}} \frac{\|v\|_M \|\boldsymbol{\tau}\|_{M_1}}{\sqrt{\vec{v}^T P_3 \vec{v}} \sqrt{\vec{\boldsymbol{\tau}}^T P_2 \vec{\boldsymbol{\tau}}}} \leq \gamma \sqrt{c_1 C_1}.$$

$\square$

The next lemma presents abstract estimates for the spectrum of a matrix of the special dual-dual form.

Lemma 2 *There holds*

$$-\frac{\sigma_M^2+\eta_N^2}{2\lambda_1}-\sqrt{\frac{(\sigma_M^2+\eta_N^2)^2}{4\lambda_1^2}+\eta_N^2}\le \mu_{-M}, \tag{24}$$

$$\mu_{-1}\le \max\left\{\frac{\lambda_L}{2}-\sqrt{\frac{\lambda_L^2}{4}+\sigma_m^2}, -\eta_1\right\}, \tag{25}$$

$$\min\left\{\lambda_1, -\frac{\sigma_M^2+\eta_1^2}{2\lambda_1}+\sqrt{\frac{(\sigma_M^2+\eta_1^2)^2}{4\lambda_1^2}+\eta_1^2}\right\}\le \mu_1, \tag{26}$$

$$\mu_{L+N}\le \max\left\{\frac{\lambda_L}{2}+\sqrt{\frac{\lambda_L^2}{4}+\sigma_M^2}, \frac{\lambda_1}{2}+\sqrt{\frac{\lambda_1^2}{4}+\sigma_M^2+\eta_N^2}\right\}. \tag{27}$$

Proof. Let us assume that μ is an eigenvalue of $\tilde{\mathcal{A}}$. Then there exists $(\vec{\mathbf{s}},\vec{\boldsymbol{\tau}},\vec{v})\in \mathbf{R}^L\times\mathbf{R}^M\times\mathbf{R}^N$ which is not zero such that

$$\begin{aligned}\tilde{A}_1\vec{\mathbf{s}}+\tilde{B}_1^T\vec{\boldsymbol{\tau}} &= \mu\vec{\mathbf{s}}, &(28)\\ \tilde{B}_1\vec{\mathbf{s}}+\tilde{B}^T\vec{v} &= \mu\vec{\boldsymbol{\tau}}, &(29)\\ \tilde{B}\vec{\boldsymbol{\tau}} &= \mu\vec{v}. &(30)\end{aligned}$$

First let us bound the negative eigenvalues of $\tilde{\mathcal{A}}$. Thus we assume $\mu<0$ and then $\vec{\boldsymbol{\tau}}\neq 0$ by (28) and (30) since $\tilde{A}_1$ is positive definite. If $\mu<0$ then $\tilde{A}_1-\mu I$ is positive definite as well. By using (28) we obtain

$$\vec{\mathbf{s}}=-(\tilde{A}_1-\mu I)^{-1}\tilde{B}_1^T\vec{\boldsymbol{\tau}}$$

and (29) then gives

$$\tilde{B}_1(\tilde{A}_1-\mu I)^{-1}\tilde{B}_1^T\vec{\boldsymbol{\tau}}+\mu\vec{\boldsymbol{\tau}}-\tilde{B}^T\vec{v}=0.$$

Inserting $\vec{v}=\frac{1}{\mu}\tilde{B}\vec{\boldsymbol{\tau}}$ from (30) and multiplying by $\mu\vec{\boldsymbol{\tau}}^T$ this yields

$$\mu\vec{\boldsymbol{\tau}}^T\tilde{B}_1(\tilde{A}_1-\mu I)^{-1}\tilde{B}_1^T\vec{\boldsymbol{\tau}}+\mu^2\vec{\boldsymbol{\tau}}^T\vec{\boldsymbol{\tau}}-\vec{\boldsymbol{\tau}}^T\tilde{B}^T\tilde{B}\vec{\boldsymbol{\tau}}=0. \tag{31}$$

Therefore, since $\mu<0$ and $(\tilde{A}_1-\mu I)^{-1}>0$,

$$0\ge \mu\sigma_M^2(\lambda_1-\mu)^{-1}+\mu^2-\eta_N^2.$$

We multiply this inequality by $-(\lambda_1-\mu)$, which is negative, and obtain

$$0\le \mu^3-\mu^2\lambda_1-\mu(\sigma_M^2+\eta_N^2)+\lambda_1\eta_N^2\le -\mu^2\lambda_1-\mu(\sigma_M^2+\eta_N^2)+\lambda_1\eta_N^2, \tag{32}$$

i.e.

$$0 \geq \mu^2 + \mu \frac{\sigma_M^2 + \eta_N^2}{\lambda_1} - \eta_N^2$$

which, for negative μ, is equivalent to

$$\mu \geq -\frac{\sigma_M^2 + \eta_N^2}{2\lambda_1} - \sqrt{\frac{(\sigma_M^2 + \eta_N^2)^2}{4\lambda_1^2} + \eta_N^2}.$$

