Conjugate gradient method for dual-dual mixed formulations *

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Abstract

We deal with the iterative solution of linear systems arising from so-called dual-dual mixed finite element formulations. The linear systems are of a two-fold saddle point structure; they are indefinite and ill-conditioned. We define a special inner product that makes matrices of the two-fold saddle point structure, after a specific transformation, symmetric and positive definite. Therefore, the conjugate gradient method with this special inner product can be used as iterative solver. For a model problem, we prove that simple scalings lead to a solution procedure which requires $O(h^{-1})$ iterations. Here, h is the mesh size of the underlying dual-dual mixed method. Moreover, we propose a preconditioner which leads to a bounded number of iterations. Numerical experiments for our model problem confirming the theoretical results are also reported.

Key words. Mixed finite elements, dual-dual variational formulation, conjugate gradient method.

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1 Introduction

As it is well-known, mixed finite element methods often lead to symmetric but indefinite linear systems. For standard variational formulations they represent a saddle point problem. Here we deal with linear systems of a two-fold saddle point structure. These systems arise, e.g., when using a combined dual-mixed finite element method with a Dirichlet-to-Neumann mapping to solve exterior transmission problems [9], or in elastostatics when coupling the primal mixed finite element method and the boundary element method [3]. In this paper we consider, as a model problem, the dual-dual mixed finite element method, which introduces the gradient as third explicit unknown, to solve linear second order elliptic equations in divergence form [7].

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There are several iterative methods for the solution of linear systems with single saddle point structure. For instance, preconditioned Krylov subspace methods can be efficiently applied to symmetric indefinite systems, see, e.g., [1, 12, 2, 13, 14, 11]. We also mention the Uzawa algorithm which is an iterative method for the corresponding positive definite Schur complement system. Finally, there is the method of Bramble and Pasciak [4] who solve an equivalent transformed system by using the conjugate gradient (CG) method with a special inner product.

Instead of investigating Krylov subspace methods for dual-dual type linear systems, for which we refer to [8], we here follow the ideas of Bramble and Pasciak. We define a specific transformation for linear systems of dual-dual type and present an inner product which makes the transformed system matrix symmetric and positive definite. Therefore, the CG method can be applied as iterative solver. Here, we have to deal with ill-conditioned transformed matrices and therefore, preconditioning strategies are in order. For our model problem we deal with two different approaches. Our first preconditioning method is implemented via the specific transformation. That means, no additional preconditioner is applied to the final transformed linear system. However, for this method it appears that either the transformed matrix is uniformly well-conditioned but the CG method may fail, or the transformed matrix is ill-conditioned when convergence of the CG method has to be ensured. In the latter case, this method behaves just as if no preconditioning strategy is applied. This investigation leads to the conclusion that only a preconditioner, which is not incorporated into the transformation process (but is applied after the transformation), can result in a reliable and efficient iterative solver, i.e. convergence is guaranteed and the number of iterations is independent of the mesh size.

An outline of this paper is as follows. In §2 we consider in an abstract way linear systems which are of dual-dual structure. We introduce the specific transformation and define a bilinear form which is proved to be an inner product under certain assumptions. We show that the transformed system matrix is symmetric and positive definite with respect to the special inner product and we give a precise estimate for its extreme eigenvalues (Theorem 1). In §3 we apply this procedure to solve linear systems arising from our model problem. First the model problem and its discretization are briefly introduced. Then, in §3.2, we consider in detail the condition number of the transformed linear system with and without additional preconditioner. For the best of our methods, we prove that the number of iterations of the CG method, which are necessary to solve the system up to a given accuracy, is independent of the mesh size h (Theorem 6). Finally, in §3.3, we present some numerical results which underline our theory.

2 Transformation of dual-dual type linear systems

The structure of the linear system we have to deal with looks like

$$\mathcal{A}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} := \begin{pmatrix} A_1 & B_1^* & 0 \\ B_1 & 0 & B^* \\ 0 & B & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}. \tag{1}$$

Here, A_1 is symmetric positive definite and the matrices B_1 and B have full rank. The solution $(x_1, x_2, x_3)^T$ is sought within a vector space $S^1 \times S^2 \times S^3$ with

$$L := \dim S^1 \geq M := \dim S^2 \geq N := \dim S^3.$$

The matrix $\mathcal{A}$ is symmetric and indefinite, it has $L+N$ positive and M negative eigenvalues, see [8]. We transform the linear system into an equivalent one which is symmetric and positive definite with respect to a certain bilinear form (Theorem 1). Therefore, due to that result, the conjugate gradient method can be used to solve the transformed linear system.

First, let us introduce step by step the transformation of the linear system. Then we define the special inner product and prove some technical results before we state the main result of this section (Theorem 1).

Let us denote a preconditioner for A_1 by A_0. Multiplying the first row of (1) by A_0^{-1}, subtracting the second row from B_1 times the new first row, and multiplying the last row by $-I$ we obtain

$$\begin{pmatrix} A_0^{-1}A_1 & A_0^{-1}B_1^* & 0 \\ B_1A_0^{-1}(A_1 - A_0) & B_1A_0^{-1}B_1^* & -B^* \\ 0 & -B & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} A_0^{-1}f_1 \\ B_1A_0^{-1}f_1 - f_2 \\ -f_3 \end{pmatrix}.$$

This system can be written in the form

$$\begin{pmatrix} \mathcal{M}_1 & \mathcal{B}^* \\ \mathcal{B} & 0 \end{pmatrix} \begin{pmatrix} X \\ x_3 \end{pmatrix} = \begin{pmatrix} F \\ -f_3 \end{pmatrix} \tag{2}$$

where

$$\mathcal{M}_1 := \begin{pmatrix} A_0^{-1}A_1 & A_0^{-1}B_1^* \\ B_1A_0^{-1}(A_1 - A_0) & B_1A_0^{-1}B_1^* \end{pmatrix}, \quad \mathcal{B} := \begin{pmatrix} 0 & -B \end{pmatrix} \tag{3}$$

and

$$X := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad F := \begin{pmatrix} A_0^{-1}f_1 \\ B_1A_0^{-1}f_1 - f_2 \end{pmatrix}.$$

Let us repeat the analogous row manipulations to the system (2). In order to do so we need to take a preconditioner $\mathcal{M}_0$ for $\mathcal{M}_1$. As given by Lemma 1 below, $\mathcal{M}_1$ is spectrally equivalent (with respect to a special inner product) to the matrix

$$\tilde{\mathcal{M}}_1 := \begin{pmatrix} I & 0 \\ 0 & B_1A_1^{-1}B_1^* \end{pmatrix}. \tag{4}$$

It therefore suffices to consider a preconditioner for $\mathcal{M}_1$ which is of the special form

$$\mathcal{M}_0 := \begin{pmatrix} \rho I & 0 \\ 0 & M_0 \end{pmatrix}$$

where ρ is a positive number and M_0 is an appropriate symmetric positive definite matrix for the vector space S^2 with inner product $(\cdot, \cdot)$.

Now we transform the system (2). We multiply the first row by $\mathcal{M}_0^{-1}$ and subtract the second row from $\mathcal{B}$ times the new first row. Then we obtain the system

$$\mathcal{M}\begin{pmatrix} X \\ x_3 \end{pmatrix} := \begin{pmatrix} \mathcal{M}_0^{-1}\mathcal{M}_1 & \mathcal{M}_0^{-1}\mathcal{B}^* \\ \mathcal{B}\mathcal{M}_0^{-1}(\mathcal{M}_1 - \mathcal{M}_0) & \mathcal{B}\mathcal{M}_0^{-1}\mathcal{B}^* \end{pmatrix}\begin{pmatrix} X \\ x_3 \end{pmatrix} = \begin{pmatrix} \mathcal{M}_0^{-1}F \\ \mathcal{B}\mathcal{M}_0^{-1}F + f_3 \end{pmatrix}, \tag{5}$$

in short form

$$\mathcal{M}x = \mathcal{F}$$

where, in detail,

$$\mathcal{M} = \begin{pmatrix} \frac{1}{\rho}A_0^{-1}A_1 & \frac{1}{\rho}A_0^{-1}B_1^* & 0 \\ M_0^{-1}B_1A_0^{-1}(A_1 - A_0) & M_0^{-1}B_1A_0^{-1}B_1^* & -M_0^{-1}B^* \\ -BM_0^{-1}B_1A_0^{-1}(A_1 - A_0) & B - BM_0^{-1}B_1A_0^{-1}B_1^* & BM_0^{-1}B^* \end{pmatrix} \tag{6}$$

and

$$\mathcal{F} = \begin{pmatrix} \frac{1}{\rho}A_0^{-1}f_1 \\ M_0^{-1}(B_1A_0^{-1}f_1 - f_2) \\ BM_0^{-1}(f_2 - B_1A_0^{-1}f_1) + f_3 \end{pmatrix}.$$

Now we introduce a bilinear form on $S^1 \times S^2 \times S^3$ which becomes an inner product if some conditions are satisfied. This inner product then makes the system matrix $\mathcal{M}$ symmetric positive definite.

First, we define a bilinear form on $S^1 \times S^2$ by

$$\left[\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right]_1 := (A_1u_1, v_1) - (A_0u_1, v_1) + (u_2, v_2) \tag{7}$$

where $(\cdot,\cdot)$ is the inner product on S^1 or S^2, as appropriate. Under certain assumptions on A_0, which will be given below, $[\cdot,\cdot]_1$ is an inner product on $S^1 \times S^2$ and, moreover, $\mathcal{M}_1$ is symmetric positive definite with respect to $[\cdot,\cdot]_1$, see Lemma 1.

Using this bilinear form on $S^1 \times S^2$ we define the following bilinear form on $S^1 \times S^2 \times S^3$:

$$\begin{aligned}\left[\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right] := & \left[\mathcal{M}_1\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right]_1 - \left[\mathcal{M}_0\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right]_1 \\ & +(u_3, v_3)\end{aligned} \tag{8}$$

Here, $(\cdot,\cdot)$ is also used as the inner product on S^3, like on S^1 and S^2 before.

Let us make the following assumptions.

(A1) There exist positive constants α_0, α_1 such that

$$\alpha_0(A_1u_1, u_1) \leq (A_0u_1, u_1) \leq \alpha_1(A_1u_1, u_1)$$

for all $u_1 \in S^1$.

(A2) $\alpha_1 < 1$

We then conclude that

$$0 < (1-\alpha_1)(A_1 u_1, u_1) \leq ((A_1 - A_0)u_1, u_1) \leq \alpha(A_1 u_1, u_1)$$

holds for all $u_1 \in S^1 \setminus \{0\}$ where $\alpha := 1 - \alpha_0$.

As shown by Bramble and Pasciak the following results hold.

Lemma 1 [4] *Let the assumptions* (A1) *and* (A2) *hold. Then*
(i) the bilinear form $[\cdot,\cdot]_1$ *is an inner product on* $S^1 \times S^2$*,*
(ii) $\mathcal{M}_1$ *is symmetric positive definite with respect to* $[\cdot,\cdot]_1$*, and*
(iii) $\mathcal{M}_1$ *is spectrally equivalent to* $\tilde{\mathcal{M}}_1$*, see* (4)*. More precisely, there holds*

$$\lambda_0 \left[\tilde{\mathcal{M}}_1 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right]_1 \leq \left[\mathcal{M}_1 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right]_1 \leq \lambda_1 \left[\tilde{\mathcal{M}}_1 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right]_1$$

for any $u_1 \in S^1$ *and* $u_2 \in S^2$ *where*

$$\lambda_0 = \left(1 + \frac{\alpha}{2} + \sqrt{\alpha + \frac{\alpha^2}{4}}\right)^{-1} \quad \textit{and} \quad \lambda_1 = \frac{1+\sqrt{\alpha}}{1-\alpha}.$$

The assumptions (A1), (A2) are concerned with the preconditioner A_0 for A_1. They are used to handle the matrix $\mathcal{M}_1$ which is a transformation of the indefinite block $A = \begin{pmatrix} A_1 & B_1^* \\ B_1 & 0 \end{pmatrix}$ of $\mathcal{A}$. We repeat the procedure at the next level, i.e. we deal with the indefinite system (2). The transformed system matrix is $\mathcal{M}$, and we consider the preconditioner $\mathcal{M}_0$ for $\mathcal{M}_1$. However, since we already know that $\mathcal{M}_1$ is spectrally equivalent to $\tilde{\mathcal{M}}_1$, we do not specify the spectral equivalence of $\mathcal{M}_0$ and $\mathcal{M}_1$. Instead, it is more convenient to deal with the matrix $\tilde{\mathcal{M}}_1$ which is much simpler than $\mathcal{M}_1$.

