The optimal rate of convergence of the p-version of the boundary element method in two dimensions

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Abstract

We introduce the Jacobi-weighted Besov and Sobolev spaces in the one-dimensional setting. In the framework of these spaces, we analyze lower and upper bounds for approximation errors in the p-version of the boundary element method for hypersingular and weakly singular integral operators on polygons. We prove the optimal rate of convergence for the p-version in the energy norms of $\tilde{H}^{1/2}$ and $\tilde{H}^{-1/2}$, respectively.

Key words: p-version, boundary element method, Jacobi-weighted Besov and Sobolev spaces, optimal rate of convergence.
AMS Subject Classification: 65N38

1 Introduction

The p-version of the boundary element method (BEM), which achieves the accuracy by increasing degrees of piecewise polynomials on a fixed mesh, and the h and the h-p version are the three basic approaches of the BEM. The first theoretical paper on the p-version was published by Stephan and Suri [21] in 1989. Since then, theory and algorithms of the p and h-p versions of the BEM have rapidly developed, and have been applied to various scientific and engineering problems. For an overview, we refer to [19, 20].

It is known that the p-version of the BEM converges at least as fast as the h-version, and it converges about twice as fast as the h-version if solutions have singularities. For the Laplace equation with Dirichlet or Neumann boundary conditions and its equivalent boundary integral equations of the first kind with weakly singular or hypersingular operators, the solutions on polygonal domains have singularities of the r^γ-type at vertices, r denoting the distance to a vertex. Singularities of this type severely affect the performance of the boundary element (BE)

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solutions as well as of the finite element (FE) solutions. Obviously, the p-version has an advantage over the h-version in the presence of singularities. For singularities of $r^{1/2}$-type (that appear in Neumann problems at the end points of open curves), Stephan and Suri [21] proved an upper bound $O(p^{-1})$ for the approximation error in $H^{1/2}$ and an order of convergence $O(p^{-1}\sqrt{\log p})$ in the energy norm. They gave a proof based on an expansion in Chebyshev polynomials that works especially for the case r^{γ}, $\gamma = 1/2$. Later, in [22], they studied the hp-version with quasi-uniform meshes and proved convergence (in energy norm) of the order $O(p^{-2\gamma+\epsilon})$ with arbitrary $\epsilon > 0$ for singularities r^{γ}, $\gamma > 0$. Here, the appearing constant in the upper bound depends in an unspecified way on ϵ. Schwab and Suri [18] improved the upper bound for polynomial approximation of singular functions of the r^{γ}-type, $\gamma > 1/2$, on $I = (-1, 1)$ by removing the parameter ϵ in this bound measured in the $H^{1/2}(I)$-norm. This upper bound seems optimal but is only valid for $\gamma > 1/2$, which implies that the singular function is in $H^1(I)$. However, it does not lead to optimal error estimates for the BE solution of the p-version on polygonal boundaries (open or closed).

The error estimates and the approximation theory of the p-version of the BEM has been analyzed and developed in the framework of Sobolev spaces. But it has been aware for long time that the usual Sobolev spaces are not appropriate to analyze the p-version of the FEM, and even worse for the p-version of the BEM. It is worth indicating that the upper estimates obtained so far for the BEM are weaker than those for the FEM, because the energy norms for integral equations of the first kind are $\tilde{H}^{1/2}$ and $\tilde{H}^{-1/2}$. This may result in an additional loss in the rate of convergence when using the spaces $H^{1/2}$ and $H^{-1/2}$ for the analysis and then generalizing to the energy spaces. The energy space for elliptic problems of the second order in one dimension is H^1, and the Legendre projection was used to establish the optimal convergence of the FEM in [11]. But these results can not be applied to the BE solutions of integral equations.

The upper bound known for the error of the BE solution in the energy norm is not sharp, and the lower bound of the error, e.g. for elliptic problems on polygonal domains, has never been addressed. Therefore, the optimal convergence of the p-version of the BEM has not been established yet, and it seems impossible to prove it in the current framework. We need to design appropriate function spaces and develop the approximation theory for the p-version of the BEM in the framework of such spaces, including the optimal convergence and other important issues of the p-version. In particular, these spaces should be consistent with the spaces $H^{1/2}$ and $\tilde{H}^{1/2}$ so that we are able to avoid a loss in the approximation in the $\tilde{H}^{1/2}$- and $\tilde{H}^{-1/2}$-norms.

Recently, Babuška and Guo [1, 2, 3, 4] showed that the Jacobi-weighted Besov and Sobolev spaces are the most appropriate spaces for the analysis of the p-version of the FEM. The structure of the spaces and approximability of functions in such spaces are adequately analyzed in [1]. The optimal convergence of the p-version of the FEM for elliptic problems on polygonal domains is proved in the framework of these spaces in [2]. In [3] inverse approximation theorems are given in the Jacobi-weighted Besov spaces, which predict regularity of solutions based on asymptotic rate of convergence of the FE solution. Meanwhile, the investigation on Jacobi approximation was done in the spectral method [12, 13]. These papers laid down a solid mathematical foundation for further development of the p-version of the FEM and the spectral method. It is expected that all of these results should be established in parallel for the p-version of the BEM in the same mathematical framework, with a proper treatment of error estimates in the $\tilde{H}^{1/2}$- and $\tilde{H}^{-1/2}$-norms.

In this paper, we will introduce the Jacobi-weighted Sobolev $H^{s,\beta}(I)$ and Besov spaces $B_\nu^{s,\beta}(I)$, $\nu \geq 0$, in the one-dimensional setting, and establish the optimal convergence of the BEM in the framework of these Jacobi-weighted spaces. This will be the first one of a series devoted to a new and effective approximation theory for the p-version of the BEM in the new mathematical framework, including the optimal convergence in the energy norms, and other fundamental issues such as the inverse approximation theorems and adaptive p-version of the BEM, which will be addressed in forthcoming papers.

In order to prove the optimal convergence, we shall derive the approximability of functions in the Jacobi-weighted Besov spaces, including upper and lower bounds for the approximation error in Jacobi-weighted Sobolev norms. For the optimal rate of convergence in the $\tilde{H}^{1/2}$- and $\tilde{H}^{-1/2}$-norms, we shall establish the equivalence between the spaces $H^{1/2}(I)$ and $H^{1/2,-1/2}(I)$, and between the spaces $\tilde{H}^{1/2}(I)$ and $\tilde{H}^{1/2,-1/2}(I)$. Such equivalence is a unique feature in the one-dimensional setting, which allows us to prove optimal error estimates in the energy norm and in the $H^{1/2}(I)$-norm. The singular functions of r^γ- and $r^\gamma \log^\nu r$-type are proved to be in the spaces $B^{2\gamma+1/2,-1/2}(I)$ and $B_\nu^{2\gamma+1/2,-1/2}(I)$, respectively. After proving the lower bounds for the approximation error for these singular functions using asymptotic expansion of the error, we are able to establish the optimal error estimates for polynomial approximation in the $\tilde{H}^{1/2}(I)$-norm for $\gamma > 0$. The convergence is of order $O(p^{-2\gamma} \log^{\nu^*} p)$ with $\nu^* = \nu - 1$ if $\nu > 0$ and γ is an integer, and $\nu^* = \nu$ otherwise. We apply these results on each element of the boundary element mesh and argue properly in the $\tilde{H}^{1/2}$- and $H^{1/2}$-norms for an elliptic boundary value problem on a polygonal domain. This allows us to establish the optimal rate of convergence of the p-version of the BEM for hypersingular integral operators. These results then are used to prove an optimal rate of convergence of the p-version of the BEM for weakly singular operators as well.

An outline of this paper is as follows. In Section 2 we introduce the Jacobi-weighted Besov and Sobolev spaces, establish the approximability of functions in these spaces, and prove the equivalence between the spaces $\tilde{H}^{1/2}(I)$, $H^{1/2}(I)$ and the Jacobi-weighted Sobolev spaces $\tilde{H}^{1/2,-1/2}(I)$, $H^{1/2,-1/2}(I)$. The optimal upper bounds for the approximation error of singular functions of r^γ-type are given in Section 3. Correspondingly, those for singular functions of $r^\gamma \log^\nu r$-type with $\nu > 0$ are studied in Section 4. The lower bounds for the approximation error of singular functions of $r^\gamma \log^\nu r$-type, $\nu \geq 0$, and the optimal approximation results for these functions are presented in Section 5. Finally, in Section 6, we apply our results to an elliptic boundary value problem on a polygonal domain with Dirichlet and Neumann boundary conditions, and prove the optimal convergence of the BE solution of the p-version.

2 Jacobi-weighted Besov and Sobolev spaces in one dimension

In the following I denotes the interval $(-1,1)$. Let $\alpha \geq 0$ be an integer and $\beta > -1$ a real number. We introduce a weight function with parameters α and β by

$$W_{\alpha,\beta}(x) = (1-x^2)^{\alpha+\beta}$$

and define spaces $H^{k,\beta}(I)$ for integers $k \geq 0$ as the closure of C^∞ functions with respect to the weighted norm

$$\|u\|_{H^{k,\beta}(I)} = \left(\sum_{\alpha=0}^{k} \int_I |u^{(\alpha)}|^2 W_{\alpha,\beta}(x)\,dx\right)^{1/2} = \left(\sum_{\alpha=0}^{k} \int_I |u^{(\alpha)}|^2 (1-x^2)^{\alpha+\beta}\,dx\right)^{1/2}.$$

The semi-norm involving only the highest derivative $u^{(k)}$ is denoted by $|u|_{H^{k,\beta}(I)}$. Note that the Jacobi polynomials $p_m^{(\beta,\beta)}$ and their derivatives $\frac{d^\alpha}{dx^\alpha} p_m^{(\beta,\beta)}(x)$ are orthogonal with respect to the weight $W_{\alpha,\beta}$.

For real $s > 0$ the space $H^{s,\beta}(I)$ is defined by interpolation: Let l and k be two integers with $l < k$ and $s = (1-\theta)l + \theta k$ for $\theta \in (0,1)$. Then

$$H^{s,\beta}(I) = (H^{l,\beta}(I), H^{k,\beta}(I))_{\theta,2}$$

with norm

$$\|u\|_{H^{s,\beta}(I)} = \left(\int_0^\infty \left(t^{-\theta} K(t,u)\right)^2 \frac{dt}{t}\right)^{1/2} \tag{2.1}$$

for

$$K(t,u) = \inf_{u=v+w} \left(\|v\|_{H^{l,\beta}(I)} + t\|w\|_{H^{k,\beta}(I)}\right). \tag{2.2}$$

The spaces $H^{s,\beta}(I)$ are referred to as Jacobi-weighted Sobolev spaces. Interpolating differently we obtain the so-called Jacobi-weighted Besov spaces:

$$B^{s,\beta}(I) = (H^{l,\beta}(I), H^{k,\beta}(I))_{\theta,\infty}$$

with norm

$$\|u\|_{B^{s,\beta}(I)} = \sup_{t>0} t^{-\theta} K(t,u).$$

Let $P_p(I)$ denote the space of polynomials up to degree p on I. We note the following approximation properties for functions in $H^{s,\beta}(I)$ and $B^{s,\beta}(I)$.

Theorem 2.1 *For $s \geq 0$ let $u \in H^{s,\beta}(I)$ and let $u_p \in P_p(I)$ be the projection of u in $H^{0,\beta}(I)$. Then, for $r \leq s$, there holds*

$$\|u - u_p\|_{H^{r,\beta}(I)} \leq c p^{-(s-r)} \|u\|_{H^{s,\beta}(I)}. \tag{2.3}$$

If, for $s > 0$, $u \in B^{s,\beta}(I)$ then with $u_p \in P_p(I)$ from before there holds for $r < s$

$$\|u - u_p\|_{H^{r,\beta}(I)} \leq c p^{-(s-r)} \|u\|_{B^{s,\beta}(I)}. \tag{2.4}$$

Proof. The assertions for integer s were proved in [4, Theorems 2.1, 2.2], and for non-integer s the assertions can be proved by interpolation. $\square$

Remark 2.1 *Due to the properties of the Jacobi polynomials, if $u \in H^{k,\beta}(I)$ with $k \geq 0$, its projection on $P_p(I)$ in the space $H^{0,\beta}(I)$, is the projection of u in the space $H^{l,\beta}(I)$ for all integer $l \leq k$, and*

$$|u|^2_{H^{l,\beta}(I)} = |u_p|^2_{H^{l,\beta}(I)} + |u - u_p|^2_{H^{l,\beta}(I)}.$$

For $u \in H^{k,\beta}(I)$ with $k \geq 0$, there is the Jacobi expansion in $H^{0,\beta}(I)$:

$$u(x) = \sum_{m=0}^{\infty} c_m P_m^{\beta,\beta}(x).$$

Then

$$u_p(x) = \sum_{m=0}^{p} c_m P_m^{\beta,\beta}(x)$$

is referred to as Jacobi projection. For $\beta = 0$ the corresponding Jacobi projection is the Legendre projection, and for $\beta = -1/2$ it is the Chebyshev projection.

In the boundary element analysis, the approximation errors are measured in the norms of $\tilde{H}^{1/2}(I)$ and $H^{1/2}(I)$,

$$H^{1/2}(I) = (L^2(I), H^1(I))_{1/2,2}, \quad \tilde{H}^{1/2}(I) = (L^2(I), H_0^1(I))_{1/2,2}.$$

$\tilde{H}^{1/2}(I)$ is the energy space for hypersingular operators, $\tilde{H}^{-1/2}(I)$ and $H^{-1/2}(I)$ are the dual spaces of $H^{1/2}(I)$ and $\tilde{H}^{1/2}(I)$, in which one analyzes the error for weakly singular operators. Hence, it is essential to explore the relation between these spaces and the corresponding Jacobi-weighted spaces. To this end, we introduce the space

$$H_0^{1,-1/2}(I) = \{u \in H^{1,-1/2}(I) \mid u(\pm 1) = 0\},$$

and the interpolation space

$$\tilde{H}^{1/2,-1/2}(I) = \left(H^{0,-1/2}(I), H_0^{1,-1/2}(I)\right)_{1/2,2}$$

with norm analogously to (2.1) for

$$K(t,u) = \inf_{u=v+w, v\in H^{0,-1/2}(I), w\in H_0^{1,-1/2}(I)} \left(\|v\|_{H^{0,-1/2}(I)} + t\|w\|_{H^{1,-1/2}(I)}\right).$$

Here, we implicitly used the continuity of functions in $H^{1,-1/2}(I)$ due to Lemma 2.1.