This proves (24). Let us show the estimate (25). We consider the cases $\boldsymbol{\tau} \in \ker(\tilde{B})$ and $\boldsymbol{\tau} \notin \ker(\tilde{B})$. In the first case we conclude that $\vec{\boldsymbol{\tau}}^T \tilde{B}_1 \tilde{B}_1^T \vec{\boldsymbol{\tau}} \geq \sigma_m^2 \vec{\boldsymbol{\tau}}^T \vec{\boldsymbol{\tau}}$ by the definition of σ_m. Note that $\sigma_m > 0$ by Lemma 1. Therefore, by again using (31) and remembering that $\mu < 0$ we find that there holds

$$0 \leq \mu \sigma_m^2 (\lambda_L - \mu)^{-1} + \mu^2.$$

Analogously as before, we multiply this inequality by the negative number $-(\lambda_L - \mu)$ and obtain

$$0 \geq \mu^3 - \mu^2 \lambda_L - \mu \sigma_m^2 = \mu \left(\mu^2 - \mu \lambda_L - \sigma_m^2 \right),$$

i.e.

$$0 \leq \mu^2 - \mu \lambda_L - \sigma_m^2.$$

For $\mu < 0$ this gives

$$\mu \leq \frac{\lambda_L}{2} - \sqrt{\frac{\lambda_L^2}{4} + \sigma_m^2}. \tag{33}$$

Now, for $\boldsymbol{\tau} \notin \ker(\tilde{B})$, there holds $\vec{\boldsymbol{\tau}}^T \tilde{B}^T \tilde{B} \vec{\boldsymbol{\tau}} \geq \eta_1^2 \vec{\boldsymbol{\tau}}^T \vec{\boldsymbol{\tau}}$ and from (31) we obtain

$$0 \leq \mu^2 - \eta_1^2,$$

i.e., for negative μ,

$$\mu \leq -\eta_1. \tag{34}$$

Combining (33) and (34) we obtain (25).

It remains to bound the positive eigenvalues of $\tilde{\mathcal{A}}$. First let us consider the case $\mu > 0$ and $\boldsymbol{\tau} \in \ker(\tilde{B})$. Then, due to (30), there holds $\vec{v} = 0$ and the eigenvalue problem (28)–(30) reduces to

$$\begin{aligned} \tilde{A}_1 \vec{\mathbf{s}} + \tilde{B}_1^T \vec{\boldsymbol{\tau}} &= \mu \vec{\mathbf{s}}, \\ \tilde{B}_1 \vec{\mathbf{s}} &= \mu \vec{\boldsymbol{\tau}}. \end{aligned} \tag{35}$$

By combining (21) and Lemma 1 we know that $\tilde{B}_1$ satisfies an inf-sup condition on $\ker(\tilde{B})$. Therefore, we can use the results of Rusten and Winther [22, Lemma 2.1] to estimate the positive eigenvalues μ of the eigenvalue problem (35). This yields

$$\lambda_1 \leq \mu \leq \frac{\lambda_L}{2} + \sqrt{\frac{\lambda_L^2}{4} + \sigma_M^2} \quad \left(\mu > 0, \boldsymbol{\tau} \in \ker(\tilde{B}) \right). \tag{36}$$

Now let us assume that $\mu > 0$ and $\boldsymbol{\tau} \notin \ker(\tilde{B})$. Evidently we then also have $\vec{\boldsymbol{\tau}} \neq 0$. Moreover, we assume that $\mu \notin [\lambda_1, \lambda_L]$. Then $\tilde{A}_1 - \mu I$ is invertible and (31) holds in this case as well:

$$\mu \vec{\boldsymbol{\tau}}^T \tilde{B}_1 (\tilde{A}_1 - \mu I)^{-1} \tilde{B}_1^T \vec{\boldsymbol{\tau}} + \mu^2 \vec{\boldsymbol{\tau}}^T \vec{\boldsymbol{\tau}} - \vec{\boldsymbol{\tau}}^T \tilde{B}^T \tilde{B} \vec{\boldsymbol{\tau}} = 0 \quad \Big(\mu \notin [\lambda_1, \lambda_L], \boldsymbol{\tau} \notin \ker(\tilde{B})\Big). \tag{37}$$

We consider the cases $0 < \mu < \lambda_1$ and $\mu > \lambda_L$ separately. If $0 < \mu < \lambda_1$ then $\tilde{A}_1 - \mu I$ is positive definite and (37) yields

$$0 \leq \mu \sigma_M^2 (\lambda_1 - \mu)^{-1} + \mu^2 - \eta_1^2.$$

Multiplying this by $-(\lambda_1 - \mu)$, which is negative, we obtain

$$0 \geq \mu^3 - \mu^2 \lambda_1 - \mu(\sigma_M^2 + \eta_1^2) + \eta_1^2 \lambda_1 > -\mu^2 \lambda_1 - \mu(\sigma_M^2 + \eta_1^2) + \eta_1^2 \lambda_1,$$

i.e.