The assumptions are the following.

(A3) There exist positive constants β_0, β_1 such that

$$\beta_0[\tilde{\mathcal{M}}_1 U, U]_1 \leq [\mathcal{M}_0 U, U]_1 \leq \beta_1[\tilde{\mathcal{M}}_1 U, U]_1$$

for all $U \in S^1 \times S^2$.

(A4) $\beta_1/\lambda_0 < 1$, where λ_0 is given in Lemma 1.

Note that Assumption (A3) makes sense because $\mathcal{M}_0 = \begin{pmatrix} \rho I & 0 \\ 0 & M_0 \end{pmatrix}$ is symmetric positive definite with respect to $[\cdot,\cdot]_1$ since M_0 is symmetric positive definite.

Then we obtain the following lemma.

Lemma 2 *Let the assumptions* (A1)–(A4) *be satisfied. Then* $\mathcal{M}_1 - \mathcal{M}_0$ *is positive definite with respect to* $[\cdot,\cdot]_1$. *More precisely there holds*

$$(1-\frac{\beta_1}{\lambda_0})[\mathcal{M}_1U,U]_1 \le [(\mathcal{M}_1-\mathcal{M}_0)U,U]_1 \le (1-\frac{\beta_0}{\lambda_1})[\mathcal{M}_1U,U]_1$$

for all $U \in S^1 \setminus \{0\}$ *where* λ_0 *and* λ_1 *are given in Lemma* 1.

Proof. The proof is a straight forward combination of Assumption (A3) and Lemma 1. Assumption (A4) ensures the positive definiteness of $\mathcal{M}_1 - \mathcal{M}_0$ with respect to $[\cdot,\cdot]_1$. □

The next lemma comprises already an important part of the main result by showing that $[\cdot,\cdot]$ is an inner product and that the transformed matrix $\mathcal{M}$ is symmetric with respect to $[\cdot,\cdot]$. These facts are implicitly used in Theorem 1 below.

Lemma 3 *Let the assumptions* (A1)–(A4) *be satisfied. Then*
(i) $[\cdot,\cdot]$ *is an inner product on* $S^1 \times S^2 \times S^3$ *and*
(ii) $\mathcal{M}$ *is symmetric with respect to* $[\cdot,\cdot]$.

Proof. (i) By definition there holds for $(U,u_3)^T = (u_1,u_2,u_3)^T$ and $(V,v_3)^T = (v_1,v_2,v_3)^T \in S^1 \times S^2 \times S^3$

$$\left[\begin{pmatrix} U \\ u_3 \end{pmatrix}, \begin{pmatrix} V \\ v_3 \end{pmatrix}\right] = [\mathcal{M}_1U,V]_1 - [\mathcal{M}_0U,V]_1 + (u_3,v_3).$$

Therefore, $[\cdot,\cdot]$ is symmetric since $\mathcal{M}_1$ and $\mathcal{M}_0$ are symmetric with respect to $[\cdot,\cdot]_1$ by Lemma 1 and by assumption, respectively. Moreover, since $\mathcal{M}_1-\mathcal{M}_0$ is positive definite with respect to $[\cdot,\cdot]_1$ by Lemma 2, the bilinear form $[\cdot,\cdot]$ is positive definite on $S^1 \times S^2 \times S^3$.

(ii) By definition of $\mathcal{M}$ there holds

$$\begin{aligned}
\left[\mathcal{M}\begin{pmatrix} U \\ u_3 \end{pmatrix}, \begin{pmatrix} V \\ v_3 \end{pmatrix}\right] &= \left[\begin{pmatrix} \mathcal{M}_0^{-1}\mathcal{M}_1U + \mathcal{M}_0^{-1}\mathcal{B}^*u_3 \\ \mathcal{B}\mathcal{M}_0^{-1}(\mathcal{M}_1-\mathcal{M}_0)U + \mathcal{B}\mathcal{M}_0^{-1}\mathcal{B}^*u_3 \end{pmatrix}, \begin{pmatrix} V \\ v_3 \end{pmatrix}\right] \\
&= \Big[(\mathcal{M}_1-\mathcal{M}_0)(\mathcal{M}_0^{-1}\mathcal{M}_1U + \mathcal{M}_0^{-1}\mathcal{B}^*u_3), V\Big]_1 \\
&\quad + (\mathcal{B}\mathcal{M}_0^{-1}(\mathcal{M}_1-\mathcal{M}_0)U + \mathcal{B}\mathcal{M}_0^{-1}\mathcal{B}^*u_3, v_3) \\
&= \Big[(\mathcal{M}_1-\mathcal{M}_0)\mathcal{M}_0^{-1}\mathcal{M}_1U, V\Big]_1 + \Big[(\mathcal{M}_1-\mathcal{M}_0)\mathcal{M}_0^{-1}\mathcal{B}^*u_3, V\Big]_1 \\
&\quad + (\mathcal{B}\mathcal{M}_0^{-1}(\mathcal{M}_1-\mathcal{M}_0)U, v_3) + (\mathcal{B}\mathcal{M}_0^{-1}\mathcal{B}^*u_3, v_3).
\end{aligned}$$

Obviously, the terms

$$\Big[(\mathcal{M}_1-\mathcal{M}_0)\mathcal{M}_0^{-1}\mathcal{M}_1U, V\Big]_1 \quad \text{and} \quad (\mathcal{B}\mathcal{M}_0^{-1}\mathcal{B}^*u_3, v_3) = (BM_0^{-1}B^*u_3, v_3)$$

are symmetric since $\mathcal{M}_1$ and $\mathcal{M}_0$ are symmetric with respect to $[\cdot,\cdot]_1$ and since M_0^{-1} is symmetric with respect to $(\cdot,\cdot)$. It therefore remains to check the symmetry of

$$\Big[(\mathcal{M}_1-\mathcal{M}_0)\mathcal{M}_0^{-1}\mathcal{B}^*u_3, V\Big]_1 + (\mathcal{B}\mathcal{M}_0^{-1}(\mathcal{M}_1-\mathcal{M}_0)U, v_3).$$

We obtain

$$
\begin{aligned}
& \Big[(\mathcal{M}_1-\mathcal{M}_0)\mathcal{M}_0^{-1}\mathcal{B}^*u_3, V\Big]_1 + (\mathcal{B}\mathcal{M}_0^{-1}(\mathcal{M}_1-\mathcal{M}_0)U, v_3) \\
= & \left[\mathcal{M}_1 \begin{pmatrix} 0 \\ -M_0^{-1}B^* \end{pmatrix} u_3, V\right]_1 - [\mathcal{B}^*u_3, V]_1 + (\mathcal{B}\mathcal{M}_0^{-1}\mathcal{M}_1 U, v_3) - (\mathcal{B}U, v_3) \\
= & \left[-\begin{pmatrix} A_0^{-1}B_1^*M_0^{-1}B^*u_3 \\ B_1A_0^{-1}B_1^*M_0^{-1}B^*u_3 \end{pmatrix}, V\right]_1 - [\mathcal{B}^*u_3, V]_1 + (\mathcal{B}\mathcal{M}_0^{-1}\mathcal{M}_1 U, v_3) - (\mathcal{B}U, v_3) \\
= & -(A_1A_0^{-1}B_1^*M_0^{-1}B^*u_3, v_1) + (B_1^*M_0^{-1}B^*u_3, v_1) - (B_1A_0^{-1}B_1^*M_0^{-1}B^*u_3, v_2) + (B^*u_3, v_2) \\
& -(BM_0^{-1}B_1A_0^{-1}(A_1-A_0)u_1, v_3) - (BM_0^{-1}B_1A_0^{-1}B_1^*u_2, v_3) + (Bu_2, v_3) \\
= & -(A_1A_0^{-1}B_1^*M_0^{-1}B^*u_3, v_1) - (BM_0^{-1}B_1A_0^{-1}A_1u_1, v_3) \\
& +(BM_0^{-1}B_1u_1, v_3) + (B_1^*M_0^{-1}B^*u_3, v_1) \\
& -(B_1A_0^{-1}B_1^*M_0^{-1}B^*u_3, v_2) - (BM_0^{-1}B_1A_0^{-1}B_1^*u_2, v_3) \\
& +(B^*u_3, v_2) + (Bu_2, v_3)
\end{aligned}
$$

where each of the couples in the last expression is symmetric. Therefore, the proof of the lemma is finished. $\square$

The next technical result will be used in the proof of Theorem 1.

Lemma 4 *For $(U, u_3)^T = (u_1, u_2, u_3)^T \in S^1 \times S^2 \times S^3$ we consider the representation*

$$
\begin{pmatrix} U \\ u_3 \end{pmatrix} = \begin{pmatrix} U_0 \\ 0 \end{pmatrix} + \begin{pmatrix} U_H \\ u_3 \end{pmatrix}
$$

where U_H is uniquely defined by

$$
\mathcal{M}_1 U_H + \mathcal{B}^* u_3 = 0.
$$

Then there holds

$$
\left[\mathcal{M}\begin{pmatrix} U \\ u_3 \end{pmatrix}, \begin{pmatrix} U \\ u_3 \end{pmatrix}\right] = \left[\mathcal{M}\begin{pmatrix} U_0 \\ 0 \end{pmatrix}, \begin{pmatrix} U_0 \\ 0 \end{pmatrix}\right] + \left[\mathcal{M}\begin{pmatrix} U_H \\ u_3 \end{pmatrix}, \begin{pmatrix} U_H \\ u_3 \end{pmatrix}\right] \tag{9}
$$

for any $(U, u_3) \in S^1 \times S^2 \times S^3$.

Proof. We only need to check that

$$
\left[\mathcal{M}\begin{pmatrix} U_0 \\ 0 \end{pmatrix}, \begin{pmatrix} U_H \\ u_3 \end{pmatrix}\right] = 0.
$$

Since $\mathcal{M}$ is symmetric with respect to $[\cdot, \cdot]$ the result then follows.

Using the relation $U_H = -\mathcal{M}_1^{-1}\mathcal{B}^* u_3$ we find that there holds

$$\begin{aligned}
\left[\mathcal{M}\begin{pmatrix} U_0 \\ 0 \end{pmatrix}, \begin{pmatrix} U_H \\ u_3 \end{pmatrix}\right] &= \left[\begin{pmatrix} \mathcal{M}_0^{-1}\mathcal{M}_1 U_0 \\ \mathcal{B}\mathcal{M}_0^{-1}(\mathcal{M}_1 - \mathcal{M}_0)U_0 \end{pmatrix}, \begin{pmatrix} U_H \\ u_3 \end{pmatrix}\right] \\
&= [\mathcal{M}_1\mathcal{M}_0^{-1}\mathcal{M}_1 U_0, U_H]_1 - [\mathcal{M}_1 U_0, U_H]_1 \\
&\quad +(\mathcal{B}\mathcal{M}_0^{-1}\mathcal{M}_1 U_0, u_3) - (\mathcal{B}U_0, u_3) \\
&= [\mathcal{M}_1\mathcal{M}_0^{-1}\mathcal{M}_1 U_0, -\mathcal{M}_1^{-1}\mathcal{B}^* u_3]_1 - [\mathcal{M}_1 U_0, -\mathcal{M}_1^{-1}\mathcal{B}^* u_3]_1 \\
&\quad +(\mathcal{B}\mathcal{M}_0^{-1}\mathcal{M}_1 U_0, u_3) - (\mathcal{B}U_0, u_3) \\
&= -[\mathcal{M}_0^{-1}\mathcal{M}_1 U_0, \mathcal{B}^* u_3]_1 + [U_0, \mathcal{B}^* u_3]_1 \\
&\quad +(\mathcal{B}\mathcal{M}_0^{-1}\mathcal{M}_1 U_0, u_3) - (\mathcal{B}U_0, u_3).
\end{aligned}$$

But $[U_0, \mathcal{B}^* u_3]_1 = (\mathcal{B}U_0, u_3)$ and $[\mathcal{M}_0^{-1}\mathcal{M}_1 U_0, \mathcal{B}^* u_3]_1 = (\mathcal{B}\mathcal{M}_0^{-1}\mathcal{M}_1 U_0, u_3)$ since $\mathcal{B}$ is of the form $(0 \ \ -B)$. Therefore,

$$\left[\mathcal{M}\begin{pmatrix} U_0 \\ 0 \end{pmatrix}, \begin{pmatrix} U_H \\ u_3 \end{pmatrix}\right] = 0$$

and the proof is finished. □

Let us define the following block-diagonal matrix:

$$\tilde{\mathcal{M}} := \begin{pmatrix} I & 0 \\ 0 & \mathcal{B}\mathcal{M}_1^{-1}\mathcal{B}^* \end{pmatrix}.$$

For what follows, it is of great importance to know that the symmetric part of $\tilde{\mathcal{M}}$ is positive definite with respect to $[\cdot,\cdot]$. This is proved by the next lemma.