Lemma 2.1 *$H^{1,-1/2}(I) \subset C^0(\bar{I})$, and*

$$\|u\|_{C^0(\bar{I})} \leq C\|u\|_{H^{1,-1/2}(I)}.$$

Proof. For any $a, b \in [-1,1]$ and $u \in H^{1,-1/2}(I)$ there holds

$$\begin{aligned} |u(a) - u(b)| &= |\int_a^b u'(t)\,dt| \\ &\leq (\int_a^b (1-t^2)^{-1/2}\,dt)^{1/2}(\int_a^b |u'(t)|^2(1-t^2)^{1/2}\,dt)^{1/2} \\ &= |\arccos(b) - \arccos(a)|^{1/2}|u|_{H^{1,-1/2}(I)} \end{aligned}$$

which implies that u is continuous on $\bar{I}$. Furthermore, we have

$$|u(a)| \leq |u(x)| + |\int_a^x u'(t)dt|$$

and

$$\begin{aligned} \int_{-1}^1 |u(a)|^2\,dx &\leq C\left(\int_{-1}^1 |u(x)|^2\,dx + \int_{-1}^1 (\int_a^x |u'(t)|^2 dt)\,dx\right) \\ &\leq C\left(\|u\|^2_{H^{0,-1/2}(I)} + \int_{-1}^1 (\int_{-1}^1 (1-t^2)^{-1/2}\,dt \int_{-1}^1 |u'(t)|^2(1-t^2)^{1/2}\,dt)\,dx\right) \\ &\leq C\left(\|u\|^2_{H^{0,-1/2}(I)} + |u|^2_{H^{1,-1/2}(I)}\right). \end{aligned}$$

which leads to the assertion of the lemma. □

We now establish the equivalence between the spaces with and without Jacobi weights.

Theorem 2.2

$$H^{1/2,-1/2}(I) \cong H^{1/2}(I), \qquad \tilde{H}^{1/2,-1/2}(I) \cong \tilde{H}^{1/2}(I),$$

i.e. there exist constants c_1, $c_2 > 0$ such that

$$c_1\|u\|_{H^{1/2}(I)} \leq \|u\|_{H^{1/2,-1/2}(I)} \leq c_2\|u\|_{H^{1/2}(I)} \tag{2.5}$$

and

$$c_1\|u\|_{\tilde{H}^{1/2}(I)} \leq \|u\|_{\tilde{H}^{1/2,-1/2}(I)} \leq c_2\|u\|_{\tilde{H}^{1/2}(I)}. \tag{2.6}$$

Proof. Let $v \in H^{0,-1/2}(I)$ and $w \in H^{1,-1/2}(I)$, and let $\hat{v}(\xi) := v(\cos\xi)$ and $\hat{w}(\xi) := w(\cos\xi)$. Then there hold

$$\|v\|^2_{H^{0,-1/2}(I)} = \int_{-1}^1 |v(x)|^2 \frac{1}{\sqrt{1-x^2}}\,dx = \int_0^\pi |\hat{v}(\xi)|^2\,d\xi = \|\hat{v}\|^2_{L^2(0,\pi)}$$

and

$$|w|^2_{H^{1,-1/2}(I)} = \int_{-1}^{1} |w'(x)|^2 \sqrt{1-x^2}\, dx = \int_0^\pi |\hat{w}'(\xi)|^2\, d\xi = |\hat{w}|^2_{H^1(0,\pi)}.$$

This implies that $\hat{u}(\xi) := u(\cos\xi) \in H^{1/2}(0,\pi)$ if and only if $u \in H^{1/2,-1/2}(I)$ by interpolation, and $\|u\|_{H^{1/2,-1/2}(I)} = \|\hat{u}\|_{H^{1/2}(0,\pi)}$.

On the other hand, from [5, Lemma 3.1] we know that there exist constants c_1, $c_2 > 0$ such that

$$c_1\|\hat{u}\|_{H^{1/2}(0,\pi)} \le \|u\|_{H^{1/2}(I)} \le c_2\|\hat{u}\|_{H^{1/2}(0,\pi)}$$

which implies (2.5).

We next prove the equivalence between $\tilde{H}^{1/2,-1/2}(I)$ and $\tilde{H}^{1/2}(I)$. Obviously, $w \in H_0^{1,-1/2}(I)$ if and only if $\hat{w} \in H_0^1(0,\pi)$. This yields immediately $\hat{u} \in \tilde{H}^{1/2}(0,\pi)$ if and only if $u \in \tilde{H}^{1/2,-1/2}(I)$ and $\|u\|_{\tilde{H}^{1/2,-1/2}(I)} = \|\hat{u}\|_{\tilde{H}^{1/2}(0,\pi)}$.

An equivalent norm in $\tilde{H}^{1/2}(0,\pi)$ can be written like

$$\|\hat{u}\|^2_{\tilde{H}^{1/2}(0,\pi)} \cong \|\hat{u}\|^2_{H^{1/2}(0,\pi)} + \int_0^{\pi/2} \frac{|\hat{u}(\xi)|^2}{\xi}\, d\xi + \int_{\pi/2}^{\pi} \frac{|\hat{u}(\xi)|^2}{\pi-\xi}\, d\xi.$$

Note that

$$\int_0^{\pi/2} \frac{|\hat{u}(\xi)|^2}{\xi}\, d\xi = \int_0^1 \frac{|\hat{u}(x)|^2}{1-x} \frac{1-x}{(1-x^2)^{1/2}\arccos x}\, dx$$

and

$$\lim_{x\to 1} \frac{1-x}{(1-x^2)^{1/2}\arccos x} = 1/2.$$

Therefore, we have

$$\int_0^{\pi/2} \frac{|\hat{u}(\xi)|^2}{\xi}\, d\xi \cong \int_0^1 \frac{|u(x)|^2}{1-x}\, dx.$$

Similarly, one has

$$\int_{\pi/2}^{\pi} \frac{|\hat{u}(\xi)|^2}{\pi-\xi}\, d\xi \cong \int_{-1}^0 \frac{|u(x)|^2}{1+x}\, dx.$$

Hence there holds

$$\|\hat{u}\|_{\tilde{H}^{1/2}(0,\pi)} \cong \|u\|_{\tilde{H}^{1/2}(I)}$$

which leads to (2.6). □

Let us consider a function $\hat{u}(\xi) = u(\cos\xi)$ with cosine expansion

$$\hat{u}(\xi) = \sum_{k=0}^{\infty} a_k \cos(k\xi). \tag{2.7}$$

This leads to the Chebyshev expansion of $u \in H^{0,-1/2}(I)$,

$$u(x) = \sum_{k=0}^{\infty} a_k T_k(x) \tag{2.8}$$

with $T_k(x) = \cos(k \arccos x)$. We note that the Chebyshev polynomials T_k are normalized differently than the corresponding Jacobi polynomials $P_k^{-1/2,-1/2}$. By the usual definition one has $T_k(1) = 1$ and

$$P_k^{\beta,\beta}(1) = \binom{k+\beta}{k} := \frac{\Gamma(k+\beta+1)}{\Gamma(k+1)\Gamma(\beta+1)}.$$

Therefore, from the Rodríguez formula

$$P_k^{\beta,\beta}(x) = \frac{(-1)^k}{2^k k!}(1-x^2)^{-\beta}\frac{d^k}{dx^k}(1-x^2)^{k+\beta}$$

we deduce that

$$T_k(x) = (-1)^k \frac{\Gamma(1/2)}{\Gamma(k+1/2)2^k}(1-x^2)^{1/2}\frac{d^k}{dx^k}(1-x^2)^{k-1/2}. \tag{2.9}$$

Corollary 2.1 *For $u \in H^{1/2}(I)$ with expansions (2.7) and (2.8), there holds the equivalence of norms*

$$\|u\|^2_{H^{1/2}(I)} \cong \|u\|^2_{H^{1/2,-1/2}(I)} \cong \sum_{k=0}^{\infty} a_k^2(1+k^2)^{1/2}.$$

Proof. By Theorem 2.2, there holds

$$\|u\|^2_{H^{1/2,-1/2}(I)} \cong \|u\|^2_{H^{1/2}(I)} \cong \|\hat{u}\|^2_{H^{1/2}(0,\pi)},$$

and it is well known that

$$\|\hat{u}\|^2_{H^{1/2}(0,\pi)} \cong \pi a_0^2 + \frac{\pi}{2}\sum_{k=1}^{\infty} a_k^2(1+k^2)^{1/2}.$$

Then the assertion follows immediately. □

3 Approximability of singular functions of x^γ-type

Now we consider the approximability of functions that behave singularly at one endpoint of $I = (-1,1)$, e.g. at $x = -1$. First step is considering singularities of the form $u(x) = (1+x)^\gamma$. Later we extend our investigation to singularities including logarithmic contributions.

First we prove regularity within weighted Sobolev and Besov spaces.

Theorem 3.1 *Let $u(x) = (1+x)^\gamma$ with $\gamma > 0$. Then, for $\beta > -1$, $u \in H^{s,\beta}(I)$ for any positive real number $s < 2\gamma + \beta + 1$ and $u \in B^{2\gamma+\beta+1,\beta}(I)$.*

Proof. The claim that $u \in H^{s,\beta}(I)$ for integer $s < 2\gamma+\beta+1$ follows directly from the definition of $H^{s,\beta}(I)$. For real s, $H^{s,\beta}(I)$ is defined by interpolating between $H^{l,\beta}(I)$ and $H^{k,\beta}(I)$ for integers $0 \le l < s < k$. Then we proceed as in the proof of [4, Theorem 2.3] for the weighted Besov norm. We decompose $u = v + w$ with $v = \varphi_\delta u$ and $w = (1-\varphi_\delta)u$ where, for $\delta < 1$, φ_δ is a C^∞-function with $\varphi_\delta(x) = 1$ for $-1 < x < -1+\delta/2$ and $\varphi_\delta(x) = 0$ for $x > -1+\delta$. One finds

$$\|v\|_{H^{l,\beta}(J)} \le C\delta^{\gamma+\frac{1}{2}(1+\beta-l)}$$

and

$$\|w\|_{H^{k,\beta}(J)} \le C\delta^{\gamma+\frac{1}{2}(1+\beta-k)}.$$

Therefore,

$$K(t,u) \le C\left(\delta^{\gamma+\frac{1}{2}(1+\beta-l)} + t\,\delta^{\gamma+\frac{1}{2}(1+\beta-k)}\right).$$

Analogously as in [4] we select $\delta = t^{\frac{2}{k-l}}$ and $\theta = \frac{s-l}{k-l}$. Then

$$\int_0^1 t^{-2\theta} K(t,u)^2 \frac{dt}{t} \le C\int_0^1 t^{-2\theta}\delta^{(2\gamma+1+\beta-l)}\frac{dt}{t} = C\int_0^1 t^{\frac{2(2\gamma+1+\beta-s)}{k-l}-1}\,dt$$

which is finite as long as $2\gamma+1+\beta-s > 0$, i.e., $s < 2\gamma+1+\beta$. Now,

$$\int_1^\infty t^{-2\theta}K(t,u)^2\frac{dt}{t} \le \int_1^\infty t^{-2\theta}\|u\|^2_{H^{l,\beta}(J)}\frac{dt}{t} \le C\|u\|^2_{H^{l,\beta}(J)}$$

for any $\theta > 0$. Therefore,

$$\|u\|^2_{H^{s,\beta}(J)} = \int_0^\infty t^{-2\theta}K(t,u)^2\frac{dt}{t}$$

is finite for $s < 2\gamma+1+\beta$.

With the decomposition $u = v + w$ from above and selecting $\delta = t^{\frac{2}{k-l}}$ and $\theta = \frac{2\gamma+1+\beta-l}{k-l}$, we obtain

$$\sup_{0<t<1} t^{-\theta}K(t,u) < \infty$$

and there holds

$$\sup_{t>1} t^{-\theta}K(t,u) \le \|u\|_{H^{l,\beta}(J)}.$$

Thus, $u \in B^{s,\beta}(J)$ with $s = (1-\theta)l + \theta k = 2\gamma+1+\beta$, which finishes the proof of the theorem. □

Theorem 3.2 *For the function $u(x) = (1+x)^\gamma$ with $\gamma > 0$ and Jacobi projection u_p on $P_p(I)$ there holds for $r < 2\gamma+1+\beta$*

$$\|u-u_p\|_{H^{r,\beta}(I)} \le cp^{-(2\gamma+1+\beta-r)}\|u\|_{B^{2\gamma+1+\beta,\beta}(I)}. \tag{3.1}$$

Proof. Since $u \in B^{2\gamma+1+\beta,\beta}(I)$ by Theorem 3.1, the assertion follows from Theorem 2.1. $\square$

Remark 3.1 *The theorem gives the error of the approximation for any $\beta > -1$, which covers general Jacobi projections. For $\beta = 0$, the Legendre projection gives a sharp error estimate in the $L^2(I)$-norm,*

$$\|u - u_p\|_{L^2(I)} \leq cp^{-(2\gamma+1)}\|u\|_{B^{2\gamma+1,0}(I)}, \tag{3.2}$$

and for $\beta = -1/2$, the Chebyshev projection ($\beta = -1/2$) will give us the approximation error in $H^{1/2}(I)$,

$$\|u - u_p\|_{H^{1/2}(I)} \cong \|u - u_p\|_{H^{1/2,-1/2}(I)} \leq cp^{-2\gamma}\|u\|_{B^{2\gamma+1/2,-1/2}(I)}. \tag{3.3}$$

Because the $\tilde{H}^{1/2}(I)$-norm represents the energy, we are, in particular, interested in the error measured in the $\tilde{H}^{1/2}(I)$-norm. To this end the choice of β can only be $-1/2$.

In order to approximate a function u within $\tilde{H}^{1/2}(I)$, i.e. subject to essential boundary conditions at ± 1, we approximate $v = (1-x^2)^{-1}u$ by a polynomial v_p and define $(1-x^2)v_p$ as the approximation of u. More precisely, for $v \in B^{s,3/2}(I)$, $s > 0$, let $v_p(x)$ be the Jacobi projection of v on $P_p(I)$ with Jacobi weight $\beta = 3/2$, and introduce an operator E by

$$E(v) = (1-x^2)(v - v_p). \tag{3.4}$$

Then we have the following lemma on the operator E.

Lemma 3.1 *For $s > 0$, the operator E defined in (3.4) with $p \geq 0$ is an operator: $B^{s,3/2}(I) \to H^{0,-1/2}(I)$, and its norm, denoted by $\|E\|_{s\to 0}$, satisfies*

$$\|E\|_{s\to 0} \leq cp^{-s}. \tag{3.5}$$

For $s > 1$, E is an operator: $B^{s,3/2}(I) \to H_0^{1,-1/2}(I)$, and its norm, denoted by $\|E\|_{s\to 1}$, satisfies

$$\|E\|_{s\to 1} \leq cp^{-(s-1)}. \tag{3.6}$$

Proof. By Theorem 2.1,

$$\|v - v_p\|_{H^{0,3/2}(I)} \leq cp^{-s}\|v\|_{B^{s,3/2}(I)}.$$

Note that

$$\|E(v)\|_{H^{0,-1/2}(I)} = \|v - v_p\|_{H^{0,3/2}(I)}$$

which leads to (3.5).