$$0 < \mu^2 + \mu \frac{\sigma_M^2 + \eta_1^2}{\lambda_1} - \eta_1^2.$$

For $\mu > 0$ this means

$$\mu > -\frac{\sigma_M^2 + \eta_1^2}{2\lambda_1} + \sqrt{\frac{(\sigma_M^2 + \eta_1^2)^2}{4\lambda_1^2} + \eta_1^2} \quad \Big(0 < \mu < \lambda_1, \boldsymbol{\tau} \notin \ker(\tilde{B})\Big)$$

which, together with (36), proves (26).

Now, for $\mu > \lambda_L$, $\tilde{A}_1 - \mu I$ is negative definite and (37) gives

$$0 \geq \mu \sigma_M^2 (\lambda_1 - \mu)^{-1} + \mu^2 - \eta_N^2.$$

Multiplying this by the positive number $-(\lambda_1 - \mu)$ we conclude that there holds

$$0 \geq \mu^3 - \mu^2 \lambda_1 - \mu(\sigma_M^2 + \eta_N^2) + \eta_N^2 \lambda_1 > \mu \left(\mu^2 - \mu \lambda_1 - (\sigma_M^2 + \eta_N^2)\right)$$

i.e.

$$0 > \mu^2 - \mu \lambda_1 - (\sigma_M^2 + \eta_N^2).$$

For $\mu > 0$ we therefore obtain

$$\mu < \frac{\lambda_1}{2} + \sqrt{\frac{\lambda_1^2}{4} + \sigma_M^2 + \eta_N^2} \quad \Big(\mu > \lambda_L, \boldsymbol{\tau} \notin \ker(\tilde{B})\Big)$$

and, together with (36), this proves (27). $\square$

Combining Lemmata 1 and 2 we have shown that there hold the following bounds for the extremum eigenvalues of $\tilde{\mathcal{A}}$.

Theorem 5 *Let the assumptions* (8) *through* (11), (14), (15), (19) *and* (20) *hold. Further, let* $0 < \lambda_1 \leq \cdots \leq \lambda_L$ *be the eigenvalues of* $\tilde{A}_1$. *Then there hold for the extremum eigenvalues* μ_{-M}, μ_{-1}, μ_1, *and* μ_{L+N} *of* $\tilde{\mathcal{A}}$ *the estimates*

$$-2\sqrt{\frac{(\gamma_1^2 \frac{c_1\lambda_L}{\alpha_0} + \gamma^2 c_1 C_1)^2}{4\lambda_1^2}} + \gamma^2 c_1 C_1 \leq \mu_{-M},$$

$$\mu_{-1} \leq \max\left\{\frac{\lambda_L}{2} - \sqrt{\frac{\lambda_L^2}{4} + \beta_1^2 \frac{c_0\lambda_1}{\alpha_1}}, -\beta\sqrt{c_0 C_0}\right\}$$

$$\min\left\{\lambda_1, -\frac{\sigma_M^2 + \eta_1^2}{2\lambda_1} + \sqrt{\frac{(\sigma_M^2 + \eta_1^2)^2}{4\lambda_1^2} + \eta_1^2}\right\} \leq \mu_1$$

with

$$\beta_1\sqrt{\frac{c_0\lambda_1}{\alpha_1}} \leq \sigma_M \leq \gamma_1\sqrt{\frac{c_1\lambda_L}{\alpha_0}}, \quad \beta\sqrt{c_0 C_0} \leq \eta_1 \leq \gamma\sqrt{c_1 C_1},$$

and

$$\mu_{L+N} \leq \frac{\lambda_L}{2} + \sqrt{\frac{\lambda_L^2}{4} + \gamma_1^2 \frac{c_1\lambda_L}{\alpha_0} + \gamma^2 c_1 C_1}.$$

In the above theorem we do not make the bound for μ_1 more precise. Inserting lower and upper bounds for the singular values could result in a trivial bound for μ_1. However, asymptotic bounds (for $h \to 0$) depending on λ_1 and λ_L can be given.

Theorem 6 *Let the assumptions of Theorem* 5 *hold and assume that all the appearing constants are independent of* h. *Moreover, we assume that the maximum eigenvalue* λ_L *of* $\tilde{A}_1$ *is bounded for* $h \to 0$, *which can be satisfied by a simple scaling. Then the eigenvalues of the preconditioned matrix* $\tilde{\mathcal{A}}$ *lie in the union of intervals which behave asymptotically like*

$$[-O(1), -O(\lambda_1)] \cup [O(\lambda_1), O(1)] \quad (h \to 0).$$

The number of iterations of the preconditioned MINRES which are required to solve (17) *up to a given accuracy is bounded by* $O(1/\lambda_1)$.