Lemma 5 *There holds*

$$\left[\tilde{\mathcal{M}}\begin{pmatrix} U \\ u_3 \end{pmatrix}, \begin{pmatrix} U \\ u_3 \end{pmatrix}\right] = \left[\begin{pmatrix} I & 0 \\ 0 & B(B_1 A_1^{-1} B_1^*)^{-1} B^* \end{pmatrix}\begin{pmatrix} U \\ u_3 \end{pmatrix}, \begin{pmatrix} U \\ u_3 \end{pmatrix}\right]$$

for any $(U, u_3) \in S^1 \times S^2 \times S^3$ *and therefore, the symmetric part of* $\tilde{\mathcal{M}}$ *with respect to* $[\cdot,\cdot]$ *is positive definite.*

Proof. By the definition of $\tilde{\mathcal{M}}$ there holds

$$\left[\tilde{\mathcal{M}}\begin{pmatrix} U \\ u_3 \end{pmatrix}, \begin{pmatrix} U \\ u_3 \end{pmatrix}\right] = \left[\begin{pmatrix} U \\ 0 \end{pmatrix}, \begin{pmatrix} U \\ 0 \end{pmatrix}\right] + u_3^T \mathcal{B}\mathcal{M}_1^{-1}\mathcal{B}^* u_3.$$

For the last term we obtain

$$u_3^T \mathcal{B}\mathcal{M}_1^{-1}\mathcal{B}^* u_3 = \begin{pmatrix} 0 \\ B^* u_3 \end{pmatrix}^T \mathcal{M}_1^{-1}\begin{pmatrix} 0 \\ B^* u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ B^* u_3 \end{pmatrix}^T \begin{pmatrix} v \\ w \end{pmatrix} = u_3^T B w$$

where (v, w) is defined by

$$\mathcal{M}_1 \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} A_0^{-1}A_1 & A_0^{-1}B_1^* \\ B_1A_0^{-1}(A_1 - A_0) & B_1A_0^{-1}B_1^* \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ B^*u_3 \end{pmatrix}.$$

By the last relation there holds

$$w = (B_1A_0^{-1}B_1^* - B_1A_0^{-1}(A_1 - A_0)A_1^{-1}A_0A_0^{-1}B_1^*)^{-1}B^*u_3 = (B_1A_1^{-1}B_1^*)^{-1}B^*u_3$$

and therefore,

$$u_3^T \mathcal{B}\mathcal{M}_1^{-1}\mathcal{B}^*u_3 = u_3^T B(B_1A_1^{-1}B_1^*)^{-1}B^*u_3.$$

By assumption B_1 and B are of full rank and A_1 is symmetric positive definite. Therefore, $B(B_1A_1^{-1}B_1^*)^{-1}B^*$ is symmetric positive definite and the proof is finished. □

The main result of this section is the positive definiteness with respect to $[\cdot,\cdot]$ of the transformed system matrix $\mathcal{M}$. Due to this result the conjugate gradient method can be used for the solution of (5) which is equivalent to solving (1).

Theorem 1 *Let the assumptions* (A1)–(A4) *be satisfied. Then $\mathcal{M}$ is symmetric positive definite with respect to $[\cdot,\cdot]$ and there hold the following inequalities:*

$$\Lambda_0 \left[\tilde{\mathcal{M}} \begin{pmatrix} U \\ u_3 \end{pmatrix}, \begin{pmatrix} U \\ u_3 \end{pmatrix}\right] \le \left[\mathcal{M} \begin{pmatrix} U \\ u_3 \end{pmatrix}, \begin{pmatrix} U \\ u_3 \end{pmatrix}\right] \le \Lambda_1 \left[\tilde{\mathcal{M}} \begin{pmatrix} U \\ u_3 \end{pmatrix}, \begin{pmatrix} U \\ u_3 \end{pmatrix}\right] \tag{10}$$

for any $(U, u_3) \in S^1 \times S^2 \times S^3$ where

$$\Lambda_0 = \left(1 + \frac{1}{2}(1 - \frac{\beta_0}{\lambda_1}) + \sqrt{\frac{1}{4}(1 - \frac{\beta_0}{\lambda_1})^2 + 1 - \frac{\beta_0}{\lambda_1}}\right)^{-1} \quad \textit{and} \quad \Lambda_1 = \frac{\lambda_1}{\beta_0}\left(1 + \sqrt{1 - \frac{\beta_0}{\lambda_1}}\right).$$

Proof. The symmetry of $\mathcal{M}$ is proved by Lemma 3 and, once the lower bound in (10) is proved, the positive definiteness of $\mathcal{M}$ follows from Lemma 5. It therefore remains to prove (10). For this we adapt the proof of Theorem 1 in [4] to our situation.

Let $(U, u_3)^T = (u_1, u_2, u_3)^T \in S^1 \times S^2 \times S^3$ be given. As in Lemma 4 we consider the representation

$$\begin{pmatrix} U \\ u_3 \end{pmatrix} = \begin{pmatrix} U_0 \\ 0 \end{pmatrix} + \begin{pmatrix} U_H \\ u_3 \end{pmatrix}$$

where U_H is defined by

$$\mathcal{M}_1 U_H + \mathcal{B}^* u_3 = 0.$$

Let us collect the following technical results which are due to the relation $U_H = -\mathcal{M}_1^{-1}\mathcal{B}^*u_3$. There holds

$$[\mathcal{M}_1 U_H, U_H]_1 = [\mathcal{M}_1\mathcal{M}_1^{-1}\mathcal{B}^*u_3, \mathcal{M}_1^{-1}\mathcal{B}^*u_3]_1 = [\mathcal{M}_1^{-1}\mathcal{B}^*u_3, \mathcal{B}^*u_3]_1 \tag{11}$$

and

$$\left[\mathcal{M}\begin{pmatrix} U_H \\ u_3 \end{pmatrix}, \begin{pmatrix} U_H \\ u_3 \end{pmatrix}\right] = \left[\begin{pmatrix} \mathcal{M}_0^{-1}\mathcal{M}_1 U_H + \mathcal{M}_0^{-1}\mathcal{B}^* u_3 \\ \mathcal{B}\mathcal{M}_0^{-1}(\mathcal{M}_1 - \mathcal{M}_0)U_H + \mathcal{B}\mathcal{M}_0^{-1}\mathcal{B}^* u_3 \end{pmatrix}, \begin{pmatrix} U_H \\ u_3 \end{pmatrix}\right] \tag{12}$$
$$= \left[\begin{pmatrix} 0 \\ \mathcal{B}\mathcal{M}_0^{-1}(\mathcal{M}_1 - \mathcal{M}_0)(-\mathcal{M}_1^{-1}\mathcal{B}^* u_3) + \mathcal{B}\mathcal{M}_0^{-1}\mathcal{B}^* u_3 \end{pmatrix}, \begin{pmatrix} U_H \\ u_3 \end{pmatrix}\right]$$
$$= (\mathcal{B}\mathcal{M}_1^{-1}\mathcal{B}^* u_3, u_3).$$

We also note that there holds

$$\left[\mathcal{M}\begin{pmatrix} U_0 \\ 0 \end{pmatrix}, \begin{pmatrix} U_0 \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} \mathcal{M}_0^{-1}\mathcal{M}_1 U_0 \\ \mathcal{B}\mathcal{M}_0^{-1}(\mathcal{M}_1 - \mathcal{M}_0)U_0 \end{pmatrix}, \begin{pmatrix} U_0 \\ 0 \end{pmatrix}\right] \tag{13}$$
$$= [\mathcal{M}_1\mathcal{M}_0^{-1}\mathcal{M}_1 U_0, U_0]_1 - [\mathcal{M}_1 U_0, U_0]_1$$
$$= [(\mathcal{M}_1 - \mathcal{M}_0)\mathcal{M}_0^{-1}(\mathcal{M}_1 - \mathcal{M}_0)U_0, U_0]_1 + [(\mathcal{M}_1 - \mathcal{M}_0)U_0, U_0]_1.$$

Now let us prove the lower boundedness of $\mathcal{M}$ by $\tilde{\mathcal{M}}$. An upper bound for $\tilde{\mathcal{M}}$ is as follows:

$$\left[\tilde{\mathcal{M}}\begin{pmatrix} U \\ u_3 \end{pmatrix}, \begin{pmatrix} U \\ u_3 \end{pmatrix}\right] = [(\mathcal{M}_1 - \mathcal{M}_0)U, U]_1 + (\mathcal{B}\mathcal{M}_1^{-1}\mathcal{B}^* u_3, u_3) \tag{14}$$
$$\le (1+\gamma)[(\mathcal{M}_1 - \mathcal{M}_0)U_0, U_0]_1 + (1+\frac{1}{\gamma})[(\mathcal{M}_1 - \mathcal{M}_0)U_H, U_H]_1$$
$$+(\mathcal{B}\mathcal{M}_1^{-1}\mathcal{B}^* u_3, u_3)$$

which holds for any positive number γ. Combination of Lemma 2 and (11) yields

$$[(\mathcal{M}_1 - \mathcal{M}_0)U_H, U_H]_1 \le (1 - \frac{\beta_0}{\lambda_1})[\mathcal{M}_1 U_H, U_H]_1 = (1 - \frac{\beta_0}{\lambda_1})[\mathcal{M}_1^{-1}\mathcal{B}^* u_3, \mathcal{B}^* u_3]_1 \tag{15}$$

and, therefore,

$$\left[\tilde{\mathcal{M}}\begin{pmatrix} U \\ u_3 \end{pmatrix}, \begin{pmatrix} U \\ u_3 \end{pmatrix}\right] \le (1+\gamma)[(\mathcal{M}_1 - \mathcal{M}_0)U_0, U_0]_1 + (1 - \frac{\beta_0}{\lambda_1})(1+\frac{1}{\gamma})[\mathcal{M}_1^{-1}\mathcal{B}^* u_3, \mathcal{B}^* u_3]_1$$
$$+(\mathcal{B}\mathcal{M}_1^{-1}\mathcal{B}^* u_3, u_3)$$
$$= (1+\gamma)[(\mathcal{M}_1 - \mathcal{M}_0)U_0, U_0]_1$$
$$+ \left(1 + (1 - \frac{\beta_0}{\lambda_1})(1+\frac{1}{\gamma})\right)(\mathcal{B}\mathcal{M}_1^{-1}\mathcal{B}^* u_3, u_3). \tag{16}$$

Here we used the relation

$$[\mathcal{M}_1^{-1}\mathcal{B}^* u_3, \mathcal{B}^* u_3]_1 = (\mathcal{B}\mathcal{M}_1^{-1}\mathcal{B}^* u_3, u_3) \tag{17}$$

which holds since $\mathcal{B}$ has the special form $\begin{pmatrix} 0 & -B \end{pmatrix}$. Now, using (9), (13), and (12), we obtain

$$\left[\mathcal{M}\begin{pmatrix} U \\ u_3 \end{pmatrix}, \begin{pmatrix} U \\ u_3 \end{pmatrix}\right] = \left[\mathcal{M}\begin{pmatrix} U_0 \\ 0 \end{pmatrix}, \begin{pmatrix} U_0 \\ 0 \end{pmatrix}\right] + \left[\mathcal{M}\begin{pmatrix} U_H \\ u_3 \end{pmatrix}, \begin{pmatrix} U_H \\ u_3 \end{pmatrix}\right]$$
$$\ge [(\mathcal{M}_1 - \mathcal{M}_0)U_0, U_0]_1 + (\mathcal{B}\mathcal{M}_1^{-1}\mathcal{B}^* u_3, u_3). \tag{18}$$