For $s > 1$, then by Theorem 2.1,

$$\|v - v_p\|_{H^{1,3/2}(I)} \leq cp^{-(s-1)}\|v\|_{B^{s,3/2}(I)}.$$

Note that

$$
\begin{aligned}
|E(v)|^2_{H^{1,-1/2}(I)} &= \int_{-1}^{1} (1-x^2)^{1/2} \left| \Big((1-x^2)(v-v_p) \Big)' \right|^2 dx \\
&\le c \left(\int_{-1}^{1} (1-x^2)^{3/2+1} |v' - v_p'|^2 \, dx + \int_{-1}^{1} (1-x^2)^{1/2} |v - v_p|^2 \, dx \right) \\
&= c \left(|v - v_p|^2_{H^{1,3/2}(I)} + \|v - v_p\|^2_{H^{0,1/2}(I)} \right). \qquad (3.7)
\end{aligned}
$$

By Theorem 2.1 the first term on the right hand side of (3.7) is bounded by

$$
|v - v_p|_{H^{1,3/2}(I)} \le c p^{-(s-1)} \|v\|_{B^{s,3/2}(I)}. \qquad (3.8)
$$

For the second term we proceed as follows. For a fixed $\delta \in (0,1)$ we write

$$
\|v - v_p\|^2_{H^{0,1/2}(I)} = \Big(\int_{-1}^{-1+\delta} + \int_{-1+\delta}^{1-\delta} + \int_{1-\delta}^{1} \Big) (1-x^2)^{1/2} |v - v_p|^2 \, dx. \qquad (3.9)
$$

The middle term is bounded by

$$
\int_{-1+\delta}^{1-\delta} (1-x^2)^{1/2} |v - v_p|^2 \, dx \le c \int_{-1+\delta}^{1-\delta} (1-x^2)^{3/2} |v - v_p|^2 \, dx \le c \|v - v_p\|^2_{H^{0,3/2}(I)}. \qquad (3.10)
$$

To estimate the first integral we use a $C_0^\infty(\mathbf{R})$ cut-off function χ with $\chi(x) = 1$ for $x \in (-1, -1+\delta)$ and $\chi(x) = 0$ for $x > 1 - \delta$. We find

$$
\int_{-1}^{-1+\delta} (1-x^2)^{1/2} |v - v_p|^2 \, dx \le c \int_{-1}^{\infty} (1+x)^{1/2} |\chi(x)(v - v_p)|^2 \, dx. \qquad (3.11)
$$

Recall the following Hardy inequality 330 from [15]:

$$
\int_0^\infty x^{-r} |F(x)|^p \, dx \le \left(\frac{p}{r-1} \right)^p \int_0^\infty x^{-r} |x f(x)|^p \, dx \qquad (3.12)
$$

where $F(x) = \int_x^\infty f(t)\, dt$ for $r < 1$, and $F(x) = \int_0^x f(t)\, dt$ for $r > 1$. Applying this inequality with $p = 2$, $r = -1/2$, and $F(x) = \chi(x)(v(x) - v_p(x))$ we have

$$
\begin{aligned}
&\int_{-1}^{\infty} (1+x)^{1/2} |\chi(x)(v - v_p)|^2 \, dx \\
&\le c \int_{-1}^{\infty} (1+x)^{3/2+1} \left(|\chi(x)(v' - v_p')|^2 + |\chi'(x)(v - v_p)|^2 \right) dx \\
&\le c \left(\int_{-1}^{1-\delta} (1+x)^{3/2+1} |v' - v_p'|^2 dx + \int_{-1+\delta}^{1-\delta} |v - v_p|^2 \, dx \right) \\
&\le c \left(\int_{-1}^{1} (1-x^2)^{3/2+1} |v' - v_p'|^2 \, dx + \int_{-1}^{1} (1-x^2)^{3/2} |v - v_p|^2 \, dx \right) \\
&= c \|v - v_p\|^2_{H^{1,3/2}(I)}. \qquad (3.13)
\end{aligned}
$$

The third integral on the right hand side of (3.9) can be estimated analogously. Therefore, combining (3.9)-(3.13) and applying Theorem 2.1 once more, we obtain

$$\|v - v_p\|_{H^{0,1/2}(I)} \leq c\|v - v_p\|_{H^{1,3/2}(I)} \leq cp^{-s}\|v\|_{B^{s,3/2}(I)}.$$

Combination with (3.7) and (3.8) yields

$$|E(v)|_{H^{1,-1/2}(I)} \leq cp^{-(s-1)}\|v\|_{B^{s,3/2}(I)}.$$

Therefore, $E(v) \in H^{1,-1/2}(I)$, $E(v)(\pm 1) = 0$ by construction, and we have (3.6) with bound $cp^{-(s-1)}$. This finishes the proof. $\square$

By interpolation we obtain the following important result which is a key point in the proof of the optimal upper bound for the approximation error in $\tilde{H}^{1/2}(I)$, cf. Theorem 3.4.

Corollary 3.1 *For $s > 1/2$, the operator E defined in (3.4) with $p \geq 0$ is an operator: $B^{s,3/2}(I) \to \tilde{H}^{1/2}(I)$, and its norm, denoted by $\|E\|_{s\to 1/2}$, satisfies*

$$\|E\|_{s\to 1/2} \leq cp^{-(s-1/2)}.$$

Proof. First note that, for $s > 1/2$, there holds

$$(B^{s-1/2,3/2}(I), B^{s+1/2,3/2}(I))_{1/2,2} = H^{s,3/2}(I). \tag{3.14}$$

Indeed, for a couple (A_0, A_1) of compatible Banach spaces the reiteration theorem gives

$$((A_0, A_1)_{\theta_0,r_0}, (A_0, A_1)_{\theta_1,r_1})_{\lambda,q} = (A_0, A_1)_{(1-\lambda)\theta_0+\lambda\theta_1,q}.$$

Here, $0 < \theta_0 \leq \theta_1 < 1$, $1 \leq r_0, r_1, q \leq \infty$, $0 \leq \lambda \leq 1$. Choosing $\epsilon > 0$ such that $s-1/2-\epsilon > 0$ and $A_0 = H^{s-1/2-\epsilon,3/2}(I)$, $A_1 = H^{s+1/2+\epsilon,3/2}(I)$, we obtain with $\theta_0 = \epsilon/(1+2\epsilon)$, $\theta_1 = (1+\epsilon)/(1+2\epsilon)$, $\lambda = 1/2$, $r_0 = r_1 = \infty$, $q = 2$ the following relations:

$$\begin{aligned}
(A_0, A_1)_{\theta_0,r_0} &= (H^{s-1/2-\epsilon,3/2}(I), H^{s+1/2+\epsilon,3/2}(I))_{\frac{\epsilon}{1+2\epsilon},\infty} = B^{s-1/2,3/2}(I),\\
(A_0, A_1)_{\theta_1,r_1} &= (H^{s-1/2-\epsilon,3/2}(I), H^{s+1/2+\epsilon,3/2}(I))_{\frac{1+\epsilon}{1+2\epsilon},\infty} = B^{s+1/2,3/2}(I),\\
(A_0, A_1)_{(1-\lambda)\theta_0+\lambda\theta_1,q} &= (H^{s-1/2-\epsilon,3/2}(I), H^{s+1/2+\epsilon,3/2}(I))_{1/2,2} = H^{s,3/2}(I).
\end{aligned}$$

Then relation (3.14) follows.

Now, from Lemma 3.1 and (3.14), we obtain by interpolation for $s > 1/2$

$$\begin{aligned}
E:\ & H^{s,3/2}(I) = (B^{s-1/2,3/2}(I), B^{s+1/2,3/2}(I))_{1/2,2}\\
\to\ & (H^{0,-1/2}(I), H_0^{1,-1/2}(I))_{1/2,2} = \tilde{H}^{1/2}(I)
\end{aligned} \tag{3.15}$$

with bound

$$\|E\| \leq \|E\|_{s-1/2\to 0}^{1/2}\|E\|_{s+1/2\to 1}^{1/2} \leq cp^{-(s-1/2)}.$$

Now interpolating (3.15) with $q = \infty$ we obtain by again using the reiteration theorem (and selecting $\epsilon \in (0, s - 1/2)$)

$$\begin{aligned} E:\ B^{s,3/2}(I) &= (H^{s-\epsilon,3/2}(I), H^{s+\epsilon,3/2}(I))_{1/2,\infty} \\ &\to (\tilde{H}^{1/2}(I), \tilde{H}^{1/2}(I))_{1/2,\infty} = \tilde{H}^{1/2}(I) \end{aligned}$$

with bound

$$\|E\|_{s\to 1/2} \le c\left(p^{-(s-\epsilon-1/2)}p^{-(s+\epsilon-1/2)}\right)^{1/2} = cp^{-(s-1/2)}.$$

□

In order to apply properties of the operator E to the singular function $(1+x)^\gamma$, we need to study the relationship between the spaces $B^{s,\beta+2}(I)$ and $B^{s,\beta}(I)$.

Lemma 3.2 *Let $u(x) = (1-x^2)v(x)$. If $v \in H^{k,\beta+2}(I), k \ge 0, \beta > -1$, then $u \in H^{k,\beta}(I)$, and there are constant c_1 and c_2 independent of u and v, such that*

$$c_1\|v\|_{H^{k,\beta+2}(I)} \le \|u\|_{H^{k,\beta}(I)} \le c_2\|v\|_{H^{k,\beta+2}(I)} \tag{3.16}$$

Furthermore, if $v \in B^{s,\beta+2}(I), s > 0, \beta > -1$, then $u \in B^{s,\beta}(I)$, and

$$\|u\|_{B^{s,\beta}(I)} \le c\|v\|_{B^{s,\beta+2}(I)} \tag{3.17}$$

with c independent of u and v.

Proof. It is obvious that $\|u\|_{H^{0,\beta}(I)} = \|v\|_{H^{0,\beta+2}(I)}$. For $k \ge 1$,

$$u^{(k)} = (1-x^2)v^{(k)} - 2kxv^{(k-1)} - k(k-1)v^{(k-2)}, \tag{3.18}$$

The third term on the right hand side of (3.18) does not appear if $k = 1$. This gives

$$\begin{aligned} \int_{-1}^{1} |u^{(k)}|^2(1-x^2)^{k+\beta}dx \le c\int_{-1}^{1}\Big(|v^{(k)}|^2(1-x^2)^{k+\beta+2} + |v^{(k-1)}|^2(1-x^2)^{k+\beta} \\ +|v^{(k-2)}|^2(1-x^2)^{k-2+\beta+2}\Big)\,dx. \end{aligned} \tag{3.19}$$

We need to deal with the second term on the right hand side of (3.19). Note that

$$\int_{-1}^{1}|v^{(k-1)}|^2(1-x^2)^{k+\beta}dx = (\int_{-1}^{-1+\delta} + \int_{-1+\delta}^{1-\delta} + \int_{1-\delta}^{1})|v^{(k-1)}|^2(1-x^2)^{k+\beta}dx. \tag{3.20}$$

The second term of (3.20) is bounded by

$$\int_{-1+\delta}^{1-\delta}|v^{(k-1)}|^2(1-x^2)^{k+\beta}dx \le c\int_{-1}^{1}|v^{(k-1)}|^2(1-x^2)^{k-1+\beta+2}dx. \tag{3.21}$$

To estimate the first term, we use a $C^\infty(\mathbf{R})$ cut-off function $\chi(x)$ where $\chi(x) = 1$ for $x \in (-1, -1+\delta)$ and $\chi(x) = 0$ for $x > 1-\delta, \delta \in (0,1)$. Then we have

$$\int_{-1}^{-1+\delta} |v^{(k-1)}|^2 (1-x^2)^{k+\beta} dx \le C \int_{-1}^{\infty} |\chi(x) v^{(k-1)}|^2 (1+x)^{k+\beta} dx. \tag{3.22}$$

Applying the Hardy inequality (3.12) with $p = 2$, $r = -(k+\beta) < 1$, $F(x) = \int_x^\infty f(t)dt$ and $f(x) = \chi(x) v^{(k-1)}(x)$, we have

$$\begin{aligned}
\int_{-1}^{\infty} |\chi v^{(k-1)}|^2 (1+x)^{(k+\beta)}\, dx &\le C \int_{-1}^{\infty} |(\chi v^{(k-1)})'|^2 (1+x)^{(k+\beta+2)}\, dx \\
&\le C \int_{-1}^{\infty} \left(|\chi v^{(k)}|^2 + |\chi' v^{(k-1)}|^2 \right) (1+x)^{k+\beta+2}\, dx \\
&\le C \left(\int_{-1}^{1-\delta} |v^{(k)}|^2 (1+x)^{k+\beta+2}\, dx + \int_{-1+\delta}^{1-\delta} |v^{(k-1)}|^2 (1+x)^{k+\beta+2}\, dx \right) \\
&\le C \int_{-1}^{1} \left(|v^{(k)}|^2 (1-x^2)^{k+\beta+2} + |v^{(k-1)}|^2 (1-x^2)^{k-1+\beta+2} \right) dx \\
&= C \left(|v|^2_{H^{k,\beta+2}(I)} + |v|^2_{H^{k-1,\beta+2}(I)} \right),
\end{aligned} \tag{3.23}$$

which together with (3.22) yields

$$\int_{-1}^{-1+\delta} |v^{(k-1)}|^2 (1-x^2)^{k+\beta}\, dx \le C \|v\|^2_{H^{k,\beta+2}(I)}. \tag{3.24}$$

Similarly it can be proved that

$$\int_{1-\delta}^{1} |v^{(k-1)}|^2 (1-x^2)^{k+\beta}\, dx \le C \|v\|^2_{H^{k,\beta+2}(I)}. \tag{3.25}$$

Combining (3.19)-(3.25), we obtain $u \in H^{k,\beta}(I)$ and the second inequality of (3.16).

We shall prove the first inequality of (3.16) by induction. For $k = 0$, it is true, i.e.

$$\|v\|_{H^{0,\beta+2}(I)} = \|u\|_{H^{0,\beta}(I)}.$$

Suppose the inequality holds for $0 \le l < k$. Due to (3.18), there holds for $k \ge 1$

$$v^{(k)} = (1-x^2)^{-1} u^{(k)} + 2kx(1-x^2)^{-1} v^{(k-1)} + k(k-1)(1-x^2)^{-1} v^{(k-2)},$$

and by the assumption of induction, we have

$$\begin{aligned}
\int_{-1}^{1} |v^{(k)}|^2 (1-x^2)^{k+\beta+2}\, dx &\le C \int_{-1}^{1} \Big(|u^{(k)}|^2 (1-x^2)^{k+\beta} + |v^{(k-1)}|^2 (1-x^2)^{k+\beta} \\
&\quad + |v^{(k-2)}|^2 (1-x^2)^{k-2+\beta+2} \Big)\, dx \\
&\le C \Big(|u|^2_{H^{k,\beta}(I)} + \|u\|^2_{H^{k-2,\beta}(I)} \\
&\quad + \int_{-1}^{1} |v^{(k-1)}|^2 (1-x^2)^{k+\beta}\, dx \Big)
\end{aligned} \tag{3.26}$$

Using the arguments in (3.20)-(3.23), we have

$$
\begin{aligned}
&\int_{-1}^{1} |v^{(k-1)}|^2 (1-x^2)^{k+\beta}\,dx \\
&\leq\; M\left(\int_{-1}^{-1+\delta} |v^{(k)}|^2 (1-x^2)^{k+\beta+2}\,dx + \int_{1-\delta}^{1} |v^{(k)}|^2 (1-x^2)^{k+\beta+2}\,dx + \|v\|^2_{H^{k-1,\beta+2}(I)}\right).
\end{aligned}
$$

Selecting δ small enough such that

$$
\left(\int_{-1}^{-1+\delta} + \int_{1-\delta}^{1}\right) |v^{(k)}|^2 (1-x^2)^{k+\beta+2}\,dx \leq \frac{1}{2CM}|v|^2_{H^{k,\beta+2}(I)}
$$

this leads, together with (3.26), to

$$
c_1\|v\|_{H^{k,\beta+2}(I)} \leq \|u\|_{H^{k,\beta}(I)}.
$$

Thus we complete the induction.