Proof. In the case λ_1 is bounded from below by a positive constant then the form of the spectrum directly follows from Theorem 5. When $\lambda_1 \to 0$ for $h \to 0$ then the asymptotic bounds for μ_{-1}, μ_1, and μ_{L+N} again hold by Theorem 5. Indeed,

$$\frac{\lambda_L}{2} - \sqrt{\frac{\lambda_L^2}{4} + \beta_1^2 \frac{c_0\lambda_1}{\alpha_1}} = \frac{\lambda_L}{2}\left(1 - \sqrt{1 + 4\beta_1^2 \frac{c_0\lambda_1}{\alpha_1\lambda_L^2}}\right) \simeq \frac{\lambda_L}{2}\left(1 - (1 + 2\beta_1^2 \frac{c_0\lambda_1}{\alpha_1\lambda_L^2})\right) = -\beta_1^2 \frac{c_0\lambda_1}{\alpha_1\lambda_L}$$

giving $-\mu_{-1} \geq O(\lambda_1/\lambda_L)$. The bound for μ_1 follows analogously and the bound for μ_{L+N} is obvious.

In order to prove the asymptotic bound for μ_{-M} we recall the proof of (24). It appears that the second estimate in (32) is very crude if $\lambda_1 \to 0$. Actually, if $\lambda_1 \to 0$, then asymptotically there holds, instead of (32),

$$0 \le \mu^3 - \mu^2\lambda_1 - \mu(\sigma_M^2 + \eta_N^2) + \lambda_1\eta_N^2 \simeq \mu^3 - \mu(\sigma_M^2 + \eta_N^2) = \mu(\mu^2 - (\sigma_M^2 + \eta_N^2))$$

if μ is bounded from above by a negative constant. For negative μ this requires $\mu \ge -\sqrt{\sigma_M^2 + \eta_N^2}$ asymptotically for $h \to 0$. Since σ_M and η_N are bounded from above the boundedness of μ_{-M} from below is proved. The bound on the number of iterations follows from Theorem 3. $\square$

3.3 A simple numerical example

In the following we present a simple numerical example to demonstrate that Theorem 6 provides asymptotically sharp results in the case of bounded eigenvalues μ_{-M} and μ_{L+N}. More precisely, we choose matrices A_1, B_1, and B such that Lemma 1 holds. Then, provided λ_L is bounded, Lemma 2 and Theorem 6 coincide. The example is the following:

$$\mathcal{A}_6 := \begin{pmatrix} h^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & h & 0 & h & 0 & 0 \\ 0 & 0 & h & 0 & \sqrt{h} & 0 \\ 0 & h & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{h} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

that means

$$A_1 = \begin{pmatrix} h^2 & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & h \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & h & 0 \\ 0 & 0 & \sqrt{h} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

Therefore, one obviously has

$$\lambda_1 = h^2, \quad \lambda_L = h, \quad \sigma_1 = \sigma_m = h, \quad \sigma_M = \sqrt{h}, \quad \text{and} \quad \eta_1 = \eta_N = 1$$

with $L = 3$, $m = 1$, $M = 2$, and $N = 1$. In particular, Lemma 1 is satisfied. By Theorem 6 we expect the spectrum of $\mathcal{A}_6$ to be asymptotically covered by the set

$$[-a, -bh] \cup [ch^2, d] \tag{38}$$

where a, b, c, and d are positive constants independent of h. Indeed, as shown by Figure 1, μ_{-2} and μ_4 are bounded for $h \to 0$ whereas μ_1 and μ_{-1} tend to 0. From the structure of $\mathcal{A}_6$ it is clear that h^2 is an eigenvalue and obviously, this is μ_1. However, the dependence of μ_{-1} on h is not directly clear. To check this dependence we consider Figure 2 where a double logarithmic scale is used. As said before, $\mu_1 = O(h^2)$ and it appears that $-\mu_{-1} = O(h)$. This underlines that the covering set (38) for the spectrum of $\mathcal{A}_6$ is asymptotically sharp (for $h \to 0$) and therefore Theorem 6 is sharp for this example as well.