Hence, the lower bound for $\mathcal{M}$ follows by combining (16) and (18) and by choosing

$$\gamma = \frac{1}{2}(1-\frac{\beta_0}{\lambda_1}) + \sqrt{\frac{1}{4}(1-\frac{\beta_0}{\lambda_1})^2 + 1 - \frac{\beta_0}{\lambda_1}}.$$

It remains to prove the upper bound for $\mathcal{M}$. First we estimate

$$\left[\mathcal{M}\begin{pmatrix} U_0 \\ 0 \end{pmatrix}, \begin{pmatrix} U_0 \\ 0 \end{pmatrix}\right].$$

In order to do so we need an intermediate result. Using Lemma 2, Lemma 1(iii), and Assumption (A3), we find that there holds

$$\begin{aligned}
[(\mathcal{M}_1 - \mathcal{M}_0)^{-1} U, U]_1 &\geq \frac{1}{1-\beta_0/\lambda_1}[\mathcal{M}_1^{-1} U, U]_1 \geq \frac{1}{\lambda_1}\frac{1}{1-\beta_0/\lambda_1}[\tilde{\mathcal{M}}_1^{-1} U, U]_1 \\
&\geq \frac{\beta_0}{\lambda_1 - \beta_0}[\mathcal{M}_0^{-1} U, U]_1.
\end{aligned}$$

Therefore, using this estimate and (13), we obtain

$$\begin{aligned}
\left[\mathcal{M}\begin{pmatrix} U_0 \\ 0 \end{pmatrix}, \begin{pmatrix} U_0 \\ 0 \end{pmatrix}\right] &= [(\mathcal{M}_1 - \mathcal{M}_0)\mathcal{M}_0^{-1}(\mathcal{M}_1 - \mathcal{M}_0)U_0, U_0]_1 + [(\mathcal{M}_1 - \mathcal{M}_0)U_0, U_0]_1 \\
&\leq \frac{\lambda_1 - \beta_0}{\beta_0}[(\mathcal{M}_1 - \mathcal{M}_0)U_0, U_0]_1 + [(\mathcal{M}_1 - \mathcal{M}_0)U_0, U_0]_1 \\
&= \frac{\lambda_1}{\beta_0}[(\mathcal{M}_1 - \mathcal{M}_0)U_0, U_0]_1 \\
&\leq \frac{\lambda_1}{\beta_0}(1+\gamma)[(\mathcal{M}_1 - \mathcal{M}_0)U, U]_1 + \frac{\lambda_1}{\beta_0}(1+\frac{1}{\gamma})[(\mathcal{M}_1 - \mathcal{M}_0)U_H, U_H]_1
\end{aligned}$$

for any positive number γ. By (15) we then obtain together with the relation (17)

$$\left[\mathcal{M}\begin{pmatrix} U_0 \\ 0 \end{pmatrix}, \begin{pmatrix} U_0 \\ 0 \end{pmatrix}\right] \leq \frac{\lambda_1}{\beta_0}(1+\gamma)[(\mathcal{M}_1 - \mathcal{M}_0)U, U]_1 + (1-\frac{\beta_0}{\lambda_1})\frac{\lambda_1}{\beta_0}(1+\frac{1}{\gamma})(\mathcal{B}\mathcal{M}_1^{-1}\mathcal{B}^* u_3, u_3).$$

Therefore, by Lemma 4 and (12), we eventually obtain the estimate

$$\begin{aligned}
\left[\mathcal{M}\begin{pmatrix} U \\ u_3 \end{pmatrix}, \begin{pmatrix} U \\ u_3 \end{pmatrix}\right] &= \left[\mathcal{M}\begin{pmatrix} U_0 \\ 0 \end{pmatrix}, \begin{pmatrix} U_0 \\ 0 \end{pmatrix}\right] + \left[\mathcal{M}\begin{pmatrix} U_H \\ u_3 \end{pmatrix}, \begin{pmatrix} U_H \\ u_3 \end{pmatrix}\right] \\
&\leq \frac{\lambda_1}{\beta_0}(1+\gamma)[(\mathcal{M}_1 - \mathcal{M}_0)U, U]_1 \\
&\quad + \left(1 + (\frac{\lambda_1}{\beta_0} - 1)(1+\frac{1}{\gamma})\right)(\mathcal{B}\mathcal{M}_1^{-1}\mathcal{B}^* u_3, u_3).
\end{aligned}$$

Comparing this bound with the relation (14) the upper bound for $\mathcal{M}$ then follows by choosing

$$\gamma = \sqrt{1 - \frac{\beta_0}{\lambda_1}}.$$

□

3 Application to a dual-dual mixed method

3.1 The model problem

We now present a model problem which leads to linear systems of the dual-dual form (1). In order to solve those systems we perform the transformation $\mathcal{A} \to \mathcal{M}$ as described in §2 and use the conjugate gradient method with the special inner product $[\cdot,\cdot]$, see Section 3.2. A straight forward transformation leads to a system matrix that is ill-conditioned and, thus, requires a large number of CG-iterations. In Section 3.2.1 we study a special transformation (i.e., special preconditioners A_0, $\mathcal{M}_0$) which leads to a bounded number of iterations as long as the Assumptions (A1)–(A4) are satisfied. However, for this special transformation, (A1)–(A4) can not be uniformly satisfied. We therefore also study the preconditioned conjugate gradient method that is based on the first transformation that uniformly satisfies (A1)–(A4). This PCG-method requires only a bounded number of iterations and uses a preconditioner that is sparse and therefore cheap.

First let us describe the model problem. Let Ω be a polygonal domain in $\mathbf{R}^2$ with boundary $\Gamma := \partial\Omega$. Then, given $f \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma)$ and a matrix valued continuous function $\boldsymbol{\kappa}$, we consider the non-homogeneous Dirichlet problem: *Find* $u \in H^1(\Omega)$ *such that*

$$\begin{aligned} -\mathrm{div}\,(\boldsymbol{\kappa}\nabla u) &= f \quad \textit{in}\ \Omega \\ u &= g \quad \textit{on}\ \Gamma. \end{aligned} \tag{19}$$

Here, we assume that $\boldsymbol{\kappa}$ is symmetric and that there exists $C > 0$ such that

$$C\|\xi\|^2 \le \sum_{i,j=1}^{2} \kappa_{ij}(x)\xi_i\xi_j \quad \forall \xi := (\xi_1,\xi_2) \in \mathbf{R}^2, \quad \forall x \in \bar{\Omega}, \tag{20}$$

with κ_{ij} being the entries of $\boldsymbol{\kappa}$.

The standard mixed finite element method for (19) requires first the definition of the flux $\boldsymbol{\sigma} := \boldsymbol{\kappa}\nabla u$ as an auxiliary unknown. Then, using that $\boldsymbol{\kappa}$ is invertible (because of (20)), the integration by parts procedure is applied to the relation $\nabla u = \boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}$. Following [7] we introduce the additional explicit unknown $\boldsymbol{\theta} := \nabla u$, see also [10, 5]. However, instead of proceeding as in [10, 5], we use the variational formulation as a dual-dual operator equation as in [7], see also [6].

Let $X_1 := [L^2(\Omega)]^2$, $M_1 := H(\mathrm{div};\Omega)$, $X := X_1 \times M_1$, $M := L^2(\Omega)$, and define the bounded linear operators $\mathbf{A}_1 : X_1 \to X_1'$, $\mathbf{B}_1 : X_1 \to M_1'$, $\mathbf{A} : X \to X'$ and $\mathbf{B} : M_1 \to M'$, and the functionals $\mathbf{F}_1 \in X_1'$, $\mathbf{G}_1 \in M_1'$ and $\mathbf{G} \in M'$, as follows:

$$(\mathbf{A}_1(\boldsymbol{\theta}),\boldsymbol{\zeta})' := \int_\Omega \boldsymbol{\kappa}\boldsymbol{\theta}\cdot\boldsymbol{\zeta}\,dx, \quad (\mathbf{B}_1(\boldsymbol{\theta}),\boldsymbol{\tau})' := -\int_\Omega \boldsymbol{\theta}\cdot\boldsymbol{\tau}\,dx,$$

$$(\mathbf{A}(\boldsymbol{\theta},\boldsymbol{\sigma}),(\boldsymbol{\zeta},\boldsymbol{\tau}))' := (\mathbf{A}_1(\boldsymbol{\theta}),\boldsymbol{\zeta})' + (\mathbf{B}_1(\boldsymbol{\zeta}),\boldsymbol{\sigma})' + (\mathbf{B}_1(\boldsymbol{\theta}),\boldsymbol{\tau})',$$

$$(\mathbf{B}(\boldsymbol{\sigma}),v)' := -\int_\Omega v\,\mathrm{div}\boldsymbol{\sigma}\,dx, \quad (\mathbf{F}_1,\boldsymbol{\zeta})' := 0,$$

and

$$(\mathbf{G}_1,\boldsymbol{\tau})' := -\langle g,\boldsymbol{\tau}\cdot\nu\rangle_{L^2(\Gamma)}, \quad (\mathbf{G},v)' := \int_\Omega fv\,dx$$

for all $(\boldsymbol{\theta}, \boldsymbol{\sigma})$, $(\boldsymbol{\zeta}, \boldsymbol{\tau}) \in X$ and for all $v \in M$, where $(\cdot, \cdot)'$ stands for the duality pairing induced by the operators appearing in each case. Further, let $\mathbf{B}_1^* : M_1 \to X_1'$ and $\mathbf{B}^* : M \to M_1'$ be the adjoints of $\mathbf{B}_1$ and $\mathbf{B}$, respectively, and let $\mathbf{O}$ denote the null operator. It follows that $\mathbf{A}$ can be equivalently defined as

$$\mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\sigma}) := \begin{pmatrix} \mathbf{A}_1 & \mathbf{B}_1^* \\ \mathbf{B}_1 & \mathbf{O} \end{pmatrix} \begin{pmatrix} \boldsymbol{\theta} \\ \boldsymbol{\sigma} \end{pmatrix} \in X' := X_1' \times M_1'. \tag{21}$$

Then, the problem (19) can be reformulated as the following operator equation (see [7]): *Find* $((\boldsymbol{\theta}, \boldsymbol{\sigma}), u) \in X \times M$ *such that*

$$\begin{pmatrix} \mathbf{A} & \tilde{\mathbf{B}}^* \\ \tilde{\mathbf{B}} & \mathbf{O} \end{pmatrix} \begin{pmatrix} (\boldsymbol{\theta}, \boldsymbol{\sigma}) \\ u \end{pmatrix} = \begin{pmatrix} \mathbf{F} \\ \mathbf{G} \end{pmatrix}, \tag{22}$$

where $\mathbf{F} := (\mathbf{F}_1, \mathbf{G}_1) \in X' := X_1' \times M_1'$ and $\tilde{\mathbf{B}} : X \to M'$ is given by $\tilde{\mathbf{B}} := (\mathbf{O} \quad \mathbf{B})$.

The equation (22) constitutes the so-called *dual-dual mixed formulation* of our model problem (19) since the operator $\mathbf{A}$ itself has the dual-type structure given by (21). This problem is uniquely solvable:

Theorem 2 [7, Theorem 4] *There exists a unique solution* $((\boldsymbol{\theta}, \boldsymbol{\sigma}), u) \in X \times M$ *of the dual-dual mixed formulation* (22).

Now we recall the Galerkin procedure from [7] for the approximate solution of (22). Let $\mathcal{T}_h$ be a regular triangulation of Ω made up of triangles T of diameter h_T such that $h := \sup_{T \in \mathcal{T}_h} h_T$ and $\bar{\Omega} = \cup\{T : T \in \mathcal{T}_h\}$. Next, we consider the canonical triangle with vertices $\hat{P}_1 = (0,0)^T$, $\hat{P}_2 = (1,0)^T$ and $\hat{P}_3 = (0,1)^T$ as a reference triangle $\hat{T}$, and introduce the family of bijective affine mappings $\{F_T\}_{T \in \mathcal{T}_h}$, such that $F_T(\hat{T}) = T$. It is well-known that $F_T(\hat{x}) = B_T \hat{x} + b_T$ for all $\hat{x} \in \hat{T}$, where the square matrix B_T of order 2 and $b_T \in \mathbf{R}^2$ depend only on the vertices of T.