If $v \in B^{s,\beta+2}(I)$, and $u(x) = (1-x^2)v(x)$, we have

$$
\begin{aligned}
K(t,u) &= \inf_{u=u_1+u_2,\,u_1\in H^{0,\beta}(I),\,u_2\in H^{k,\beta}(I)} \left(\|u_1\|_{H^{0,\beta}(I)} + t\|u_2\|_{H^{k,\beta}(I)}\right) \\
&\leq \inf_{v=v_1+v_2,\,v_1\in H^{0,\beta+2}(I),\,v_2\in H^{k,\beta+2}(I)} \left(\|v_1\|_{H^{0,\beta+2}(I)} + t\|v_2\|_{H^{k,\beta+2}(I)}\right) \\
&= K(t,v),
\end{aligned}
$$

which implies $u \in B^{s,\beta}(I)$ and (3.17). □

The inequality inverse to (3.17) is not true in general, but holds for functions in a subset of $B^{s,\beta}(I)$. Let $\hat{H}^{k,\beta}(I)$ be the set of functions $u(x) = (1-x^2)v(x)$ with $v(x) \in H^{k,\beta+2}(I)$, $k \geq 0$, $\beta > -1$. Due to Lemma 3.2, $\hat{H}^{k,\beta}(I)$ is a complete subspace of $H^{k,\beta}(I)$. By $\hat{B}^{s,\beta}(I)$, we denote the interpolation space

$$
\hat{B}^{s,\beta}(I) = \left(\hat{H}^{l,\beta}(I), \hat{H}^{k,\beta}(I)\right)_{\theta,\infty}
$$

with $s = (1-\theta)l + \theta k$, $k > l \geq 0$, $0 < \theta < 1$. It is easy to see that $\hat{B}^{s,\beta}(I) \subset B^{s,\beta}(I)$ for $s > 0$ and $\beta > -1$, and

$$
\|u\|_{\hat{B}^{s,\beta}(I)} \geq \|u\|_{B^{s,\beta}(I)}.
$$

The next lemma will establish an isomorphism between $B^{s,\beta+2}(I)$ and $\hat{B}^{s,\beta}(I)$ under the mapping $F:\ B^{s,\beta+2}(I) \to \hat{B}^{s,\beta}(I)$ with

$$
F(v) = (1-x^2)v.
$$

Lemma 3.3 *Let $u \in \hat{B}^{s,\beta}(I)$ with $s > 0$ and $\beta > -1$, and let $v = F^{-1}(u) = (1-x^2)^{-1}u$. Then $v \in B^{s,\beta+2}(I)$, and*

$$\|v\|_{B^{s,\beta+2}(I)} \leq c_1 \|u\|_{\hat{B}^{s,\beta}(I)}.$$

Vice versa, if $v \in B^{s,\beta+2}(I)$, then $u = (1-x^2)v \in \hat{B}^{s,\beta}(I)$, and

$$\|u\|_{\hat{B}^{s,\beta}(I)} \leq c_2 \|u\|_{B^{s,\beta+2}(I)}.$$

Proof. Let u_1 and u_2 form an arbitrary decomposition of u, with $u_1 \in \hat{H}^{0,\beta}(I) = H^{0,\beta}(I)$ and $u_2 \in \hat{H}^{k,\beta}(I)$, $k > s$. Then, due to the definition of the space $\hat{H}^{k,\beta}(I)$, $v_1 = (1-x^2)^{-1}u_1 \in H^{0,\beta+2}(I)$ and $v_2 = (1-x^2)^{-1}u_2 \in H^{k,\beta+2}(I)$. Note that the functor satisfies

$$\begin{aligned}
\hat{K}(t,u) &= \inf_{u=u_1+u_2, u_1\in H^{0,\beta}(I), u_2 \in \hat{H}^{k,\beta}(I)} \left(\|u_1\|_{H^{0,\beta}(I)} + t\|u_2\|_{H^{k,\beta}(I)}\right) \\
&= \inf_{v=v_1+v_2, v_1\in H^{0,\beta+2}(I), v_2 \in H^{k,\beta+2}(I)} \left(\|(1-x^2)v_1\|_{H^{0,\beta}(I)} + t\|(1-x^2)v_2\|_{H^{k,\beta}(I)}\right) \\
&\geq \frac{1}{c_1} \inf_{v=v_1+v_2} \left(\|v_1\|_{H^{0,\beta+2}(I)} + t\|v_2\|_{H^{k,\beta+2}(I)}\right) = \frac{1}{c_1} K(t,v)
\end{aligned}$$

where we made use of Lemma 3.2 for the last estimate. This implies $v \in B^{s,\beta+2}(I)$, and

$$\|v\|_{B^{s,\beta+2}(I)} \leq c_1 \|u\|_{\hat{B}^{s,\beta}(I)}.$$

If $v \in B^{s,\beta+2}(I)$, $v_1 \in H^{0,\beta+2}(I)$ and $v_2 \in H^{k,\beta+2}(I)$, $k > s$, form a decomposition of v, i.e. $v = v_1 + v_2$, then, by Lemma 3.2, $u_1 = (1-x^2)v_1 \in H^{0,\beta}(I)$ and $u_2 = (1-x^2)v_2 \in \hat{H}^{k,\beta}(I)$. Furthermore, u_1 and u_2 form a decomposition of $u = (1-x^2)v$, and

$$\begin{aligned}
\hat{K}(t,u) &= \inf_{u=w_1+w_2, w_1\in H^{0,\beta}(I), w_2 \in \hat{H}^{k,\beta}(I)} \left(\|w_1\|_{H^{0,\beta}(I)} + t\|w_2\|_{H^{k,\beta}(I)}\right) \\
&\leq \|u_1\|_{H^{0,\beta}(I)} + t\|u_2\|_{H^{k,\beta}(I)} \\
&= \|(1-x^2)v_1\|_{H^{0,\beta}(I)} + t\|(1-x^2)v_2\|_{H^{k,\beta}(I)} \\
&\leq c_2 \left(\|v_1\|_{H^{0,\beta+2}(I)} + t\|v_2\|_{H^{k,\beta+2}(I)}\right),
\end{aligned}$$

where again we used Lemma 3.2 in the last step. Since v_1 and v_2 are an arbitrary decomposition of v, we have

$$\hat{K}(t,u) \leq c_2 \inf_{v=v_1+v_2} \left(\|v_1\|_{H^{0,\beta+2}(I)} + t\|v_2\|_{H^{k,\beta+2}(I)}\right) \leq c_2 K(t,v)$$

which implies $u \in \hat{B}^{s,\beta}(I)$ and

$$\|u\|_{\hat{B}^{s,\beta}(I)} \leq c_2 \|v\|_{B^{s,\beta+2}(I)}.$$

$\square$

Theorem 3.3 *Let $u(x) = (1+x)^\gamma \chi(x)$, and let $v(x) = (1-x^2)^{-1}u(x)$ where $\chi(x)$ is a $C^\infty(\mathbf{R})$ cut-off function such that $\chi(x) = 1$ for $-1 < x < -1+\delta_0/2$ and $\chi(x) = 0$ for $x > 1-\delta_0$ for some $\delta_0 \in (0,1)$. Then $u(x) \in \hat{B}^{2\gamma+1/2,-1/2}(I)$, $v(x) \in B^{2\gamma+1/2,3/2}(I)$, and there holds*

$$\|v\|_{B^{2\gamma+1/2,3/2}(I)} \cong \|u\|_{\hat{B}^{2\gamma+1/2,-1/2}(I)}. \tag{3.27}$$

Proof. Let $\varphi_\delta(x)$ be a $C^\infty(\mathbf{R})$ cut-off function where $0 < \delta < \delta_0$ such that $\varphi_\delta(x) = 1$ for $-1 < x < -1+\delta/2$ and $\varphi_\delta(x) = 0$ for $x > 1-\delta$. We decompose $u = u_1 + u_2$ with $u_1 = \varphi_\delta u(x)$. Then $u_1 \in H^{0,-1/2}(I)$ and $u_2 \in \hat{H}^{k,-1/2}(I)$ for any $k > 2\gamma + 1/2$. Therefore

$$\begin{aligned}\hat{K}(t,u) &= \inf_{u=w_1+w_2;\, w_1\in H^{0,-1/2}(I), w_2\in \hat{H}^{k,-1/2}(I)} \left(\|w_1\|_{H^{0,-1/2}(I)} + t\|w_2\|_{\hat{H}^{k,-1/2}(I)}\right)\\ &\le \left(\|u_1\|_{H^{0,-1/2}(I)} + t\|u_2\|_{\hat{H}^{k,-1/2}(I)}\right).\end{aligned}$$

Arguing as in the proof of Theorem 3.1 we have

$$\|u_1\|_{H^{0,-1/2}(I)} \le C\delta^{\gamma+1/4}$$

and

$$\|u_2\|_{H^{k,-1/2}(I)} \le C\delta^{\gamma+1/2(1/2-k)}.$$

Selecting $\delta = t^{\frac{2}{k}}$ and $\theta = \frac{2\gamma+1/2}{k}$, we obtain

$$\|u\|_{B^{2\gamma+1/2,-1/2}(I)} = \sup_{t>0} t^{-\theta} K(t,u) < \infty.$$

By Lemma 3.3 we have $v \in B^{2\gamma+1/2,3/2}(I)$, and

$$\|v\|_{B^{2\gamma+1/2,3/2}(I)} \cong \|u\|_{\hat{B}^{2\gamma+1/2,-1/2}(I)}$$

□

Now we are in the position to prove an optimal upper bound for the approximation error in $\tilde{H}^{1/2}(I)$ for the Jacobi projection of the singular function of $(1+x)^\gamma$-type.

Theorem 3.4 *Let $u(x) = (1+x)^\gamma \chi(x)$ with $\gamma > 0$ and $\chi(x) \in C^\infty(I)$ such that $\chi(x) = 1$ for $-1 < x < -1+\delta_0/2$ and $\chi(x) = 0$ for $-1+\delta_0 < x < 1$. Here, $\delta_0 < 1$ is a positive constant. Then, for $p \ge 2$, there exists a polynomial $\psi \in P_p(I)$ such that $\psi(\pm1) = u(\pm1) = 0$, and*

$$\|u-\psi\|_{\tilde{H}^{1/2}(I)} \le cp^{-2\gamma}\|u\|_{\hat{B}^{2\gamma+1/2,-1/2}(I)}.$$

Proof. Let $v(x) = (1-x^2)^{-1}u(x)$. By Theorem 3.3, $u \in \hat{B}^{2\gamma+1/2,-1/2}(I)$ and $v \in B^{2\gamma+1/2,3/2}(I)$. Let v_{p-2} denote the Jacobi projection of v onto $P_{p-2}(I)$ with $p \ge 2$ and Jacobi weight $\beta = 3/2$. Further we define $\psi \in P_p(I)$ by $\psi(x) = (1-x^2)v_p(x)$. Then $u(\pm1) = \psi(\pm1) = 0$, and

$$u(x) - \psi(x) = (1-x^2)(v(x) - v_{p-2}(x)) = E(v)(x) \quad (x \in I)$$

where E is the operator defined in (3.4). Applying Corollary 3.1 with $s = 2\gamma + 1/2 > 1/2$ we have

$$\|E(v)\|_{\tilde{H}^{1/2}(I)} \leq cp^{-2\gamma}\|v\|_{B^{2\gamma+1/2,3/2}(I)}.$$

This together with (3.27) leads to

$$\|u - \psi\|_{\tilde{H}^{1/2}(I)} = \|E(v)\|_{\tilde{H}^{1/2}(I)} \leq cp^{-2\gamma}\|u\|_{\hat{B}^{2\gamma+1/2,-1/2}(I)}.$$

□

4 Modified Jacobi-weighted Besov space and approximability of singular functions of $x^\gamma \log^\nu x$-type

Let

$$u(x) = (1+x)^\gamma \log^\nu(1+x) \tag{4.1}$$

with $\gamma > 0$ and integer $\nu \geq 0$. To prove optimal approximation results we have to modify the weighted Besov space $B^{s,\beta}(I)$, as it is done in [4]. For $\nu > 0$, $l < k$, $0 < \theta < 1$, $s = (1-\theta)l + \theta k$ we define

$$B_\nu^{s,\beta}(I) = (H^{l,\beta}(I), H^{k,\beta}(I))_{\theta,\infty,\nu}$$

with norm

$$\|u\|_{B_\nu^{s,\beta}(I)} = \sup_{t>0} \frac{t^{-\theta}}{(1+|\log t|)^\nu} K(t,u)$$

where the functor $K(t,u)$ is defined in (2.2).

Remark 4.1 *The modified Jacobi-weighted Besov space $B_\nu^{s,\beta}(I)$ is not an exact interpolation space, but a uniform interpolation space, for which the reiteration theorem and other properties of exact interpolation spaces do not hold. A partial reiteration theorem (Theorem 3.2 of [4]) implies that the space $B_\nu^{s,\beta}(I)$ does not depend on the individual values of l and k, but the linear combination $(1-\theta)l + \theta k$, and that l and k can be non-integers.*

It is easy to verify that the function given in (4.1) belongs to $B^{s,\beta}(I)$ with $s = 2\gamma + 1 + \beta - \epsilon$ and $\epsilon > 0$ arbitrary. This leads to a loss of $O(p^\epsilon)$ in the upper bound of the approximation error. An analysis in the modified space $B_\nu^{s,\beta}(I)$ can remove the ϵ in s, therefore avoiding the loss in the approximation.

Theorem 4.1 *Let u be the singular function given by (4.1) with $\gamma > 0$ and integer $\nu \geq 0$. Then $u \in B_\nu^{2\gamma+1+\beta,\beta}(I)$ if γ is non-integer, or if γ is an integer and $\nu = 0$. If γ is an integer and $\nu \geq 1$ then $u \in B_{\nu-1}^{2\gamma+1+\beta,\beta}(I)$. Here, $\beta > -1$ is arbitrary.*

Proof. The proof for $\nu = 0$ is given by Theorem 3.1. When $\nu \geq 1$ we follow the proof of [4, Theorem 2.5]. As in the proof of Theorem 3.1 we use the cut-off function φ_δ. Then, for $u = v + w$ with $v = \varphi_\delta u$ and $w = (1-\varphi_\delta)u$, we obtain

$$\|v\|_{H^{l,\beta}(I)} \leq C\delta^{\gamma+\frac{1}{2}(1+\beta-l)}|\log\delta|^\nu$$

and

$$\|w\|_{H^{k,\beta}(I)} \leq C\delta^{\gamma+\frac{1}{2}(1+\beta-k)}|\log\delta|^\nu$$

for all integers l, k with $l < 2\gamma+1+\beta < k$. Choosing $\delta = t^{\frac{2}{k-l}}$ for $0 < t < 1$ and $\theta = \frac{2\gamma+1+\beta-l}{k-l}$ this proves

$$\sup_{0<t<1} \frac{t^{-\theta}K(t,u)}{(1+|\log t|)^\nu} \leq C.$$

There also holds

$$\sup_{t>1} \frac{t^{-\theta}K(t,u)}{(1+|\log t|)^\nu} \leq \|u\|_{H^{l,\beta}(I)}$$

such that $u \in B_\nu^{2\gamma+1+\beta,\beta}(J)$.