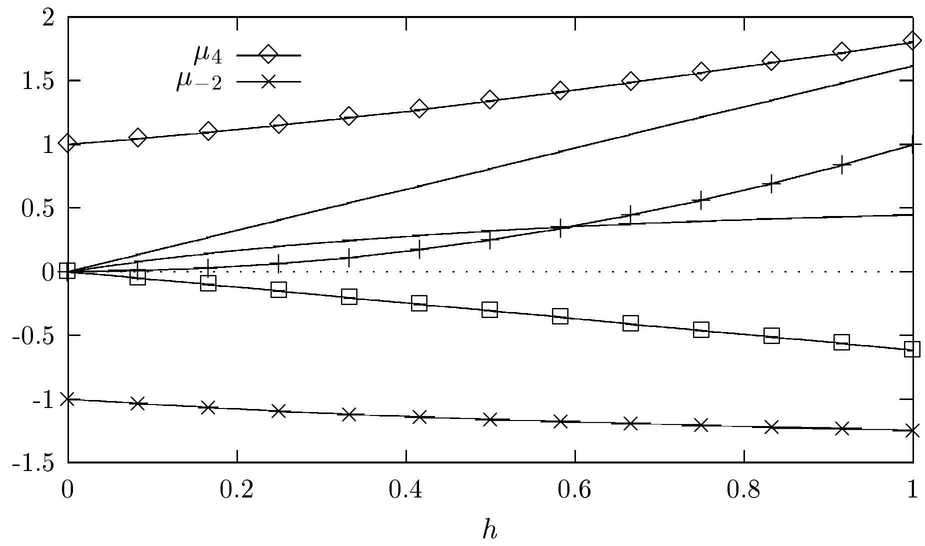

Figure 1: The spectrum of $\mathcal{A}_6$.

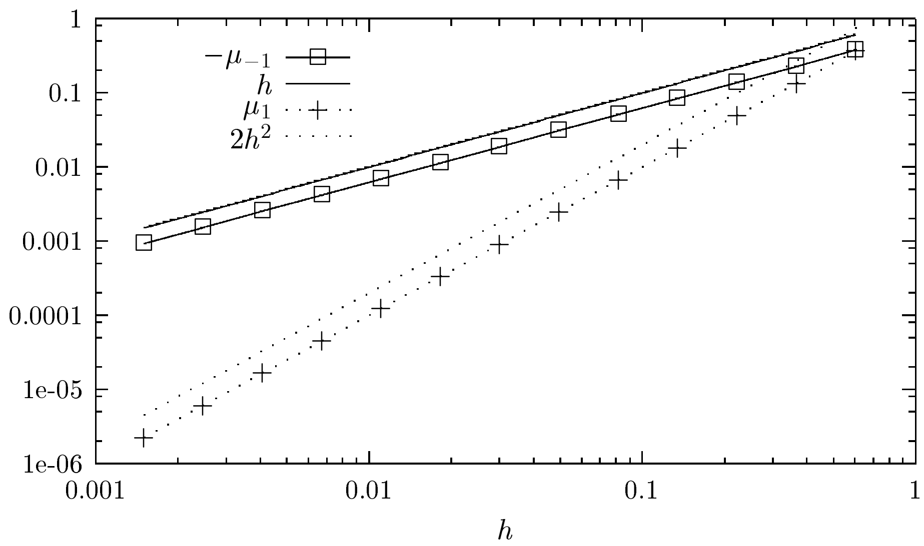

Figure 2: Testing the inner bounds for the spectrum of $\mathcal{A}_6$.

4 Application to the exterior transmission problem

We now apply the abstract results of §3 to the Galerkin system (7) of the exterior transmission problem.

First let us recall the notation for this case. The operators $\mathbf{A}_1$, $\mathbf{B}_1$, $\mathbf{B}$, and the functional $\mathbf{H}$ are defined by

$$[\mathbf{A}_1(\boldsymbol{\theta},\xi),(\boldsymbol{\zeta},\phi)] := \int_{\Omega_1} (\kappa\boldsymbol{\theta})\cdot\boldsymbol{\zeta}\,dx + \int_{\Omega_2} \boldsymbol{\theta}\cdot\boldsymbol{\zeta}\,dx + 2\langle\phi,\mathbf{W}\xi\rangle_{L^2(\Gamma)},$$

$$[\mathbf{B}_1(\boldsymbol{\theta},\xi),\boldsymbol{\tau}] := -\int_{\Omega} \boldsymbol{\theta}\cdot\boldsymbol{\tau}\,dx + \langle\xi,\boldsymbol{\tau}\cdot\boldsymbol{\nu}\rangle_{L^2(\Gamma)},$$

$$[\mathbf{B}(\boldsymbol{\sigma}),v] := -\int_{\Omega} v\,\mathrm{div}\,\boldsymbol{\sigma}\,dx,$$

$$[\mathbf{H},v] := \int_{\Omega} fv\,dx$$

for all $((\boldsymbol{\theta},\xi),\boldsymbol{\sigma})$, $((\boldsymbol{\zeta},\phi),\boldsymbol{\tau}) \in X_1\times M_1$ and for any $v\in M$. Here, $X_1 := (L^2(\Omega))^2\times H_0^{1/2}(\Gamma)$, $M_1 := H_0(\mathrm{div};\Omega)$, and $M := L^2(\Omega)$. The right hand side functionals $\mathbf{F}$ and $\mathbf{G}$ in (12) are 0. For the definition of the finite element spaces $X_{1,h} = X_{1,h}^{\boldsymbol{\theta}}\times X_{1,h}^{\xi}$, $M_{1,h}$, and M_h see §2.2.