We consider the lowest order Raviart-Thomas spaces. For each triangle $T \in \mathcal{T}_h$ let

$$\mathcal{RT}_0(T) := \{\boldsymbol{\tau} : \boldsymbol{\tau} = |\det(B_T)|^{-1} B_T \,\hat{\boldsymbol{\tau}} \circ F_T^{-1}, \ \hat{\boldsymbol{\tau}} \in \mathcal{RT}_0(\hat{T})\},$$

where

$$\mathcal{RT}_0(\hat{T}) := \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} \right\}.$$

Then, we define the finite element subspaces for the unknowns $\boldsymbol{\theta}$ and $\boldsymbol{\sigma}$, respectively, as follows:

$$X_{1,h} := \left\{ \boldsymbol{\zeta} \in [L^2(\Omega)]^2 : \ \boldsymbol{\zeta}|_T \in \mathcal{RT}_0(T) \quad \forall T \in \mathcal{T}_h \right\} \tag{23}$$

and

$$M_{1,h} := \{\boldsymbol{\tau} \in H(\mathrm{div}; \Omega) : \ \boldsymbol{\tau}|_T \in \mathcal{RT}_0(T) \quad \forall T \in \mathcal{T}_h\}. \tag{24}$$

Next, we put

$$X_h := X_{1,h} \times M_{1,h},$$

and consider the piecewise constant functions as the finite element subspace for the unknown u, i.e.

$$M_h := \{v \in L^2(\Omega) : \ v|_T \text{ is constant } \forall T \in \mathcal{T}_h\}. \tag{25}$$

Then, the Galerkin scheme for the continuous problem (22) reads: *Find* $((\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h), u_h) \in X_h \times M_h$ *such that*

$$\begin{array}{lllll} (\mathbf{A}_1(\boldsymbol{\theta}_h), \boldsymbol{\zeta})' & + \ (\mathbf{B}_1(\boldsymbol{\zeta}), \boldsymbol{\sigma}_h)' & & = & 0 \\ (\mathbf{B}_1(\boldsymbol{\theta}_h), \boldsymbol{\tau})' & & + \ (\mathbf{B}(\boldsymbol{\tau}), u_h)' & = & (\mathbf{G}_1, \boldsymbol{\tau})' \\ & (\mathbf{B}(\boldsymbol{\sigma}_h), v)' & & = & (\mathbf{G}, v)' \end{array} \tag{26}$$

for all $((\boldsymbol{\zeta}, \boldsymbol{\tau}), v) \in X_h \times M_h$.

The convergence of the Galerkin scheme is as follows.

Theorem 3 [7, Theorems 5, 6] *There exists a unique solution* $((\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h), u_h) \in X_h \times M_h$ *of the Galerkin system* (26). *Let* $((\boldsymbol{\theta}, \boldsymbol{\sigma}), u)$ *be the unique solution of* (22). *Assume that* $\boldsymbol{\theta}|_T \in [H^1(T)]^2 \ \forall T \in \mathcal{T}_h$, $\boldsymbol{\sigma} \in [H^1(\Omega)]^2$, $\operatorname{div} \boldsymbol{\sigma} \in H^1(\Omega)$ *and* $u \in H^1(\Omega)$. *Then, there exists* $C > 0$ *which is independent of* h *such that*

$$\|((\boldsymbol{\theta}, \boldsymbol{\sigma}), u) - ((\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h), u_h)\| \le Ch \left\{ \sum_{T \in \mathcal{T}_h} \|\boldsymbol{\theta}\|^2_{[H^1(T)]^2} + \|\boldsymbol{\sigma}\|^2_{[H^1(\Omega)]^2} + \|\operatorname{div} \boldsymbol{\sigma}\|^2_{H^1(\Omega)} + \|u\|^2_{H^1(\Omega)} \right\}^{1/2}.$$

3.2 Conjugate gradient method

Now let us study the conjugate gradient method for the solution of the linear system (26). This system is exactly of the form (1) where the matrices A_1, B_1, and B correspond to discretizations of the operators $\mathbf{A}_1$, $\mathbf{B}_1$, and $\mathbf{B}$, respectively. In the following we use plain symbols for matrices which correspond to operators with the analogous bold symbol. For a coefficient vector which belongs to a certain function we use the same symbol as for the function.

Given preconditioners A_0 for A_1 and $\mathcal{M}_0$ for $\mathcal{M}_1$ (as defined in (3)) we transform the linear system (26) into the form

$$\mathcal{M} \begin{pmatrix} X \\ x_3 \end{pmatrix} := \begin{pmatrix} \mathcal{M}_0^{-1}\mathcal{M}_1 & \mathcal{M}_0^{-1}\mathcal{B}^* \\ \mathcal{B}\mathcal{M}_0^{-1}(\mathcal{M}_1 - \mathcal{M}_0) & \mathcal{B}\mathcal{M}_0^{-1}\mathcal{B}^* \end{pmatrix} \begin{pmatrix} X \\ x_3 \end{pmatrix} = \begin{pmatrix} \mathcal{M}_0^{-1}F \\ \mathcal{B}\mathcal{M}_0^{-1}F + f_3 \end{pmatrix} \tag{27}$$

given by (5), and use the conjugate gradient method (CG) with the special inner product $[\cdot,\cdot]$ (8) as iterative solver. In order to characterize the convergence of the CG method we need to collect some estimates for the eigenvalues and singular values of A_1, B_1, and B.

Let us denote the eigenvalues of A_1 by

$$0 < \lambda_1 \le \lambda_1 \le \cdots \le \lambda_L.$$

We also need the singular values of B_1,

$$0 < \sigma_1 \le \sigma_2 \le \cdots \le \sigma_M$$

and those of B,

$$0 < \eta_1 \leq \eta_2 \leq \cdots \leq \eta_N.$$

These values depend on the scaling of the basis functions in use.

Lemma 6 *Let the basis functions of all the three ansatz spaces $X_{1,h}$, $M_{1,h}$ and M_h be scaled to L^∞-norm being $O(h^{-1})$. Then there exist generic positive constants c_0 and c_1 being independent of h such that there holds*

$$c_0 \leq \lambda_1 \leq \lambda_L \leq c_1, \tag{28}$$

$$c_0 \leq \sigma_1 \leq \sigma_M \leq c_1, \tag{29}$$

and

$$c_0 \leq \eta_1 \leq \eta_N \leq c_1 h^{-1}. \tag{30}$$

Proof. Since A_1 and B_1 are simple Gram matrices for the spaces $X_{1,h} \times X_{1,h}$ and $X_{1,h} \times M_{1,h}$, respectively, the estimates (28) and (29) are obvious. The proof of (30) is also straight forward, cf. the general result [8, Lemma 1]. For the convenience of the reader we recall the short proof. First we note that $\mathbf{B}$ is bounded and that it satisfies an inf-sup condition (see [7]): There exists $\beta > 0$ which is independent of h such that

$$\sup_{\boldsymbol{\tau} \in M_{1,h} \setminus \{0\}} \frac{(\mathbf{B}(\boldsymbol{\tau}), v)'}{\|\boldsymbol{\tau}\|_{M_1}} \geq \beta \|v\|_M \quad \text{for any } v \in M_h.$$

Now, taking the scalings within $M_{1,h}$, M_h and the inf-sup condition for $\mathbf{B}$ into account, we obtain

$$\begin{aligned}
\eta_1 &= \min_{v \in M_h \setminus \{0\}} \max_{\boldsymbol{\tau} \in M_{1,h} \setminus \{0\}} \frac{v^T B \boldsymbol{\tau}}{\sqrt{v^T v} \sqrt{\boldsymbol{\tau}^T \boldsymbol{\tau}}} \\
&= \min_{v \in M_h \setminus \{0\}} \max_{\boldsymbol{\tau} \in M_{1,h} \setminus \{0\}} \frac{(\mathbf{B}(\boldsymbol{\tau}), v)'}{\|v\|_M \|\boldsymbol{\tau}\|_{M_1}} \frac{\|v\|_M \|\boldsymbol{\tau}\|_{M_1}}{\sqrt{v^T v} \sqrt{\boldsymbol{\tau}^T \boldsymbol{\tau}}} \\
&\geq \min_{v \in M_h \setminus \{0\}} \max_{\boldsymbol{\tau} \in M_{1,h} \setminus \{0\}} \frac{(\mathbf{B}(\boldsymbol{\tau}), v)'}{\|v\|_M \|\boldsymbol{\tau}\|_{M_1}} \frac{\|v\|_{L^2(\Omega)} \|\boldsymbol{\tau}\|_{L^2(\Omega) \times L^2(\Omega)}}{\sqrt{v^T v} \sqrt{\boldsymbol{\tau}^T \boldsymbol{\tau}}} \geq \beta c.
\end{aligned}$$

For an upper bound of η_N we write

$$\eta_N = \max_{v \in M_h \setminus \{0\}} \max_{\boldsymbol{\tau} \in M_{1,h} \setminus \{0\}} \frac{(\mathbf{B}(\boldsymbol{\tau}), v)'}{\|v\|_M \|\boldsymbol{\tau}\|_{M_1}} \frac{\|v\|_M \|\boldsymbol{\tau}\|_{M_1}}{\sqrt{v^T v} \sqrt{\boldsymbol{\tau}^T \boldsymbol{\tau}}}.$$

By the inverse property of the basis functions within $M_{1,h}$ there holds

$$\|\boldsymbol{\tau}\|_{M_1}^2 = \|\boldsymbol{\tau}\|_{L^2(\Omega) \times L^2(\Omega)}^2 + \|\operatorname{div} \boldsymbol{\tau}\|_{L^2(\Omega)}^2 \leq (1 + h^{-2}) \|\boldsymbol{\tau}\|_{L^2(\Omega) \times L^2(\Omega)}^2$$

and thus, using the boundedness of $\mathbf{B}$,

$$\eta_N \leq (1 + h^{-2})^{1/2} \|\mathbf{B}\| \max_{v \in M_h \setminus \{0\}} \max_{\boldsymbol{\tau} \in M_{1,h} \setminus \{0\}} \frac{\|v\|_{L^2(\Omega)} \|\boldsymbol{\tau}\|_{L^2(\Omega) \times L^2(\Omega)}}{\sqrt{v^T v} \sqrt{\boldsymbol{\tau}^T \boldsymbol{\tau}}} \leq c h^{-1}.$$

Therefore, the proof of the lemma is finished. □

Now the main result, characterizing the convergence of the CG method when only scalings for A_0 and M_0 are used, is as follows.

Theorem 4 *Let the basis functions of all the three spaces $X_{1,h}$, $M_{1,h}$ and M_h be scaled such that their L^∞-norms are $O(h^{-1})$. Then there exist constants μ, ρ, ω such that with*

$$A_0 := \mu I \in \mathbf{R}^{L\times L}, \quad M_0 := \omega I \in \mathbf{R}^{M\times M}, \quad \mathcal{M}_0 := \begin{pmatrix} \rho I & 0 \\ 0 & M_0 \end{pmatrix} \in \mathbf{R}^{(L+M)\times(L+M)}$$

there holds

$$\Lambda_0 \left[\begin{pmatrix} \mathbf{U} \\ u \end{pmatrix}, \begin{pmatrix} \mathbf{U} \\ u \end{pmatrix} \right] \le \left[\mathcal{M} \begin{pmatrix} \mathbf{U} \\ u \end{pmatrix}, \begin{pmatrix} \mathbf{U} \\ u \end{pmatrix} \right] \le \Lambda_1 h^{-2} \left[\begin{pmatrix} \mathbf{U} \\ u \end{pmatrix}, \begin{pmatrix} \mathbf{U} \\ u \end{pmatrix} \right]$$

for any $(\mathbf{U}, u) \in X_h \times M_h$. Here, Λ_0, Λ_1 are positive constants being independent of h, I denotes the identity matrix of generic size and A_0, $\mathcal{M}_0$ (together with A_1 and $\mathcal{M}_1$) define the inner product $[\cdot,\cdot]$, cf. (8), (7). Moreover, choosing A_0 and $\mathcal{M}_0$ as above, the number of iterations of the conjugate gradient method with inner product $[\cdot,\cdot]$ for the solution of (27) is bounded by $O(h^{-1})$.