For integer $\gamma > 0$ we represent u as $u = v + w$ with

$$w = x^\gamma \log^\nu(x+\delta)$$

and

$$v = x^\gamma(\log^\nu x - \log^\nu(x+\delta)).$$

As in [1] one obtains for $k > 2\gamma+1+\beta$

$$\|w\|_{H^{k,\beta}(I)} \leq C\delta^{\gamma+\frac{1}{2}(1+\beta-k)}|\log\delta|^{\nu-1}$$

and

$$\|v\|_{H^{0,\beta}(I)} \leq C\delta^{\gamma+\frac{1}{2}(1+\beta)}|\log\delta|^{\nu-1}$$

such that

$$K(t,u) \leq C|\log\delta|^{\nu-1}\left(\delta^{\gamma+\frac{1}{2}(1+\beta)} + t\delta^{\gamma+\frac{1}{2}(1+\beta-k)}\right).$$

Choosing $\delta = t^{\frac{2}{k}}$ and $\theta = \frac{2\gamma+1+\beta}{k}$ this proves

$$\sup_{t>0} \frac{t^{-\theta}K(t,u)}{(1+|\log t|)^{\nu-1}} \leq C.$$

Therefore, $u \in B_{\nu-1}^{2\gamma+1+\beta,\beta}(I)$. □

Theorem 4.2 *Let $u \in B^{s,\beta}_{\nu}(I)$ for $s > 0$, $\nu \geq 0, \beta > -1$. Then there exists a polynomial $\varphi \in P_p(I)$ such that for real $r < s$*

$$\|u - \varphi\|_{H^{r,\beta}(I)} \leq Cp^{-(s-r)}(1 + \log p)^{\nu} \|u\|_{B^{s,\beta}_{\nu}(I)}$$

where C is independent of p.

Proof. The proof of Theorem 2.6 in [1] in two dimensions holds in one dimension as well. There, only integers r are considered. For the convenience of the reader we recall the main steps and give the modification of the proof to real r.

By the partial reiteration theorem mentioned in Remark 4.1, we can define $B^{s,\beta}_{\nu}(I)$ equivalently by interpolating between $H^{r,\beta}(I)$ and $H^{k,\beta}(I)$ for real $r < s$ and integer $k > s$:

$$B^{s,\beta}_{\nu}(I) = (H^{r,\beta}(I), H^{k,\beta}(I))_{\theta,\infty,\nu}$$

with $\theta = \frac{s-r}{k-r}$ such that $(1-\theta)r + \theta k = s$. Then, by definition of $B^{s,\beta}_{\nu}(I)$, there exist $v \in H^{r,\beta}(I)$ and $w \in H^{k,\beta}(I)$ such that $u = v + w$ and

$$\|v\|_{H^{r,\beta}(I)} + t\|w\|_{H^{k,\beta}(I)} \leq Ct^{\theta}(1 + |\log t|)^{\nu} \|u\|_{B^{s,\beta}_{\nu}(I)} \quad \forall t > 0. \tag{4.2}$$

Approximating u by the projection w_p of w onto $P_p(I)$ in $H^{0,\beta}(I)$ and using the estimate (2.3) in Theorem 2.1 we obtain

$$\|u - w_p\|_{H^{r,\beta}(I)} \leq \|v\|_{H^{r,\beta}(I)} + \|w - w_p\|_{H^{r,\beta}(I)} \leq \|v\|_{H^{r,\beta}(I)} + cp^{-(k-r)}\|w\|_{H^{k,\beta}(I)}. \tag{4.3}$$

Taking $t = p^{-(k-r)}$ combination of (4.2) and (4.3) yields

$$\begin{aligned} \|u - w_p\|_{H^{r,\beta}(I)} &\leq c\left(\|v\|_{H^{r,\beta}(I)} + t\|w\|_{H^{k,\beta}(I)}\right) \\ &\leq ct^{\theta}(1 + |\log t|)^{\nu}\|u\|_{B^{s,\beta}_{\nu}(I)} \leq cp^{-(s-r)}(1 + \log p)^{\nu}\|u\|_{B^{s,\beta}_{\nu}(I)} \end{aligned}$$

which proves the assertion. □

Corollary 4.1 *For the function u given by (4.1) there exists a polynomial $\varphi \in P_p(I)$ such that there holds for $r = 0, 1/2$*

$$\|u - \varphi\|_{H^r(I)} \leq cp^{-2\gamma-1+2r}(1 + \log p)^{\nu^*}\|u\|_{B^{2\gamma+1-r,-r}_{\nu^*}(I)}.$$

Here, $\nu^* = \nu$ if γ is non-integer or integer and $\nu = 0$, and $\nu^* = \nu - 1$ if γ is an integer and $\nu \geq 1$.

Proof. Due to Theorem 4.1, $u \in B^{2\gamma+1+\beta,\beta}_{\nu^*}(I)$ for $\beta > -1$. By Theorem 4.2 with $\beta = r = 0$ we know that there exists $\varphi \in P_p(I)$ such that

$$\|u-\varphi\|_{H^{0,0}(I)} = \|u-\varphi\|_{L^2(I)} \le cp^{-2\gamma-1}(1+\log p)^{\nu^*}\|u\|_{B^{2\gamma+1,0}_{\nu^*}(I)},$$

and for $\beta = -1/2, r = 1/2$ there holds

$$\|u-\varphi\|_{H^{1/2}(I)} \cong \|u-\varphi\|_{H^{1/2,-1/2}(I)} \le cp^{-2\gamma}(1+\log p)^{\nu^*}\|u\|_{B^{2\gamma+1/2,-1/2}_{\nu^*}(I)}.$$

□

In order to prove an optimal upper bound in $\tilde{H}^{1/2}(I)$ for the approximation of a singular function with logarithmic terms we need to prove technical results similar to those in Section 3. The next lemma corresponds to Corollary 3.1.

Lemma 4.1 *For $s > 1/2$, the operator E defined in (3.4) with $p \ge 0$ is an operator: $B^{s,3/2}_{\nu}(I) \to \tilde{H}^{1/2}(I)$, and its norm, denoted by $\|E\|_{\nu,s\to 1/2}$, satisfies*

$$\|E\|_{\nu,s\to 1/2} \le cp^{-(s-1/2)}(1+\log p)^{\nu}.$$

Proof. From Lemma 3.1 and (3.14) we obtain that, for $s > 1/2$, E is an operator

$$\begin{aligned} E:\ H^{s,3/2}(I) &= (B^{s-1/2,3/2}(I), B^{s+1/2,3/2}(I))_{1/2,2} \\ &\to\ (H^{0,-1/2}(I), H^{1,-1/2}_0(I))_{1/2,2} = \tilde{H}^{1/2}(I) \end{aligned} \tag{4.4}$$

with bound

$$\|E\| \le \|E\|^{1/2}_{s-1/2\to 0}\|E\|^{1/2}_{s+1/2\to 1} \le cp^{-(s-1/2)}. \tag{4.5}$$

Due to the partial reiteration theorem mentioned in Remark 4.1, there holds for $\epsilon \in (0, s-1/2)$

$$(H^{s-\epsilon,3/2}(I), H^{s+\epsilon,3/2}(I))_{1/2,\infty,\nu} = B^{s,3/2}_{\nu}(I).$$

Therefore, for given $v \in B^{s,3/2}_{\nu}(I)$ there exist $v_1 \in H^{s-\epsilon,3/2}(I)$ and $v_2 \in H^{s+\epsilon,3/2}(I)$ such that $v = v_1 + v_2$ and

$$\|v_1\|_{H^{s-\epsilon,3/2}(I)} + t\|v_2\|_{H^{s+\epsilon,3/2}(I)} \le ct^{1/2}(1+|\log t|^{\nu})\|v\|_{B^{s,3/2}_{\nu}(I)} \qquad \forall t > 0. \tag{4.6}$$

Application of (4.5) gives

$$\begin{aligned} \|E(v)\|_{\tilde{H}^{1/2}(I)} &\le \|E(v_1)\|_{\tilde{H}^{1/2}(I)} + \|E(v_2)\|_{\tilde{H}^{1/2}(I)} \\ &\le cp^{-(s-\epsilon-1/2)}\|v_1\|_{H^{s-\epsilon,3/2}(I)} + cp^{-(s+\epsilon-1/2)}\|v_2\|_{H^{s+\epsilon,3/2}(I)}. \end{aligned}$$

Selecting $t = p^{-2\epsilon}$ in (4.6) then yields

$$\|E(v)\|_{\tilde{H}^{1/2}(I)} \le cp^{-(s-1/2)}(1+|\log p|^{\nu})\|v\|_{B^{s,3/2}_{\nu}(I)}$$

for any $v \in B_\nu^{s,3/2}(I)$. This proves the lemma. □

To apply the operator E to singular functions of $(1+x)^\gamma \log^\nu(1+x)$-type, we need to study the relation between the spaces $B_\nu^{s,3/2}(I)$ and $B_\nu^{s,-1/2}(I)$. We proceed analogously as in the previous section and introduce an interpolation space by

$$\hat{B}_\nu^{s,\beta}(I) = \left(\hat{H}^{l,\beta}(I), \hat{H}^{k,\beta}(I)\right)_{\theta,\infty,\nu}.$$

Here, $s = (1-\theta)l + \theta k$, $k > l \geq 0$, $0 < \theta < 1$ and $\nu \geq 0$, where $\hat{H}^{k,\beta}(I)$ is the subspace of $H^{k,\beta}(I)$ defined in the previous section. It is easy to see that for $s > 0$, $\nu \geq 0$ and $\beta > -1$,

$$\|u\|_{\hat{B}_\nu^{s,\beta}(I)} \geq \|u\|_{B_\nu^{s,\beta}(I)}.$$

As analogues to Lemma 3.2 and Theorem 3.3, we have following lemma and theorem regarding the equivalence between the spaces $\hat{B}_\nu^{2\gamma+1/2,-1/2}(I)$ and $B_\nu^{2\gamma+1/2,3/2}(I)$ under the mapping F defined by

$$F(v) = (1-x^2)v.$$

Lemma 4.2 *Let $u(x) = (1-x^2)v(x)$. If $v \in B_\nu^{s,\beta+2}(I)$, $k \geq 0$, $\beta > -1$, $\nu \geq 0$, then $u \in B_\nu^{s,\beta}(I)$, and*

$$\|u\|_{B_\nu^{s,\beta}(I)} \leq c\|v\|_{B_\nu^{s,\beta+2}(I)}$$

with c being independent of u and v.

Theorem 4.3 *Let $u(x) = (1+x)^\gamma \log^\nu(1+x)\chi(x)$ and $v(x) = (1-x^2)^{-1}u(x)$, where χ is a $C^\infty(\mathbf{R})$ cut-off function such that $\chi(x) = 1$ for $-1 < x < -1+\delta_0/2$ and $\chi(x) = 0$ for $x > 1-\delta_0$ for some $\delta_0 \in (0,1)$. Then $u \in \hat{B}_{\nu^*}^{2\gamma+1/2,-1/2}(I)$, $v \in B_{\nu^*}^{2\gamma+1/2,3/2}(I)$, and there holds*

$$\|v\|_{B_{\nu^*}^{2\gamma+1/2,3/2}(I)} \cong \|u\|_{\hat{B}_{\nu^*}^{2\gamma+1/2,-1/2}(I)}. \tag{4.7}$$

The proofs of Lemma 4.2 and Theorem 4.3 are similar to those of Lemma 3.2 and Theorem 3.3, and are omitted.

Now we are able to prove optimal approximation for the singular function of $(1+x)^\gamma \log^\nu(1+x)$-type for $\gamma > 0$ in the norm of $\tilde{H}^{1/2}(I)$.

Theorem 4.4 *Let $u(x) = (1+x)^\gamma \log^\nu(1+x)\chi(x)$ with $\gamma > 0$ and integer $\nu > 0$. Here, $\chi(x) \in C^\infty(I)$ such that $\chi(x) = 1$ for $-1 < x < -1+\delta_0/2$ and $\chi(x) = 0$ for $-1+\delta_0 < x < 1$ for a positive constant $\delta_0 < 1$. Then, for $p \geq 2$, there exists a polynomial $\psi \in P_p(I)$ such that $\psi(\pm 1) = u(\pm 1) = 0$, and*

$$\|u - \psi\|_{\tilde{H}^{1/2}(I)} \leq cp^{-2\gamma}(1+\log p)^{\nu^*}\|u\|_{\hat{B}_{\nu^*}^{2\gamma+1/2,-1/2}(I)}.$$

Here, $\nu^* = \nu$ if γ is non-integer or integer and $\nu = 0$, and $\nu^* = \nu - 1$ if γ is an integer and $\nu \geq 1$.

Proof. The proof is analogous to that of Theorem 3.4. We define $v(x) = (1-x^2)^{-1}u(x)$. By Theorem 4.3, $u \in \hat{B}^{2\gamma+1/2,-1/2}_{\nu^*}(I)$ and $v \in B^{2\gamma+1/2,3/2}_{\nu^*}(I)$. Let v_{p-2} denote the Jacobi projection of v onto $P_{p-2}(I)$ with $p \geq 2$ and Jacobi weight $\beta = 3/2$. Further we define $\psi \in P_p(I)$ by $\psi(x) = (1-x^2)v_p(x)$. Then $u(\pm 1) = \psi(\pm 1) = 0$, and

$$u(x) - \psi(x) = (1-x^2)(v(x) - v_{p-2}(x)) = E(v)(x) \quad (x \in I)$$

where E is the operator defined in (3.4). Applying Lemma 4.1 with $s = 2\gamma + 1/2 > 1/2$ we have

$$\|E(v)\|_{\tilde{H}^{1/2}(I)} \leq cp^{-2\gamma}(1+\log p)^{\nu^*}\|v\|_{B^{2\gamma+1/2,3/2}_{\nu^*}(I)}.$$

This together with (4.7) leads to

$$\|u(x) - \psi(x)\|_{\tilde{H}^{1/2}(I)} = \|E(v)\|_{\tilde{H}^{1/2}(I)} \leq cp^{-2\gamma}(1+\log p)^{\nu^*}\|u\|_{\hat{B}^{2\gamma+1/2,-1/2}_{\nu^*}(I)}.$$

□

5 Asymptotically exact bounds for the approximation error of singular functions of $x^\gamma \log^\nu x$-type

Now we analyze asymptotically exact bounds for approximations in $H^{1/2}(I)$ of singular functions. To this end we need estimates for the coefficients of their Chebyshev expansions. For singularities without logarithmic term ($\nu = 0$) we analyze the exact asymptotic behavior of the coefficients in Lemma 5.1. Then, the asymptotically exact bounds for the approximation errors in $H^{1/2}(I)$ are given in Theorems 5.1 and 5.2.

The following lemma is the counterpart of the Legendre expansion studied in [11] (for singular point $\xi = -1$).