The discrete Galerkin solution to (5) is given by the solution of the linear system (7) which is

$$\begin{array}{llll}[\mathbf{A}_1(\boldsymbol{\theta}_h,\xi_h),(\boldsymbol{\zeta},\phi)] & +\,[\mathbf{B}_1^*(\boldsymbol{\sigma}_h),(\boldsymbol{\zeta},\phi)] & & = \;0,\\ [\mathbf{B}_1(\boldsymbol{\theta}_h,\xi_h),\boldsymbol{\tau}] & & +\,[\mathbf{B}^*(u_h),\boldsymbol{\tau}] & = \;0,\\ & [\mathbf{B}(\boldsymbol{\sigma}_h),v] & & = \;[\mathbf{H},v]\end{array} \tag{39}$$

for all $(\boldsymbol{\zeta},\phi,\boldsymbol{\tau},v)\in X_{1,h}^{\boldsymbol{\theta}}\times X_{1,h}^{\xi}\times M_{1,h}\times M_h$.

First let us deal with the case when no preconditioner is in use. Then, $P_1 = I\in\mathbf{R}^{L\times L}$, $P_2 = I\in\mathbf{R}^{M\times M}$, $P_3 = I\in\mathbf{R}^{N\times N}$ in (18), and from (19), (20) and from the definition of $\lambda_1,\ldots,\lambda_L$ as the eigenvalues of $\tilde{A}_1 = P_1^{-1/2}A_1P_1^{-1/2} = A_1$ it is clear that the scaling of basis functions influences the convergence of the MINRES. We choose the scalings such that the outer bounds of the spectrum of $\mathcal{A}$ are independent of h. Then, Theorem 3 is directly applicable giving a bound for the number of iterations of the MINRES.

The scalings are taken as follows.

- $X_{1,h}^{\boldsymbol{\theta}}$: the space is piecewise of the Raviart-Thomas type of order zero. Taking the basis functions as given in (6) on the reference element, we scale such that the L^∞-norm of the basis functions is $O(h^{-1})$.

- $X_{1,h}^{\xi}$: this space consists of piecewise linear functions with respect to the uniform partitions of $[0,2\pi]$. We scale to $O(1)$.

- M_1: this is the Raviart-Thomas space of lowest order. We scale to $O(1)$.

- M: the space of piecewise constant functions on Ω. We scale to $O(h^{-1})$.

We obtain the following result.

Theorem 7 *The eigenvalues of the stiffness matrix in* (39) *are covered by the set*

$$[-a, -bh^3] \cup [ch^3, d]$$

where the positive constants a, b, c, and d do not depend on h. The number of iterations of the minimum residual method which are required to solve (39) *up to a given accuracy is bounded by* $O(h^{-3})$.

Proof. First we note that $\mathbf{B}_1$ and $\mathbf{B}$ are bounded, cf. [13]. Further, $\mathbf{A}_1$ is bounded since κ and $\mathbf{W}$ are bounded, and $\mathbf{A}_1$ is also positive definite by assumption on κ and since $\mathbf{W}$ is, cf. (2). Here, all the bounds are independent of h. From [13, Lemmata 5.6, 5.8] we know that $\mathbf{B}$ and $\mathbf{B}_1$ both satisfy the required inf-sup conditions. More precisely, there holds

$$\sup_{\boldsymbol{\sigma} \in M_{1,h} \setminus \{0\}} \frac{[\mathbf{B}(\boldsymbol{\sigma}), v]}{\|\boldsymbol{\sigma}\|_{M_1}} \geq \beta \|v\|_M \quad \text{for all } v \in M_h$$

where $\beta > 0$ does not depend on h. Moreover, the kernel of $\mathbf{B}|_{M_{1,h} \to M_h}$ is

$$\tilde{M}_{1,h} = \{\boldsymbol{\tau} \in M_{1,h} : \operatorname{div} \boldsymbol{\tau} = 0 \text{ in } \Omega\}$$

(see [13, Lemma 5.7]) and there exists $\beta_1 > 0$, independent of h, such that

$$\sup_{(\boldsymbol{\theta}, \xi) \in X_{1,h} \setminus \{0\}} \frac{[\mathbf{B}_1(\boldsymbol{\theta}, \xi), \boldsymbol{\tau}]}{\|(\boldsymbol{\theta}, \xi)\|_{X_1}} \geq \beta_1 \|\boldsymbol{\tau}\|_{M_1} \quad \text{for all } \boldsymbol{\tau} \in \tilde{M}_{1,h}.$$