Proof. By Lemma 6 the mass matrix A_1 is uniformly spectrally equivalent to the identity matrix $I \in \mathbf{R}^{L\times L}$. Therefore, there exists a constant $\mu > 0$ such that Assumptions (A1), (A2) hold for $A_0 := \mu I$ and constants α_0, α_1 in (A1) being independent of h. Thus, in particular, Lemma 1(iii) holds with λ_0 and λ_1 being independent of h. Now, again by Lemma 6, $B_1 A_1^{-1} B_1^*$ is uniformly spectrally equivalent to $I \in \mathbf{R}^{M\times M}$. We conclude, that there exist constants ρ, $\omega > 0$ such that Assumptions (A3), (A4) hold for $\mathcal{M}_0 = \begin{pmatrix} \rho I & 0 \\ 0 & \omega I \end{pmatrix}$ and constants β_0, β_1 being independent of h. Eventually, we know that Theorem 1 holds with constants Λ_0 and Λ_1 that are also independent of h. Combining Theorem 1 and Lemma 5 we conclude that $\mathcal{M}$ is uniformly spectrally equivalent to $\begin{pmatrix} I & 0 \\ 0 & B(B_1 A_1^{-1} B_1^*)^{-1} B^* \end{pmatrix}$. Due to Lemma 6 the minimum eigenvalue of $B(B_1 A_1^{-1} B_1^*)^{-1} B^*$ is bounded from below by a positive constant which is independent of h and the maximum eigenvalue of $B(B_1 A_1^{-1} B_1^*)^{-1} B^*$ may grow like $O(h^{-2})$. Therefore, the spectral condition number of $\mathcal{M}$ behaves like $O(h^{-2})$ and the number of iterations of the CG method is bounded by $O(h^{-1})$. □

3.2.1 Improved transformation

By Theorem 4 we expect an increase of the iteration numbers of the CG method when the mesh size h is reduced to improve the approximation given by the mixed finite element solution. Therefore, a preconditioner is necessary to bound the iteration numbers. Since we already use preconditioners A_0 and $\mathcal{M}_0$ to construct the transformation of the original linear system, one could ask for preconditioners A_0 and $\mathcal{M}_0$ that lead to a well conditioned system matrix (and thus making an additional preconditioner for the transformed matrix superfluous). It turns out that such preconditioners exist but, on the other hand, they do not satisfy Assumptions (A1)–(A4) uniformly (i.e., the constants α_0, α_1, β_0, β_1 do not satisfy (A1)–(A4) for all h). Although this strategy, therefore, is not fully satisfying, it seems to be natural and we present the results which are, nevertheless, instructive.

In Section 3.2.2 below we will present a preconditioner for the matrix $\mathcal{M}$ (additionally to A_0 and $\mathcal{M}_0$) which leads to uniformly bounded iteration numbers and allows for uniformly satisfying Assumptions (A1)–(A4).

Let us recall the situation of Theorem 4. The increase in the condition number of $\mathcal{M}$ is due to the singular values of B depending on h. Considering the block $BM_0^{-1}B^*$ of $\mathcal{M}$ (cf. (6)) and remembering that $B = -\mathrm{div}$, an obvious choice for an appropriate preconditioner M_0 is, instead of a simple scaling, a matrix which is equivalent to the inner product within $M_{1,h} \subset H(\mathrm{div};\Omega)$. Indeed, in that case it turns out that the condition number of $\mathcal{M}$ does not depend on h, as long as A_0 and $\mathcal{M}_0$ are chosen such that the bilinear form $[\cdot,\cdot]$ is positive definite.

Let us take a closer look at this situation. We retain the scalings of the basis functions in $X_{1,h}$ and M_h, and require that M_0 is uniformly spectrally equivalent to the $H(\mathrm{div};\Omega)$-inner product in $M_{1,h}$, i.e.

$$c_0 \boldsymbol{\tau}^T M_0 \boldsymbol{\tau} \le \|\boldsymbol{\tau}\|^2_{H(\mathrm{div};\Omega)} \le c_1 \boldsymbol{\tau}^T M_0 \boldsymbol{\tau} \quad \text{for any } \boldsymbol{\tau} \in M_{1,h}, \tag{31}$$

where c_0 and c_1 do not depend on h. We then consider preconditioners $A_0 = \mu I$ (as before) and $\mathcal{M}_0$ of the form

$$\mathcal{M}_0 := \begin{pmatrix} \rho I & 0 \\ 0 & \omega M_0 \end{pmatrix} \in \mathbf{R}^{(L+M)\times(L+M)}$$

where ρ and ω are positive numbers.

As in the previous theorem, Assumptions (A1), (A2) do not cause problems. The parameter μ can be chosen independently of h since A_1 is uniformly spectrally equivalent to the identity. However, since M_0 is uniformly spectrally equivalent to the $H(\mathrm{div};\Omega)$-inner product in $M_{1,h}$, the preconditioner $\mathcal{M}_0$ cannot be uniformly spectrally equivalent to $\tilde{\mathcal{M}}_1$ (cf. (4)) which is spectrally equivalent to the identity. Thus, the numbers β_0 and β_1 in Assumption (A3) cannot both be independent of h. Nevertheless, let us assume that the parameters ρ and ω are chosen in such a way that

$$\beta_0((A_1 - A_0)u_1, u_1) \le \rho((A_1 - A_0)u_1, u_1) \le \beta_1((A_1 - A_0)u_1, u_1) \quad \text{for any } u_1 \in X_{1,h}$$

and

$$\beta_0(B_1 A_1^{-1} B_1^* u_2, u_2) \le \omega(M_0 u_2, u_2) \le \beta_1(B_1 A_1^{-1} B_1^* u_2, u_2) \quad \text{for any } u_2 \in M_{1,h} \tag{32}$$

hold with $\beta_1/\lambda_0 < 1$. Then, both (A3) and (A4) are satisfied and $[\cdot,\cdot]$ is an inner product. Obviously, also ρ can be chosen independently of h. On the other hand, by the scaling to $O(h^{-1})$ in $M_{1,h}$ and by the inverse property, there holds

$$u_2^T u_2 \simeq \|u_2\|^2_{L^2(\Omega)\times L^2(\Omega)} \le \|u_2\|^2_{H(\mathrm{div};\Omega)} \le c(1+h^{-2})\|u_2\|^2_{L^2(\Omega)\times L^2(\Omega)} \simeq h^{-2} u_2^T u_2$$

for any $u_2 \in M_{1,h}$. Since $u_2^T M_0 u_2 \simeq \|u_2\|^2_{H(\mathrm{div};\Omega)}$ by assumption there holds $\lambda_{\min}(M_0) = O(1)$ and $\lambda_{\max}(M_0) = O(h^{-2})$. Now, keeping ω fixed, the parameter β_1 in (32) must eventually increase if the mesh size is reduced since the maximum eigenvalue of $B_1 A_1^{-1} B_1^*$ is bounded by Lemma 6. Therefore, the bilinear form $[\cdot,\cdot]$ may become indefinite if we fix ω and then, the CG method may fail.

Theorem 5 *Let the basis functions of $X_{1,h}$ and M_h be scaled such that their L^∞-norms are $O(h^{-1})$. Moreover, assume that M_0 is given such that* (31) *holds for constants c_0 and c_1 that are independent of h. Let the preconditioners A_0 and $\mathcal{M}_0$ be of the form*

$$A_0 := \mu I \in \mathbf{R}^{L\times L}, \quad \mathcal{M}_0 := \begin{pmatrix} \rho I & 0 \\ 0 & \omega M_0 \end{pmatrix} \in \mathbf{R}^{(L+M)\times(L+M)}$$

such that Assumptions (A1)–(A4) *are satisfied. The numbers μ and ρ can be chosen independently of h and ω has to be sufficiently small, asymptotically $\omega = O(h^2)$ is sufficient. Then there holds*

$$\Lambda_0 \left[\begin{pmatrix} \mathbf{U} \\ u \end{pmatrix}, \begin{pmatrix} \mathbf{U} \\ u \end{pmatrix} \right] \le \left[\mathcal{M} \begin{pmatrix} \mathbf{U} \\ u \end{pmatrix}, \begin{pmatrix} \mathbf{U} \\ u \end{pmatrix} \right] \le \frac{\Lambda_1}{\omega} \left[\begin{pmatrix} \mathbf{U} \\ u \end{pmatrix}, \begin{pmatrix} \mathbf{U} \\ u \end{pmatrix} \right]$$

for any $(\mathbf{U}, u) \in X_h \times M_h$. Here the positive constants Λ_0, Λ_1 are independent of h. The conjugate gradient method may fail if ω is not small enough. Otherwise, the number of iterations of the conjugate gradient method with inner product $[\cdot,\cdot]$ for the solution of (27) *is bounded by $O(\omega^{-1/2})$.*

Proof. We avoid making explicit use of the parameters β_0 and β_1 in Assumption (A3). Instead, we only assume that $A_1 - A_0$ and $\mathcal{M}_1 - \mathcal{M}_0$ are positive definite, i.e., that $[\cdot,\cdot]$ is an inner product. This can be ensured by choosing the parameters μ, ρ and ω appropriately. As mentioned in the proof of Theorem 4 μ and ρ can be chosen independently of h. Even though the parameters β_0 and β_1 may depend on h (or on ω) we can use Theorem 1 to obtain the bound Λ_0. In fact, $\Lambda_0 \ge \frac{2}{3+\sqrt{5}}$ by setting $\beta_0 = 0$. It remains to bound Λ_1. There holds

$$\begin{aligned}
\left[\mathcal{M} \begin{pmatrix} \mathbf{U} \\ u \end{pmatrix}, \begin{pmatrix} \mathbf{U} \\ u \end{pmatrix} \right] &= \left[\begin{pmatrix} \mathcal{M}_0^{-1}\mathcal{M}_1 \mathbf{U} + \mathcal{M}_0^{-1}\mathcal{B}^* u \\ \mathcal{B}\mathcal{M}_0^{-1}(\mathcal{M}_1 - \mathcal{M}_0)\mathbf{U} + \mathcal{B}\mathcal{M}_0^{-1}\mathcal{B}^* u \end{pmatrix}, \begin{pmatrix} \mathbf{U} \\ u \end{pmatrix} \right] \\
&= \left[(\mathcal{M}_1 - \mathcal{M}_0)(\mathcal{M}_0^{-1}\mathcal{M}_1\mathbf{U} + \mathcal{M}_0^{-1}\mathcal{B}^* u), \mathbf{U} \right]_1 \\
&\quad + (\mathcal{B}\mathcal{M}_0^{-1}(\mathcal{M}_1 - \mathcal{M}_0)\mathbf{U} + \mathcal{B}\mathcal{M}_0^{-1}\mathcal{B}^* u, u) \\
&= \left[(\mathcal{M}_1 - \mathcal{M}_0)\mathcal{M}_0^{-1}\mathcal{M}_1\mathbf{U}, \mathbf{U} \right]_1 + \left[(\mathcal{M}_1 - \mathcal{M}_0)\mathcal{M}_0^{-1}\mathcal{B}^* u, \mathbf{U} \right]_1 \\
&\quad + (\mathcal{B}\mathcal{M}_0^{-1}(\mathcal{M}_1 - \mathcal{M}_0)\mathbf{U}, u) + (\mathcal{B}\mathcal{M}_0^{-1}\mathcal{B}^* u, u) \\
&= \left[(\mathcal{M}_1 - \mathcal{M}_0)\mathcal{M}_0^{-1}\mathcal{M}_1\mathbf{U}, \mathbf{U} \right]_1 + 2\left[(\mathcal{M}_1 - \mathcal{M}_0)\mathcal{M}_0^{-1}\mathcal{B}^* u, \mathbf{U} \right]_1 \\
&\quad + (\mathcal{B}\mathcal{M}_0^{-1}\mathcal{B}^* u, u) \\
&= \left[(\mathcal{M}_1 - \mathcal{M}_0)\mathcal{M}_0^{-1}\mathcal{M}_1\mathbf{U}, \mathbf{U} \right]_1 + 2\left[\mathcal{M}_0^{-1/2}\mathcal{B}^* u, \mathcal{M}_0^{-1/2}(\mathcal{M}_1 - \mathcal{M}_0)\mathbf{U} \right]_1 \\
&\quad + (\mathcal{B}\mathcal{M}_0^{-1}\mathcal{B}^* u, u) \\
&\le \left[(\mathcal{M}_1 - \mathcal{M}_0)\mathcal{M}_0^{-1}\mathcal{M}_1\mathbf{U}, \mathbf{U} \right]_1 + \left[(\mathcal{M}_1 - \mathcal{M}_0)\mathcal{M}_0^{-1}(\mathcal{M}_1 - \mathcal{M}_0)\mathbf{U}, \mathbf{U} \right]_1
\end{aligned}$$