Lemma 5.1 *For the function $u(x) = (1+x)^\gamma$ with $\gamma > 0$ let $\sum_{k=0}^\infty a_k T_k$ be its Chebyshev expansion. Then there holds for $k > 0$*

$$a_k = \frac{2^{\gamma+1}}{\pi}\Gamma(1/2)\Gamma(\gamma+1/2)\frac{\gamma(\gamma-1)\cdot\ldots\cdot(\gamma-k+1)}{\Gamma(\gamma+k+1)}$$

and, for non-integer γ,

$$|a_k| \sim k^{-2\gamma-1} \quad (k \to \infty).$$

Proof. Let us denote the factor in the Rodríguez formula (2.9) by

$$t_k := (-1)^k \frac{\Gamma(1/2)}{\Gamma(k+1/2)2^k}.$$

Using the orthogonal properties of the Chebyshev polynomials and formula (2.9) we obtain

$$a_k = \frac{2}{\pi}\int_{-1}^{1} u(x)T_k(x)(1-x^2)^{-1/2}\,dx = \frac{2}{\pi}t_k\int_{-1}^{1} u(x)\frac{d^k}{dx^k}(1-x^2)^{k-1/2}\,dx.$$

Since for $m = 1,\dots,k$

$$\lim_{x\to\pm1}(1+x)^{\gamma-m+1}[(1-x^2)^{k-1/2}]^{(k-m)} = 0$$

integration by parts (and using $u(x) = (1+x)^\gamma$) gives

$$\begin{aligned} a_k &= (-1)^k\frac{2}{\pi}t_k\gamma(\gamma-1)\cdot\ldots\cdot(\gamma-k+1)\int_{-1}^{1}(1+x)^{\gamma-k}(1-x^2)^{k-1/2}\,dx \\ &= (-1)^k\frac{2}{\pi}t_k\gamma(\gamma-1)\cdot\ldots\cdot(\gamma-k+1)\int_{-1}^{1}(1+x)^{\gamma-1/2}(1-x)^{k-1/2}\,dx. \end{aligned} \tag{5.1}$$

To calculate the integral we transform $x = \cos\xi$. This gives

$$\begin{aligned} \int_{-1}^{1}(1+x)^{\gamma-1/2}(1-x)^{k-1/2}\,dx &= \int_0^\pi (1+\cos\xi)^\gamma(1-\cos\xi)^k\,d\xi \\ &= 2\int_0^{\pi/2}(1+\cos2\xi)^\gamma(1-\cos2\xi)^k\,d\xi \\ &= 2^{\gamma+k+1}\int_0^{\pi/2}\cos^{2\gamma}(\xi)\sin^{2k}(\xi)\,d\xi. \end{aligned} \tag{5.2}$$

For the last integral one obtains (see, e.g., [7, §1.1.3.4, 10.])

$$\int_0^{\pi/2}\cos^{2\gamma}(\xi)\sin^{2k}(\xi)\,d\xi = \frac{\Gamma(\gamma+1/2)\Gamma(k+1/2)}{2\Gamma(\gamma+k+1)}. \tag{5.3}$$

Combining (5.1), (5.2), (5.3) we obtain

$$a_k = (-1)^k\frac{2^{\gamma+k+1}}{\pi}t_k\gamma(\gamma-1)\cdot\ldots\cdot(\gamma-k+1)\frac{\Gamma(\gamma+1/2)\Gamma(k+1/2)}{\Gamma(\gamma+k+1)}$$

which gives the assertion by substituting t_k. For non-integer γ the asymptotic behavior $|a_k| \sim k^{-2\gamma-1}$ follows from the property

$$\frac{\Gamma(k+\alpha)}{\Gamma(k+\beta)} = k^{\alpha-\beta}\left(1+O(1/k)\right) \qquad (k\to\infty), \tag{5.4}$$

see [11, Lemma 1]. □

The next theorem gives an asymptotically exact error estimate in $H^{1/2}(I)$ for the Chebyshev projection of singular functions without logarithmic term.

Theorem 5.1 *For the function $u(x) = (1+x)^\gamma$ with non-integer $\gamma > 0$ and Chebyshev projection u_p on $P_p(I)$ there holds*

$$\|u - u_p\|_{H^{1/2}(I)} \cong p^{-2\gamma}\left(1 + O(p^{-1})\right).$$

Proof. For $u(x) = (1+x)^\gamma$ with Chebyshev expansion $u = \sum_{k=0}^{\infty} a_k T_k$ and projection $u_p = \sum_{k=0}^{p} a_k T_k$ we obtain with Corollary 2.1 and Lemma 5.1

$$\begin{aligned}\|u - u_p\|_{H^{1/2}(I)} &\cong (\sum_{k=p+1}^{\infty} a_k^2 (1+k^2)^{1/2})^{1/2} \\ &\cong \left(\sum_{k=p+1}^{\infty} k^{-4\gamma-1}(1 + O(k^{-1}))\right)^{1/2} \cong p^{-2\gamma}\left(1 + O(p^{-1})\right)\end{aligned}$$

which is the assertion. □

Now we extend our results to singularities with logarithmic contributions.

Theorem 5.2 *For the function $u_\nu(x) = (1+x)^\gamma \log^\nu(1+x)$ with $\gamma > 0$ and Chebyshev projection $\phi_\gamma^\nu \in P_p(I)$ there holds*

$$\|u_\nu - \phi_\gamma^\nu\|_{H^{1/2}(I)} \cong E_\nu(\gamma, p) p^{-2\gamma}\left(1 + O(p^{-1})\right) \tag{5.5}$$

with

$$E_\nu^2(\gamma, p) = \frac{1}{4\gamma} \sum_{s=0}^{2\nu} C_{2\nu-s}(\gamma) \log^s p.$$

Here, $C_{\nu-s}(\gamma)$, $s = 0, \dots, \nu$, are analytic functions for $\gamma > 0$ and $C_0(\gamma) = C_1(\gamma) = 0$ for integer γ. The appearing constants in the equivalence (5.5) *do not depend on γ, ν, and p.*

Proof. We use the idea presented in the proof of Theorem 2.8 of [4]. First we consider in some details the case $\nu = 1$ (and $\nu = 0$) and then mention the extension to general integer $\nu \geq 1$.

Defining $u_0(x) = (1+x)^\gamma$ we obtain

$$u_1(x) := \frac{d}{d\gamma} u_0(x) = (1+x)^\gamma \log(1+x).$$

With regard to Lemma 5.1, u_0 has a Chebyshev expansion $u_0 = \sum_{k=0}^{\infty} a_k T_k$ with

$$a_k = \frac{2^{\gamma+1}}{\pi} \Gamma(1/2)\Gamma(\gamma + 1/2) \frac{\gamma(\gamma-1) \cdot \ldots \cdot (\gamma - k + 1)}{\Gamma(\gamma + k + 1)} \quad (k > 0). \tag{5.6}$$

Defining its Chebyshev approximation $\phi_\gamma^0 := \sum_{k=0}^{p} a_k(\gamma) T_k$ there holds by Theorem 5.1

$$\|u_0 - \phi_\gamma^0\|_{H^{1/2}(I)} \cong p^{-2\gamma}\left(1 + O(p^{-1})\right) \quad (\gamma \text{ non-integer}, \nu = 0) \tag{5.7}$$

and the approximation is exact for integer γ if p is sufficiently large. (5.7) gives the assertion for $\nu = 0$. Now, $u_1 \in H^{1/2}(I)$ has the Chebyshev expansion

$$u_1 = \sum_{k=0}^{\infty} a'_k(\gamma) T_k \quad \left(a'_k(\gamma) := \frac{d}{d\gamma} a_k(\gamma) \right)$$

and Chebyshev approximation $\phi^1_\gamma = \sum_{k=0}^{p} a'_k(\gamma) T_k$. By Corollary 2.1 we obtain

$$\|u_1 - \phi^1_\gamma\|^2_{H^{1/2}(I)} \cong \sum_{k=p+1}^{\infty} \left(a'_k(\gamma)\right)^2 (1+k^2)^{1/2}.$$

We need to estimate the coefficients $a'_k(\gamma)$. By (5.6) the numbers a_k can be written like

$$a_k(\gamma) = (-1)^{k-k^*} \tilde{C}_0(\gamma) \frac{\Gamma(k-\gamma)}{\Gamma(\gamma+k+1)} \quad (k > \gamma)$$

and

$$\tilde{C}_0(\gamma) = \frac{2^{\gamma+1}}{\pi} \Gamma(1/2)\Gamma(\gamma+1/2)\gamma(\gamma-1)\cdot\ldots\cdot(\gamma-k^*).$$

Here, k^* is the maximum integer less than or equal to γ. For $k > \gamma$, $\tilde{C}_0$ does not depend on k. We obtain

$$(-1)^{k-k^*} a'_k(\gamma) = \tilde{C}'_0(\gamma) \frac{\Gamma(k-\gamma)}{\Gamma(\gamma+k+1)} + \tilde{C}_0(\gamma) \frac{\Gamma'(k-\gamma)}{\Gamma(\gamma+k+1)} + \tilde{C}_0(\gamma) \frac{\Gamma(k-\gamma)\Gamma'(\gamma+k+1)}{\Gamma(\gamma+k+1)^2}.$$

Using (5.4) and the relation $\Gamma'(z) = \Gamma(z)\left(\log z + O(z^{-1})\right)$ for $z > 1$ (see [4, Proof of Theorem 2.8]) we obtain

$$(-1)^{k-k^*} a'_k(\gamma) = \tilde{C}'_0(\gamma) k^{-2\gamma-1} \left(1 + O(k^{-1})\right) + \tilde{C}_0(\gamma) k^{-2\gamma-1} (\log k) \left(1 + O(k^{-1})\right).$$

Thus

$$\begin{aligned} \|u_1 - \phi^1_\gamma\|^2_{H^{1/2}(I)} &\cong \sum_{k=p+1}^{\infty} \left(a'_k(\gamma)\right)^2 (1+k^2)^{1/2} \\ &= \sum_{k=p+1}^{\infty} \left(\tilde{C}'_0(\gamma) k^{-2\gamma-1} + \tilde{C}_0(\gamma) k^{-2\gamma-1} (\log k)\right)^2 \left(1 + O(k^{-1})\right)^2 (1+k^2)^{1/2} \\ &= \sum_{k=p+1}^{\infty} \left(\tilde{C}'_0(\gamma) + \tilde{C}_0(\gamma)(\log k)\right)^2 k^{-4\gamma-1} \left(1 + O(k^{-1})\right)^2 \end{aligned}$$

such that

$$\|u_1 - \phi^1_\gamma\|_{H^{1/2}(I)} \cong E_1(\gamma, p) p^{-2\gamma} \left(1 + O(p^{-1})\right) \quad (\gamma \text{ non-integer}, \nu = 1) \tag{5.8}$$

with

$$
\begin{aligned}
E_1^2(\gamma,p) &= \frac{\tilde{C}_0^2(\gamma)}{4\gamma}\log^2 p + \frac{2\tilde{C}_0^2(\gamma)+8\gamma\tilde{C}_0(\gamma)\tilde{C}_0'(\gamma)}{16\gamma^2}\log p \\
&\quad + \frac{2\tilde{C}_0^2(\gamma)+8\gamma\tilde{C}_0(\gamma)\tilde{C}_0'(\gamma)+16\gamma^2\tilde{C}_0'(\gamma)^2}{64\gamma^3} \\
&= \frac{1}{4\gamma}\sum_{s=0}^{2} C_{2-s}(\gamma)\log^s p,
\end{aligned}
$$

where $C_{2-s}(\gamma)$, $s=0,1,2$, are analytic functions for $\gamma>0$. We note that $C_0(\gamma)=\tilde{C}_0^2(\gamma)=0$ for integer γ such that

$$
\|u_1-\phi_\gamma^1\|_{H^{1/2}(I)} \cong \frac{C_0'(\gamma)^2}{4\gamma}p^{-2\gamma}\Big(1+O(p^{-1})\Big) \quad (\gamma \text{ integer}, \nu=1). \tag{5.9}
$$

(5.8) and (5.9) prove the assertion for $\nu=1$. For $\nu>1$ one gets analogously

$$
a_k^{(\nu)} = \frac{\tilde{C}_0(\gamma)\log^\nu k+\tilde{C}_1(\gamma)\log^{\nu-1}k+\cdots+\tilde{C}_\nu(\gamma)}{k^{2\gamma+1}}\Big(1+O(k^{-1})\Big)
$$

such that

$$
\|u_\nu-\phi_\gamma^\nu\|_{H^{1/2}(I)} = E_\nu(\gamma,p)p^{-2\gamma}\Big(1+O(p^{-1})\Big)
$$

with

$$
E_\nu^2(\gamma,p) = \frac{1}{4\gamma}\sum_{s=0}^{2\nu} C_{2\nu-s}(\gamma)\log^s p.
$$

Again, $C_{2\nu-s}(\gamma)$, $s=0,\dots,2\nu$, are analytic functions for $\gamma>0$ and $C_0(\gamma)=C_1(\gamma)=0$ for integer γ. This finishes the proof of the theorem. $\square$

6 Rate of convergence of the p-version of the BEM

In this section we demonstrate how the best approximation results for singular functions in the previous sections can be applied to estimate approximation errors of the p-version of the boundary element method. For simplicity we only consider the Laplace equation on polygonal domains with Dirichlet or Neumann boundary conditions and piecewise analytic data. The solutions of these problems exhibit singularities of $x^\gamma\log^\nu x$-type with $\gamma>1/2$ and $\nu\le 1$. However, our results are applicable without modifications to problems with stronger singularities ($\gamma>0$, $\nu\ge 0$) which appear, e.g., when considering mixed boundary conditions and singular right hand sides.

We consider boundary integral equations for solving Dirichlet and Neumann boundary value problems for the Laplacian in a polygonal domain Ω with boundary Γ. The vertices of Ω are denoted by t_j ($1\le j\le J$, $t_{J+1}=t_1$) and Γ^j is the open edge connecting t_j and t_{j+1}. The internal angle at t_j is ω_j. Let us assume that Γ has conformal radius less than one; this can always be achieved by an appropriate scaling. Then the problems under consideration are as follows:

Dirichlet problem: For given $f \in H^{1/2}(\Gamma)$ find $u \in H^1(\Omega)$ such that

$$\Delta u = 0 \quad \text{in} \quad \Omega, \qquad u = f \quad \text{on} \quad \Gamma. \tag{6.1}$$

Neumann problem: For given $g \in H^{-1/2}(\Gamma)$ with $\int_\Gamma g\, ds = 0$ find $u \in H^1(\Omega)$ such that

$$\Delta u = 0 \quad \text{in} \quad \Omega, \qquad \frac{\partial u}{\partial n} = g \quad \text{on} \quad \Gamma. \tag{6.2}$$

$\frac{\partial u}{\partial n}$ denotes the normal derivative of u with respect to the outer normal n. It is well-known that problems (6.1) and (6.2) can be converted into boundary integral equations of the first kind on Γ, cf. [8]. With $v = u|_\Gamma, \psi = \frac{\partial u}{\partial n}|_\Gamma$ we have for (6.1) and (6.2), respectively,

$$V\psi = (\frac{1}{2}I + K)f \quad \text{ on } \Gamma, \tag{6.3}$$

$$Wv = (\frac{1}{2}I - K')g \quad \text{ on } \Gamma. \tag{6.4}$$

It is also well-known that there exist unique solutions $\psi \in H^{-1/2}(\Gamma)$ of (6.3) and

$$v \in H_0^{1/2}(\Gamma) = \left\{ w \in H^{1/2}(\Gamma); \int_\Gamma w\, ds = 0 \right\}$$

of (6.4). Here, the appearing integral operators are defined by (for $\phi \in H^{-1/2}(\Gamma), w \in H^{1/2}(\Gamma)$)

$$V\phi(x) := -\frac{1}{2\pi}\int_\Gamma \phi(y)\log|x-y|\, ds_y, \quad Kw(x) := -\frac{1}{2\pi}\int_\Gamma w(y)\frac{\partial}{\partial n_y}\log|x-y|\, ds_y$$

$$K'\phi(x) := -\frac{1}{2\pi}\int_\Gamma \phi(y)\frac{\partial}{\partial n_x}\log|x-y|\, ds_y, \quad Ww(x) := \frac{1}{2\pi}\frac{\partial}{\partial n_x}\int_\Gamma w(y)\frac{\partial}{\partial n_y}\log|x-y|\, ds_y.$$

To study the convergence properties of the p-version of the boundary element method for solving (6.3) and (6.4) we need to prescribe the regularity of the exact solutions ψ and v. For simplicity we only consider piecewise analytic data. Then, appearing singularities are of the form $r^\gamma \log^\nu r$ (r distance to a corner) with $\nu = 0, 1$. Of course, our theoretical approximation results from the previous sections allow for treating also problems with singular right hand sides f and boundary conditions g [17]. Now, for piecewise analytic data, the solutions ψ and v behave singularly at the corners of the polygon and are smooth elsewhere.