We also need to prove the inf-sup condition (16) of $\mathbf{B}_1$ on the whole space $M_{1,h}$. This condition was used to count the numbers of positive and negative eigenvalues of the stiffness matrix $\mathcal{A}$. There holds

$$\sup_{\mathbf{t} = (\boldsymbol{\theta}, \xi)^T \in X_{1,h} \setminus \{0\}} \frac{[\mathbf{B}_1(\mathbf{t}), \boldsymbol{\tau}]}{\|\mathbf{t}\|_{X_1}} \geq \sup_{\boldsymbol{\theta} \in X^{\boldsymbol{\theta}}_{1,h} \setminus \{0\}} \frac{-\int_\Omega \boldsymbol{\tau} \cdot \boldsymbol{\theta} \, dx}{\|\boldsymbol{\theta}\|_{(L^2(\Omega))^2}} = \|\boldsymbol{\tau}\|_{(L^2(\Omega))^2} \quad \text{for all } \boldsymbol{\tau} \in M_{1,h},$$

see also [13, Lemma 5.8]. By the inverse property we can estimate

$$\|\boldsymbol{\tau}\|^2_{M_1} = \|\boldsymbol{\tau}\|^2_{(L^2(\Omega))^2} + \|\operatorname{div} \boldsymbol{\tau}\|^2_{(L^2(\Omega))^2} \leq (1 + ch^{-2}) \|\boldsymbol{\tau}\|^2_{(L^2(\Omega))^2}$$

where the constant c does not depend on h. Therefore,

$$\sup_{\mathbf{t} = (\boldsymbol{\theta}, \xi)^T \in X_{1,h} \setminus \{0\}} \frac{[\mathbf{B}_1(\mathbf{t}), \boldsymbol{\tau}]}{\|\mathbf{t}\|_{X_1}} \geq (1 + ch^{-2})^{-1/2} \|\boldsymbol{\tau}\|^2_{M_1} \quad \text{for all } \boldsymbol{\tau} \in M_{1,h},$$

i.e., the inf-sup condition (16) holds with a number $\beta_0 = O(h)$.

We now check the eigenvalues of the stiffness matrix $\mathcal{A}$. Obviously, A_1 is a 2-block diagonal matrix with one block belonging to the space $X^{\boldsymbol{\theta}}_{1,h}$ with $(L^2(\Omega))^2$-inner product and the other block belonging to the space $X^{\xi}_{1,h}$ with inner product given by the hypersingular operator $\mathbf{W}$. The first block is well-conditioned and has, due to the scaling $O(h^{-1})$, a positive spectrum with bounds not depending on h. On the other hand the spectrum of the latter block is asymptotically covered by $(O(h), O(1))$. This estimate can be seen as follows. By the inverse property of the basis functions in $X^{\xi}_{1,h}$ we obtain

$$\|\xi\|^2_{L^2(\Gamma)} \le \|\xi\|^2_{H^{1/2}(\Gamma)} \le ch^{-1}\|\xi\|^2_{L^2(\Gamma)} \quad \text{for all } \xi \in X^{\xi}_{1,h}.$$

By the scaling of basis functions in $X^{\xi}_{1,h}$, $\|\xi\|^2_{L^2(\Gamma)}$ is equivalent to h times the l_2-norm squared of the corresponding coefficient vector. We conclude that there holds

$$ch \le \frac{\langle \xi, \mathbf{W}\xi\rangle_{L^2(\Gamma)}}{\vec{\xi}^T\vec{\xi}} \le C \quad \forall \xi \in X^{\xi}_{1,h}$$

for constants $c, C > 0$ which are independent of h. This gives the required estimate for the spectrum of the stiffness matrix for the hypersingular operator. Of course, this estimate is well-known. We have given it for completeness here. Therefore, $\lambda_1 = O(h)$ and $\lambda_L = O(1)$.

Now we make use of the chosen scalings to estimate the numbers c_0, c_1, C_0, and C_1 in (19) and (20). In the following c and C are generic positive constants that do not depend on h. By the scaling $O(1)$ in $M_{1,h}$ and the inverse property we obtain

$$\begin{aligned} ch^2\vec{\boldsymbol{\tau}}^T\vec{\boldsymbol{\tau}} \le \|\boldsymbol{\tau}\|^2_{(L^2(\Omega))^2} &\le \|\boldsymbol{\tau}\|^2_{(L^2(\Omega))^2} + \|\operatorname{div}\boldsymbol{\tau}\|^2_{L^2(\Omega)} = \|\boldsymbol{\tau}\|^2_{M_1} \\ &\le (1 + Ch^{-2})\|\boldsymbol{\tau}\|^2_{(L^2(\Omega))^2} \le C\vec{\boldsymbol{\tau}}^T\vec{\boldsymbol{\tau}} \quad \text{for any } \boldsymbol{\tau} \in M_{1,h}, \end{aligned} \tag{40}$$

and by the scaling $O(h^{-1})$ in M_h there holds

$$c\vec{v}^T\vec{v} \le \|v\|^2_{L^2(\Omega)} = \|v\|^2_M \le C\vec{v}^T\vec{v} \quad \text{for any } v \in M_h,$$

i.e.