$$
\begin{aligned}
& +2(\mathcal{B}\mathcal{M}_0^{-1}\mathcal{B}^* u, u) \\
\leq\ & 2\left[(\mathcal{M}_1 - \mathcal{M}_0)\mathcal{M}_0^{-1}\mathcal{M}_1\mathbf{U}, \mathbf{U}\right]_1 + 2(\mathcal{B}\mathcal{M}_0^{-1}\mathcal{B}^* u, u).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Lambda_1 &\leq 2 \max_{\mathbf{U}\in X_{1,h}\times M_{1,h}\setminus\{0\}} \frac{\left[(\mathcal{M}_1 - \mathcal{M}_0)\mathcal{M}_0^{-1}\mathcal{M}_1\mathbf{U}, \mathbf{U}\right]_1}{[(\mathcal{M}_1 - \mathcal{M}_0)\mathbf{U}, \mathbf{U}]_1} + 2 \max_{u\in M_h\setminus\{0\}} \frac{(\mathcal{B}\mathcal{M}_0^{-1}\mathcal{B}^* u, u)}{(u,u)} \\
&= 2\lambda_{\max}(\mathcal{M}_0^{-1}\mathcal{M}_1) + \frac{2}{\omega}\lambda_{\max}(BM_0^{-1}B^*) \qquad (33)
\end{aligned}
$$

where $\lambda_{\max}$ denotes the respective maximum eigenvalue. By using Lemma 6 and the estimate $\lambda_{\min}(M_0) = O(1)$ we obtain

$$
\lambda_{\max}(BM_0^{-1}B^*) = O(1),
$$

and therefore, it remains to bound $\lambda_{\max}(\mathcal{M}_0^{-1}\mathcal{M}_1)$. There holds

$$
\begin{aligned}
\lambda_{\max}(\mathcal{M}_0^{-1}\mathcal{M}_1) &= \lambda_{\max}(\mathcal{M}_0^{-1/2}\mathcal{M}_1\mathcal{M}_0^{-1/2}) \\
&= \lambda_{\max}\begin{pmatrix} \frac{1}{\rho}A_0^{-1}A_1 & \frac{1}{\sqrt{\omega\rho}}A_0^{-1}B_1^*M_0^{-1/2} \\ \frac{1}{\sqrt{\omega\rho}}M_0^{-1/2}B_1A_0^{-1}(A_1 - A_0) & \frac{1}{\omega}M_0^{-1/2}B_1A_0^{-1}B_1^*M_0^{-1/2} \end{pmatrix} \\
&=: \lambda_{\max}\begin{pmatrix} A & R^* \\ L & D \end{pmatrix}.
\end{aligned}
$$

We have to bound from above the eigenvalues λ of the problem

$$
\begin{aligned}
Ax + R^*y &= \lambda x, \\
Lx + Dy &= \lambda y
\end{aligned}
\qquad (34)
$$

with symmetric positive definite matrices A and D (note that A_0^{-1} is only a scaling in the present situation). Using $y = -(D - \lambda I)^{-1}Lx$, we obtain

$$
Ax - \lambda x = R^*(D - \lambda I)^{-1}Lx.
$$

If $x = 0$ then $\lambda \leq \lambda_{\max}(D)$. Therefore, let us assume that $x \neq 0$ and $\lambda > \lambda_{\max}(D)$. Then, $D - \lambda I$ is negative definite and there holds

$$
\begin{aligned}
\lambda_{\max}(A) - \lambda &\geq \max_{x\neq 0} \frac{x^T R^*(D - \lambda I)^{-1}Lx}{x^T x} \\
&= \frac{1}{\omega\rho} \max_{x\neq 0} \frac{x^T A_0^{-1}B_1^*M_0^{-1/2}(D - \lambda)^{-1}M_0^{-1/2}B_1A_0^{-1}(A_1 - A_0)x}{x^T x} \\
&\geq \frac{1}{\omega\rho} \max_{x\neq 0,\, y\neq 0} \frac{x^T A_0^{-1}B_1^*M_0^{-1/2}(D - \lambda)^{-1}M_0^{-1/2}B_1A_0^{-1}y}{\sqrt{x^T x}\sqrt{y^T(A_1 - A_0)^{-2}y}}.
\end{aligned}
$$

Since A_0^{-1}, $A_1 - A_0$ and $B_1^* B_1$ are all bounded from above, the bounds being independent of h, there exists a constant $c > 0$ (depending on ρ) such that

$$\begin{aligned}
\lambda_{\max}(A) - \lambda &\geq \frac{c}{\omega} \max_{x \neq 0,\, y \neq 0} \frac{x^T M_0^{-1/2} (D - \lambda)^{-1} M_0^{-1/2} y}{\sqrt{x^T x}\sqrt{y^T y}} \\
&= \frac{c}{\omega} \max_{x \neq 0} \frac{x^T M_0^{-1/2} (D - \lambda)^{-1} M_0^{-1/2} x}{x^T x} \\
&= \frac{c}{\omega} \max_{x \neq 0,\, y \neq 0} \frac{x^T (\omega^{-1} M_0^{-1/2} B_1 A_0^{-1} B_1^* M_0^{-1/2} - \lambda)^{-1} y}{x^T M_0 x} \\
&\geq c \frac{\Big(\lambda_{\max}(B_1 A_0^{-1} B_1^*) \lambda_{\min}(M_0)^{-1} - \omega\lambda\Big)^{-1}}{\lambda_{\min}(M_0)}.
\end{aligned}$$

Here, for the first equality, we made use of the fact that the maximum singular value of a symmetric matrix is nothing but the maximum eigenvalue of that matrix. Now, since $\lambda_{\min}(M_0) = O(1)$ as previously seen, there exists another constant $0 < C < \omega\lambda$ such that

$$\lambda_{\max}(A) - \lambda \geq c(C - \omega\lambda)^{-1},$$

i.e.

$$\lambda^2 - \lambda(C\omega^{-1} + \lambda_{\max}(A)) - c\omega^{-1} \leq 0$$

and therefore, including the assumptions on λ from above,

$$\begin{aligned}
\lambda_{\max}(\mathcal{M}_0^{-1}\mathcal{M}_1) &= \max\{\lambda : \lambda \text{ is eigenvalue of } (34)\} \\
&\leq \max\{O(\omega^{-1}), \lambda_{\max}(D)\} \\
&= \frac{1}{\omega} \max\{O(1), \lambda_{\max}(M_0^{-1/2} B_1 A_0^{-1} B_1^* M_0^{-1/2})\}
\end{aligned}$$

which is bounded independently of h. This is what was left to be proved, cf. (33), and the proof of the theorem is finished. □

Remark. *If we do not fix ω in the preconditioner $\mathcal{M}_0$ of the previous theorem but choose $\omega = O(h^2)$ such that* (A3) *and* (A4) *are uniformly satisfied, then $\beta_0 = O(h^2)$ in* (A3), *see also* (32). *In that case Theorem* 5 *and Theorem* 4 *coincide since $\Lambda_0 = O(1)$ and $\Lambda_1 = O(\beta_0^{-1})$ in Theorem* 1.

Theorem 5 clearly shows for our model problem what the influence of the preconditioners A_0 and $\mathcal{M}_0$ to the convergence of the CG method is. It is obvious that these preconditioners are primarily designed to ensure convergence and not to speed up the convergence of the CG method in a robust way. The impact of A_0 and $\mathcal{M}_0$ on the convergence rate is only pre-asymptotic. One is therefore led to the task of preconditioning the final matrix $\mathcal{M}$ to improve the convergence of the CG method. This is subject of the next section.

3.2.2 Preconditioned conjugate gradient method

Now we study the preconditioned conjugate gradient method for the solution of the transformed linear system for our model problem. That means we not only take preconditioners A_0 and $\mathcal{M}_0$ to define the transformation and the special inner product $[\cdot,\cdot]$, but we use an additional preconditioner for the transformed matrix $\mathcal{M}$. However, we note that A_0 and $\mathcal{M}_0$ for our model problem are simple scalings that are very cheap to implement and the additional preconditioner is sparse and cheap as well.

In view of Theorems 1 and 4 it is already clear what an efficient preconditioner looks like. Since $\mathcal{M}$ is uniformly spectrally equivalent to $\tilde{\mathcal{M}}$ (if (A1)–(A4) are uniformly satisfied) one only needs to find a preconditioner for $B(B_1A_1^{-1}B_1^*)^{-1}B^*$, cf. Lemma 5. But since $B_1A_1^{-1}B_1^*$ is uniformly well conditioned we simply can take the sparse matrix BB^* as a preconditioner for this block. We then obtain an efficient preconditioned iterative method for the solution of (27) (and of (26)). Since all the matrices which need to be inverted in this procedure, i.e. A_0, M_0, and P, are either just scalings or sparse and since we obtain a bounded number of iterations, the complexity of this method is of the same order as that of the preconditioned minimum residual method as proposed in [8] (for a different model problem).

Theorem 6 *Let the basis functions of all the three spaces $X_{1,h}$, $M_{1,h}$ and M_h be scaled such that their L^∞-norms are $O(h^{-1})$. Further, let us take a preconditioner*

$$\mathcal{P} := \begin{pmatrix} I & 0 \\ 0 & P \end{pmatrix}$$

with $I \in \mathbf{R}^{(L+M)\times(L+M)}$ and $P \in \mathbf{R}^{N\times N}$ such that P is uniformly spectrally equivalent to BB^*. Then there exist constants μ, ρ, ω such that with

$$A_0 := \mu I \in \mathbf{R}^{L\times L}, \quad M_0 := \omega I \in \mathbf{R}^{M\times M}, \quad \mathcal{M}_0 := \begin{pmatrix} \rho I & 0 \\ 0 & M_0 \end{pmatrix} \in \mathbf{R}^{(L+M)\times(L+M)}$$

there holds

$$\Lambda_0 \left[\mathcal{P}\begin{pmatrix} \mathbf{U} \\ u \end{pmatrix}, \begin{pmatrix} \mathbf{U} \\ u \end{pmatrix}\right] \le \left[\mathcal{M}\begin{pmatrix} \mathbf{U} \\ u \end{pmatrix}, \begin{pmatrix} \mathbf{U} \\ u \end{pmatrix}\right] \le \Lambda_1 \left[\mathcal{P}\begin{pmatrix} \mathbf{U} \\ u \end{pmatrix}, \begin{pmatrix} \mathbf{U} \\ u \end{pmatrix}\right]$$

for any $(\mathbf{U},u) \in X_h \times M_h$. Λ_0 and Λ_1 are positive constants being independent of h. Moreover, choosing A_0, $\mathcal{M}_0$ and the preconditioner $\mathcal{P}$ as above, the number of iterations of the preconditioned conjugate gradient method with inner product $[\cdot,\cdot]$ for the solution of (27) is bounded.