To make this statement precise let us consider a partition of unity $(\chi_1, \dots, \chi_J)$ where χ_j is the restriction of a $C_0^\infty(\mathbf{R}^2)$ function to Γ such that $\chi_j = 1$ in a neighborhood of the vertex t_j and $\operatorname{supp}(\chi_j) \subset \Gamma^{j-1} \cup \{t_j\} \cup \Gamma^j$ ($\Gamma^0 = \Gamma^J$). In this way we may write any function φ on Γ like

$$\varphi = \sum_{j=1}^{J} (\varphi_-, \varphi_+)\chi_j$$

where a pair (φ_-, φ_+) corresponds to φ on $\Gamma^{j-1} \cup \{t_j\} \cup \Gamma^j$ with

$$\varphi_- = \varphi|_{\Gamma^{j-1}} \quad \text{and} \quad \varphi_+ = \varphi|_{\Gamma^j}.$$

From [16] (see also [10]) we cite the following result.

Proposition 6.1 *Let $\alpha_{jk} := k\frac{\pi}{\omega_j}$ (integer $k \geq 1$, $j = 1, \ldots, J$) and, for $t \geq 1/2$, let n be an integer with $n + 1 > \frac{\omega_j}{\pi}(t - 1/2) \geq n$.*

(i)If f is a piecewise analytic function, then there exists a function ψ_0 with $\psi_0|_{\Gamma^j} \in H^{t-1}(\Gamma^j)$ such that, for the solution ψ of (6.3), there holds

$$\psi = \sum_{j=1}^{J} \sum_{k=1}^{n} ((\psi_{jk})_-, (\psi_{jk})_+) \, \chi_j + \psi_0.$$

Here

$$\begin{aligned} (\psi_{jk})_\pm(x) &= c|x - t_j|^{\alpha_{jk}-1} \quad (\alpha_{jk} \textit{ is not an integer}), \\ (\psi_{jk})_\pm(x) &= c_1|x - t_j|^{\alpha_{jk}-1} + c_2|x - t_j|^{\alpha_{jk}-1} \log|x - t_j| \quad (\alpha_{jk} \textit{ is an integer}). \end{aligned}$$

(ii) If g is a piecewise analytic function, then there exists a function v_0 with $v_0|_{\Gamma^j} \in H^t(\Gamma^j)$ such that, for the solution v of (6.4), there holds

$$v = \sum_{j=1}^{J} \sum_{k=1}^{n} ((v_{jk})_-, (v_{jk})_+) \, \chi_j + v_0.$$

Here

$$\begin{aligned} (v_{jk})_\pm(x) &= c|x - t_j|^{\alpha_{jk}} \quad (\alpha_{jk} \textit{ is not an integer}), \\ (v_{jk})_\pm(x) &= c_1|x - t_j|^{\alpha_{jk}} + c_2|x - t_j|^{\alpha_{jk}} \log|x - t_j| \quad (\alpha_{jk} \textit{ is an integer}). \end{aligned}$$

The constants c, c_1 and c_2 above are generic.

For the p-version of the boundary element method we define piecewise polynomial spaces. Let Γ be decomposed into straight line pieces Γ_j, $j = 1, \ldots, n$, i.e. the corners of the polygon Γ coincide with endpoints of some elements and the mesh $\Gamma_h := \{\Gamma_1, \ldots, \Gamma_n\}$ covers Γ. For a conforming approximation of the solution v of (6.4) we need continuous basis functions whereas the solution ψ of (6.3) can be approximated by discontinuous functions. For a given integer $p > 0$ we define

$$S_p^1(\Gamma_h) := \{v \in H^1(\Gamma);\ v|_{\Gamma_j} \in P_p(\Gamma_j), j = 1, \ldots, n\}$$

and

$$S_{p-1}^0(\Gamma_h) := \{v \in L^2(\Gamma);\ v|_{\Gamma_j} \in P_{p-1}(\Gamma_j), j = 1, \ldots, n\}.$$

Here, $P_p(\Gamma_j)$ denotes the set of polynomials on Γ_j (with respect to the arc length) up to degree p. Then there holds $S_p^1(\Gamma_h) \subset H^{1/2}(\Gamma)$ and $S_{p-1}^0(\Gamma_h) \subset H^{-1/2}(\Gamma)$. However, the condition of integral mean zero is not satisfied by the functions in $S_p^1(\Gamma_h)$. This condition is most conveniently incorporated by a Lagrangian multiplier, see, e.g., [9]. The Galerkin schemes are as follows.

Find $\psi_p \in S_p^0(\Gamma_h)$ $(p \geq 0)$ satisfying

$$\langle V\psi_p, \phi\rangle_{L^2(\Gamma)} = \langle(\frac{1}{2}I + K)f, \phi\rangle_{L^2(\Gamma)} \quad \forall\, \phi \in S_p^0(\Gamma_h), \tag{6.5}$$

and find $v_p \in S_p^1(\Gamma_h)$ $(p > 0)$ and a real number a satisfying

$$\begin{aligned} \langle Wv_p, w\rangle_{L^2(\Gamma)} + \langle w, a\rangle_{L^2(\Gamma)} &= \langle(\frac{1}{2}I - K')g, w\rangle_{L^2(\Gamma)} \quad \forall\, w \in S_p^1(\Gamma_h), \\ \langle v_p, 1\rangle_{L^2(\Gamma)} &= 0. \end{aligned} \tag{6.6}$$

The boundary integral operators $V:\ H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and $W:\ H_0^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ are continuous and positive definite. Therefore, conforming Galerkin schemes for (6.3) and (6.4) converge quasi-optimally in the energy norm. That means for the Galerkin solutions ψ_p, v_p and the true solutions ψ, v there hold

$$\|\psi - \psi_p\|_{H^{-1/2}(\Gamma)} \leq c_1 \inf\{\|\psi - \phi\|_{H^{-1/2}(\Gamma)};\ \phi \in S_p^0(\Gamma_h)\} \tag{6.7}$$

and

$$\|v - v_p\|_{H^{1/2}(\Gamma)} \leq c_2 \inf\{\|v - w\|_{H^{1/2}(\Gamma)};\ w \in S_p^1(\Gamma_h)\} \tag{6.8}$$

where the constants $c_1, c_2 > 0$ are independent of the respective polynomials degrees p and the meshes Γ_h.

In the following we apply our approximation results of the preceding sections to the boundary element Galerkin methods for our model problems. First we present an upper bound for the approximation error concerning the hypersingular integral equation.

Theorem 6.1 *Let v be the exact solution of* (6.4) *with piecewise analytic g, and let v_p be the BE approximation defined by* (6.6). *There holds*

$$\|v - v_p\|_{\tilde{H}^{1/2}(\Gamma)} \leq Cp^{-2\pi/\omega^*}. \tag{6.9}$$

Here, $\omega^* = \max\{\omega_j;\ j = 1, \ldots, J\}$ and the constant C does not depend on p.

Proof. By the quasi-optimality of the Galerkin scheme (6.8) we only need to define an element $w \in S_p^1(\Gamma_h)$ such that $\|v - w\|_{\tilde{H}^{1/2}(\Gamma)}$ satisfies the bound stated by the theorem.

By construction of the ansatz space $S_p^1(\Gamma_h)$, each edge Γ^j is covered by elements which we denote by $\{\Gamma_1^j, \Gamma_2^j, \ldots, \Gamma_{n_j}^j\}$. By Proposition 6.1(ii), on Γ_1^j we can represent the solution v of (6.4) like

$$v(x) = v_{11}(x) + v_{12}(x) + v_{10}(x) \tag{6.10}$$

with $v_{10} \in H^{s_1}(\Gamma_1^j)$, $s_1 = 3\pi/\omega_j + 1/2 - \epsilon$ $(\epsilon > 0)$ and singularities

$$v_{11}(x) := c_{11}|x - t_j|^{\pi/\omega_j}\chi_j + c_{12}|x - t_j|^{\pi/\omega_j}\log|x - t_j|\chi_j,$$

$$v_{12}(x) := c_{13}|x-t_j|^{2\pi/\omega_j}\chi_j + c_{14}|x-t_j|^{2\pi/\omega_j}\log|x-t_j|\chi_j.$$

The constant c_{12} vanishes for non-integer $\alpha_{j1} = \pi/\omega_j$, and c_{14} vanishes for non-integer $\alpha_{j2} = 2\pi/\omega_j$. χ_j is the C^∞ cut-off function. Accordingly, on the element $\Gamma^j_{n_j}$, v can be represented by a decomposition

$$v(x) = v_{21}(x) + v_{22}(x) + v_{20}(x)$$

with $v_{20} \in H^{s_2}(\Gamma^j_{n_j})$, $s_2 = 3\pi/\omega_{j+1} + 1/2 - \epsilon$ ($\epsilon > 0$) and singularities

$$v_{21}(x) := c_{21}|x-t_{j+1}|^{\pi/\omega_{j+1}}\chi_{j+1} + c_{22}|x-t_{j+1}|^{\pi/\omega_{j+1}}\log|x-t_{j+1}|\chi_{j+1},$$

$$v_{22}(x) := c_{23}|x-t_{j+1}|^{2\pi/\omega_{j+1}}\chi_{j+1} + c_{24}|x-t_{j+1}|^{2\pi/\omega_{j+1}}\log|x-t_{j+1}|\chi_{j+1}.$$

By Theorems 3.4 and 4.4 there exist, along the first element Γ^j_1, polynomials w_{11} and w_{12} of degree p such that $v_{11} = w_{11}$ and $v_{12} = w_{12}$ in the endpoints of Γ^j_1, and

$$\|v_{11} - w_{11}\|_{\tilde{H}^{1/2}(\Gamma^j_1)} \le cp^{-2\pi/\omega_j}, \tag{6.11}$$

$$\|v_{12} - w_{12}\|_{\tilde{H}^{1/2}(\Gamma^j_1)} \le cp^{-4\pi/\omega_j}. \tag{6.12}$$

Note that c_{12} in the representation of v_{11} is non-zero only for integer π/ω_j, in which case $\nu^* = \nu - 1 = 0$ in Theorem 4.4. If π/ω_j is non-integer such that $c_{12} = 0$, then $\nu^* = \nu = 0$ such that the estimate (6.11) does not depend on the logarithmic singularities for our example. The same argument applies to (6.12).

For the smooth remainder v_{10} of v on Γ^j_1 we find by standard results [21] a polynomial w_{10} of degree p that interpolates v_{10} at the endpoints of the element such that

$$\|v_{10} - w_{10}\|_{\tilde{H}^{1/2}(\Gamma^j_1)} \le c(\epsilon)p^{-s_1+1/2}, \quad s_1 = 3\pi/\omega_j + 1/2 - \epsilon, \quad \epsilon > 0. \tag{6.13}$$

Analogously on the element $\Gamma^j_{n_j}$, we find polynomials w_{21} and w_{22} of degree p which coincide with v_{21} and v_{22}, respectively, at the endpoints of the element such that there holds

$$\|v_{21} - w_{21}\|_{\tilde{H}^{1/2}(\Gamma^j_{n_j})} \le cp^{-2\pi/\omega_{j+1}}, \tag{6.14}$$

$$\|v_{22} - w_{22}\|_{\tilde{H}^{1/2}(\Gamma^j_{n_j})} \le cp^{-4\pi/\omega_{j+1}}. \tag{6.15}$$

Also, there exists a polynomial w_{20} of degree p that interpolates v_{20} at the endpoints of $\Gamma^j_{n_j}$ such that

$$\|v_{20} - w_{20}\|_{\tilde{H}^{1/2}(\Gamma^j_{n_j})} \le c(\epsilon)p^{-s_2+1/2}, \quad s_2 = 3\pi/\omega_{j+1} + 1/2 - \epsilon, \quad \epsilon > 0. \tag{6.16}$$

On the remainder $\Gamma^j \setminus (\Gamma^j_1 \cup \Gamma^j_{n_j})$ the solution v of (6.4) is analytic. Therefore, by Theorem 2.1 there is a polynomial w^0_0 of degree p such that

$$\|v - w^0_0\|_{H^l(\Gamma^j \setminus (\Gamma^j_1 \cup \Gamma^j_{n_j}))} \le c(\alpha)p^{-\alpha+l} \tag{6.17}$$

for any $\alpha > s + l = \max\{s_1, s_2\} + l > 0$. Now let $\bar{v}$ be the linear interpolant of $v - w_0^0$ in the endpoints of $\Gamma^j \setminus (\Gamma_1^j \cup \Gamma_{n_j}^j)$. By using the equivalence of norms

$$\|\cdot\|_{\tilde{H}^{1/2}(I)} \cong \|\cdot\|_{H^{1/2}(I)} + \|(1-x^2)^{-1/2} \cdot \|_{L^2(I)}$$

we obtain

$$\begin{aligned} \|v - (w_0^0 + \bar{v})\|_{\tilde{H}^{1/2}(\Gamma^j \setminus (\Gamma_1^j \cup \Gamma_{n_j}^j))} \leq\ & c \Bigg(\|v - w_0^0\|_{H^{1/2}(\Gamma^j \setminus (\Gamma_1^j \cup \Gamma_{n_j}^j))} + \|\bar{v}\|_{H^{1/2}(\Gamma^j \setminus (\Gamma_1^j \cup \Gamma_{n_j}^j))} \\ & + \| \prod_{1 \leq i \leq 2} |x - b_i|^{-1/2} (v - (w_0^0 + \bar{v}))\|_{L^2(\Gamma^j \setminus (\Gamma_1^j \cup \Gamma_{n_j}^j))} \Bigg) \end{aligned} \tag{6.18}$$

where b_i, $i = 1, 2$, are the two endpoints of $\Gamma^j \setminus (\Gamma_1^j \cup \Gamma_{n_j}^j)$ and c is a positive constant independent of v and p.