$$c_0 = O(h^2), \quad c_1 = O(1), \quad C_0 = O(1), \quad \text{and} \quad C_1 = O(1).$$

By Lemma 1 we obtain the following bounds for the singular values of B_1 and B:

$$O(h^{3/2}) \le \sigma_m \le \sigma_M \le O(1), \quad O(h) \le \eta_1 \le \eta_N \le O(1).$$

The assertion of the theorem concerning the bounds for the eigenvalues μ_{-1}, μ_1, and μ_{L+N} then follows from Theorem 5. However, as already mentioned in the proof of Theorem 6, the bound for μ_{-M} given by Theorem 5 is not sharp if $\lambda_1 \to 0$. Instead, in this case, there holds the asymptotic bound

$$\mu_{-M} \ge -\sqrt{\sigma_M^2 + \eta_N^2}$$

which is bounded from below since $\sigma_M = O(1)$ and $\eta_N = O(1)$, as we have seen above. Therefore, the assertion on the spectrum of $\mathcal{A}$ is proved and the bound on the number of iterations of the MINRES follows from Theorem 3. $\square$

From the previous theorem it is obvious that an efficient solution of (39) by the MINRES requires a preconditioner. Otherwise, the convergence of the MINRES deteriorates with decreasing mesh size. From the theory of §3 it is also clear that the block of the stiffness matrix corresponding to the hypersingular integral operator needs to be preconditioned and that we need to consider, instead of a pure scaling, a preconditioner for the space $M_{1,h}$ which is equivalent to the $H(\mathrm{div};\Omega)$-inner product. Then, provided the basis functions in the spaces $X^{\boldsymbol{\theta}}_{1,h}$ and M_h are properly scaled as indicated before, the preconditioned MINRES for the solution of (39) can be very efficient.

Here we do not go into details concerning special preconditioners for the hypersingular operator. We only mention that there are several possibilities, e.g., multigrid methods [27, 28], two and multilevel additive Schwarz methods [24], and wavelet-based techniques [8, 9, 25, 26]. Therefore, there are different efficient techniques which yield a bounded spectrum for the discretization of the hypersingular operator. Then, recalling the proof of Theorem 7, our main result is the following.

Theorem 8 *Let*

$$\mathcal{P} := \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & P_1^{\xi} & 0 & 0 \\ 0 & 0 & P_2 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

be a preconditioner for $\mathcal{A}$, yielding the preconditioned symmetric matrix $\tilde{\mathcal{A}}$, where $P_1^{\xi} \in \mathbf{R}^{l \times l}$ with $l = \dim X^{\xi}_{1,h}$, $P_2 \in \mathbf{R}^{M \times M}$ and the remaining blocks $I = \mathrm{diag}\,(1, \dots, 1)$ are of the correct size. Further, assume that P_1^{ξ} is uniformly spectrally equivalent to the block of $\mathcal{A}$ which belongs to the hypersingular operator, and that P_2 is uniformly spectrally equivalent to the $H(\mathrm{div};\Omega)$-inner product in $M_{1,h}$. We choose the basis functions in $X^{\boldsymbol{\theta}}_{1,h}$ and M_h to be scaled to $O(h^{-1})$. Then, the spectrum of $\tilde{\mathcal{A}}$ is covered by the set

$$[-a, -b] \cup [c, d]$$

where the positive constants a, b, c, and d do not depend on h. Moreover, in that case, the number of iterations of the preconditioned minimum residual method for the solution of (39) *is bounded.*

Proof. The proof of Theorem 7 with some modifications holds in this case as well. We only have to mention that, due to the preconditioner for the block belonging to the hypersingular operator, there holds $\lambda_1 = O(1)$ and $\lambda_L = O(1)$ and, instead of (40), we now have for generic positive constants c and C

$$c\vec{\boldsymbol{\tau}}^T P_2 \vec{\boldsymbol{\tau}} \le \|\boldsymbol{\tau}\|^2_{M_1} \le C \vec{\boldsymbol{\tau}}^T P_2 \vec{\boldsymbol{\tau}} \quad \text{for any } \boldsymbol{\tau} \in M_{1,h}.$$

Therefore, the number c_0 in (19) is bounded. The boundedness of the numbers c_1, C_0, and C_1 in (19) and (20) has already been checked in the proof of Theorem 7. Thus, the proof of the theorem is finished by Theorem 6. $\square$

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