Proof. Concerning the details for choosing the parameters μ, ρ and ω we refer to the proof of Theorem 4. Therefore, the proof is finished by Theorem 1 since

$$\begin{aligned} \left[\tilde{\mathcal{M}}\begin{pmatrix} \mathbf{U} \\ u \end{pmatrix}, \begin{pmatrix} \mathbf{U} \\ u \end{pmatrix}\right] &= \left[\begin{pmatrix} I & 0 \\ 0 & B(B_1A_1^{-1}B_1^*)^{-1}B^* \end{pmatrix}\begin{pmatrix} \mathbf{U} \\ u \end{pmatrix}, \begin{pmatrix} \mathbf{U} \\ u \end{pmatrix}\right] \\ &\simeq \left[\begin{pmatrix} I & 0 \\ 0 & BB^* \end{pmatrix}\begin{pmatrix} \mathbf{U} \\ u \end{pmatrix}, \begin{pmatrix} \mathbf{U} \\ u \end{pmatrix}\right] \simeq \left[\mathcal{P}\begin{pmatrix} \mathbf{U} \\ u \end{pmatrix}, \begin{pmatrix} \mathbf{U} \\ u \end{pmatrix}\right] \end{aligned}$$

for all $\mathbf{U} \in X_{1,h} \times M_{1,h}$ and $u \in M_h$ by Lemmata 5, 6 and since $BB^* \simeq P$ by assumption. Here, $\simeq$ means spectral equivalence of the terms involved. $\square$

3.3 Numerical results

For the computational implementation of (26) we choose the finite element subspaces according to (23), (24) and (25). Let M and N be the number of edges and the number of triangles, respectively, of the triangulation $\mathcal{T}_h$, and let $L = 3N$. Then, we let $\{\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, ..., \boldsymbol{\theta}_L\}$, $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, ..., \boldsymbol{\sigma}_M\}$ and $\{u_1, u_2, ..., u_N\}$ be bases of $X_{1,h}$, $M_{1,h}$ and M_h, respectively. In particular, if $\{e_1, e_2, ..., e_M\}$ denote the edges of $\mathcal{T}_h$, the functions $\boldsymbol{\sigma}_j$ can be characterized by the relation

$$\boldsymbol{\sigma}_j \in M_{1,h} \quad \text{and} \quad \boldsymbol{\sigma}_j|_{e_i} \cdot \nu_i = c_j \delta_{ij} \quad \forall i, j \in \{1, 2, ..., M\},$$

where the c_j are scaling constants and ν_i denotes the unit normal on the edge e_i (in a previously chosen direction). In addition, if $\{T_1, T_2, ..., T_N\}$ denote the triangles of $\mathcal{T}_h$, we can take u_i such that $u_i|_{T_j} = \hat{c}_i\, \delta_{ij}$ for all $i, j \in \{1, 2, ..., N\}$, where the $\hat{c}_i$ are also scaling constants. We scale all the basis functions to $O(h^{-1})$.

We find that

$$A_1 = (a_{ij})_{L \times L} \quad \text{with} \quad a_{ij} := \int_\Omega \boldsymbol{\kappa} \boldsymbol{\theta}_i \cdot \boldsymbol{\theta}_j \, dx,$$

$$B_1 = (b_{ij}^{(1)})_{M \times L} \quad \text{with} \quad b_{ij}^{(1)} := -\int_\Omega \boldsymbol{\sigma}_i \cdot \boldsymbol{\theta}_j \, dx,$$

$$B = (b_{ij})_{N \times M} \quad \text{with} \quad b_{ij} := -\hat{c}_i \int_{T_i} \operatorname{div} \boldsymbol{\sigma}_j \, dx = \begin{cases} -\hat{c}_i \operatorname{div}(\boldsymbol{\sigma}_j)|T_i| \text{ if} & e_j \subset \bar{T}_i, \\ 0 & \text{otherwise}, \end{cases}$$

$$G_1 = (g_i^{(1)})_{M \times 1} \quad \text{with} \quad g_i^{(1)} := -\int_\Gamma g \boldsymbol{\sigma}_i \cdot \nu \, ds = \begin{cases} -(\boldsymbol{\sigma}_i \cdot \nu_i) \int_{e_i} g \, ds & \text{if } e_i \subset \Gamma, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$G = (g_i)_{N \times 1} \quad \text{with} \quad g_i := \hat{c}_i \int_{T_i} f \, dx.$$

For our numerical example we choose $\Omega = (0,1) \times (0,1)$ and take the right hand side functions f and g in (19) such that $u(x_1, x_2) = 1/(x_1 + x_2 + 1)$ and $\boldsymbol{\kappa} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Of course, the choice of the right hand side does not influence the properties of the stiffness matrix. Moreover, we consider uniform triangular meshes.

For our computations we have to specify the preconditioners A_0 and $\mathcal{M}_0 = \begin{pmatrix} \rho I & 0 \\ 0 & M_0 \end{pmatrix}$. For Theorem 4 we have to take $A_0 = \mu I$ and $M_0 = \omega I$. The parameters μ, ρ and ω have to be chosen sufficiently small such that (A1)–(A4) are satisfied. In practice, one can perform a power method to estimate the needed parameters. This can be done for a rather coarse mesh size since the behavior of the spectrum of the appearing matrices is quite stable. In our actual experiments we just tested some parameters and checked the positive definiteness of the inner product $[\cdot,\cdot]$

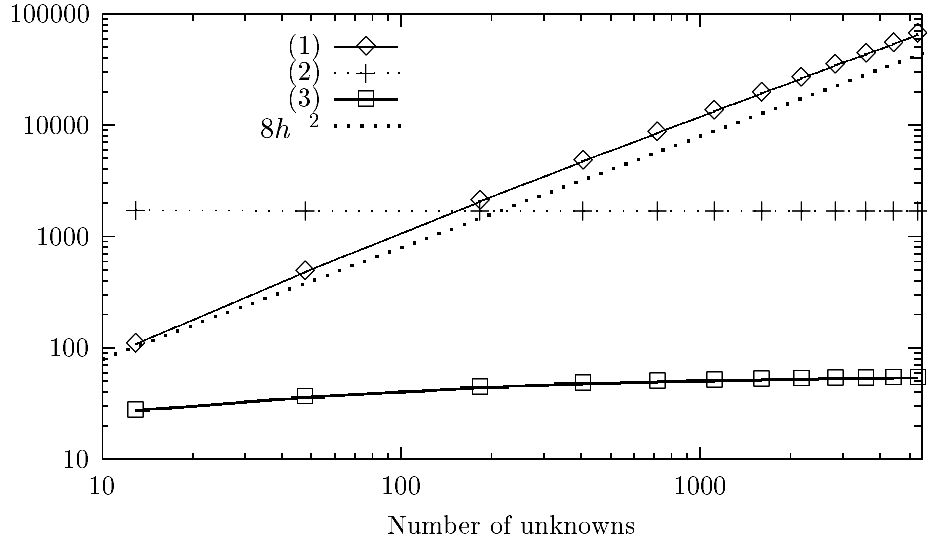

Figure 1: The condition numbers of $\mathcal{M}$: (1) pure scalings for A_0, M_0, (2) pure scaling for A_0, $H(\text{div};\Omega)$-inner product for M_0, (3) pure scalings for A_0, M_0 plus preconditioner BB^*.

during the computations. The choice $\mu = 0.3$, $\rho = 0.7$ and $\omega = 0.06$ performed well. To check the results of Theorem 5 we keep A_0 and ρ from before and choose M_0 to be the discrete $H(\text{div};\Omega)$-inner product in $M_{1,h}$. Here we have to choose a rather small number ω to scale M_0. In our experiments the factor $\omega = 0.001$ is sufficient. Finally, to check Theorem 6, we keep A_0 and $\mathcal{M}_0$ from Theorem 4 and simply take BB^* as additional preconditioner for the CG method. Actually, an application of this preconditioner requires only the solution of a sparse linear system. Figure 1 shows the condition numbers of the transformed stiffness matrices (plus preconditioner in the last case) in a double logarithmic scale. As given by Theorems 4, 5, 6 the spectral condition numbers increase like $O(h^{-2})$ in the first case and are bounded in the latter cases. Actually, it appears that in this range of unknowns the second method (with $H(\text{div};\Omega)$-inner product) does not break down. Nevertheless, as expected from the theory, the condition numbers for the third method (with preconditioner) are much smaller than for the second method. For the second method we expect by Theorem 5 a condition number of order $O(\omega^{-1})$, and actually they are of the order $\omega^{-1} = 1000$. Table 1 gives the extreme eigenvalues for all three cases. As proved by Theorems 4, 5, 6 the minimum eigenvalues are bounded from below for all three methods. Indeed, this is already clear from the general Theorem 1 (by setting $\beta_0 = 0$). The magnitude of the condition number hinges only on the size of the maximum eigenvalue.

Table 2 presents the iteration numbers of the CG method which are required to reduce the initial residual in discrete l^2-norm by the factor 10^{-6}. They are quite large for the first method and appear to be bounded for the other two methods. However, the second method may break

		(1)		(2)		(3)	
dim	$1/h$	$\lambda_{\min}$	$\lambda_{\max}$	$\lambda_{\min}$	$\lambda_{\max}$	$\lambda_{\min}$	$\lambda_{\max}$
48	2	0.9912	481	0.9776	1670	0.8935	32.28
184	4	0.9878	2056	0.9783	1670	0.8742	38.68
408	6	0.9869	4712	0.9784	1670	0.8624	41.28
720	8	0.9864	8440	0.9784	1670	0.8549	42.57
1120	10	0.9861	13236	0.9785	1670	0.8500	43.36
1608	12	0.9859	19100	0.9785	1670	0.8466	43.90
2184	14	0.9857	26032	0.9785	1670	0.8442	44.28
2848	16	0.9856	34030	0.9785	1670	0.8424	44.56
3600	18	0.9855	43096	0.9785	1670	0.8410	44.78
4440	20	0.9855	53228	0.9785	1670	0.8399	44.95
5368	22	0.9854	64427	0.9785	1670	0.8390	45.09

Table 1: The extreme eigenvalues of $\mathcal{M}$: (1) pure scalings for A_0, M_0, (2) pure scaling for A_0, $H(\mathrm{div};\Omega)$-inner product for M_0, (3) pure scalings for A_0, M_0 plus preconditioner BB^*. dim is the number of unknowns $L+M+N$.

down if the mesh size is further reduced. The number of iterations for the first method increases asymptotically like $O(h^{-1})$ as it is confirmed by Figure 2 where a double logarithmic scale is used. This is exactly what is expected due to Theorem 4.

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dim	$1/h$	(1)	(2)	(3)
48	2	24	20	15
184	4	73	32	33
408	6	131	36	37
720	8	177	42	40
1120	10	223	43	40
1608	12	270	44	41
2184	14	316	44	42
2848	16	364	45	42
3600	18	410	45	41
4440	20	458	45	41
5368	22	504	45	41
6384	24	552	45	41
7488	26	598	45	41

Table 2: The numbers of iterations of the CG method to reduce the initial residual by 10^{-6}: (1) pure scalings for A_0, M_0, (2) pure scaling for A_0, $H(\mathrm{div};\Omega)$-inner product for M_0, (3) pure scalings for A_0, M_0 plus preconditioner BB^*. dim is the number of unknowns $L+M+N$.

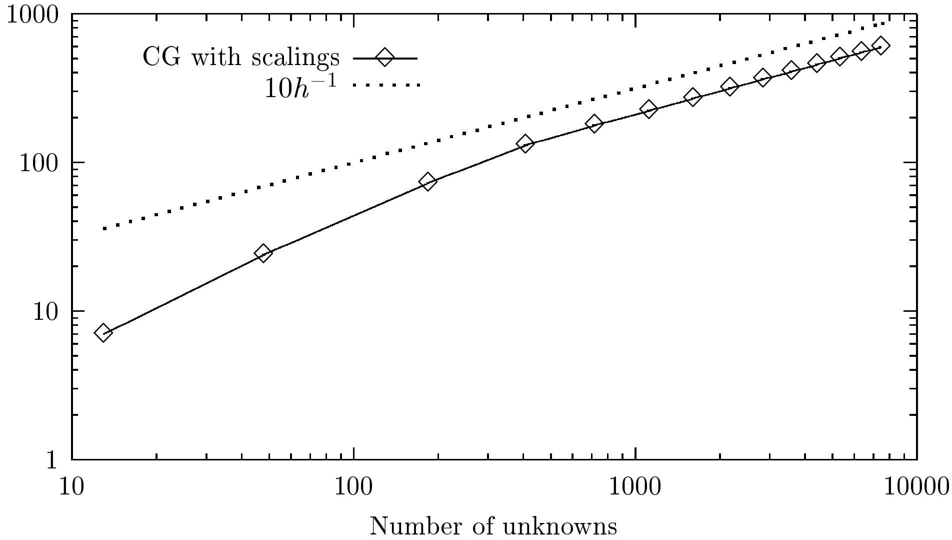

Figure 2: The number of iterations of the CG method to reduce the initial residual by 10^{-6} (no preconditioner, scalings for A_0 and M_0).

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