By the embedding theorem of Sobolev spaces, we have for $m = 1/2, 1$

$$\|\bar{v}\|_{H^m(\Gamma^j \setminus (\Gamma_1^j \cup \Gamma_{n_j}^j))} \leq c \max\{|\bar{v}(b_1)|, |\bar{v}(b_2)|\} \leq c\|v - w_0^0\|_{H^1(\Gamma^j \setminus (\Gamma_1^j \cup \Gamma_{n_j}^j))} \tag{6.19}$$

Also we have

$$\| \prod_{1 \leq i \leq 2} |x - b_i|^{-1/2} (v - (w_0^0 + \bar{v}))\|_{L^2(\Gamma^j \setminus (\Gamma_1^j \cup \Gamma_{n_j}^j))} \leq c\{\|v - w_0^0\|_{H^1(\Gamma^j \setminus (\Gamma_1^j \cup \Gamma_{n_j}^j))} + \|\bar{v}\|_{H^1(\Gamma^j \setminus (\Gamma_1^j \cup \Gamma_{n_j}^j))}\} \tag{6.20}$$

Combination of (6.18) with (6.17), (6.19) and (6.20) yields with $w_0 := w_0^0 + \bar{v}$

$$\|v - w_0\|_{\tilde{H}^{1/2}(\Gamma^j \setminus (\Gamma_1^j \cup \Gamma_{n_j}^j))} \leq c(\alpha) p^{-\alpha+1} \leq c(\alpha) p^{-s}. \tag{6.21}$$

The approximation $w^j \in S_p^1(\Gamma_h)|_{\Gamma^j}$ is constructed by the pieces defined above. If $n_j \geq 3$ then we define

$$w^j := \begin{cases} w_{11} + w_{12} + w_{10} & \text{on } \Gamma_1^j \\ w_0 & \text{on } \Gamma^j \setminus (\Gamma_1^j \cup \Gamma_{n_j}^j) \\ w_{21} + w_{22} + w_{20} & \text{on } \Gamma_{n_j}^j \end{cases}$$

Combining (6.11)-(6.21) we obtain

$$\begin{aligned} & \|v - w^j\|^2_{\tilde{H}^{1/2}(\Gamma^j)} \\ \leq\ & c \left(\sum_{i=0}^{2} \|v_{1i} - w_{1i}\|^2_{\tilde{H}^{1/2}(\Gamma_1^j)} + \|v - w_0\|^2_{\tilde{H}^{1/2}(\Gamma^j \setminus (\Gamma_1^j \cup \Gamma_{n_j}^j))} + \sum_{i=0}^{2} \|v_{2i} - w_{2i}\|^2_{\tilde{H}^{1/2}(\Gamma_{n_j}^j)} \right) \\ \leq\ & c \max\{p^{-4\pi/\omega_i};\ i = j, j+1\}. \end{aligned}$$

It is straightforward to find appropriate approximations in the case of less elements ($n_j = 1, 2$). Now, proceeding in the same way an all the edges Γ^j of Γ, we define $w := w^j$ on Γ^j and conclude that there holds

$$\|v - w\|_{\tilde{H}^{1/2}(\Gamma)} \leq c \left(\sum_{j=1}^{J} \|v - w^j\|^2_{\tilde{H}^{1/2}(\Gamma^j)} \right)^{1/2} \leq c \max\{p^{-2\pi/\omega_i};\ i = 1, \ldots, J\}$$

which proves the theorem. $\square$

Now we present a lower bound for the error of the Galerkin approximation of the hypersingular integral equation (6.4).

Theorem 6.2 *Let v be the exact solution of* (6.4) *with piecewise analytic g, and let v_p be the BE approximation defined by* (6.6). *Suppose that the most severe singularity $|x-t_{j_0}|^{\pi/\omega_{j_0}}$ or $|x-t_{j_0}|^{\pi/\omega_{j_0}}\log|x-t_{j_0}|$ occurs at some vertex t_{j_0} of Ω with $\omega_{j_0}=\omega^*=\max\{\omega_j;\ j=1,\dots,J\}$. Then there holds*

$$\|v-v_p\|_{\tilde H^{1/2}(\Gamma)} \geq cp^{-2\pi/\omega^*} \tag{6.22}$$

with $c>0$ independent of p.

Proof. We may assume that $j_0=1, \omega_1=\omega^*$ and that Γ_1^1 is the corner element at the vertex t_1. There holds

$$\|v-v_p\|_{\tilde H^{1/2}(\Gamma)} \cong \|v-v_p\|_{H^{1/2}(\Gamma)} \geq \|v-v_p\|_{H^{1/2}(\Gamma_1^1)}.$$

By Proposition 6.1, $v=v_1+v_2+v_0$ on Γ_1^1 with

$$v_1(r)=c_1r^{\pi/\omega_1}+c_2r^{\pi/\omega_1}\log r,$$

$$v_2(r)=c_3r^{2\pi/\omega_1}+c_4r^{2\pi/\omega_1}\log r,$$

and

$$v_0(r)\in H^t(\Gamma_1^1)\ \forall\, t=3\pi/\omega_1+1/2-\epsilon.$$

Here, $\epsilon>0$ is arbitrary, and r denotes the distance to the corner t_1. Note that in Proposition 6.1 the singular functions v_{11} and v_{12} are associated with cut-off functions, but v_1 and v_2 are plain singular functions. However, both representations differ only by C^∞-perturbations which become a part of v_0. Now we assume that the assertion of the theorem does not hold. Therefore, there exists a piecewise polynomial $w\in S_p^1(\Gamma_h)$ and a function $\delta(p)$ such that

$$\|v-w\|_{H^{1/2}(\Gamma_1^1)} \leq \|v-w\|_{H^{1/2}(\Gamma)} \leq cp^{-2\pi/\omega_1}\delta(p), \quad \delta(p)\to 0\ (p\to\infty). \tag{6.23}$$

By Theorems 3.4 and 4.4, there exists a polynomial w_2 of degree p such that

$$\|v_2-w_2\|_{H^{1/2}(\Gamma_1^1)} \leq cp^{-4\pi/\omega_1}.$$

Moreover, by a standard approximation argument, there exists a polynomial w_0 of degree p such that

$$\|v_0-w_0\|_{H^{1/2}(\Gamma_1^1)} \leq cp^{-(t-1/2)}.$$

Therefore, combining the last two estimates and the assumption (6.23) we obtain

$$\begin{aligned}\|v_1-(w-w_2-w_0)\|_{H^{1/2}(\Gamma_1^1)} &\leq \|v-w\|_{H^{1/2}(\Gamma_1^1)}+\|v_2-w_2\|_{H^{1/2}(\Gamma_1^1)}+\|v_0-w_0\|_{H^{1/2}(\Gamma_1^1)}\\ &\leq cp^{-2\pi/\omega_1}\left(\delta(p)+p^{-2\pi/\omega_1}+p^{-(t-1/2-2\pi/\omega_1)}\right)\end{aligned}$$

which, by choosing $t \in (2\pi/\omega_1 + 1/2, 3\pi/\omega_1 + 1/2)$, leads to a contradiction to Theorems 5.1 and 5.2.

$\square$

The next corollary combines the two preceding theorems.

Corollary 6.1 *Let v be the exact solution of* (6.4) *with piecewise analytic g, and let v_p be the BE approximation defined by* (6.6). *Suppose that the most severe singularity $|x - t_{j_0}|^{\pi/\omega_{j_0}}$ or $|x - t_{j_0}|^{\pi/\omega_{j_0}} \log|x - t_{j_0}|$ occurs at some vertex t_{j_0} of Ω with $\omega_{j_0} = \omega^* = \max\{\omega_j;\ j = 1, \dots, J\}$. Then there holds the optimal error estimate*

$$cp^{-2\pi/\omega^*} \le \|v - v_p\|_{\tilde{H}^{1/2}(\Gamma)} \le Cp^{-2\pi/\omega^*}, \tag{6.24}$$

with $C > 0$ and $c > 0$ independent of p.

Making use of the approximation results for the hypersingular integral equation we prove analogous results for the weakly singular integral equation (6.3).

Theorem 6.3 *Let ψ be the exact solution of* (6.3) *with piecewise analytic f, and let ψ_p be the BE approximation defined by* (6.5). *Suppose that the most severe singularity $|x - t_{j_0}|^{\pi/\omega_{j_0}-1}$ or $|x - t_{j_0}|^{\pi/\omega_{j_0}-1} \log|x - t_{j_0}|$ occurs at some vertex t_{j_0} of Ω with $\omega_{j_0} = \omega^* = \max\{\omega_j;\ j = 1, \dots, J\}$. Then there holds*

$$cp^{-2\pi/\omega^*} \le \|\psi - \psi_p\|_{\tilde{H}^{-1/2}(\Gamma)} \le Cp^{-2\pi/\omega^*}. \tag{6.25}$$

with $C > 0$ and $c > 0$ independent of p.

Proof. We apply the results of Theorems 6.1 and 6.2 by considering antiderivatives of ψ, which have singularities like the solution v of (6.4). This technique has been used previously by Stephan and Suri [21].

First we proof the upper bound for $\|\psi - \psi_p\|_{\tilde{H}^{-1/2}(\Gamma)}$. By the quasi-optimality of the Galerkin scheme (6.7) we only need to define an element $\phi \in S^0_p(\Gamma_h)$ such that $\|\psi - \phi\|_{\tilde{H}^{-1/2}(\Gamma)}$ satisfies the upper bound stated by the theorem.

We consider an arbitrary element Γ^j_i of an edge Γ^j. This element is identified with the interval $I_h = (0, h)$ where h is the length of Γ^j_i. We define

$$u(x) = \int_0^x (\psi - \bar{\psi})(t)\, dt \quad (x \in I_h) \tag{6.26}$$

where $\bar{\psi} = \frac{1}{h}\int_0^h \psi(t)\, dt$. Then u vanishes at the endpoints of I_h. Moreover $u \in \tilde{H}^{1/2}(I_h)$, see [14, Lemma 3]. If the element Γ^j_i is not a corner element then u is analytic and there exists a polynomial w_i of degree p such that

$$\|u - w_i\|_{\tilde{H}^{1/2}(\Gamma^j_i)} \le c(\alpha)p^{-\alpha} \tag{6.27}$$

for any $\alpha > 0$. Now, if Γ_i^j is a corner element, e.g. $i = 1$, then by Proposition 6.1(i) and standard calculation

$$\begin{aligned} u(x) &= \int_0^x (\psi - \bar{\psi})(t)\,dt \\ &= c_1|x - t_j|^{\alpha_{j1}} + c_2|x - t_j|^{\alpha_{j1}} \log|x - t_j| \\ &+ c_3|x - t_j|^{\alpha_{j2}} + c_4|x - t_j|^{\alpha_{j2}} \log|x - t_j| + u_0(x) \end{aligned}$$

with $\alpha_{ji} = i\pi/\omega_j$ and a function $u_0 \in H^s(\Gamma_1^j)$, $s = 3\pi/\omega_j + 1/2 - \epsilon$, $\epsilon > 0$. As in the proof of Theorem 6.1, e.g. by using Theorem 4.4, there exists a polynomial w_1 of degree p that interpolates u at the endpoints of Γ_1^j such that

$$\|u - w_1\|_{\tilde{H}^{1/2}(\Gamma_1^j)} \le cp^{-2\pi/\omega_j}. \tag{6.28}$$

By [22, Lemma 3.4] differentiation is a mapping

$$\tilde{H}^{1/2}(I_h) \to \tilde{H}^{-1/2}(I_h)$$

with norm being independent of h. Therefore, defining the polynomial $\phi_i = w_i' + \bar{\psi}$ of degree $p-1$ on Γ_i^j we obtain

$$\|\psi - \phi_i\|_{\tilde{H}^{-1/2}(\Gamma_i^j)} \le c\|u - w_i\|_{\tilde{H}^{1/2}(\Gamma_i^j)}.$$

Estimates (6.27) and (6.28) give the needed approximation result on a single element. We define a piecewise polynomial ϕ of degree $p-1$ on Γ by piecing together the local constructions and the upper bound for $\|\psi - \phi\|_{\tilde{H}^{-1/2}(\Gamma)}$ follows as in the proof of Theorem 6.1.

Now, to prove the lower bound for $\|\psi - \psi_p\|_{\tilde{H}^{-1/2}(\Gamma)}$, we assume that there exists a piecewise polynomial ϕ of degree $p-1$ such that

$$\|\psi - \phi\|_{\tilde{H}^{-1/2}(\Gamma)} \le cp^{-2\pi/\omega^*}\delta(p), \quad \delta(p) \to 0 \ (p \to \infty). \tag{6.29}$$

Analogously to (6.26) we define with $\bar{\psi} = \frac{1}{|\Gamma|}\int_\Gamma \psi\,ds$ the function

$$u(s) = \int_0^s (\psi - \bar{\psi})(t)\,dt,$$

s being the arc length of Γ, e.g. starting at the vertex t_1. By [14, Lemma 3] the antiderivative operator is continuous as a mapping from $H_0^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$ ($H_0^{-1/2}(\Gamma)$ is the space of $H^{-1/2}(\Gamma)$-functions with integral mean zero). Therefore, defining the piecewise polynomial $w(s) = \int_0^s (\phi(t) - \bar{\phi})\,dt$ of degree p (with $\bar{\phi} = \frac{1}{|\Gamma|}\int_\Gamma \phi\,ds$), we obtain

$$\begin{aligned} \|u - w\|_{\tilde{H}^{1/2}(\Gamma)} &\le c\|\psi - \bar{\psi} - (\phi - \bar{\phi})\|_{\tilde{H}^{-1/2}(\Gamma)} \\ &\le c\|\psi - \phi\|_{\tilde{H}^{-1/2}(\Gamma)} + c\frac{1}{|\Gamma|}\left|\int_\Gamma (\psi - \phi)\,ds\right| \|1\|_{\tilde{H}^{-1/2}(\Gamma)}. \\ &\le c(1 + \|1\|_{\tilde{H}^{-1/2}(\Gamma)}\frac{\|1\|_{\tilde{H}^{1/2}(\Gamma)}}{|\Gamma|})\|\psi - \phi\|_{\tilde{H}^{-1/2}(\Gamma)}. \end{aligned} \tag{6.30}$$

The function u being an antiderivative of ψ possesses corner singularities which are exactly of the type given in Proposition 6.1(ii). Without loss of generality we may assume that the most severe sigularity $|x-t_1|^{\pi/\omega^*}$ or $|x-t_1|^{\pi/\omega^*}\log|x-t_1|$ occurs at the vertex t_1 of Ω with $\omega_1=\omega^*=\max\{\omega_j;\ j=1,\ldots,J\}$. Further we assume that Γ_1^1 is the corner element at this vertex. Repeating the arguments of Theorem 6.2 we have

$$\|u-w\|_{H^{1/2}(\Gamma_1^1)}\leq cp^{-2\pi/\omega^*}\delta(p)$$

which contradicts Theorems 5.1 and 5.2. Thus, the assumption (6.29) does not hold and the proof of the theorem is finished. $\square$

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